ANALYTICAL LINE GEOMETRY OF THE PLANE.

by

Lucy Taft Dougherty, B.S.,

Washburn, 1895.

Submitted to the Department
of Mathematics and the
Faculty of the Graduate School
of the University of Kansas
in partial fulfillment of the
requirements for the degree of
Master of Arts.

Approved by:

W. G. Mitchell
Instructor in Charge

U. H. Ashton
Chairman of Department

May 10, 1927.

CONTENTS

ANALYTICAL LINE GEOMETRY of the PLANE

Introduction

A plane is two-dimensional when considered as an assemblage of points. Two independent coordinates determine a point in the plane, and one first degree equation between those coordinates determines the set of all points on one line # in the plane. The equation is called the equation of the line.

A plane is also two-dimensional when considered as an assemblage of lines. Two independent coordinates determine a line in the plane, and one first degree equation between those coordinates determines the set of all lines on one point in the plane. The equation is called the equation of the point.

This dual relationship has been studied in various ways. It is the purpose of this paper to discuss a system by means of which the geometry of the plane can be studied by an analysis based on the line as the fundamental element.

<u>Line</u>, in this paper, is limited by definition to straight line.

In order to do this, it was found necessary to devise a system of line coordinates. This was done by dualizing the Cartesian rectangular system. The exact method used is shown in Note A, page 57.

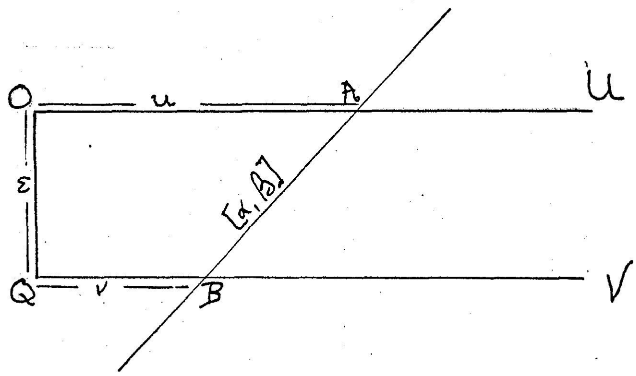
Sometime after this work was started, when the system of coordinates had alre^a^dy been devised, the writer discovered in the Jahrbuch[1], a review of an article by Dr. Karl Schwering, which had been published in the Schlömilch Zeitschrift[2], explaining a similar system of line coordinates. Some years later, in 1884, Dr. Schwering had published a geometry[3] based on his system. Neither the Zeitschrift, nor Dr. Schwering's book, were in the University of Kansas library, and only about ten days before the completion of this paper, the magazine was obtained from the John Crerar library, Chicago, and the book from the Harvard University library.

1 Jahrbuch d. Fortschritte d. Mathematik.,Berlin, 1878, vol. 8, page 414.

2 Zeitschrift f. Mathematik u. Physik, Leipzig, 1876, vol. 21, page 278-286

3 Schwering, Karl. Theorie u. Auswendung d. Liniencoordinaten in d. analytische Geometrie der Ebene. Leipzig, 1884.

Dr. Schwering's system of coordinates differs slightly from that devised by the writer. He lays down two lines which he calls axes; OU and QV cut by a perpendicular QO of length e.



Then the coordinates are u = OA, v = QB. There is no "origin" in Dr, Schwering's system, and his "axes" are not the dual of the Cartesian. The reason is apparent since he says in explanation that he does not claim his system is the exact dual to the Cartesian system, but that it is a system by which the geometry of the line as an element can be studied independently of the idea of points on the line.

A year later, in volume 23 of the Schlomilch Zeitschrift, Dr. V. Schelling, in a note, points out that the system is an exact dual of the Cartesian rectangular system. The parallel he sets up resembles much the method originally used by the writer, and shown in Note A.

Dr. Schwering has done much more work than the writer on the conics, taking up tangents, asymptotes, poles and polars, and devoting one chapter to higher plane curves, including among others the cardioid, cycloid, and lemniscate, deriving equations for them in terms of his coordinates. Maxima and minima and singular points are worked out by the Calculus.

Dr. Schwering has not worked out a set of equa- of the line dual to those of the point. He does have an equation in "normal form", but does not make use of it in finding distances along the axis. ("Axis" is here used as in this paper, and is the join designated by e in Dr. Schwering's work.)

He derives the equations of the conics through focal relations, and has the line of foci coincide with e. This makes the parameters of his equations different from those used by the writer; but the general form of the conic equation is the same, and the same relations between the coefficients hold to give the equations of the different conics.

On the last page of Dr. Schwering's book, he has equations for transformations of coordinates identical with those the writer worked out and used in deriving the general equations of the conics. The derivations of the transformation equations are quite different.

The proof given on page 57 ff that the general equation of the second degree in line coordinates is a conic, is taken from Dr. Schwering, and is the only proof of his used by the writer.

The method used in this paper of studying the relations of lines and points by dualizing the ordinary equations of the line, and the relations between points and lines, can be carried on into conics, and it is quite likely a simultaneous analytical procedure that would apply interchangeably to lines and points might result. Dr, Schwering, apparently, had not anything of that kind in mind.

In the Addenda are noted some of the supplementary suggestions that have come to mind during the study. The limits of this paper have prohibited further work that gave promise of being interesting.

Section I

Points and Lines

1. A point is considered as made up of all the lines on it, and is said to be generated by the rotation of any one of its lines. The position from which the generating line starts is called the <u>initial</u> line of the point; the position where it stops, the <u>terminal</u> line.

2. A point may be considered as <u>directed</u> when there is established on it an initial line, a unit of angle, and a positive direction of rotation.

3. The movement of a line on a point generates an angle. The notation $\widehat{ab}$ is used to denote the angle generated by rotation from position <u>a</u> to position <u>b</u>. If $l_i, (i = 0,1,2, ---)$, stands for any position of a line on a given point, the angle $\widehat{l_0 l_2}$ is given by

$$\widehat{l_0 l_2} = \widehat{l_0 l_1} + \widehat{l_1 l_2} \qquad (1)$$

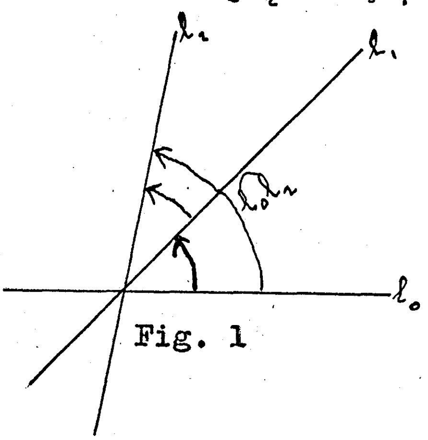


Fig. 1

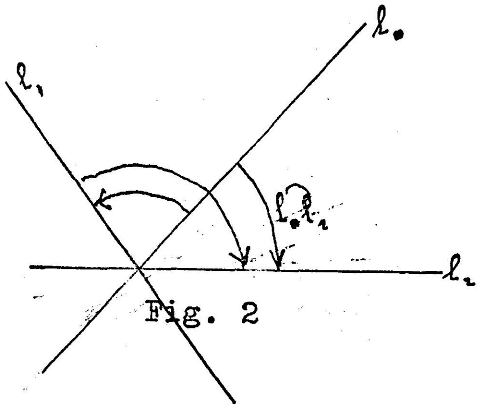


Fig. 2

Since by (1) $\widehat{l_1 l_2} + \widehat{l_2 l_1} = 0,$

then $$\widehat{l_1 l_2} = -\widehat{l_2 l_1} \qquad (2)$$

$\widehat{l_1 l_2}$ is positive if the direction of rotation from l_1 to l_2 agrees with the positive direction established on the point.

4. Similarly, as a point P_1 moves along a line l to position P_2 it generates the join P_1P_2. If P_0 stands for any position of the point, the join P_1P_2 is given by the equation

$$P_1P_2 = P_1P_0 + P_0P_2 \qquad (3)$$

also $$P_1P_2 = -P_2P_1 \qquad (4)$$

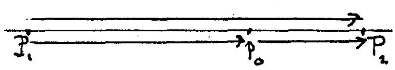


Fig. 3

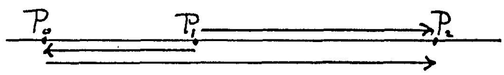


Fig. 4

The Cartesian conventions as to directions on both lines and points will be followed, unless otherwise noted.

Section II

System of Coordinates.

5. Let U and V be any two points, and o the join of U and V. Let l be any line in a plane with o, and designate by B and A the points on l

whose orthogonal projections on o are U and V, respectively.

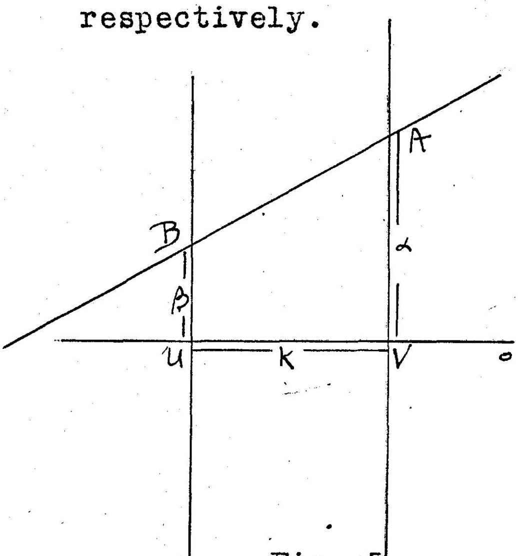


Fig. 5

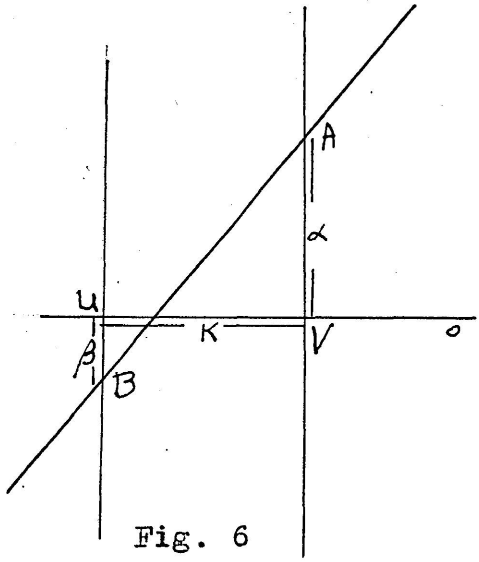


Fig. 6

Then if $VA = \alpha$, $UB = \beta$, the numbers α, β are called the coordinates of l. (The method of arriving at this system is shown in Note A.) U and V are called the centers of coordinates, or origins, and o, the line on U and V is called the axis of coordinates.

The perpendiculars to o at U and V, are the lines of reference. The axis is to be considered positive from U to V, and the length of the join UV will be designated by k.

It will be seen that this system in reality uses angles as line coordinates, but the angles are expressed in terms of their tangents:

$$\tan VUA = \frac{\alpha}{k}\ ; \qquad \tan UVB = \frac{\beta}{k}$$

This system determines all the lines in the plane except those parallel to the lines of reference; that is, the lines on the point at infinity. A discussion of these lines will be found in paragraph 8.

Then, in this system, any finite pair of numbers α and β, determine uniquely, a line <u>l</u>. The line will be designated $[\alpha, \beta]$. Conversely, a line <u>l</u>, not parallel to the lines of reference, determines uniquely a pair of numbers.

Section III

Angles of lines. Slopes of points.

6. <u>Lemma</u> . Any line <u>l</u>, whose coordinates are $[\alpha, \beta]$ cuts the axis <u>o</u> in the angle whose tangent is :

$$\tan \widehat{ol} = \frac{\alpha - \beta}{k} \ . \qquad (1)$$

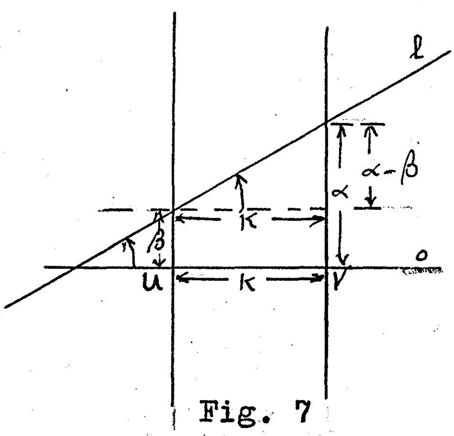


Fig. 7

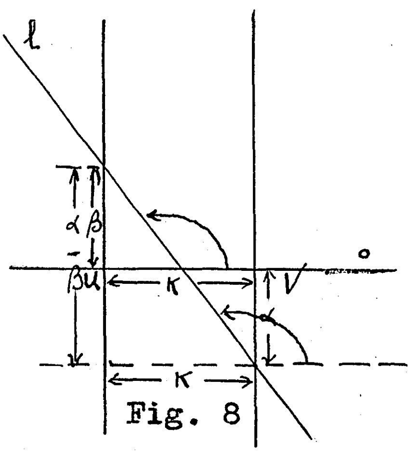


Fig. 8

The truth of the lemméa is seen from the figures.

