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# BOOTSTRAPPING FORWARD SELECTION WITH BIC

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# BOOTSTRAPPING FORWARD SELECTION WITH BIC

by

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B.S., Southern Illinois University, 2016

A Research Paper
Submitted in Partial Fulfillment of the Requirements for the
Master of Science

Department of Mathematics
in the Graduate School
Southern Illinois University Carbondale
May, 2018

RESEARCH PAPER APPROVAL

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in the field of Mathematics

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April 3, 2018

AN ABSTRACT OF THE RESEARCH PAPER OF

CHARLES MURPHY, for the Master of Science degree in MATHEMATICS, presented on APRIL 3, 2018, at Southern Illinois University Carbondale.

TITLE: BOOTSTRAPPING FORWARD SELECTION WITH BIC

MAJOR PROFESSOR: Dr. David J. Olive

This paper presents a method for bootstrapping the multiple linear regression model $Y = \beta_1 + \beta_2 x_2 + \cdots + \beta_p x_p + e$ using forward selection with the BIC criterion.

KEY WORDS: Bootstrap; Confidence Region; Forward Selection; Prediction Interval.

ACKNOWLEDGMENTS

I would like to take this opportunity to thank my research advisor, Dr. David Olive for overseeing my Master's project and Dr. Michael Sullivan and Dr. Seyed Samadi for sitting on my committee. I would also like to thank all of the other professors who have encouraged and challenged me over the past 5.5 years here to be the best student I could be. Finally, I would like to thank my family for their neverending love and support. I certainly couldn't have achieved all that I have without them!

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# CHAPTER 1

# INTRODUCTION

Suppose that the response variable $Y_i$ and at least one predictor variable $x_{i,j}$ are quantitative with $x_{i,1} \equiv 1$. Let $\boldsymbol{x}_i^T = (x_{i,1}, ..., x_{i,p})$ and $\boldsymbol{\beta} = (\beta_1, ..., \beta_p)^T$ where $\beta_1$ corresponds to the intercept. Then the multiple linear regression (MLR) model is

$$Y_i = \beta_1 + x_{i,2}\beta_2 + \cdots + x_{i,p}\beta_p + e_i = \boldsymbol{x}_i^T \boldsymbol{\beta} + e_i \tag{1.1}$$

for $i = 1, ..., n$. This model is also called the full model. Here $n$ is the sample size, and assume that the random variables $e_i$ are independent and identically distributed (iid) with variance $V(e_i) = \sigma^2$. In matrix notation, these $n$ equations become

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{e}, \tag{1.2}$$

where $\boldsymbol{Y}$ is an $n \times 1$ vector of dependent variables, $\boldsymbol{X}$ is an $n \times p$ matrix of predictors, $\boldsymbol{\beta}$ is a $p \times 1$ vector of unknown coefficients, and $\boldsymbol{e}$ is an $n \times 1$ vector of unknown errors. The $i$th fitted value $\hat{Y}_i = \boldsymbol{x}_i^T \hat{\boldsymbol{\beta}}$ and the $i$th residual $r_i = Y_i - \hat{Y}_i$ where $\hat{\boldsymbol{\beta}}$ is an estimator of $\boldsymbol{\beta}$. Ordinary least squares (OLS) is often used for inference if $n/p$ is large.

Variable selection is the search for a subset of predictor variables that can be deleted without important loss of information. Following Olive and Hawkins (2005), a *model for variable selection* can be described by

$$\boldsymbol{x}^T \boldsymbol{\beta} = \boldsymbol{x}_S^T \boldsymbol{\beta}_S + \boldsymbol{x}_E^T \boldsymbol{\beta}_E = \boldsymbol{x}_S^T \boldsymbol{\beta}_S \tag{1.3}$$

where $\boldsymbol{x} = (\boldsymbol{x}_S^T, \boldsymbol{x}_E^T)^T$, $\boldsymbol{x}_S$ is an $a_S \times 1$ vector, and $\boldsymbol{x}_E$ is a $(p - a_S) \times 1$ vector. Given that $\boldsymbol{x}_S$ is in the model, $\boldsymbol{\beta}_E = \boldsymbol{0}$ and $E$ denotes the subset of terms that can be eliminated given that the subset $S$ is in the model. Let $\boldsymbol{x}_I$ be the vector of $a$ terms from a candidate subset indexed by $I$, and let $\boldsymbol{x}_O$ be the vector of the remaining predictors (out of the candidate submodel). Suppose that $S$ is a subset of $I$ and that model (1.3) holds. Then

$$\boldsymbol{x}^T \boldsymbol{\beta} = \boldsymbol{x}_S^T \boldsymbol{\beta}_S = \boldsymbol{x}_S^T \boldsymbol{\beta}_S + \boldsymbol{x}_{I/S}^T \boldsymbol{\beta}_{(I/S)} + \boldsymbol{x}_O^T \boldsymbol{0} = \boldsymbol{x}_I^T \boldsymbol{\beta}_I \tag{1.4}$$

where $\boldsymbol{x}_{I/S}$ denotes the predictors in $I$ that are not in $S$. Since this is true regardless of the values of the predictors, $\boldsymbol{\beta}_O = \boldsymbol{0}$ if $S \subseteq I$.

Forward selection forms a sequence of submodels $I_1, ..., I_M$ where $I_j$ uses $j$ predictors including the constant. Let $I_1$ use $x_1^* = x_1 \equiv 1$: the model has a constant but no nontrivial predictors. To form $I_2$, consider all models $I$ with two predictors including $x_1^*$. Compute $Q_2(I) = SSE(I) = RSS(I) = \boldsymbol{r}^T(I)\boldsymbol{r}(I) = \sum_{i=1}^n r_i^2(I) = \sum_{i=1}^n (Y_i - \hat{Y}_i(I))^2$. Let $I_2$ minimize $Q_2(I)$ for the $p-1$ models $I$ that contain $x_1^*$ and one other predictor. Denote the predictors in $I_2$ by $x_1^*, x_2^*$. In general, to form $I_j$ consider all models $I$ with $j$ predictors including variables $x_1^*, ..., x_{j-1}^*$. Compute $Q_j(I) = \boldsymbol{r}^T(I)\boldsymbol{r}(I) = \sum_{i=1}^n r_i^2(I) = \sum_{i=1}^n (Y_i - \hat{Y}_i(I))^2$. Let $I_j$ minimize $Q_j(I)$ for the $p-j+1$ models $I$ that contain $x_1^*, ..., x_{j-1}^*$ and one other predictor not already selected. Denote the predictors in $I_j$ by $x_1^*, ..., x_j^*$. Continue in this manner for $j = 2, ..., p$ where $n \geq 10p$ and $p$ is fixed.

When there is a sequence of $p$ submodels, the final submodel $I_d$ needs to be selected. Let the candidate model $I$ contains $a$ terms, including a constant. For example, let $\boldsymbol{x}_I$ and $\hat{\boldsymbol{\beta}}_I$ be $a \times 1$ vectors. Then there are many criteria used to select the final submodel $I_d$. For a given data set, $p, n$, and $\hat{\sigma}^2$ act as constants, and a criterion below may add a constant or be divided by a positive constant without changing the subset $I_{min}$ that minimizes the criterion.

Let criteria $C_S(I)$ have the form

$$C_S(I) = SSE(I) + aK_n\hat{\sigma}^2.$$

These criteria need a good estimator of $\sigma^2$. The criterion $C_p(I) = AIC_S(I)$ uses $K_n = 2$ while the $BIC_S(I)$ criterion uses $K_n = \log(n)$. Typically $\hat{\sigma}^2$ is the OLS full model

$$MSE = \sum_{i=1}^n \frac{r_i^2}{n-p}$$

when $n/p$ is large. Then $\hat{\sigma}^2 = MSE$ is a $\sqrt{n}$ consistent estimator of $\sigma^2$ under mild conditions by Su and Cook (2012).

The following criterion are described in Burnham and Anderson (2004), but still need $n/p$ large. $AIC$ is due to Akaike (1973) and $BIC$ to Schwarz (1978).

$$AIC(I) = n \log\left(\frac{SSE(I)}{n}\right) + 2a, \quad \text{and}$$

$$BIC(I) = n \log\left(\frac{SSE(I)}{n}\right) + a \log(n).$$

Let $I_{min}$ be the submodel that minimizes the criterion using variable selection with OLS. Following Nishi (1984), the probability that model $I_{min}$ from $C_p$ or $AIC$ underfits goes to zero as $n \to \infty$. If $\hat{\boldsymbol{\beta}}_I$ is $a \times 1$, form the $p \times 1$ vector $\hat{\boldsymbol{\beta}}_{I,0}$ from $\hat{\boldsymbol{\beta}}_I$ by adding 0s corresponding to the omitted variables. Since fewer than $2^p$ regression models $I$ contain the true model, and each such model gives a $\sqrt{n}$ consistent estimator $\hat{\boldsymbol{\beta}}_{I,0}$ of $\boldsymbol{\beta}$, the probability that $I_{min}$ picks one of these models goes to one as $n \to \infty$. Hence $\hat{\boldsymbol{\beta}}_{I_{min},0}$ is a $\sqrt{n}$ consistent estimator of $\boldsymbol{\beta}$ under model (1.3). See Pelawa Watagoda and Olive (2018) and Olive (2017a: p. 123, 2017b: p. 176).

Chapter 2 considers mixture distributions. Chapter 3 shows that a bootstrap confidence region can be formed by applying a prediction region to the bootstrap sample, and Chapter 4 gives a simulation.

# CHAPTER 2

## MIXTURE DISTRIBUTIONS

Mixture distributions are useful for variable selection since asymptotically $\hat{\boldsymbol{\beta}}_{I_{min},0}$ is a mixture distribution of $\hat{\boldsymbol{\beta}}_{I_j,0}$ where $S \subseteq I_j$. See Equation (1.3). A random vector $\boldsymbol{u}$ has a mixture distribution if $\boldsymbol{u}$ equals a random vector $\boldsymbol{u}_j$ with probability $\pi_j$ for $j = 1, ..., J$.

Definition 1. The distribution of a $g \times 1$ random vector $\boldsymbol{u}$ is a mixture distribution if the cumulative distribution function (cdf) of $\boldsymbol{u}$ is

$$F_{\boldsymbol{u}}(\boldsymbol{t}) = \sum_{j=1}^{J} \pi_j F_{\boldsymbol{u}_j}(\boldsymbol{t}) \tag{2.1}$$

where the probabilities $\pi_j$ satisfy $0 \le \pi_j \le 1$ and $\sum_{j=1}^{J} \pi_j = 1$, $J \ge 2$, and $F_{\boldsymbol{u}_j}(\boldsymbol{t})$ is the cdf of a $g \times 1$ random vector $\boldsymbol{u}_j$. Then $\boldsymbol{u}$ has a mixture distribution of the $\boldsymbol{u}_j$ with probabilities $\pi_j$.

Theorem 1. Suppose $E(h(\boldsymbol{u}))$ and the $E(h(\boldsymbol{u}_j))$ exist. Then

$$E(h(\boldsymbol{u})) = \sum_{j=1}^{J} \pi_j E[h(\boldsymbol{u}_j)]. \tag{2.2}$$

Hence

$$E(\boldsymbol{u}) = \sum_{j=1}^{J} \pi_j E[\boldsymbol{u}_j], \tag{2.3}$$

and $Cov(\boldsymbol{u}) = E(\boldsymbol{u}\boldsymbol{u}^T) - E(\boldsymbol{u})E(\boldsymbol{u}^T) = E(\boldsymbol{u}\boldsymbol{u}^T) - E(\boldsymbol{u})[E(\boldsymbol{u})]^T =$

$\sum_{j=1}^{J} \pi_j E[\boldsymbol{u}_j \boldsymbol{u}_j^T] - E(\boldsymbol{u})[E(\boldsymbol{u})]^T =$

$$\sum_{j=1}^{J} \pi_j Cov(\boldsymbol{u}_j) + \sum_{j=1}^{J} \pi_j E(\boldsymbol{u}_j)[E(\boldsymbol{u}_j)]^T - E(\boldsymbol{u})[E(\boldsymbol{u})]^T. \tag{2.4}$$

If $E(\boldsymbol{u}_j) = \boldsymbol{\theta}$ for $j = 1, ..., J$, then $E(\boldsymbol{u}) = \boldsymbol{\theta}$ and

$$Cov(\boldsymbol{u}) = \sum_{j=1}^{J} \pi_j Cov(\boldsymbol{u}_j).$$

This theorem is easy to prove if the $\boldsymbol{u}_j$ are continuous random vectors with (joint) probability density functions (pdfs) $f_{\boldsymbol{u}_j}(\boldsymbol{t})$. Then $\boldsymbol{u}$ is a continuous random vector with pdf

$$f_{\boldsymbol{u}}(\boldsymbol{t}) = \sum_{j=1}^{J} \pi_j f_{\boldsymbol{u}_j}(\boldsymbol{t}), \quad \text{and}$$

$$E(h(\boldsymbol{u})) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h(\boldsymbol{t}) f_{\boldsymbol{u}}(\boldsymbol{t}) d\boldsymbol{t} = \sum_{j=1}^{J} \pi_j \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h(\boldsymbol{t}) f_{\boldsymbol{u}_j}(\boldsymbol{t}) d\boldsymbol{t} = \sum_{j=1}^{J} \pi_j E[h(\boldsymbol{u}_j)]$$

where $E[h(\boldsymbol{u}_j)]$ is the expectation with respect to the random vector $\boldsymbol{u}_j$. Note that

$$E(\boldsymbol{u})[E(\boldsymbol{u})]^T = \sum_{j=1}^{J} \sum_{k=1}^{J} \pi_j \pi_k E(\boldsymbol{u}_j)[E(\boldsymbol{u}_k)]^T. \tag{2.5}$$

Definition 2. The *population mean* of a random $p \times 1$ vector $\boldsymbol{X} = (X_1, ..., X_p)^T$ is

$$E(\boldsymbol{X}) = (E(X_1), ..., E(X_p))^T$$

and the $p \times p$ *population covariance matrix*

$$Cov(\boldsymbol{X}) = E(\boldsymbol{X} - E(\boldsymbol{X}))(\boldsymbol{X} - E(\boldsymbol{X}))^T = (\sigma_{ij}).$$

That is, the $ij$ entry of $Cov(\boldsymbol{X})$ is $Cov(X_i, X_j) = \sigma_{ij}$.

Note that $Cov(\boldsymbol{X})$ is a symmetric positive semidefinite matrix. The following results are useful. If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are $p \times 1$ random vectors, $\boldsymbol{a}$ a conformable constant vector, and $\boldsymbol{A}$ and $\boldsymbol{B}$ are conformable constant matrices, then

$$E(\boldsymbol{a} + \boldsymbol{X}) = \boldsymbol{a} + E(\boldsymbol{X}) \ \text{ and } \ \mathrm{E}(\boldsymbol{X} + \boldsymbol{Y}) = \mathrm{E}(\boldsymbol{X}) + \mathrm{E}(\boldsymbol{Y}) \tag{2.6}$$

and

$$E(\boldsymbol{A}\boldsymbol{X}) = \boldsymbol{A}E(\boldsymbol{X}) \ \text{ and } \ \mathrm{E}(\boldsymbol{A}\boldsymbol{X}\boldsymbol{B}) = \boldsymbol{A}\mathrm{E}(\boldsymbol{X})\boldsymbol{B}. \tag{2.7}$$

Thus

$$Cov(\boldsymbol{a} + \boldsymbol{A}\boldsymbol{X}) = Cov(\boldsymbol{A}\boldsymbol{X}) = \boldsymbol{A}Cov(\boldsymbol{X})\boldsymbol{A}^T. \tag{2.8}$$

For the multivariate normal (MVN) distribution $\boldsymbol{X} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Then $E(\boldsymbol{X}) = \boldsymbol{\mu}$ and

$$Cov(\boldsymbol{X}) = \boldsymbol{\Sigma}.$$

# CHAPTER 3

## BOOTSTRAPPING CONFIDENCE REGIONS

Inference will consider bootstrap confidence intervals and bootstrap confidence regions for bootstrap hypothesis testing. Applying the shorth prediction interval and the Olive (2013) prediction region to the bootstrap sample will give the bootstrap confidence intervals and regions.

