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Analytical Formulas for Risk Assessment for a Class of Problems where Risk Depends on Three Interrelated Variables

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ABSTRACT

We derive general analytical formulas for assessing risks in a problem domain where the risk depends on three interrelated variables. More specifically, we derive general analytical formulas for propagating beliefs in a network where three binary variables, A, B and C, are related to a fourth binary variable Z through an 'AND' relationship. In addition, we assume that variables A, B and C are interrelated in that a change in one variable may affect the value of each of the other two. The analytical formulas derived in this article determine the overall belief and plausibility that Z is true or not true, given that we have beliefs on variables A, B and/or C.

To demonstrate the importance of the general results, we use the results to develop models applicable to three real-world situations. The first model can aid external auditors in assessing the quality of an audit client's internal audit function to determine the extent to which the internal auditor's work can be relied on in the conduct of a financial audit while the second can aid in assessing the risk of impaired auditor independence when conducting a financial statement audit. The third model can be used to assess the risk of management fraud in financial reporting. Assessment of such risks is of critical importance to external auditors, regulators, and the investing public. Analytical formulas to help address these types of important business and economic problems have not been available prior to these derivations.

Key Words: Risk Assessment, Belief Propagation, Dempster-Shafer Theory of Belief Functions, Interacting Variables, Fraud Risk Assessment Model, Auditor's Independent Risk Assessment Model, Internal Audit Function Assessment Model

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I. INTRODUCTION

In the business world, failure to recognize and assess risks can result in significant costs to the public. In a financial statement audit, for example, it is important that the auditor assess the risk of financial statement fraud. However, as evidenced from the many cases of fraudulent financial reporting, auditors face significant challenges in assessing such risks adequately.[1] The inability to assess fraud risk adequately has cost the auditing profession and the investing public billions of dollars. This is evidenced by the demise of one of the world's largest accounting and auditing firms, Arthur Andersen, and the failures of companies such as Enron and WorldCom. According to Cotton [5], shareholders lost $460 billion in the five fraud cases of Enron, Global Crossing, Qwest, WorldCom, and Tyco alone. The cost is much more if the indirect costs of fraudulent financial reporting behaviors are considered, such as the loss of public trust in the auditing profession and reduced confidence in the capital market system that is the engine of the global economy.

The auditing profession has been aware of the need to identify and assess the risk of financial statement fraud for some time. In 2002 the American Institute of Certified Public Accountants (AICPA) published *Statement on Auditing Standards No. 99*, *Consideration of Fraud in a Financial Statement Audit* [3], which requires a pre-audit assessment of the risk of fraud by the independent auditor as well as a continuous assessment update as a financial

[1] For example, Enron, Global Crossing, Qwest, WorldCom, and Tyco. See [5] for more examples.

statement audit progresses. This statement indicates that three conditions generally are present when fraud occurs: [3, ¶7]:

1. management or other employees have an *incentive* or are under pressure which provides a reason to commit fraud,
2. circumstances exist that provide an *opportunity* for a fraud to be perpetrated, such as the absence of controls, ineffective controls, or the ability of management to override existing controls, and
3. those involved are able to *rationalize* committing a fraudulent act.

Logically however, if any one, two, or all of these conditions are absent then fraud should not occur. These three factors are known as “fraud triangle” factors [17].

The main purpose of this article is to derive general analytical formulas for assessing risks in a problem domain where the risk depends on three interrelated variables such as in the case of fraud. This problem context is quite general and applies to several other important business risk- assessment contexts such as auditor independence and the quality of the internal audit function.

For example, lack of auditor independence is a critical risk requiring assessment. Auditor independence risk is defined as the risk that threats to auditor independence, to the extent that they are not mitigated by safeguards, compromise or can reasonably be expected to compromise, an auditor’s ability to make unbiased audit decisions about the financial statements of a specific client [9]. In testimony before the U.S. Securities and Exchange Commission (SEC), Ralph Whitworth, Managing Member, Relational Investors LLC argued that “[A]uditor independence goes to the very essence of our capital markets, and its linked inextricably to the efficiencies of our capitalist system" [18]. Turner et al. [15, 34] argue that the risk of compromised independence depends on three interrelated variables: Incentives, Opportunity and Integrity. These three factors are similar to fraud triangle factors discussed earlier.

Another example of the general three-variable problem is assessing the risk of the internal audit function not being of high quality. Internal auditing is a key function within most large organizations that is intended to monitor and improve the operating effectiveness and efficiencies of the organization it serves. Krishnamoorthy [10] has analyzed the quality [‘strength’] of the internal audit function as a function of three interrelated variables: Competence, Work Performance, and Objectivity. Again, one can use the general formulas developed in this article to assess the risk of the internal audit function not being of high quality.

Usually, the degree to which factors affecting a specific type of risk are present or absent is not known with certainty. Thus, we use the Dempster-Shafer (D-S) theory of belief functions to model the uncertainties associated with the items of evidence pertaining to these variables [19, 36]. Under the D-S theory of belief functions, risk is defined by the plausibility function [32]. In this article, we derive analytical formulas for propagating beliefs in a network of four interacting binary variables; a risk variable and three other interrelated variables that can affect the risk variable. As part of our derivation, we use the Shenoy and Shafer [23, 24] approach for propagating beliefs through the network to derive the general formulas.

To illustrate our solution for this class of risk assessment problems, we derive general analytical formulas for propagating beliefs in a network where three binary variables, A, B and C, are related to a fourth binary variable Z through a logical ‘AND’ relationship. In addition, we assume that variables A, B and C may be interrelated in that a change in one variable may affect the value of each of the other two. The analytical formulas derived determine the overall beliefs and plausibilities that Z is true or not true, given that we have beliefs about variables A, B and C.

As noted above, such formulas provide analytical models for assessing risks in several important real world problems as discussed in Section IV.

The remainder of this paper is divided into four sections. The next section introduces belief functions while Section III develops the analytical formulas by combining seven sets of belief functions using Shenoy and Shafer [24]. Section IV discusses three real world applications of the general formulas in assessing fraud risk in financial reporting, assessing the auditor's independence risk in assurance services, and assessing the strength or quality of the internal audit function by the external auditor. Section V provides the overall study conclusions. Finally, Appendix A provides the proof of Theorem 1, and Appendix B provides the proof of Corollary 1 proposed in Section III.

II. INTRODUCTION TO THE DEMPSTER-SHAFER THEORY OF BELIEF FUNCTIONS

The D-S theory of belief functions is based on the work of Dempster [6] during the 1960s and the work of Shafer during the 1970s [19, see also 20, 21, 22, 23]. In fact, the D-S theory of belief functions is a generalization of Bayesian theory. To clarify the distinction between the two frameworks, let us consider a variable X with q possible mutually exclusive and exhaustive sets of values[2]: $x_1, x_2, \ldots, x_q$. This set of values defines the *frame* of X. Let us denote this frame by the symbol $\Theta_X = \{x_1, x_2, \ldots x_q\}$. Suppose we do not know the true state of variable X, i.e., we do not know what value X will take. In such a situation, under probability framework we assign probability mass, $P(x_i)$, to each single element, x_i, of the frame Θ_X in such a way that sum of these probability masses equals one, i.e., $\sum_{i=1}^{q} P(x_i) = 1$, where $1 \geq P(x_i) \geq 0$. Under the DS theory, we assign belief mass to all the possible subsets of the frame, Θ_X, i.e., to all the singletons, all the subsets of two, all the subsets of three, and so on to the entire frame Θ_X. The belief mass assigned to a subset, say Y, can be denoted by m(Y), and the sum of these belief masses equals one, i.e., $\sum_{Y \subseteq \Theta_X} m(Y) = 1$, where $1 \geq m(Y) \geq 0$. By definition, the belief mass on the empty set is zero. i.e., $m(\varnothing) = 0$. Shafer [19] calls this set of belief masses the *basic probability assignment function*; we will call it the *m-values* or *belief mass function* or simply the *mass function*. As one can see from the above definition of the mass function, the D-S theory reduces to a probability framework if m-values for all the subsets except the singletons are zero.

In more conceptual terms, the basic algebra of belief functions is relatively simple and begins with developing beliefs about an assertion or issue based on items of evidence pertaining to that assertion or issue. For example, when evaluating a general assertion, say assertion A, evidence E1 may provide, in general, some support that assertion A is true, i.e., '*a*' is true, and some support that A is not true, i.e., '~*a*' is true. In terms of the *mass function* we can write these assessments as $m_{E1}(\{a\})$ and $m_{E1}(\{\sim a\})$ respectively. Lack of knowledge about whether A is true or not true is represented by $m_{E1}(\{a,\sim a\})$, such that the sum of the three m-values is one. i.e., $m_{E1}(\{a\}) + m_{E1}(\{\sim a\}) + m_{E1}(\{a,\sim a\}) = 1$.

[2] We use the upper case letter for the name of the variable and lower case letter for its values. For example, if Z is the name of a binary variable then 'z', and '~z', respectively, represent the two possible values of Z being true or false. The frame of a variable is denoted by the symbol Θ with the variable as a subscript. For example, the frame of variable Z is denoted by $\Theta_Z = \{z, \sim z\}$.

Belief Functions

The belief in a subset, say Y, represents the total belief that Y is true and is the sum of the m-values defined at Y and the m-values defined on any subsets contained in Y. Mathematically, it can be expressed as:

$$\mathrm{Bel}(\mathrm{Y}) = \sum_{\mathrm{G} \subseteq \mathrm{Y}} \mathrm{m}(\mathrm{G}) .$$

For our example above, the belief that assertion A is true based on evidence E1 is given by $\mathrm{Bel}_{\mathrm{E1}}(\{a\}) = \mathrm{m}_{\mathrm{E1}}(\{a\})$, the belief that assertion A is not true is given by $\mathrm{Bel}_{\mathrm{E1}}(\{\sim a\}) = \mathrm{m}_{\mathrm{E1}}(\{\sim a\})$, and the lack of belief about assertion A is given by $\mathrm{Bel}_{\mathrm{E1}}(\{a,\sim a\}) = \mathrm{m}_{\mathrm{E1}}(\{a,\sim a\})$. A belief of one in a statement represents certainty similar to a value of one for probability in a statement. However, a belief of zero in a statement represents ignorance while a zero probability represents impossibility.

Plausibility Functions

The plausibility in a subset, say Y, determines the maximum possible belief one could assign to Y based on the current evidence and the assumption that all the future evidence will be in favor of supporting the subset Y. In mathematical terms, this definition can be written as:

$$\mathrm{Pl}(\mathrm{Y}) = \sum_{\mathrm{G} \cap \mathrm{Y} \neq \varnothing} \mathrm{m}(\mathrm{G}) .$$

For our example of assertion A described earlier, the plausibility that '*a*' is true based on the evidence E1 is given by $\mathrm{Pl}_{\mathrm{E1}}(\{a\}) = \mathrm{m}_{\mathrm{E1}}(\{a\}) + \mathrm{m}_{\mathrm{E1}}(\{a,\sim a\})$, and the plausibility that '$\sim a$' is true is given by $\mathrm{Pl}_{\mathrm{E1}}(\{\sim a\}) = \mathrm{m}_{\mathrm{E1}}(\{\sim a\}) + \mathrm{m}_{\mathrm{E1}}(\{a,\sim a\})$.

Dempster's Rule of Combination

Dempster's rule [19] is used to combine independent items of evidence from multiple sources. For combining two sets of mass functions defined on the same frame, one can write Dempster's rule as:

$$m(Y) = \sum_{Y_1 \cap Y_2 = Y} m_1(Y_1)m_2(Y_2)/K, \text{ where } K = 1 - \sum_{Y_1 \cap Y_2 = \varnothing} m_1(Y_1)m_2(Y_2).$$

K represents the renormalization constant defined above as one minus the conflict.

To illustrate the concepts, let us consider our example of assertion A and the evidence E1 that yield a set of m-values represented by $m_{E1}(\{a\})$, $m_{E1}(\{\sim a\})$, and $m_{E1}(\{a,\sim a\})$. Consider a second source of evidence, E2, with the following mass function: $m_{E2}(a)$, $m_{E2}(\sim a)$, and $m_{E2}(\{a,\sim a\})$. The combined mass function using Dempster's rule is given as:

$$m_{E12}(\{a\}) = [m_{E1}(\{a\})m_{E2}(\{a\}) + m_{E1}(\{a\})m_{E2}(\{a,\sim a\}) + m_{E1}(\{a,\sim a\})m_{E2}(\{a\})]/K_{E12},$$

$$m_{E12}(\{\sim a\}) = [m_{E1}(\{\sim a\})m_{E2}(\{\sim a\}) + m_{E1}(\{\sim a\})m_{E2}(\{a,\sim a\}) + m_{E1}(\{a,\sim a\})m_{E2}(\{\sim a\})]/K_{E12},$$

$$m_{E12}(\{a,\sim a\}) = [m_{E1}(\{a,\sim a\})m_{E2}(\{a,\sim a\})]/K_{E12},$$

where K_{E12} is the renormalization constant defined as:

$$K_{E12} = 1 - [m_{E1}(\{a\})m_{E2}(\{\sim a\}) + m_{E1}(\{\sim a\})m_{E2}(\{a\})].$$

The second term in K_{E12} represents conflict between the two sets of beliefs pertaining to assertion A.

III. ANALYTICAL FORMULAS

In this section, we develop the analytical formulas for propagating beliefs in the network of binary variables shown in Figure 1 from variables A, B and C to the variable Z. Variables A, B and C are related to Z through a logical 'AND' relationship. In addition, in our derivation of the general formula we consider two-way relationships among the variables A, B and C. In other words, variable A is connected to B through a relationship depicted by R_1, B is connected to C through a relationship depicted by R_2, and C is connected to A through a relationship depicted by R_3. These relationships are bidirectional and are elaborated later in this section. We consider one item of evidence for each variable A, B and C as depicted in Figure 1. However, one can extend the present approach to the case where there is more than one item of evidence for each variable by using Dempster's rule to combine the multiple items of evidence for each variable as described in [28].