7. To find the angle between two lines in terms of the coordinates of the lines.

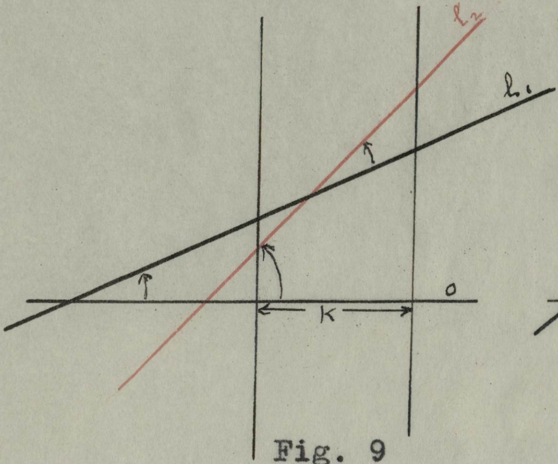


Fig. 9

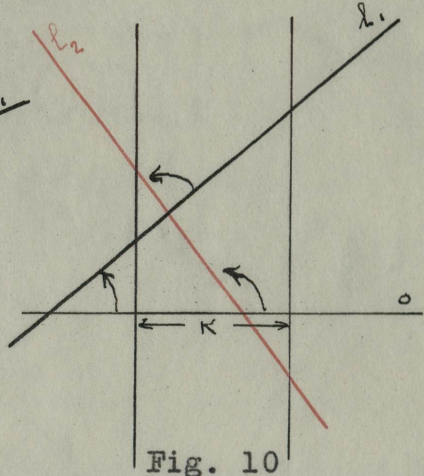


Fig. 10

Let the two lines be $l_1 : [\alpha_1, \beta_1]$ and $l_2 : [\alpha_2, \beta_2]$ and let them cut the axis <u>o</u> as in the figures.

Then $\widehat{l_1 l_2} = \widehat{l_1 o} + \widehat{o l_2} = \widehat{o l_2} - \widehat{o l_1}$

Then by lemma,

$$\tan \widehat{l_1 l_2} = \frac{\frac{\alpha_2 - \beta_2}{k} - \frac{\alpha_1 - \beta_1}{k}}{1 + \frac{\alpha_2 - \beta_2}{k} \cdot \frac{\alpha_1 - \beta_1}{k}}$$

$$= \frac{k\left[(\alpha_2 - \beta_2) - (\alpha_1 - \beta_1)\right]}{k^2 + (\alpha_2 - \beta_2)(\alpha_1 - \beta_1)} \qquad (2)$$

If l_1 and l_2 are parallel, $\tan \widehat{l_1 l_2} = 0$.

Then $\alpha_2 - \beta_2 = \alpha_1 - \beta_1$ or $\alpha_2 - \alpha_1 = \beta_2 - \beta_1$ (3)

This may be expressed as a corollary.

Corollary I. If two lines are parallel, their coordinates are of the form $[\alpha_0, \beta_0]$ and $[\alpha_0 + m, \beta_0 + m]$, when <u>m</u> is any finite number.

If l_1 and l_2 are perpendicular, $\tan \widehat{l_1 l_2}$ is infinite, and this may be expressed as a second corollary.

Corollary II. The condition that two lines $[\alpha_1, \beta_1]$ and $[\alpha_2, \beta_2]$ are perpendicular, is

$$(\alpha_1 - \beta_1)(\alpha_2 - \beta_2) + k^2 = 0 \qquad (4)$$

8. <u>Slope of a point</u>. In point geometry, the slope of a line is the tangent of the angle it makes with the positive end of the X-axis. In particular, if the line is on the origin, <u>O</u>, it is the ratio of the distance of any point on the line from the X-axis, to its distance from the Y-axis.

Corresponding to this, the slope of a point on the axis <u>o</u> is defined to be the ratio of its distance from the U-origin, to its distance from the V-origin.

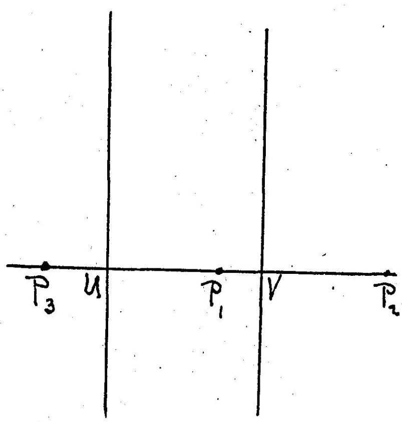


Fig. 11

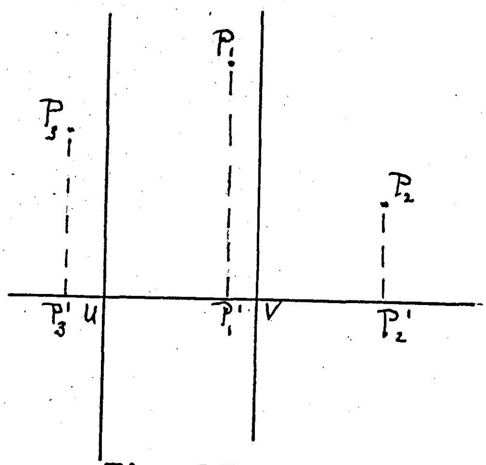


Fig. 12

Taking the distances in each case as maesured from the origins, the slope of any point P_i, ($i = 1,2,3$--) on the axis, is

$$\mu = UP_i : VP_i \qquad \text{(Fig. 11)} \qquad (5a)$$

If P_i is not on the axis, take the orthogonal projection of P_i on the axis, P_i'. Then the slope of P_i is defined to be

$$\mu = UP_i' : VP_i' \qquad (5b)$$

It is to be noted that a point between the lines of reference has a negative slope, while one without the lines, has a positive slope.

Also the slope of the point determines the point as on a particular line of the point at infinity. As in the usual geometry, the slope of the line determines the line as on a particular point of the line at infinity.

Therefore, lines parallel to the lines of reference are determined by the slope of the point where they meet the axis. That is, these are the lines on the point at infinity, and are the exact dual to the points on the line at infinity, and are determined in the dual way.

9. <u>Slope of a point on two given lines</u>.

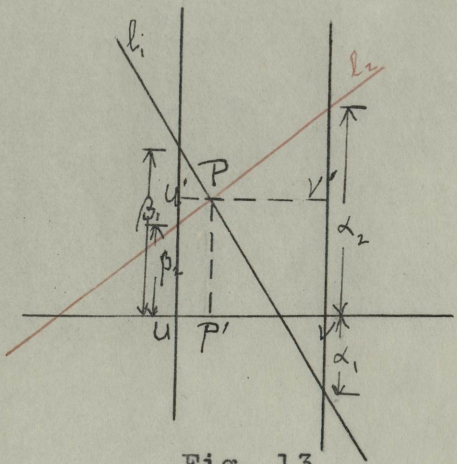


Fig. 13

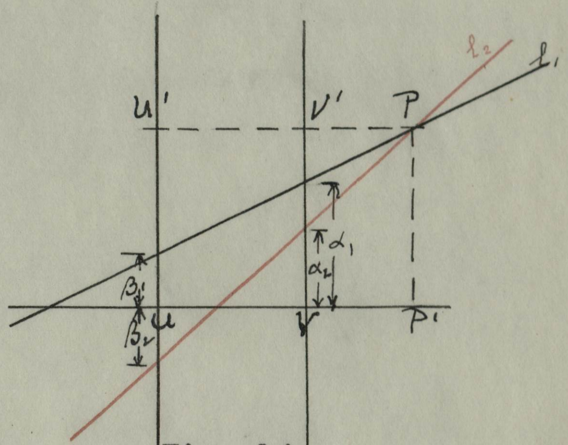


Fig. 14

Given point P on two lines $[\alpha_1, \beta_1]$ and $[\alpha_2, \beta_2]$. Then by proportional lines of similar triangles,

$$\frac{\beta_2 - \beta_1}{\alpha_2 - \alpha_1} = \frac{U'P}{V'P} = \frac{UP'}{VP'} = \mu$$

Then, the slope of a point on two lines $[\alpha_1, \beta_1]$ and $[\alpha_2, \beta_2]$ is

$$\mu = \frac{\beta_2 - \beta_1}{\alpha_2 - \alpha_1} \qquad (6)$$

If the lines are $[\alpha_0, 0]$ and $[0, \beta_0]$, the slope in terms of the intercepts of the point is

$$\mu = -\frac{\beta_0}{\alpha_0} \qquad (7)$$

10. <u>Parallel points</u>. In ordinary point geometry, the points on the line at infinity are distinguished from each other by the slope of the line

joining the point to the origin.

In the line coordinate system, the lines on the point at infinity are distinguished from each other by the slope of the point in which the line meets the axis.

In point geometry, lines which have the same slope are parallel.

Definition. Parallel points are points which have the same slope.

This definition fixes parallel points as those on the same line parallel to the lines of reference. The point at infinity in the plane, is the point on which the lines of reference intersect. All lines on the point at infinity have the coordinates $[\infty, \infty]$. Two points might then be defined as parallel, if their join is parallel to the lines of refsrencs. (See also paragraph 24.)

11. Perpendicular points. Corresponding to the condition that two lines are perpendicular if the product of their slopes is - 1, two points may be defined as perpendicular if the product of their slopes is - 1. That is, two points P_1 and P_2

are perpendicular if

$$\mu_1 \mu_2 = -1 \tag{8a}$$

or, by paragraph 8, (5b), if

$$\frac{UP_1'}{VP_1'} = -\frac{VP_2'}{UP_2'} \tag{8b}$$

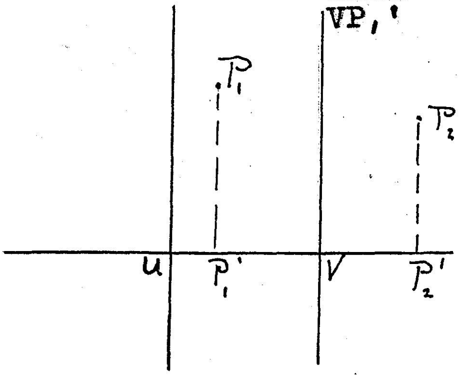


Fig. 15

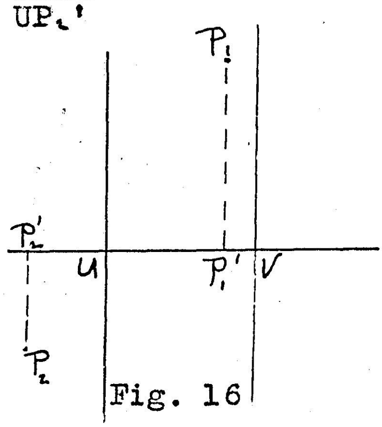


Fig. 16

That is, if two points are perpendicular, one lies within, and the other without the lines of reference. When any two points have been determined to bo perpendicular, every point on the same line with one of them, parallel to the lines of reference, is perpendicular to every point on the corresponding line through the other. (See also paragraph 25.)

Section IV

Equation of a point. Length of joins.

12. <u>Two line form of the equation of a point</u>. A point is uniquely determined by two lines.

Let $l_1 : [u_1, v_1]$ and $l_2 : [u_2, v_2]$ be two given lines on P and let $l : [u, v]$ be any other line on P.

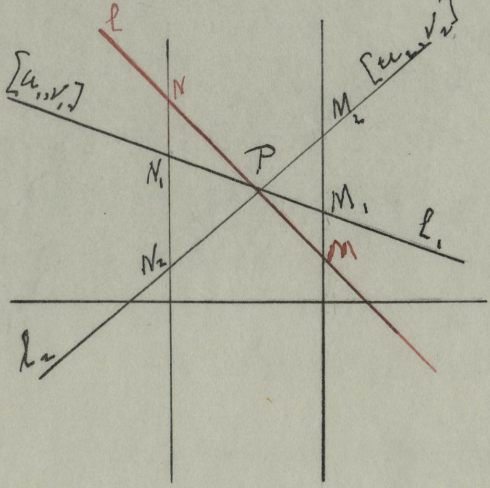


Fig. 17

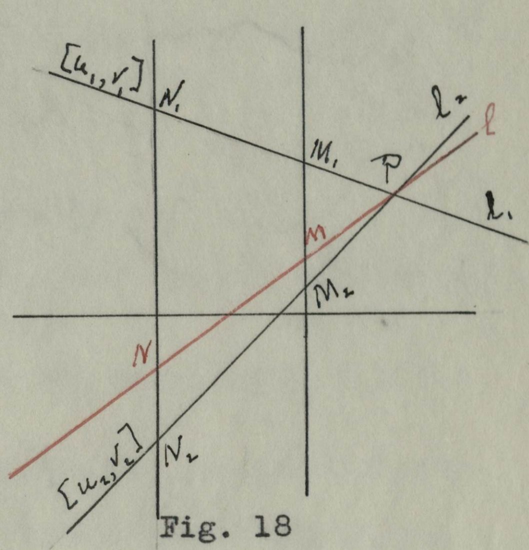


Fig. 18

The condition that l shall be on the point with l_1 and l_2 is

$$\frac{N_1N}{M_1M} = \frac{N_1P}{M_1P} = \frac{N_1N_2}{M_1M_2}$$

Translating this into coordinates,

$$\frac{v - v_1}{u - u_1} = \frac{v_2 - v_1}{u_2 - u_1} \qquad \text{(1a)}$$

This may be put into the form,

$$\begin{vmatrix} u & v & 1 \\ u_1 & v_1 & 1 \\ u_2 & v_2 & 1 \end{vmatrix} = 0 \qquad \text{(1b)}$$

and may be called the <u>two</u> <u>line</u> form of the point equation.

