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A FETI-DP TYPE DOMAIN DECOMPOSITION ALGORITHM FOR THREE-DIMENSIONAL INCOMPRESSIBLE STOKES EQUATIONS*

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Abstract. The FETI-DP (dual-primal finite element tearing and interconnecting) algorithms, proposed by the authors in [*SIAM J. Numer. Anal.*, 51 (2013), pp. 1235–1253] and [*Internat. J. Numer. Methods Engrg.*, 94 (2013), pp. 128–149] for solving incompressible Stokes equations, are extended to three-dimensional problems. A new analysis of the condition number bound for using the Dirichlet preconditioner is given. The algorithm and analysis are valid for mixed finite elements with both continuous and discontinuous pressures. An advantage of this new analysis is that the numerous coarse level velocity components, required in the previous analysis to enforce the divergence-free subdomain boundary velocity conditions, are no longer needed. This greatly reduces the size of the coarse level problem in the algorithm, especially for three-dimensional problems. The coarse level velocity space can be chosen as simple as those coarse spaces for solving scalar elliptic problems corresponding to each velocity component. Both the Dirichlet and lumped preconditioners are analyzed using the same framework in this new analysis. Their condition number bounds are proved to be independent of the number of subdomains for fixed subdomain problem size. Numerical experiments in both two and three dimensions, using mixed finite elements with both continuous and discontinuous pressures, demonstrate the convergence rate of the algorithms.

Key words. domain decomposition, incompressible Stokes, FETI-DP, BDDC, divergence-free

AMS subject classifications. 65F10, 65N30, 65N55

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1. Introduction.

Mixed finite elements are often used to solve incompressible Stokes and Navier–Stokes equations. Continuous pressures have been used in many mixed finite elements, e.g., the well-known Taylor–Hood finite elements [27]. However, most domain decomposition methods require that the pressure be discontinuous when they are used to solve the indefinite linear systems arising from such mixed finite element discretizations; see, e.g., [4, 5, 6, 10, 11, 16, 19, 21, 23, 24, 29, 30]. Several domain decomposition algorithms allow one to use continuous pressures, e.g., Klawonn and Pavarino [16], Goldfeld [9], Šístek et al. [25], Benhassine and Bendali [1], and Kim and Lee [14], even though no convergence rate analysis of those algorithms is known for the continuous pressure case.

Recently, the authors [20, 31] proposed and analyzed a FETI-DP (dual-primal finite element tearing and interconnecting) type domain decomposition algorithm for solving the incompressible Stokes equation in two dimensions. Both discontinuous and continuous pressures can be used in the mixed finite element discretization. In both cases, the indefinite system of linear equations can be reduced to a symmetric positive semidefinite system. Therefore, the preconditioned conjugate gradient method can be applied, and a scalable convergence rate of the algorithm has been proved.

The lumped and Dirichlet preconditioners have been studied in [20] and [31], respectively. For the lumped preconditioner it was shown both experimentally and

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analytically in [20] that the coarse level space can be chosen the same as for solving scalar elliptic problems corresponding to each velocity component to achieve a scalable convergence rate. Similar observations for the lumped preconditioner were also pointed out earlier by Kim and Lee [13, 12] and Kim, Lee, and Park [15], even though their studies are only for using discontinuous pressures.

For the Dirichlet preconditioner studied in [31], a distinctive feature is the application of subdomain discrete harmonic extensions in the preconditioner. In other existing FETI-DP and BDDC (balancing domain decomposition by constraints) algorithms (cf. [19, 21]), subdomain discrete Stokes extensions have been used, and the coarse level velocity space has to contain sufficient components to enforce divergence-free subdomain boundary velocity conditions. Those complicated and numerous coarse level velocity components, especially for three-dimensional problems as discussed in [21], are not needed for the implementation of the Dirichlet preconditioner in [31]. But they are still required in [31] just for the analysis, where subdomain Stokes extensions were used, to obtain a scalable condition number bound.

In this paper, we provide a new analysis for the algorithms in [20, 31], which can analyze both the lumped and Dirichlet preconditioners in the same framework. It does not use any subdomain Stokes extensions, and those additional coarse level velocity components to enforce divergence-free subdomain boundary velocity conditions are no longer needed. For both the lumped and Dirichlet preconditioners, the coarse level space can be chosen as simple as those coarse spaces for solving scalar elliptic problems corresponding to each velocity component. This greatly simplifies the requirements on the coarse level space for the case of the Dirichlet preconditioner, especially in three dimensions. This paper is presented in the context of solving three-dimensional problems; the same approach can be applied to two-dimensional problems as well.

The remainder of this paper is organized as follows. The finite element discretization of the incompressible Stokes equation is introduced in section 2. A domain decomposition approach is described in section 3, and the system is reduced to a symmetric positive semidefinite problem in section 4. A few preliminary results used in the condition number bound estimates are given in section 5. The lumped and Dirichlet preconditioners are introduced in section 6, and the condition number bounds of the preconditioned systems are established in section 7. Finally, numerical results of solving the incompressible Stokes equation in both two and three dimensions are given in section 8 to demonstrate the convergence rate of the algorithm.

2. Finite element discretization. We consider solving the following incompressible Stokes problem on a bounded, three-dimensional polyhedral domain Ω with a Dirichlet boundary condition:

$$\begin{cases} -\Delta \mathbf{u}^* + \nabla p^* = \mathbf{f} & \text{in } \Omega, \\ -\nabla \cdot \mathbf{u}^* = 0 & \text{in } \Omega, \\ \mathbf{u}^* = \mathbf{u}^*_{\partial\Omega} & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where the boundary velocity $\mathbf{u}^*_{\partial\Omega}$ satisfies the compatibility condition $\int_{\partial\Omega} \mathbf{u}^*_{\partial\Omega} \cdot \mathbf{n} = 0$. For simplicity, we assume that $\mathbf{u}^*_{\partial\Omega} = \mathbf{0}$ without loss of generality.

The weak solution of (2.1) is given by the following: find $\mathbf{u}^* \in \left(H_0^1(\Omega)\right)^3 = \{\mathbf{v} \in (H^1(\Omega))^3 \mid \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega\}$ and $p^* \in L^2(\Omega)$ such that

$$\begin{cases} a(\mathbf{u}^*, \mathbf{v}) + b(\mathbf{v}, p^*) = (\mathbf{f}, \mathbf{v}) & \forall \mathbf{v} \in \left(H_0^1(\Omega)\right)^3, \\ b(\mathbf{u}^*, q) = 0 & \forall q \in L^2(\Omega), \end{cases} \tag{2.2}$$

where $a(\mathbf{u}^*, \mathbf{v}) = \int_\Omega \nabla\mathbf{u}^* \cdot \nabla\mathbf{v}$, $b(\mathbf{u}^*, q) = -\int_\Omega (\nabla\cdot\mathbf{u}^*)q$, $(\mathbf{f}, \mathbf{v}) = \int_\Omega \mathbf{f}\cdot\mathbf{v}$. We note that the solution of (2.2) is not unique, with the pressure p^* different up to an additive constant.

A mixed finite element is used to solve (2.2). In this paper we apply a mixed finite element with continuous pressures, e.g., the Taylor–Hood type mixed finite elements. The same algorithm and analysis can be applied to mixed finite elements with discontinuous pressures as well; see [31]. Denote the velocity finite element space by $\mathbf{W} \subset \left(H_0^1(\Omega)\right)^3$ and the pressure finite element space by $Q \subset L^2(\Omega)$. The finite element solution $(\mathbf{u}, p) \in \mathbf{W} \bigoplus Q$ of (2.2) satisfies

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix}, \tag{2.3}$$

where A, B, and $\mathbf{f}$ represent, respectively, the restrictions of $a(\cdot,\cdot)$, $b(\cdot,\cdot)$, and $(\mathbf{f},\cdot)$ to the finite-dimensional spaces $\mathbf{W}$ and Q. We use the same notation in this paper to represent both a finite element function and the vector of its nodal values.

The coefficient matrix in (2.3) is rank deficient even though A is symmetric positive definite. $\mathrm{Ker}(B^T)$, the kernel of B^T, contains all constant pressures in Q. $\mathrm{Im}(B)$, the range of B, is orthogonal to $\mathrm{Ker}(B^T)$ and consists of all vectors in Q with zero average. For a general right-hand side vector $(\mathbf{f}, g)$ in (2.3), the existence of solution requires that $g \in \mathrm{Im}(B)$, i.e., g has zero average; for the right-hand side given in (2.3), $g = 0$, and the solution always exists. When the pressure is considered in the quotient space $Q/\mathrm{Ker}(B^T)$, the solution is unique. In this paper, when $q \in Q/\mathrm{Ker}(B^T)$, we always assume that q has zero average.

Let h represent the characteristic diameter of the mixed elements. We assume that the mixed finite element space $\mathbf{W} \times Q$ is inf-sup stable in the sense that there exists a positive constant β, independent of h, such that

$$\sup_{\mathbf{w}\in\mathbf{W}} \frac{\langle q, B\mathbf{w}\rangle^2}{\langle \mathbf{w}, A\mathbf{w}\rangle} \geq \beta^2 \langle q, Zq\rangle \qquad \forall q \in Q/\mathrm{Ker}(B^T); \tag{2.4}$$

cf. [3, Chapter III, section 7]. Here, as used throughout this paper, $\langle\cdot,\cdot\rangle$ represents the inner (or semi-inner) product of two vectors. The matrix Z represents the mass matrix defined on the pressure finite element space Q; i.e., for any $q \in Q$, $\|q\|_{L^2}^2 = \langle q, Zq\rangle$. It is easy to see (cf. [28, Lemma B.31]) that Z is spectrally equivalent to $h^3 I$ for three-dimensional problems; i.e., there exist positive constants c and C such that

$$ch^3 I \leq Z \leq Ch^3 I, \tag{2.5}$$

where I represents the identity matrix. Here, as in other places in this paper, c and C represent generic positive constants which are independent of h and the subdomain diameter H (described in the following section).

3. A nonoverlapping domain decomposition approach. The domain Ω is decomposed into N nonoverlapping polyhedral subdomains Ω_i, $i = 1, 2, \ldots, N$. Each subdomain is the union of a bounded number of elements, with the diameter of the subdomain in the order of H. We use Γ to represent the subdomain interface which contains all the subdomain boundary nodes shared by neighboring subdomains; we assume that the subdomain meshes have matching nodes across Γ. Γ is composed of subdomain faces, which are regarded as open subsets of Γ shared by two subdomains; subdomain edges, which are regarded as open subsets of Γ shared by more than two subdomains; and subdomain vertices, which are end points of edges.

The velocity and pressure finite element spaces $\mathbf{W}$ and Q are decomposed into

$$\mathbf{W} = \mathbf{W}_I \bigoplus \mathbf{W}_\Gamma, \quad Q = Q_I \bigoplus Q_\Gamma,$$

where $\mathbf{W}_I$ and Q_I are direct sums of independent subdomain interior velocity spaces $\mathbf{W}_I^{(i)}$ and interior pressure spaces $Q_I^{(i)}$, respectively, i.e.,

$$\mathbf{W}_I = \bigoplus_{i=1}^{N} \mathbf{W}_I^{(i)}, \quad Q_I = \bigoplus_{i=1}^{N} Q_I^{(i)}.$$

$\mathbf{W}_\Gamma$ and Q_Γ are subdomain interface velocity and pressure spaces, respectively. All functions in $\mathbf{W}_\Gamma$ and Q_Γ are continuous across Γ; their degrees of freedom are shared by neighboring subdomains.

