$\Gamma(x)$ AND RELATED FUNCTIONS.

by

Paul Eberhart.

Bachelor of Science. Washburn College.
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Approved by

C. H. Ashton

Department of Mathematics.

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TABLE OF CONTENTS.

Page

INTRODUCTION.

The function $\Gamma(x)$ as a solution of the difference equation

$$y(x+1) = x\,y(x) \qquad (1)$$

has been investigated at length by Batchelder[1] and others. The second principal solution, $\overline{\Gamma}(x)$, has not been so completely investigated, although Batchelder does derive expressions for it in terms of the gamma function. Our object here is to find an expression for $\overline{\Gamma}(x)$ in the form of an infinite product directly from the difference equation and to study various related functions.

To do this we will make use of various results obtained by Batchelder. First, we will use the theorem on the solution of the general linear homogeneous equation of the first order stated by Batchelder[2] as follows:

"The linear homogeneous difference equation

$$y(x+1) - r(x)y(x) = 0 \qquad (2)$$

where the rational function $r(x)$ may be written

$$r(x) = x^{\mu}\left(c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + - - - - \right)$$

where μ is an integer and $c_0 \neq 0$, is satisfied formally by

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1. Batchelder, Paul M. "An Introduction to Linear Difference Equations." 1927.
2. Batchelder. Page 40.

the series

$$S(x) = x^{\mu x}\, c_0^x\, e^{-\mu x}\, x^{\frac{c_1}{c_0} - \frac{1}{2}\mu}\, (s_0 + \frac{s_1}{x} + - - -); \qquad (3)$$

there exist also two analytic solutions

$$h(x) = \lim_{n \doteq \infty} \frac{1}{r(x)} \frac{1}{r(x+1)} \cdots\cdots \frac{1}{r(x+n)}\, T(x+n+1), \qquad (4)$$

$$g(x) = \lim_{n \doteq \infty} r(x-1)r(x-2) \cdots\cdots r(x-n)\, T(x-n), \qquad (5)$$

where $t(x)$ is the sum of the first k terms of $S(x)$.

The solution $h(x)$ is analytic throughout the plane except for poles at the zeros of $r(x)$ and points congruent on the left; it vanishes at the poles of $r(x)$ and points congruent on the left; it is represented asymptotically by $S(x)$ in the sector $-\pi < \arg x < \pi$. The solution $g(x)$ is analytic throughout the plane except for poles at points congruent on the right to the poles of $r(x)$; it vanishes at points congruent on the right to the zeros of $r(x)$; it is represented asymptotically by $S(x)$ in the sector $0 < \arg x < 2\pi$. The solutions $h(x)$ and $g(x)$ are called the first principal solution and the second principal solution respectively. They are uniquely determined apart from a constant factor, by their asymptotic properties."

We will also frequently use the following relations between $\bar{\Gamma}(x)$ and $\Gamma(x)$.

$$\bar{\Gamma}(x) = (1 - e^{2\pi i x})\, \Gamma(x) \qquad (6)$$

$$\bar{\Gamma}(x) = \frac{-2\pi i e^{\pi i x}}{\Gamma(1-x)} \qquad (7)$$

THE FUNCTION $\overline{\Gamma}(x)$.

Solution of the Difference Equation.

Since the first particular solution, $\Gamma(x)$ of equation (1) comes from $h(x)$ in the general theorem, the second principal solution, $\overline{\Gamma}(x)$ must come from $g(x)$. We have $r(x)=x$, $\mu=1$, $c_0=1$, $c_1=c_2= - - - - - =0$, and

$$S(x) = x^{x-\frac{1}{2}}\, e^{-x}\, (s_0 + \frac{s_1}{x} + - - - - -). \qquad (8)$$

Using the first term of $S(x)$ as $T(x)$, as Batchelder has shown we may do, we have

$$T(x-n) = (x-n)^{x-n-\frac{1}{2}}\, e^{-x+n}\, s_0.$$

Therefore we have

$$g(x) = \overline{\Gamma}(x) = \lim_{n \doteq \infty} (x-1)(x-2) \cdots\cdots (x-n)(x-n)^{x-n-\frac{1}{2}} e^{-x+n} s_0.$$

To evaluate this limit, we divide by $\overline{\Gamma}(0)$.

$$\overline{\Gamma}(0) = \lim_{n \doteq \infty} (-1)(-2) \cdots\cdots (-n)(-n)^{-n-\frac{1}{2}}\, e^{n}\, s_0$$

and $$\frac{\overline{\Gamma}(x)}{\overline{\Gamma}(0)} = \lim_{n \doteq \infty} \frac{(x-1)(x-2) \cdots\cdots (x-n)(x-n)^{x-n-\frac{1}{2}}\, e^{-x+n}\, s_0}{(-1)\,(-2) \cdots\cdots (-n)(-n)^{-n-\frac{1}{2}}\, e^{n}\, s_0}$$

or $$\frac{\overline{\Gamma}(x)}{\overline{\Gamma}(0)} = \lim_{n \doteq \infty} \frac{(1-x)(2-x) \cdots (n-x)}{1 \quad 2 \quad \cdots \quad n} \left(\frac{x-n}{n}\right)^{x} \left(\frac{x-n}{-n}\right)^{-n-\frac{1}{2}} n^{x} e^{-x}$$