Consider predicting a future test random variable $Z_f$ given iid training data $Z_1, ..., Z_n$. A large sample $100(1-\delta)\%$ prediction interval (PI) for $Z_f$ has the form $[\hat{L}_n, \hat{U}_n]$ where $P(\hat{L}_n \leq Z_f \leq \hat{U}_n) \to 1-\delta$ as the sample size $n \to \infty$. The shorth($c$) estimator is useful for making prediction intervals. Let $Z_{(1)}, ..., Z_{(n)}$ be the order statistics of $Z_1, ..., Z_n$. Then let the shortest closed interval containing at least $c$ of the $Z_i$ be

$$\mathrm{shorth(c)} = [\mathrm{Z_{(s)}}, \mathrm{Z_{(s+c-1)}}]. \tag{3.1}$$

Let $\lceil x \rceil$ be the smallest integer $\geq x$, e.g., $\lceil 7.7 \rceil = 8$. Let

$$k_n = \lceil n(1-\delta) \rceil. \tag{3.2}$$

Frey (2013) showed that for large $n\delta$ and iid data, the shorth($k_n$) PI has maximum undercoverage $\approx 1.12\sqrt{\delta/n}$, and used the shorth($c$) estimator as the large sample $100(1-\delta)\%$ PI where

$$c = \min(n, \lceil n[1-\delta + 1.12\sqrt{\delta/n}\ ]\ \rceil). \tag{3.3}$$

Example 1. Given below were votes for preseason 1A basketball poll from Nov. 22, 2011 WSIL News where the 778 was a typo: the actual value was 78. As shown below, finding shorth(3) from the ordered data is simple. If the outlier was corrected, shorth(3) = [76,78].

```
111   89   778   78   76
```

```
order data: 76 78 89 111 778

                 13 = 89 - 76

                    33 = 111 - 78

                       689 = 778 - 89
shorth(3) = [76,89]
```

We also want to use bootstrap tests. Consider testing $H_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ versus $H_1 : \boldsymbol{\theta} \neq \boldsymbol{\theta}_0$ where $\boldsymbol{\theta}_0$ is a known $g \times 1$ vector. Given training data $\boldsymbol{z}_1, ..., \boldsymbol{z}_n$, a large sample $100(1-\delta)\%$ confidence region for $\boldsymbol{\theta}$ is a set $\mathcal{A}_n$ such that $P(\boldsymbol{\theta} \in \mathcal{A}_n) \to 1 - \delta$ as $n \to \infty$. Then reject $H_0$ if $\boldsymbol{\theta}_0$ is not in the confidence region $\mathcal{A}_n$. For model (1.1), let $\boldsymbol{\theta} = \boldsymbol{A}\boldsymbol{\beta}$ where $\boldsymbol{A}$ is a known full rank $g \times p$ matrix with $1 \leq g \leq p$.

To bootstrap a confidence region, Mahalanobis distances and prediction regions will be useful. Consider predicting a future test value $\boldsymbol{z}_f$, given past training data $\boldsymbol{z}_1, ..., \boldsymbol{z}_n$ where the $\boldsymbol{z}_i$ are $g \times 1$ random vectors. A *large sample* $100(1-\delta)\%$ *prediction region* is a set $\mathcal{A}_n$ such that $P(\boldsymbol{z}_f \in \mathcal{A}_n) \to 1 - \delta$ as $n \to \infty$. Let the $g \times 1$ column vector $T$ be a multivariate location estimator, and let the $g \times g$ symmetric positive definite matrix $\boldsymbol{C}$ be a dispersion estimator. Then the $i$th *squared sample Mahalanobis distance* is the scalar

$$D_i^2 = D_i^2(T, \boldsymbol{C}) = D^2_{\boldsymbol{z}_i}(T, \boldsymbol{C}) = (\boldsymbol{z}_i - T)^T \boldsymbol{C}^{-1}(\boldsymbol{z}_i - T) \tag{3.4}$$

for each observation $\boldsymbol{z}_i$. Notice that the Euclidean distance of $\boldsymbol{z}_i$ from the estimate of center $T$ is $D_i(T, \boldsymbol{I}_g)$ where $\boldsymbol{I}_g$ is the $g \times g$ identity matrix. The classical Mahalanobis distance $D_i$ uses $(T, \boldsymbol{C}) = (\overline{\boldsymbol{z}}, \boldsymbol{S})$, the sample mean and sample covariance matrix where

$$\overline{\boldsymbol{z}} = \frac{1}{n}\sum_{i=1}^n \boldsymbol{z}_i \quad \text{and} \quad \boldsymbol{S} = \frac{1}{\mathrm{n}-1}\sum_{\mathrm{i}=1}^{\mathrm{n}} (\boldsymbol{z}_\mathrm{i} - \overline{\boldsymbol{z}})(\boldsymbol{z}_\mathrm{i} - \overline{\boldsymbol{z}})^\mathrm{T}. \tag{3.5}$$

Let $q_n = \min(1 - \delta + 0.05, 1 - \delta + g/n)$ for $\delta > 0.1$ and

$$q_n = \min(1 - \delta/2, 1 - \delta + 10\delta g/n), \quad \text{otherwise.} \tag{3.6}$$

If $1-\delta < 0.999$ and $q_n < 1-\delta+0.001$, set $q_n = 1-\delta$. Let

$$c = \lceil n q_n \rceil. \tag{3.7}$$

Let $(T, \boldsymbol{C}) = (\overline{\boldsymbol{z}}, \boldsymbol{S})$, and let $D_{(U_n)}$ be the $100q_n$th sample quantile of the $D_i$. Then the Olive (2013) large sample $100(1-\delta)\%$ nonparametric prediction region for a future value $\boldsymbol{z}_f$ given iid data $\boldsymbol{z}_1, ..., , \boldsymbol{z}_n$ is

$$\{\boldsymbol{z} : D^2_{\boldsymbol{z}}(\overline{\boldsymbol{z}}, \boldsymbol{S}) \leq D^2_{(U_n)}\}, \tag{3.8}$$

while the classical large sample $100(1-\delta)\%$ prediction region is

$$\{\boldsymbol{z} : D^2_{\boldsymbol{z}}(\overline{\boldsymbol{z}}, \boldsymbol{S}) \leq \chi^2_{g,1-\delta}\}. \tag{3.9}$$

Definition 3. Suppose that data $\boldsymbol{x}_1, ..., \boldsymbol{x}_n$ has been collected and observed. Often the data is a random sample (iid) from a distribution with cdf $F$. The *empirical distribution* is a discrete distribution where the $\boldsymbol{x}_i$ are the possible values, and each value is equally likely. If $\boldsymbol{w}$ is a random variable having the empirical distribution, then $p_i = P(\boldsymbol{w} = \boldsymbol{x}_i) = 1/n$ for $i = 1, ..., n$. The *cdf of the empirical distribution* is denoted by $F_n$.

Example 2. Let $\boldsymbol{w}$ be a random variable having the empirical distribution given by Definition 3. Show that $E(\boldsymbol{w}) = \overline{\boldsymbol{x}} \equiv \overline{\boldsymbol{x}}_n$ and $Cov(\boldsymbol{w}) = \dfrac{n-1}{n}\boldsymbol{S} \equiv \dfrac{n-1}{n}\boldsymbol{S}_n$.

Solution: Recall that for a discrete random vector, the population expected value $E(\boldsymbol{w}) = \sum \boldsymbol{x}_i p_i$ where $\boldsymbol{x}_i$ are the values that $\boldsymbol{w}$ takes with positive probability $p_i$. Similarly, the population covariance matrix

$$Cov(\boldsymbol{w}) = E[(\boldsymbol{w} - E(\boldsymbol{w}))(\boldsymbol{w} - E(\boldsymbol{w}))^T] = \sum (\boldsymbol{x}_i - E(\boldsymbol{w}))(\boldsymbol{x}_i - E(\boldsymbol{w}))^T p_i.$$

Hence

$$E(\boldsymbol{w}) = \sum_{i=1}^n \boldsymbol{x}_i \frac{1}{n} = \overline{\boldsymbol{x}},$$

and

$$Cov(\boldsymbol{w}) = \sum_{i=1}^n (\boldsymbol{x}_i - \overline{\boldsymbol{x}})(\boldsymbol{x}_i - \overline{\boldsymbol{x}})^T \frac{1}{n} = \frac{n-1}{n}\boldsymbol{S}. \ \square$$

Example 3. If $W_1, ..., W_n$ are iid from a distribution with cdf $F_W$, then the empirical cdf $F_n$ corresponding to $F_W$ is given by

$$F_n(y) = \frac{1}{n} \sum_{i=1}^{n} I(W_i \leq y)$$

where the indicator $I(W_i \leq y) = 1$ if $W_i \leq y$ and $I(W_i \leq y) = 0$ if $W_i > y$. Fix $n$ and $y$. Then $nF_n(y) \sim$ binomial $(n, F_W(y))$. Thus $E[F_n(y)] = F_W(y)$ and $V[F_n(y)] = F_W(y)[1 - F_W(y)]/n$. By the central limit theorem,

$$\sqrt{n}(F_n(y) - F_W(y)) \xrightarrow{D} N(0, F_W(y)[1 - F_W(y)]).$$

Thus $F_n(y) - F_W(y) = O_P(n^{-1/2})$, and $F_n$ is a reasonable estimator of $F_W$ if the sample size $n$ is large.

Suppose there is data $\boldsymbol{w}_1, ..., \boldsymbol{w}_n$ collected into an $n \times p$ matrix $\boldsymbol{W}$. Let the statistic $T_n = t(\boldsymbol{W}) = T(F_n)$ be computed from the data. Suppose the statistic estimates $\boldsymbol{\theta} = T(F)$, and let $t(\boldsymbol{W}^*) = t(F_n^*) = T_n^*$ indicate that $t$ was computed from an iid sample from the empirical distribution $F_n$: a sample $\boldsymbol{w}_1^*, ..., \boldsymbol{w}_n^*$ of size $n$ was drawn with replacement from the observed sample $\boldsymbol{w}_1, ..., \boldsymbol{w}_n$. This notation is used for von Mises differentiable statistical functions in large sample theory. See Serfling (1980, ch. 6). The *empirical bootstrap* or *nonparametric bootstrap* or *naive bootstrap* draws $B$ samples of size $n$ from the rows of $\boldsymbol{W}$, e.g. from the empirical distribution of $\boldsymbol{w}_1, ..., \boldsymbol{w}_n$. Then $T_{jn}^*$ is computed from the $j$th bootstrap sample for $j = 1, ..., B$.

Example 4. Suppose the data is 1, 2, 3, 4, 5, 6, 7. Then $n = 7$ and the sample median $T_n$ is 4. Using $R$, we drew $B = 2$ bootstrap samples (samples of size $n$ drawn with replacement from the original data) and computed the sample median $T_{1,n}^* = 3$ and $T_{2,n}^* = 4$.

```
b1 <- sample(1:7,replace=T)

b1

[1] 3 2 3 2 5 2 6
```

```
median(b1)
[1] 3
b2 <- sample(1:7,replace=T)
b2
[1] 3 5 3 4 3 5 7
median(b2)
[1] 4
```

The bootstrap has been widely used to estimate the population covariance matrix of the statistic $Cov(T_n)$, for testing hypotheses, and for obtaining confidence regions (often confidence intervals). An iid sample $T_{1n}, ..., T_{Bn}$ of size $B$ of the statistic would be very useful for inference, but typically we only have one sample of data and one value $T_n = T_{1n}$ of the statistic. Often $T_n = t(\boldsymbol{w}_1, ..., \boldsymbol{w}_n)$, and the bootstrap sample $T^*_{1n}, ..., T^*_{Bn}$ is formed where $T^*_{jn} = t(\boldsymbol{w}^*_{j1}, ..., \boldsymbol{w}^*_{jn})$. The *residual bootstrap* is often useful for additive error regression models of the form $Y_i = m(\boldsymbol{x}_i) + e_i = \hat{m}(\boldsymbol{x}_i) + r_i = \hat{Y}_i + r_i$ for $i = 1, ..., n$ where the $i$th residual $r_i = Y_i - \hat{Y}_i$. Let $\boldsymbol{Y} = (Y_1, ..., Y_n)^T$, $\boldsymbol{r} = (r_1, ..., r_n)^T$, and let $\boldsymbol{X}$ be an $n \times p$ matrix with $i$th row $\boldsymbol{x}_i^T$. Then the fitted values $\hat{Y}_i = \hat{m}(\boldsymbol{x}_i)$, and the residuals are obtained by regressing $\boldsymbol{Y}$ on $\boldsymbol{X}$. Here the errors $e_i$ are iid, and it would be useful to be able to generate $B$ iid samples $e_{1j}, ..., e_{nj}$ from the distribution of $e_i$ where $j = 1, ..., B$. If the $m(\boldsymbol{x}_i)$ were known, then we could form a vector $\boldsymbol{Y}_j$ where the $i$th element $Y_{ij} = m(\boldsymbol{x}_i) + e_{ij}$ for $i = 1, ..., n$. Then regress $\boldsymbol{Y}_j$ on $\boldsymbol{X}$. Instead, draw samples $r^*_{1j}, ..., r^*_{nj}$ with replacement from the residuals, then form a vector $\boldsymbol{Y}^*_j$ where the $i$th element $Y^*_{ij} = \hat{m}(\boldsymbol{x}_i) + r^*_{ij}$ for $i = 1, ..., n$. Then regress $\boldsymbol{Y}^*_j$ on $\boldsymbol{X}$.

The Olive (2017ab, 2018ab) prediction region method obtains a confidence region for $\boldsymbol{\theta}$ by applying the nonparametric prediction region (3.8) to the bootstrap sample $T^*_1, ..., T^*_B$, and the theory for the method is sketched below. Let $\overline{T}^*$ and $\boldsymbol{S}^*_T$ be the sample mean and sample covariance matrix of the bootstrap sample. Assume $n\boldsymbol{S}^*_T \xrightarrow{P} \boldsymbol{\Sigma}_A$. See Machado and Parente (2005) for regularity conditions for this assumption. Following Bickel and Ren

(2001), let the vector of parameters $\boldsymbol{\theta} = T(F)$, the statistic $T_n = T(F_n)$, and $T^* = T(F_n^*)$ where $F$ is the cdf of iid $\boldsymbol{x}_1, ..., \boldsymbol{x}_n$, $F_n$ is the empirical cdf, and $F_n^*$ is the empirical cdf of $\boldsymbol{x}_1^*, ..., \boldsymbol{x}_n^*$, a sample from $F_n$ using the nonparametric bootstrap. If $\sqrt{n}(F_n - F) \xrightarrow{D} \boldsymbol{z}_F$, a Gaussian random process, and if $T$ is sufficiently smooth (has a Hadamard derivative $\dot{T}(F)$), then $\sqrt{n}(T_n - \boldsymbol{\theta}) \xrightarrow{D} \boldsymbol{u}$ and $\sqrt{n}(T_i^* - T_n) \xrightarrow{D} \boldsymbol{u}$ with $\boldsymbol{u} = \dot{T}(F)\boldsymbol{z}_F$. Olive (2017b) used these results to show that if $\boldsymbol{u} \sim N_g(\boldsymbol{0}, \boldsymbol{\Sigma}_A)$, then $\sqrt{n}(\overline{T}^* - T_n) \xrightarrow{D} \boldsymbol{0}$, $\sqrt{n}(T_i^* - \overline{T}^*) \xrightarrow{D} \boldsymbol{u}$, $\sqrt{n}(\overline{T}^* - \boldsymbol{\theta}) \xrightarrow{D} \boldsymbol{u}$, and that the prediction region method large sample $100(1-\delta)\%$ confidence region for $\boldsymbol{\theta}$ is

$$\{\boldsymbol{w} : (\boldsymbol{w} - \overline{T}^*)^T [\boldsymbol{S}_T^*]^{-1} (\boldsymbol{w} - \overline{T}^*) \leq D^2_{(U_B)}\} = \{\boldsymbol{w} : D^2_{\boldsymbol{w}}(\overline{T}^*, \boldsymbol{S}_T^*) \leq D^2_{(U_B)}\} \tag{3.10}$$

where $D^2_{(U_B)}$ is computed from $D_i^2 = (T_i^* - \overline{T}^*)^T [\boldsymbol{S}_T^*]^{-1} (T_i^* - \overline{T}^*)$ for $i = 1, ..., B$. Note that the corresponding test for $H_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ rejects $H_0$ if $(\overline{T}^* - \boldsymbol{\theta}_0)^T [\boldsymbol{S}_T^*]^{-1} (\overline{T}^* - \boldsymbol{\theta}_0) > D^2_{(U_B)}$. The prediction region method for testing $H_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ versus $H_1 : \boldsymbol{\theta} \neq \boldsymbol{\theta}_0$ is simple. Let $\hat{\boldsymbol{\theta}}$ be a consistent estimator of $\boldsymbol{\theta}$ and make a bootstrap sample $\boldsymbol{w}_i = \hat{\boldsymbol{\theta}}_i^* - \boldsymbol{\theta}_0$ for $i = 1, ..., B$. Make the nonparametric prediction region (3.10) for the $\boldsymbol{w}_i$ and fail to reject $H_0$ if $\boldsymbol{0}$ is in the prediction region (if $D_{\boldsymbol{0}} \leq D_{(U_B)}$), reject $H_0$ otherwise.