To formulate the domain decomposition algorithm, we introduce a partially subassembled subdomain interface velocity space $\widetilde{\mathbf{W}}_\Gamma$,

$$\widetilde{\mathbf{W}}_\Gamma = \mathbf{W}_\Delta \bigoplus \mathbf{W}_\Pi = \left(\bigoplus_{i=1}^{N} \mathbf{W}_\Delta^{(i)} \right) \bigoplus \mathbf{W}_\Pi.$$

$\mathbf{W}_\Pi$ is the continuous, coarse level, primal velocity space which is typically spanned by subdomain vertex nodal basis functions and/or by interface edge/face-cutoff functions with constant nodal values on each edge/face or with values of positive weights on these edges/faces. The primal, coarse level velocity degrees of freedom are shared by neighboring subdomains. The complementary space $\mathbf{W}_\Delta$ is the direct sum of independent subdomain dual interface velocity spaces $\mathbf{W}_\Delta^{(i)}$, which correspond to the remaining subdomain interface velocity degrees of freedom and are spanned by basis functions which vanish at the primal degrees of freedom. Thus, an element in $\widetilde{\mathbf{W}}_\Gamma$ typically has a continuous primal velocity component and a discontinuous dual velocity component.

It is well known that, for domain decomposition algorithms, the coarse space $\mathbf{W}_\Pi$ should be sufficiently rich to achieve a scalable convergence rate. On the other hand, a large coarse level problem will certainly degrade the parallel performance of the algorithm. Therefore it is important to keep the size of the coarse level problem as small as possible. When the Dirichlet preconditioner was used in the FETI-DP algorithm for solving the incompressible Stokes equation [19] and similarly in the BDDC algorithm [21], subdomain discrete Stokes extensions were used, and $\mathbf{W}_\Pi$ has to contain sufficient subdomain interface components such that functions in $\mathbf{W}_\Delta$ have zero flux across the subdomain boundaries. Such requirements lead to a large coarse level velocity space, especially for three-dimensional problems; cf. [21].

In [31], a FETI-DP type algorithm is proposed for solving two-dimensional incompressible Stokes problems. A distinctive feature of the Dirichlet preconditioner used in that algorithm is the application of subdomain discrete harmonic extensions, instead of subdomain discrete Stokes extensions. As a result, the divergence-free subdomain boundary velocity conditions are not needed in that algorithm. However, the analysis, given in [31] for the Dirichlet preconditioner, still uses subdomain Stokes extensions and requires the same type of coarse level velocity space as discussed in [21] to establish a scalable condition number bound estimate. In this paper, a new analysis is offered, and it is sufficient for $\mathbf{W}_\Pi$ to be spanned just by the subdomain vertex nodal basis functions and subdomain edge-cutoff functions corresponding to each velocity component, as for solving three-dimensional scalar elliptic problems; cf. [28, section 6.4.2].

The functions $\mathbf{w}_\Delta$ in $\mathbf{W}_\Delta$ are in general not continuous across Γ. To enforce their continuity, we define a Boolean matrix B_Δ of the form

$$B_\Delta = \begin{bmatrix} B_\Delta^{(1)} & B_\Delta^{(2)} & \cdots & B_\Delta^{(N)} \end{bmatrix},$$

constructed from $\{0, 1, -1\}$. On each row of B_Δ, there are only two nonzero entries, 1 and -1, corresponding to one velocity degree of freedom shared by two neighboring subdomains, such that for any $\mathbf{w}_\Delta$ in $\mathbf{W}_\Delta$, each row of $B_\Delta \mathbf{w}_\Delta = 0$ implies that these two degrees of freedom from the two neighboring subdomains are the same. We note that, in three dimensions, a velocity degree of freedom on a subdomain edge is shared by more than two subdomains, e.g., by four subdomains. In this case, a minimum of three continuity constraints can be applied to enforce the continuity of this velocity degree of freedom among the four subdomains, which corresponds to the use of nonredundant Lagrange multipliers. In this paper, the fully redundant Lagrange multipliers are used, which means, e.g., for a subdomain edge velocity degree of freedom shared by four subdomains, six Lagrange multipliers are used to enforce all of the six possible continuity constraints among them; cf. [28, section 6.3.1].

We denote the range of B_Δ applied on $\mathbf{W}_\Delta$ by Λ, the vector space of the Lagrange multipliers. Solving the original fully assembled linear system (2.3) is then equivalent to the following: find $(\mathbf{u}_I,\ p_I,\ \mathbf{u}_\Delta,\ \mathbf{u}_\Pi,\ p_\Gamma,\ \lambda) \in \mathbf{W}_I \bigoplus Q_I \bigoplus \mathbf{W}_\Delta \bigoplus \mathbf{W}_\Pi \bigoplus Q_\Gamma \bigoplus \Lambda$ such that

$$\begin{bmatrix} A_{II} & B_{II}^T & A_{I\Delta} & A_{I\Pi} & B_{\Gamma I}^T & 0 \\ B_{II} & 0 & B_{I\Delta} & B_{I\Pi} & 0 & 0 \\ A_{\Delta I} & B_{I\Delta}^T & A_{\Delta\Delta} & A_{\Delta\Pi} & B_{\Gamma\Delta}^T & B_\Delta^T \\ A_{\Pi I} & B_{I\Pi}^T & A_{\Pi\Delta} & A_{\Pi\Pi} & B_{\Gamma\Pi}^T & 0 \\ B_{\Gamma I} & 0 & B_{\Gamma\Delta} & B_{\Gamma\Pi} & 0 & 0 \\ 0 & 0 & B_\Delta & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Delta \\ \mathbf{u}_\Pi \\ p_\Gamma \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{f}_I \\ 0 \\ \mathbf{f}_\Delta \\ \mathbf{f}_\Pi \\ 0 \\ 0 \end{bmatrix}, \tag{3.1}$$

where the subblocks in the coefficient matrix represent the restrictions of A and B in (2.3) to appropriate subspaces. The leading three-by-three block can be made block diagonal with each diagonal block corresponding to one subdomain.

The coefficient matrix in (3.1) is singular. The trivial null space vectors are those with λ in the null space of B_Δ^T and with the rest of the components of the vector equal to zero. Such singularity, due to the rank deficiency of B_Δ, need not be a concern, since the Lagrange multiplier vector λ will be confined in Λ, the range of B_Δ. The only meaningful basis vector in the null space of (3.1) corresponds to the one-dimensional null space of the original incompressible Stokes system (2.3) and is specified in Lemmas 3.1–3.4.

We first need to introduce a positive scaling factor $\delta^\dagger(x)$ for each node x on Γ. We let $\mathcal{N}_x$ be the number of subdomains sharing x, and we define $\delta^\dagger(x) = 1/\mathcal{N}_x$. Given such scaling factors at the subdomain interface nodes, we can define a scaled operator $B_{\Delta,D}$. We note that each row of B_Δ has only two nonzero entries, 1 and -1, connecting two neighboring subdomains sharing a node x on Γ. Multiplying each entry by the scaling factor $\delta^\dagger(x)$ gives us $B_{\Delta,D}$. Namely,

$$B_{\Delta,D} = \begin{bmatrix} D_\Delta B_\Delta^{(1)} & D_\Delta B_\Delta^{(2)} & \cdots & D_\Delta B_\Delta^{(N)} \end{bmatrix},$$

where D_Δ is a diagonal matrix and contains $\delta^\dagger(x)$ on its diagonal. We also see from the definition of $B_{\Delta,D}$ that the scalings on all the Lagrange multipliers related to the same subdomain interface node are the same, from which we have the following lemma.

LEMMA 3.1. *The null space of B_Δ^T is the same as the null space of $B_{\Delta,D}^T$; the range of B_Δ is the same as the range of $B_{\Delta,D}$.*

LEMMA 3.2. *For any $\lambda \in \Lambda$, $B_\Delta B_{\Delta,D}^T \lambda = B_{\Delta,D} B_\Delta^T \lambda = \lambda$.*

Proof. The equality $B_\Delta B_{\Delta,D}^T \lambda = \lambda$ can be found directly in [28, page 175]. A similar approach is used here to prove $B_{\Delta,D} B_\Delta^T \lambda = \lambda$. Since Λ is the range of B_Δ applied on $\mathbf{W}_\Delta$, let us denote $\lambda = B_\Delta \mathbf{w}_\Delta$ for a certain $\mathbf{w}_\Delta \in \mathbf{W}_\Delta$. Then just look at one entry of λ, e.g., the Lagrange multiplier connecting two neighboring subdomain velocity degrees of freedom, represented by $w^{(i)}(x)$ and $w^{(j)}(x)$, of $\mathbf{w}_\Delta$ at a subdomain interface boundary node x between Ω_i and Ω_j. The value of that Lagrange multiplier equals $w^{(i)}(x) - w^{(j)}(x)$ (or $w^{(j)}(x) - w^{(i)}(x)$ depending on the choice of signs in B_Δ). On the other hand, the corresponding entry of $B_{\Delta,D} B_\Delta^T B_\Delta \mathbf{w}_\Delta$ equals

$$\begin{aligned}
&\delta^\dagger(x) \left(B_\Delta^T B_\Delta \mathbf{w}_\Delta\right)^{(i)}(x) - \delta^\dagger(x) \left(B_\Delta^T B_\Delta \mathbf{w}_\Delta\right)^{(j)}(x) \\
&= \delta^\dagger(x) \sum_{k \in \mathcal{N}_x} \left(w^{(i)}(x) - w^{(k)}(x)\right) - \delta^\dagger(x) \sum_{k \in \mathcal{N}_x} \left(w^{(j)}(x) - w^{(k)}(x)\right) \\
&= \sum_{k \in \mathcal{N}_x} \delta^\dagger(x) \left(w^{(i)}(x) - w^{(j)}(x)\right) = w^{(i)}(x) - w^{(j)}(x),
\end{aligned}$$

where, in the last step, we used the fact that $\sum_{k \in \mathcal{N}_x} \delta^\dagger(x) = 1$. ∎

LEMMA 3.3. *Let $1_{p_I} \in Q_I$, $1_{p_\Gamma} \in Q_\Gamma$ represent vectors with value 1 on each entry. Then*

$$[B_{I\Delta}^T \quad B_{\Gamma\Delta}^T] \begin{bmatrix} 1_{p_I} \\ 1_{p_\Gamma} \end{bmatrix} = B_\Delta^T \lambda, \tag{3.2}$$

where

$$\lambda = B_{\Delta,D} [B_{I\Delta}^T \quad B_{\Gamma\Delta}^T] \begin{bmatrix} 1_{p_I} \\ 1_{p_\Gamma} \end{bmatrix} \in \Lambda. \tag{3.3}$$

Proof. The left-hand side of (3.2) contains face integrals of the normal component of the dual subdomain interface velocity finite element basis functions across the subdomain interface. For a face velocity degree of freedom, which is shared by two neighboring subdomains, the face integrals of their normal components on the two neighboring subdomains have opposite signs, since their normal directions are opposite. This pair of opposite values can then be represented by the product of B_Δ^T and a Lagrange multiplier with value equal to the face integral of the corresponding basis function.