We can evaluate the terms $\underset{n\doteq\infty}{\text{Lim}} \left(\frac{x-n}{n}\right)^x$ and $\underset{n\doteq\infty}{\text{Lim}} \left(\frac{x-n}{-n}\right)^{-n-\frac{1}{2}}$ as follows:

$$\underset{n\doteq\infty}{\text{Lim}} \left(\frac{x-n}{-n}\right)^{-n-\frac{1}{2}} = \underset{n\doteq\infty}{\text{Lim}} \frac{1}{\left(1-\frac{x}{n}\right)^{n+\frac{1}{2}}} = \frac{1}{e^{-x}} = e^x. \quad (9)$$

$$\underset{n\doteq\infty}{\text{Lim}} \left(\frac{x-n}{n}\right)^x = (-1)^x$$

Then $$\frac{\overline{\Gamma}(x)}{\overline{\Gamma}(0)} = \underset{n\doteq\infty}{\text{Lim}} \frac{(1-x)(2-x)\cdots\cdots(n-x)\,n^x\,(-1)^x}{\lfloor n}$$

To determine the constant $\overline{\Gamma}(0)$ we make use of equation (7):

$$\overline{\Gamma}(x) = \frac{-2\pi i e^{\pi i x}}{\Gamma(1-x)} \quad (7)$$

Since $\Gamma(1)=1$, we have

$$\overline{\Gamma}(0) = -2\pi i.$$

Therefore $$\overline{\Gamma}(x) = 2\pi i(-1)^{x+1} \underset{n\doteq\infty}{\text{Lim}} \frac{(1-x)(2-x)\cdots(n-x)n^x}{\lfloor n}. \quad (10)$$

By the general theorem, $\overline{\Gamma}(x)$ is analytic throughout the plane, having first order zeros at 1, 2, 3, $\cdots$, n, $\cdots$. It has no poles in the finite plane, but has an essentially singular point at infinity.

The infinite product for $\overline{\Gamma}(x)$ may also be written in the form

$$\overline{\Gamma}(x) = 2\pi i(-1)^{x+1} \prod_{n=1}^{\infty} \left(\frac{n-x}{n}\right)\left(\frac{n+1}{n}\right)^x$$

$$\overline{\Gamma}(x) = 2\pi i(-1)^{x+1} \prod_{n=1}^{\infty} \left(1 - \frac{x}{n}\right)\left(1 + \frac{1}{n}\right)^{x} . \qquad (11)$$

A second infinite product for $\overline{\Gamma}(x)$ analogous to the second form for $\Gamma(x)$ may be obtained as follows:

$$\overline{\Gamma}(x) = 2\pi i(-1)^{x+1} \operatorname*{Lim}_{n=\infty} \frac{(1-x)(2-x)\cdots(n-x)n^{x}}{\lfloor n} \qquad (10)$$

$$\overline{\Gamma}(x) = 2\pi i(-1)^{x+1} \operatorname*{Lim}_{m=\infty} e^{x \log m} \prod_{n=1}^{m} \frac{n+x}{n}$$

Using the definition of γ given by

$$\gamma = \operatorname*{Lim}_{n=\infty} \left(1 + \tfrac{1}{2} + - - - - + \frac{1}{n-1} - \log n\right)$$

this equation may be written

$$\overline{\Gamma}(x) = 2\pi i(-1)^{x+1} e^{-x} \prod_{n=1}^{\infty} \left(1 - \frac{x}{n}\right) e^{\frac{x}{n}} . \qquad (12)$$

A Second Method of Solution.

The function $\Gamma(x)$ can also be found without making use of the asymptotic series S(x), or T(x). We have seen that g(x) has zeros at 1,2, ... Therefore we proceed to set up a product having these zero points and multiply it by exponentials (which have no zeros) in order to make it converge and satisfy the difference equation

$$y(x+1) - x\,y(x). \qquad (1)$$

We can do this, since by the factor theorem of Weierstrass[1] a function is determined by its zero points and poles, except for an exponential (or constant) multiplier.

The product

$$\prod_{n=1}^{\infty} \left(1 - \frac{x}{n}\right) e^{\frac{x}{n}}$$

converges and has the required zero points. To make it satisfy the difference equation, we multiply it by an exponential $e^{G(x)}$ and determine the form of G(x). We then have

$$\Gamma(x) = e^{G(x)} \prod_{n=1}^{\infty} \left(1 - \frac{x}{n}\right) e^{\frac{x}{n}}$$

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1. Weierstrass, K. "Ueber die Theorie der analytischen Facultaten." Journ. fur Math. Vol.51. pp.1-60. 1856.

where $G(x)$ is to be chosen so that

$$\Gamma(x+1) = x\,\Gamma(x).$$

Then $$e^{G(x+1)} \prod_{n=1}^{\infty} \left(1 - \frac{x+1}{n}\right) e^{\frac{x+1}{n}} = x\, e^{G(x)} \prod_{n=1}^{\infty} \left(1 - \frac{x}{n}\right) e^{\frac{x}{n}}$$