The modified Bickel and Ren (2001) large sample $100(1-\delta)\%$ confidence region is

$$\{\boldsymbol{w} : (\boldsymbol{w} - T)^T [\boldsymbol{S}_T^*]^{-1} (\boldsymbol{w} - T_n) \leq D^2_{(U_B,T)}\} = \{\boldsymbol{w} : D^2_{\boldsymbol{w}}(T_n, \boldsymbol{S}_T^*) \leq D^2_{(U_B,T)}\} \tag{3.11}$$

where $D^2_{(U_B,T)}$ is computed from $D_i^2 = (T_i^* - T_n)^T [\boldsymbol{S}_T^*]^{-1} (T_i^* - T_n)$.

The Pelawa Watagoda and Olive (2018) hybrid large sample $100(1-\delta)\%$ confidence region shifts the hyperellipsoid (3.10) to be centered at $T$ instead of $\overline{T}^*$:

$$\{\boldsymbol{w} : (\boldsymbol{w} - T_n)^T [\boldsymbol{S}_T^*]^{-1} (\boldsymbol{w} - T_n) \leq D^2_{(U_B)}\} = \{\boldsymbol{w} : D^2_{\boldsymbol{w}}(T_n, \boldsymbol{S}_T^*) \leq D^2_{(U_B)}\}. \tag{3.12}$$

Hyperellipsoids (3.10) and (3.12) have the same volume since they are the same region shifted to have a different center. The ratio of the volumes of regions (3.10) and (3.11) is

$$\left( \frac{D_{(U_B)}}{D_{(U_B,T)}} \right)^g . \tag{3.13}$$

Consider testing $H_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ versus $H_0 : \boldsymbol{\theta} \neq \boldsymbol{\theta}_0$ where $\boldsymbol{\theta}$ is $g \times 1$. For example, let $\boldsymbol{A}$ be a $g \times p$ matrix with full rank $g$, $\boldsymbol{\theta} = \boldsymbol{A\beta}$, $\boldsymbol{\theta}_0 = \boldsymbol{0}$, and $T_n = \boldsymbol{A}\hat{\boldsymbol{\beta}}_{I_{min},0}$. This section gives some theory for the bagging estimator $\overline{T}^*$, also called the smoothed bootstrap estimator. The theory may be useful for hypothesis testing after model selection if $n/p$ is large. Empirically, bootstrapping with the bagging estimator often outperforms bootstrapping with $T_n$. See Efron (2014). See Büchlmann and Yu (2002) and Friedman and Hall (2007) for theory and references for the bagging estimator.

If i) $\sqrt{n}(T_n - \boldsymbol{\theta}) \xrightarrow{D} \boldsymbol{u}$, then under regularity conditions, ii) $\sqrt{n}(T_i^* - T_n) \xrightarrow{D} \boldsymbol{u}$, iii) $\sqrt{n}(\overline{T}^* - \boldsymbol{\theta}) \xrightarrow{D} \boldsymbol{u}$, iv) $\sqrt{n}(T_i^* - \overline{T}^*) \xrightarrow{D} \boldsymbol{u}$, and v) $n\boldsymbol{S}_T^* \xrightarrow{P} Cov(\boldsymbol{u})$.

Suppose i) and ii) hold with $E(\boldsymbol{u}) = \boldsymbol{0}$ and $Cov(\boldsymbol{u}) = \boldsymbol{\Sigma}_{\boldsymbol{u}}$. With respect to the bootstrap sample, $T_n$ is a constant and the $\sqrt{n}(T_i^* - T_n)$ are iid for $i = 1, ..., B$. Let $\sqrt{n}(T_i^* - T_n) \xrightarrow{D} \boldsymbol{v}_i \sim \boldsymbol{u}$ where the $\boldsymbol{v}_i$ are iid with the same distribution as $\boldsymbol{u}$. Fix $B$. Then the average of the $\sqrt{n}(T_i^* - T_n)$ is

$$\sqrt{n}(\overline{T}^* - T_n) \xrightarrow{D} \frac{1}{B}\sum_{i=1}^{B} \boldsymbol{v}_i \sim AN_g\left(\boldsymbol{0}, \frac{\boldsymbol{\Sigma}_{\boldsymbol{u}}}{B}\right)$$

where $\boldsymbol{z} \sim AN_g(\boldsymbol{0}, \boldsymbol{\Sigma})$ is an asymptotic multivariate normal approximation. Hence as $B \to \infty$, $\sqrt{n}(\overline{T}^* - T_n) \xrightarrow{P} \boldsymbol{0}$, and iii) and iv) hold. If $B$ is fixed and $\boldsymbol{u} \sim N_g(\boldsymbol{0}, \boldsymbol{\Sigma}_{\boldsymbol{u}})$, then

$$\frac{1}{B}\sum_{i=1}^{B} \boldsymbol{v}_i \sim N_g\left(\boldsymbol{0}, \frac{\boldsymbol{\Sigma}_{\boldsymbol{u}}}{B}\right) \text{ and } \sqrt{\mathrm{B}}\sqrt{\mathrm{n}}(\overline{\mathrm{T}}^* - \mathrm{T_n}) \xrightarrow{\mathrm{D}} \mathrm{N_g}(\boldsymbol{0}, \boldsymbol{\Sigma}_{\boldsymbol{u}}).$$

Hence the prediction region method gives a large sample confidence region for $\boldsymbol{\theta}$ provided that the sample percentile $\hat{D}^2_{1-\delta}$ of the $D^2_{T_i^*}(\overline{T}^*, \boldsymbol{S}_T^*) = \sqrt{n}(T_i^* - \overline{T}^*)^T (n\boldsymbol{S}_T^*)^{-1} \sqrt{n}(T_i^* - \overline{T}^*)$ is a consistent estimator of the percentile $D^2_{n,1-\delta}$ of the random variable $D^2_{\boldsymbol{\theta}}(\overline{T}^*, \boldsymbol{S}_T^*) = \sqrt{n}(\boldsymbol{\theta} - \overline{T}^*)^T (n\boldsymbol{S}_T^*)^{-1} \sqrt{n}(\boldsymbol{\theta} - \overline{T}^*)$ in that $\hat{D}^2_{1-\delta} - D^2_{n,1-\delta} \xrightarrow{P} 0$. Since iii) and iv) hold, the sample percentile will be consistent under much weaker conditions than v) if $\boldsymbol{\Sigma}_{\boldsymbol{u}}$ is nonsingular. For example, if $(n\boldsymbol{S}_T^*)^{-1} = \boldsymbol{\Sigma}_{\boldsymbol{u}}^{-1} + \boldsymbol{C} + o_p(1)$ for some $g \times g$ constant matrix $\boldsymbol{C}$. Olive (2017b $\oint$ 5.3.3) proved that the prediction region method gives a large sample confidence region under the much stronger conditions of v) and $\boldsymbol{u} \sim N_g(\boldsymbol{0}, \boldsymbol{\Sigma}_{\boldsymbol{u}})$, but the above proof is simpler.

Now suppose that $T_n$ is equal to the estimator $T_{jn}$ with probability $\pi_{jn}$ for $j = 1, ..., J$ where $\sum_j \pi_{jn} = 1$, $\pi_{jn} \to \pi_j$ as $n \to \infty$, and $\sqrt{n}(T_{jn} - \boldsymbol{\theta}) \xrightarrow{D} \boldsymbol{u}_j$ with $E(\boldsymbol{u}_j) = \boldsymbol{0}$ and $Cov(\boldsymbol{u}_j) = \boldsymbol{\Sigma}_j$. Then the cumulative distribution function (cdf) of $T_n$ is $F_{T_n}(\boldsymbol{z}) = \sum_j \pi_{jn} F_{T_{jn}}(\boldsymbol{z})$ where $F_{T_{jn}}(\boldsymbol{z})$ is the cdf of $T_{jn}$. Hence

$$\sqrt{n}(T_n - \boldsymbol{\theta}) \xrightarrow{D} \boldsymbol{u} \tag{3.14}$$

where the cdf of $\boldsymbol{u}$ is $F_{\boldsymbol{u}}(\boldsymbol{z}) = \sum_j \pi_j F_{\boldsymbol{u}_j}(\boldsymbol{z})$ and $F_{\boldsymbol{u}_j}(\boldsymbol{z})$ is the cdf of $\boldsymbol{u}_j$. Thus $\boldsymbol{u}$ is a mixture distribution of the $\boldsymbol{u}_j$ with probabilities $\pi_j$, $E(\boldsymbol{u}) = \boldsymbol{0}$, and $Cov(\boldsymbol{u}) = \boldsymbol{\Sigma}_{\boldsymbol{u}} = \sum_j \pi_j \boldsymbol{\Sigma}_j$.

For the bootstrap, suppose that $T_i^*$ is equal to $T_{ij}^*$ with probability $\rho_{jn}$ for $j = 1, ..., J$ where $\sum_j \rho_{jn} = 1$, and $\rho_{jn} \to \pi_j$ as $n \to \infty$. Let $B_{jn}$ count the number of times $T_i^* = T_{ij}^*$ in the bootstrap sample. Then the bootstrap sample $T_1^*, ..., T_B^*$ can be written as

$$T_{1,1}^*, ..., T_{B_{1n},1}^*, ..., T_{1,J}^*, ..., T_{B_{Jn},J}^*$$

where the $B_{jn}$ follow a multinomial distribution and $B_{jn}/B \xrightarrow{P} \rho_{jn}$ as $B \to \infty$. Conditionally on the $B_{jn}$ and with respect to the bootstrap sample, the $T_{ij}^*$ are independent. Denote $T_{1j}^*, ..., T_{B_{jn},j}^*$ as the $j$th bootstrap component of the bootstrap sample with sample mean $\overline{T}_j^*$ and sample covariance matrix $\boldsymbol{S}_{T,j}^*$. Then

$$\overline{T}^* = \frac{1}{B} \sum_{i=1}^{B} T_i^* = \sum_j \frac{B_{jn}}{B} \frac{1}{B_{jn}} \sum_{i=1}^{B_{jn}} T_{ij}^* = \sum_j \hat{\rho}_{jn} \overline{T}_j^*.$$

Suppose $\sqrt{n}(T_i^* - E(T^*)) \xrightarrow{D} \boldsymbol{v}_i \sim \boldsymbol{v}$ where $E(\boldsymbol{v}) = \boldsymbol{0}$, $Cov(\boldsymbol{v}) = \boldsymbol{\Sigma}_{\boldsymbol{v}}$, and $E(T^*) = \sum_j \rho_{jn} E(T_{ij}^*)$ where often $E(T_{ij}^*) = T_{jn}$. With respect to the data distribution, suppose $\sqrt{n}(E(T^*) - \boldsymbol{\theta}) \xrightarrow{D} \boldsymbol{w}$. Then by an argument similar to the one given for when $T_n$ is not from a mixture distribution, $\sqrt{n}(\overline{T}^* - E(T^*)) \xrightarrow{P} \boldsymbol{0}$, $\sqrt{n}(T_i^* - \overline{T}^*) \xrightarrow{D} \boldsymbol{v}$, and $\sqrt{n}(\overline{T}^* - \boldsymbol{\theta}) \xrightarrow{D} \boldsymbol{w}$.

Assume $T_1, ..., T_B$ are iid with nonsingular covariance matrix $\boldsymbol{\Sigma}_{T_n}$. Then the large sample $100(1-\delta)\%$ prediction region $R_p = \{\boldsymbol{w} : D_{\boldsymbol{w}}^2(\overline{T}, \boldsymbol{S}_T) \le \hat{D}_{(U_B)}^2\}$ centered at $\overline{T}$

contains a future value of the statistic $T_f$ with probability $1-\delta_B \to 1-\delta$ as $B \to \infty$. Hence the region $R_c = \{\boldsymbol{w} : D^2_{\boldsymbol{w}}(T_n, \boldsymbol{S}_T) \le \hat{D}^2_{(U_B)}\}$ centered at a randomly selected $T_n$ contains $\overline{T}$ with probability $1-\delta_B$. If i) holds with $E(\boldsymbol{u}) = \boldsymbol{0}$ and $Cov(\boldsymbol{u}) = \boldsymbol{\Sigma_u}$, then for fixed $B$,

$$\sqrt{n}(\overline{T} - \boldsymbol{\theta}) \xrightarrow{D} \frac{1}{B}\sum_{i=1}^{B} \boldsymbol{v}_i \sim AN_g\left(\boldsymbol{0}, \frac{\boldsymbol{\Sigma_u}}{B}\right).$$

Hence $(\overline{T} - \boldsymbol{\theta}) = O_P((nB)^{-1/2})$, and $\overline{T}$ gets arbitrarily close to $\boldsymbol{\theta}$ compared to $T_n$ as $B \to \infty$. Hence $R_c$ is a large sample $100(1-\delta)\%$ confidence region for $\boldsymbol{\theta}$ as $n, B \to \infty$. We also need $(n\boldsymbol{S}_T)^{-1}$ to be "not too ill conditioned."

With a mixture distribution, the bootstrap sample shifts the data cloud to be centered at $\overline{T}^*$ where $\sqrt{n}(\overline{T}^* - \sum_j \rho_{jn} T_{jn}) \xrightarrow{P} \boldsymbol{0}$. The $T_{jn}$ are computed from the same data set and hence correlated. Suppose $\sqrt{n}(T_n - \boldsymbol{\theta}) \xrightarrow{D} \boldsymbol{u}$, $\sqrt{n}(\overline{T}^* - \boldsymbol{\theta}) \xrightarrow{D} \boldsymbol{w}$, and $(n\boldsymbol{S}^*_T)^{-1}$ is "not too ill conditioned." Then

$$D_1^2 = D^2_{T_i^*}(\overline{T}^*, \boldsymbol{S}^*_T) = \sqrt{n}(T_i^* - \overline{T}^*)^T (n\boldsymbol{S}^*_T)^{-1} \sqrt{n}(T_i^* - \overline{T}^*),$$

$$D_2^2 = D^2_{\boldsymbol{\theta}}(T_n, \boldsymbol{S}^*_T) = \sqrt{n}(T_n - \boldsymbol{\theta})^T (n\boldsymbol{S}^*_T)^{-1} \sqrt{n}(T_n - \boldsymbol{\theta}), \quad \text{and}$$

$$D_3^2 = D^2_{\boldsymbol{\theta}}(\overline{T}^*, \boldsymbol{S}^*_T) = \sqrt{n}(\overline{T}^* - \boldsymbol{\theta})^T (n\boldsymbol{S}^*_T)^{-1} \sqrt{n}(\overline{T}^* - \boldsymbol{\theta})$$

are well behaved in that there exist cutoffs $\hat{D}^2_{i,1-\delta}$ that would result in good confidence regions for $i = 2$ and 3. Heuristically, for a mixture distribution, the deviation $\overline{T}^* - \boldsymbol{\theta}$ tends to be smaller on average than the deviations $T_n - \boldsymbol{\theta} \approx T_i^* - \overline{T}^*$, while the deviation $T_i^* - T_n$ tends to be larger than the other three deviations, on average. Hence $\hat{D}^2_{2,1-\delta} = D^2_{(U_B)}$ gives coverage close to the nominal coverage for prediction region (3.12), but cutoffs $\hat{D}^2_{3,1-\delta} = D^2_{(U_B)}$ and $\hat{D}^2_{2,1-\delta} = D^2_{(U_B,T)}$ are slightly too large, and prediction regions (3.10) and (3.11) tend to have coverage slightly higher than the nominal coverage $1-\delta$ if $n$ and $B$ are large. In simulations for $n \ge 20p$, the coverage tends to get close to $1-\delta$ for $B \ge \max(400, 50p)$ so that $\boldsymbol{S}^*_T$ is a good estimator of $Cov(T^*)$.