Now we consider a subdomain edge velocity degree of freedom, which is shared by more than two subdomains, e.g., by four subdomains Ω_i, Ω_j, Ω_k, and Ω_l. A two-dimensional projection of such an edge node is shown in Figure 1, where the third direction points outward directly and the dashed lines represent the element boundaries. $\mathcal{F}_{ij}$, $\mathcal{F}_{jk}$, $\mathcal{F}_{kl}$, and $\mathcal{F}_{li}$ represent the common element faces connected to this edge node on the subdomain interfaces; e.g., $\mathcal{F}_{ij}$ represents the element faces sharing this node between Ω_i and Ω_j, while the elements on Ω_i and Ω_k have no common face. Denote the integration of the normal component of this velocity basis

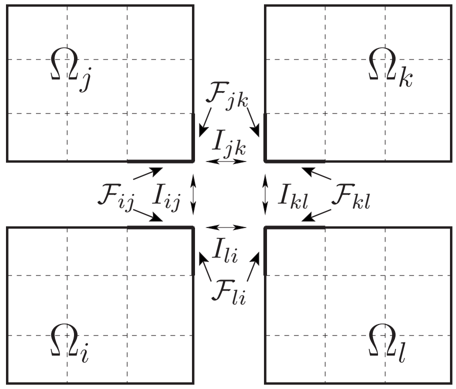

FIG. 1. *Illustration on a subdomain edge interface degree of freedom.*

function on $\mathcal{F}_{ij}$, $\mathcal{F}_{jk}$, $\mathcal{F}_{kl}$, and $\mathcal{F}_{li}$ by I_{ij}, I_{jk}, I_{kl}, and I_{li}, respectively, with a certain chosen normal direction for each element face. Then the entries of the left-hand side vector in (3.2) corresponding to this edge velocity degree of freedom on the four subdomains Ω_i, Ω_j, Ω_k, and Ω_l can be represented by $I_{ij}+I_{li}$, $-I_{ij}+I_{jk}$, $-I_{jk}-I_{kl}$, and $I_{kl}-I_{li}$, respectively, where the choice of positive and negative signs is due to the fact that the normal directions to the same element faces shared by two neighboring subdomains are the opposite of each other. Take I_{ij}, I_{jk}, I_{kl}, I_{li} as the four Lagrange multiplier values as illustrated in Figure 1. Then the four subdomain face integral values, $I_{ij}+I_{li}$, $-I_{ij}+I_{jk}$, $-I_{jk}-I_{kl}$, and $I_{kl}-I_{li}$, can be represented as the product of the corresponding B_Δ^T and a Lagrange multiplier vector which contains the four Lagrange multiplier values and zero values elsewhere.

The above has just shown that the left-hand side of (3.2) can be represented by the product of B_Δ^T with a Lagrange multiplier vector λ. If λ is not in Λ, i.e., not in the range of B_Δ, then it can always be written as the sum of its components in Λ and in the null space of B_Δ^T. Then we just take its component in Λ as λ, which does not change the product $B_\Delta^T\lambda$. By multiplying $B_{\Delta,D}$ by both sides of (3.2) and using Lemma 3.2, we have (3.3). ∎

LEMMA 3.4. *The basis vector in the null space of* (3.1)*, corresponding to the one-dimensional null space of the original incompressible Stokes system* (2.3)*, is*

$$\left(\mathbf{0},\quad 1_{p_I},\quad \mathbf{0},\quad \mathbf{0},\quad 1_{p_\Gamma},\quad -B_{\Delta,D}[B_{I\Delta}^T \quad B_{\Gamma\Delta}^T]\begin{bmatrix} 1_{p_I} \\ 1_{p_\Gamma} \end{bmatrix}\right). \tag{3.4}$$

Proof. Since the null space of (2.3) consists of all constant pressures, substituting the vector (3.4) into (3.1) gives zero blocks on the right-hand side, except at the third block where

$$\mathbf{f}_\Delta = [B_{I\Delta}^T \quad B_{\Gamma\Delta}^T]\begin{bmatrix} 1_{p_I} \\ 1_{p_\Gamma} \end{bmatrix} - B_\Delta^T B_{\Delta,D}[B_{I\Delta}^T \quad B_{\Gamma\Delta}^T]\begin{bmatrix} 1_{p_I} \\ 1_{p_\Gamma} \end{bmatrix}, \tag{3.5}$$

which also equals zero from (3.2) and (3.3) in Lemma 3.3. ∎

4. A reduced symmetric positive semidefinite system. The system (3.1) can be reduced to a Schur complement problem for the variables (p_Γ, λ). Since the leading four-by-four block of the coefficient matrix in (3.1) is invertible, the variables

$(\mathbf{u}_I,\ p_I,\ \mathbf{u}_\Delta,\ \mathbf{u}_\Pi)$ can be eliminated, and we obtain

$$G \begin{bmatrix} p_\Gamma \\ \lambda \end{bmatrix} = g, \tag{4.1}$$

where

$$G = B_C \widetilde{A}^{-1} B_C^T, \qquad g = B_C \widetilde{A}^{-1} \begin{bmatrix} \mathbf{f}_I \\ 0 \\ \mathbf{f}_\Delta \\ \mathbf{f}_\Pi \end{bmatrix}, \tag{4.2}$$

with

$$\widetilde{A} = \begin{bmatrix} A_{II} & B_{II}^T & A_{I\Delta} & A_{I\Pi} \\ B_{II} & 0 & B_{I\Delta} & B_{I\Pi} \\ A_{\Delta I} & B_{I\Delta}^T & A_{\Delta\Delta} & A_{\Delta\Pi} \\ A_{\Pi I} & B_{I\Pi}^T & A_{\Pi\Delta} & A_{\Pi\Pi} \end{bmatrix} \quad \text{and} \quad B_C = \begin{bmatrix} B_{\Gamma I} & 0 & B_{\Gamma\Delta} & B_{\Gamma\Pi} \\ 0 & 0 & B_\Delta & 0 \end{bmatrix}. \tag{4.3}$$

We can see that $-G$ is the Schur complement of the coefficient matrix of (3.1) with respect to the last two row blocks, i.e.,

$$\begin{bmatrix} I & 0 \\ -B_C\widetilde{A}^{-1} & I \end{bmatrix} \begin{bmatrix} \widetilde{A} & B_C^T \\ B_C & 0 \end{bmatrix} \begin{bmatrix} I & -\widetilde{A}^{-1}B_C^T \\ 0 & I \end{bmatrix} = \begin{bmatrix} \widetilde{A} & 0 \\ 0 & -G \end{bmatrix}.$$

From the Sylvester law of inertia, namely, the number of positive, negative, and zero eigenvalues of a symmetric matrix is invariant under a change of coordinates, we can see that the number of zero eigenvalues of G is the same as the number of zero eigenvalues (with multiplicity counted) of the original coefficient matrix of (3.1), and all other eigenvalues of G are positive. Therefore G is symmetric positive semidefinite. The basis vectors of the null space of G also inherit those from the null space of (3.1), and the only interesting basis vector is

$$\left(1_{p_\Gamma}, \quad -B_{\Delta,D}[B_{I\Delta}^T \quad B_{\Gamma\Delta}^T] \begin{bmatrix} 1_{p_I} \\ 1_{p_\Gamma} \end{bmatrix} \right), \tag{4.4}$$

which is derived from Lemma 3.4. The other null space vectors of G are all vectors with λ in the null space of B_Δ^T and $p_\Gamma = 0$. The range of G contains all vectors orthogonal to those null vectors. Denote $X = Q_\Gamma \bigoplus \Lambda$, where, as defined earlier, Λ is the range of B_Δ. Then the range of G, denoted by R_G, is the subspace of X orthogonal to (4.4), i.e.,

$$R_G = \left\{ \begin{bmatrix} g_{p_\Gamma} \\ g_\lambda \end{bmatrix} \in X \;\middle|\; g_{p_\Gamma}^T 1_{p_\Gamma} - g_\lambda^T \left(B_{\Delta,D}[B_{I\Delta}^T \quad B_{\Gamma\Delta}^T] \begin{bmatrix} 1_{p_I} \\ 1_{p_\Gamma} \end{bmatrix} \right) = 0 \right\}. \tag{4.5}$$

The restriction of G to its range R_G is positive definite. The fact that the solution of (3.1) always exists for any given $(\mathbf{f}_I,\ \mathbf{f}_\Delta,\ \mathbf{f}_\Pi)$ on the right-hand side implies that the solution of (4.1) exists for any g defined by (4.2). Therefore $g \in R_G$. When the conjugate gradient (CG) method is applied to solve (4.1) with zero initial guess, all the iterates are in the Krylov subspace generated by G and g, which is also a subspace of R_G, where the CG method cannot break down. After obtaining $(p_\Gamma,\ \lambda)$ from solving (4.1), the other components $(\mathbf{u}_I,\ p_I,\ \mathbf{u}_\Delta,\ \mathbf{u}_\Pi)$ in (3.1) are obtained by back substitution.

In the rest of this section, we discuss the implementation of multiplying G by a vector. The main operation is the product of $\widetilde{A}^{-1}$ with a vector; cf. (4.2). We denote

$$A_{rr} = \begin{bmatrix} A_{II} & B_{II}^T & A_{I\Delta} \\ B_{II} & 0 & B_{I\Delta} \\ A_{\Delta I} & B_{I\Delta}^T & A_{\Delta\Delta} \end{bmatrix}, \quad A_{\Pi r} = A_{r\Pi}^T = \begin{bmatrix} A_{\Pi I} & B_{I\Pi}^T & A_{\Pi\Delta} \end{bmatrix}, \quad f_r = \begin{bmatrix} \mathbf{f}_I \\ 0 \\ \mathbf{f}_\Delta \end{bmatrix},$$

and define the Schur complement

$$S_\Pi = A_{\Pi\Pi} - A_{\Pi r} A_{rr}^{-1} A_{r\Pi},$$

which is symmetric positive definite from the Sylvester law of inertia. S_Π defines the coarse level problem in the algorithm. The product

$$\begin{bmatrix} A_{II} & B_{II}^T & A_{I\Delta} & A_{I\Pi} \\ B_{II} & 0 & B_{I\Delta} & B_{I\Pi} \\ A_{\Delta I} & B_{I\Delta}^T & A_{\Delta\Delta} & A_{\Delta\Pi} \\ A_{\Pi I} & B_{I\Pi}^T & A_{\Pi\Delta} & A_{\Pi\Pi} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{f}_I \\ 0 \\ \mathbf{f}_\Delta \\ \mathbf{f}_\Pi \end{bmatrix}$$

can then be represented by

$$\begin{bmatrix} A_{rr}^{-1} f_r \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} -A_{rr}^{-1} A_{r\Pi} \\ I_\Pi \end{bmatrix} S_\Pi^{-1} \left(\mathbf{f}_\Pi - A_{\Pi r} A_{rr}^{-1} f_r \right),$$

which requires solving the coarse level problem once and independent subdomain Stokes problems with Neumann type boundary conditions twice.