Dividing by the second side, and taking the limit of m terms:

$$\frac{e^{G(x+1)}}{x\, e^{G(x)}} \operatorname*{Lim}_{m=\infty} \frac{\prod_{n=1}^{m} \frac{n-x-1}{n} e^{\frac{x+1}{n}}}{\prod_{n=1}^{m} \frac{n-x}{n} e^{\frac{x}{n}}} = 1$$

$$e^{G(x+1)-G(x)} = x \operatorname*{Lim}_{m=\infty} \prod_{n=1}^{m} \frac{n-x}{n-x-1} e^{-\frac{1}{n}}$$

$$= \operatorname*{Lim}_{m=\infty} \frac{x(1-x)(2-x)\cdots(m-x)}{(-x)(1-x)\cdots(m-1-x)} e^{-\sum_{n=1}^{m} \frac{1}{n}}$$

$$= - \operatorname*{Lim}_{m=\infty} e^{\log (m-x) - \sum_{n=1}^{m} \frac{1}{n}}$$

$$= - \operatorname*{Lim}_{m=\infty} e^{\log (m+1) - \sum_{n=1}^{m} \frac{1}{n}} e^{\log \frac{m-x}{m+1}}$$

$$= - e^{-\gamma} \operatorname*{Lim}_{m=\infty} \frac{m-x}{m+1} = - e^{-\gamma}$$

$$e^{G(x+1)-G(x)} = e^{\log (-1) - \gamma}$$

We have then to solve the difference equation

$$G(x+1) - G(x) = \log (-1) - \gamma$$

The general solution is

$$G(x) = S\left[\log(-1) - \gamma\right] + c$$

$$G(x) = x\left[\log(-1) - \gamma\right] + c.$$

And $$e^{G(x)} = e^{x \log(-1)}\, e^{-\gamma x}\, e^{c} = (-1)^{x}\, e^{-x\gamma} e^{c}$$

Hence $$\overline{\Gamma}(x) = (-1)^{x}\, e^{-\gamma x}\, e^{c} \prod_{n=1}^{\infty} \left(1 - \frac{x}{n}\right) e^{\frac{x}{n}}$$

To determine the constant e^{c}, we have, setting $x = 0$,

$$\overline{\Gamma}(0) = e^{c}$$

But we have shown that

$$\overline{\Gamma}(0) = -2\pi i.$$

Therefore

$$\overline{\Gamma}(x) = 2\pi i(-1)^{x+1}\, e^{-\gamma x} \prod_{n=1}^{\infty} \left(1 - \frac{x}{n}\right) e^{\frac{x}{n}} \qquad (12)$$

which is the second form as obtained above.

If the function is found by this method, the other product form can easily be found in the same way as we change from one to the other in the gamma function.

Relation of $\Gamma(x)$ and Sin x.

As for the gamma function, we should expect to find a relation between $\Gamma(x)$ and the trigonometric functions. The function sin x has zeros at all positive and negative integers and at zero. $\Gamma(x)$ has zeros at all positive integers. $\Gamma(1-x)$ has zeros at zero and the negative integers. Consequently we are led to form the product

$$\Gamma(x)\,\Gamma(1-x)$$

and find its relation to sin x.

We have, using the first form of the infinite product

$$\Gamma(x) = 2\pi i(-1)^{x+1} \prod_{n=1}^{\infty} \left(1 - \frac{x}{n}\right)\left(1 + \frac{1}{n}\right)^{x} \qquad (11)$$

$$\Gamma(1-x) = -x\,\Gamma(-x) = -x\; 2\pi i(-1)^{-x+1} \prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right)\left(1 + \frac{1}{n}\right)^{-x}$$

Therefore $$\Gamma(x)\,\Gamma(1-x) = 4\pi^2 x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right)$$

But we have that $$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right)$$

or $$\frac{\sin \pi x}{\pi} = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right)$$

Hence $$\overline{\Gamma}(x)\overline{\Gamma}(1-x) = 4\pi \sin \pi x \qquad (13)$$

From this last relation we obtain

$$\overline{\Gamma}(\tfrac{1}{2})\overline{\Gamma}(\tfrac{1}{2}) = 4\pi \sin \tfrac{1}{2}\pi = 4\pi.$$

$$\overline{\Gamma}(\tfrac{1}{2}) = 2\sqrt{\pi}.$$

This is the same value that is obtained from the relation

$$\overline{\Gamma}(x) = (1 - e^{2\pi i x})\,\Gamma(x) \qquad (6)$$

$$\overline{\Gamma}(\tfrac{1}{2}) = 2\,\Gamma(\tfrac{1}{2}) = 2\sqrt{\pi}.$$

Another Relation Between $\Gamma(x)$ and $\overline{\Gamma}(x)$.

From the relation between $\overline{\Gamma}(x)$ and sin x as derived above, we are able to obtain another relation between $\Gamma(x)$ and $\overline{\Gamma}(x)$.

We have

$$\overline{\Gamma}(x)\,\overline{\Gamma}(1-x) = 4\pi \sin \pi x. \qquad (13)$$

Also $$\sin x = \frac{e^{ix} - e^{-ix}}{2i} = \frac{e^{2ix} - 1}{2ie^{ix}}$$

Hence $$\overline{\Gamma}(x)\,\overline{\Gamma}(1-x) = 4\pi \frac{1 - e^{2\pi ix}}{-2ie^{\pi ix}}$$

But $$\Gamma(x) = \frac{1}{1 - e^{2\pi ix}}\,\overline{\Gamma}(x) \qquad (6)$$

$$\Gamma(x)\,\overline{\Gamma}(1-x) = \frac{4\pi}{-2ie^{\pi ix}}$$

or $$\Gamma(x) = \frac{2\,ie^{-\pi ix}}{\overline{\Gamma}(1-x)} \qquad (14)$$

The Multiplication Theorem for $\bar{\Gamma}(x)$.