13. Line-slope form of the equation of a point

If equation (1a) paragraph 12, is written

$$v - v_1 = \frac{v_2 - v_1}{u_2 - u_1} \cdot (u - u_1), \quad \text{and, as}$$

in (6), paragraph 9, $\frac{v_2 - v_1}{u_2 - u_1} = \mu$, then the line-slope form of the point equation is

$$v - v_1 = \mu(u - u_1) \qquad (2)$$

14. Intercept form of the equation of a point

If in equation (1b), paragraph 12, the particular lines $[\alpha_0, 0]$ and $[0, \beta_0]$ are substituted for $[u_1, v_1]$ and $[u_2, v_2]$, (1b) becomes

$$\begin{vmatrix} u & v & 1 \\ \alpha_0 & 0 & 1 \\ 0 & \beta_0 & 1 \end{vmatrix} = 0$$

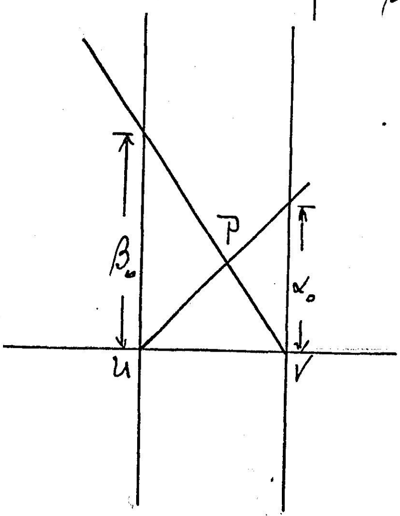


Fig. 19

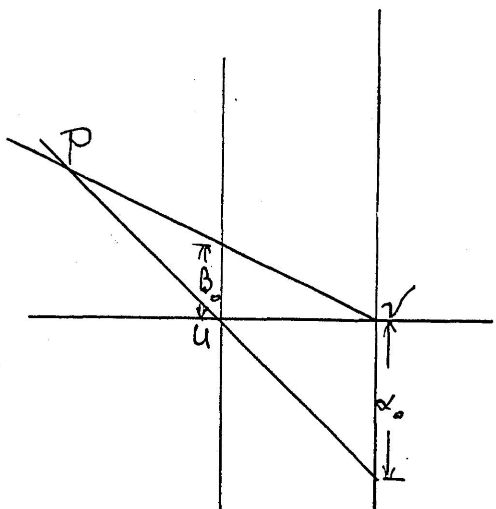


Fig. 20

It is seen from figures 19 and 20, that α_0 and β_0 are the intercepts which lines on P and the two origins, U and V cut from the V- and U-lines of reference, respectively. For this reason, this is called the intercept equation of a point. That it is in the same form as the intercept equation of a line in Cartesian coordinates, is seen when it is written

$$\frac{u}{\alpha_0} + \frac{v}{\beta_0} = 1 \qquad (3)$$

15. Slope-intercept form of the equation of a point. If in equation (2), paragraph 13, the particular line $[0, \beta_0]$ is substituted for $[u_1, v_1]$, the equation becomes

$$v = \mu u + \beta_0 \qquad (4a)$$

If $[\alpha_0, 0]$ is used as the line, the equation becomes

$$v = \mu(u - \alpha_0) \qquad (4b)$$

These two are the slope-intercept forms.

16. The general linear equation

In the preceding discussion it has appeared that the equation of a point determined by any two straight lines, or by one line and the slope of the point, is of the first degree.

Consider now any equation of the first degree, to determine whether it must represent a point. Take

$$Au + Bv + C = 0.$$

If $B \neq 0$, $v = -\frac{A}{B}\cdot u - \frac{C}{B}$, and $\frac{A}{B}$ and $\frac{C}{B}$ can take any values.

Since this equation is in the form

$v = \mu u + \beta_o$, where μ and β_o can take any values, $Au + Bv + C = 0$ is the equation for the point for which $\mu = -\frac{A}{B}$ and $\beta_o = -\frac{C}{B}$.

If $B = 0$, the equation becomes $Au + C = 0$.

By (5b), paragraph 8, $\mu = \frac{UP_1'}{VP_1'}$, so $\frac{UP_1'}{VP_1'} = -\frac{A}{B}$, and when $B = 0$, $VP_1' = 0$. Then $Au + C = 0$ is the point on the V-reference line where $u = -\frac{C}{A}$.

This proves completely that the general equation of the first degree, has always a point as its locus.

17. <u>Equations of special points</u>

Let $Au + Bv + C = 0$ be the equation of any point. By paragraph 16, the equation of a point on the V-reference line is $Au + C = 0$. (5a)

The equation of the V-origin is $u = 0$. (5b)

Similarly, the equation of a point on the U-reference line is $Bv + C = 0.$ (6a)

The equation of the U-origin is $v = 0.$ (6b)

If $Au + Bv + C = 0$ be the equation of a point on the axis, $[0,0]$, then $C = 0$. The equation of a point on the axis is $Au + Bv = 0.$ (7)

At the point midway between the origins, on the axis, $UP = -VP$. That is, $\mu = -1 = -\frac{A}{B}$, so the mid point of the axis is $u + v = 0$ (8)

If A and B have the same sign, the slope is negative, and the point is between the reference lines. If A and B have opposite signs, the slope is positive, and the point is outside the reference lines. (See also paragraph 8).

If $\mu = 1$, $A = -B$, and the equation becomes $u - v = -\frac{C}{A}$, and the point is on a line of the infinite point, infinitely removed. The infinitely distant point of the axis is $u - v = 0$ (9).

18. <u>The length of the join of two points</u>

<u>Lemma:</u> Let P be a point whose equation is $\frac{u}{\alpha_0} + \frac{v}{\beta_0} = 1$, P' its projection on the axis, and

h the distance from the axis to the point. The length UV = k. Then,

$$UP' = \frac{k\beta_o}{\alpha_o+\beta_o} \; ; \; P'V = \frac{k\,\alpha_o}{\alpha_o+\beta_o} \; ; \; h = \frac{\alpha_o\beta_o}{\alpha_o+\beta_o}$$

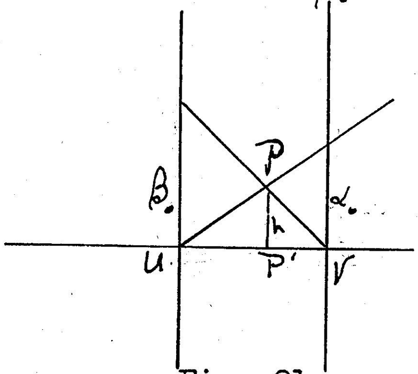


Fig. 21

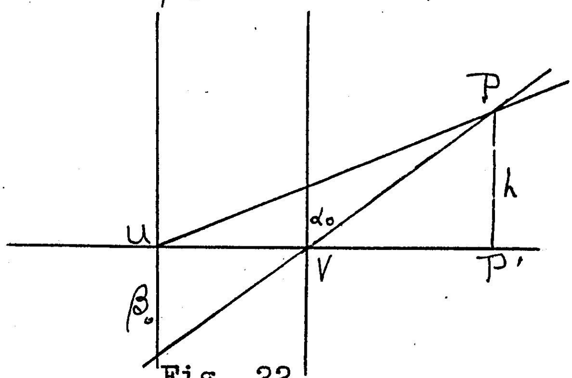


Fig. 22

Proof:

$$h:\alpha_o = UP' : UV \; ; \qquad h:\beta_o = P'V : UV$$

$$\text{Then } h\left[\frac{1}{\alpha_o} + \frac{1}{\beta_o}\right] = \frac{UP' + P'V}{UV} = 1$$

$$h = \frac{\alpha_o\beta_o}{\alpha_o+\beta_o} \qquad (10)$$

$$UP' = \frac{kh}{\alpha_o} = \frac{k\,\beta_o}{\alpha_o+\beta_o} \qquad (11)$$

$$P'V = \frac{kh}{\beta_o} = \frac{k\,\alpha_o}{\alpha_o+\beta_o} \qquad (12)$$

<u>To find the distance between any two points.</u>

Let the two points be

$$P_1 : \frac{u}{\alpha_1} + \frac{v}{\beta_1} = 1, \text{ and } P_2 : \frac{u}{\alpha_2} + \frac{v}{\beta_2} = 1.$$

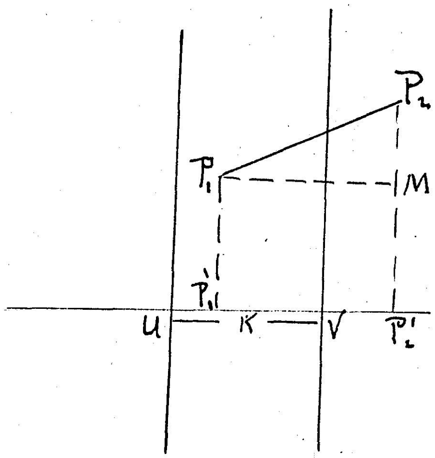


Fig. 23

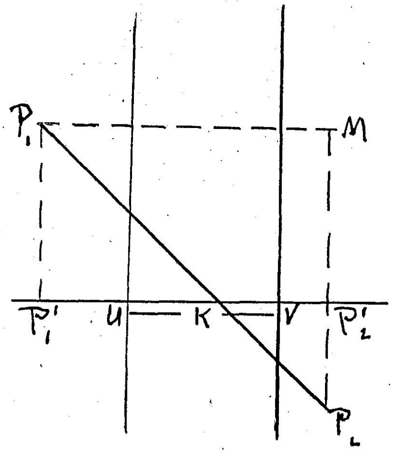


Fig. 24

In figures 23 and 24,

$$P_1P_2 = \sqrt{\overline{P_1M}^2 + \overline{MP_2}^2}$$

$$P_1M = P_1'V + VP_2' \; ; \qquad MP_2 = h_2 - h_1$$

$$= \frac{K\alpha_1}{\alpha_1+\beta_1} - \frac{K\alpha_2}{\alpha_2+\beta_2} \qquad = \frac{\alpha_2\beta_2}{\alpha_2+\beta_2} - \frac{\alpha_1\beta_1}{\alpha_1+\beta_1}$$

$$= \frac{K(\alpha_1\beta_2 - \alpha_2\beta_1)}{(\alpha_1+\beta_1)(\alpha_2+\beta_2)} \qquad = \frac{\alpha_2\beta_2(\alpha_1+\beta_1) - \alpha_1\beta_1(\alpha_2+\beta_2)}{(\alpha_1+\beta_1)(\alpha_2+\beta_2)}$$

Then $$P_1P_2 = \frac{\sqrt{K^2(\alpha_1\beta_2-\alpha_2\beta_1)^2 + [\alpha_2\beta_2(\alpha_1+\beta_1) - \alpha_1\beta_1(\alpha_2+\beta_2)]^2}}{(\alpha_1+\beta_1)(\alpha_2+\beta_2)} \quad (13)$$

when α_1, β_1, α_2, β_2 are the intercepts of the points P_1 and P_2. For another formula, see paragraph 21.

19. Coordinates of a line ~~point~~ on a point ~~line~~ parallel to a given line. Given a point P: $Au + Bv + C = 0$, and a line $[u_1, v_1]$. The coordinates of a line parallel to $[u_1, v_1]$ are $[u_1 + m, v_1 + m]$, by paragraph 7, Corollary 1. Note, *m* is measured like *u* and *v* on

the lines of reference or parallel to them, and from the axis toward the point.

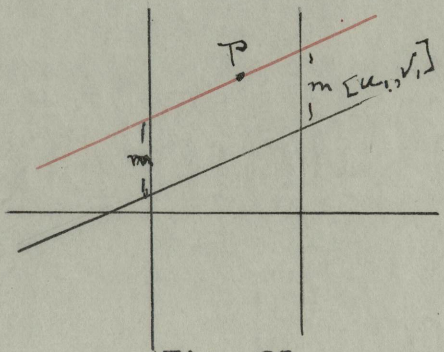


Fig. 25

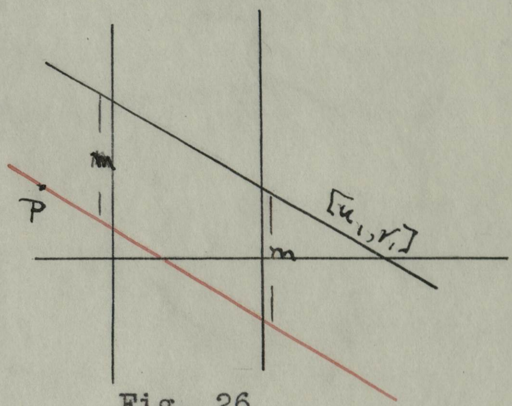


Fig. 26

Since the line $[u_1 + m, v_1 + m]$ is on P,

$$A(u_1 + m) + B(v_1 + m) + C = 0.$$

Solving, $$m = -\frac{Au_1 + Bv_1 + C}{A + B} \qquad (14)$$

The coordinates of the parallel line are then,

$$\left[\frac{Bu_1 - Bv_1 - C}{A + B}, \quad \frac{-Au_1 + Av_1 - C}{A + B}\right]$$

The value of <u>m</u> in (14) is in the form of the equation of the form of the equation of the point, with the coordinates of the line substituted, if the equation of the point is first put in the form

$$\frac{Au + Bv + C}{A + B} = 0.$$

This suggests a normal form of the point equation.

20. <u>Normal form of the equation of a point</u>.

<u>Definition</u>: <u>The normal of a point</u> is the perpendicular from the axis to the point. <u>h</u>, in the figures 21 and 22, is the normal of point P.

Let P be the point whose equation is

$$\frac{u}{\alpha_0} + \frac{v}{\beta_0} = 1$$

Clear the equation of fractions, and divide by $\alpha_0 + \beta_0$ to put the equation in the form

$$\frac{\beta_0}{\alpha_0 + \beta_0} u + \frac{\alpha_0}{\alpha_0 + \beta_0} v - \frac{\alpha_0 \beta_0}{\alpha_0 + \beta_0} = 0$$

and write in place of this, the simple form,

$$au + bv + c = 0. \qquad (15)$$

By the conditions of the substitution, $a + b = 1$. This is the normal form of the equation of a point. With the help of (10), (11), and (12), paragraph 18, this can be interpreted geometrically:

$$UP' = ka \qquad (16)$$

$$P'V = kb \qquad (17)$$

$$h = -c \qquad (18)$$

To put the general equation of the first degree in normal form, divide by $A + B$.

$$\frac{A}{A + B} u + \frac{B}{A + B} v + \frac{C}{A + B} = 0.$$

The sum of the coefficients of u and v is unity.