----- Figure 1 about here -----

As mentioned earlier, we use the Dempster-Shafer theory of belief functions to represent the uncertainties in the strength of evidence pertaining to individual variables A, B and C. Let us consider the following set of *mass functions* to represent the beliefs at these variables:

The beliefs at variable A: $\mathrm{m_A}(\{a\}) = \mathrm{m_A^+}$, $\mathrm{m_A}(\{\sim a\}) = \mathrm{m_A^-}$, $\mathrm{m_A}(\{a,\sim a\}) = \mathrm{m_A^\Theta}$. (1)

The beliefs at variable B: $\mathrm{m_B}(\{b\}) = \mathrm{m_B^+}$, $\mathrm{m_B}(\{\sim b\}) = \mathrm{m_B^-}$, $\mathrm{m_B}(\{b,\sim b\}) = \mathrm{m_B^\Theta}$. (2)

The beliefs at variable C: $\mathrm{m_C}(\{c\}) = \mathrm{m_C^+}$, $\mathrm{m_C}(\{\sim c\}) = \mathrm{m_C^-}$, $\mathrm{m_C}(\{c,\sim c\}) = \mathrm{m_C^\Theta}$. (3)

The interrelationships between A and B, between B and C, and between A and C, are assumed to be of the following form:

Relationship between A and B: $\mathrm{m_{AB}}(\{ab,\sim a\sim b\}) = \mathrm{r_1}$, $\mathrm{m_{AB}}(\{ab,a\sim b,\sim ab,\sim a\sim b\}) = 1-\mathrm{r_1}$. (4)

Relationship between B and C: $\mathrm{m_{BC}}(\{bc,\sim b\sim c\}) = \mathrm{r_2}$, $\mathrm{m_{BC}}(\{bc,b\sim c,\sim bc,\sim b\sim c\}) = 1-\mathrm{r_2}$. (5)

Relationship between A and C: $\mathrm{m_{AC}}(\{ac,\sim a\sim c\}) = \mathrm{r_3}$, $\mathrm{m_{AC}}(\{ac, a\sim c, \sim ac, \sim a\sim c\}) = 1-\mathrm{r_3}$. (6)

Various m-values and the interrelationships are defined in Table 1.

----- Table 1 about here -----

These relationships imply that if one variable, say A, is true then variable B also is true with a belief given by the corresponding strength of the relationship represented by r_1 and variable C is true with a belief given by the corresponding strength of the relationship represented by r_2, assuming there is no other belief defined for variables B and C. In addition, if one variable is false then the other variables also are false, again with a belief given by the corresponding strength of the relationships. The values of the strength of each of the relationships, r_i's, lie between zero and 1 where a zero value means there is no relationship between the two variables. A value of one for a relationship implies that if one variable is true with a given degree of belief then the related variable also is true with the same degree of belief assuming that there is no other belief defined for the related variable. For example, if A is true with a belief of, say 0.9 (i.e., Bel($\{a\}$) = 0.9) and we assume that there is no relationship between B and C (i.e., $r_2 = 0$) and there are no beliefs from any other source at B and C, then B will be true with a belief of 0.9 and C will be true with a belief of 0.9 if $r_1 = 1$ and $r_3 = 1$. Also, under the above condition (i.e., $r_1 = 1$ and $r_3 = 1$), if A is not true with a belief of, say 0.9 (i.e., Bel($\sim a$) = 0.9) then B will also be not true with a belief of 0.9, and C will not be true with a belief of 0.9.

Such relationships are quite common in real world situations as discussed in Section IV in detail. For example, even though management of a company may appear to have high integrity, if incentives exist for management to benefit from misrepresenting financial information, their ethics may be compromised to the point of committing financial statement fraud to achieve those incentives. Similarly, if management's integrity is compromised, then incentives and/or opportunities may be created to benefit from committing fraud. On the other hand, if there are no incentives to benefit from committing fraud or no opportunities available, then management will behave appropriately and not commit fraud. These interrelationships can be modeled using the above relationships.

The logical relationship ‘AND’ between Z and the variables A, B and C is expressed in terms of the following *mass function* (see [30] for details):

$$m_{ZABC}(\Theta_{ZABC}) = 1.0. \tag{7}$$

where $\Theta_{ZABC} = \{zabc, \sim zab\sim c, \sim za\sim bc, \sim z\sim abc, \sim za\sim b\sim c, \sim z\sim ab\sim c, \sim z\sim a\sim bc, \sim z\sim a\sim b\sim c\}$.

In the present problem, we have seven *mass functions*, three corresponding to the variables A, B and C, (i.e., m_A, m_B and m_C) and four representing the interrelationships, m_{AB}, m_{BC}, m_{AC} and m_{ZABC}, as given in (4)-(7). To derive the analytical formulas for the *mass function* at variable Z, we need to combine all seven *mass functions* and marginalize[3] the result to variable Z:

$$m_{Z\leftarrow ABC} = (m_A \oplus m_B \oplus m_C \oplus m_{AB} \oplus m_{AC} \oplus m_{BC} \oplus m_{ZABC})^{\downarrow Z}, \tag{8}$$

where $m_{Z\leftarrow ABC}$ represents the *mass function* at Z propagated from variables A, B and C, the symbol $\oplus$ denotes the combination of beliefs, i.e., *mass functions*, using Dempster’s rule, and the symbol $\downarrow Z$ represents the process of marginalization of the combined mass function within the parenthesis to the frame of variable Z. We express these results through the following theorem.

[3] The marginalization process in D-S theory is similar to the marginalization process in probability theory. For example, suppose we have a probability distribution over two variables A and B and we want the distribution over just one variable, say A. The second variable B can be eliminated by summing the probabilities over variable B to obtain the probability distribution over A. Similarly, under D-S theory, if we have a mass function defined over the joint space of variables A and B, then we can obtain the mass function defined just over variable A by summing the mass function over the variable B.

Theorem 1: For a binary variable Z that is related to three other binary variables, A, B and C, through the logical relationship 'AND', and where the variables A, B and C are interrelated, the *mass function* propagated to Z from variables A, B and C is given by the following expressions:

$$
\begin{aligned}
m_{Z\leftarrow ABC}(\{\varnothing\}) = {} & [r_2 + r_1r_3(1-r_2)]\, m_A^{\Theta}(m_B^{+}\, m_C^{-} + m_B^{-}\, m_C^{+}) + [r_3 + r_1r_2(1-r_3)]\, m_B^{\Theta}(m_A^{+}\, m_C^{-} + m_A^{-}\, m_C^{+}) \\
& + [r_1 + r_2r_3(1-r_1)]\, m_C^{\Theta}(m_A^{+}\, m_B^{-} + m_A^{-}\, m_B^{+}) + (r_1 + r_2 - r_1r_2)(m_A^{+}\, m_B^{-}\, m_C^{+} + m_A^{-}\, m_B^{+}\, m_C^{-}) \\
& + (r_2 + r_3 - r_2r_3)(m_A^{+}\, m_B^{+}\, m_C^{-} + m_A^{-}\, m_B^{-}\, m_C^{+}) + (r_1 + r_3 - r_1r_3)(m_A^{-}\, m_B^{+}\, m_C^{+} + m_A^{+}\, m_B^{-}\, m_C^{-}),
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
m_{Z\leftarrow ABC}(\{z\}) = {} & m_A^{+}\, m_B^{+}\, m_C^{+} + (r_1 + r_2 - r_1r_2)\, m_A^{+}\, m_B^{\Theta}\, m_C^{+} \\
& + (r_1 + r_3 - r_1r_3)\, m_A^{\Theta}\, m_B^{+}\, m_C^{+} + (r_2 + r_3 - r_2r_3)\, m_A^{+}\, m_B^{+}\, m_C^{\Theta} \\
& + (r_1r_2 + r_2r_3 + r_1r_3 - 2r_1r_2r_3)(m_A^{+}\, m_B^{\Theta}\, m_C^{\Theta} + m_A^{\Theta}\, m_B^{+}\, m_C^{\Theta} + m_A^{\Theta}\, m_B^{\Theta}\, m_C^{+}),
\end{aligned} \tag{10}
$$

$$
m_{Z\leftarrow ABC}(\{\sim z\}) = 1 - m_{Z\leftarrow ABC}(\{\varnothing\}) - (m_A^{+} + m_A^{\Theta})(m_B^{+} + m_B^{\Theta})(m_C^{+} + m_C^{\Theta}), \tag{11}
$$

$$
\begin{aligned}
m_{Z\leftarrow ABC}(\{z, \sim z\}) = {} & m_A^{\Theta}\, m_B^{\Theta}\, m_C^{\Theta} + (1-r_1)(1-r_2)\, m_A^{+}\, m_B^{\Theta}\, m_C^{+} \\
& + (1-r_1)(1-r_3)\, m_A^{\Theta}\, m_B^{+}\, m_C^{+} + (1-r_2)(1-r_3)\, m_A^{+}\, m_B^{+}\, m_C^{\Theta} \\
& + (1 - r_1r_2 - r_2r_3 - r_1r_3 + 2r_1r_2r_3)(m_A^{+}\, m_B^{\Theta}\, m_C^{\Theta} + m_A^{\Theta}\, m_B^{+}\, m_C^{\Theta} + m_A^{\Theta}\, m_B^{\Theta}\, m_C^{+}).
\end{aligned} \tag{12}
$$

Proof of Theorem 1: See Appendix A for the proof.

By definition, the beliefs in 'z' and '~z', i.e., Bel({z}) and Bel({~z}), are respectively equal to the normalized m-values, m({z}) and m({~z}). The normalization constant K is defined as:

$$
K = 1 - m_{Z\leftarrow ABC}(\varnothing), \tag{13}
$$

Using (9) and (13), one can obtain the following expression for K:

$$
\begin{aligned}
K = 1 & - [r_2 + r_1r_3(1-r_2)]\, m_A^{\Theta}(m_B^{+}\, m_C^{-} + m_B^{-}\, m_C^{+}) - [r_3 + r_1r_2(1-r_3)]\, m_B^{\Theta}(m_A^{+}\, m_C^{-} + m_A^{-}\, m_C^{+}) \\
& - [r_1 + r_2r_3(1-r_1)]\, m_C^{\Theta}(m_A^{+}\, m_B^{-} + m_A^{-}\, m_B^{+}) - (r_1 + r_2 - r_1r_2)(m_A^{+}\, m_B^{-}\, m_C^{+} + m_A^{-}\, m_B^{+}\, m_C^{-}) \\
& - (r_2 + r_3 - r_2r_3)(m_A^{+}\, m_B^{+}\, m_C^{-} + m_A^{-}\, m_B^{-}\, m_C^{+}) - (r_1 + r_3 - r_1r_3)(m_A^{-}\, m_B^{+}\, m_C^{+} + m_A^{+}\, m_B^{-}\, m_C^{-}).
\end{aligned} \tag{14}
$$

Using the definitions of $Bel_{Z\leftarrow ABC}(\{z\})$ and $Bel_{Z\leftarrow ABC}(\{\sim z\})$, and (10)-(12) and (14), we obtain the following expressions for the beliefs:

$$\begin{aligned}\mathrm{Bel}_{Z\leftarrow ABC}(\{z\}) = \mathrm{m}_{Z\leftarrow ABC}(\{z\})/K = [\, &\mathrm{m}_A^{+}\ \mathrm{m}_B^{+}\ \mathrm{m}_C^{+} + (r_1 + r_2 - r_1r_2)\,\mathrm{m}_A^{+}\ \mathrm{m}_B^{\Theta}\ \mathrm{m}_C^{+} \\ &+ (r_1 + r_3 - r_1r_3)\,\mathrm{m}_A^{\Theta}\ \mathrm{m}_B^{+}\ \mathrm{m}_C^{+} + (r_2 + r_3 - r_2r_3)\,\mathrm{m}_A^{+}\ \mathrm{m}_B^{+}\ \mathrm{m}_C^{\Theta} \\ &+ (r_1r_2 + r_2r_3 + r_1r_3 - 2r_1r_2r_3)(\mathrm{m}_A^{+}\ \mathrm{m}_B^{\Theta}\ \mathrm{m}_C^{\Theta} + \mathrm{m}_A^{\Theta}\ \mathrm{m}_B^{+}\ \mathrm{m}_C^{\Theta} + \mathrm{m}_A^{\Theta}\ \mathrm{m}_B^{\Theta}\ \mathrm{m}_C^{+})]/K.\end{aligned} \tag{15}$$

$$\mathrm{Bel}_{Z\leftarrow ABC}(\{\sim z\}) = \mathrm{m}_{Z\leftarrow ABC}(\{\sim z\})/K = 1 - (\mathrm{m}_A^{+} + \mathrm{m}_A^{\Theta})(\mathrm{m}_B^{+} + \mathrm{m}_B^{\Theta})(\mathrm{m}_C^{+} + \mathrm{m}_C^{\Theta})/K, \tag{16}$$

By definition, the plausibility in 'z' is given by $\mathrm{Pl}_{Z\leftarrow ABC}(z) = 1 - \mathrm{Bel}_{Z\leftarrow ABC}(\sim z)$, which yields the following expression:

$$\mathrm{Pl}_{Z\leftarrow ABC}(\{z\}) = 1 - \mathrm{Bel}_{Z\leftarrow ABC}(\{\sim z\}) = (\mathrm{m}_A^{+} + \mathrm{m}_A^{\Theta})(\mathrm{m}_B^{+} + \mathrm{m}_B^{\Theta})(\mathrm{m}_C^{+} + \mathrm{m}_C^{\Theta})/K. \tag{17}$$