To examine the bagging estimator, assume that each bootstrap component satisfies vi) $\sqrt{n}(T_{jn} - \boldsymbol{\theta}) \xrightarrow{D} \boldsymbol{u}_j \sim N_g(\boldsymbol{0}, \boldsymbol{\Sigma}_j)$, vii) $\sqrt{n}(T^*_{ij} - T_{jn}) \xrightarrow{D} \boldsymbol{u}_j$, viii) $\sqrt{n}(\overline{T}^*_j - \boldsymbol{\theta}) \xrightarrow{D} \boldsymbol{u}_j$, ix) $\sqrt{n}(T^*_{ij} - \overline{T}^*_j) \xrightarrow{D} \boldsymbol{u}_j$, x) $n\boldsymbol{S}^*_{T,j} \xrightarrow{P} \boldsymbol{\Sigma}_j$, and xi) $\sqrt{n}(T_{jn} - \overline{T}^*_j) \xrightarrow{P} \boldsymbol{0}$ as $B_{jn} \to \infty$ and $n \to \infty$.

Consider the random vectors

$$Z_n = \sum_j \frac{B_{jn}}{B} T_{jn} \quad \text{and} \quad \mathrm{W_n} = \sum_{\mathrm{j}} \rho_{\mathrm{jn}} \mathrm{T_{jn}}.$$

By xi)

$$\sqrt{n}(Z_n - \overline{T}^*) = \sqrt{n}(\sum_j \frac{B_{jn}}{B} T_{jn} - \overline{T}^*) = \sum_j \frac{B_{jn}}{B} \sqrt{n}(T_{jn} - \overline{T}^*_j) \xrightarrow{P} \mathbf{0}.$$

Also, $\sqrt{n}(Z_n - \boldsymbol{\theta}) - \sqrt{n}(W_n - \boldsymbol{\theta}) =$

$$\sum_j \left( \frac{B_{jn}}{B} - \rho_{jn} \right) \sqrt{n}(T_{jn} - \boldsymbol{\theta}) = \sum_j O_P(1) O_P(n^{-1/2}) \xrightarrow{P} \mathbf{0}.$$

Assume the $\boldsymbol{u}_{nj} = \sqrt{n}(T_{jn} - \boldsymbol{\theta}) \xrightarrow{D} \boldsymbol{u}_j$ are such that

$$\sqrt{n}(W_n - \boldsymbol{\theta}) = \sum_j \rho_{jn} \sqrt{n}(T_{jn} - \boldsymbol{\theta}) \xrightarrow{D} \boldsymbol{w} = \sum_j \pi_j \boldsymbol{u}_j.$$

Note that $E(\boldsymbol{w}) = \mathbf{0}$ and $Cov(\boldsymbol{w}) = \boldsymbol{\Sigma}_{\boldsymbol{w}} = \sum_j \sum_k \pi_j \pi_k Cov(\boldsymbol{u}_j, \boldsymbol{u}_k)$. Hence

$$\sqrt{n}(\overline{T}^* - \boldsymbol{\theta}) \xrightarrow{D} \boldsymbol{w}. \tag{3.15}$$

Since $\boldsymbol{w}$ is a weighted mean of the $\boldsymbol{u}_j \sim N_g(\mathbf{0}, \boldsymbol{\Sigma}_j)$, a normal approximation is $\boldsymbol{w} \approx N_g(\mathbf{0}, \boldsymbol{\Sigma}_{\boldsymbol{w}})$. The approximation is exact if the $\boldsymbol{u}_j$ with positive $\pi_j$ have a joint multivariate normal distribution.

Now consider variable selection for model (1.1) with $\boldsymbol{\theta} = \boldsymbol{A}\boldsymbol{\beta}$ where $\boldsymbol{A}$ is a known full rank $g \times p$ matrix with $1 \leq g \leq p$. Olive (2017a: p. 128, 2018a) showed that the prediction region method can simulate well for the $p \times 1$ vector $\hat{\boldsymbol{\beta}}_{I_{min},0}$. Assume $p$ is fixed, $n \geq 20p$, and that the error distribution is unimodal and not highly skewed. The response plot and residual plot are plots with $\hat{Y} = \boldsymbol{x}^T \hat{\boldsymbol{\beta}}$ on the horizontal axis and $Y$ or $r$ on the vertical axis, respectively. Then the plotted points in these plots should scatter in roughly even bands about the identity line (with unit slope and zero intercept) and the $r = 0$ line, respectively. If the error distribution is skewed or multimodal, then much larger sample sizes may be needed.

For the nonparametric bootstrap, cases are sampled with replacement, and the above conditions hold since each component bootstraps correctly. For the residual bootstrap, we use the fitted values and residuals from the OLS full model, but fit $\hat{\boldsymbol{\beta}}$ for a method such as forward selection, lasso, et cetera. Consider forward selection where each component uses a $\hat{\boldsymbol{\beta}}_{I_j}$. Let $\hat{\boldsymbol{Y}} = \hat{\boldsymbol{Y}}_{OLS} = \boldsymbol{X}\hat{\boldsymbol{\beta}}_{OLS} = \boldsymbol{HY}$ be the fitted values from the OLS full model where $\boldsymbol{H} = \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T$. Let $\boldsymbol{r}^W$ denote an $n \times 1$ random vector of elements selected with replacement from the OLS full model residuals. Following Freedman (1981) and Efron (1982, p. 36), $\boldsymbol{Y}^* = \boldsymbol{X}\hat{\boldsymbol{\beta}}_{OLS} + \boldsymbol{r}^W$ follows a standard linear model where the elements $r_i^W$ of $\boldsymbol{r}^W$ are iid from the empirical distribution of the OLS full model residuals $r_i$. Hence

$$E(r_i^W) = \frac{1}{n}\sum_{i=1}^n r_i = 0, \quad V(r_i^W) = \sigma_n^2 = \frac{1}{n}\sum_{i=1}^n r_i^2 = \frac{n-p}{n}MSE,$$

$$E(\boldsymbol{r}^W) = \boldsymbol{0}, \text{ and } \operatorname{Cov}(\boldsymbol{Y}^*) = \operatorname{Cov}(\boldsymbol{r}^{\mathrm{W}}) = \sigma_{\mathrm{n}}^2\boldsymbol{I}_{\mathrm{n}}.$$

Then $\hat{\boldsymbol{\beta}}^*_{I_j} = (\boldsymbol{X}_{I_j}^T\boldsymbol{X}_{I_j})^{-1}\boldsymbol{X}_{I_j}^T\boldsymbol{Y}^* = \boldsymbol{D}_j\boldsymbol{Y}^*$ with $Cov(\hat{\boldsymbol{\beta}}^*_{I_j}) = \sigma_n^2(\boldsymbol{X}_{I_j}^T\boldsymbol{X}_{I_j})^{-1}$ and $E(\hat{\boldsymbol{\beta}}^*_{I_j}) = (\boldsymbol{X}_{I_j}^T\boldsymbol{X}_{I_j})^{-1}\boldsymbol{X}_{I_j}^T E(\boldsymbol{Y}^*) = (\boldsymbol{X}_{I_j}^T\boldsymbol{X}_{I_j})^{-1}\boldsymbol{X}_{I_j}^T\boldsymbol{HY} = \hat{\boldsymbol{\beta}}_{I_j}$ since $\boldsymbol{HX}_{I_j} = \boldsymbol{X}_{I_j}$. The expectations are with respect to the bootstrap distribution where $\hat{\boldsymbol{Y}}$ acts as a constant.

For the above residual bootstrap with forward selection and $C_p$, let $T_n = \boldsymbol{A}\hat{\boldsymbol{\beta}}_{I_{min},0}$ and $T_{jn} = \boldsymbol{A}\hat{\boldsymbol{\beta}}_{I_j,0} = \boldsymbol{AD}_{j,0}\boldsymbol{Y}$ where $\boldsymbol{D}_{j,0}$ adds rows of zeroes to $\boldsymbol{D}_j$ corresponding to the $x_i$ not in $I_j$. If $S \subseteq I_j$, then $\sqrt{n}(\hat{\boldsymbol{\beta}}_{I_j} - \boldsymbol{\beta}_{I_j}) \xrightarrow{D} N_{a_j}(\boldsymbol{0}, \sigma^2\boldsymbol{V}_j)$ and $\sqrt{n}(\hat{\boldsymbol{\beta}}_{I_j,0} - \boldsymbol{\beta}) \xrightarrow{D} \boldsymbol{u}_j \sim N_p(\boldsymbol{0}, \sigma^2\boldsymbol{V}_{j,0})$ where $\boldsymbol{V}_{j,0}$ adds columns and rows of zeroes corresponding to the $x_i$ not in $I_j$. Then under regularity conditions, (3.14) and (3.15) hold where $\sqrt{n}(\sum_j \rho_{jn}T_{jn} - \boldsymbol{\theta}) \xrightarrow{D} \boldsymbol{w}$, and the sum is over $j : S \subseteq I_j$. Thus $E(T^*) = \sum_j \rho_{jn}\boldsymbol{A}\hat{\boldsymbol{\beta}}_{I_j,0}$ and $\boldsymbol{S}_T^*$ is a consistent estimator of $Cov(T^*)$

$$= \sum_j \rho_{jn}Cov(T_{jn}^*) + \sum_j \rho_{jn}\boldsymbol{A}\hat{\boldsymbol{\beta}}_{I_j,0}\hat{\boldsymbol{\beta}}_{I_j,0}^T\boldsymbol{A}^T - E(T^*)[E(T^*)]^T$$

where asymptotically the sum is over $j : S \subseteq I_j$. If $\boldsymbol{\theta}_0 = \boldsymbol{0}$, then $n\boldsymbol{S}_T^* = \boldsymbol{\Sigma}_A + O_P(1)$ where

$$nCov(T_n) \xrightarrow{P} \boldsymbol{\Sigma}_A = \sum_j \sigma^2\pi_j\boldsymbol{A}\boldsymbol{V}_{j,0}\boldsymbol{A}^T.$$

Then $(n\boldsymbol{S}_T^*)^{-1}$ tends to be "well behaved" if $\boldsymbol{\Sigma}_A$ is nonsingular. The prediction region (3.10) bootstraps $T_n$, but uses $\overline{T}^*$ to increase the coverage for moderate samples.

Some special cases are also interesting. Suppose $\pi_d = 1$ so $\boldsymbol{u} \sim \boldsymbol{u}_d \sim N_p(\boldsymbol{0}, \boldsymbol{\Sigma}_d)$. This occurs for $C_p$ if $a_S = p$ so $S$ is the full model, and for methods like BIC that choose $I_S$ with probability going to one. Knight and Fu (2000) had similar bootstrap results for this case. Next, if for each $\pi_j > 0$, $\boldsymbol{A}\boldsymbol{u}_j \sim N_g(\boldsymbol{0}, \boldsymbol{A}\boldsymbol{\Sigma}_j\boldsymbol{A}^T) = N_g(\boldsymbol{0}, \boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{A}^T)$, then
$\boldsymbol{A}\boldsymbol{u} \sim N_g(\boldsymbol{0}, \boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{A}^T)$.

In the simulations where $S$ is not the full model, inference with forward selection with $I_{min}$ using $C_p$ appears to be more precise than inference with the OLS full model if $n \geq 20p$ and $B \geq 50p$. Higher than nominal coverage can occur because of the zero padding. It is possible that $\boldsymbol{S}_T^*$ is singular if a column of the bootstrap sample is equal to $\boldsymbol{0}$.

Examining $\hat{\boldsymbol{\beta}}_S$ and $\hat{\boldsymbol{\beta}}_E$ is informative for $I_{min}$. See Equation (1.3). First assume that the nontrivial predictors are orthogonal or uncorrelated with zero mean so $\boldsymbol{X}^T\boldsymbol{X}/n$ $\rightarrow diag(d_1, ..., d_p)$ as $n \rightarrow \infty$ where each $d_i > 0$. Then $\hat{\boldsymbol{\beta}}_S$ has the same multivariate normal limiting distribution for $I_{min}$ and for the OLS full model. The bootstrap distribution for $\hat{\boldsymbol{\beta}}_E$ is a mixture of zeros and a distribution that would produce a confidence region for $\boldsymbol{A}\boldsymbol{\beta}_E = \boldsymbol{0}$ that has asymptotic coverage of $\boldsymbol{0}$ equal to $100(1-\delta)\%$. Hence the asymptotic coverage is greater than the nominal coverage provided that $\boldsymbol{S}_T^*$ in nonsingular with probability going to one (e.g., $p - a_S$ is small), where $T = \boldsymbol{A}\hat{\boldsymbol{\beta}}_{E,I_{min},0}$. For uncorrelated predictors with zero mean, the number of bootstrap samples $B \geq 50p$ may work well for the shorth confidence intervals and for testing $\boldsymbol{A}\boldsymbol{\beta}_S = \boldsymbol{0}$.

In the simulations for forward selection, coverages did not change much as the $\rho$ was increased from zero to near one, where $\rho$ was the correlation between any two nontrivial predictors. Under model (1.3), we still have that $\hat{\boldsymbol{\beta}}_{I_j,0}$ is a $\sqrt{n}$ consistent asymptotically normal estimator of $\boldsymbol{\beta} = (\boldsymbol{\beta}_S^T, \boldsymbol{\beta}_E^T)^T$ where $\boldsymbol{\beta}_E = \boldsymbol{0}$. Hence the limiting distribution of $\sqrt{n}(\hat{\boldsymbol{\beta}}_{I_{min},0} - \boldsymbol{\beta})$ is a mixture of $N_p(\boldsymbol{0}, \sigma^2\boldsymbol{V}_{j,0})$ distributions, and the limiting distribution of $\sqrt{n}(\hat{\beta}_{i,I_{min},0} - \beta_i)$ is a mixture of $N(0, \sigma_{ij}^2)$ distributions. For a $\beta_i$ that is a component of

$\boldsymbol{\beta}_S$, the symmetric mixture distribution has a pdf. Then the simulated shorth confidence intervals have coverage near the nominal coverage if $n$ and $B$ are large enough.

Note that there are several important variable selection models, including the model given by Equation (1.3). Another model is $\boldsymbol{x}^T\boldsymbol{\beta} = \boldsymbol{x}_{S_i}^T\boldsymbol{\beta}_{S_i}$ for $i = 1, ..., J$. Then there are $J \geq 2$ competing "true" nonnested submodels where $\boldsymbol{\beta}_{S_i}$ is $a_{S_i} \times 1$. For example, suppose the $J = 2$ models have predictors $x_1, x_2, x_3$ for $S_1$ and $x_1, x_2, x_4$ for $S_2$. Then $x_3$ and $x_4$ are likely to be selected and omitted often by forward selection for the $B$ bootstrap samples. Hence omitting all predictors $x_i$ that have a $\beta^*_{ij} = 0$ for at least one of the bootstrap samples $j = 1, ..., B$ could result in underfitting, e.g. using just $x_1$ and $x_2$ in the above $J = 2$ example. If $n$ and $B$ are large enough, the singleton set $\{\boldsymbol{0}\}$ could still be the "100%" confidence region for a vector $\boldsymbol{\beta}_O$.

Suppose the predictors $x_i$ have been standardized. Then another important regression model has the $\beta_i$ taper off rapidly, but no coefficients are equal to zero. For example, $\beta_i = e^{-i}$ for $i = 1, ..., p$.

For $g = 1$, the percentile method uses an interval that contains $U_B \approx k_B = \lceil B(1-\delta) \rceil$ of the $T_i^*$ from a bootstrap sample $T_1^*, ..., T_B^*$ where the statistic $T_n$ is an estimator of $\theta$ based on a sample of size $n$. Note that the squared Mahalanobis distance $D_\theta^2 = (\theta - \overline{T^*})^2/S_T^{2*} \leq D_{(U_B)}^2$ is equivalent to $\theta \in [\overline{T^*} - S_T^* D_{(U_B)}, \overline{T^*} + S_T^* D_{(U_B)}]$, which is an interval centered at $\overline{T^*}$ just long enough to cover $U_B$ of the $T_i^*$. Hence the prediction region method is a special case of the percentile method if $g = 1$. Efron (2014) used a similar large sample $100(1-\delta)\%$ confidence interval assuming that $\overline{T}^*$ is asymptotically normal. The Frey (2013) shorth($c$) interval (3.1) (with $c$ given by (3.3)) applied to the $T_i^*$ is recommended since the shorth confidence interval can be much shorter than the Efron (2014) or prediction region method confidence intervals if $g = 1$. The shorth confidence interval is a practical implementation of the Hall (1988) shortest bootstrap interval based on all possible bootstrap samples. Note that if $\sqrt{n}(T_n - \theta) \xrightarrow{D} \boldsymbol{u}$ and $\sqrt{n}(T_i^* - \theta) \xrightarrow{D} \boldsymbol{u}$ where $\boldsymbol{u}$ has a symmetric probability density function, then the shorth confidence interval is asymptotically equivalent to the usual percentile method confidence interval that uses the central proportion of the bootstrap sample.