The value of m in (14), paragraph 19, may now be written

$$m = -(au_1 + bv_1 + c), \quad a + b = 1. \qquad (19)$$

21. The formula for the distance between two points can be much simplified by the use of this equation.

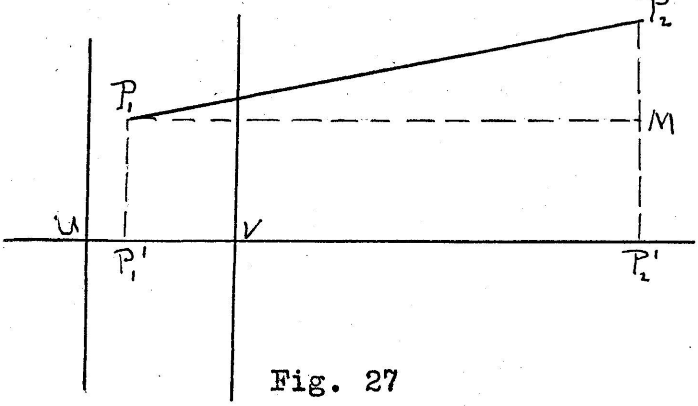


Fig. 27

Let P_1 be $a_1u + b_1v + c = 0$, and

P_2 be $a_2u + b_2v + c = 0$, with the condition

$a_1 + b_1 = a_2 + b_2 = 1.$

In the figure, $P_1M = P_1'P_2' = P_1'U + UP_2' = P_1'V + VP_2'$

$$= -ka_1 + ka_2 = kb_1 - kb_2$$

$$MP_2 = h_2 - h_1 = -c_2 + c_1$$

Then $P_1P_2 = \sqrt{(c_2 - c_1)^2 + k^2(a_2 - a_1)^2}$ (20a)

$$= \sqrt{(c_2 - c_1)^2 + k^2(b_1 - b_2)^2} \qquad (20b)$$

22. Distance from a point to a line.

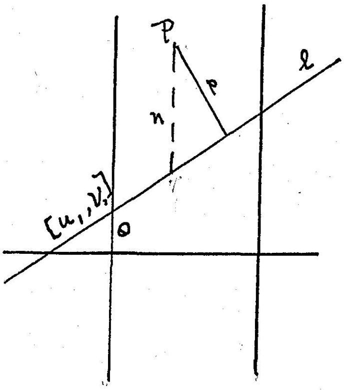


Fig. 28

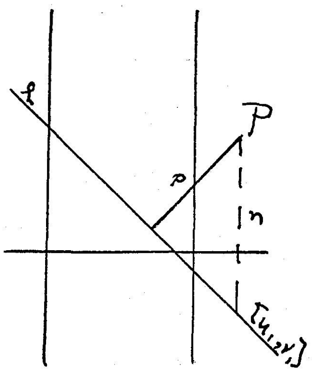


Fig. 29

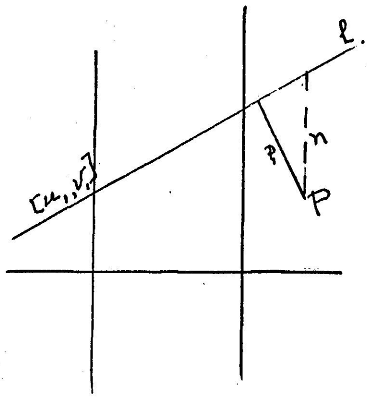


Fig. 30

Given: Point P: $au + bv + c = 0$, $a + b = 1$, and line l: $[u_1, v_1]$. Let p be the perpendicular from P to the line l, and n the distance from P to the line, measured parallel to the lines of reference.

Then $n = -m = au_1 + bv_1 + c$, by (19), paragraph 20. Let the angle between the line and the axis be θ. Then the angle $\widehat{np} = \pm\theta$ or $\pm(\pi - \theta)$

$$\tan\theta = \frac{u_1 - v_1}{k}; \quad \cos\theta = \frac{k}{\pm\sqrt{k^2 + (u_1 - v_1)^2}}$$

$$p = n\cos\theta = \frac{k(au_1 + bv_1 + c)}{\pm\sqrt{k^2 + (u_1 - v_1)^2}} \qquad (21)$$

Section V

Relations of points to each other

23. Systems of points. An equation of the first degree in u and v, which contains an arbitrary constant, will represent an infinite number of points. These points are said to form a system. An equation which represents all the points satisfying a given condition, must contain an arbitrary comstant, for there is an infinite number of points satisfying a single condition. Hence a single geometric condition defines a system of points.

24. Systems of parallel points. In paragraph 10, parallel points are defined as points that have the smme slope.

Let P be the point whose equation is $Au + Bv + C = 0$. The slope of P is, then, $-\frac{A}{B}$, by paragraph 16, and the equation $Au + Bv + \lambda = 0$ represents points parallel to P since they have the same slope.

Also $Au + Bv + \lambda = 0$ represents all the point s parallel to P. For any such point is determined by a line $[u_1, v_1]$ and the slope $-\frac{A}{B}$. Suppose $[u_1, v_1]$ is one of the lines on $Au + Bv + \lambda = 0$; then $Au_1 + Bv_1 + \lambda = 0$, and $\lambda = -(Au_1 + Bv_1)$. That is, λ may be so chosen that the locus $Au + Bv + \lambda = 0$ is on any line $[u_1, v_1]$.

Therefore the locus of the system of points parallel to $Au + Bv + C = 0$ is $Au + Bv + \lambda = 0$, where λ can take any values.

The condition for parallel points can also be expressed: Any two points,

$$A_1 u + B_1 v + C_1 = 0$$

and $$A_2 u + B_2 v + C_2 = 0$$ are

parallel if and only if, $A_1 : A_2 = B_1 : B_2$ or, this may be written, if $\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = 0$. (1)

25. <u>Systems of perpendicular points</u>. Perpendicular points are defined in paragraph 11, as two points, the product of whose slopes is -1. So by reasoning exactly similar to that of paragraph 24, the equation of the system of points perpendicular to $Au + Bv + C = 0$, is $Bu - Av + \lambda_2 = 0$.

The condition that two points be perpendicular can also be expressed: Any two points,

$$A_1 u + B_1 v + C_1 = 0,$$

and $A_2 u + B_2 v + C_2 = 0$ are perpendicular,

if and only if, $A_1 : B_1 = -B_2 : A_2$, or if,

$$A_1 A_2 + B_1 B_2 = 0. \qquad (2)$$

26. <u>Systems of points on the line common to two points</u>. The system of points on the line common to the two points P_1 : $A_1 u + B_1 v + C_1 = 0$ (a)

and P_2: $A_2 u + B_2 v + C = 0$, (b)

is expressed by the equation

$$A_1 u + B_1 v + C_1 + \lambda (A_2 u + B_2 v + C_2) = 0 \qquad (c)$$

where λ is an arbitrary constant.

Proof: Let $l_1 : [u_1, v_1]$ be the line common to P_1 and P_2. Then u_1 and v_1 satisfy equations (a) and (b) above, and therefore equation (c). So

$$\lambda = -\frac{A_1 u_1 + B_1 v_1 + C_1}{A_2 u_1 + B_2 v_1 + C_2}.$$ That is, λ may be so chosen that equation (c) is any point on the line common to P_1 and P_2.

27. <u>Condition that three points are on a line</u>. The condition that three points are on a line, is evidently that the coordinates of the line satisfy the equations of the three points.

Let the equations of the three points be

$$A_i u + B_i v + C_i = 0, \quad (i = 1, 2, 3)$$

Then
$$A_1 u_1 + B_1 v_1 + C_1 = 0$$
$$A_2 u_1 + B_2 v_1 + C_2 = 0$$
$$A_3 u_1 + B_3 v_1 + C_3 = 0, \quad \text{must all be true.}$$

The condition for the simultaneous solution of these three equations is:

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = 0 \qquad (3)$$

28. <u>Equation of a point midway between two points.</u>

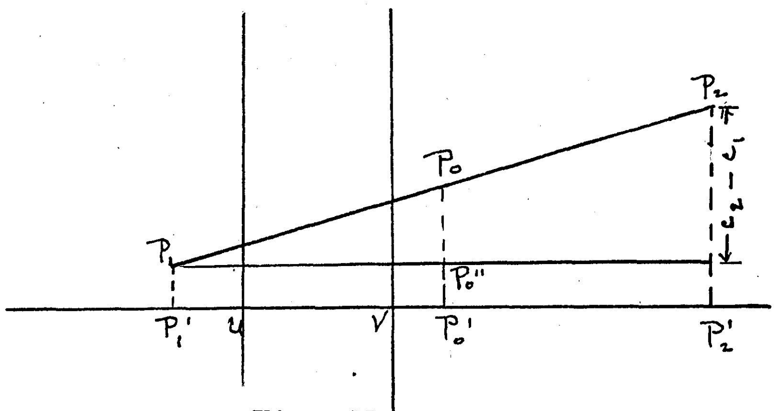


Fig. 31

Let P_1: $a_1 u + b_1 v + c_1 = 0, \quad (a_1 + b_1 = 1)$

and P_2: $a_2 u + b_2 v + c_2 = 0, \quad (a_2 + b_2 = 1)$

be two points, and let the equation of P_0 midway between P_1 and P_2 be:

$$a_0 u + b_0 v + c_0 = 0, \quad (a_0 + b_0 = 1).$$

Then $c_0 = P_0P'_0 = \frac{1}{2}(c_1 + c_2)$ and

$ka_0 = UP'_0$ by (18) and (16), paragraph 20. Then $ka_0 = UP'_1 + P'_1P'_0$

$$= UP'_1 + \tfrac{1}{2}P'_1P'_2$$

$$P'_1P'_2 = P'_1U + UV + VP'_2$$

$$= k[-a_1 + (1 - b_2)]$$

$$= k(-a_1 + a_2)$$

Then $ka_0 = ka_1 + \frac{1}{2}k(-a_1 + a_2)$

$$a_0 = \tfrac{1}{2}(a_1 + a_2)$$

Similarly $b_0 = \frac{1}{2}(b_1 + b_2)$

Substituting, and retaining the fractions, that the equation may be in normal form, the equation of the point midway between $a_1u + b_1v + c_1 = 0$ and $a_2u + b_2v + c_2 = 0$, is

$$\tfrac{1}{2}(a_1 + a_2)u + \tfrac{1}{2}(b_1 + b_2)v + \tfrac{1}{2}(c_1 + c_2) = 0 \quad (4)$$

To find the equation of a point that divides P_1P_2 in the ratio $m_1:m_2$ the work is similar, with

$$P'_1P'_0 = \frac{m_1}{m_1 + m_2} \cdot P'_1P'_2$$

$$a_0 = \frac{m_2a_1 + m_1a_2}{m_1 + m_2} \qquad b_0 = \frac{m_2b_1 + m_1b_2}{m_1 + m_2}$$

In figure 31, $P''_0P_0 : c_2 - c_1 = m_1 : m_1 + m_2$

$$c_0 = P''_0P_0 + c_1 = \frac{m_2c_1 + m_1c_2}{m_1 + m_2}$$

So the equation of the point dividing the distance from $a_1u + b_1v + c_1 = 0$, to $a_2u + b_2v + c_2 = 0$, in the ratio $m_1 : m_2$ is

$$\frac{(m_2a_1 + m_1a_2)u}{m_1 + m_2} + \frac{(m_2b_1 + m_1b_2)v}{m_1 + m_2} + \frac{m_2c_1 + m_1c_2}{m_1 + m_2} = 0 \qquad (5)$$

Section VI

The area of a triangle

29, <u>Area of a triangle</u>. First, in terms of the coefficients of the equations of its vertices.

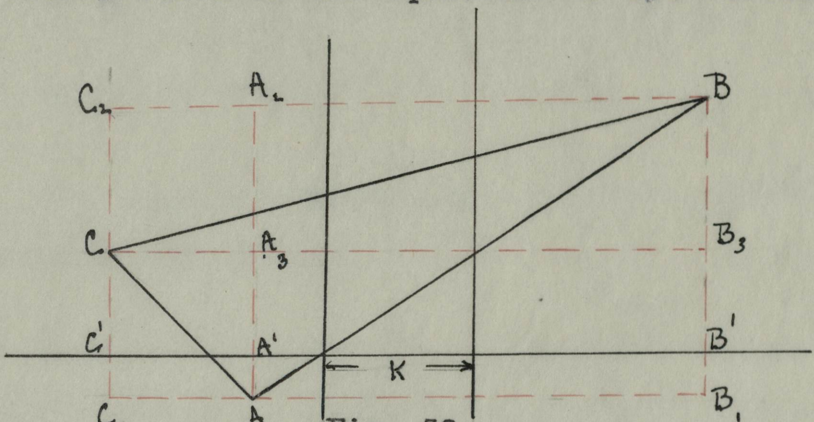


Fig. 32

Let A be $a_1u + b_1v + c_1 = 0$, $a_1 + b_1 = 1$

B be $a_2u + b_2v + c_2 = 0$, $a_2 + b_2 = 1$

C be $a_3u + b_3v + c_3 = 0$, $a_3 + b_3 = 1$

Then, by geometric considerations, and the substitutions of paragraph 20, (16), (17), and (18),

$\triangle ABC = \square C_1B_1BC_2 - [\triangle AB_1B + \triangle BC_2C + \triangle CC_1A]$

Then, substituting values,

$$\triangle ABC = (B_1B'+B'B)(C'U+UB') - \tfrac{1}{2}[(B_1B'+B'B)(A'U+UB')$$
$$+ (CC'+C'C_2)(C'U+UB') + (C_1C'+C'C)(C'U+UA')]$$
$$= (c_1 - c_2)(-ka_3 + ka_2) - \tfrac{1}{2}[(c_1 - c_2)(-ka_1 + ka_2)$$
$$+ (c_3 - c_2)(-ka_3 + ka_2) + (c_1 - c_3)(-ka_3 + ka_1)]$$
$$= \tfrac{1}{2}k[a_1c_3 + a_2c_1 + a_3c_2 - a_1c_2 - a_2c_3 - a_3c_1]$$
$$= \tfrac{1}{2}k\begin{vmatrix} a_1 & 1 & c_1 \\ a_2 & 1 & c_2 \\ a_3 & 1 & c_3 \end{vmatrix} = \tfrac{1}{2}k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & c_3 & c_3 \end{vmatrix} \qquad (1)$$