The plausibility in '~z' is expressed as:

$$\begin{aligned}\mathrm{Pl}_{Z\leftarrow ABC}(\{\sim z\}) = 1 - [\, &\mathrm{m}_A^{+}\ \mathrm{m}_B^{+}\ \mathrm{m}_C^{+} + (r_1 + r_2 - r_1r_2)\,\mathrm{m}_A^{+}\ \mathrm{m}_B^{\Theta}\ \mathrm{m}_C^{+} \\ &+ (r_1 + r_3 - r_1r_3)\,\mathrm{m}_A^{\Theta}\ \mathrm{m}_B^{+}\ \mathrm{m}_C^{+} + (r_2 + r_3 - r_2r_3)\,\mathrm{m}_A^{+}\ \mathrm{m}_B^{+}\ \mathrm{m}_C^{\Theta} \\ &+ (r_1r_2 + r_2r_3 + r_1r_3 - 2r_1r_2r_3)(\mathrm{m}_A^{+}\ \mathrm{m}_B^{\Theta}\ \mathrm{m}_C^{\Theta} + \mathrm{m}_A^{\Theta}\ \mathrm{m}_B^{+}\ \mathrm{m}_C^{\Theta} + \mathrm{m}_A^{\Theta}\ \mathrm{m}_B^{\Theta}\ \mathrm{m}_C^{+})]/K\end{aligned} \tag{18}$$

Since the plausibilities in 'a', 'b', and 'c' are defined as: $\mathrm{Pl}_A(a) = (\mathrm{m}_A^{+} + \mathrm{m}_A^{\Theta})$, $\mathrm{Pl}_B(b) = (\mathrm{m}_B^{+} + \mathrm{m}_B^{\Theta})$, and $\mathrm{Pl}_C(c) = (\mathrm{m}_C^{+} + \mathrm{m}_C^{\Theta})$, we obtain from (18):

$$\mathrm{Pl}_{Z\leftarrow ABC}(\{z\}) = \mathrm{Pl}_A(a)\mathrm{Pl}_B(b)\mathrm{Pl}_C(c)/K. \tag{19}$$

Discussion of Theorem 1 Results

The results of Theorem 1 are comprehensible. For example, the conflict term, $\mathrm{m}_{Z\leftarrow ABC}(\varnothing)$ in (9) consists of 12 components. The first six components arise from situations where one variable has non-zero m-values on its frame; the second variable has non-zero m-values in its support; and the third variable has a non-zero m-value for its negation, hence the conflict. The conflict is clear in the other six components also. Three components are such that two variables have non-zero m-values in their support and the third has an m-value for its negation, while in the case of other three components, one variable has a non-zero m-value in its support and the other two have non-zero m-values against them being true.

The belief in '~z', i.e., $\text{Bel}_{Z\leftarrow ABC}(\sim z)$, also is comprehensible. Since the three variables, A, B and C, are related to variable Z through a logical 'AND', one expects '~z' to be true when '~*a*' is true, or '~*b*' is true, or '~*c*' is true. In probability framework, one can write this as:

$$P(\sim z) = P(\sim a \text{ or } \sim b \text{ or } \sim c) = 1 - P(a)P(b)P(c) = 1 - (1-P(\sim a))(1-P(\sim b))(1-P(\sim c)),$$

which is equivalent to $\text{Bel}(\{\sim z\}) = 1-(1-m(\{\sim a\}))(1-m(\{\sim b\}))(1-m(\{\sim c\}))/K$ in (16). This reasoning also supports the formula for plausibility in 'z' as the product of three plausibilities, $\text{Pl}_A(a)$, $\text{Pl}_B(b)$, and $\text{Pl}_C(c)$ in (19). As discussed later, plausibility Pl(z) determines the risk associated with Z that it is true, even though there may not be any belief that Z is true [32].

The expressions in Equations (15), (16), and (19) are important results. As shown in the application section, these expressions can be used to model risks and beliefs in the following situations. 1) The belief and plausibility that fraud exists in a financial audit, 2) the belief and plausibility that the auditor is not independent from an audit client, and 3) the belief and plausibility that the internal audit function does not produce high quality work. In the rest of this section, we discuss special cases of Theorem 1.

Special Cases

Case 1. No Interrelationships, i.e., $r_1 = r_2 = r_3 = 0$

Here we discuss a case where all the interrelationships among the three variables, A, B and C are assumed not to exist, i.e., $r_1 = r_2 = r_3 = 0$. First, we express the beliefs in 'z' and '~z' in terms of Corollary 1 given below and then discuss the results.

Corollary 1: For $r_1 = r_2 = r_3 = 0$, the beliefs propagated to Z from variables A, B and C are given by the following formulas given that variable Z is related to variables, A, B and C, through the logical relationship 'AND':

$$\text{Bel}_{Z\leftarrow ABC}(\{z\}) = m_A^+ \, m_B^+ \, m_C^+, \tag{20}$$

$$\mathrm{Bel}_{Z\leftarrow ABC}(\{\sim z\}) = 1 - (m_A^+ + m_A^\Theta)(m_B^+ + m_B^\Theta)(m_C^+ + m_C^\Theta), \tag{21}$$

Proof of Corollary 1: See Appendix B

Equations (20)-(21) are a special case of Equations (15) and (16), where there are no interrelationships among the variables A, B and C, (i.e., $r_1 = r_2 = r_3 = 0$). It can be seen from (14) that the normalization constant K, reduces to 1 under this condition and the expressions for beliefs in (15) and (16) reduce to (20) and (21), respectively. From (20), one can write the belief in '~*z*' that it is true in the following form:[4]

$$\mathrm{Bel}_{Z\leftarrow ABC}(\{\sim z\}) = 1 - (1-\mathrm{Bel}_A(\{\sim a\}))(1-\mathrm{Bel}_B(\{\sim b\}))(1-\mathrm{Bel}_C(\{\sim c\})).$$

The above relationship is intuitive and as discussed earlier, is equivalent to the following relationship among the variables under the probability framework:

$$P(\sim z) = P(\sim a \text{ or } \sim b \text{ or } \sim c) = 1 - P(a)P(b)P(c) = 1 - (1-P(\sim a))(1-P(\sim b))(1-P(\sim c)).$$

The belief that 'z' is true, i.e., $\mathrm{Bel}_{Z\leftarrow ABC}(\{z\})$ is non-zero, results only under the condition that m_A^+, m_B^+, and m_C^+ are non-zero simultaneously. This is an intuitive result. Since A, B and C are related to Z through the logical 'AND', 'z' is true under only one condition that '*a*', '*b*', and '*c*' are true at the same time. This means that the belief that 'z' is true is equal to the product of the three beliefs, $\mathrm{Bel}_A(\{a\})$, $\mathrm{Bel}_B(\{b\})$ and $\mathrm{Bel}_C(\{c\})$. However, as one can see from (15), if the interrelationships are non-zero, then $\mathrm{Bel}_{Z\leftarrow ABC}(\{z\})$ is non-zero even if only one variable has a non-zero $m_{..}^+$. This result has practical implications, as we will show in the next section. For example, it is argued and supported empirically [3, 11] that the presence of the following three factors: Incentive, Attitude, and Opportunity, must exist for management to commit fraud.

However, under strong interrelationships among the three factors, even if only one factor is present, the belief that fraud may exist can be high.

Case 2. All Interrelationships are of the Same Strength

Here we assume that $r_1 = r_2 = r_3 = r$. For this case the normalization constant K, and the beliefs propagated to Z from variables A, B and C are given by the following expressions using (14)-(16):

$$\begin{aligned} K = 1 - [r + r^2 - r^3)][\, & m_A^\Theta (m_B^+ m_C^- + m_B^- m_C^+) + m_B^\Theta (m_A^+ m_C^- + m_A^- m_C^+) \\ & + m_C^\Theta (m_A^+ m_B^- + m_A^- m_B^+)] - (2r - r^2)[m_A^+ m_B^- m_C^+ + m_A^- m_B^+ m_C^- \\ & + m_A^+ m_B^+ m_C^- + m_A^- m_B^- m_C^+ + m_A^- m_B^+ m_C^+ + m_A^+ m_B^- m_C^-], \end{aligned} \tag{22}$$

$$\begin{aligned} \mathrm{Bel}_{Z\leftarrow ABC}(\{z\}) = [\, & m_A^+ m_B^+ m_C^+ + (2r - r^2)[m_A^+ m_B^\Theta m_C^+ + m_A^\Theta m_B^+ m_C^+ + m_A^+ m_B^+ m_C^\Theta] \\ & + r^2(3 - 2r)(m_A^+ m_B^\Theta m_C^\Theta + m_A^\Theta m_B^+ m_C^\Theta + m_A^\Theta m_B^\Theta m_C^+)]/K, \end{aligned} \tag{23}$$

$$\mathrm{Bel}_{Z\leftarrow ABC}(\{\sim z\}) = 1 - (m_A^+ + m_A^\Theta)(m_B^+ + m_B^\Theta)(m_C^+ + m_C^\Theta)/K,. \tag{24}$$

From (22) we can see that the normalization constant K starts with a value of 1 at r = 0, decreases as r increases, and is smallest at r =1. However, if we choose any two variables, say B and C, to have no knowledge about their presence or absence, i.e., $m_B^\Theta = 1$ and $m_C^\Theta = 1$, then the normalization constant K equals 1 for all values of r, and the beliefs reduce to:

$$\mathrm{Bel}_{Z\leftarrow ABC}(\{z\}) = r^2(3 - 2r)\, m_A^+, \text{ and } \mathrm{Bel}_{Z\leftarrow ABC}(\{\sim z\}) = m_A^-.$$

This is an interesting result. Usually, under an 'AND' relationship and in the absence of any interrelationships (i.e., r = 0), when $m_B^+ = 0$, and $m_C^+ = 0$, one expects $\mathrm{Bel}_{Z\leftarrow ABC}(\{z\}) = 0$, which

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[4] Since $(m_A^+ + m_A^\Theta) = 1 - m_A^- = 1 - \mathrm{Bel}_A(\{\sim a\})$, $(m_B^+ + m_B^\Theta) = 1 - m_B^- = 1 - \mathrm{Bel}_B(\{\sim b\})$, $(m_C^+ + m_C^\Theta) = 1 - m_C^- = 1 - \mathrm{Bel}_C(\{\sim c\})$, we obtain
$\mathrm{Bel}_{Z\leftarrow ABC}(\{\sim z\}) = 1 - (1 - \mathrm{Bel}_A(\{\sim a\}))(1 - \mathrm{Bel}_B(\{\sim b\}))\,(1 - \mathrm{Bel}_C(\{\sim c\}))$.

is what we get from the above result. However, if we assume strong interrelationships (say, r = 1) among the variables A, B and C, $\mathrm{Bel}_{Z\leftarrow ABC}(\{z\}) = m_A^+$, which makes logical sense. Because of the strong interrelationship, even though two of the three factors, say B and C, have zero belief masses in support of the corresponding variables, the belief in 'z' is simply equal to the m-value for '*a*'. This result has important practical implications in assessing fraud risk as we show in the next section.

Let us consider another situation where we have no knowledge about the presence or absence of just one variable, say C, i.e., $m_C^\Theta = 1$. The normalization constant K, and the beliefs are given by the following expressions (see (22) – (24)):

$$K = 1-(r+r^2-r^3)(m_A^+\, m_B^- + m_A^-\, m_B^+), \tag{25}$$

$$\mathrm{Bel}_{Z\leftarrow ABC}(\{z\}) = [(2r-r^2)\, m_A^+\, m_B^+ + r^2(3-2r)(m_A^+\, m_B^\Theta + m_A^\Theta\, m_B^+)]/K, \tag{26}$$

$$\mathrm{Bel}_{Z\leftarrow ABC}(\{\sim z\}) = 1-(m_A^+ + m_A^\Theta)(m_B^+ + m_B^\Theta)/K. \tag{27}$$

Equations (26) and (27) again show that if the interrelationships are non-zero, even if we have no information about the presence or absence of one of the variables, but do have beliefs about the presence or absence of the other two variables, a non-zero belief for 'z' is provided because of the interrelationships.

Case 3. No Knowledge about the Presence of All the Three Factors but Partial Knowledge About Their Absence

In this case, we assume that we have no belief that the three factors A, B and C are present, i.e., $m_A^+ = m_B^+ = m_C^+ = 0$, and m_A^-, m_B^- and m_C^- are greater than zero. For these values, there is no conflict and thus, the renormalization constant K in (14) becomes 1 for any strength of the interrelationships and the beliefs and plausibilities for Z from (15) and (16) reduce to:

$$\mathrm{Bel}_{Z\leftarrow ABC}(\{z\}) = 0,\ \mathrm{Bel}_{Z\leftarrow ABC}(\{\sim z\}) = 1-(1-m_A^-)(1-m_B^-)(1-m_C^-),$$

$$\mathrm{Pl}_{Z\leftarrow ABC}(\{z\}) = (1-m_A^-)(1-m_B^-)(1-m_C^-) = \mathrm{Pl}(\{a\})\mathrm{Pl}(\{b\})\mathrm{Pl}(\{c\}), \text{ and } \mathrm{Pl}_{Z\leftarrow ABC}(\{\sim z\}) = 1.$$

Again, the above results make intuitive sense. Since the mass values in support of all the three factors are zero, the m-value for '*z*' is zero also even if the interrelationships are strongest, i.e., all r's = 1. The plausibility that Z is true is simply a product of three plausibilities for '*a*', '*b*' and '*c*'. Such a result is of a great value to the auditor because of its simplicity, especially when the auditor is planning an audit where fraud is suspected as briefly discussed in the next section.