Note that correction factors $b_n \to 1$ are used in large sample confidence intervals and tests if the limiting distribution is N(0,1) or $\chi^2_p$, but a $t_{d_n}$ or $pF_{p,d_n}$ cutoff is used: $t_{d_n,1-\delta}/z_{1-\delta} \to 1$ and $pF_{p,d_n,1-\delta}/\chi^2_{p,1-\delta} \to 1$ if $d_n \to \infty$ as $n \to 1$. Using correction factors for prediction intervals and bootstrap confidence regions improves the performance for moderate sample size $n$.

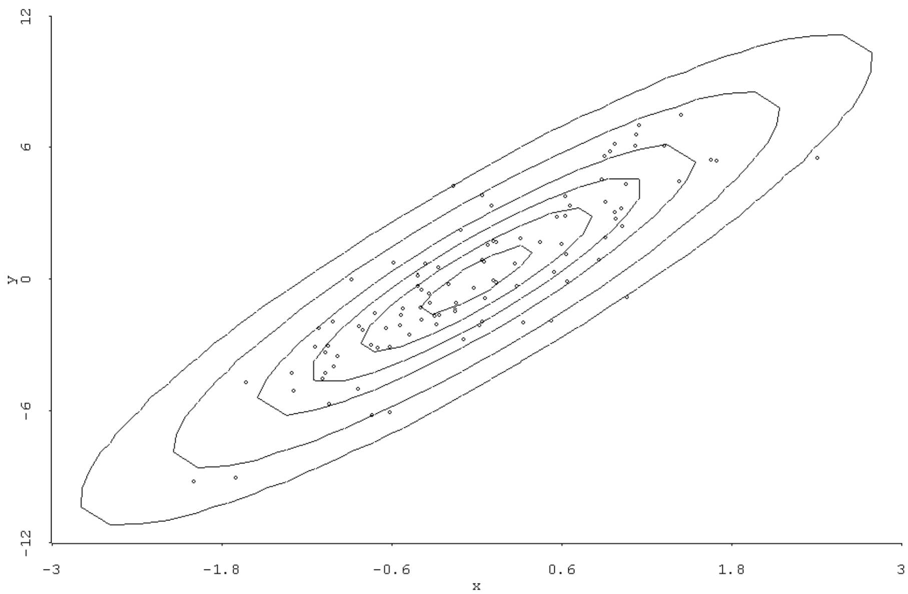
# CHAPTER 4

## EXAMPLE AND SIMULATIONS

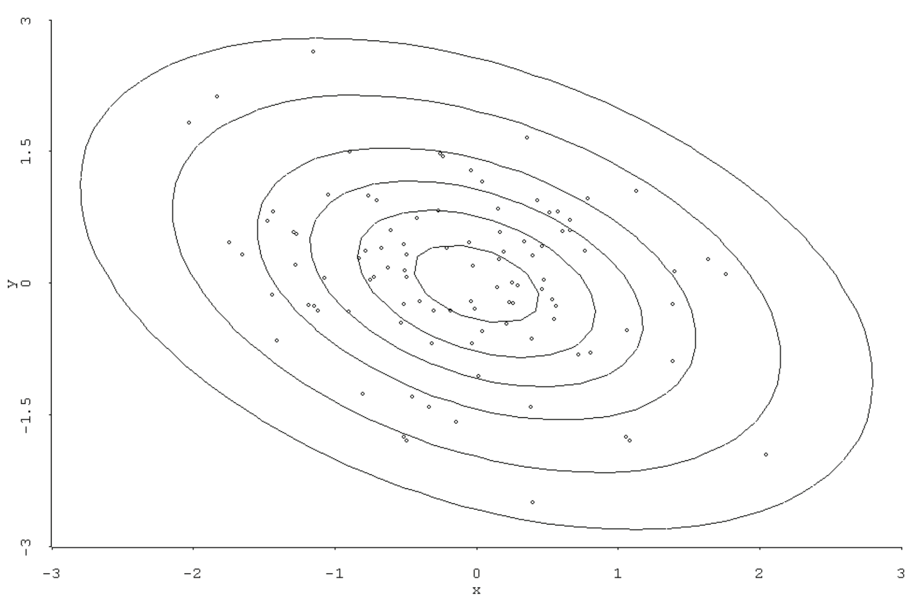
Figure 1 shows 10%, 30%, 50%, 70%, 90% and 98% prediction regions for a future value of $T_f$ for two multivariate normal distributions. The plotted points are iid $T_1, ..., T_B$ with $B = 100$.

Example. The Hebbler (1847) data was collected from $n = 26$ districts in Prussia in 1843. We will study the relationship between $Y =$ the *number of women married to civilians* in the district with the predictors $x_1 =$ constant, $x_2 = pop =$ the *population of the district in 1843*, $x_3 = mmen =$ the *number of married civilian men* in the district, $x_4 = mmilmen =$ *number of married men in the military* in the district, and $x_5 = milwmn =$ the *number of women married to husbands in the military* in the district. Sometimes the person conducting the survey would not count a spouse if the spouse was not at home. Hence $Y$ and $X_3$ are highly correlated but not equal. Similarly, $x_4$ and $x_5$ are highly correlated but not equal. We expect that $Y = x_3 + e$ is a good model. Forward selection with BIC selected the model a constant and *mmen.*

Let $\boldsymbol{x} = (1\ \boldsymbol{u}^T)^T$ where $\boldsymbol{u}$ is the $(p-1) \times 1$ vector of nontrivial predictors. In the simulations, for $i = 1, ..., n$, we generated $\boldsymbol{w}_i \sim N_{p-1}(\boldsymbol{0}, \boldsymbol{I})$ where the $m = p - 1$ elements of the vector $\boldsymbol{w}_i$ are iid N(0,1). Let the $m \times m$ matrix $\boldsymbol{A} = (a_{ij})$ with $a_{ii} = 1$ and $a_{ij} = \psi$ where $0 \leq \psi < 1$ for $i \neq j$. Then the vector $\boldsymbol{u}_i = \boldsymbol{A}\boldsymbol{w}_i$ so that $Cov(\boldsymbol{u}_i) = \boldsymbol{\Sigma}_{\boldsymbol{u}} = \boldsymbol{A}\boldsymbol{A}^T = (\sigma_{ij})$ where the diagonal entries $\sigma_{ii} = [1 + (m-1)\psi^2]$ and the off diagonal entries $\sigma_{ij} = [2\psi + (m-2)\psi^2]$. Hence the correlations are $cor(x_i, x_j) = \rho = (2\psi + (m-2)\psi^2)/(1 + (m-1)\psi^2)$ for $i \neq j$ where $x_i$ and $x_j$ are nontrivial predictors. If $\psi = 1/\sqrt{cp}$, then $\rho \to 1/(c+1)$ as $p \to \infty$ where $c > 0$. As $\psi$ gets close to 1, the predictor vectors cluster about the line in the direction of $(1, ..., 1)^T$. Let $Y_i = 1 + 1x_{i,2} + \cdots + 1x_{i,k+1} + e_i$ for $i = 1, ..., n$. Hence $\boldsymbol{\beta} = (1, .., 1, 0, ..., 0)^T$ with $k+1$ ones and $p - k - 1$ zeros. The zero mean errors $e_i$ were iid from five distributions: i)




)




b)

Figure 4.1. Prediction Regions

N(0,1), ii) $t_3$, iii) EXP(1) - 1, iv) uniform$(-1,1)$, and v) 0.9 N(0,1) + 0.1 N(0,100). Only distribution iii) is not symmetric.

A small simulation was done using $B = \max(1000, n, 20p)$ and 5000 runs. So an observed coverage in [0.94, 0.96] gives no reason to doubt that the CI or confidence region has the nominal coverage of 0.95. The simulation used $p = 4, 6, 7, 8,$ and 10; $n = 25p$ and $50p$, $\psi = 0, 1/\sqrt{p}$, and 0.9; and $k = 1$ and $p - 2$.

When $\psi = 0$, the full model least squares confidence intervals for $\beta_i$ should have length near $2t_{96,0.975}\sigma/\sqrt{n} \approx 2(1.96)\sigma/10 = 0.392\sigma$ when the iid zero mean errors have variance $\sigma^2$. The simulation computed the Frey shorth$(c)$ interval for each $\beta_i$ and used bootstrap confidence regions to test whether first $k+1$ $\beta_i = 1$ and the last $p-k-1$ $\beta_i = 0$. The nominal coverage was 0.95 with $\delta = 0.05$. Observed coverage between 0.94 and 0.96 would suggest coverage is close to the nominal value.

The regression models used the residual bootstrap on the forward selection estimator $\hat{\boldsymbol{\beta}}_{I_{min},0}$ with BIC. Table 1 gives results for when the iid errors $e_i \sim N(0,1)$. Two rows for each model giving the observed confidence interval coverages and average lengths of the confidence intervals. The last six columns give results for the tests. The the length and coverage = P(fail to reject $H_0$) for the interval $[0, D_{(U_B)}]$ or $[0, D_{(U_B),T}]$ where $D_{(U_B)}$ or $D_{(U_B),T}$ is the cutoff for the confidence region. Volumes of the confidence regions can be compared using (3.13). The first two lines of the table correspond to the $R$ output shown below, with $g = 2$.

```
library(leaps);Y <- marry[,3]; X <- marry[,-3]
temp<-regsubsets(X,Y,method="forward")
out<-summary(temp)
out$bic
[1] -239.4149 -236.3515 -233.1085 -229.8540
Selection Algorithm: forward
         pop mmen mmilmen milwmn
```

```
1  ( 1 ) " " "*"  " "     " "
2  ( 1 ) " " "*"  "*"     " "
3  ( 1 ) "*" "*"  "*"     " "
4  ( 1 ) "*" "*"  "*"     "*"

record coverages and ``lengths" for
b1, b2, bp-1, bp, pm0,  hyb0,   BR0,  pm1,   hyb1,  BR1,
library(leaps)
bicbootsim(n=100,p=4,k=1,nruns=5000,type=1,psi=0)

$cicov
 [1] 0.9478 0.9478 0.9996 0.9998 0.9992 0.9918 0.9996 0.9408 0.9418 0.9422
$avelen
 [1] 0.3948321 0.3973231 0.2153983 0.2145764 3.4006284 3.4006282 3.6963001
 [8] 2.4501023 2.4501437 2.45612555
$beta
[1] 1 1 0 0
$k
[1] 1
```

Table 4.1. Bootstrapping OLS Forward Selection with BIC Type 1

| n,p,k,$\psi$ | $\beta_1$ | $\beta_2$ | $\beta_{p-1}$ | $\beta_p$ | pm0 | hyb0 | br0 | pm1 | hyb1 | br1 |
|---|---|---|---|---|---|---|---|---|---|---|
| 100,4,1,0 | 0.9478 | 0.9478 | 0.9996 | 0.9998 | 0.9992 | 0.9918 | 0.9996 | 0.9408 | 0.9418 | 0.9422 |
| len | 0.3948 | 0.3973 | 0.2154 | 0.2146 | 3.4006 | 3.4006 | 3.6963 | 2.4501 | 2.4501 | 2.4561 |
| 100,4,2,0 | 0.9396 | 0.9466 | 0.9462 | 0.9998 | 0.9998 | 0.9682 | 0.9998 | 0.9326 | 0.9326 | 0.9320 |
| len | 0.3950 | 0.3975 | 0.3984 | 0.2195 | 1.8434 | 1.8434 | 2.0855 | 2.8003 | 2.8003 | 2.8047 |
| 100,4,1,$1/\sqrt{p}$ | 0.9452 | 0.9742 | 1.0000 | 0.9998 | 0.9992 | 0.9960 | 1.0000 | 0.9764 | 0.9772 | 0.9838 |
| len | 0.3958 | 0.6261 | 0.3595 | 0.3573 | 3.4424 | 3.4424 | 3.7143 | 2.5574 | 2.5574 | 2.7219 |
| 100,4,2,$1/\sqrt{p}$ | 0.9442 | 0.9596 | 0.9618 | 0.9998 | 0.9996 | 0.9740 | 0.9996 | 0.9774 | 0.9774 | 0.9834 |
| len | 0.3962 | 0.6512 | 0.6500 | 0.3681 | 1.8300 | 1.8300 | 2.0645 | 2.9388 | 2.9388 | 3.0528 |
| 100,4,1,0.9 | 0.9428 | 0.9486 | 0.9976 | 0.9978 | 1.0000 | 0.8894 | 1.0000 | 0.9604 | 0.9272 | 0.9576 |
| len | 0.3956 | 2.1746 | 1.9488 | 1.9684 | 2.7434 | 2.7434 | 2.9890 | 2.5333 | 2.5333 | 2.6716 |
| 100,4,2,0.9 | 0.9466 | 0.9110 | 0.9104 | 0.9990 | 0.9990 | 0.8854 | 0.9994 | 0.9920 | 0.9826 | 0.9948 |
| len | 0.3968 | 2.3035 | 2.2987 | 2.1084 | 2.4007 | 2.4007 | 2.7693 | 3.2150 | 3.2150 | 3.4741 |
| 175,7,1,0 | 0.9514 | 0.9452 | 0.9998 | 0.9998 | 1.0000 | 1.0000 | 1.0000 | 0.9422 | 0.9432 | 0.9436 |
| len | 0.2945 | 0.3045 | 0.1334 | 0.1354 | 5.1894 | 5.1894 | 5.3111 | 2.4342 | 2.4343 | 2.4500 |
| 175,7,5,0 | 0.9498 | 0.9234 | 0.9226 | 0.9212 | 0.9222 | 0.9252 | 0.9994 | 0.9994 | 0.9492 | 0.9994 |
| len | 0.3004 | 0.3011 | 0.3021 | 0.1242 | 1.5442 | 1.5542 | 1.7002 | 3.6042 | 3.6042 | 3.6212 |
| 175,7,1,$1/\sqrt{p}$ | 0.9498 | 0.9234 | 0.9226 | 0.9212 | 0.9222 | 0.9252 | 0.9994 | 0.9994 | 0.9492 | 0.9994 |
| len | 0.2991 | 0.4386 | 0.1918 | 0.1958 | 5.0443 | 5.0443 | 5.2423 | 2.4869 | 2.4869 | 2.5553 |
| 175,7,5,$1/\sqrt{p}$ | 0.9498 | 0.9234 | 0.9226 | 0.9212 | 0.9222 | 0.9252 | 0.9994 | 0.9614 | 0.9614 | 0.9678 |
| len | 0.3001 | 0.4419 | 0.4119 | 0.2188 | 1.5232 | 1.5232 | 1.6532 | 3.5999 | 3.5999 | 3.6423 |
| 175,7,1,0.9 | 0.9450 | 0.9208 | 0.9996 | 0.9996 | 1.0000 | 0.9998 | 0.9998 | 0.9182 | 0.8652 | 0.9182 |
| len | 0.2992 | 2.0704 | 1.5432 | 1.5433 | 4.5547 | 4.5547 | 4.7647 | 2.5887 | 2.5887 | 2.6196 |
| 175,7,5,0.9 | 0.9498 | 0.9234 | 0.9226 | 0.9212 | 0.9222 | 0.9252 | 0.9994 | 0.9722 | 0.9270 | 0.9730 |
| len | 0.3038 | 2.5443 | 2.5379 | 1.6935 | 1.7776 | 1.7776 | 1.9763 | 4.2855 | 4.2855 | 4.5140 |

Table 4.2. Bootstrapping OLS Forward Selection with BIC Type 1(cont.)