Since $a_i + b_i = 1$

30. Area of a triangle. In terms of the coordinates of its sides.

In Figure 32, let BC, CA, and AB have the coordinates $[u_1, v_1]$, $[u_2, v_2]$, and $[u_3, v_3]$, respectively. Then, by substituting in the point equations,

BC, on B and C, gives $\begin{cases} a_2u_1 + b_2v_1 + c_2 = 0 & (a) \\ a_3u_1 + b_3v_1 + c_3 = 0 & (b) \end{cases}$

CA, on C and A, gives $\begin{cases} a_3u_2 + b_3v_2 + c_3 = 0 & (c) \\ a_1u_2 + b_1v_2 + c_1 = 0 & (d) \end{cases}$

AB, on A and B, gives $\begin{cases} a_1u_3 + b_1v_3 + c_1 = 0 & (e) \\ a_2u_3 + b_2v_3 + c_2 = 0 & (f) \end{cases}$

Solve for a_1 and b_1 in (d) and (e),

$$a_1 = \frac{c_1(v_2 - v_3)}{u_2v_3 - u_3v_2} \; ; \qquad b_1 = \frac{c_1(u_3 - u_2)}{u_2v_3 - u_3v_2}$$

Similarly,

$$a_2 = \frac{c_2(v_3 - v_1)}{u_3 v_1 - u_1 v_3} \quad ; \qquad b_2 = \frac{c_2(u_1 - u_3)}{u_3 v_1 - u_1 v_3}$$

$$a_3 = \frac{c_3(v_1 - v_2)}{u_1 v_2 - u_2 v_1} \quad ; \qquad b_3 = \frac{c_3(u_2 - u_1)}{u_1 v_2 - u_2 v_1}$$

Now, since $a_i + b_i = 1$, the values of c_i can be found:

$$c_1 = \frac{u_2 v_3 - u_3 v_2}{u_3 - u_2 + v_2 - v_3}$$

$$c_2 = \frac{u_3 v_1 - u_1 v_3}{u_1 - u_3 + v_3 - v_1}$$

$$c_3 = \frac{u_1 v_2 - u_2 v_1}{u_2 - u_1 + v_1 - v_2}$$

By substituting these values in determinant (1), and expanding, then reducing back to detreminant form, the area may be written:

$$S = \frac{\frac{1}{2}k \begin{vmatrix} u_1 & v_1 & 1 \\ u_2 & v_2 & 1 \\ u_3 & v_3 & 1 \end{vmatrix}^2}{\begin{vmatrix} 1 & u_2 - v_2 \\ 1 & u_3 - v_3 \end{vmatrix} \cdot \begin{vmatrix} 1 & u_3 - v_3 \\ 1 & u_1 - v_1 \end{vmatrix} \cdot \begin{vmatrix} 1 & u_1 - v_1 \\ 1 & u_2 - v_2 \end{vmatrix}} \qquad (2)$$

Section VII

Transformation of Coordinates

31. Methods of transforming coordinates.

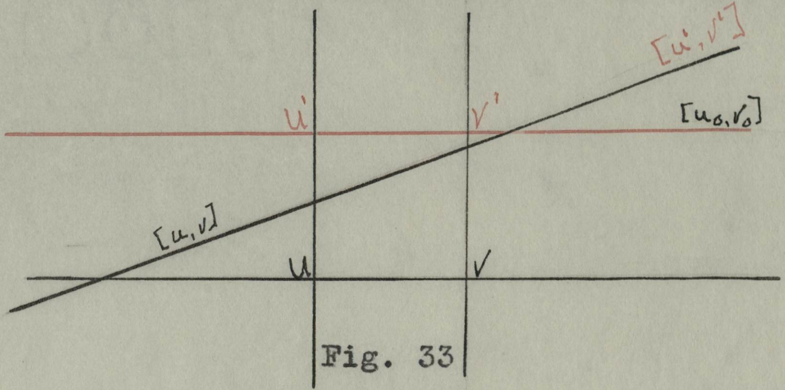


Fig. 33

I If the origins are moved parallel to UV, and in such a way that UU' = VV' = m = -c, then the equations of the new origins are:

$U' : v + c = 0, \quad V' : u + c = 0.$

The coordinates of a line [u,v] referred to the new axes, are: $u' = u - m, \quad v' = v - m$. (1a)

Then $u = u' + m, \quad v = v' + m$. (1b)

A point P: $Au + Bv + C = 0$, becomes

$$Au' + Bv' + Am + Bm + C = 0 .$$

II. If the origins are moved on the same axis, so that the length UV = k , becomes U'V' = k', ($k' \lessgtr k$), let the equations of the new origins be:

$U' : \quad au + bv = 0, \qquad a + b = 1$

$V' : \quad du + ev = 0, \qquad d + e = 1$

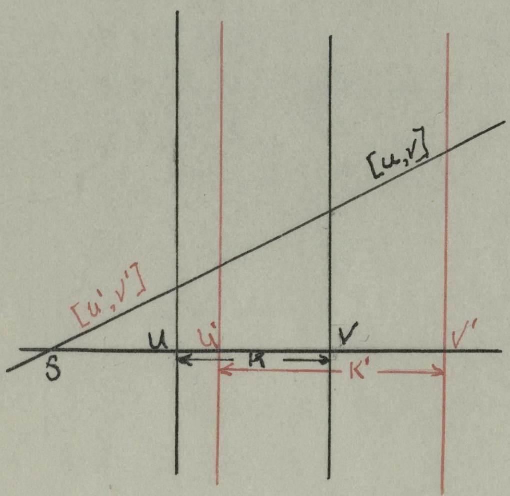


Fig. 34

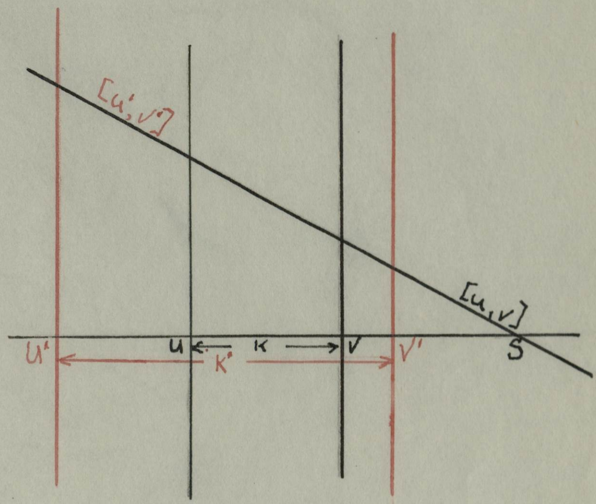


Fig. 35

Lengths on the axis, by paragraph 20, (16) and (17),

$UU' = ka, \qquad U'V = kb, \qquad UV = k$

$UV' = kd, \qquad V'V = ke, \qquad U'V' = k'$

$k' = U'V' = U'U + UV' = -ka + kd = k(d - a)$

In either figure,

$u' : u = SV' : SV \; ; \qquad v' : v = SU' : SU$

Then,

$u' - u : u = VV' : SV \; ; \quad v' - v : v = UU' : SU$

Also,

$SV : u = (SV - k) : v \; ; \quad SU : v = (SU + k) : u$

$$SV = \frac{ku}{u - v} \qquad\qquad SU = \frac{kv}{u - v}$$

Substituting these values,

$$(u' - u) : u = -ke(u - v) : ku$$

$$(v' - v) : v = +ka(u - v) : kv.$$

Solving for u' and v',

$$u' = (1 - e)u + ev \; ; \qquad v' = au + (1 - a)v$$

$$= du + ev \qquad\qquad = au + bv \qquad (2a)$$

Solving equations (2a) for u and v,

$$u = \frac{-bu' + ev'}{ae - bd} \qquad v = \frac{au' - dv'}{ae - bd}$$

$$= \frac{au' - dv' - (u' - v')}{a - d} \qquad = \frac{au' - dv'}{a - d} \qquad (2b)$$

To effect transformations I and II as one, the points can be taken:

U' : $au + bv + c = 0$, V' : $du + ev + c = 0$.

First shift to origins $au + bv = 0$, $du + ev = 0$, then translate through $-c$. The new coordinates become:

$$u' = du + ev + c \, , \qquad v' = au + bv + c \qquad (3)$$

III. If the axis and origins are moved to any arbitrary position in the plane.

Let the arbitrary points to which U and V are to be moved be:

U': $au + bv + c = 0$, $(a + b = 1)$

V': $du + ev + f = 0$, $(d + e = 1)$

and let the coordinates of the new axis with reference to the old axis, be $[u_o, v_o]$.

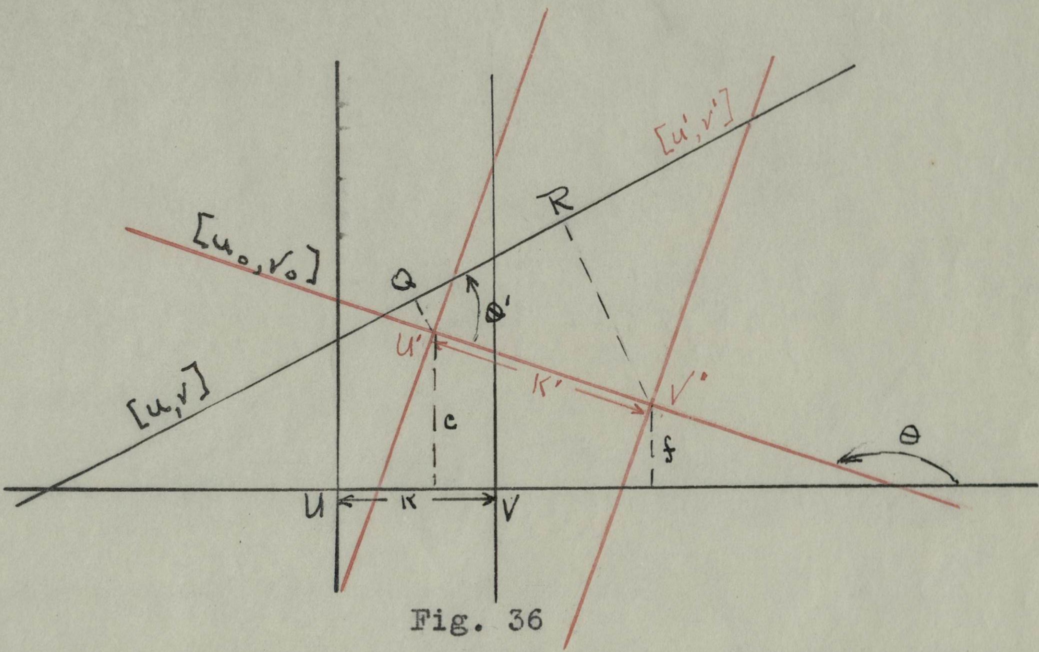


Fig. 36

In the figure,

$$\tan \alpha = \frac{u_o - v_o}{k} = \frac{f - c}{ka - kd} \quad ; \quad u_o - v_o = \frac{f - c}{a - d}$$

(Since $c = -h$, by paragraph 20, (18), then $f - c$ is positive.)

α' is the angle from $[u_o, v_o]$ to $[u, v]$, so by paragraph 7, (2),

$$\tan \phi' = \frac{k[(u_0 - v_0) - (u - v)]}{k^2 + (u_0 - v_0)(u - v)}$$

$$= \frac{k[(f - c) - (a - d)(u - v)]}{k^2(a - d) + (f - c)(u - v)} = \frac{u' - v'}{k'} \qquad \text{(a)}$$

Drop perpendiculars $U'Q$ and $V'R$ on $[u', v']$, then by paragraph 22, (21),

$$\frac{u'}{v'} = \frac{V'R}{U'Q} = \frac{du + ev + f}{au + bv + c}$$

$$\frac{u'}{u' - v'} = \frac{du + ev + f}{(d-a)(u-v) + (f-c)} \qquad \text{(b)}$$

$$\frac{v'}{u' - v'} = \frac{au + bv + c}{(d-a)(u-v) + (f-c)} \qquad \text{(c)}$$

From (a), (b), (c),

$$u' = \frac{kk'(du + ev + f)}{k^2(a-d) + (f-c)(u-v)} \; ; \; v' = \frac{kk'(au + bv + c)}{k^2(a-d) + (f-c)(u-v)} \qquad (4)$$

The solution for u and v in terms of u' and v',

$$u = -\frac{k^2(a-d)[(u'-v') - (au'-dv')] + (f-c)(cu'-fv') + kk'[(c-f) - (cd-fa)]}{kk'(a-d) + (f-c)(u'-v')}$$

$$v = \frac{k^2(a-d)(au'-dv') - (f-c)(cu'-fv') + kk'(cd-fa)}{kk'(a-d) + (f-c)(u'-v')} \qquad (5)$$

Note that the substitutions for u' and v' are in terms of the new <u>origins</u>. In the Cartesian transformations, the values of x' and y' are in terms of the new <u>axes</u>.

Section VIII

Equations of the Conics

32. <u>Equation of a circle</u>. A circle is generated by a line which turns about a point at a constant distance from the point.