Case 4. No Information on One Variable and No Relationship with the Other Two Variables

For this case, let us assume that we do not have any information on variable B, i.e., m_B^Θ = 1, and also assume that there is no relationship between variables A and B, or between B and C, i.e., $r_1 = r_2 = 0$. Substituting the above values in (14-18), we obtain the following expressions for belief and plausibility in *z* and ~z:

$$\mathrm{Bel}_{Z\leftarrow ABC}(\{z\}) = 0,$$

$$\mathrm{Bel}_{Z\leftarrow ABC}(\{\sim z\}) = 1-(1-m_A^-)(1-m_C^-)/K,$$

$$\mathrm{Pl}_{Z\leftarrow ABC}(\{z\}) = (1-m_A^-)(1-m_C^-)/K = \mathrm{Pl}(\{a\})\mathrm{Pl}(\{c\})/K,$$

$$\mathrm{Pl}_{Z\leftarrow ABC}(\{\sim z\}) = 1,$$

where $K = 1 - r_3(m_A^+ m_C^- + m_A^- m_C^+)$.

The above results are logical. Since we do not have any knowledge about the presence or absence of variable B and since there is no relationship between A and B or B and C, knowing about the presence or absence of either A or C or both, does not affect B. Thus, the belief in *z*, i.e., $\mathrm{Bel}_{Z\leftarrow ABC}(\{z\})$, should be zero because of the logical 'AND' relationship: $z = a\wedge b\wedge c$. This is what we get for this case for $\mathrm{Bel}_{Z\leftarrow ABC}(\{z\})$ as shown above.

Case 5. No Information on One Variable and No Relationship between the Other Two Variables

For this case, let us assume we have no information on variable B, i.e., $m_B^{\Theta} = 1$, and also assume there is no relationship between variables A and C, i.e., $r_3 = 0$. Substituting the above values in (14-18), we obtain the following expressions for belief and plausibility in *z* and ~*z*:

$$\mathrm{Bel}_{Z\leftarrow A,B,C}(\{z\}) = [(r_1 + r_2 - r_1r_2)m_A^+m_C^+ + r_1r_2(m_A^+m_C^{\Theta} + m_A^{\Theta}m_C^+)]/K,$$

$$\mathrm{Bel}_{Z\leftarrow ABC}(\{\sim z\}) = 1 - (1 - m_A^-)(1 - m_C^-)/K,$$

$$\mathrm{Pl}_{Z\leftarrow ABC}(\{z\}) = (1 - m_A^-)(1 - m_C^-)/K = \mathrm{Pl}(\{a\})\mathrm{Pl}(\{c\})/K,$$

$$\mathrm{Pl}_{Z\leftarrow ABC}(\{\sim z\}) = 1 - [(r_1 + r_2 - r_1r_2)m_A^+m_C^+ + r_1r_2(m_A^+m_C^{\Theta} + m_A^{\Theta}m_C^+)]/K,$$

where $K = 1 - r_1r_2(m_A^+ m_C^- + m_A^- m_C^+)$.

This case is more interesting than the previous case. Since variable B is related to both A and C ($r_1>0$ and $r_2>0$), knowing about the presence or absence of A and C tells us about the presence or absence of B. Thus, though we have no direct knowledge about the presence or absence of B, the knowledge of the presence or absence of A and C results in a non-zero belief in *z* and/or ~*z*. In fact, this belief is higher if both r_1 and r_2 are greater than zero and increases with the increase in their strengths.

Another interesting result is that the conflict term in K arises because of the two-way interaction; knowledge about A gives us the knowledge about B through r_1 and then tells us about C through r_2. Similarly, knowledge about C tells us about B through r_2 and tells us about A thought r_1. Thus, even though there is no direct link between A and C ($r_3 = 0$) in the present case, because of the interrelationships between A and B and between B and C, we have non-zero conflict.

The above result is of a great significance in assessing fraud risk since all three fraud factors, an Incentive to commit fraud (I), an Attitude to commit fraud (D), and the Opportunity to commit fraud (O) must be present for management to commit fraud. If one assumes no relationship between the two factors (variables) I and O, and there is no knowledge about D (management attitude to commit fraud) because of the difficulties in measuring attitude, fraud still may be believed to be possible. That is, the belief about the risk of fraud can be greater than zero because knowledge that both an incentive and opportunity exist creates a belief that management may have an attitude to commit fraud, even through there is no direct knowledge about management's attitude toward fraud.

IV. APPLICATIONS

Here we illustrate three important applications of the general results presented in Theorem 1. The main purpose of presenting these applications is to show the importance of the general results derived in the present paper. The first application deals with an assessment by the external auditor of belief and plausibility that an audit client's internal audit function is not of high enough quality to allow the external auditor to rely on the work of that internal auditor. The second application deals with assessing the belief and plausibility that in an audit engagement the auditor is not independent of the client. The third application deals with the assessment of belief and plausibility that a company's management may have committed fraud in reporting financial results. In addition to using the general results for assessing the above beliefs and plausibilities by the auditing profession, regulators such as the Security and Exchange Commission (SEC) can assess from a regulator's perspective the beliefs and plausibilities that fraud may exist or that an auditor is not independent in an engagement.

1. Application to Internal Audit Function Quality

The first application of the general results of Theorem 1 deals with the assessment of the quality of the internal audit function. The Sarbanes-Oxley Act of 2002 (hereafter SOX), requires management of publicly-traded companies to document, evaluate, and report on the effectiveness of internal controls over financial reporting and that the independent auditor evaluate and opine on management's assessment of such controls. SOX also requires companies covered by the Act to maintain an internal audit function. That is, each company must employ non-independent internal auditors whose function is the examination and appraisal of both controls and performance. This requirement also may increase the independent auditors' reliance on the work of internal auditors when performing an integrated audit now required under Audit Standard No. 2 [16].

For independent auditors to rely on work performed by an internal auditor, the independent auditor must assess the quality of the internal audit function [16] as to whether it is of high quality or not.[5] The Public Company Accounting Oversight Board [16] contends that the considerable flexibility that external auditors have in using the work of the internal auditor should encourage companies to develop high-quality internal audit functions, especially to reduce the cost of documentation and evaluation of internal controls. The external auditor will be able to rely more extensively on the internal audit function if they perceive the quality of the internal audit function to be high [16]. Even prior to SOX, *Statement on Auditing Standards No. 65* [2] outlined various ways independent auditors could enhance the efficiency and effectiveness of an independent audit by relying on the work of internal auditors.

[5] We denote the variable that the internal audit function is of high quality by the symbol H and the two values by 'h' and '~'h, respectively, representing that H is true and not true.

There is a substantial body of accounting literature that focuses on the external auditor's assessment of the quality of internal audit function [1, 4, 8, 10, 12, 13, 14, 25, 26, 27]. The main finding of these studies is that the quality of an internal audit function depends on three quality factors—Competence (P), Work performance (W), and Internal Auditor Objectivity (J). Competence deals with academic and professional qualifications. Work Performance deals with the quality of work, such as assessment of internal controls, risk assessment, and substantive procedures performed by the internal auditor. The Internal Auditor Objectivity deals with how independent internal auditors are in terms of evaluating and reporting weaknesses in the internal control systems. The presence of these three factors is found to be essential for the internal audit function to be of high quality. The literature also has identified interrelationships among these factors[6] [7, 10]. Thus, the problem of assessing the quality of internal audit function is similar to that of assessing whether fraud is present or that an auditor is not independent.

The three factors P, W and J, are related to the internal audit function through the logical "AND" relationship. The "AND" relationship between the quality of internal audit function and the three factors P, W and J implies that $h = p \wedge w \wedge j$, which implies that the quality of the internal audit function is high if and only if the internal auditor is competent (p), the internal auditor's work performance is of high quality (w), and the internal auditor is objective (j). Thus, the problem of assessing the quality of audit function is equivalent to assessing whether variable Z is present in Figure 1, i.e., $h = p \wedge w \wedge j$ is equivalent to the relationship $z = a \wedge b \wedge c$ (Compare Figure 2 with Figure 1).

----- Figure 2 about here -----

[6] In prior research, these factors have been assumed to be binary in nature, i.e., whether the factor is present or absent, or whether the internal audit function is of high quality (h) or is not of high quality ($\sim h$).

Thus, we can write the belief that the internal audit function is of high quality (h) given that we have knowledge about the presence or absence of the factors, Competence (P), Work performance (W), and Objectivity (J), in terms mass functions by using (14) and (15) and replacing 'a' by 'p', 'b' by 'w', and 'c' by 'j':

$$\begin{aligned}\mathrm{Bel}(\{h\}) = [\,& m_P^+ m_W^+ m_J^+ + (r_1 + r_2 - r_1r_2) m_P^+ m_W^\Theta m_J^+ \\ & + (r_1 + r_3 - r_1r_3) m_P^\Theta m_W^+ m_J^+ + (r_2 + r_3 - r_2r_3) m_P^+ m_W^+ m_J^\Theta \\ & + (r_1r_2 + r_2r_3 + r_1r_3 - 2r_1r_2r_3)(m_P^+ m_W^\Theta m_J^\Theta + m_P^\Theta m_W^+ m_J^\Theta + m_P^\Theta m_W^\Theta m_J^+)]/K. \end{aligned} \tag{28}$$

Where K is defined as:

$$\begin{aligned} K = 1 &- [r_2 + r_1r_3(1 - r_2)] m_P^\Theta (m_W^+ m_J^- + m_W^- m_J^+) - [r_3 + r_1r_2(1 - r_3)] m_W^\Theta (m_P^+ m_J^- + m_P^- m_J^+) \\ &- [r_1 + r_2r_3(1 - r_1)] m_J^\Theta (m_P^+ m_W^- + m_P^- m_W^+) - (r_1 + r_2 - r_1r_2)(m_P^+ m_W^- m_J^+ + m_P^- m_W^+ m_J^-) \\ &- (r_2 + r_3 - r_2r_3)(m_P^+ m_W^+ m_J^- + m_P^- m_W^- m_J^+) - (r_1 + r_3 - r_1r_3)(m_P^- m_W^+ m_J^+ + m_P^+ m_W^- m_J^-). \end{aligned} \tag{29}$$

Various m-values and the interrelationships are defined in Table 1.

Equation (28) is the general expression for the belief that the internal audit function is of high quality. If we assume that there is no relationships among the quality factors, i.e., all r's are zero, then the belief that the internal audit function is of high quality, Bel($\{h\}$), is simply equal to $m_P^+ m_W^+ m_J^+$. This implies that the internal audit function will be of high quality under only one condition—the internal auditor is competent (i.e., $m_p(p) \equiv m_P^+ > 0$), the work performance is of high quality (i.e., $m_W(w) \equiv m_W^+ > 0$), and the internal auditor is objective ((i.e., $m_J(j) \equiv m_J^+ > 0$). Because of the limited space in the current article, we do not discuss various scenarios of (28). Interested readers should see Desai et al. [7] who provide a detailed discuss of the assessment of the internal audit function under belief function for various scenarios.

2. Application to Auditor Independence Impairment

In this section, we demonstrate the use of the general results of Theorem 1 to assess the belief and plausibility that an auditor is not independent from an audit client. For an auditor to be independent, he/she must not exhibit bias favoring the clients representations included in financial statements when such representations may not be appropriate under accepted accounting rules or governmental regulations. Figure 3 represents a diagram of the variables that determine whether auditor is independent (N). This diagram is based on the auditing literature (see e.g., [3, 15, 35]) that suggests that an auditor may not be independent if and only if all three factors, Incentive (I), Attitude (D), and Opportunity (O) are present. In other words, the auditor will not maintain independence if and only if the auditor has an incentive to gain from being not independent, has an attitude to be not independent, and has an opportunity to be not independent. This relationship can be written as $n = i \wedge d \wedge o$ which is equivalent to the relationship $z = a \wedge b \wedge c$ (Compare Figure 2 with Figure 1).

----- Figure 3 about here -----

As we see, Figure 3 is very similar to Figure 2 except that we have two items of evidence for each variable I, D and O, whereas we have only one item of evidence for each variable, P, W and J, in Figure 2. Of the two items of evidence pertaining to each variable in Figure 3, one determines the impact of threats that increase the presence of the corresponding variable and the other supports the negation of the related variable. The formulas for beliefs and plausibilities that the auditor is independent or not independent can be derived directly from (15)-(18) by substituting 'N' for 'Z', 'I' for 'A', 'D' for 'B', and 'O' for 'C'. However, since we have two items of evidence for each of the three variables, I, D and O, we first need to determine the total belief mass function at each of the variables by combining them using Dempster's rule. We then use (15)-(18) to determine the beliefs and plausibilities as to whether the auditor is independent or not. For a detailed discussion, we refer readers to [15].

3. Application to Assessing Belief and Plausibility in Fraud

As discussed earlier, the American Institute of Certified Public Accountants (AICPA) published Statement of Auditing Standards No. 99, *Consideration of Fraud in a Financial Statement Audit* (SAS No. 99) [3] requiring auditors during an audit to assess the risk of fraud in financial statements prepared by management. The statement provides a detailed discussion on factors that if present may be indicators that fraud is present. These fraud risk factors generally are classified into three categories known as fraud triangle factors: Incentive (I), Attitude (D), and Opportunity (O). In other words, management may commit fraud in financial statements if all of the following three conditions exist: there is an incentive for management to commit fraud, management lacks integrity or has an attitude conducive to committing fraud, and there is an opportunity to commit fraud. SAS No. 99 also indicates that safeguards may exist that reduce the possibility of the presence of the above fraud risk factors and that such safeguards should be evaluated as to effectiveness. Figure 4 represents a diagram of the interrelationship of the three conditions I, D and O with a fourth variable F, representing the assertion that management fraud is present, along with the interrelationships among themselves.