From "Gauss's multiplication theorem" for $\Gamma(x)$

$$\Gamma(nx) = \frac{n^{nx-\frac{1}{2}}}{(2\pi)^{\frac{n-1}{2}}} \prod_{k=0}^{n-1} \Gamma\left(x + \frac{k}{n}\right) \qquad (15)$$

and the relation between $\bar{\Gamma}(x)$ and $\Gamma(x)$ given by eq. (7), we can obtain the multiplication theorem for $\bar{\Gamma}(x)$.

Substituting (nx) for x in eq. (7), we obtain

$$\bar{\Gamma}(nx) = \frac{-2\pi i e^{\pi i n x}}{\Gamma(1-nx)} \qquad (16)$$

Substituting (-x) for x in eq. (15), we also obtain

$$\Gamma(-nx) = \frac{n^{-nx-\frac{1}{2}}}{(2\pi)^{\frac{n-1}{2}}} \prod_{k=0}^{n-1} \Gamma\left(-x + \frac{k}{n}\right)$$

If now we set (-nx) = z, we have from the difference equation for $\Gamma(x)$

$$\Gamma(-nx) = \Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{\Gamma(1-nx)}{-nx}$$

Hence $$\Gamma(1-nx) = \frac{-nx\, n^{-nx-\frac{1}{2}}}{(2\pi)^{\frac{n-1}{2}}} \prod_{k=0}^{n-1} \Gamma\left(-x + \frac{k}{n}\right) \qquad (17)$$

We have previously obtained the relation

$$\Gamma(x) = \frac{2\pi i e^{-\pi i x}}{\Gamma(1-x)} \tag{14}$$

Substituting $\left(-x + \frac{k}{n}\right)$ for x,

$$\Gamma\left(-x + \frac{k}{n}\right) = \frac{2\pi i e^{\pi i\left(x - \frac{k}{n}\right)}}{\Gamma\left(1 + x - \frac{k}{n}\right)}$$

The product in eq. (17) then becomes

$$\prod_{k=0}^{n-1} \Gamma\left(-x + \frac{k}{n}\right) = \left[2\pi i e^{\pi i x}\right]^n \prod_{k=0}^{n-1} \frac{e^{-\pi i \frac{k}{n}}}{\Gamma\left(1 + x - \frac{k}{n}\right)} \tag{18}$$

Equation (17) now becomes

$$\Gamma(1-nx) = -x\, n^{-nx+\frac{1}{2}} \left[2\pi i e^{\pi i x}\right]^n \prod_{k=0}^{n-1} \frac{e^{-\pi i \frac{k}{n}}}{\Gamma\left(1 + x - \frac{k}{n}\right)}$$

Using this expression for $\Gamma(1-nx)$, eq. (16) becomes

$$\Gamma(nx) = \frac{-2\pi i e^{\pi i n x} (2\pi)^{\frac{n-1}{2}}}{-x n^{-nx+\frac{1}{2}}} \frac{1}{\left[2\pi i e^{\pi i x}\right]^n} \prod_{k=0}^{n-1} e^{\pi i \frac{k}{n}} \Gamma\left(1 + x - \frac{k}{n}\right)$$

$$\Gamma(nx) = \frac{n^{nx-\frac{1}{2}}}{(2\pi)^{\frac{n-1}{2}}\, i^{n-1}} \frac{1}{x} \prod_{k=0}^{n-1} e^{\pi i \frac{k}{n}} \Gamma\left(1 + x - \frac{k}{n}\right)$$

Since $$\prod_{k=0}^{n-1} \Gamma\left(1 + x - \frac{k}{n}\right) = \prod_{k=1}^{n} \Gamma\left(x + \frac{k}{n}\right) .$$

$$i^{n-1} = (-1)^{\frac{n-1}{2}}$$

$$\prod_{k=0}^{n-1} e^{\pi i \frac{k}{n}} = - \prod_{k=1}^{n} e^{\pi i \frac{k}{n}} = -e^{\frac{k(n+1)}{2}\pi i}$$

we obtain as a final form for $\Gamma(nx)$

$$\Gamma(nx) = \frac{n^{nx-\frac{1}{2}}}{(2\pi)^{\frac{n-1}{2}}} \frac{(-1)}{x} e^{\frac{k(n+1)}{2}\pi i} \prod_{k=1}^{n} \Gamma\left(x + \frac{k}{n}\right). \qquad (19)$$

The Logarithmic Derivative of $\overline{\Gamma}(x)$.

We now proceed to find an expression for the logarithmic derivative of $\overline{\Gamma}(x)$. From the equation

$$\overline{\Gamma}(x) = 2\pi i(-1)^{x+1}\, e^{-\gamma x} \prod_{n=1}^{\infty} \frac{n-x}{n}\, e^{\frac{x}{n}} \qquad (12)$$

we obtain by taking the logarithm of both sides

$$\log \overline{\Gamma}(x) = \log 2\pi i + (x+1)\log(-1) - \gamma x + \sum_{n=1}^{\infty}\left(\frac{x}{n} + \log\frac{n-x}{n}\right)$$

Differentiating, and denoting $\overline{\Gamma}'(x)/\overline{\Gamma}(x)$ by $\overline{\Psi}(x)$

$$\overline{\Psi}(x) = \frac{\overline{\Gamma}'(x)}{\overline{\Gamma}(x)} = \log(-1) - \gamma + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n-x}\right)$$

$$\overline{\Psi}(x) = \log(-1) - \gamma + \sum_{n=1}^{\infty}\left(\frac{1}{x-n} + \frac{1}{n}\right)$$

For the multiple valued term

$$\log(-1) = \log 1 + i\ \mathrm{am}(-1) = i(2n+1)\pi$$

we may take πi, $(n=0)$, since $\overline{\Psi}(x)$ exists in the sector $0 < \mathrm{am}\ x < 2\pi$. The positive axis of reals is excluded since $\overline{\Psi}(x)$ has poles at the positive integers.

$$\overline{\Psi}(x) = \pi i - \gamma + \sum_{n=1}^{\infty}\left(\frac{1}{x-n} + \frac{1}{n}\right) \qquad (20)$$

$\overline{\Psi}(x)$ Defined by its Difference Equation.