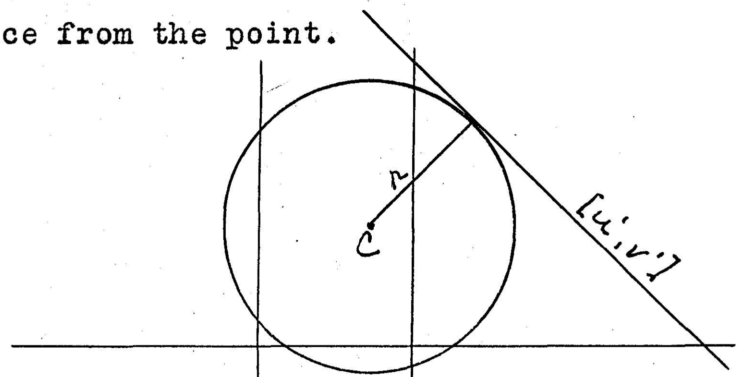


Fig. 37

Let the point C, be $au + bv + c = 0$, $(a + b = 1)$, and the line which generates the circle, be $[u',v']$, the distance from the point to the line be <u>r</u>. Then by (21), paragraph 22,

$$r = \frac{k(au' + bv' + c)}{\pm\sqrt{k^2 + (u'-v')^2}}$$

That is, by squaring and dropping the primes,

$$r^2(u-v)^2 - k^2(au + bv + c)^2 + k^2r^2 = 0 \qquad (1)$$

or, collecting like powers,

$$(k^2a^2 - r^2)u^2 + 2[k^2a(1-a) + r^2]uv + [k^2(1-a)^2 - r^2]v^2$$
$$+ 2k^2cau + 2k^2c(1-a)v + k^2(c^2 - r^2) = 0. \qquad (1a)$$

If the axis is so placed as to be the diameter of the circle, with the lines of reference, therefore, tangents, the center becomes of the form $au + bv = 0$, $(a + b = 1,\ a = b)$, that is the center is the mid-point of UV, $u + v = 0$, and $k = 2r$, so the equation of the circle, under these conditions becomes,

$$uv = r^2 \qquad (2)$$

33. Equations of central conics. Apollonius, Book III, proposition 66,[1] states the following:

"If in a central conic, parallel tangents be drawn at the extremities of a fixed diameter, and if both tangents be met by any variable tangent, the rectangle under the intercepts on the parallel tangents is constant, being equal to the square on half the parallel diameter; that is, the diameter conjugate to that joining the points of contact."

This property has been used by the writer in the derivation of the equations of the ellipse and the hyperbola. The tangents are taken perpendicular to the diameter, the major axis; The conjugate diameter is, then, the minor axis.

[1] Heath's Edition, page cxxx. (See bibliography.)

34. Equation of an ellipse.

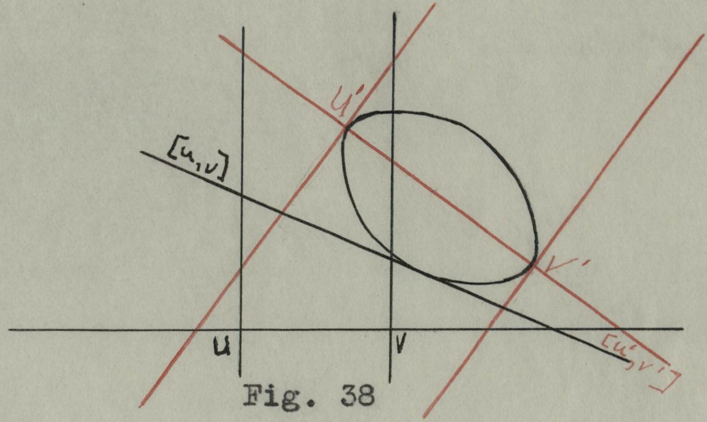


Fig. 38

Let the ellipse have the ends of its major axis,

U': $au + bv + c = 0$ $(a + b = 1)$

V': $du + ev + f = 0$ $(d + e = 1)$

the points being referred to the axis UV.

We can then consider U'V' as the axis of a new set of coordinates, and the intercepts by any tangent on the lines of reference, which are the tangents at the vertices, are $V'A = u'$, and $U'B = v'$; $k' = 2A$. If $2B$ is the minor axis, Apollonius's theorem gives $u'v' = B^2$. (See paragraph 33.)

That would be the equation if referred to U'V' as axis. By equations (4), paragraph 31, when referred to UV as axis,

$$u' = \frac{kk'(du + ev + f)}{k^2(a-d)+(f-c)(u-v)} \; ; \; v' = \frac{kk'(au + bv + c)}{k^2(a-d)+(f-c)(u-v)}$$

Putting k' equal to 2A, and substituting these values in equation (1), $u'v' = B^2$, gives

$$\frac{4A^2k^2(au + bv + c)(du + ev + f)}{[k^2(a-d) + (f-c)(u-v)]^2} = B^2$$

This reduces to

$$[4A^2k^2ad - B^2(f-c)^2]u^2 + 2[2A^2k^2(1-2ad) + B^2(f-c)^2]uv$$
$$+ [4A^2k^2ad - B^2(f-c)^2]v^2 + 2k^2[2A^2(fa+cd) - B^2(a-d)(f-c)]u$$
$$+ 2k^2[2A^2\{(f+c)-(fa+cd)\} + B^2(a-d)(f-c)]v$$
$$+ k^2[4A^2cf - B^2k^2(a-d)^2] = 0 \qquad (3)$$

This is the most general form of the equation of an ellipse. The equation is in simplest form when U' and V' are at U and V, $k = k'$; that is, $k = 2A$. Then U': $au + bv + c = 0$, is $v = 0$; and V': $du + ev + f = 0$, is $u = 0$; that is, $a = c = 0$, $b = 1$; $e = f = 0$, $d = 1$. Substituting these values in the general equation (3), it becomes

$$uv = B^2 \qquad (4)$$

This is the same form as that used at the start when the ellipse was referred to the origins U' and V'.

<u>Corollary</u> If $A = B$ in the equation of the ellipse,

we have the equation of a circle, in terms of the equations of the points at the ends of a diameter, instead of in terms of the center, and a given radius, as in paragraph 32. The equation becomes:

$$\left[4k^2ad - (f-c)^2\right]u^2 + 2\left[2k^2(1-2ad) + (f-c)^2\right]uv$$
$$+ \left[4k^2ad - (f-c)^2\right]v^2 + 2k^2(f+c)u + 2k^2(f+c)v$$
$$+ k^2\left[4fc - k^2(a-d)^2\right] = 0 \qquad (5)$$

35. Equation of a hyperbola.

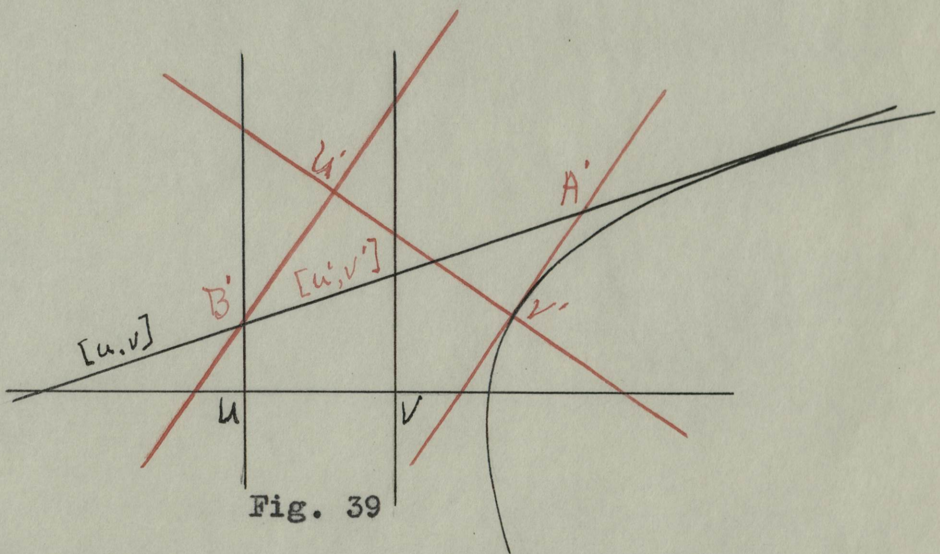


Fig. 39

Let the hyperbola have the ends of transverse axis,

U': $au + bv + c = 0$, $(a + b = 1)$

V': $du + ev + f = 0$, $(d + e = 1)$, the points being referred to axis UV. We can then, as in the case of the ellipse, consider $U'V'$ as the axis of a new set of coordinates. $A'B'$ is any tangent cutting the lines of reference, which are the tangents at the vertices. The intercepts are, then, $V'A' = u'$, $U'B' = v'$; also $k' = 2A$. Call the conjugate axis 2B.

Then $u'v' = -B^2$, since u' and v' have opposite signs. Substituting, as in the derivation of the equation of an ellipse, the equation of the hyperbola becomes:

$$\begin{aligned}&[4A^2k^2ad + B^2(f-c)^2]u^2 + 2[2A^2k^2(1-2ad) - B^2(f-c)^2]uv \\ &+ [4A^2k^2ad + B^2(f-c)^2]v^2 + 2k^2[2A^2(fa+cd)+B^2(a-d)(f-c)]u \\ &+ 2k^2[2A^2\{(f+c) - (fa+cd)\} - B^2(a-d)(f-c)]v \\ &+ k^2[4A^2cf + B^2k^2(a-d)^2] = 0 \qquad (6)\end{aligned}$$

This is the most general form of the equation of the hyperbola. The equation is in simplest form when U' and V' are at U and V, $k' = k$, that is, $k = 2A$.

U': $au + bv + c = 0$, becomes $v = 0$; $a = c = 0$, $b = 1$.

V': $du + ev + f = 0$, becomes $u = 0$; $b = f = 0$, $d = 1$.

Substituting these values in equation (6), it becomes:

$$uv = -B^2 \qquad (7)$$

36. <u>The equation of a parabola</u>. A parabola may be defined as a conic, having the following property: The perpendicular from the focus of a parabola on any tangent, meets it on the tangent at the vertex.[1]

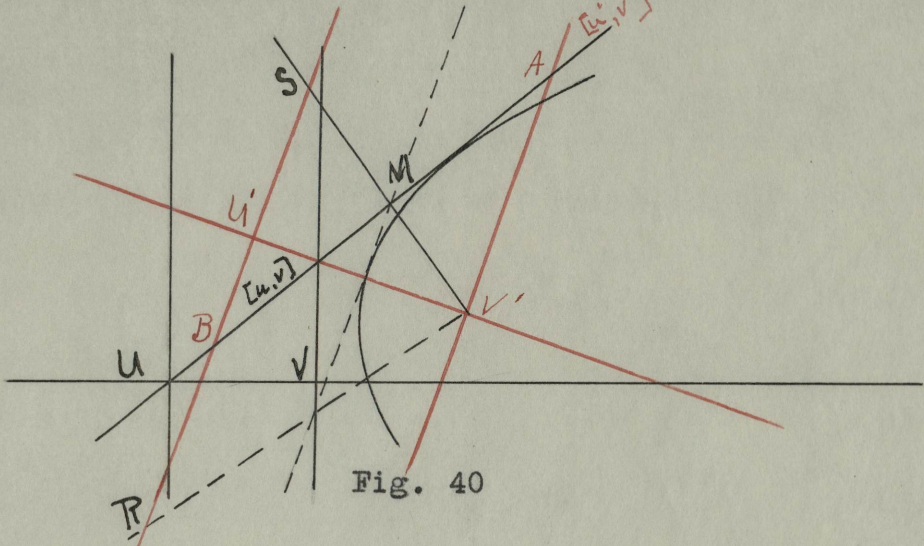


Fig. 40

Let V': du + ev + f = 0, be at the focus of the parabola, and U': au + bv + c = 0, be the point where the axis of the parabola meets the directrix. We can consider U'V' the axis of a new set of co-oridnates; then V'A = u', U'B = v', the U-reference line is the directrix, and the V-reference line is the line on the focus, perpendicular to the axis. U'V' = 2p.

Using the definition above, let M be the point where the perpendicular from the focus meets the tan-

[1] Ashton, Analytic Geometry, Article 78, (14)

gent. Extend V'M to meet U'B at S . Draw V'R parallel to AB . Then, by congruent triangles,

$$BS = V'A = u' ; \qquad U'S = v' + u',$$

$$U'R = U'B + BR = U'B + AV' = v' - u' .$$

Since SV'R is a right angle,

$$\overline{U'V'}^2 = RU' \cdot U'S; \quad \text{or,} \quad (u' + v')(u' - v') = k'^2$$

That is, $u'^2 - v'^2 = k'^2 = 4p^2.$

To put this into terms of the coordinates referred to axis UV, substitute the values of u' and v' from paragraph 31, (4), and the equation becomes:

$$\frac{k^2k'^2\left[(du+ev+f) + (au+bv+c)\right]\cdot\left[(du+ev+f) - (au+bv+c)\right]}{\left[k^2(a-d) + (f-c)(u-v)\right]^2} = k'^2$$

This reduces to:

$$\left[k^2(d^2-a^2) - (f-c)^2\right]u^2 + 2(f-c)^2uv + \left[k^2(a^2-d^2) - (f-c)^2\right]v^2$$
$$+ 2k^2\left[2fd-fa-cd\right]u + 2k^2\left[(f-c) + (2f-c)(a-d)\right]v + k^2\left[(f^2-c^2)\right.$$
$$\left.- k^2(a-d)^2\right] = 0. \qquad (8)$$

In the simplest form, U' and V' are at U and V, so $a = c = 0$, $b = 1$; $e = f = 0$, $d = 1$;
$k = k' = 2p$.