5. Preliminary results.

Denote

$$\widetilde{\mathbf{W}} = \mathbf{W}_I \bigoplus \widetilde{\mathbf{W}}_\Gamma = \mathbf{W}_I \bigoplus \mathbf{W}_\Delta \bigoplus \mathbf{W}_\Pi. \tag{5.1}$$

For any $\mathbf{w}$ in $\widetilde{\mathbf{W}}$, denote its restriction to subdomain Ω_i by $\mathbf{w}^{(i)}$. A subdomainwise H^1-seminorm can be defined for functions in $\widetilde{\mathbf{W}}$ by

$$|\mathbf{w}|_{H^1}^2 = \sum_{i=1}^N |\mathbf{w}^{(i)}|_{H^1(\Omega_i)}^2.$$

We also define

$$\widetilde{V} = \mathbf{W}_I \bigoplus Q_I \bigoplus \mathbf{W}_\Delta \bigoplus \mathbf{W}_\Pi$$

and its subspace

$$\widetilde{V}_0 = \left\{ v = (\mathbf{w}_I,\ p_I,\ \mathbf{w}_\Delta,\ \mathbf{w}_\Pi) \in \widetilde{V} \;\middle|\; B_{II}\mathbf{w}_I + B_{I\Delta}\mathbf{w}_\Delta + B_{I\Pi}\mathbf{w}_\Pi = 0 \right\}. \tag{5.2}$$

For any $v = (\mathbf{w}_I,\ p_I,\ \mathbf{w}_\Delta,\ \mathbf{w}_\Pi) \in \widetilde{V}_0$, we have $\mathbf{w} = (\mathbf{w}_I,\ \mathbf{w}_\Delta,\ \mathbf{w}_\Pi) \in \widetilde{\mathbf{W}}$. Then

$$\langle v, v\rangle_{\widetilde{A}} = \begin{bmatrix} \mathbf{w}_I \\ \mathbf{w}_\Delta \\ \mathbf{w}_\Pi \end{bmatrix}^T \begin{bmatrix} A_{II} & A_{I\Delta} & A_{I\Pi} \\ A_{\Delta I} & A_{\Delta\Delta} & A_{\Delta\Pi} \\ A_{\Pi I} & A_{\Pi\Delta} & A_{\Pi\Pi} \end{bmatrix} \begin{bmatrix} \mathbf{w}_I \\ \mathbf{w}_\Delta \\ \mathbf{w}_\Pi \end{bmatrix}$$

$$(5.3) \qquad = \sum_{i=1}^{N} \begin{bmatrix} \mathbf{w}_I^{(i)} \\ \mathbf{w}_\Delta^{(i)} \\ \mathbf{w}_\Pi^{(i)} \end{bmatrix}^T \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} & A_{I\Pi}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} & A_{\Delta\Pi}^{(i)} \\ A_{\Pi I}^{(i)} & A_{\Pi\Delta}^{(i)} & A_{\Pi\Pi}^{(i)} \end{bmatrix} \begin{bmatrix} \mathbf{w}_I^{(i)} \\ \mathbf{w}_\Delta^{(i)} \\ \mathbf{w}_\Pi^{(i)} \end{bmatrix} = \sum_{i=1}^{N} \left| \begin{bmatrix} \mathbf{w}_I^{(i)} \\ \mathbf{w}_\Delta^{(i)} \\ \mathbf{w}_\Pi^{(i)} \end{bmatrix} \right|^2_{H^1(\Omega_i)}$$

$$= |\mathbf{w}|^2_{H^1},$$

where the superscript $^{(i)}$ is used to represent the restrictions of corresponding vectors and matrices to subdomain Ω_i. We can see from (5.3) that for any $v \in \widetilde{V}_0$, the value $\langle v, v\rangle_{\widetilde{A}}$ is independent of its pressure component p_I. $\langle \cdot, \cdot\rangle_{\widetilde{A}}$ defines a semi-inner product on $\widetilde{V}_0$; $\langle v, v\rangle_{\widetilde{A}} = 0$ if and only if the velocity component of v is constant on Ω and is in fact zero due to the zero boundary condition on $\partial\Omega$, while its pressure component can be arbitrary.

Denote

$$(5.4) \qquad \widetilde{B} = \begin{bmatrix} B_{II} & B_{I\Delta} & B_{I\Pi} \\ B_{\Gamma I} & B_{\Gamma\Delta} & B_{\Gamma\Pi} \end{bmatrix};$$

cf. (3.1). The following lemma on the stability of $\widetilde{B}$ can be found in [20, Lemma 5.1].

LEMMA 5.1. *For any* $\mathbf{w} \in \widetilde{\mathbf{W}}$ *and* $q \in Q$, $\langle \widetilde{B}\mathbf{w}, q\rangle \le |\mathbf{w}|_{H^1} \|q\|_{L^2}$.

The following lemma will also be used and can be found in [10, Lemma 2.3].

LEMMA 5.2. *Let* $(\mathbf{u}, p) \in \mathbf{W} \bigoplus Q$ *satisfy*

$$(5.5) \qquad \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ g \end{bmatrix},$$

where A *and* B *are as in* (2.3), $\mathbf{f} \in \mathbf{W}$, *and* $g \in Im(B) \subset Q$. *Let* β *be the inf-sup constant specified in* (2.4). *Then*

$$\|\mathbf{u}\|_A \le \|\mathbf{f}\|_{A^{-1}} + \frac{1}{\beta}\|g\|_{Z^{-1}},$$

where Z *is the mass matrix defined in section* 2.

6. Jump operators and preconditioners. We first define certain jump operators across the subdomain interface Γ, which will be used for the analysis of the preconditioners.

Denote the restriction operator from $\widetilde{V}$ onto $\mathbf{W}_\Delta$ by $\widetilde{R}_\Delta$; i.e., for any $v = (\mathbf{w}_I,\ p_I,\ \mathbf{w}_\Delta,\ \mathbf{w}_\Pi) \in \widetilde{V}$, $\widetilde{R}_\Delta v = \mathbf{w}_\Delta$. Define $P_{D,L} : \widetilde{V} \to \widetilde{V}$ by

$$P_{D,L} = \widetilde{R}_\Delta^T B_{\Delta,D}^T B_\Delta \widetilde{R}_\Delta.$$

Following this definition, given any $v = (\mathbf{w}_I,\ p_I,\ \mathbf{w}_\Delta,\ \mathbf{w}_\Pi) \in \widetilde{V}$, the dual velocity component of $P_{D,L}v$, on any subdomain interface node x in subdomain Ω_i, is given by (cf. [28, equation (6.70)])

$$\left(\widetilde{R}_\Delta\left(P_{D,L}v\right)\right)^{(i)}(x) = \sum_{j \in \mathcal{N}_x} \delta^\dagger(x)\left(\mathbf{w}_\Delta^{(i)}(x) - \mathbf{w}_\Delta^{(j)}(x)\right),$$

which represents the so-called jump of the dual velocity component $\mathbf{w}_\Delta$ across the subdomain interface Γ. All other components of $P_{D,L}v$ equal zero. We also have

$$\begin{aligned}\langle P_{D,L}v, P_{D,L}v\rangle_{\widetilde{A}} &= (\widetilde{R}_\Delta^T B_{\Delta,D}^T B_\Delta \widetilde{R}_\Delta v)^T \widetilde{A}(\widetilde{R}_\Delta^T B_{\Delta,D}^T B_\Delta \widetilde{R}_\Delta v)\\ &= \left\langle B_{\Delta,D}^T B_\Delta \mathbf{w}_\Delta, B_{\Delta,D}^T B_\Delta \mathbf{w}_\Delta\right\rangle_{A_{\Delta\Delta}}.\end{aligned} \tag{6.1}$$

Together with (5.3), we have the following lemma, which can be found in [22, section 6.1].

LEMMA 6.1. *There exist a constant C and a function $\Phi_L(H/h)$ such that for all $v \in \widetilde{V}_0$, $\langle P_{D,L}v, P_{D,L}v\rangle_{\widetilde{A}} \le C\Phi_L(H/h)\langle v,v\rangle_{\widetilde{A}}$. Here $\Phi_L(H/h) = (H/h)(1+\log(H/h))$ when the coarse level space is spanned by the subdomain vertex nodal basis functions and subdomain edge-cutoff functions corresponding to each velocity component.*

When applying $P_{D,L}$ to a vector, the jump of the dual subdomain interface velocities is extended by zero to the interior of subdomains. To improve the stability of the jump operator, the jump can be extended to the interior of subdomains by subdomain discrete harmonic extension. We define a Schur complement operator $H_\Delta^{(i)} : \mathbf{W}_\Delta^{(i)} \to \mathbf{W}_\Delta^{(i)}$ by, for any $\mathbf{u}_\Delta^{(i)} \in \mathbf{W}_\Delta^{(i)}$,

$$\begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix} \begin{bmatrix} \mathbf{u}_I^{(i)} \\ \mathbf{u}_\Delta^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ H_\Delta^{(i)}\mathbf{u}_\Delta^{(i)} \end{bmatrix}. \tag{6.2}$$

To multiply $H_\Delta^{(i)}$ by a vector $\mathbf{u}_\Delta^{(i)}$, a subdomain elliptic problem on Ω_i with given boundary velocity $\mathbf{u}_\Delta^{(i)}$ and $\mathbf{u}_\Pi^{(i)} = \mathbf{0}$ needs to be solved.

Using $H_\Delta^{(i)}$, we define the second jump operator, $P_{D,D} : \widetilde{V} \to \widetilde{V}$, as follows: for any given $v = (\mathbf{w}_I,\ p_I,\ \mathbf{w}_\Delta,\ \mathbf{w}_\Pi) \in \widetilde{V}$, the subdomain interior velocity part of $P_{D,D}v$ on each subdomain Ω_i is taken as $\mathbf{u}_I^{(i)}$ in the solution of (6.2), with given subdomain boundary velocity $\mathbf{u}_\Delta^{(i)} = B_{\Delta,D}^{(i)^T} B_\Delta \mathbf{w}_\Delta$. Here $B_{\Delta,D}^{(i)^T}$ represents restriction of $B_{\Delta,D}^T$ on subdomain Ω_i and is a map from Λ to $\mathbf{W}_\Delta^{(i)}$. The other components of $P_{D,D}v$ are kept as zero. Therefore

$$\begin{aligned}\langle P_{D,D}v, P_{D,D}v\rangle_{\widetilde{A}} &= \sum_{i=1}^N \begin{bmatrix} \mathbf{u}_I^{(i)} \\ \mathbf{u}_\Delta^{(i)} \end{bmatrix}^T \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix} \begin{bmatrix} \mathbf{u}_I^{(i)} \\ \mathbf{u}_\Delta^{(i)} \end{bmatrix}\\ &= \sum_{i=1}^N \mathbf{u}_\Delta^{(i)^T} H_\Delta^{(i)} \mathbf{u}_\Delta^{(i)} = \sum_{i=1}^N \mathbf{w}_\Delta^T B_\Delta^T B_{\Delta,D}^{(i)} H_\Delta^{(i)} B_{\Delta,D}^{(i)^T} B_\Delta \mathbf{w}_\Delta\\ &= \sum_{i=1}^N \left|\left[\begin{matrix} B_{\Delta,D}^{(i)^T} B_\Delta \mathbf{w}_\Delta \\ 0 \end{matrix}\right]\right|_{H^{1/2}(\partial\Omega^i)}^2 \le C\Phi_D(H/h)\sum_{i=1}^N \left|\left[\begin{matrix} \mathbf{w}_\Delta^{(i)} \\ \mathbf{w}_\Pi^{(i)} \end{matrix}\right]\right|_{H^{1/2}(\partial\Omega^i)}^2\\ &\le C\Phi_D(H/h)\sum_{i=1}^N \left|\left[\begin{matrix} \mathbf{w}_I^{(i)} \\ \mathbf{w}_\Delta^{(i)} \\ \mathbf{w}_\Pi^{(i)} \end{matrix}\right]\right|_{H^1(\Omega_i)}^2 = C\Phi_D(H/h)|\mathbf{w}|_{H^1(\Omega_i)}^2.\end{aligned} \tag{6.3}$$

The first inequality in (6.3) is a well-established result; cf. [28, Lemma 6.36]. Since for any $v \in \widetilde{V}_0$, $\langle v,v\rangle_{\widetilde{A}} = |\mathbf{w}|_{H^1(\Omega_i)}^2$ (cf. (5.3)), we have the following lemma.

LEMMA 6.2. *There exist a constant C and a function $\Phi_D(H/h)$ such that for all $v \in \widetilde{V}_0$, $\langle P_{D,D}v, P_{D,D}v\rangle_{\widetilde{A}} \le C\Phi_D(H/h)\langle v,v\rangle_{\widetilde{A}}$. Here $\Phi_D(H/h) = (1+\log(H/h))^2$*

when the coarse level space is spanned by the subdomain vertex nodal basis functions and subdomain edge-cutoff functions corresponding to each velocity component.