From the equation

$$\overline{\Gamma}(x+1) = x\,\overline{\Gamma}(x)$$

we obtain by logarithmic differentiation

$$\frac{\overline{\Gamma}'(x+1)}{\overline{\Gamma}(x+1)} = \frac{1}{x} + \frac{\overline{\Gamma}'(x)}{\overline{\Gamma}(x)}$$

or

$$\overline{\Psi}(x+1) = \frac{1}{x} + \overline{\Psi}(x) \qquad (21)$$

Therefore $\overline{\Psi}(x)$ is a solution of the difference equation

$$y(x+1) - y(x) = \frac{1}{x} \qquad (22)$$

We now proceed to find a solution of this equation and to determine the additive constant (or periodic function) so that the solution is identical with $\overline{\Psi}(x)$.

The expansionsto the right and to the left are given by

$$y_r(x) = -\frac{1}{x} - \frac{1}{x+1} - \frac{1}{x+2} - - - - -$$

$$y_l(x) = \frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3} + - - - -$$

We have then, the two symbolic solutions:

$$y_r(x) = -\sum_{n=0}^{\infty} \frac{1}{x+n}$$

$$y_1(x) = \sum_{n=1}^{\infty} \frac{1}{x-n}$$

It is evident that neither of these solutions converge, so we attempt to find an additive constant which will make the sums finite. It is evident that the series

$$-\sum_{n=0}^{\infty}\left(\frac{1}{x+n} - \frac{1}{n+1}\right) = -\sum_{n=0}^{\infty} \frac{1-x}{(x+n)(n+1)}$$

converges for all values of x except $x = 0, -1, -2, \text{-----}$

The function defined by the series

$$-\sum_{n=0}^{\infty}\left(\frac{1}{x+n} - \frac{1}{n+1}\right)$$

is then the solution of the difference equation, as may be readily seen by substitution.

It is also evident that the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{x-n} + \frac{1}{n}\right)$$

converges except for $x = 1, 2, 3, - - -$, and satisfies the difference equation.

We may then write

$$\overline{\Psi}(x) = C_1 - \sum_{n=0}^{\infty}\left(\frac{1}{x+n} - \frac{1}{n+1}\right) \qquad (23)$$

and $$\overline{\Psi}(x) = C_1 + \sum_{n=0}^{\infty} \left(\frac{1}{x-n} + \frac{1}{n}\right) \qquad (24)$$

where C_1 and C_2 are constants or periodic functions, determined so that these functions coincide with $\overline{\Psi}(x)$.

From the relation

$$\overline{\Gamma}(x) = (1 - e^{2\pi i x})\,\Gamma(x) \qquad (6)$$

we obtain by logarithmic differentiation

$$\overline{\Psi}(x) = \frac{2\pi i}{1 - e^{-2\pi i x}} + \Psi(x) \qquad (25)$$

Also we have the following expression for the logarithmic derivative of the gamma function:

$$\Psi(x) = -\gamma - \sum_{n=0}^{\infty} \left(\frac{1}{x+n} - \frac{1}{n+1}\right)$$

Hence

$$\overline{\Psi}(x) = C_1 - \sum_{n=0}^{\infty} \left(\frac{1}{x+n} - \frac{1}{n+1}\right) = \frac{2\pi i}{1-e^{-2\pi i x}} - \gamma - \sum_{n=0}^{\infty} \left(\frac{1}{x+n} - \frac{1}{n+1}\right)$$

Therefore $$C_1 = \frac{2\pi i}{1 - e^{-2\pi i x}} - \gamma$$

and $$\overline{\Psi}(x) = \frac{2\pi i}{1 - e^{-2\pi i x}} - \gamma - \sum_{n=0}^{\infty} \left(\frac{1}{x+n} - \frac{1}{n+1}\right) \qquad (26)$$

To determine C_2 we have, by eq. (25)

$$\overline{\Psi}(x) = \frac{2\pi i}{1 - e^{-2\pi i x}} + \Psi(x) \qquad (25)$$

and in particular, $\overline{\Psi}(\frac{1}{2}) = \pi i + \Psi(\frac{1}{2})$.

from the equation

$$\Psi(x) = -\gamma - \sum_{n=0}^{\infty}\left(\frac{1}{x+n} - \frac{1}{n+1}\right) = -\gamma - \sum_{n=0}^{\infty}\frac{1-x}{(x+n)(n+1)}$$

we obtain $$\Psi(\tfrac{1}{2}) = -\gamma - \sum_{n=0}^{\infty}\frac{1}{(1+2n)(n+1)}$$

or $$\Psi(\tfrac{1}{2}) = -\gamma - \left[1 + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 5} + - - - -\right]$$