Substituting these values, the equation becomes:

$$u^2 - v^2 = k^2 = 4p^2. \qquad (9)$$

37. Equation of the point of tangency.

Since the equations of all the conics as derived, are of the second in the coordinates of the tangents, they may all be written as the general equation of the second degree:

$$Au^2 + 2Huv + Bv^2 + 2Gu + 2Fv + C = 0.$$

The slope of the tangent at any point is,

$$\frac{dv}{du} = -\frac{Au + Hv + G}{Hu + Bv + F} .$$

By the line-slope formula, paragraph 13, (2), the tangent whose coordinates are $[u_1, v_1]$ will meet the curve in the point

$$\frac{v - v_1}{u - u_1} = -\frac{Au_1 + Hv_1 + G}{Hu_1 + Bv_1 + F} .$$

This reduces to:

$$Au_1 u + H(u_1 v + v_1 u) + Bv_1 v + Gu + Fv$$
$$- (Au_1^2 + 2Hu_1 v_1 + Bv_1^2 + Gu_1 + Fv_1) = 0.$$

Since $[u_1, v_1]$ is on the curve,

$$Au_1^2 + 2Hu_1 v_1 + Bv_1^2 + Gu_1 + Fv_1 = -C - Gu_1 - Fv_1 ,$$

so the equation of the point of tangency to the conic becomes:

$$Au_1 u + H(u_1 v + v_1 u) + Bv_1 v + G(u + u_1) + F(v + v_1) + C = 0$$

38. Summary.

The most general form of the equation of all conics, in terms of the coordinates of a generating tangent, is seen to be that of the general equation of the second degree, in the coordinates:

$$Au^2 + 2Huv + Bv^2 + 2Gu + 2Fv + C = 0.$$

The Circle:

$$(k^2a^2 - r^2)u^2 + 2[k^2a(1-a) + r^2]uv + [k^2(1-a)^2 - r^2]v^2$$
$$+ 2k^2cau + 2k^2c(1-a)v + k^2(c^2 - r^2) = 0. \qquad (1a)$$

or:
$$[4k^2ad - (f-c)^2]u^2 + 2[2k^2(1-2ad) + (f-c)^2]uv$$
$$+[4k^2ad - (f-c)^2]v^2 + 2k^2(f+c)(u+v) + [k^2\{4fc - k^2(a-d)^2\}] = 0 \qquad (5)$$

The Ellipse:

$$[4A^2k^2ad - B^2(f-c)^2]u^2 + 2[2A^2k^2(1-2ad) + B^2(f-c)^2]uv$$
$$+[4A^2k^2ad - B^2(f-c)^2]v^2 + 2k^2[2A^2(fa+cd) - B^2(a-d)(f-c)]u$$
$$+ 2k^2[2A^2\{(f+c) - (fa+cd)\} + B^2(a-d)(f-c)]v$$
$$+ k^2[4A^2fc - B^2k^2(a-d)^2] = 0. \qquad (3)$$

The Hyperbola:

$$[4A^2k^2ad + B^2(f-c)^2]u^2 + 2[2A^2k^2(1-2ad) - B^2(f-c)^2]uv$$
$$+[4A^2k^2ad + B^2(f-c)^2]v^2 + 2k^2[2A^2(fa+cd) + B^2(a-d)(f-c)]u$$
$$+ 2k^2[2A^2\{(f+c) - (fa+cd)\} - B^2(a-d)(f-c)]v$$
$$+ k^2[4A^2fc + B^2k^2(a-d)^2] = 0. \qquad (6)$$

The Parabola:

$$[k^2(d^2 - a^2) - (f-c)^2]u^2 + [k^2(a^2 - d^2) - (f-c)^2]v^2$$
$$+ 2(f-c)^2uv + 2k^2[2fd-fa-cd]u + 2k^2[(f-c) + (2f-c)(a-d)]v$$
$$+ k^2[(f^2 - c^2) - k^2(a-d)^2] = 0. \qquad (8)$$

If the general equation, $F[u,v] = 0$, can be separated into two factors, each will be linear, and therefore the equation of a point. The conic then, degenerates into two points, if:

$$\begin{vmatrix} A & H & F \\ H & B & G \\ F & G & C \end{vmatrix} = 0.$$

The analytic work necessary to derive this result, is of course identical with that found for the corresponding proof in an ordinary analytic geometry, and is not repeated here.

By examination of the relation between the coefficients in the equations of the conics, it develops that for the:

Ellipse,	$A + 2H + B = 4A^2k^2$
Circle,	$A + 2H + B = 4k^2$
Hyperbola,	$A + 2H + B = 4A^2k^2$
Parabola,	$A + 2H + B = 0$

That is, the sum of the coefficients of the second degree terms is greater than zero for ellipse, circle, and hyperbola, and equal to zero for the parabola. If the curve is a conic, these conditions hold.

It is still to be proved that the general equation of the second degree, always represents a conic.

This has been proved by Dr. Schwering[1] and the summary of his proof is given here.

39. <u>General equation of the second degree</u>.

This discussion is a translation of the principal part of Dr. Schwering's proof, edited somewhat.

Take as the general equation of the second degree, $a_{11}u^2 + 2a_{12}uv + a_{22}v^2 + 2a_{13}u + 2a_{23}v + a_{33} = 0.$ (1)

If we put with it the equation of a point,

$$Au + Bv + C = 0, \qquad (2)$$

there are in general, two solutions, $[u_1, v_1]$ and $[u_2, v_2]$ so, in general there can be two tangents drawn from point (2) to the curve (1).

Let one tangent to the curve be $[u_0, v_0]$. If there is a parallel tangent, its coordinates will be

1 Schwering. Liniencoordinaten, pp.45-51.

$[u_0 + \alpha, v_0 + \alpha]$. If u_0, v_0 are finite,

$$[2a_{11}u_0 + 2a_{12}(u_0 + v_0) + 2a_{22}v_0 + 2a_{13} + 2a_{23}] + \alpha(a_{11} + 2a_{12} + a_{22}) = 0$$

Then α has a definite, finite value, except when

$$a_{11} + 2a_{12} + a_{22} = 0. \qquad (3)$$

If u_0, v_0 are infinite, (that is, if $[u_0, v_0]$ is parallel to the lines of reference; if only one of the coordinates is infinite, the tangent is one of the lines of reference. ed.), if their quotient m is finite, the result is the same, for in that case it must be that $a_{11}m^2 + 2a_{12}m + a_{22} = 0$, and this equation gives two values for m, which may be identical. So in general, there are two tangents parallel to the lines of reference. But, in case (3) holds true, $m = 1$, and then we have the line parallel to the lines of reference at an infinite distance, and since we can identify the equation with that derived for the parabola, we now state that if

$a_{11} + 2a_{12} + a_{22} = 0$, curve (1) is a parabola. (Discussion omitted.)

If $a_{11} + 2a_{12} + a_{22} \neq 0$, see if the curve can be put in the form:

$$(Au + Bv + C)(Du + Ev + F)e^2 = m[e^2 + (u - v)^2] \qquad (5)$$

($A + B = D + E = 1$, e is the join of the origins.).

Set $$e^2 = a_{11} + 2a_{12} + a_{22} \neq 0. \tag{6}$$

Then, equating coefficients between (1) and (5),

$$\left.\begin{aligned} ADe^2 - m &= a_{11} \\ BEe^2 - m &= a_{22} \\ (AE + BD)e^2 + 2m &= 2a_{12} \\ (AF + CD)e^2 - &= 2a_{13} \\ (BF + CE)e^2 &= 2a_{23} \\ CFe^2 - me^2 &= a_{33} \end{aligned}\right\} \tag{7}$$

The sum of the first three is e^2, which justifies (6). Also

$$(F + C)e^2 = 2a_{13} + 2a_{23} \tag{8}$$

$$CFe^2 = me^2 + a_{33} \tag{8a}$$

Similarly, $$(A + D)e^2 = 2a_{12} + 2a_{11} \tag{9}$$

$$ADe^2 = m + a_{11} \tag{9a}$$

For any given value of <u>m</u> one can find F and C, A and D, ~~and~~ as roots of quadratics. But these values are dependent, for:

$$F + C = \frac{2(a_{13} + a_{23})}{e^2}$$

$$AF + CD = \frac{2a_{13}}{e^2}$$

Then $$\left.\begin{aligned} C(A - D)e^2 &= 2(Aa_{13} + Aa_{23} - a_{13}) \\ F(A - D)e^2 &= 2(-Da_{13} - Da_{23} + a_{13}) \end{aligned}\right. \tag{10}$$

Subtracting, and substituting from (9) for A + D,

$$(C - F)(A - D)e^4 = 4(a_{11} + a_{12})(a_{13} + a_{23}) - 4a_{13}e^2 \quad (11)$$

From (8) and (8a),

$$(C - F)^2 e^4 = 4(a_{13} + a_{23})^2 - 4e^4 m - 4e^2 a_{33}$$

Similarly,

$$(A - D)^2 e^4 = 4(a_{12} + a_{13})^2 - 4e^4 m - 4e^2 a_{11}$$

Combining the last two equations with (11),

$$[(a_{11} + a_{12})(a_{13} + a_{23}) - a_{23}e^2]^2$$
$$= [(a_{11} + a_{12})^2 - e^4 m - e^2 a_{11}][(a_{13} + a_{23})^2 - e^4 m - e^2 a_{33}] \quad (12)$$

By the substitution of (6), this reduces to

$$e^4 m^2 - m[e^2 a_{12}^2 - e^2 a_{11} a_{22} - e^2 a_{33}^2 + (a_{13} + a_{23})^2]$$
$$- (a_{11}a_{23}^2 + a_{22}a_{13}^2 + a_{33}a_{12}^2 - 2a_{12}a_{13}a_{23} - a_{11}a_{22}a_{33}) = 0 \quad (13)$$

This equation has two real roots, as we shall proceed to show.

$$\text{Let: } \Delta = a_{11}a_{23}^2 + a_{22}a_{13}^2 + a_{33}a_{12}^2 - 2a_{12}a_{13}a_{23} - a_{11}a_{22}a_{33} \quad (14)$$

Then one can easily verify the following:

$$\left.\begin{aligned} &(a_{11}a_{23} - a_{13}a_{12})^2 - (a_{11}a_{22} - a_{12}^2)(a_{11}a_{33} - a_{13}^2) = a_{11}\Delta \\ &(a_{22}a_{13} - a_{12}a_{23})^2 - (a_{11}a_{22} - a_{12}^2)(a_{22}a_{33} - a_{23}^2) = a_{22}\Delta. \\ &(a_{11}a_{23} - a_{13}a_{12})(a_{22}a_{13} - a_{12}a_{23}) \\ &\qquad - (a_{11}a_{22} - a_{12}^2)(a_{13}a_{23} - a_{12}a_{33}) = -a_{12}\Delta \end{aligned}\right\} (15)$$

By adding the first two, and subtracting the third multiplied by two, this reduces to:

$$\left[(a_{11} + a_{12})a_{23} - (a_{12} + a_{22})a_{13}\right]^2 - (a_{11}a_{22} - a_{12}^2)\left[e^2 a_{33} - (a_{13} + a_{23})^2\right] = e^2\Delta \qquad (16)$$

Let us set,

$$(a_{11} + a_{12})a_{23} - (a_{12} + a_{22})a_{13} = p$$

$$a_{11}a_{22} - a_{12}^2 = q$$

(16) then becomes

$$p^2 - q\left[e^2 a_{33} - (a_{13} + a_{23})^2\right] = e^2\Delta$$

and (13) becomes

$$e^4 m^2 + m\left[e^2 q + \frac{p^2 - e^2\Delta}{q}\right] - \Delta = 0,$$

and the discriminant is:

$$\frac{(p^2 + e^2q^2 - e^2\Delta)^2 + 4e^4q^2\Delta}{q^2}.$$

Since this is the same as

$$\frac{(p^2 - e^2q^2 - e^2\Delta)^2 + 4e^2p^2q^2}{q^2},$$

it is a positive quantity, and <u>m</u> can have two values:

$$m = \frac{-p^2 - e^2q^2 + e^2\Delta \pm \sqrt{(p^2 - e^2q^2 - e^2\Delta)^2 + 4e^2p^2q^2}}{2e^4q}$$

Substitute in the $(C - F)^2$ equation:

$$(C - F)^2 e^4 = \frac{4(e^2\Delta - p^2)}{q} - 4e^4 m$$

Then,

$$(C-F)^2 e^4 = \frac{-p^2 + e^2q^2 + e^2\Delta \pm \sqrt{(p^2 - e^2q^2 - e^2\Delta)^2 + 4e^2p^2q^2}}{q}$$

Since in these roots $|b^2 - 4ac| > b^2$, one value of $(C - F)$ is positive, and one negative. Hence there is one value of $\underline{m}$ for which $(C - F)$ is real. Then C and F, and by (11), A and D also, are real.

Therefore equation (1) can be factored into form (5) in one and only one way, if the factors are to be real.

But this form is a conic, either ellipse or hyperbola, according to whether the value of $\underline{m}$ that makes $(C - F)$ real, is positive or negative.

The foci are:

$$Au + Bv + C = 0$$
$$Du + Ev + F = 0 .$$

The value of $\underline{m}$ is b^2 for the ellipse, and $-b^2$ for the hyperbola. The value of $\underline{m}$ which cannot be used, gives two imaginary foci, which, however, we exclude from this discussion. (Dr. Schwering's exclusion.)

If $\Delta = 0$, then $m = 0$, and the conic degenerates to a pair of points.

Note A

The coordinate system used in this paper, was obtained as the dual of the Cartesian system, by the following parallels:

Cartesian Point-coor.	Dual Line-coor.
1. Assume two lines on a point, with a given angle between them. (Having assumed the axes, the origin and the angle are determined.)	1. Assume two points on a a line with a given join between them. (Having assumed the origins, the axis and the join are determined.)
2. Choose arbitrarily a direction on the axis in the order of the points.	2. Choose arbitraily a direction on the origins in order of the lines.
3. Choose an arbitrary unit length on the axes.	3. Choose an arbitrary unit angle on the origins.
4. To fix a point, (a,b) lay off a units on the X-axis, and draw a line parallel to the Y-axis; then lay off b units on the Y-	4. To fix a line, $[\alpha, \beta]$ lay off α units on the U-origin, and draw a line on the V-origin perpendicular to o; then β units on the V-

axis and draw a line parallel to the X-axis. The point determined by the two lines parallel to the Y- and X-axes, respectively, at the distances a and b from the axes, is the point (a,b).

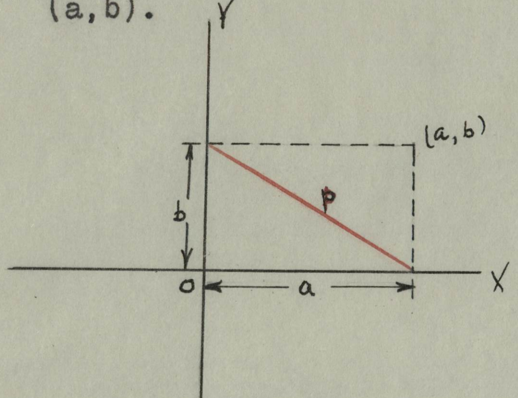


Fig. 41

origin and draw a line on the U-origin perpendicular to o. The line determined by the two points parallel (see Par. 10) to V- and U-origins, respectively, at the angles α and β from the origins, is the line $[\alpha, \beta]$.