----- Figure 4 about here -----

Although SAS No. 99 provides a detailed description of fraud risk factors associated with various fraud triangle factors, it does not provide any guidance on how to assess and aggregate the impacts of these factors on the presence or absence of fraud. To develop a complete fraud risk assessment model as shown in Figure 4, we consider two items of evidence for each fraud triangle variable similar to Figure 3 considered for the auditor independence impairment case. One item of evidence pertains to fraud risk factors related to the corresponding fraud triangle variable. For example, management may have bonus plans and other perquisites tied to financial performance. This factor may create an incentive for management to commit fraud. Such pieces of information are treated in our model as one item of evidence as fraud risks pertaining to the corresponding variable. In Figure 4, E_{TI}, E_{TD}, and E_{TO} represent evidence about threat factors pertaining to incentive (I), attitude (D), and opportunity (O), respectively. The other item of evidence depicts preventative controls or safeguards related to the fraud triangle variable. For example, the organization may have an active board of directors and an effective audit committee to control management behavior related to incentives to commit fraud or there may be strong internal accounting controls in place to reduce opportunities for management to commit fraud. Such factors can reduce the likelihood of the presence of the corresponding fraud triangle variable. In Figure 4, E_{SI}, E_{SD}, and E_{SO}, respectively, represent evidence about safeguard factors pertaining to incentive (I), attitude (D), and opportunity (O).

For the fraud variable (F) we consider three items of evidence. One item of evidence represented by E_{PI} is based on prior information known to the auditor. The second item of evidence, E_{OP}, depends on traditional, non-fraud-oriented audit procedures termed 'Other Procedures'. The third, E_{FP}, represents any fraud-specific forensic procedures performed by the auditor. Each of these items of evidence provides some degree of belief about whether the corresponding variable is present or absent.

Again, we assume that each variable takes two values; the variable is either present or not present. For example, F represents the variable that fraud exists in the financial statements and '*f*' represents its value that fraud is present and '$\sim f$' represents that fraud is not present. As discussed earlier, according to SAS No. 99, variable F is related to the three variables, I, D and O through a logical 'AND' relationship. In other words, fraud is present if and only if all the three factors, I, D and O are present, i.e., $f = i \wedge d \wedge o$, or $\sim f = \sim i \vee \sim d \vee \sim o$. These relationships are similar to the relationships considered in the derivation of the beliefs and plausibilities for variable Z propagated from three variables A, B and C in Figure 1. In fact, '*z*' is equivalent to '*f*', and '*a*', *'b'*, and *'c'* are equivalent to *'i', 'd'* and *'o'*, respectively. Thus, we can write the normalized mass function propagated from the three variables, I, D and O directly from (9-12) as:

$$\begin{aligned} m_{F\leftarrow IDO}(\{f\}) = [\, & m_I^+ m_D^+ m_O^+ + (r_1 + r_2 - r_1r_2) m_I^+ m_D^\Theta m_O^+ \\ & + (r_1 + r_3 - r_1r_3) m_I^\Theta m_D^+ m_O^+ + (r_2 + r_3 - r_2r_3) m_I^+ m_D^+ m_O^\Theta \\ & + (r_1r_2 + r_2r_3 + r_1r_3 - 2r_1r_2r_3)(m_I^+ m_D^\Theta m_O^\Theta + m_I^\Theta m_D^+ m_O^\Theta + m_I^\Theta m_D^\Theta m_O^+)]/K_F, \end{aligned} \quad (30)$$

$$m_{F\leftarrow IDO}(\{\sim f\}) = 1 - (m_I^+ + m_I^\Theta)(m_D^+ + m_D^\Theta)(m_O^+ + m_O^\Theta)/K_F, \quad (31)$$

$$\begin{aligned} m_{F\leftarrow IDO}(\{f, \sim f\}) = [\, & m_I^\Theta m_D^\Theta m_O^\Theta + (1-r_1)(1-r_2)\, m_I^+ m_D^\Theta m_O^+ \\ & + (1-r_1)(1-r_3)\, m_I^\Theta m_D^+ m_O^+ + (1-r_2)(1-r_3)\, m_I^+ m_D^+ m_O^\Theta \\ & + (1 - r_1r_2 - r_2r_3 - r_1r_3 + 2r_1r_2r_3)\,(m_I^+ m_D^\Theta m_O^\Theta + m_I^\Theta m_D^+ m_O^\Theta + m_I^\Theta m_D^\Theta m_O^+)]/K_F. \end{aligned} \quad (32)$$

Where K_F is given below:

$$\begin{aligned} K_F = 1 & - [r_2 + r_1r_3(1-r_2)]\, m_I^\Theta (m_D^+ m_O^- + m_D^- m_O^+) - [r_3 + r_1r_2(1-r_3)]\, m_D^\Theta (m_I^+ m_O^- + m_I^- m_O^+) \\ & - [r_1 + r_2r_3(1-r_1)]\, m_O^\Theta (m_I^+ m_D^- + m_I^- m_D^+) - (r_1 + r_2 - r_1r_2)(m_I^+ m_D^- m_O^+ + m_I^- m_D^+ m_O^-) \\ & - (r_2 + r_3 - r_2r_3)(m_I^+ m_D^+ m_O^- + m_I^- m_D^- m_O^+) - (r_1 + r_3 - r_1r_3)(m_I^- m_D^+ m_O^+ + m_I^+ m_D^- m_O^-). \end{aligned} \quad (33)$$

Various m-values and the interrelationships are defined in Table 1.

The relationships r_1, r_2 and r_3, respectively, represent the relationships between Incentives (I) and Attitude (D), between Attitude (D) and Opportunities (O), and between Incentive (I) and Opportunities (O). As seen in Figure 4, the mass function at each variable, I, D and O, is the combination of two mass functions; one from the fraud risk factors, and the other from the safeguard factors. Thus, the following expressions define the three mass functions:[7]

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[7] We use Dempster's rule to combine the two sets of mass functions, one from the threat factors denoted by m_T and the other from the safeguard factors denoted by m_S. In general, we assume that fraud risk factors may provide non-zero values for m^+, m^- and m^Θ, for the corresponding variable. However, for the safeguard factors, we assume that they yield non-zero values for only m^- and m^Θ. In other words, the safeguard factors only negate the presence of the corresponding fraud triangle factor.

$$m_I^+ = m_{TI}^+(1-m_{SI}^-)/K_I,\ m_I^- = (m_{TI}^- + m_{SI}^-\ m_{TI}^{\Theta})/K_I,\ m_I^{\Theta} = m_{TI}^{\Theta}(1-m_{SI}^-)/K_I, \tag{34}$$

where $K_I = 1-\ m_{TI}^+\ m_{SI}^-$.

$$m_D^+ = m_{TD}^+(1-m_{SD}^-)/K_D, m_D^- = (m_{TD}^- + m_{SD}^-\ m_{TD}^{\Theta})/K_D,\ m_D^{\Theta} = m_{TD}^{\Theta}(1-m_{SD}^-)/K_D, \tag{35}$$

where $K_D = 1 - m_{TD}^+\ m_{SD}^-$.

$$m_O^+ = m_{TO}^+(1-m_{SO}^-)/K_O,\ m_O^- = (m_{TO}^- + m_{SO}^-\ m_{TO}^{\Theta})/K_O,\ m_O^{\Theta} = m_{TO}^{\Theta}(1-m_{SO}^-)/K_O, \tag{36}$$

where $K_O = 1 - m_{TO}^+\ m_{SO}^-$.

The three mass functions defined at variable F due to the three items of evidence, E_{PI}, E_{OP} and E_{FP} depicted in Figure 4, are represented by $m_{PI}(\{f\})$, $m_{PI}(\{\sim f\})$, $m_{PI}(\{f,\sim f\})$; $m_{OP}(\{f\})$, $m_{OP}(\{\sim f\})$, $m_{OP}(\{f,\sim f\})$; and $m_{FP}(\{f\})$, $m_{FP}(\{\sim f\})$, $m_{FP}(\{f,\sim f\})$, respectively. To determine the overall belief and plausibility that fraud exists, we combine the four sets of mass functions at variable F, three directly defined at F as defined above by m_{PI}, m_{OP}, and m_{FP}, and the fourth denoted by $m_{F\leftarrow IDO}$, propagated from variables I, D and O, as defined in (30)-(30). We use again Dempster's rule to combine the above four sets of mass functions and obtain the following expressions[8] for the total belief and total plausibility in fraud (f):

$$Bel_T(\{f\}) = 1 - [1-m_{PI}(\{f\})][1-m_{OP}(\{f\})][1-m_{FP}(\{f\})][1-m_{F\leftarrow IDO}(\{f\})]/K_T, \tag{37}$$

$$Pl_T(\{f\}) = [1-m_{PI}(\{\sim f\})][1-m_{OP}(\{\sim f\})][1-m_{FP}(\{\sim f\})][1-m_{F\leftarrow IDO}(\{\sim f\})]/K_T. \tag{38}$$

The symbol K_T is given by:

$$K_T = \prod_i (1-m_i(\{f\})) + \prod_i (1-m_i(\{\sim f\})) - \prod_i m_i(\{f,\sim f\}), \tag{39}$$

where $i \in \{PI, OP, FP, F\leftarrow IDO\}$.

[8] For binary variables, Dempster's rule can be simplified yielding directly the expressions in (37) and (38) (see Srivastava [28] for details).

The total belief that fraud exists in (37) and the total plausibility of fraud in (38) are of interest when investigating fraud. Srivastava and Shafer [32] argue that the plausibility of financial statements containing serious misstatements is the appropriate measure of overall audit risk. Similar to Srivastava and Shafer, we define the total plausibility of fraud to be the fraud risk. Thus, the expression in (38) represents the overall fraud risk after combining all the evidence. To express the overall fraud risk formula in (38) in terms of individual risks or plausibilities that incentives are present, attitude is present, and opportunities are present, we need to make the following simplifications.

We know from (31) that $[1 - m_{F \leftarrow IDO}(\{\sim f\})] = (m_I^{+} + m_I^{\Theta})(m_D^{+} + m_D^{\Theta})(m_O^{+} + m_O^{\Theta})/K_F$, which by definition equals to $Pl_I(i)Pl_A(a)Pl_O(o)/K_F$. Also, we know from (34)-(36) that there are two items of evidence pertaining to each variable I, A and O, and thus the plausibility that each variable present is given by the product of two plausibilities that the variable is present, one due to the threat factors and other due to the failure of safeguards. In other words, $Pl_I(i) = Pl_{TI}(i)Pl_{SI}(i)/K_I$, $Pl_A(a) = Pl_{TA}(a)Pl_{SA}(a)/K_A$, and $Pl_O(o) = Pl_{TO}(o)Pl_{SO}(o)/K_O$. In addition, we know that

$$(1-m_{PI}(\{\sim f\}))(1-m_{OP}(\{\sim f\}))(1-m_{FP}(\{\sim f\}))/K_T = Pl_{PI}(\{f\})Pl_{OP}(\{f\})Pl_{FP}(\{f\})/K_T.$$

Thus, using (38) and the above simplifications, we can express the fraud risk (FR) formula in terms of the individual plausibility functions as:

$$FR = \left(\frac{Pl_{PI}(\{f\})Pl_{OP}(\{f\})Pl_{FP}(\{f\})}{K_T K_F}\right)\cdot\left(\frac{Pl_{TI}(\{i\})Pl_{SI}(\{i\})}{K_I}\right)\cdot\left(\frac{Pl_{TD}(\{d\})Pl_{SD}(\{d\})}{K_D}\right)\cdot\left(\frac{Pl_{TO}(\{o\})Pl_{SO}(\{o\})]}{K_O}\right). \qquad (40)$$

The above expression represents the overall fraud risk given all the evidence in Figure 4. Srivastava et al. [31] discuss this risk model in detail and contrast it with a Bayesian-based model to demonstrate the usefulness of the belief function model. We do not plan to discuss all the special cases of (40) here; rather we refer readers to Srivastava et al. [31].

V. CONCLUSION

In conclusion, we have derived analytical formulas for the overall beliefs on a binary variable Z resulting from beliefs on three binary variables A, B and C that are related to variable Z through an 'AND' relationship under the assumption that these three variables are interrelated. The general results are presented in Theorem 1 along with a special case presented in Corollary 1. Several other special cases are presented to demonstrate the importance of the results in Theorem 1.

Importantly, under the assumption that there are no interrelationships between the three variables, A, B and C, we show that the general formulas (see Corollary 1) reduce to the results obtained directly from Proposition 1 of Srivastava et al. [33]. In addition, we demonstrate applications of the general formulas in three important areas. 1) assessment of the quality of the internal audit function by the external auditor to determine the appropriate level of reliance on the work of internal auditor, 2) assessment of the auditor's independence risk in a financial statement audit, and 3) assessment of fraud risk in financial reporting.

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Appendix A

Proof of Theorem 1

The proof of Theorem 1 is straightforward but computationally very cumbersome. Basically, we want to combine seven *mass functions* as given in (8) and marginalize (see footnote 3) the resulting *mass function* (i.e., m-values) to variable Z. Since the combination of *mass functions* is known to be commutative and associative (see, e.g., Shafer [19]), one can chose any order to combine the above *mass functions*. We chose the following sequence for combining the *mass functions*:

$$m_{Z\leftarrow ABC} = ((((((m_{AB}\oplus m_{AC})\oplus m_{BC})\oplus m_{ZABC})\oplus m_{A})^{\downarrow ZBC}\oplus m_{B})^{\downarrow ZC}\oplus m_{C})^{\downarrow Z} \qquad \text{(A1)}$$

In other words, we first combine the two *mass functions*, m_{AB} and m_{AC}, defined in (4) and (6), respectively, and denote the resulting *mass function* by m_1, i.e., $m_1 = (m_{AB}\oplus m_{AC})$. Next, we combine m_1 with m_{BC} given in (5) and obtain the following *mass function* denoted by $m_2 = (m_1\oplus m_{BC}) = ((m_{AB}\oplus m_{AC})\oplus m_{BC})$. In the third step, we combine m_2 with m_{ZABC} given in (7) and obtain the *mass function* denoted by $m_3 = (m_2\oplus m_{ZABC}) = (((m_{AB}\oplus m_{AC})\oplus m_{BC})\oplus m_{ZABC})$. In the fourth step, we combine m_3 with m_A given in (1), and marginalize the resulting *mass function* to the frame of ZBC by eliminating variable A. This process yields the following *mass function* denoted by $m_4 = ((((m_{AB}\oplus m_{AC})\oplus m_{BC})\oplus m_{ZABC})\oplus m_{A})^{\downarrow ZBC}$. Next, we combine m_4 with m_B given in (2) and marginalize the resulting m-values to the frame of ZC by eliminating variable B. This process yields the following *mass function*: $m_5 = (m_4\oplus m_B)^{\downarrow ZC}$. Finally, we combine m_5 with m_C given in (3) and marginalize the resulting *mass function* to the frame of Z by eliminating C to obtain the desired result: $m_{Z\leftarrow ABC} = (m_5\oplus m_C)^{\downarrow Z} = ((((((m_{AB}\oplus m_{AC})\oplus m_{BC})\oplus m_{ZABC})\oplus m_A)^{\downarrow ZBC}\oplus m_B)^{\downarrow ZC}\oplus m_C)^{\downarrow Z}$. These steps are described below in detail.