Remark 6.3. We note that the coarse spaces in Lemmas 6.1 and 6.2 are not necessarily the coarse space of minimal size needed to achieve a scalable bound $\Phi_L(H/h)$ and $\Phi_D(H/h)$, respectively; several other choices can be found in [28, section 6.4.2]. As for the edge-cutoff function in Lemma 6.2, the proof given in (6.3) requires the edge-cutoff function of each velocity component. Meanwhile, enriching the coarse space may improve those bounds. For the case of $\Phi_L(H/h)$ in Lemma 6.1, it has been proved that the bound can be improved to H/h for two dimensions with additional edge-cutoff functions in the coarse space [22] and for three dimensions with additional face-cutoff functions in the coarse space [13, 12]. But, to the best of our knowledge, no such improvements have been proved in the literature for the case of $\Phi_D(H/h)$ in Lemma 6.2 in either two or three dimensions.

To introduce the preconditioners, we write G, defined in (4.2) and (4.3), in a two-by-two block structure. Denote the first row of B_C by

$$\widetilde{B}_\Gamma = [B_{\Gamma I} \quad 0 \quad B_{\Gamma\Delta} \quad B_{\Gamma\Pi}],$$

and note that $\widetilde{R}_\Delta$ is the restriction operator from $\widetilde{V}$ onto $\mathbf{W}_\Delta$. Then G can be written as

$$G = \begin{bmatrix} G_{p_\Gamma p_\Gamma} & G_{p_\Gamma \lambda} \\ G_{\lambda p_\Gamma} & G_{\lambda\lambda} \end{bmatrix}, \tag{6.4}$$

where

$$G_{p_\Gamma p_\Gamma} = \widetilde{B}_\Gamma \widetilde{A}^{-1} \widetilde{B}_\Gamma^T, \qquad G_{p_\Gamma \lambda} = \widetilde{B}_\Gamma \widetilde{A}^{-1} \widetilde{R}_\Delta^T B_\Delta^T,$$
$$G_{\lambda p_\Gamma} = B_\Delta \widetilde{R}_\Delta \widetilde{A}^{-1} \widetilde{B}_\Gamma^T, \qquad G_{\lambda\lambda} = B_\Delta \widetilde{R}_\Delta \widetilde{A}^{-1} \widetilde{R}_\Delta^T B_\Delta^T.$$

We consider a block diagonal preconditioner for (4.1). As for two-dimensional problems, the first diagonal block $G_{p_\Gamma p_\Gamma}$ of G can be shown spectrally equivalent to $h^3 I_{p_\Gamma}$, where I_{p_Γ} is the identity matrix of the same dimension as $G_{p_\Gamma p_\Gamma}$; see [20, 31]. Therefore, in the following block diagonal preconditioners, the inverse of $G_{p_\Gamma p_\Gamma}$ is approximated by $\alpha h^{-3} I_{p_\Gamma}$. Here α is a given constant. We will show in the next section that α has only a minor effect on the condition number bound of the preconditioned operator and its value is typically taken as 1; cf. Remark 7.7. We introduce α in the preconditioner just for the convenience in the numerical experiments to demonstrate the convergence rates of the proposed algorithm.

The inverse of the second diagonal block $B_\Delta \widetilde{R}_\Delta \widetilde{A}^{-1} \widetilde{R}_\Delta^T B_\Delta^T$ can be approximated by the lumped block

$$M_{\lambda,L}^{-1} = B_{\Delta,D} \widetilde{R}_\Delta \widetilde{A} \widetilde{R}_\Delta^T B_{\Delta,D}^T. \tag{6.5}$$

This leads to the following lumped preconditioner for solving (4.1):

$$M_L^{-1} = \begin{bmatrix} \alpha h^{-3} I_{p_\Gamma} & \\ & M_{\lambda,L}^{-1} \end{bmatrix}. \tag{6.6}$$

Applying subdomain discrete harmonic extensions in the preconditioning step, we have the following Dirichlet preconditioner:

$$M_D^{-1} = \begin{bmatrix} \alpha h^{-3} I_{p_\Gamma} & \\ & M_{\lambda,D}^{-1} \end{bmatrix}, \tag{6.7}$$

where

$$M_{\lambda,D}^{-1} = B_{\Delta,D} H_\Delta B_{\Delta,D}^T. \tag{6.8}$$

Here $H_\Delta : \mathbf{W}_\Delta \to \mathbf{W}_\Delta$ represents the direct sum of $H_\Delta^{(i)}$, $i = 1, \ldots, N$, defined in (6.2).

Remark 6.4. The lumped and Dirichlet preconditioners (6.6) and (6.7) have the same feature as those in the original FETI-DP algorithms [7, 8, 17] developed for solving elliptic problems. The only difference here is the additional diagonal block in these two preconditioners corresponding to the pressure variables; removing all the pressure variables and their related blocks in the algorithms studied in this paper will just lead to the original FETI-DP algorithms.

We can see from Lemma 3.1 that both $M_{\lambda,L}^{-1}$ and $M_{\lambda,D}^{-1}$ are symmetric positive definite when restricted on Λ. Therefore both the lumped and Dirichlet preconditioners M_L^{-1} and M_D^{-1} are symmetric positive definite in the range of G.

7. Condition number bounds. In the following, we use the same framework to establish the condition number bounds for both the lumped and Dirichlet preconditioned operators $M_L^{-1}G$ and $M_D^{-1}G$. Let M^{-1}, M_λ^{-1}, P_D, and Φ represent M_L^{-1}, $M_{\lambda,L}^{-1}$, $P_{D,L}$, Φ_L for the lumped preconditioner case, and M_D^{-1}, $M_{\lambda,D}^{-1}$, $P_{D,D}$, Φ_D for the Dirichlet preconditioner case, respectively, as needed in the proofs.

When the CG method is applied to solving the preconditioned system

$$M^{-1}Gx \;=\; M^{-1}g, \tag{7.1}$$

with zero initial guess, all iterates belong to the Krylov subspace generated by the operator $M^{-1}G$ and the vector $M^{-1}g$, which is a subspace of the range of $M^{-1}G$. We denote the range of $M^{-1}G$ by $R_{M^{-1}G}$ and note that both preconditioners are symmetric positive definite in the range of G. We have the following lemma; cf. [31, Lemma 6].

LEMMA 7.1. *The CG method applied to solving* (7.1) *with zero initial guess cannot break down.*

Proof. We just need to show that for any $0 \neq x \in R_{M^{-1}G}$, $\langle x, Gx\rangle \neq 0$, i.e., to show that $Gx \neq 0$. Let $0 \neq x = M^{-1}Gy$ for a certain $y \in X$ and $y \neq 0$. Then $Gx = GM^{-1}Gy$, which cannot be zero since $Gy \neq 0$ and $y^T GM^{-1}Gy \neq 0$. ∎

The following lemma will be used to provide the upper eigenvalue bound of the preconditioned operator. It is similar to [20, Lemma 6.4] and [31, Lemmas 8 and 11].

LEMMA 7.2. *There exist positive constants C_1 and C_2 such that for all $v \in \widetilde{V}_0$,*

$$\left\langle M^{-1}B_C v, B_C v\right\rangle \le (C_1\alpha + C_2\Phi(H/h)) \left\langle \widetilde{A}v, v\right\rangle,$$

where $\Phi(H/h)$ is defined in Lemmas 6.1 *and* 6.2*, respectively.*

Proof. Given $v = (\mathbf{w}_I,\ q_I,\ \mathbf{w}_\Delta,\ \mathbf{w}_\Pi) \in \widetilde{V}_0$, let $g_{p_\Gamma} = B_{\Gamma I}\mathbf{w}_I + B_{\Gamma\Delta}\mathbf{w}_\Delta + B_{\Gamma\Pi}\mathbf{w}_\Pi$. From (4.3), (6.5)–(6.8), (6.1), and (6.3), we have

$$\begin{aligned}\left\langle M^{-1}B_C v, B_C v\right\rangle &= \alpha h^{-3}\left\langle g_{p_\Gamma}, g_{p_\Gamma}\right\rangle + \left(B_\Delta \widetilde{R}_\Delta v\right)^T M_\lambda^{-1} B_\Delta \widetilde{R}_\Delta v\\ &= \alpha h^{-3}\left\langle g_{p_\Gamma}, g_{p_\Gamma}\right\rangle + \left\langle P_D v, P_D v\right\rangle_{\widetilde{A}}\\ &\le \alpha h^{-3}\left\langle g_{p_\Gamma}, g_{p_\Gamma}\right\rangle + C_2\Phi(H/h)\left\langle v, v\right\rangle_{\widetilde{A}}, \end{aligned} \tag{7.2}$$

where we used Lemmas 6.1 and 6.2 for the last inequality. It is sufficient to bound the first term of the right-hand side in the above inequality.

Since $v \in \widetilde{V}_0$, we have $B_{II}\mathbf{w}_I + B_{I\Delta}\mathbf{w}_\Delta + B_{I\Pi}\mathbf{w}_\Pi = 0$; cf. (5.2). Then

$$\begin{aligned}\langle g_{p_\Gamma}, g_{p_\Gamma}\rangle &= \begin{bmatrix} B_{II}\mathbf{w}_I + B_{I\Delta}\mathbf{w}_\Delta + B_{I\Pi}\mathbf{w}_\Pi \\ B_{\Gamma I}\mathbf{w}_I + B_{\Gamma\Delta}\mathbf{w}_\Delta + B_{\Gamma\Pi}\mathbf{w}_\Pi \end{bmatrix}^T \begin{bmatrix} B_{II}\mathbf{w}_I + B_{I\Delta}\mathbf{w}_\Delta + B_{I\Pi}\mathbf{w}_\Pi \\ B_{\Gamma I}\mathbf{w}_I + B_{\Gamma\Delta}\mathbf{w}_\Delta + B_{\Gamma\Pi}\mathbf{w}_\Pi \end{bmatrix} \\ &= \left\langle \widetilde{B}\mathbf{w}, \widetilde{B}\mathbf{w} \right\rangle,\end{aligned}$$

where $\widetilde{B}$ is defined in (5.4) and $\mathbf{w} = (\mathbf{w}_I,\ \mathbf{w}_\Delta,\ \mathbf{w}_\Pi) \in \widetilde{\mathbf{W}}$. From (2.5) and the stability of $\widetilde{B}$ (cf. Lemma 5.1), we have

$$\begin{aligned}h^{-3}\langle g_{p_\Gamma}, g_{p_\Gamma}\rangle &= h^{-3}\left\langle \widetilde{B}\mathbf{w}, \widetilde{B}\mathbf{w} \right\rangle \le C \left\langle \widetilde{B}\mathbf{w}, \widetilde{B}\mathbf{w} \right\rangle_{Z^{-1}} = C \max_{q\in Q} \frac{\langle \widetilde{B}\mathbf{w}, q\rangle^2}{\langle q, q\rangle_Z} \\ &\le C_1 \max_{q\in Q} \frac{|\mathbf{w}|^2_{H^1}\|q\|^2_{L^2}}{\|q\|^2_{L^2}} = C_1|\mathbf{w}|^2_{H^1} = C_1\langle v, v\rangle_{\widetilde{A}},\end{aligned} \tag{7.3}$$

where for the last equality, we used (5.3). ∎

The following lemma will be used to provide the lower eigenvalue bound of the preconditioned operator. In [31, Lemmas 9 and 12], the lower eigenvalue bounds for the lumped and Dirichlet preconditioners were analyzed differently. In the analysis of the Dirichlet preconditioner, subdomain discrete Stokes extensions were used. Such extensions require enforcing the same type of divergence-free subdomain boundary velocity conditions as discussed in [21], even though they are not necessary for implementing the algorithm. The new proof given in the next lemma works for both the lumped and Dirichlet preconditioners. It does not use the subdomain Stokes extensions, and those additional subdomain divergence-free boundary conditions are no longer needed. For both types of preconditioners, the coarse level velocity space can be chosen as simple as those coarse spaces for solving scalar elliptic problems corresponding to each velocity component.