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 250,10,1,0 | 0.9495 | 0.9413 | 1.0000 | 1.0000 | 0.9995 | 0.9986 | 1.0000 | 0.9388 | 0.9378 | 0.9398 |
| len | 0.2512 | 0.2516 | 0.1062 | 0.1069 | 6.2092 | 6.2092 | 6.4014 | 2.4322 | 2.4322 | 2.4577 |
| 250,10,1,1/$\sqrt{p}$ | 0.9438 | 0.9706 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9636 | 0.9636 | 0.9752 |
| len | 0.2505 | 0.3492 | 0.1428 | 0.1405 | 6.1386 | 6.1386 | 6.3393 | 2.4222 | 2.4222 | 2.6282 |
| 250,10,1,0.9 | 0.9434 | 0.9078 | 1.0000 | 0.9998 | 0.9998 | 1.0000 | 1.0000 | 0.8762 | 0.8156 | 0.8776 |
| len | 0.2503 | 1.9279 | 1.1796 | 1.1844 | 5.6734 | 5.6734 | 5.9500 | 2.5454 | 2.5454 | 2.6776 |
| 250,10,8,0.9 | 0.9464 | 0.9164 | 0.9224 | 0.9160 | 0.9198 | 0.9188 | 0.9154 | 0.9014 | 0.6748 | 0.8820 |
| len | 0.2556 | 2.3371 | 2.3157 | 1.2916 | 1.5965 | 1.5965 | 1.8022 | 4.7095 | 4.7095 | 5.1978 |
| 300,6,1,0 | 0.9494 | 0.9476 | 1.0000 | 1.0000 | 1.0000 | 0.9996 | 1.0000 | 0.9454 | 0.9462 | 0.9460 |
| len | 0.2298 | 0.2307 | 0.0888 | 0.0892 | 4.2355 | 4.2355 | 5.0332 | 2.3323 | 2.3326 | 2.7442 |
| 300,6,4,0 | 0.9454 | 0.9528 | 0.9468 | 0.9484 | 0.9502 | 0.9996 | 0.9996 | 0.9818 | 0.9452 | 0.9456 |
| len | 0.2300 | 0.2307 | 0.2342 | 0.0966 | 1.2304 | 1.2304 | 1.4493 | 3.3326 | 3.3326 | 3.3418 |
| 300,6,1,1/$\sqrt{p}$ | 0.9498 | 0.9816 | 1.0000 | 1.0000 | 1.0000 | 0.9998 | 1.0000 | 0.9998 | 0.9740 | 0.9746 |
| len | 0.2290 | 0.3354 | 0.1270 | 0.1292 | 4.8107 | 4.8107 | 5.0252 | 2.7557 | 2.7757 | 2.6979 |
| 300,6,4,1/$\sqrt{p}$ | 0.9466 | 0.9570 | 0.9498 | 0.9552 | 0.9536 | 1.0000 | 1.0000 | 0.9692 | 0.9696 | 0.9736 |
| len | 0.2300 | 0.3473 | 0.3470 | 0.1397 | 1.3121 | 1.3121 | 1.4444 | 3.2214 | 3.2214 | 3.4970 |
| 300,6,1,0.9 | 0.9470 | 0.9384 | 1.0000 | 0.9998 | 1.0000 | 0.9998 | 0.9976 | 0.9692 | 0.9978 | 0.9252 |
| len | 0.2297 | 1.7048 | 1.1010 | 1.1001 | 4.0613 | 4.0613 | 4.2221 | 2.4661 | 2.4661 | 2.6089 |
| 300,6,4,0.9 | 0.9530 | 0.9286 | 0.9292 | 0.9998 | 0.9998 | 0.9696 | 0.9998 | 0.9230 | 0.8770 | 0.9242 |
| len | 0.2315 | 2.1667 | 2.1546 | 1.2109 | 1.6445 | 1.6445 | 1.8543 | 3.7622 | 3.7622 | 4.1773 |
| 400,8,1,0 | 0.9488 | 0.9554 | 1.0000 | 0.9998 | 0.9998 | 1.0000 | 1.0000 | 0.9434 | 0.9450 | 0.9466 |
| len | 0.1987 | 0.1989 | 0.0716 | 0.0773 | 5.7824 | 5.7824 | 5.9973 | 2.5753 | 2.5753 | 2.6002 |
| 400,8,6,0 | 0.9468 | 0.9544 | 0.9444 | 1.0000 | 1.0000 | 0.9836 | 1.0000 | 0.9418 | 0.9428 | 0.9428 |
| len | 0.1994 | 0.1998 | 0.1999 | 0.0732 | 1.5532 | 1.5532 | 1.6665 | 3.1142 | 3.1142 | 3.2001 |

Table 4.3. Bootstrapping OLS Forward Selection with BIC Type 1(cont.)

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 400,8,1,1/$\sqrt{p}$ | 0.9488 | 0.9808 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9682 | 0.9712 | 0.9796 |
| len | 0.1986 | 0.2812 | 0.0933 | 0.1077 | 5.2232 | 5.2232 | 5.3090 | 2.4112 | 2.4112 | 2.6968 |
| 400,8,6,1/$\sqrt{p}$ | 0.9496 | 0.9554 | 0.9518 | 0.9518 | 0.9468 | 0.9444 | 0.9518 | 0.9604 | 0.9600 | 0.9644 |
| len | 0.1993 | 0.2854 | 0.2766 | 0.1043 | 1.2985 | 1.2985 | 1.3117 | 3.7884 | 3.7884 | 3.8965 |
| 400,8,1,0.9 | 0.9464 | 0.9438 | 1.0000 | 0.9996 | 1.0000 | 0.9852 | 0.9966 | 0.9154 | 0.8384 | 0.9162 |
| len | 0.1990 | 1.6517 | 0.8133 | 0.8129 | 5.0542 | 5.0542 | 5.2327 | 2.4098 | 2.4098 | 2.5056 |
| 400,8,6,0.9 | 0.9456 | 0.9108 | 0.9144 | 1.0000 | 1.0000 | 0.9666 | 1.0000 | 0.8976 | 0.7948 | 0.8774 |
| len | 0.2010 | 2.0326 | 2.0226 | 0.9537 | 1.4552 | 1.4552 | 1.6662 | 4.1299 | 4.1299 | 4.5120 |

Table 4.4. Bootstrapping OLS Forward Selection with BIC Type 2

| n,p,k,$\psi$ | $\beta_1$ | $\beta_2$ | $\beta_{p-1}$ | $\beta_p$ | pm0 | hyb0 | br0 | pm1 | hyb1 | br1 |
|---|---|---|---|---|---|---|---|---|---|---|
| 100,4,1,0 | 0.9478 | 0.9484 | 0.9998 | 0.9998 | 0.9998 | 0.9936 | 0.9998 | 0.9518 | 0.9508 | 0.9522 |
| len | 0.6652 | 0.6778 | 0.3656 | 0.3615 | 3.3553 | 3.3554 | 3.6589 | 2.4765 | 2.4765 | 2.4848 |
| 100,4,2,0 | 0.9396 | 0.9348 | 0.9404 | 1.0000 | 1.0000 | 0.9704 | 1.0000 | 0.9450 | 0.9472 | 0.9498 |
| len | 0.6582 | 0.6884 | 0.6892 | 0.3687 | 1.8572 | 1.8572 | 2.1113 | 2.8692 | 2.8692 | 2.8835 |
| 100,4,1,1/$\sqrt{p}$ | 0.9414 | 0.9646 | 1.0000 | 0.9998 | 0.9996 | 0.9880 | 1.0000 | 0.9700 | 0.9556 | 0.9746 |
| len | 0.6580 | 1.1045 | 0.6889 | 0.6885 | 3.3182 | 3.3182 | 3.6039 | 2.5784 | 2.5784 | 2.7611 |
| 100,4,2,1/$\sqrt{p}$ | 0.9436 | 0.9334 | 0.9328 | 0.9998 | 0.9998 | 0.9652 | 0.9996 | 0.9706 | 0.9448 | 0.9672 |
| len | 0.6628 | 1.2667 | 1.2675 | 0.7583 | 2.2142 | 2.2412 | 2.5027 | 3.0391 | 3.0390 | 3.2039 |
| 100,4,1,0.9 | 0.9436 | 0.9336 | 0.9980 | 0.9966 | 0.9984 | 0.9286 | 0.9998 | 0.9754 | 0.9712 | 0.9790 |
| len | 0.6602 | 3.4802 | 3.3721 | 3.4000 | 3.0051 | 3.0051 | 3.2116 | 2.6446 | 2.6446 | 2.7773 |
| 100,4,2,0.9 | 0.9464 | 0.9034 | 0.9072 | 0.9976 | 0.9982 | 0.8782 | 0.9996 | 0.9892 | 0.9796 | 0.9922 |
| len | 0.6548 | 3.2829 | 3.3038 | 3.1870 | 2.0573 | 2.0574 | 2.4191 | 3.0484 | 3.0484 | 3.2597 |
| 175,7,1,0 | 0.9452 | 0.9448 | 0.9998 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9416 | 0.9408 | 0.9418 |
| len | 0.5017 | 0.5115 | 0.2307 | 0.2320 | 5.0702 | 5.0702 | 5.2886 | 2.4680 | 2.4680 | 2.4764 |
| 175,7,5,0 | 0.9386 | 0.9492 | 0.9446 | 1.0000 | 1.0000 | 0.9774 | 1.0000 | 0.9498 | 0.9480 | 0.9498 |
| len | 0.5009 | 0.5129 | 0.5117 | 0.2394 | 1.5668 | 1.5668 | 1.7745 | 3.6649 | 3.6649 | 3.6723 |
| 175,7,1,1/$\sqrt{p}$ | 0.9472 | 0.9702 | 1.0000 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9748 | 0.9748 | 0.9818 |
| len | 0.4998 | 0.7591 | 0.3363 | 0.3399 | 5.0069 | 5.0069 | 5.2248 | 2.5067 | 2.5067 | 2.6653 |
| 175,7,5,1/$\sqrt{p}$ | 0.9448 | 0.9352 | 0.9414 | 1.0000 | 1.0000 | 0.9776 | 1.0000 | 0.9834 | 0.9778 | 0.9846 |
| len | 0.5070 | 0.8026 | 0.8053 | 0.3565 | 1.5584 | 1.5584 | 1.7535 | 3.8685 | 3.8685 | 3.9477 |
| 175,7,1,0.9 | 0.9440 | 0.8778 | 0.9998 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9696 | 0.9638 | 0.9724 |
| len | 0.5001 | 3.1455 | 2.7716 | 2.7721 | 4.6978 | 4.6978 | 4.9545 | 2.6784 | 2.6784 | 2.8209 |
| 175,7,5,0.9 | 0.9392 | 0.8666 | 0.8654 | 1.0000 | 1.0000 | 0.9574 | 1.0000 | 0.9986 | 0.9960 | 0.9992 |
| len | 0.5063 | 3.5194 | 3.5408 | 2.7855 | 1.9122 | 1.9122 | 2.1371 | 4.4440 | 4.4440 | 4.7823 |

Table 4.5. Bootstrapping OLS Forward Selection with BIC Type 2(cont.)

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 250,10,1,0 | 0.9422 | 0.9490 | 0.9996 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9444 | 0.9448 | 0.9464 |
| len | 0.4239 | 0.4310 | 0.1768 | 0.1777 | 6.1685 | 6.1685 | 6.3665 | 2.4669 | 2.4669 | 2.4745 |
| 250,10,1,1/$\sqrt{p}$ | 0.9474 | 0.9738 | 0.9998 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9636 | 0.9662 | 0.9756 |
| len | 0.4203 | 0.5964 | 0.2409 | 0.2441 | 6.0990 | 6.0990 | 6.3031 | 2.4802 | 2.4802 | 2.6346 |
| 250,10,1,0.9 | 0.9412 | 0.8430 | 0.9998 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9350 | 0.9206 | 0.9360 |
| len | 0.4244 | 2.7286 | 2.1710 | 2.1765 | 5.7059 | 5.7059 | 5.9675 | 2.6515 | 2.6516 | 2.7959 |
| 250,10,8,0.9 | 0.9424 | 0.8612 | 0.8580 | 0.9998 | 0.9998 | 0.9458 | 0.9996 | 0.9974 | 0.9812 | 0.9958 |
| len | 0.4269 | 3.2147 | 3.2002 | 2.1729 | 1.6402 | 1.6402 | 1.8454 | 5.1395 | 5.1395 | 5.5527 |
| 300,6,1,0 | 0.9476 | 0.9454 | 1.0000 | 1.0000 | 0.9998 | 0.9996 | 1.0000 | 0.9488 | 0.9486 | 0.9490 |
| len | 0.3874 | 0.3920 | 0.1589 | 0.1566 | 4.7395 | 4.7395 | 4.9914 | 2.3908 | 2.3908 | 2.4545 |
| 300,6,4,0 | 0.9466 | 0.9472 | 0.9484 | 0.9998 | 0.9996 | 0.9822 | 0.9998 | 0.9552 | 0.9568 | 0.9564 |
| len | 0.3885 | 0.3966 | 0.3946 | 0.1554 | 1.3390 | 1.3390 | 1.4854 | 3.4049 | 3.4049 | 3.4086 |
| 300,6,1,1/$\sqrt{p}$ | 0.9512 | 0.9796 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9752 | 0.9742 | 0.9840 |
| len | 0.3874 | 0.5748 | 0.2257 | 0.2247 | 4.7823 | 4.7823 | 5.0229 | 2.5368 | 2.5368 | 2.6983 |
| 300,6,4,1/$\sqrt{p}$ | 0.9478 | 0.9474 | 0.9522 | 0.9506 | 0.9532 | 0.9998 | 0.9998 | 0.9788 | 0.9804 | 0.9838 |
| len | 0.3878 | 0.6050 | 0.6064 | 0.2379 | 1.3026 | 1.3026 | 1.4389 | 3.5264 | 3.5264 | 3.5912 |
| 300,6,1,0.9 | 0.9504 | 0.9004 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 1.0000 | 0.9398 | 0.8990 | 0.9386 |
| len | 0.3905 | 2.2815 | 1.8054 | 1.8165 | 4.1676 | 4.1676 | 4.3697 | 2.5782 | 2.5782 | 2.6982 |
| 300,6,4,0.9 | 0.9524 | 0.8918 | 0.8924 | 0.9996 | 0.9996 | 0.9672 | 0.9996 | 0.9886 | 0.9752 | 0.9912 |
| len | 0.3906 | 2.9795 | 2.9661 | 1.9804 | 1.7049 | 1.7049 | 1.9000 | 4.0476 | 4.0476 | 4.4212 |
| 400,8,1,0 | 0.9496 | 0.9474 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9434 | 0.9440 | 0.9444 |
| len | 0.3373 | 0.3407 | 0.1241 | 0.1247 | 5.7576 | 5.7576 | 5.9650 | 2.4643 | 2.4643 | 2.4682 |
| 400,8,6,0 | 0.9448 | 0.9506 | 0.9576 | 1.0000 | 1.0000 | 0.9864 | 1.0000 | 0.9542 | 0.9542 | 0.9542 |
| len | 0.3379 | 0.3420 | 0.3419 | 0.1275 | 1.2489 | 1.2489 | 1.3696 | 3.8506 | 3.8506 | 3.8540 |

Table 4.6. Bootstrapping OLS Forward Selection with BIC Type 2(cont.)