Step 1:

In this step, we want to compute $m_1 = (m_{AB}\oplus m_{AC})$, i.e., combine m_{AB} with m_{AC}. This is achieved by first extending m_{AB} and m_{AC} to the frame, Θ_{ABC} = {*abc, ab~c, a~bc, ~abc, a~b~c, ~ab~c, ~a~bc, ~a~b~c*}, through vacuous extension[9] and then combine the two *mass functions* using Dempster's rule. We obtain the following *mass function* after extending m_{AB} and m_{AC} onto the frame Θ_{ABC}:

$$m_{AB}(\{ab,\sim a\sim b\}) = m_{AB}(\{abc,\ ab\sim c,\ \sim a\sim bc,\ \sim a\sim b\sim c\}) = r_1,$$

$$m_{AB}(\{ab,\ a\sim b,\ \sim ab,\ \sim a\sim b\}) = m_{AB}(\Theta_{ABC}) = 1- r_1, \qquad \text{(A2)}$$

and

$$m_{AC}(\{ac,\sim a\sim c\}) = m_{AC}(\{abc,\ a\sim bc,\ \sim ab\sim c,\ \sim a\sim b\sim c\}) = r_3,$$

$$m_{AC}(\{ac,\ a\sim c,\ \sim ac,\ \sim a\sim c\}) = m_{AC}(\Theta_{ABC}) = 1- r_3. \qquad \text{(A3)}$$

By combining the above m-values, we obtain the following *mass function* on the frame Θ_{ABC}:

$$m_1(\{abc,\ \sim a\sim b\sim c\}) = r_1r_3,$$

$$m_1(\{abc,\ ab\sim c,\ \sim a\sim bc,\ \sim a\sim b\sim c\}) = r_1(1-r_3),$$

$$m_1(\{abc,\ a\sim bc,\ \sim ab\sim c,\ \sim a\sim b\sim c\}) = (1-r_1)r_3,$$

$$m_1(\Theta_{ABC}) = (1-r_1)(1-r_3). \qquad \text{(A4)}$$

[9] *Vacuous Extension* is the process through which a *mass function* from a smaller node (having fewer variables) are extended to a *mass function* at a larger node (having a larger number of variables).

Step 2:

In this step, we combine m_1 defined in (A4) with m_{BC} again by first extending m_{BC} onto the frame Θ_{ABC}. The vacuous extension of m_{BC} onto Θ_{ABC} yields the following *mass function*:

$$m_{BC}(\{bc, \sim b\sim c\}) = m_{BC}(\{abc, \sim abc, a\sim b\sim c, \sim a\sim b\sim c\}) = r_2,$$

$$m_{BC}(\{bc, b\sim c, \sim bc, \sim b\sim c\}) = m_{BC}(\Theta_{ABC}) = 1 - r_2, \tag{A5}$$

The combination process of the two *mass functions*, one in (A4) and the other in (A5), yields the following *mass function* on the frame Θ_{ABC}:

$$m_2(\{abc, \sim a\sim b\sim c\}) = r_1r_2 + r_1r_3 + r_2r_3 - 2r_1r_2r_3,$$

$$m_2(\{abc, ab\sim c, \sim a\sim bc, \sim a\sim b\sim c\}) = r_1(1-r_2)(1-r_3),$$

$$m_2(\{abc, \sim abc, a\sim b\sim c, \sim a\sim b\sim c\}) = (1-r_1)r_2(1-r_3),$$

$$m_2(\{abc, a\sim bc, \sim ab\sim c, \sim a\sim b\sim c\}) = (1-r_1)(1-r_2)r_3,$$

$$m_2(\Theta_{ABC}) = (1-r_1)(1-r_2)(1-r_3). \tag{A6}$$

Step 3:

In Step 3, we combine the *mass function* in (A6) with m_{ZABC}. This process is straight forward because $m_{ZABC}(\Theta_{ZABC}) = 1$ for $\Theta_{ZABC} = \{zabc, \sim zab\sim c, \sim za\sim bc, \sim z\sim abc, \sim za\sim b\sim c, \sim z\sim ab\sim c, \sim z\sim a\sim bc, \sim z\sim a\sim b\sim c\}$. Combining m_2 and m_{ZABC} yields the following *mass function* on the frame Θ_{ZABC}:

$$m_3(\{zabc, \sim z\sim a\sim b\sim c\}) = r_1r_2 + r_1r_3 + r_2r_3 - 2r_1r_2r_3,$$

$$m_3(\{zabc, \sim zab\sim c, \sim z\sim a\sim bc, \sim z\sim a\sim b\sim c\}) = r_1(1-r_2)(1-r_3),$$

$$m_3(\{zabc, \sim z\sim abc, \sim za\sim b\sim c, \sim z\sim a\sim b\sim c\}) = (1-r_1)r_2(1-r_3),$$

$$m_3(\{zabc, \sim za\sim bc, \sim z\sim ab\sim c, \sim z\sim a\sim b\sim c\}) = (1-r_1)(1-r_2)r_3,$$

$$m_3(\Theta_{ZABC}) = (1-r_1)(1-r_2)(1-r_3). \tag{A7}$$

Step 4:

In Step 4, we combine the *mass function* in (A7) with m_A and marginalize the resulting *mass function* to the frame $\Theta_{ZBC} = \{zbc, \sim zbc, \sim zb\sim c, \sim z\sim bc, \sim z\sim b\sim c\}$ by eliminating variable A. Before we combine m_A with m_3 in (A7), we vacuously extend m_A onto the frame Θ_{ZABC} as follows:

$$m_A(\{a\}) = m_A(\{zabc, \sim zab\sim c, \sim za\sim bc, \sim za\sim b\sim c\}) = m_A^+,$$

$$m_A(\{\sim a\}) = m_A(\{\sim z\sim abc, \sim z\sim ab\sim c, \sim z\sim a\sim bc, \sim z\sim a\sim b\sim c\}) = m_A^-,$$

$$m_A(\{a,\sim a\}) = m_A(\Theta_{ZABC}) = m_A^{\Theta}. \tag{A8}$$

Since there are five non-zero *belief masses* for m_3 in (A7) and three non-zero *belief masses* for m_A in (A8), combining the two *mass functions* using Dempster's rule yields fifteen *belief masses* on the frame Θ_{ZABC}. However, when these fifteen *belief masses* are marginalized to the frame Θ_{ZBC} by eliminating variable A, we obtain the following *mass function* with twelve *belief masses*:

$$m_4(\{zbc\}) = (r_1r_2 + r_1r_3 + r_2r_3-2r_1r_2r_3)\, m_A^+$$

$$m_4(\{\sim z\sim b\sim c\}) = (r_1r_2 + r_1r_3 + r_2r_3-2r_1r_2r_3)\, m_A^-,$$

$$m_4(\{zbc,\sim z\sim b\sim c\}) = (r_1r_2 + r_1r_3 + r_2r_3-2r_1r_2r_3)\, m_A^{\Theta} + (1-r_1)r_2(1-r_3)\, m_A^+$$

$$m_4(\{zbc, \sim zb\sim c\}) = r_1(1-r_2)(1-r_3)\, m_A^+,$$

$$m_4(\{\sim z\sim bc ,\sim z\sim b\sim c\}) = r_1(1-r_2)(1-r_3)\, m_A^-,$$

$m_4(\{zbc, \sim z\sim bc\}) = (1-r_1)(1-r_2)r_3\, m_A^{+}$,

$m_4(\{\sim zb\sim c, \sim z\sim b\sim c\}) = (1-r_1)(1-r_2)r_3\, m_A^{-}$,

$m_4(\{\sim zbc, \sim z\sim b\sim c\}) = (1-r_1)r_2(1-r_3)\, m_A^{-}$,

$m_4(\{zbc, \sim zbc, \sim z\sim b\sim c\}) = (1-r_1)r_2(1-r_3)\, m_A^{\Theta}$,

$m_4(\{\sim zbc, \sim zb\sim c, \sim z\sim bc, \sim z\sim b\sim c\}) = (1-r_1)(1-r_2)(1-r_3)\, m_A^{-}$,

$m_4(\{zbc, \sim zb\sim c, \sim z\sim bc, \sim z\sim b\sim c\})=[r_1(1-r_2)(1-r_3)+(1-r_1)(1-r_2)r_3]\, m_A^{\Theta} + (1-r_1)(1-r_2)(1-r_3)\, m_A^{+}$,

$m_4(\{zbc, \sim zbc, \sim zb\sim c, \sim z\sim bc, \sim z\sim b\sim c\}) = (1-r_1)(1-r_2)(1-r_3)\, m_A^{\Theta}$. (A9)

Step 5:

In this step, we combine the mass function in (A9) with m_B and marginalize the resulting *mass function* to the frame $\Theta_{ZC} = \{zc, \sim zc, \sim z\sim c\}$ by eliminating variable B. In order to combine m_B with m_4, we vacuously extend m_B onto the frame $\Theta_{ZBC} = \{zbc, \sim zbc, \sim zb\sim c, \sim z\sim bc, \sim z\sim b\sim c\}$ as follows:

$m_B(\{b\}) = m_B(\{zbc, \sim zbc, \sim zb\sim c\}) = m_B^{+}$,

$m_B(\{\sim b\}) = m_B(\{\sim z\sim bc, \sim z\sim b\sim c\}) = m_B^{-}$,

$m_B(\{b,\sim b\}) = m_B(\Theta_{ZBC}) = m_B^{\Theta}$. (A10)

Combining m_B in (A10) with m_4 in (A9) using Dempster's rule[10] yields 36 *belief masses*, which is the result of multiplying 12 *belief masses* in (A9) with three *belief masses* in (A10). However, out of 36 *belief masses*, 32 are defined over the frame $\Theta_{ZBC} = \{zbc, \sim zbc, \sim zb\sim c, \sim z\sim bc, \sim z\sim b\sim c\}$ and four pertain to the empty set representing the conflicts among the two *mass*

[10] We do not re-normalize the m-values at this stage. This is done at the end after combining all the m-values.

functions denoted by $m_5(\emptyset)$. Next, we marginalize the above 32 *belief masses* to the frame Θ_{ZC} = $\{zc, \sim zc, \sim z\sim c\}$ by eliminating variable B. This process yields the following *mass function*:

$$m_5(\{zc\}) = (r_1r_2 + r_1r_3 + r_2r_3 - 2r_1r_2r_3)(m_A^+ m_B^+ + m_A^+ m_B^\Theta + m_A^\Theta m_B^+) + [(1-r_1)r_2(1-r_3) + (1-r_1)(1-r_2)r_3]\, m_A^+ m_B^+,$$

$$m_5(\{\sim z\sim c\}) = (r_1r_2 + r_1r_3 + r_2r_3 - 2r_1r_2r_3)(m_A^- m_B^- + m_A^- m_B^\Theta + m_A^\Theta m_B^-) + (1-r_1)(1-r_2)r_3(m_A^- m_B^- + m_A^- m_B^+ + m_A^- m_B^\Theta) + (1-r_1)r_2(1-r_3)(m_A^- m_B^- + m_A^+ m_B^- + m_A^\Theta m_B^-),$$

$$m_5(\{zc, \sim z\sim c) = (r_1r_2 + r_1r_3 + r_2r_3 - 2r_1r_2r_3)\, m_A^\Theta m_B^\Theta + (r_1 + r_2 - 2r_1r_2)(1-r_3)\, m_A^+ m_B^\Theta + (r_1 + r_3 - 2r_1r_3)(1-r_2)\, m_A^\Theta m_B^+ + (1-r_2)(1-r_3)\, m_A^+ m_B^+,$$

$$m_5(\{\sim zc, \sim z\sim c) = (1-r_1)(1-r_2)(1-r_3)((m_A^- m_B^- + m_A^- m_B^+ + m_A^- m_B^\Theta + m_A^+ m_B^- + m_A^\Theta m_B^-) + r_1(1-r_2)(1-r_3)\, m_A^- m_B^- + [r_1(1-r_2)(1-r_3) + (1-r_1)r_2(1-r_3)]\, m_A^- m_B^\Theta + [r_1(1-r_2)(1-r_3) + (1-r_1)(1-r_2)r_3]\, m_A^\Theta m_B^-,$$

$$m_5(\sim zc) = (1-r_1)(1-r_2)r_3\, m_A^+ m_B^- + (1-r_1)r_2(1-r_3)\, m_A^- m_B^+,$$

$$m_5(\{zc, \sim zc\}) = (1-r_1)(1-r_2)r_3\, m_A^+ m_B^\Theta + (1-r_1)r_2(1-r_3)\, m_A^\Theta m_B^+,$$

$$m_5(\Theta_{ZC}) = [r_1(1-r_2)(1-r_3) + (1-r_1)r_2(1-r_3) + (1-r_1)(1-r_2)r_3 + (1-r_1)(1-r_2)(1-r_3)]\, m_A^\Theta m_B^\Theta + (1-r_1)(1-r_2)(1-r_3)(m_A^\Theta m_B^+ + m_A^+ m_B^\Theta).$$

$$m_5(\emptyset) = (r_1 + r_2r_3 - r_1r_2r_3)(m_A^+ m_B^- + m_A^- m_B^+) \quad \text{(A11)}$$