LEMMA 7.3. *There exists a constant C such that for any nonzero $y = (g_{p_\Gamma}, g_\lambda) \in R_G$, there exists $v \in \widetilde{V}_0$, which satisfies $B_C v = y$, $\langle v, v\rangle_{\widetilde{A}} \neq 0$, and*

$$\left\langle \widetilde{A}v, v \right\rangle \le C \max\left\{1, \frac{1}{\alpha}\right\}\left(1 + \frac{1}{\beta^2}\right)\left\langle M^{-1}y, y\right\rangle.$$

Proof. Given $y = (g_{p_\Gamma}, g_\lambda) \in R_G$, take $\mathbf{u}_\Delta^{(I)} = B_{\Delta,D}^T g_\lambda$, $\mathbf{u}_\Pi^{(I)} = \mathbf{0}$, and $p^{(I)} = 0$. On each subdomain Ω_i, let $\mathbf{u}_I^{(I,i)}$ be zero for the lumped preconditioner, and let it be obtained for the Dirichlet preconditioner through the solution of (6.2) with given subdomain boundary values $\mathbf{u}_\Delta^{(i)} = \mathbf{u}_\Delta^{(I,i)}$. Let $v^{(I,i)} = (\mathbf{u}_I^{(I,i)},\ p_I^{(I,i)},\ \mathbf{u}_\Delta^{(I,i)},\ \mathbf{u}_\Pi^{(I,i)})$, the corresponding global vectors $v^{(I)} = (\mathbf{u}_I^{(I)},\ p_I^{(I)},\ \mathbf{u}_\Delta^{(I)},\ \mathbf{u}_\Pi^{(I)})$, and $\mathbf{u}^{(I)} = (\mathbf{u}_I^{(I)},\ \mathbf{u}_\Delta^{(I)},\ \mathbf{u}_\Pi^{(I)})$. Then we have

$$B_C v^{(I)} = \begin{bmatrix} B_{\Gamma I} & 0 & B_{\Gamma\Delta} & B_{\Gamma\Pi} \\ 0 & 0 & B_\Delta & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_I^{(I)} \\ p_I^{(I)} \\ \mathbf{u}_\Delta^{(I)} \\ \mathbf{u}_\Pi^{(I)} \end{bmatrix} = \begin{bmatrix} B_{\Gamma I}\mathbf{u}_I^{(I)} + B_{\Gamma\Delta}\mathbf{u}_\Delta^{(I)} + B_{\Gamma\Pi}\mathbf{u}_\Pi^{(I)} \\ g_\lambda \end{bmatrix}, \tag{7.4}$$

where we have used Lemma 3.2. Also,

$$|\mathbf{u}^{(I)}|_{H^1}^2 = \begin{bmatrix} \mathbf{u}_I^{(I)} \\ \mathbf{u}_\Delta^{(I)} \\ \mathbf{u}_\Pi^{(I)} \end{bmatrix}^T \begin{bmatrix} A_{II} & A_{I\Delta} & A_{I\Pi} \\ A_{\Delta I} & A_{\Delta\Delta} & A_{\Delta\Pi} \\ A_{\Pi I} & A_{\Pi\Delta} & A_{\Pi\Pi} \end{bmatrix} \begin{bmatrix} \mathbf{u}_I^{(I)} \\ \mathbf{u}_\Delta^{(I)} \\ \mathbf{u}_\Pi^{(I)} \end{bmatrix} = \begin{cases} |\mathbf{u}_\Delta^{(I)}|_{A_{\Delta\Delta}}^2 & \text{for the lumped preconditioner,} \\ |\mathbf{u}_\Delta^{(I)}|_{H_\Delta}^2 & \text{for the Dirichlet preconditioner.} \end{cases} \tag{7.5}$$

We consider a solution to the following fully assembled system of linear equations of the form (2.3): find $(\mathbf{u}_I^{(II)},\ p_I^{(II)},\ \mathbf{u}_\Gamma^{(II)},\ p_\Gamma^{(II)}) \in \mathbf{W}_I \bigoplus Q_I \bigoplus \mathbf{W}_\Gamma \bigoplus Q_\Gamma$ such that

$$\begin{bmatrix} A_{II} & B_{II}^T & A_{I\Gamma} & B_{\Gamma I}^T \\ B_{II} & 0 & B_{I\Gamma} & 0 \\ A_{\Gamma I} & B_{I\Gamma}^T & A_{\Gamma\Gamma} & B_{\Gamma\Gamma}^T \\ B_{\Gamma I} & 0 & B_{\Gamma\Gamma} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_I^{(II)} \\ p_I^{(II)} \\ \mathbf{u}_\Gamma^{(II)} \\ p_\Gamma^{(II)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ -B_{II}\mathbf{u}_I^{(I)} - B_{I\Delta}\mathbf{u}_\Delta^{(I)} - B_{I\Pi}\mathbf{u}_\Pi^{(I)} \\ \mathbf{0} \\ g_{p_\Gamma} - B_{\Gamma I}\mathbf{u}_I^{(I)} - B_{\Gamma\Delta}\mathbf{u}_\Delta^{(I)} - B_{\Gamma\Pi}\mathbf{u}_\Pi^{(I)} \end{bmatrix}, \tag{7.6}$$

where we know that the particularly chosen right-hand side is essentially

$$\begin{bmatrix} \mathbf{0} \\ -B_{II}\mathbf{u}_I^{(I)} - B_{I\Delta}\mathbf{u}_\Delta^{(I)} \\ \mathbf{0} \\ g_{p_\Gamma} - B_{\Gamma I}\mathbf{u}_I^{(I)} - B_{\Gamma\Delta}\mathbf{u}_\Delta^{(I)} \end{bmatrix}. \tag{7.7}$$

Since $(g_{p_\Gamma}, g_\lambda) \in R_G$, we have (cf. (4.5))

$$(-B_{I\Delta}\mathbf{u}_\Delta^{(I)})^T 1_{p_I} + (g_{p_\Gamma} - B_{\Gamma\Delta}\mathbf{u}_\Delta^{(I)})^T 1_{p_\Gamma} = g_{p_\Gamma}^T 1_{p_\Gamma} - g_\lambda^T B_{\Delta,D}\left(B_{I\Delta}^T 1_{p_I} + B_{\Gamma\Delta}^T 1_{p_\Gamma}\right) = 0.$$

Meanwhile,

$$(-B_{II}\mathbf{u}_I^{(I)})^T 1_{p_I} + (-B_{\Gamma I}\mathbf{u}_I^{(I)})^T 1_{p_\Gamma} = -\int_\Omega \left(\nabla\cdot\mathbf{u}_I^{(I)}\right) 1 = 0.$$

We have that the right-hand side vector (7.7) has zero average, which implies existence of the solution to (7.6).

Denote $\mathbf{u}^{(II)} = (\mathbf{u}_I^{(II)},\ \mathbf{u}_\Gamma^{(II)})$. Then from Lemma 5.2 and (2.5), we have

$$\begin{aligned} |\mathbf{u}^{(II)}|_{H^1}^2 &\le \frac{1}{\beta^2}\left\| \begin{bmatrix} -B_{II}\mathbf{u}_I^{(I)} - B_{I\Delta}\mathbf{u}_\Delta^{(I)} - B_{I\Pi}\mathbf{u}_\Pi^{(I)} \\ g_{p_\Gamma} - B_{\Gamma I}\mathbf{u}_I^{(I)} - B_{\Gamma\Delta}\mathbf{u}_\Delta^{(I)} - B_{\Gamma\Pi}\mathbf{u}_\Pi^{(I)} \end{bmatrix} \right\|_{Z^{-1}}^2 \\ &\le \frac{1}{\beta^2}\left\| \begin{bmatrix} B_{II}\mathbf{u}_I^{(I)} + B_{I\Delta}\mathbf{u}_\Delta^{(I)} + B_{I\Pi}\mathbf{u}_\Pi^{(I)} \\ B_{\Gamma I}\mathbf{u}_I^{(I)} + B_{\Gamma\Delta}\mathbf{u}_\Delta^{(I)} + B_{\Gamma\Pi}\mathbf{u}_\Pi^{(I)} \end{bmatrix} \right\|_{Z^{-1}}^2 + \frac{1}{\beta^2}\left\| \begin{bmatrix} 0 \\ g_{p_\Gamma} \end{bmatrix} \right\|_{Z^{-1}}^2 \\ &\le \frac{1}{\beta^2}|\mathbf{u}^{(I)}|_{H^1}^2 + \frac{C}{\beta^2 h^3}\langle g_{p_\Gamma}, g_{p_\Gamma}\rangle, \end{aligned} \tag{7.8}$$

where the bound on the first term is obtained in the same way as in (7.3).

Split the continuous subdomain interface velocity $\mathbf{u}_\Gamma^{(II)}$ into the dual part $\mathbf{u}_\Delta^{(II)}$ and the primal part $\mathbf{u}_\Pi^{(II)}$, and denote $v^{(II)} = (\mathbf{u}_I^{(II)},\ p_I^{(II)},\ \mathbf{u}_\Delta^{(II)},\ \mathbf{u}_\Pi^{(II)})$. Let $v = v^{(I)} + v^{(II)}$. Then we have from (7.6) that $v \in \widetilde{V}_0$, and

$$B_C v^{(II)} = \begin{bmatrix} B_{\Gamma I} & 0 & B_{\Gamma\Delta} & B_{\Gamma\Pi} \\ 0 & 0 & B_\Delta & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_I^{(II)} \\ p_I^{(II)} \\ \mathbf{u}_\Delta^{(II)} \\ \mathbf{u}_\Pi^{(II)} \end{bmatrix}$$
$$= \begin{bmatrix} g_{p_\Gamma} - B_{\Gamma I}\mathbf{u}_I^{(I)} - B_{\Gamma\Delta}\mathbf{u}_\Delta^{(I)} - B_{\Gamma\Pi}\mathbf{u}_\Pi^{(I)} \\ 0 \end{bmatrix}.$$

Together with (7.4), we have $B_C v = y$. From (5.3) and (7.8), we have

$$|v|^2_{\widetilde{A}} = |\mathbf{u}^{(I)} + \mathbf{u}^{(II)}|^2_{H^1} \leq |\mathbf{u}^{(I)}|^2_{H^1} + |\mathbf{u}^{(II)}|^2_{H^1} = \left(1 + \frac{1}{\beta^2}\right)|\mathbf{u}^{(I)}|^2_{H^1} + \frac{C}{\beta^2 h^3}\langle g_{p_\Gamma}, g_{p_\Gamma}\rangle$$
$$= \begin{cases} \left(1 + \frac{1}{\beta^2}\right)|\mathbf{u}_\Delta^{(I)}|^2_{A_{\Delta\Delta}} + \frac{C}{\beta^2 h^3}\langle g_{p_\Gamma}, g_{p_\Gamma}\rangle & \text{for the lumped preconditioner,} \\ \left(1 + \frac{1}{\beta^2}\right)|\mathbf{u}_\Delta^{(I)}|^2_{H_\Delta} + \frac{C}{\beta^2 h^3}\langle g_{p_\Gamma}, g_{p_\Gamma}\rangle & \text{for the Dirichlet preconditioner,} \end{cases}$$

where we used (7.5) in the last equality.