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 400,8,1,1/$\sqrt{p}$ | 0.9518 | 0.9792 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9750 | 0.9760 | 0.9834 |
| len | 0.3363 | 0.4812 | 0.1726 | 0.1686 | 5.7419 | 5.7419 | 5.9311 | 2.5085 | 2.5085 | 2.6737 |
| 400,8,6,1/$\sqrt{p}$ | 0.9482 | 0.9504 | 0.9514 | 1.0000 | 0.9998 | 0.9850 | 1.0000 | 0.9684 | 0.9694 | 0.9732 |
| len | 0.3379 | 0.4933 | 0.4916 | 0.1781 | 1.2154 | 1.2154 | 1.3313 | 3.9364 | 3.9364 | 3.9815 |
| 400,8,1,0.9 | 0.9478 | 0.8698 | 1.0000 | 1.0000 | 1.0000 | 0.9966 | 1.0000 | 0.8994 | 0.8362 | 0.8930 |
| len | 0.3369 | 2.1940 | 1.5615 | 1.5275 | 5.1174 | 5.1174 | 5.3412 | 2.5855 | 2.5855 | 2.7040 |
| 400,8,6,0.9 | 0.9522 | 0.8918 | 0.9018 | 1.0000 | 1.0000 | 0.9588 | 1.0000 | 0.9672 | 0.9232 | 0.9672 |
| len | 0.3399 | 2.7933 | 2.8184 | 1.5926 | 1.4750 | 1.4750 | 1.6518 | 4.5143 | 4.5143 | 4.9527 |

Table 4.7. Bootstrapping OLS Forward Selection with BIC Type 3

| n,p,k,$\psi$ | $\beta_1$ | $\beta_2$ | $\beta_{p-1}$ | $\beta_p$ | pm0 | hyb0 | br0 | pm1 | hyb1 | br1 |
|---|---|---|---|---|---|---|---|---|---|---|
| 100,4,1,0 | 0.9412 | 0.9504 | 1.0000 | 1.0000 | 0.9998 | 0.9954 | 0.9998 | 0.9366 | 0.9372 | 0.9372 |
| len | 0.3916 | 0.3912 | 0.2150 | 0.2145 | 3.3608 | 3.3609 | 3.3609 | 3.6704 | 2.4538 | 2.4538 |
| 100,4,2,0 | 0.9378 | 0.9458 | 0.9478 | 1.0000 | 0.9998 | 0.9708 | 0.9998 | 0.9312 | 0.9316 | 0.9332 |
| len | 0.3915 | 0.3974 | 0.3978 | 0.2179 | 1.8355 | 1.8355 | 2.0802 | 2.8209 | 2.8209 | 2.8256 |
| 100,4,1,1/$\sqrt{p}$ | 0.9420 | 0.9770 | 0.9998 | 0.9998 | 0.9992 | 0.9954 | 0.9996 | 0.9672 | 0.9680 | 0.9764 |
| len | 0.3911 | 0.6210 | 0.3507 | 0.3496 | 3.4236 | 3.4236 | 3.6593 | 2.5516 | 2.5516 | 2.7117 |
| 100,4,2,1/$\sqrt{p}$ | 0.9382 | 0.9516 | 0.9460 | 0.9996 | 0.9996 | 0.9734 | 0.9996 | 0.9678 | 0.9672 | 0.9750 |
| len | 0.3913 | 0.6547 | 0.6578 | 0.3676 | 1.8384 | 1.8384 | 2.0840 | 2.9561 | 2.9561 | 3.0718 |
| 100,4,1,0.9 | 0.9360 | 0.9464 | 0.9980 | 0.9982 | 0.9986 | 0.9034 | 0.9996 | 0.9484 | 0.9222 | 0.9508 |
| len | 0.3909 | 2.1869 | 1.9452 | 1.9543 | 2.7413 | 2.7413 | 2.9875 | 2.5190 | 2.5190 | 2.6552 |
| 100,4,2,0.9 | 0.9402 | 0.9120 | 0.9172 | 0.9984 | 0.8966 | 0.9994 | 0.9822 | 0.9734 | 0.9822 | 0.9884 |
| len | 0.3911 | 2.2675 | 2.2986 | 2.0796 | 2.3662 | 2.3662 | 2.7231 | 3.2088 | 3.2088 | 3.4783 |
| 175,7,1,0 | 0.9364 | 0.9444 | 1.0000 | 0.9998 | 1.0000 | 1.0000 | 1.0000 | 0.9354 | 0.9338 | 0.9354 |
| len | 0.2966 | 0.2988 | 0.1376 | 0.1370 | 5.0766 | 5.0766 | 5.2946 | 2.4532 | 2.4532 | 2.4612 |
| 175,7,5,0 | 0.9376 | 0.9472 | 0.9408 | 1.0000 | 1.0000 | 0.9804 | 1.0000 | 0.9342 | 0.9358 | 0.9362 |
| len | 0.2991 | 0.3019 | 0.3021 | 0.1370 | 1.5203 | 1.5203 | 1.7048 | 3.6059 | 3.6059 | 3.6094 |
| 175,7,1,1/$\sqrt{p}$ | 0.9454 | 0.9790 | 0.9998 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9622 | 0.9644 | 0.9748 |
| len | 0.2978 | 0.4381 | 0.1988 | 0.2018 | 5.0196 | 5.0196 | 5.2325 | 2.4873 | 2.4873 | 2.6507 |
| 175,7,5,1/$\sqrt{p}$ | 0.9410 | 0.9500 | 0.9490 | 0.9998 | 0.9996 | 0.9764 | 0.9996 | 0.9578 | 0.9578 | 0.9626 |
| len | 0.2987 | 0.4419 | 0.4420 | 0.2074 | 1.5506 | 1.5506 | 1.7450 | 3.6888 | 3.6888 | 3.7518 |
| 175,7,1,0.9 | 0.9444 | 0.9176 | 0.9998 | 0.9996 | 1.0000 | 0.9938 | 1.0000 | 0.9141 | 0.8626 | 0.9138 |
| len | 0.2979 | 2.0714 | 1.5217 | 1.5289 | 4.4997 | 4.4997 | 4.7467 | 2.5438 | 2.5438 | 2.6715 |
| 175,7,5,0.9 | 0.9432 | 0.9280 | 0.9208 | 0.9998 | 0.9998 | 0.9454 | 0.9998 | 0.9682 | 0.9278 | 0.9702 |
| len | 0.3015 | 2.5473 | 2.5437 | 1.6526 | 1.8231 | 1.8231 | 2.0587 | 4.1539 | 4.1539 | 4.5718 |

Table 4.8. Bootstrapping OLS Forward Selection with BIC Type 3(cont.)

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 250,10,1,0 | 0.9488 | 0.9480 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9410 | 0.9420 |
| len | 0.2495 | 0.2511 | 0.1046 | 0.1053 | 0.1006 | 6.2009 | 6.2009 | 6.3936 | 2.4516 | 2.4596 |
| 250,10,1,1/$\sqrt{p}$ | 0.9422 | 0.9764 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9640 | 0.9652 | 0.9736 |
| len | 0.2487 | 0.3473 | 0.1396 | 0.1390 | 6.1349 | 6.1349 | 6.3352 | 2.4672 | 2.4672 | 2.6317 |
| 250,10,1,0.9 | 0.9496 | 0.9038 | 0.9998 | 0.9998 | 0.9998 | 0.9992 | 1.0000 | 0.8796 | 0.8226 | 0.8812 |
| len | 0.2503 | 1.9361 | 1.1754 | 1.1703 | 5.6670 | 5.6670 | 5.9426 | 2.5317 | 2.5317 | 2.6659 |
| 250,10,8,0.9 | 0.9444 | 0.9212 | 0.9200 | 0.9996 | 0.9996 | 0.9474 | 0.9996 | 0.9114 | 0.7780 | 0.8890 |
| len | 0.2544 | 2.3389 | 2.3467 | 2.3290 | 2.3483 | 1.2941 | 1.6328 | 4.7296 | 4.7296 | 5.2096 |
| 300,6,1,0 | 0.9442 | 0.9528 | 1.0000 | 1.0000 | 1.0000 | 0.9996 | 1.0000 | 0.9424 | 0.9414 | 0.9420 |
| len | 0.2290 | 0.2302 | 0.0908 | 0.0920 | 4.7612 | 4.7612 | 5.0091 | 2.4517 | 2.4517 | 2.4550 |
| 300,6,4,0 | 0.9508 | 0.9512 | 0.9424 | 1.0000 | 0.9998 | 0.9842 | 1.0000 | 0.9388 | 0.9390 | 0.9394 |
| len | 0.2289 | 0.2300 | 0.2302 | 0.0915 | 1.3204 | 1.3204 | 1.4537 | 3.3613 | 3.3613 | 3.3637 |
| 300,6,1,1/$\sqrt{p}$ | 0.9478 | 0.9794 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9732 | 0.9748 | 0.9814 |
| len | 0.2285 | 0.3346 | 0.1296 | 0.1293 | 4.8306 | 4.8306 | 5.0424 | 2.5326 | 2.5326 | 2.6973 |
| 300,6,4,1/$\sqrt{p}$ | 0.9468 | 0.9516 | 0.9578 | 1.0000 | 0.9998 | 0.9844 | 1.0000 | 0.9652 | 0.9662 | 0.9708 |
| len | 0.2291 | 0.3470 | 0.3472 | 0.1425 | 1.3540 | 1.3504 | 1.4912 | 3.4517 | 3.4517 | 3.5168 |
| 300,6,1,0.9 | 0.9504 | 0.9386 | 0.9996 | 1.0000 | 0.9968 | 0.9734 | 0.9972 | 0.9208 | 0.8368 | 0.9228 |
| len | 0.2288 | 1.6864 | 1.0954 | 1.0822 | 4.0650 | 4.0650 | 4.2720 | 2.4643 | 2.4643 | 2.6102 |
| 300,6,4,0.9 | 0.9512 | 0.9316 | 0.9314 | 0.9994 | 0.9992 | 0.9612 | 0.9992 | 0.9292 | 0.8848 | 0.9292 |
| len | 0.2307 | 2.1519 | 2.1536 | 1.1358 | 1.6014 | 1.6014 | 1.8020 | 3.7569 | 3.7569 | 4.1761 |
| 400,8,1,0 | 0.9496 | 0.9500 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9194 | 0.9410 | 0.9412 |
| len | 0.1983 | 0.1989 | 0.0729 | 0.0727 | 5.7878 | 5.7878 | 5.9940 | 2.5409 | 2.4509 | 2.4543 |
| 400,8,6,0 | 0.9512 | 0.9504 | 0.9478 | 0.9998 | 0.9996 | 0.9856 | 0.9996 | 0.9404 | 0.9414 | 0.9412 |
| len | 0.1986 | 0.1995 | 0.1998 | 0.0737 | 1.2149 | 1.2149 | 1.3267 | 3.7947 | 3.7947 | 3.7973 |

Table 4.9. Bootstrapping OLS Forward Selection with BIC Type 3(cont.)

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 400,8,1,1/$\sqrt{p}$ | 0.9500 | 0.9814 | 1.0000 | 1.0000 | 0.9998 | 0.9998 | 1.0000 | 0.9708 | 0.9690 | 0.9780 |
| len | 0.1979 | 0.2816 | 0.0974 | 0.0994 | 5.7646 | 5.7646 | 5.9522 | 2.4991 | 2.4991 | 2.6660 |
| 400,8,6,1/$\sqrt{p}$ | 0.9496 | 0.9538 | 0.9466 | 0.9484 | 0.9560 | 0.9998 | 0.9998 | 0.9860 | 0.9998 | 0.9876 |
| len | 0.1986 | 0.2850 | 0.2851 | 0.1077 | 1.2427 | 1.2427 | 1.3585 | 3.8743 | 3.8743 | 3.9204 |
| 400,8,1,0.9 | 0.9490 | 0.9418 | 0.9970 | 0.9882 | 0.9974 | 0.9168 | 0.8432 | 1.0000 | 0.9164 | 0.9866 |
| len | 0.1983 | 1.6620 | 0.8137 | 0.8039 | 5.1044 | 5.1044 | 5.3029 | 2.4428 | 2.4428 | 2.5860 |
| 400,8,6,0.9 | 0.9500 | 0.9164 | 0.9160 | 0.9996 | 0.9996 | 0.9624 | 0.9996 | 0.8934 | 0.8014 | 0.8784 |
| len | 0.2003 | 2.0217 | 2.0326 | 0.9321 | 1.4769 | 1.4769 | 1.6566 | 4.1085 | 4.1085 | 4.6190 |

Table 4.10. Bootstrapping OLS Forward Selection with BIC Type 4

| n,p,k,$\psi$ | $\beta_1$ | $\beta_2$ | $\beta_{p-1}$ | $\beta_p$ | pm0 | hyb0 | br0 | pm1 | hyb1 | br1 |
|---|---|---|---|---|---|---|---|---|---|---|
| 100,4,1,0 | 0.9486 | 0.9518 | 0.9998 | 0.9998 | 0.9994 | 0.9992 | 0.9424 | 0.9420 | 0.9422 | 0.9422 |
| len | 0.2284 | 0.2302 | 0.1257 | 0.1243 | 3.3881 | 3.3881 | 3.6908 | 2.4449 | 2.4449 | 2.4509 |
| 100,4,2,0 | 0.9474 | 0.9466 | 0.9420 | 0.9996 | 0.9996 | 0.9716 | 0.9996 | 0.9340 | 0.9356 | .9368 |
| len | 0.2289 | 0.2307 | 0.2303 | 0.1277 | 1.8762 | 1.8762 | 2.1145 | 2.7925 | 2.7925 | 2.7964 |
| 100,4,1,1/$\sqrt{p}$ | 0.9430 | 0.9804 | 0.9994 | 0.9998 | 0.9988 | 0.9950 | 0.9990 | 0.9708 | 0.9686 | 0.9802 |
| len | 0.2284 | 0.3574 | 0.2035 | 0.2044 | 3.4673 | 3.4673 | 3.7354 | 2.5651 | 2.5651 | 2.7287 |
| 100,4,2,1/$\sqrt{p}$ | 0.9468 | 0.9560 | 0.9552 | 0.9992 | 0.9990 | 0.9678 | 0.9992 | 0.9722 | 0.9718 | 0.9770 |
| len | 0.2290 | 0.3724 | 0.3719 | 0.2141 | 1.8515 | 1.8515 | 2.0875 | 2.9239 | 2.9239 | 3.0373 |
| 100,4,1,0.9 | 0.9462 | 0.9596 | 0.9996 | 0.9992 | 0.9972 | 0.9934 | 0.9990 | 0.9504 | 0.8934 | 0.9534 |
| len | 0.2291 | 1.5321 | 1.1806 | 1.1810 | 2.9694 | 2.9694 | 3.2135 | 2.4636 | 2.4636 | 2.6350 |
| 100,4,2,0.9 | 0.9450 | 0.9350 | 0.9384 | 1.0000 | 0.9998 | 0.9672 | 0.9998 | 0.9412 | 0.9156 | 0.9498 |
| len | 0.2309 | 1.9054 | 1.9038 | 1.3513 | 2.1684 | 2.1684 | 2.4246 | 3.1921 | 3.1921 | 3.5055 |
| 175,7,1,0 | 0.9486 | 0.9502 | 0.9998 | 0.9998 | 1.0000 | 1.0000 | 1.0000 | 0.9404 | 0.9422 | 0.9432 |
| len | 0.1730 | 0.1736 | 0.0785 | 0.0786 | 5.1223 | 5.1223 | 5.3305 | 2.4481 | 2.4481 | 2.4552 |
| 175,7,5,0 | 0.9440 | 0.9516 | 0.9454 | 1.0000 | 1.0000 | 0.9786 | 1.0000 | 0.9312 | 0.9306 | 0.9310 |
| len | 0.1734 | 0.1743 | 0.1743 | 0.0825 | 1.5569 | 1.5569 | 1.7475 | 3.5505 | 3.5504 | 3.5543 |
| 175,7,1,1/$\sqrt{p}$ | 0.9494 | 0.9774 | 1.0000 | 0.9996 | 0.9996 | 0.9996 | 0.9998 | 0.9688 | 0.9676 | 0.9776 |
| len | 0.1728 | 0.2530 | 0.1153 | 0.1154 | 5.0389 | 5.0389 | 5.2498 | 2.4884 | 2.4885 | 2.6550 |
| 175,7,5,1/$\sqrt{p}$ | 0.9454 | 0.9450 | 0.9462 | 1.0000 | 1.0000 | 0.9794 | 1.0000 | 0.9558 | 0.9560 | 0.9618 |
| len | 0.1736 | 0.2552 | 0.2555 | 0.1192 | 1.5581 | 1.5581 | 1.7409 | 3.6406 | 3.6406 | 3.7037 |
| 175,7,1,0.9 | 0.9538 | 0.9664 | 1.0000 | 1.0000 | 0.9984 | 0.9898 | 0.9988 | 0.9466 | 0.8914 | 0.9500 |
| len | 0.1734 | 1.5276 | 0.8450 | 0.8397 | 4.7090 | 4.7090 | 4.9299 | 2.4204 | 2.4204 | 2.5734 |
| 175,7,5,0.9 | 0.9462 | 0.9332 | 0.9304 | 0.9998 | 0.9998 | 0.9580 | 0.9998 | 0.9142 | 0.8318 | 0.8992 |
| len | 0.1760 | 1.7847 | 1.7941 | 0.9309 | 1.7665 | 1.7665 | 1.9949 | 3.9187 | 3.9187 | 4.2770 |

Table 4.11. Bootstrapping OLS Forward Selection with BIC Type 4(cont.)