Step 6:

In Step 6, we combine the mass function in (A11) with m_C and marginalize the resulting *mass function* to the frame $\Theta_Z = \{z, \sim z\}$ by eliminating variable C. Here again, in order to combine m_C with m_5 in (A11), we vacuously extend m_C onto the frame $\Theta_{ZC} = \{zc, \sim zc, \sim z\sim c\}$ as follows:

$$m_C(\{c\}) = m_C(\{zc, \sim zc\}) = m_C^+,$$

$$m_c(\{\sim c\}) = m_C(\{\sim z\sim c\}) = m_C^-,$$

$$m_c(\{c,\sim c\}) = m_C(\Theta_{ZC}) = m_C^{\Theta}. \tag{A12}$$

Combining m_C in (A12) with m_5 in (A11) using Dempster's rule yields 24 *belief masses*. Seven out of twenty-four *belief masses* pertain to the empty set or the conflict. The marginalization process of the above 24 m-values onto the frame $\Theta_Z = \{z, \sim z\}$ by eliminating variable C, yields the following *mass function*[11] along with the conflict term denoted by $m_6(\varnothing)$:

$$\begin{aligned} m_6(\{\varnothing\}) = {} & [r_2 + r_1r_3(1-r_2)]\, m_A^{\Theta}(m_B^+ m_C^- + m_B^- m_C^+) + [r_3 + r_1r_2(1-r_3)]\, m_B^{\Theta}(m_A^+ m_C^- + m_A^- m_C^+) \\ & + [r_1 + r_2r_3(1-r_1)]\, m_C^{\Theta}(m_A^+ m_B^- + m_A^- m_B^+) + (r_1 + r_2 - r_1r_2)(m_A^+ m_B^- m_C^+ + m_A^- m_B^+ m_C^-) \\ & + (r_2 + r_3 - r_2r_3)(m_A^+ m_B^+ m_C^- + m_A^- m_B^- m_C^+) + (r_1 + r_3 - r_1r_3)(m_A^- m_B^+ m_C^+ + m_A^+ m_B^- m_C^-), \end{aligned}$$

$$\begin{aligned} m_6(\{z\}) = {} & m_A^+ m_B^+ m_C^+ + (r_1 + r_2 - r_1r_2)\, m_A^+ m_B^{\Theta} m_C^+ \\ & + (r_1 + r_3 - r_1r_3)\, m_A^{\Theta} m_B^+ m_C^+ + (r_2 + r_3 - r_2r_3)\, m_A^+ m_B^+ m_C^{\Theta} \\ & + (r_1r_2 + r_2r_3 + r_1r_3 - 2r_1r_2r_3)(m_A^+ m_B^{\Theta} m_C^{\Theta} + m_A^{\Theta} m_B^+ m_C^{\Theta} + m_A^{\Theta} m_B^{\Theta} m_C^+), \end{aligned}$$

$$m_6(\{\sim z\}) = 1 - m_6(\{\varnothing\}) - (m_A^+ + m_A^{\Theta})(m_B^+ + m_B^{\Theta})(m_C^+ + m_C^{\Theta}),$$

$$\begin{aligned} m_6(\{z, \sim z\}) = {} & m_A^{\Theta} m_B^{\Theta} m_C^{\Theta} + (1-r_1)(1-r_2)\, m_A^+ m_B^{\Theta} m_C^+ \\ & + (1-r_1)(1-r_3)\, m_A^{\Theta} m_B^+ m_C^+ + (1-r_2)(1-r_3)\, m_A^+ m_B^+ m_C^{\Theta} \\ & + (1 - r_1r_2 - r_2r_3 - r_1r_3 + 2r_1r_2r_3)(m_A^+ m_B^{\Theta} m_C^{\Theta} + m_A^{\Theta} m_B^+ m_C^{\Theta} + m_A^{\Theta} m_B^{\Theta} m_C^+). \end{aligned} \tag{A13}$$

The above mass function is not normalized and represents the desired result at variable Z, which we express as $m_{Z\leftarrow ABC}$, the mass function propagated from the variables A, B, and C to Z.

Q.E.D.

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[11] The marginalization process yields $m_6(\sim z) = m_A^- m_B^- m_C^- + m_A^- m_B^- m_C^{\Theta} + m_A^- m_B^{\Theta} m_C^- + m_A^{\Theta} m_B^- m_C^- + m_A^- m_B^{\Theta} m_C^{\Theta} + m_A^{\Theta} m_B^- m_C^{\Theta} + m_A^{\Theta} m_B^{\Theta} m_C^- + (1 - r_2 - r_1r_3 + r_1r_2r_3)\, m_A^{\Theta}(m_B^+ m_C^- + m_B^- m_C^+) + (1 - r_3 - r_1r_2 + r_1r_2r_3)\, m_B^{\Theta}(m_A^+ m_C^- + m_A^- m_C^+) + (1 - r_1 - r_2r_3 + r_1r_2r_3)\, m_C^{\Theta}(m_A^+ m_B^- + m_A^- m_B^+) + (1 - r_1 - r_2 + r_1r_2)(m_A^+ m_B^- m_C^+ + m_A^- m_B^+ m_C^-) + (1 - r_2 - r_3 + r_2r_3)(m_A^+ m_B^+ m_C^- + m_A^- m_B^- m_C^+) + (1 - r_1 - r_3 + r_1r_3)(m_A^- m_B^+ m_C^+ + m_A^+ m_B^- m_C^-)$, which can be simplified with some efforts to the expression: $m_6(\sim z) = 1 - m_6(\{\varnothing\}) - (m_A^+ + m_A^{\Theta})(m_B^+ + m_B^{\Theta})(m_C^+ + m_C^{\Theta})$.

Appendix B

Proof of Corollary 1

By definition, the beliefs in 'z' and in '$\sim z$' are equal to m-values for 'z' and '$\sim z$', respectively. These m-values can be obtained directly from Proposition 1 of Srivastava et al. [33]. Their Proposition 1 provides formulas to combine m-values propagated from sub-objectives to the main objective in an 'AND' tree. This situation is equivalent to our situation where Z is related to three variables, A, B and C, through the logical 'AND', i.e., $z = a \wedge b \wedge c$. Their Proposition 1 states that "The resultant m-values propagated from n sub-objectives (O_i, i = 1, 2, . . . n) to the main objective X in an AND-tree are given as follows (their Equations 1, 2, and 3).

$$m_{X \leftarrow \text{all O's}}(x) = \prod_{i=1}^{n} m_{O_i}(o_i),$$

$$m_{X \leftarrow \text{all O's}}(\sim x) = 1 - \prod_{i=1}^{n} [1 - m_{O_i}(\sim o_i)],$$

and

$$m_{X \leftarrow \text{all O's}}(\{x, \sim x\}) = 1 - m_{X \leftarrow \text{all O's}}(x) - m_{X \leftarrow \text{all O's}}(\sim x).$$

In the present case, we have three sub-objectives, A, B, and C, with Z being the main objective and thus, x, is z, and o_i's are a, b and c. The above formulas yield the following m-values for our case:

$$m_{Z \leftarrow ABC}(z) = m_A(a) m_B(b) m_C(c) = m_A^+ \; m_B^+ \; m_C^+, \tag{B1}$$

$$\begin{aligned} m_{Z \leftarrow ABC}(\sim z) &= 1 - (1 - m_A(\sim a))\,(1 - m_B(\sim b))(1 - m_C(\sim c)) \\ &= 1 - (m_A^+ + m_A^\Theta)(m_B^+ + m_B^\Theta)(m_C^+ + m_C^\Theta), \end{aligned} \tag{B2}$$

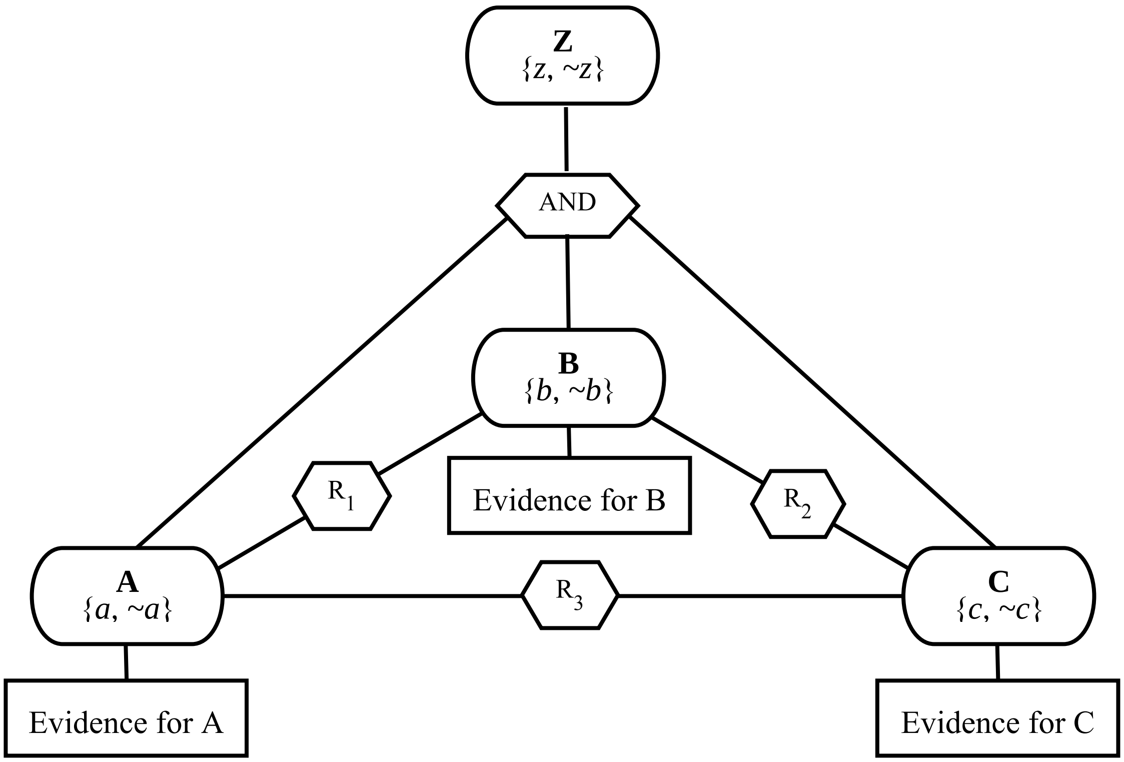
These m-values yield:

$$\text{Bel}_{Z \leftarrow ABC}(\sim z) = 1 - (m_A^+ + m_A^\Theta)(m_B^+ + m_B^\Theta)(m_C^+ + m_C^\Theta) \tag{B3}$$

$$\text{Bel}_{Z \leftarrow ABC}(z) = m_A^+ m_B^+ m_C^+ \tag{B4}$$

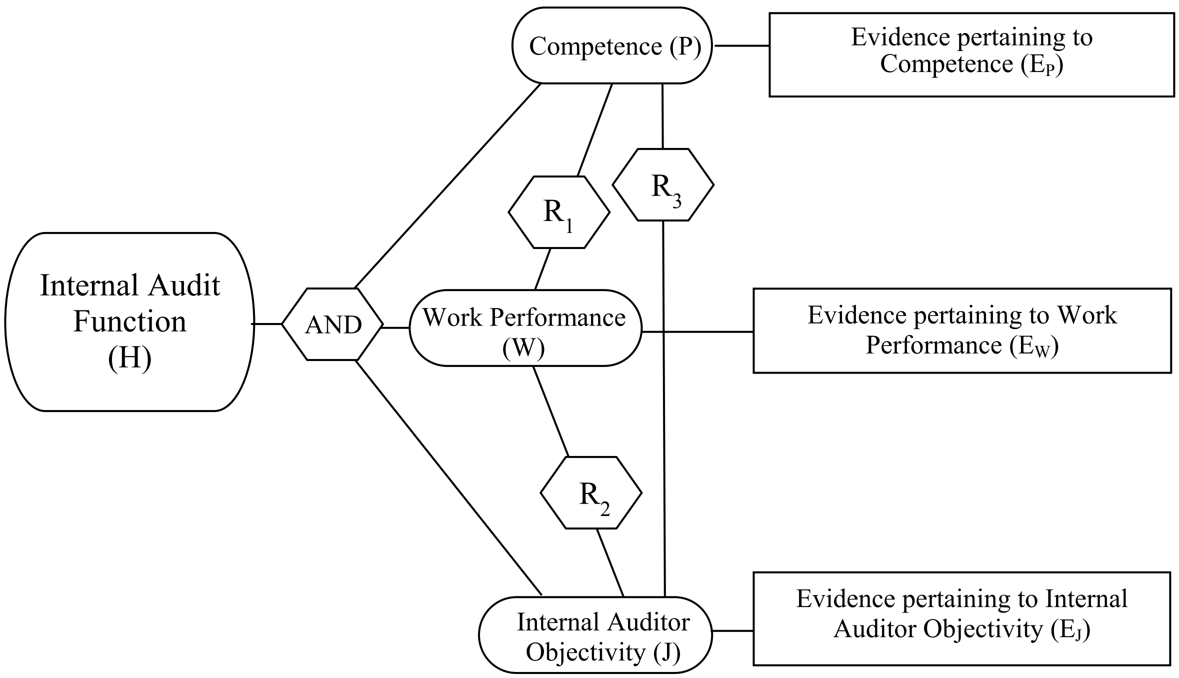
These are exactly the same beliefs given in Equations (20) and (21). **Q.E.D.**