On the other hand, we have from (6.5)–(6.8)

$$\langle M^{-1}y, y\rangle = \frac{\alpha}{h^3}\langle g_{p_\Gamma}, g_{p_\Gamma}\rangle + g_\lambda^T M_\lambda^{-1} g_\lambda$$
$$= \begin{cases} \frac{\alpha}{h^3}\langle g_{p_\Gamma}, g_{p_\Gamma}\rangle + g_\lambda^T B_{\Delta,D} A_{\Delta\Delta} B_{\Delta,D}^T g_\lambda & \text{for the lumped preconditioner,} \\ \frac{\alpha}{h^3}\langle g_{p_\Gamma}, g_{p_\Gamma}\rangle + g_\lambda^T B_{\Delta,D} H_\Delta B_{\Delta,D}^T g_\lambda & \text{for the Dirichlet preconditioner} \end{cases}$$

$$= \begin{cases} \frac{\alpha}{h^3}\langle g_{p_\Gamma}, g_{p_\Gamma}\rangle + |\mathbf{u}_\Delta^{(I)}|^2_{A_{\Delta\Delta}} & \text{for the lumped preconditioner,} \\ \frac{\alpha}{h^3}\langle g_{p_\Gamma}, g_{p_\Gamma}\rangle + |\mathbf{u}_\Delta^{(I)}|^2_{H_\Delta} & \text{for the Dirichlet preconditioner.} \end{cases}$$

It is not difficult to see that $\langle v, v\rangle_{\widetilde{A}} \neq 0$. Otherwise, all the velocity components of v would be zero; cf. (5.3), and then $B_C v$ would be zero, which conflicts with $B_C v = y$ and y being nonzero. ☐

The proofs of the following two lemmas can be found in [20, Lemmas 6.6 and 6.3].

LEMMA 7.4. *For any* $v = (\mathbf{w}_I,\ p_I,\ \mathbf{w}_\Delta,\ \mathbf{w}_\Pi) \in \widetilde{V}_0$, $B_C v \in R_G$.

LEMMA 7.5. *For any* $x \in R_{M^{-1}G}$,

$$\langle Mx, x\rangle = \max_{y \in R_G, y \neq 0} \frac{\langle y, x\rangle^2}{\langle M^{-1}y, y\rangle}.$$

The condition number bound of the preconditioned operator $M^{-1}G$ is given in the following theorem.

THEOREM 7.6. *There exist positive constants* c, C_1, *and* C_2 *such that for all* $x \in R_{M^{-1}G}$,

$$\min\{1, \alpha\}\frac{c\beta^2}{(1+\beta^2)}\langle Mx, x\rangle \leq \langle Gx, x\rangle \leq \left(C_1\alpha + C_2\Phi\left(\frac{H}{h}\right)\right)\langle Mx, x\rangle.$$

Proof. We need only prove the above inequalities for any nonzero $x \in R_{M^{-1}G}$. We know from Lemma 7.1 that

$$0 \neq \langle Gx, x\rangle = x^T B_C \widetilde{A}^{-1} B_C^T x = x^T B_C \widetilde{A}^{-1} \widetilde{A} \widetilde{A}^{-1} B_C^T x = \left\langle \widetilde{A}^{-1} B_C^T x, \widetilde{A}^{-1} B_C^T x \right\rangle_{\widetilde{A}}.$$

Therefore $\widetilde{A}^{-1} B_C^T x \neq 0$. Also note that $\widetilde{A}^{-1} B_C^T x \in \widetilde{V}_0$ and $\langle \cdot, \cdot \rangle_{\widetilde{A}}$ defines a semi-inner product on $\widetilde{V}_0$ (cf. (5.3)); then we have

$$\langle Gx, x\rangle = \max_{v \in \widetilde{V}_0, \langle v, v\rangle_{\widetilde{A}} \neq 0} \frac{\left\langle v, \widetilde{A}^{-1} B_C^T x \right\rangle_{\widetilde{A}}^2}{\langle v, v\rangle_{\widetilde{A}}} = \max_{v \in \widetilde{V}_0, \langle v, v\rangle_{\widetilde{A}} \neq 0} \frac{\langle B_C v, x\rangle^2}{\left\langle \widetilde{A} v, v \right\rangle}. \tag{7.9}$$

Lower bound. From Lemma 7.3, we know that for any nonzero $y \in R_G$, there exists $w \in \widetilde{V}_0$ such that $B_C w = y$, $\langle w, w\rangle_{\widetilde{A}} \neq 0$, $\langle \widetilde{A} w, w\rangle \leq \max\left\{1, \frac{1}{\alpha}\right\} \frac{C(1+\beta^2)}{\beta^2} \langle M^{-1} y, y\rangle$. Then from (7.9), we have

$$\langle Gx, x\rangle \geq \frac{\langle B_C w, x\rangle^2}{\left\langle \widetilde{A} w, w\right\rangle} \geq c \frac{\beta^2}{\max\left\{1, \frac{1}{\alpha}\right\}(1+\beta^2)} \frac{\langle y, x\rangle^2}{\langle M^{-1} y, y\rangle}.$$

Since y is arbitrary, using Lemma 7.5, we have

$$\langle Gx, x\rangle \geq c \frac{\beta^2}{\max\left\{1, \frac{1}{\alpha}\right\}(1+\beta^2)} \max_{y \in R_G, y \neq 0} \frac{\langle y, x\rangle^2}{\langle M^{-1} y, y\rangle} = \min\{1, \alpha\} \frac{c\beta^2}{(1+\beta^2)} \langle Mx, x\rangle .$$

Upper bound. From (7.9) and the fact that $\langle Gx, x\rangle \neq 0$, we have

$$\langle Gx, x\rangle = \max_{v \in \widetilde{V}_0, \langle v, v\rangle_{\widetilde{A}} \neq 0} \frac{\langle B_C v, x\rangle^2}{\left\langle \widetilde{A} v, v\right\rangle} = \max_{v \in \widetilde{V}_0, \langle v, v\rangle_{\widetilde{A}} \neq 0, B_C v \neq 0} \frac{\langle B_C v, x\rangle^2}{\left\langle \widetilde{A} v, v\right\rangle},$$

where the maximum need only be considered among v also satisfying $B_C v \neq 0$. Then using Lemmas 7.2, 7.4, and 7.5, we have

$$\begin{aligned} \langle Gx, x\rangle &\leq (C_1 \alpha + C_2 \Phi(H, h)) \max_{v \in \widetilde{V}_0, \langle v, v\rangle_{\widetilde{A}} \neq 0, B_C v \neq 0} \frac{\langle B_C v, x\rangle^2}{\langle M^{-1} B_C v, B_C v\rangle} \\ &\leq (C_1 \alpha + C_2 \Phi(H, h)) \max_{y \in R_G, y \neq 0} \frac{\langle y, x\rangle^2}{\langle M^{-1} y, y\rangle} = (C_1 \alpha + C_2 \Phi(H, h)) \langle Mx, x\rangle . \qquad \square \end{aligned}$$

Remark 7.7. We can see from Theorem 7.6 that, for $\alpha \geq 1$, the condition number bound of $M^{-1}G$ is proportional to $\alpha + C\Phi(H, h)$, and we should take smaller α to achieve faster convergence. When $\alpha \leq 1$, the condition number bound is proportional to $1 + C\frac{\Phi(H,h)}{\alpha}$, and we should take larger α. This explains why the value of α in (6.6) and (6.7) is typically taken as 1. We introduce α in the preconditioner just for the convenience of demonstrating the convergence rates of the proposed algorithm in the following section.

8. Numerical experiments.

We illustrate the convergence rate of the proposed algorithm by solving the incompressible Stokes problem (2.1) in both two and three dimensions, on $\Omega = [0, 1]^2$ and $\Omega = [0, 1]^3$, respectively. The right-hand side $\mathbf{f}$ is chosen such that the exact solution is

$$\mathbf{u} = \begin{bmatrix} \sin^3(\pi x) \sin^2(\pi y) \cos(\pi y) \\ -\sin^2(\pi x) \sin^3(\pi y) \cos(\pi x) \end{bmatrix}, \quad p = x^2 - y^2,$$

for two dimensions, and for three dimensions

$$\mathbf{u} = \begin{bmatrix} \sin^2(\pi x)\,(\sin(2\pi y)\sin(\pi z) - \sin(\pi y)\sin(2\pi z)) \\ \sin^2(\pi y)\,(\sin(2\pi z)\sin(\pi x) - \sin(\pi z)\sin(2\pi x)) \\ \sin^2(\pi z)\,(\sin(2\pi x)\sin(\pi y) - \sin(\pi x)\sin(2\pi y)) \end{bmatrix}, \quad p = xyz - \frac{1}{8},$$

with zero Dirichlet boundary condition. We also test a mixed boundary condition with the same right-hand side $\mathbf{f}$.

Three types of mixed finite elements are used in the experiments. They are two- and three-dimensional Q_2-Q_1 Taylor–Hood mixed finite elements with continuous pressures [2, 26] and a three-dimensional mixed finite element with discontinuous pressures, which is used in [18]. For the continuous pressure elements, in two dimensions, the velocity space contains piecewise biquadratic functions, and the pressure space contains piecewise bilinear functions; in three dimensions, the velocity space contains piecewise triquadratic functions, and the pressure space contains piecewise trilinear functions. The discontinuous pressure element is illustrated in Figure 2, where the velocity is spanned by 1, x, y, z, zx, zy on each prism and the pressure is a constant on each of the eight prisms.

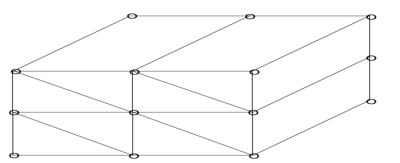

FIG. 2. *A three-dimensional mixed finite element with discontinuous pressures.*

The preconditioned CG method is used to solve (7.1); the iteration is stopped when the L^2-norm of the residual is reduced by a factor of 10^{-6}. Tables 1–6 list the minimum and maximum eigenvalues (λ_{min} and λ_{max}) of the iteration matrix $M^{-1}G$ and the iteration count (iter) for using both the lumped and Dirichlet preconditioners and in different cases. Here λ_{min} and λ_{max} are estimated by using the tridiagonal Lanczos matrix generated in the iteration.

We note that the lower eigenvalue bound of the preconditioned operator is proportional to the inf-sup constant β^2, defined in (2.4), as shown in Theorem 7.6. Estimates of β^2 can be obtained as in [9, Lemma 2.3], and their values for these three mixed elements are

$$\beta^2_{2D} = 0.0719, \qquad \beta^2_{3D,continuous} = 0.0189, \qquad \beta^2_{3D,discontinuous} = 0.1053. \tag{8.1}$$

From these values, we can expect and indeed have observed in the tables that λ_{min}, for the three-dimensional finite element with continuous pressure (Table 3), is quite small compared with the values for the two-dimensional case (Table 1) and for the three-dimensional discontinuous pressure case (Table 2).

Table 1 shows the performance for solving the two-dimensional problem. The coarse level velocity space is spanned by the subdomain corner nodal basis functions corresponding to each velocity component. Table 2 shows the performance for solving the three-dimensional problem using the discontinuous pressure mixed finite element illustrated in Figure 2. The coarse level velocity space is spanned by the subdomain vertex nodal basis functions and subdomain edge-cutoff functions corresponding to

Table 1
Solving two-dimensional problem, $\alpha = 1$ in (6.6) and (6.7).

		Errors		Lumped			Dirichlet		
$\frac{H}{h}$	#sub	$\|\mathbf{u}-\mathbf{u}_h\|_2$	$\|p-p_h\|_2$	λ_{min}	λ_{max}	iter	λ_{min}	λ_{max}	iter
8	4^2	9.50e-7	1.15e-5	0.3066	32.28	31	0.2983	4.40	18
	8^2	5.98e-8	7.00e-7	0.3067	37.25	46	0.2859	5.03	24
	16^2	3.74e-9	4.33e-8	0.3068	38.42	51	0.2556	5.28	25
	32^2	2.36e-10	3.04e-9	0.3070	38.68	51	0.2304	5.36	25
#sub	$\frac{H}{h}$	$\|\mathbf{u}-\mathbf{u}_h\|_2$	$\|p-p_h\|_2$	λ_{min}	λ_{max}	iter	λ_{min}	λ_{max}	iter
8^2	4	9.50e-7	1.15e-5	0.3024	15.91	34	0.2706	4.15	21
	8	5.98e-8	7.00e-7	0.3067	37.25	46	0.2859	5.03	24
	16	3.74e-9	4.33e-8	0.3069	85.32	62	0.2966	6.04	25
	32	2.36e-10	3.04e-9	0.3075	192.32	83	0.3070	7.19	27

Table 2
Solving three-dimensional problem using discontinuous pressure discretization, $\alpha = 1$ in (6.6) and (6.7).