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 250,10,1,0 | 0.9486 | 0.9496 | 0.9998 | 1.0000 | 0.9994 | 1.0000 | 1.0000 | 0.9436 | 0.9436 | 0.9448 |
| len | 0.1446 | 0.1450 | 0.0591 | 0.0605 | 6.2170 | 6.2170 | 6.4070 | 2.4488 | 2.4488 | 2.4565 |
| 250,10,1,1/$\sqrt{p}$ | 0.9432 | 0.9782 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9608 | 0.9628 | 0.9698 |
| len | 0.1446 | 0.2017 | 0.0826 | 0.0813 | 6.1348 | 6.1348 | 6.3360 | 2.4660 | 2.4660 | 2.6299 |
| 250,10,1,0.9 | 0.9484 | 0.9780 | 0.9998 | 0.9998 | 0.9992 | 0.9986 | 0.9994 | 0.9498 | 0.9022 | 0.9550 |
| len | 0.1449 | 1.4437 | 0.5954 | 0.6013 | 6.0119 | 6.0119 | 6.2151 | 2.4261 | 2.4261 | 2.5819 |
| 250,10,8,0.9 | 0.9498 | 0.9288 | 0.9318 | 0.9336 | 0.9998 | 0.9996 | 0.9728 | 0.9996 | 0.8174 | 0.8844 |
| len | 0.1479 | 1.5961 | 1.5993 | 0.7035 | 1.5692 | 1.5692 | 1.7678 | 4.4792 | 4.4792 | 4.8229 |
| 300,6,1,0 | 0.9516 | 0.9490 | 1.0000 | 0.9998 | 0.9998 | 0.9994 | 0.9998 | 0.9478 | 0.9466 | 0.9470 |
| len | 0.1326 | 0.1327 | 0.0524 | 0.0533 | 4.7810 | 4.7810 | 5.0298 | 2.4491 | 2.4491 | 2.4527 |
| 300,6,4,0 | 0.9508 | 0.9416 | 0.9538 | 1.0000 | 0.9996 | 0.9812 | 1.0000 | 0.9450 | 0.9438 | 0.9444 |
| len | 0.1327 | 0.1231 | 0.1332 | 0.0525 | 1.2950 | 1.2950 | 1.4244 | 3.3317 | 3.3317 | 3.3347 |
| 300,6,1,1/$\sqrt{p}$ | 0.9490 | 0.9820 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9726 | 0.9748 | 0.9836 |
| len | 0.1326 | 0.1944 | 0.0765 | 0.0757 | 4.8152 | 4.8152 | 5.0303 | 2.5320 | 2.5320 | 2.6990 |
| 300,6,4,1/$\sqrt{p}$ | 0.9562 | 0.9472 | 0.9548 | 1.0000 | 1.0000 | 0.9828 | 1.0000 | 0.9662 | 0.9654 | 0.9700 |
| len | 0.1329 | 0.2004 | 0.2006 | 0.0820 | 1.3333 | 1.3333 | 1.4697 | 3.4292 | 3.4292 | 3.4938 |
| 300,6,1,0.9 | 0.9520 | 0.9800 | 1.0000 | 0.9998 | 0.9992 | 0.9946 | 0.9992 | 0.9740 | 0.9504 | 0.9814 |
| len | 0.1327 | 1.1535 | 0.5278 | 0.5256 | 4.6690 | 4.6690 | 4.8983 | 2.5783 | 2.5783 | 2.8073 |
| 300,6,4,0.9 | 0.9530 | 0.9420 | 0.9438 | 0.9998 | 0.9998 | 0.9780 | 0.9998 | 0.9580 | 0.9250 | 0.9594 |
| len | 0.1336 | 1.4494 | 1.4438 | 0.5901 | 1.4967 | 1.4968 | 1.6719 | 3.6865 | 3.6865 | 3.9079 |
| 400,8,1,0 | 0.9512 | 0.9456 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9466 | 0.9462 | 0.9468 |
| len | 0.1148 | 0.1150 | 0.0424 | 0.0423 | 5.8070 | 5.8070 | 6.0107 | 2.4489 | 2.4489 | 2.4527 |
| 400,8,6,0 | 0.9492 | 0.9450 | 0.9480 | 1.0000 | 1.0000 | 0.9868 | 1.0000 | 0.9410 | 0.9410 | 0.9424 |
| len | 0.1149 | 0.1152 | 0.1152 | 0.0429 | 1.2320 | 1.2320 | 1.3476 | 3.7631 | 3.7631 | 3.7652 |

Table 4.12. Bootstrapping OLS Forward Selection with BIC Type 4(cont.)

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 400,8,1,1/$\sqrt{p}$ | 0.9550 | 0.9822 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9708 | 0.9696 | 0.9794 |
| len | 0.1148 | 0.1632 | 0.0580 | 0.0590 | 5.7763 | 5.7763 | 5.9636 | 2.5016 | 2.5016 | 2.6677 |
| 400,8,6,1/$\sqrt{p}$ | 0.9498 | 0.9490 | 0.9488 | 1.0000 | 1.0000 | 0.9882 | 1.0000 | 0.9618 | 0.9638 | 0.9670 |
| len | 0.1149 | 0.1648 | 0.1647 | 0.0615 | 1.2425 | 1.2425 | 1.3549 | 3.8397 | 3.8397 | 3.8867 |
| 400,8,1,0.9 | 0.9518 | 0.9850 | 1.0000 | 1.0000 | 0.9998 | 0.9994 | 0.9998 | 0.9724 | 0.9616 | 0.9840 |
| len | 0.1148 | 1.0996 | 0.4184 | 0.4083 | 5.7252 | 5.7252 | 5.9246 | 2.5917 | 2.5917 | 2.8365 |
| 400,8,6,0.9 | 0.9522 | 0.9410 | 0.9402 | 0.9998 | 0.9996 | 0.9832 | 0.9998 | 0.9658 | 0.9418 | 0.9630 |
| len | 0.1158 | 1.2818 | 1.2789 | 0.4604 | 1.3524 | 1.3524 | 1.4912 | 4.1606 | 4.1606 | 4.3391 |

Table 4.13. Bootstrapping OLS Forward Selection with BIC Type 5

| n,p,k,$\psi$ | $\beta_1$ | $\beta_2$ | $\beta_{p-1}$ | $\beta_p$ | pm0 | hyb0 | br0 | pm1 | hyb1 | br1 |
|---|---|---|---|---|---|---|---|---|---|---|
| 100,4,1,0 | 0.9432 | 0.9414 | 0.9998 | 1.0000 | 0.9992 | 0.9954 | 0.9996 | 0.9410 | 0.9288 | 0.9446 |
| len | 1.2681 | 1.3469 | 0.8314 | 0.8318 | 3.1723 | 3.1723 | 3.4764 | 2.5255 | 2.5255 | 2.5690 |
| 100,4,2,0 | 0.9422 | 0.9342 | 0.9362 | 0.9998 | 0.9998 | 0.9634 | 0.9998 | 0.9254 | 0.8716 | 0.9178 |
| len | 1.2735 | 1.3733 | 1.3800 | 0.7521 | 1.9297 | 1.9297 | 2.2047 | 2.9693 | 2.9693 | 3.1249 |
| 100,4,1,1/$\sqrt{p}$ | 0.9400 | 0.9604 | 0.9992 | 0.9992 | 0.9978 | 0.9720 | 0.9990 | 0.9536 | 0.9254 | 0.9578 |
| len | 1.2754 | 1.8040 | 1.4672 | 1.4725 | 3.0138 | 3.0138 | 3.2893 | 2.5095 | 2.5095 | 2.6738 |
| 100,4,2,1/$\sqrt{p}$ | 0.9394 | 0.9360 | 0.9330 | 1.0000 | 1.0000 | 0.9436 | 1.0000 | 0.9562 | 0.9338 | 0.9630 |
| len | 1.2724 | 2.0618 | 2.0483 | 1.4971 | 2.2581 | 2.2581 | 2.5647 | 3.0645 | 3.0645 | 3.3658 |
| 100,4,1,0.9 | 0.9332 | 0.9310 | 0.9976 | 0.9980 | 0.9986 | 0.9842 | 0.9998 | 0.9824 | 0.9834 | 0.9884 |
| len | 1.2691 | 6.8111 | 6.8183 | 6.8333 | 3.2946 | 3.2946 | 3.5151 | 2.7596 | 2.7596 | 2.9196 |
| 100,4,2,0.9 | 0.9376 | 0.9604 | 0.9014 | 0.9984 | 0.9976 | 0.9080 | 0.9970 | 0.9932 | 0.9900 | 0.9962 |
| len | 1.2677 | 6.3865 | 6.3837 | 6.3609 | 1.9602 | 1.9602 | 2.2601 | 3.3072 | 3.3072 | 3.4745 |
| 175,7,1,0 | 0.9462 | 0.9288 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9504 | 0.9466 | 0.9550 |
| len | 0.9739 | 1.0769 | 0.4739 | 0.4716 | 4.9193 | 4.9193 | 5.1507 | 2.5252 | 2.5252 | 2.5673 |
| 175,7,5,0 | 0.9424 | 0.9298 | 0.9296 | 1.0000 | 1.0000 | 0.9764 | 1.0000 | 0.9602 | 0.9208 | 0.9494 |
| len | 0.9877 | 1.1252 | 1.1210 | 0.4564 | 1.5369 | 1.5369 | 1.7446 | 3.9744 | 3.9744 | 4.0992 |
| 175,7,1,1/$\sqrt{p}$ | 0.9514 | 0.9640 | 0.9998 | 1.0000 | 0.9998 | 0.9992 | 0.9998 | 0.9468 | 0.9178 | 0.9530 |
| len | 0.9762 | 1.4899 | 0.7805 | 0.7824 | 4.7572 | 4.7572 | 4.9923 | 2.4946 | 2.4946 | 2.6505 |
| 175,7,5,1/$\sqrt{p}$ | 0.9432 | 0.9322 | 0.9386 | 0.9996 | 0.9996 | 0.9608 | 0.9996 | 0.9250 | 0.8688 | 0.9156 |
| len | 0.9937 | 1.6758 | 1.6747 | 0.8430 | 1.7289 | 1.7289 | 1.9676 | 3.9330 | 3.9330 | 4.2147 |
| 175,7,1,0.9 | 0.9444 | 0.8132 | 1.0000 | 0.9998 | 1.0000 | 0.9998 | 1.0000 | 0.9846 | 0.9836 | 0.9900 |
| len | 0.9700 | 6.0518 | 5.8893 | 5.8521 | 4.5535 | 4.5535 | 4.7975 | 2.8241 | 2.8241 | 3.0073 |
| 175,7,5,0.9 | 0.9424 | 0.7528 | 0.7412 | 0.9998 | 0.9998 | 0.9218 | 0.9998 | 0.9998 | 1.0000 | 1.0000 |
| len | 0.9724 | 5.2087 | 5.2194 | 4.8546 | 2.0096 | 2.0096 | 2.2791 | 4.5509 | 4.5509 | 4.8019 |

Table 4.14. Bootstrapping OLS Forward Selection with BIC Type 5(cont.)

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 250,10,1,0 | 0.9476 | 0.9410 | 0.9994 | 0.9998 | 1.0000 | 1.0000 | 1.0000 | 0.9596 | 0.9594 | 0.9616 |
| len | 0.8189 | 0.8923 | 0.3506 | 0.3452 | 6.1250 | 6.1250 | 6.3264 | 2.5109 | 2.5109 | 2.5361 |
| 250,10,1,1/$\sqrt{p}$ | 0.9434 | 0.9594 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9546 | 0.9338 | 0.9628 |
| len | 0.8208 | 1.2706 | 0.4983 | 0.4917 | 6.0051 | 6.0051 | 6.2162 | 2.5178 | 2.5178 | 2.6859 |
| 250,10,1,0.9 | 0.9438 | 0.7414 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9844 | 0.9844 | 0.9894 |
| len | 0.8188 | 5.0029 | 4.7703 | 5.3780 | 5.3780 | 5.6168 | 5.6168 | 2.8269 | 2.8269 | 3.0155 |
| 250,10,8,0.9 | 0.9464 | 0.7068 | 0.6962 | 0.9996 | 0.9996 | 0.9620 | 0.9996 | 1.0000 | 0.9998 | 1.0000 |
| len | 0.8264 | 4.7997 | 4.7404 | 3.9512 | 1.5842 | 1.5842 | 1.7615 | 5.6604 | 5.6604 | 5.9544 |
| 300,6,1,0 | 0.9494 | 0.9428 | 1.0000 | 1.0000 | 1.0000 | 0.9998 | 1.0000 | 0.9628 | 0.9638 | 0.9654 |
| len | 0.7523 | 0.7902 | 0.2986 | 0.2969 | 4.7093 | 4.7093 | 4.9580 | 2.4926 | 2.4926 | 2.5019 |
| 300,6,4,0 | 0.9488 | 0.9426 | 0.9360 | 1.0000 | 1.0000 | 0.9844 | 1.0000 | 0.9780 | 0.9728 | 0.9776 |
| len | 0.2300 | 0.2303 | 0.2305 | 0.2305 | 0.2304 | 0.0929 | 1.3467 | 1.3467 | 1.4834 | 3.3391 |
| 300,6,1,1/$\sqrt{p}$ | 0.9510 | 0.9690 | 1.0000 | 1.0000 | 0.9996 | 0.9998 | 0.9996 | 0.9708 | 0.9562 | 0.9782 |
| len | 0.7534 | 1.2346 | 0.5082 | 0.5040 | 4.6388 | 4.6388 | 4.8629 | 2.5457 | 2.5457 | 2.7269 |
| 300,6,4,1/$\sqrt{p}$ | 0.9428 | 0.9380 | 0.9444 | 0.9998 | 0.9998 | 0.9770 | 0.9998 | 0.9596 | 0.9306 | 0.9544 |
| len | 0.2300 | 0.3473 | 0.3473 | 0.3471 | 0.3469 | 0.1406 | 1.3240 | 1.3240 | 1.4625 | 3.4319 |
| 300,6,1,0.9 | 0.9478 | 0.8594 | 0.9996 | 0.9998 | 1.0000 | 0.9998 | 1.0000 | 0.9784 | 0.9764 | 0.9826 |
| len | 0.7515 | 4.1236 | 3.8798 | 3.8860 | 4.5491 | 4.5491 | 4.7538 | 2.7358 | 2.7358 | 2.8789 |
| 300,6,4,0.9 | 0.9464 | 0.7902 | 0.7826 | 0.9998 | 0.9998 | 0.9066 | 1.0000 | 0.9990 | 0.9976 | 0.9996 |
| len | 0.7500 | 3.8525 | 3.8329 | 3.4621 | 1.8744 | 1.8744 | 2.1509 | 4.2067 | 4.2067 | 4.4727 |
| 400,8,1,0 | 0.9526 | 0.9480 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9598 | 0.9612 | 0.9618 |
| len | 0.6511 | 0.6697 | 0.2387 | 0.2357 | 5.7319 | 5.7319 | 5.9392 | 2.4782 | 2.4782 | 2.4852 |
| 400,8,6,0 | 0.9492 | 0.9454 | 0.9438 | 1.0000 | 0.9998 | 0.9864 | 1.0000 | 0.9762 | 0.9758 | 0.9770 |
| len | 0.6546 | 0.6789 | 0.6791 | 0.2346 | 1.1894 | 1.1894 | 1.3040 | 3.9991 | 3.9991 | 4.0171 |

Table 4.15. Bootstrapping OLS Forward Selection with BIC Type 5(cont.)

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 400,8,1,1/$\sqrt{p}$ | 0.9516 | 0.9696 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.9716 | 0.9676 | 0.9798 |
| len | 0.6530 | 1.0319 | 0.3563 | 0.3519 | 5.6889 | 5.6889 | 5.8853 | 2.5461 | 2.5461 | 2.7269 |
| 400,8,6,1/$\sqrt{p}$ | 0.9478 | 0.9398 | 0.9364 | 1.0000 | 1.0000 | 0.9836 | 1.0000 | 0.9796 | 0.9608 | 0.9748 |
| len | 0.6591 | 1.1202 | 1.1170 | 0.3938 | 1.3080 | 1.3080 | 1.4506 | 4.2212 | 4.2212 | 4.3306 |
| 400,8,1,0.9 | 0.9540 | 0.8134 | 0.9996 | 1.0000 | 0.9998 | 0.9998 | 0.9998 | 0.9794 | 0.9782 | 0.9818 |
| len | 0.6503 | 3.6476 | 3.3059 | 3.3015 | 5.2450 | 5.2450 | 5.4707 | 2.7237 | 2.7237 | 2.8710 |
| 400,8,6,0.9 | 0.9488 | 0.7232 | 0.7294 | 1.0000 | 1.0000 | 0.9610 | 1.0000 | 0.9996 | 0.9992 | 0.9666 |
| len | 0.6561 | 3.9995 | 4.0096 | 2.9919 | 1.4677 | 1.4677 | 1.6283 | 5.1005 | 5.1005 | 5.4079 |

# CHAPTER 5

## CONCLUSIONS

There is massive literature on variable selection and a fairly large literature for inference after variable selection. See references in Pelawa Watagoda and Olive (2018).

Response plots of the fitted values $\hat{Y}$ versus the response $Y$ are useful for checking linearity of the MLR model and for detecting outliers. Residual plots should also be made.

The simulations were done in *R*. See R Core Team (2016). We used several *R* functions including forward selection as computed with the `regsubsets` function from the `leaps` library. The collection of Olive (2018b) *R* functions *slpack*, available from (http://lagrange.math.siu.edu/Olive/slpack.txt), has some useful functions for the inference.

The tables were made with `bicbootsim`. There was occasionally undercoverage, especially for the hybrid region and $\psi$=0.9.

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Research Paper Title:
Bootstrapping Forward Selection with BIC

Major Professor: Dr. David J. Olive