From equation (24), we obtain

$$\overline{\Psi}(\tfrac{1}{2}) = C_2 + \sum_{n=1}^{\infty}\frac{1}{n(1-2n)}$$

or $$\overline{\Psi}(\tfrac{1}{2}) = C_2 - \left[\frac{1}{1} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 5} + - - - -\right]$$

Substituting:

$$C_2 - \left[\frac{1}{1} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 5} + - -\right] = \pi i - \gamma - \left[1 + \frac{1}{2\cdot 3} + - -\right]$$

$$C_2 = \pi i - \gamma$$

$$\overline{\Psi}(x) = \pi i - \gamma + \sum_{n=1}^{\infty}\left(\frac{1}{x-n} + \frac{1}{n}\right) \qquad (27)$$

We then have two expressions for $\overline{\Psi}(x)$:

$$\overline{\Psi}(x) = \frac{2\pi i}{1 - e^{-2\pi i x}} - \gamma - \sum_{n=0}^{\infty}\left(\frac{1}{x+n} - \frac{1}{n+1}\right) \qquad (26)$$

$$\overline{\Psi}(x) = \pi i - \gamma + \sum_{n=1}^{\infty}\left(\frac{1}{x-n} + \frac{1}{n}\right) \qquad (27)$$

The second of these equations is the same form as found from the logarithmic differentiation of the infinite product for $\overline{\Gamma}(x)$.

It is evident that the second form for $\overline{\Psi}(x)$ is the more convenient. From the second form, we see that $\overline{\Psi}(x)$ has poles of the first order at $x = 1, 2, 3, - - -$. This also appears from the first form, since the term

$$\frac{2\pi i}{1 - e^{-2\pi i x}}$$

becomes infinite for these values, while the series remains finite. The first form, however, becomes indeterminate for $x = 0, -1, -2, - - -$, since both the exponential term and the series become infinite for these values but differ in sign.

Since $\overline{\Psi}(x)$ is the logarithmic derivative of $\overline{\Gamma}(x)$, we know, from function theory, that it has first order poles at the zeros of $\overline{\Gamma}(x)$ with residue (+1) since the zeros of $\overline{\Gamma}(x)$ are of the first order.

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1. Townsend, E. J. "Functions of a Complex Variable." (1915) pp. 296-297.

The Derivative of $\overline{\Psi}(x)$.

From the equation

$$\overline{\Psi}(x) = \pi i - \gamma + \sum_{n=1}^{\infty}\left(\frac{1}{x-n} + \frac{1}{n}\right) \qquad (27)$$

we obtain by differentiating

$$\frac{d}{dx}\overline{\Psi}(x) = -\sum_{n=1}^{\infty}\frac{1}{(x-n)^2}$$

$$\frac{d}{dx}\overline{\Psi}(x) = -\sum_{n=1}^{\infty}\frac{1}{(-x+n)^2} \qquad (28)$$

This series is real for real values of x. If we define, with Batchelder, a function

$$\phi(x,r,K) = \frac{1}{x^K} + \frac{r}{(x+1)^K} + \frac{r^2}{(x+2)^K} + - - - - \qquad (29)$$

we see by comparing (28) and (29) that

$$\frac{d}{dx}\psi(x) = -\phi(-x,1,2) \qquad (30)$$

By differentiating eq. (28), we likewise obtain

$$\frac{d^2}{dx^2}\overline{\Psi}(x) = -2\sum_{n=1}^{\infty}\frac{1}{(-x+n)^3}$$

$$\frac{d^2}{dx^2}\overline{\Psi}(x) = -2\,\phi(-x,1,3)$$

The function $\overline{\psi}(x)$ is a solution of the difference equation

$$y(x+1) - y(x) = \frac{1}{x} \tag{22}$$

By differentiation, we obtain

$$y'(x+1) - y'(x) = -\frac{1}{x^2} \tag{31}$$

Then $\frac{d}{dx}\overline{\psi}(x)$ is a solution of this equation, and in fact may be defined by it except for a periodic multiplier. From eq. (31), by expanding to the left, we obtain

$$y_1(x) = -\frac{1}{(x-1)^2} - \frac{1}{(x-2)^2} - \frac{1}{(x-3)^2} - - - - -$$

$$= -\sum_{n=1}^{\infty} \frac{1}{(x-n)^2}$$

Comparison with eq. (28) shows that

$$y_1(x) \equiv \frac{d}{dx}\overline{\psi}(x).$$

Value of $\overline{\Gamma}(x)$ for Real Values of x.

The value of $\overline{\Gamma}(x)$ for any particular x may be obtained by evaluating one of the products

$$\overline{\Gamma}(x) = 2\pi i(-1)^{x+1} \prod_{n=1}^{\infty} \left(1 - \frac{x}{n}\right)\left(1 + \frac{1}{n}\right)^{x} \qquad (11)$$

$$\overline{\Gamma}(x) = 2\pi i(-1)^{x+1}\, e^{-\gamma x} \prod_{n=1}^{\infty} \left(1 - \frac{x}{n}\right) e^{\frac{x}{n}} \qquad (12)$$