		Errors		Lumped			Dirichlet		
$\frac{H}{h}$	#sub	$\|\mathbf{u}-\mathbf{u}_h\|_2$	$\|p-p_h\|_2$	λ_{min}	λ_{max}	iter	λ_{min}	λ_{max}	iter
4	3^3	1.24e-2	6.15e-2	0.2509	3.72	21	0.2510	3.15	18
	4^3	6.97e-3	3.45e-2	0.2549	3.96	22	0.2504	3.17	19
	6^3	3.09e-3	1.53e-2	0.2535	4.10	23	0.2256	3.20	20
	8^3	1.74e-3	8.62e-3	0.2551	4.08	23	0.2082	3.22	20
#sub	$\frac{H}{h}$	$\|\mathbf{u}-\mathbf{u}_h\|_2$	$\|p-p_h\|_2$	λ_{min}	λ_{max}	iter	λ_{min}	λ_{max}	iter
3^3	3	2.36e-2	1.10e-1	0.2485	3.19	20	0.2467	3.02	18
	4	1.24e-2	6.15e-2	0.2509	3.72	21	0.2510	3.15	18
	6	5.12e-3	2.72e-2	0.2543	5.77	24	0.2590	3.28	19
	8	3.09e-3	1.53e-2	0.2548	8.20	28	0.2794	3.34	18

each velocity component. We take $\alpha = 1$ in both the lumped and Dirichlet preconditioners (6.6) and (6.7). These results are consistent with our theory. The error of the finite element solution is also listed in Tables 1 and 2.

Tables 3 and 4 are for solving the three-dimensional problem using the mixed finite element with continuous pressures. The coarse velocity space is spanned by the subdomain vertex nodal basis functions and subdomain edge-cutoff functions corresponding to each velocity component. In Table 3, $\alpha = 1$; in Table 4, $\alpha = 1/2$. As expected, λ_{min} is smaller than in Tables 1 and 2; cf. (8.1). However, the error of the finite element solution using continuous pressures shown in Table 3 is much better than that shown in Table 2 for using discontinuous pressures.

In Table 3, the minimum eigenvalue is independent of the mesh size for both preconditioners. The maximum eigenvalue is independent of the number of subdomains for fixed H/h; we have to admit that due to the limit of the computing resource, we are not able to experiment with smaller mesh size to show the asymptotic behavior of convergence rate more accurately. For a fixed number of subdomains, λ_{max} depends on H/h and its least squares fits by $C_1\alpha + C_2(H/h)(1 + \log(H/h))$ for the lumped preconditioner and by $C_1\alpha + C_2(1 + \log(H/h))^2$ for the Dirichlet preconditioner, as

TABLE 3
Solving three-dimensional problem using continuous pressure discretization, $\alpha = 1$ *in* (6.6) *and* (6.7).

		Errors		Lumped			Dirichlet		
$\frac{H}{h}$	#sub	$\lVert\mathbf{u}-\mathbf{u}_h\rVert_2$	$\lVert p-p_h\rVert_2$	λ_{min}	λ_{max}	iter	λ_{min}	λ_{max}	iter
	3^3	1.22e-4	2.10e-3	0.0776	9.13	56	0.0776	8.97	56
4	4^3	3.85e-5	6.99e-4	0.0775	9.35	54	0.0774	9.19	55
	6^3	6.96e-6	1.38e-4	0.0773	9.41	58	0.0773	9.23	59
	8^3	2.41e-6	5.35e-5	0.0773	9.51	57	0.0772	9.34	61
#sub	$\frac{H}{h}$	$\lVert\mathbf{u}-\mathbf{u}_h\rVert_2$	$\lVert p-p_h\rVert_2$	λ_{min}	λ_{max}	iter	λ_{min}	λ_{max}	iter
	3	4.09e-4	6.85e-3	0.0760	8.06	54	0.0760	7.96	54
3^3	4	1.22e-4	2.10e-3	0.0776	9.13	56	0.0776	8.97	56
	6	2.27e-5	4.20e-4	0.0780	11.88	53	0.0780	9.35	55
	8	6.96e-6	1.38e-4	0.0780	16.64	57	0.0780	9.44	55

TABLE 4
Solving three-dimensional problem using continuous pressure discretization, $\alpha = 1/2$ *in* (6.6) *and* (6.7).

		Lumped			Dirichlet		
$\frac{H}{h}$	#sub	λ_{min}	λ_{max}	iter	λ_{min}	λ_{max}	iter
	3^3	0.0395	7.20	59	0.0395	4.89	54
4	4^3	0.0394	8.15	66	0.0394	5.01	53
	6^3	0.0393	8.85	70	0.0393	5.03	55
	8^3	0.0393	9.09	72	0.0393	5.09	56
#sub	$\frac{H}{h}$	λ_{min}	λ_{max}	iter	λ_{min}	λ_{max}	iter
	3	0.0387	5.15	55	0.0387	4.35	53
3^3	4	0.0395	7.20	57	0.0395	4.89	54
	6	0.0397	11.70	63	0.0397	5.11	52
	8	0.0397	16.52	73	0.0397	5.17	52

guided by Theorem 7.6, are shown in Figure 3. Moreover, the convergence rate of the algorithm using the Dirichlet preconditioner is only slightly better than the rate when using the lumped preconditioner. The reason is that the upper eigenvalue bound in Theorem 7.6 depends on both α and $\Phi(H/h)$, and in this case $\alpha = 1$ dominates when H/h is small. Therefore, even though using the Dirichlet preconditioner can reduce $\Phi(H/h)$ compared with using the lumped preconditioner, this improvement cannot show up in Table 3. What shows in Table 3 for λ_{max} is essentially its dependence on α. Only for larger H/h, e.g., for $H/h = 6$ and $H/h = 8$ in Table 3, does this improvement on λ_{max} by using the Dirichlet preconditioner become visible.

To test the case when α is less dominant in the upper eigenvalue bound, we take $\alpha = 1/2$ in Table 4. Consistent with Theorem 7.6, the lower eigenvalue bounds in Table 4 become half of those in Table 3, and they are also independent of the mesh size. The upper eigenvalue bounds show the improvement by using the Dirichlet preconditioner. The least squares fits of $\lambda_{\max}$ with respect to H/h are also shown in Figure 3.

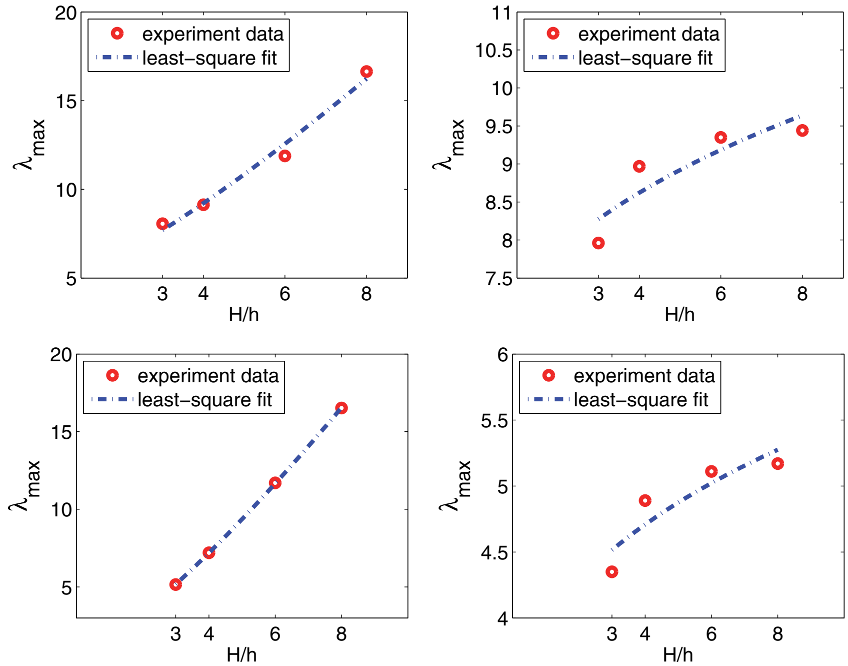

FIG. 3. *Least squares fits of* λ_{max} *in* Table 3 *(top row) and in* Table 4 *(bottom row) with respect to* H/h *by* $C_1\alpha + C_2(H/h)(1 + \log(H/h))$ *for the lumped preconditioner (left) and by* $C_1\alpha + C_2(1 + \log(H/h))^2$ *for the Dirichlet preconditioner (right).*

TABLE 5
Solving three-dimensional problem with an enriched coarse space, $\alpha = 1/2$ *in* (6.6) *and* (6.7).

		Lumped			Dirichlet		
$\frac{H}{h}$	#sub	λ_{min}	λ_{max}	iter	λ_{min}	λ_{max}	iter
4	3^3	0.0395	5.07	54	0.0395	4.89	51
	4^3	0.0394	5.51	54	0.0394	5.01	47
	6^3	0.0393	6.18	56	0.0393	5.03	50
	8^3	0.0393	6.43	59	0.0393	5.09	51
#sub	$\frac{H}{h}$	λ_{min}	λ_{max}	iter	λ_{min}	λ_{max}	iter
3^3	3	0.0388	4.47	51	0.0388	4.35	50
	4	0.0395	5.07	52	0.0395	4.89	51
	6	0.0396	7.10	57	0.0397	5.11	49
	8	0.0397	9.67	64	0.0397	5.17	48

We also test using an enriched coarse space in the algorithm with continuous pressures on the three-dimensional problem. In addition to the subdomain vertex nodal basis functions and subdomain edge-cutoff functions corresponding to each velocity component, one face-cutoff function representing the normal velocity component on each subdomain face is also included in the coarse space. The performance of the algorithm is listed in Table 5. Compared with Table 4, the iteration counts are reduced

TABLE 6
Solving three-dimensional problem with mixed boundary conditions, $\alpha = 1/2$ in (6.6) *and* (6.7).

$\frac{H}{h}$	#sub	Lumped λ_{min}	Lumped λ_{max}	Lumped iter	Dirichlet λ_{min}	Dirichlet λ_{max}	Dirichlet iter
4	3^3	0.0395	7.32	71	0.0394	4.89	62
	4^3	0.0394	8.22	80	0.0389	5.01	66
	6^3	0.0393	8.88	86	0.0297	5.10	69
	8^3	0.0391	9.10	89	0.0237	5.13	71
#sub	$\frac{H}{h}$	λ_{min}	λ_{max}	iter	λ_{min}	λ_{max}	iter
3^3	3	0.0391	5.22	67	0.0396	4.35	64
	4	0.0395	7.32	71	0.0394	4.89	62
	6	0.0397	11.91	81	0.0397	5.11	62
	8	0.0397	16.81	90	0.0397	5.17	61

for both preconditioners, and the condition number bound is improved a little bit in the lumped preconditioner case, while the improvement is too small to be seen in the Dirichlet preconditioner case; cf. Remark 6.3.

Finally, we test the algorithm solving the three-dimensional problem with a mixed boundary condition, where the normal derivative of each velocity component equals zero at the bottom face of the cube and the velocity equals zero at the other faces. We can see from Table 6 that the performance is only a bit worse than that for solving the Dirichlet boundary condition problem shown in Table 4.

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