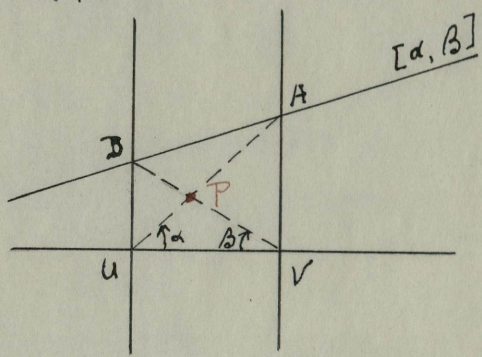


Fig. 42

As a matter of practical convenience in using a pair of numbers directly in measuerment, it has seemed better to use as the coordinates α and β , not the angles, which are the direct dual of the Cartesian system, but the lengths VA and UB , which are determined by the angles, and are propoortional to their tangents. (see paragraph 5.)

It is to preserve the Cartesian convention as to direction of lines, that the angle β is chosen as in the figure, rather than as the angle made with the positive direction of UV which would make the coordinate β negative.

As coordinates, then, we have α = VA, β = UB. The line diEections are taken as in the Cartesian system.

Note B

There are some interesting equations connecting this coordinate system with the Cartesian rectangular system.

Let the axes and origins be set up as indicated in the accompanying figures.

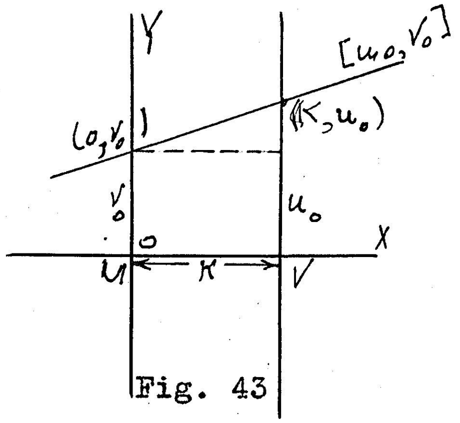


Fig. 43

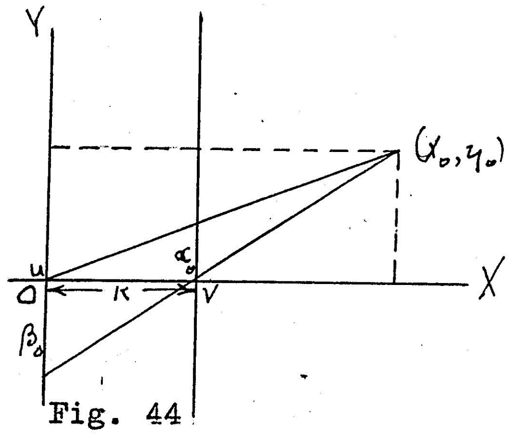


Fig. 44

Given: Line $[u_0, v_0]$ to find its equation in the Cartesian system.

$$\frac{y - v_0}{x} = \frac{u_0 - v_0}{k}$$

$$y = \frac{u_0 - v_0}{k} x + kv_0$$

Given: Point (x_0, y_0) to find its equation in the line system.

$$\frac{u}{\alpha_0} + \frac{v}{\beta_0} = 1$$

$$\frac{\alpha_0}{k} = \frac{y_0}{x_0}; \quad \frac{\beta_0}{\alpha_0} = \frac{x_0}{x_0 - k}$$

$$\frac{x_0 u}{ky_0} + \frac{(x_0 - k)v}{ky_0} = 1$$

$$v = -\frac{x_0}{x_0 - k} u + \frac{ky_0}{x_0 - k}$$

Given: Point P, whose equation is, $\frac{u}{u_0} + \frac{v}{v_0} = 1$

To find its coordinates in Cartesian system.

Given: Line l, whose equation is, $\frac{x}{x_0} + \frac{y}{y_0} = 1$

To find its coordinates in line system.

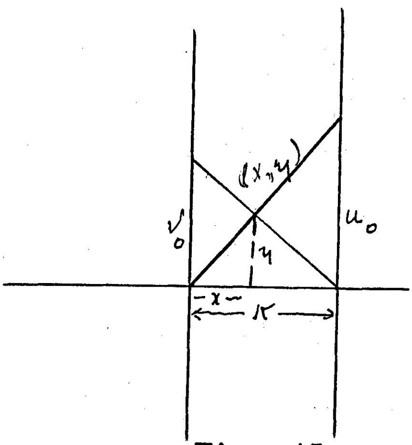


Fig. 45

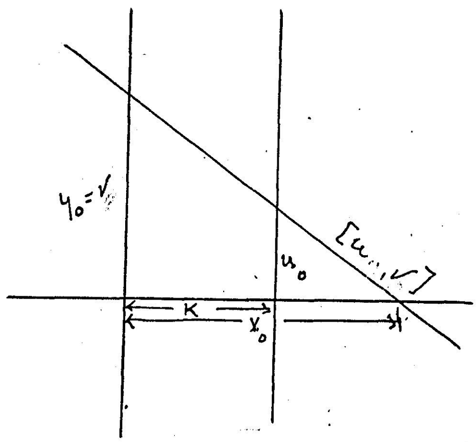


Fig. 46

From figure 45,

$$\frac{y}{u_o} = \frac{x}{k} ; \frac{y}{y_o} = \frac{k - x}{k}$$

$$\frac{v_o}{u_o} = \frac{x}{k - x}$$

$$x = \frac{kv_o}{u_o + v_o} ; y = \frac{u_o v_o}{u_o + v_o}$$

From figure 46,

$$v = y_o$$

$$\frac{u}{y_o} = \frac{x_o - k}{x_o}$$

$$u = \frac{y_o(x_o - k)}{x_o}$$

Note C

In the Cartesian system, the coordinates of any point (a,b), determine not the point only, but also a line -- the line AB in the figure 47 -- which is the join on the intercept points A: (a,0) and B: (0,b) on the axes.

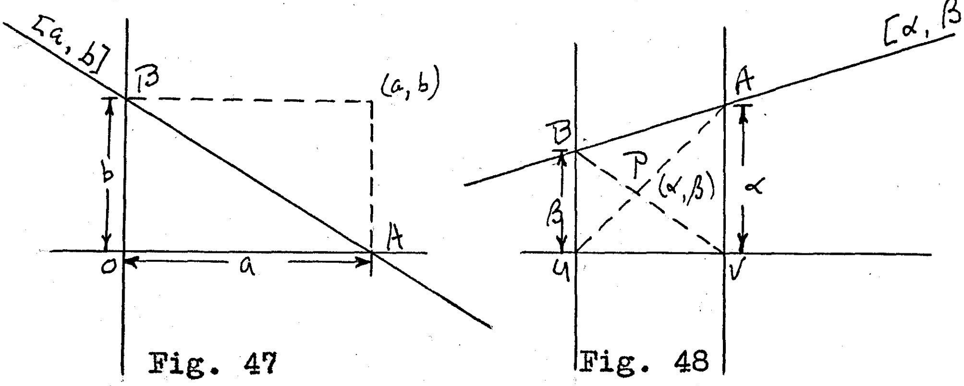


Fig. 47 Fig. 48

Likewise in the line system, the coordinates of any line [α, β] determine not only the line, but also a point -- the point P in figure 48 -- which is the intersection of the intercept lines UA: $[\alpha, 0]$ and VB: $[0, \beta]$ on the origins.

The point P and the line AB are duals, and it may be that through a study of this dualism, together with the dual line $[\alpha, \beta]$ and the point (a,b), an identical analytic geometry for point and line can be found.

Note D

The point P, of Note C, is a projective, non-perspective point with reference to the centers U and V, and when the coordinates (α, β) of the point are used in an equation of the first degree, the locus is a conic, as would be expected. A few graphs are appended, to show the method of plotting with this system.

It seems probable that further work on this will reveal a method for a graphical study of the general linear transformation, and perhaps show from the graph the relations necessary to give the different types of conics.

Note E

It must be noted that this system is a special case of the one set up by Veblen and Young in their Projective Geometry. The appended figure shows the special relations. It is an adaptation of one from Veblen and Young, in their discussion in volume I of their book, pages 169 ff.

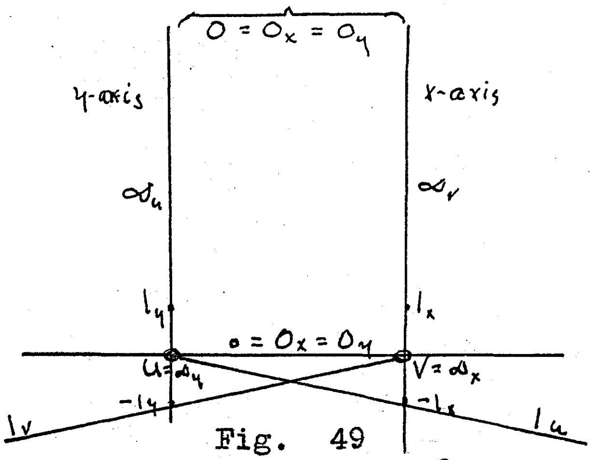


Fig. 49
[V. + Y. page 172.]

In line coordinates, the line represented by $[\alpha, \beta]$, join UV is $o = 0_u = 0_v$, The lines on U and V, (reference lines), are ∞_u and ∞_v respectively, and their intersection is P_∞.

In point coordinates, the point represented by (α, β), the origins U and V, are points on 1_∞, or, U is ∞_y and V is ∞_x, while $O = O_x = O_y$ is the intersection of the parallel lines 1_x and 1_y.

Bibliography

Apollonius of Perga: Treatise on Conic Sections. Edited by T. L. Heath, Cambridge University Press, 1896.

Ashton, Charles H. Plane and solid Analytic Geometry. Scribners, N.Y., 1902.

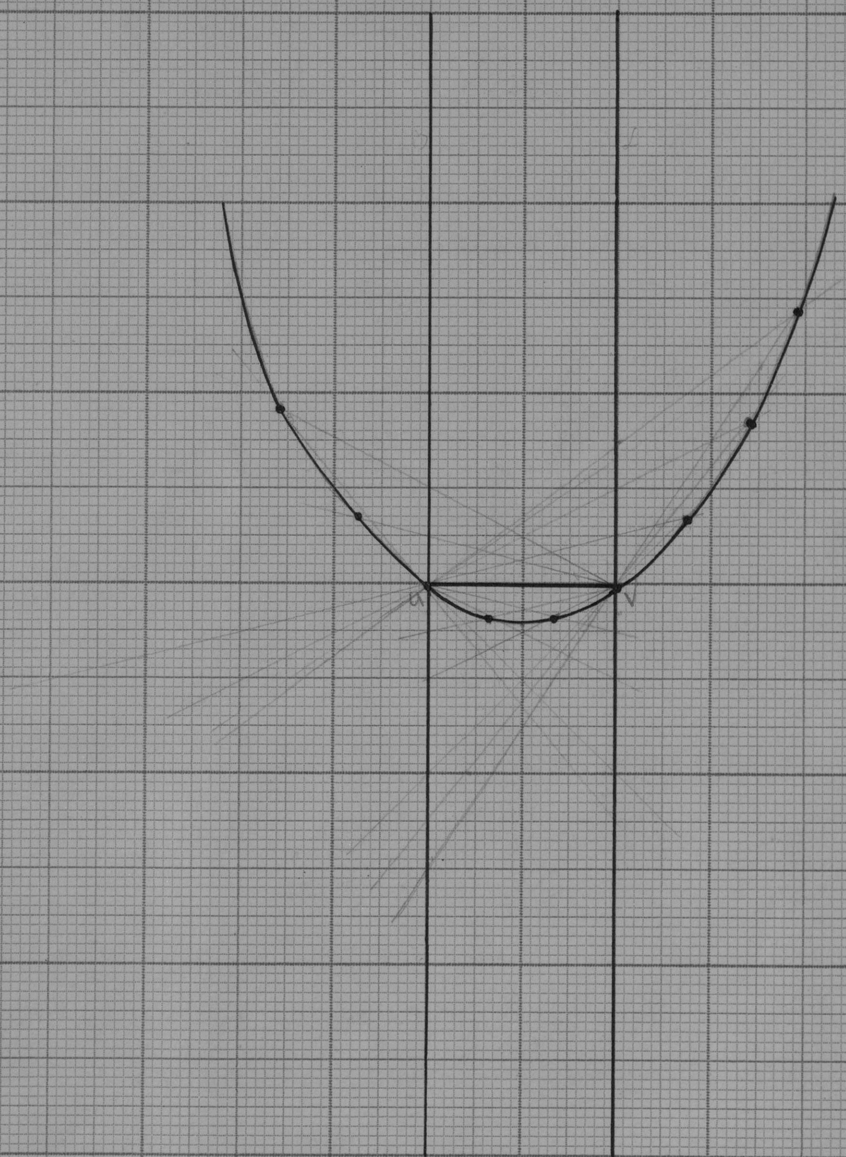
Schlegel, V. Ueber das dem Cartesischen reciproke Coordinatensystem. (in Der Zeitschrift fuer Mathematik u. Physik --Schloemilch-- Bd.23, s.195, 1876.)

Schwering, K Ueber ein besonderes Liniencoordinatensystem. (in Der Zeitschrift fuer M. u. P., --Schloemilch-- Bd. 21, s.278 ff., 1876.)

Schwering, K. Theorie und Anwendung der Liniencoordinaten in der analytische Geometrie der Ebene. Teubner, Leipzig, 1884.

Veblen and Young Projective Geometry. Volume 1, Ginn, N.Y., 1910.

$\alpha + \beta = -3$



$\alpha - \beta = 3$