However, since we have tables for the gamma function, it is easier to calculate the value of $\overline{\Gamma}(x)$ from one of the relations:

$$\overline{\Gamma}(x) = (1 - e^{2\pi i x})\,\Gamma(x) \qquad (6)$$

$$\overline{\Gamma}(x) = \frac{-2\pi i\, e^{\pi i x}}{\Gamma(1-x)} \qquad (7)$$

Having obtained the values for any unit interval, $\overline{\Gamma}(x)$ for any other x may be found by means of the difference equation:

$$\overline{\Gamma}(x+1) = x\,\overline{\Gamma}(x).$$

For real values of x, we at once see from (6) or (7) that $\overline{\Gamma}\left(\pm\frac{2n+1}{2}\right)$ is real and from eq. (7) that $\overline{\Gamma}(-n)$ is imaginary. (In all of these expressions we take n as

a positive integer.) We have already noted that $\overline{\Gamma}(+n)$ is zero. These same results might also have been obtained in another way. We have defined $\overline{\Gamma}(0)$ as $(-2\pi i)$ and we have found that $\overline{\Gamma}(\frac{1}{2}) = 2\sqrt{\pi}$, hence, applying the difference equation, we see that for negative integers $\overline{\Gamma}(x)$ is imaginary, and for other multiples of one half is real. In fact from the two values

$$\overline{\Gamma}(0) = -2\pi i$$

$$\overline{\Gamma}(\tfrac{1}{2}) = 2\sqrt{\pi}$$

and the difference equation we obtain, where n is a positive integer:

$$\overline{\Gamma}(+n) = 0.$$

$$\overline{\Gamma}(-n) = \frac{2\pi i}{(-1)^n \lfloor n}$$

$$\overline{\Gamma}\left(\frac{2n+1}{2}\right) = \frac{2n-1}{2} \cdot \frac{2n-3}{2} \cdot \cdots \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot 2\sqrt{\pi} \tag{32}$$

$$\overline{\Gamma}\left(-\frac{2n+1}{2}\right) = \frac{-2}{2n+1} \cdot \frac{-2}{2n-1} \cdot \cdots \cdot \frac{-2}{3} \cdot \frac{-2}{1} \cdot 2\sqrt{\pi}$$

For any other real value of x, other than an integer or an integral multiple of one half, we see from (6) or (7) that $\overline{\Gamma}(x)$ is complex.

Graph of $\bar{\Gamma}(x)$ for Real Values of x.

Since the value of $\bar{\Gamma}(x)$ is in general complex for real values of x, if we wish to graph $\bar{\Gamma}(x)$ for such values, we must use the entire complex plane. The previous work has given us the points on the axis of reals and on the axis of imaginaries of such a graph. We may find other points from the relation

$$\bar{\Gamma}(x) = (1 - e^{2\pi ix})\,\Gamma(x) \qquad (6)$$

The graph of $\bar{\Gamma}(x)$ for real values of x from (-4) to (+4), at intervals of one tenth, has been obtained from this relation. The value of $\Gamma(x)$ was taken from Legendre's tables[1] and the exponential factor was calculated from the relation

$$e^{2\pi ix} = \cos 2\pi x + i \sin 2\pi x.$$

$$1 - e^{2\pi ix} = (1 - \cos 2\pi x) + i \sin 2\pi x.$$

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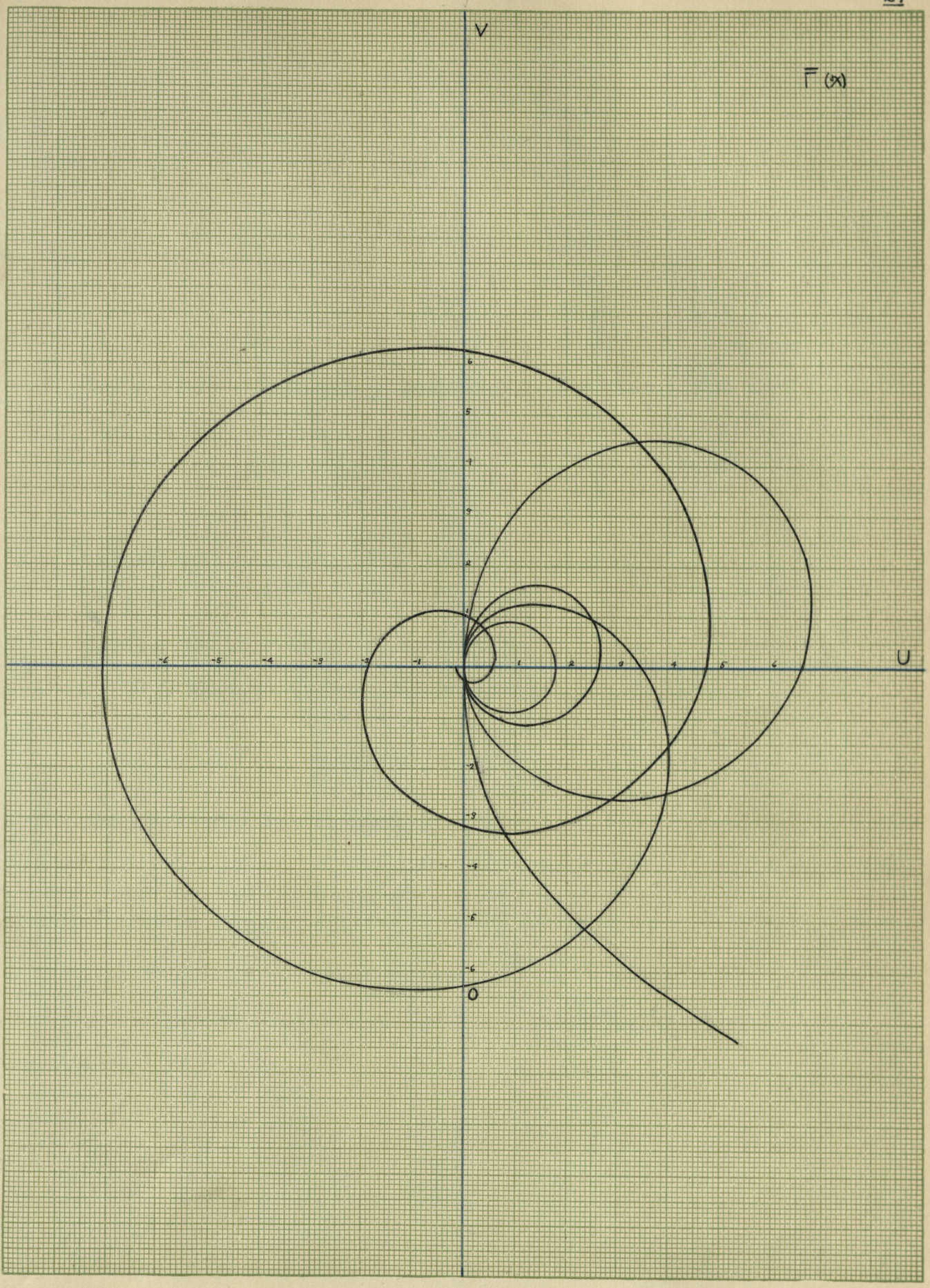
1. Legendre's Tables: "Tracts for Computers." Edited by Karl Pearsons. 1921. No. IV. "Tables of the Logarithms of the complete Γ-function to Twelve figures." (Originally computed by A.M.Legendre.)