Figure 1: **Network of Variables***



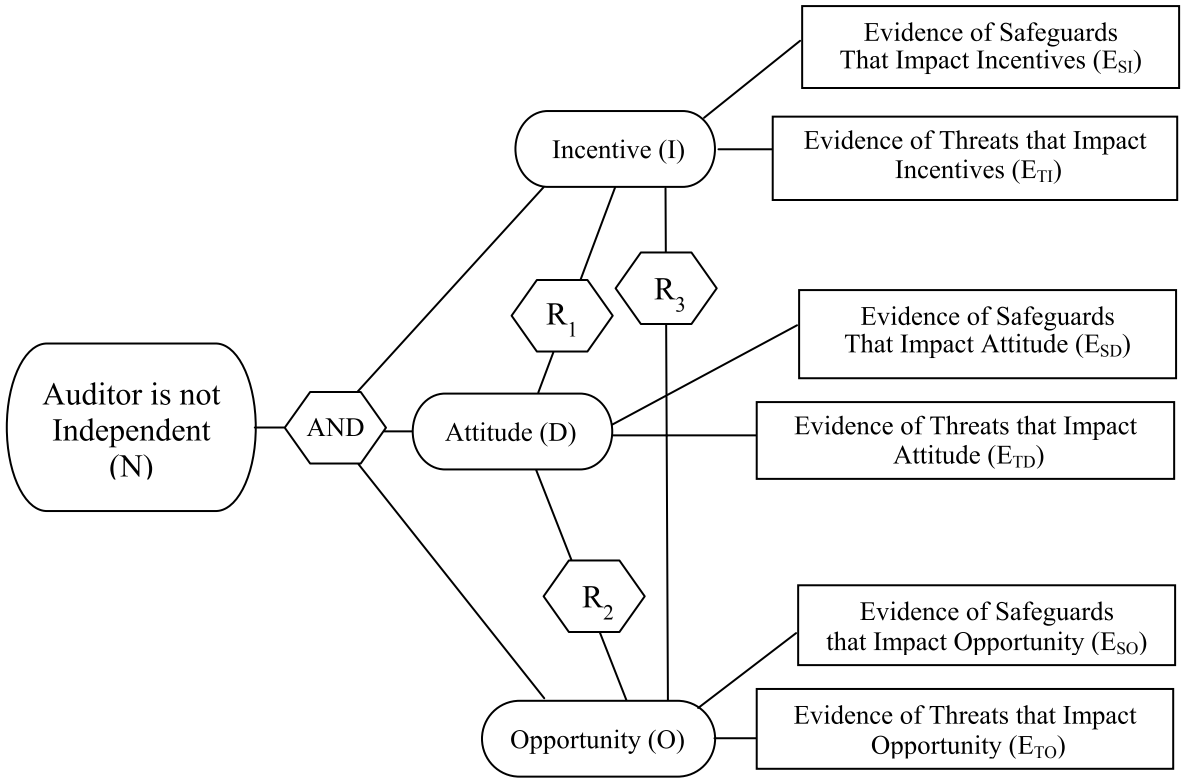
* A rounded box represents a variable, a rectangle represents an item of evidence, and a hexagonal box represents a relationship. These relationships are defined in Table 1.

Figure 2: Diagram Representing Network of Variables for Internal Audit Function Evaluation with Associated Items of Evidence*



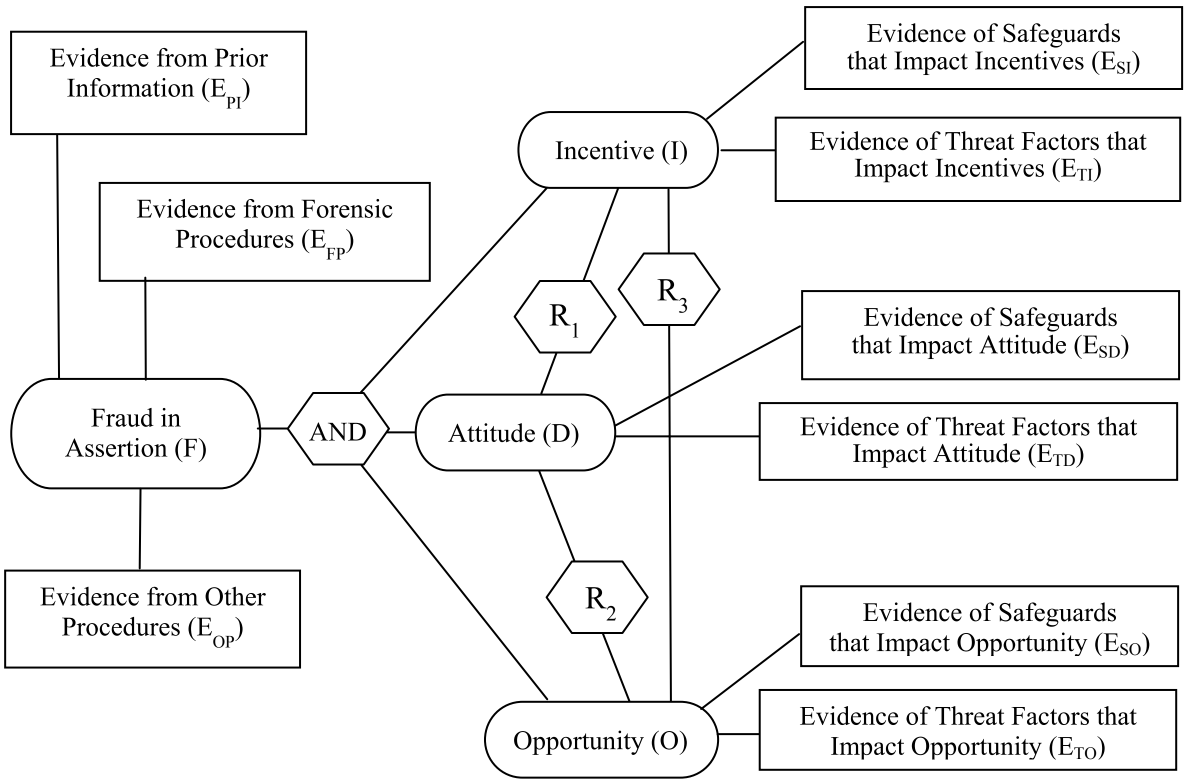
* Similar to Figures 2 and 3, a rounded box here represents a variable, a rectangle represents an item of evidence, and a hexagonal box represents a relationship. These relationships are defined in Table 1 similar to the relationships in Figure 1.

Figure 3: Diagram Representing Network of Variables for Auditor Independence Risk with Associated Items of Evidence*



* A rounded box represents a variable, a rectangle represents an item of evidence, and a hexagonal box represents a relationship. These relationships are defined in Table 1 similar to the relationships in Figure 1.

Figure 4: Diagram Representing Network of Variables with Associated Items of Evidence*



* A rounded box represents a variable, a rectangle represents an item of evidence, and a hexagonal box represents a relationship. These relationships are defined in Table 1 similar to the relationships in Figure 1.

Table 1: List of Symbols and Their Descriptions	
Symbol	Description
General Analytical Formula	
Z {*z*,~*z*}	Binary variable Z that is related to three binary variables, A, B and C, through the logical relationship 'AND' where *z* and ~*z* represent that Z is true and not true, respectively.
A {*a*,~*a*}	Binary variable A where *a* and ~*a* represent that A is true and not true, respectively.
B {*b*,~*b*}	Binary variable B where *b* and ~*b* represent that B is true and not true, respectively.
C {*c*,~*c*}	Binary variable C where *c* and ~*c* represent that C is true and not true, respectively.
R_1, r_1	R_1 denotes the relational node between A and B and r_1 represents its strength
R_2, r_2	R_2 denotes the relational node between B and C and r_2 represents its strength
R_3, r_3	R_3 denotes the relational node between A and C and r_3 represents its strength
$m_{..}(..)$	The basic belief mass (m-value) for the value of the variable in the parenthesis from the evidence represented by the subscript.
$\Theta_{..}$	This symbol represents the frame of a variable denoted by the subscript. For example, the frame of variable 'A' is represented as $\Theta_A = \{a, \sim a\}$.
m_A^+, m_B^+ and m_C^+	m-values supporting the presence of the factors A, B and C, respectively.
m_A^-, m_B^- and m_C^-	m-values negating the presence of the factors A, B and C, respectively.
m_A^Θ, m_B^Θ and m_C^Θ	m-values representing the basic beliefs on the entire frame of the variables represented by the sub-script.
$Bel_{..}(..)$	The belief that the argument in the parenthesis is true
$Pl_{..}(..)$	The plausibility that the argument in the parenthesis is true.
K	A normalization constant
$Bel_{Z\leftarrow ABC}(\{z\})$	The belief that Z is true after all beliefs from variables A, B, and C have been propagated to Z and combined.
$Bel_{Z\leftarrow ABC}(\{\sim z\})$	The belief that Z is not true after all beliefs from variables A, B, and C have been propagated to Z and combined.
$Pl_{Z\leftarrow ABC}(\{z\})$	The plausibility that Z is true after all beliefs from variables A, B, and C have been propagated to Z and combined.
$Pl_{Z\leftarrow ABC}(\{\sim z\})$	The plausibility that Z is not true after all beliefs from variables A, B, and C have been propagated to Z and combined.
Application to Assessing Belief and Plausibility in Fraud	
F {*f*,~*f*}	F represents the variable 'Fraud'. Values *f* and ~*f* represent that F is true, and not true, respectively. In other words, '*f*' represents that fraud is present and '~*f* 'that fraud is not present.
I {*i*,~*i*}	I represents the variable 'Incentive'. Values *i* and ~*i* represent that I is true and not true, respectively. In other words, *i* represents that there is an incentive and ~*i* represents that there is no incentive.
D {*d*,~*d*}	D represents the variable 'Attitude'. Values *d* and ~*d* represent that D is true and not true, respectively. In other words, *d* represents that management's attitude rationalizes the commitment of fraud, and ~*d* the opposite of *d*.
O {*o*,~*o*}	O represents the variable 'Opportunity'. Values *o* and ~*o* represent that O is true and not true, respectively. In other words, *o* represents that there is an opportunity and ~*o* represents that there is no opportunity.
m_I^+, m_D^+, m_O^+	m-values supporting the presence of the factors I, D and O, respectively.

m_I^-, m_D^-, m_O^-	m-values negating the presence of the factors I, D and O, respectively.
m_I^Θ, m_D^Θ, and m_O^Θ	m-values representing the basic beliefs on the entire frame of the variables represented by the sub-script.
E_{TI}, E_{TD}, E_{TO}	Evidence about threat factors pertaining to Incentive (I), Attitude (D), and Opportunity (O)
E_{SI}, E_{SD}, E_{SO}	Evidence about safeguard factors pertaining to Incentive (I), Attitude (D), and Opportunity (O)
E_{PI}	Evidence related to whether fraud (F) is present or not based on prior information (PI),
E_{FP}	Evidence related to whether fraud (F) is present or not obtained from forensic procedures
E_{OP}	Evidence related to whether fraud (F) is present or not from procedures other than forensic procedures
$Pl_{PI}(f)$, $Pl_{FP}(f)$, $Pl_{OP}(f)$	Plausibility of fraud based on prior information (PI), evidence from forensic procedures (FP), and evidence from other procedures (OP), respectively
$Pl_{TI}(i)$, $Pl_{TD}(d)$, $Pl_{TO}(o)$	The plausibility that an incentive exists (*i*), management may have an attitude (*d*) rationalizing fraud, and opportunities exist (*o*) because of the corresponding threat factors.
$Pl_{SI}(i)$, $Pl_{SD}(d)$, $Pl_{SO}(o)$	The plausibility that an incentive exists (*i*), management may have an attitude (*d*) rationalizing fraud, and opportunities exist (*o*) because of ineffective safeguards.
K_T, K_F, K_I, K_D, K_O,	Normalization constants
Application to Auditor Independence Impairment	
N {*n*,~*n*}	N represents the variable 'Independence Risk'. Values *n* and ~*n* represent that N is true, and not true, respectively. In other words, *n* represents that independence has been impaired and '~*n* 'that independence has not been impaired.
I {*i*,~*i*}	I represents the variable 'Incentive'. Values *i* and ~*i* represent that I is true and not true, respectively. In other words, *i* represents that a threat to independence exists in the form of an incentive and ~*i* represents that there is no threat.
D {*d*,~*d*}	D represents the variable 'Attitude'. Values *d* and ~*d* represent that D is true and not true, respectively. In other words, *d* represents that the auditor's attitude rationalizes the impairment of independence, and ~*d* the opposite of *d*.
O {*o*,~*o*}	O represents the variable 'Opportunity'. Values *o* and ~*o* represent that O is true and not true, respectively. In other words, *o* represents that a threat to independence exists in the form of an opportunity and ~*o* represents that there is no opportunity.
Application to Internal Audit Function Quality	
H {*h*,~*h*}	H represents the quality of the internal audit function. Values *h* and ~*h* represent that H is true, and not true, respectively. In other words, *h* represents that the quality of the internal audit function is high and ~*h* that quality is low.
P {*p*,~*p*}	P represents the variable 'Competence'. Values *p* and ~*p* represent that P is true and not true, respectively. In other words, *p* represents that the internal auditor is competent and ~*p* represents that the auditor is not competent.
W {*w*,~*w*}	W represents the variable 'Work Performance'. Values *w* and ~*w* represent that W is true and not true, respectively. In other words, *w* represents that the work performance of the auditor is high and ~*w* represents that the work performance is low.
J {*j*,~*j*}	J represents the variable 'Internal Auditor Objectivity'. Values *j* and ~*j* represent that J is true and not true, respectively. In other words, *j* represents that the internal auditor is objective and ~*j* represents that the auditor is not objective.