The following fundamental set was obtained

x.	$\overline{\Gamma}(x)$.
0.0	- 6.28318 i
0.1	1.81675 - 5.60484 i
0.2	3.17218 - 4.36617 i
0.3	3.91602 - 2.84510 i
0.4	4.01269 - 1.30381 i
0.5	3.54491
0.6	2.69391 + 0.87533 i
0.7	1.69918 + 1.23453 i
0.8	0.80446 + 1.10720 i
0.9	0.20409 + 0.62813 i
1.0	0.00000

In the graph on the following page the point marked 0 corresponds to the zero value of x. As x takes on positive values from 0 to 1, $\overline{\Gamma}(x)$ takes on the values along this curve to the right to the origin. As x travels on to the right on the axis of reals, $\overline{\Gamma}(x)$ takes on the values on the successive curves through the origin, becoming zero for positive integral values of x, and each curve being larger than the previous one. For negative values, the curve starts at the same point and goes off in the opposite direction, forming a spiral-like curve about the origin.



THE FUNCTION W(x).

Solution of the Difference Equation.

We have seen that $\overline{\Gamma}(x)$ is, in general, complex for real values of x. By studying the expression

$$\overline{\Gamma}(x) = 2\pi i(-1)^{x+1} \prod_{n=1}^{\infty} \left(1 - \frac{x}{n}\right)\left(1 + \frac{1}{n}\right)^{x} \qquad (10)$$

we see that the imaginary values are due to the factors $2\pi i$ and $(-1)^x$. The function defined by the infinite product is then real for real values of x.

We will now consider the difference equation

$$y(x+1) = -x\, y(x) \qquad (33)$$

and show that the infinite product

$$\prod_{n=1}^{\infty} \left(1 - \frac{x}{n}\right)\left(1 + \frac{1}{n}\right)^{x}$$

is a solution of this equation.

From the general theorem the series which satisfies equation (33) is

$$S(x) = x^{x}\,(-1)^{x}\, e^{-x}\, x^{-\frac{1}{2}} \left(s_0 + \frac{s_1}{x} + - - -\right) \qquad (34)$$

Taking T(x) as the first term of S(x) we have

$$T(x+n) = (x+n)^{x+n-\frac{1}{2}}\, e^{-x-n}\, (-1)^{x+n}\, s_o$$

$$T(x-n) = (x-n)^{x-n-\frac{1}{2}}\, e^{-x+n}\, (-1)^{x-n}\, s_o$$

The two analytic solutions of eq. (33) are then

$$h(x) = \operatorname*{Lim}_{n=\infty} \frac{1}{(-x)}\,\frac{1}{(-x+1)} \cdots \frac{1}{(-x+n-1)} (x+n)^{x+n-\frac{1}{2}} e^{-x-n} (-1)^{x+n} s_o$$

$$g(x) = \operatorname*{Lim}_{n=\infty} (-x+1)(-x+2) \cdots (-x+n)(x-n)^{x-n-\frac{1}{2}}\, e^{-x+n} (-1)^{x-n}\, s_o$$

We shall here study the second of these principal solutions and denote it by W(x). We then have

$$W(x) = \operatorname*{Lim}_{n=\infty} (1-x)(2-x) \cdots (n-x)(x-n)^{x-n-\frac{1}{2}}\, e^{-x+n} (-1)^{x-n} s_o$$

and $$W(0) = \operatorname*{Lim}_{n=\infty} 1\cdot 2\cdot 3\cdot \cdots \cdot n\, (-n)^{-n-\frac{1}{2}}\, e^{n}\, (-1)^{-n}\, s_o$$

By division

$$\frac{W(x)}{W(0)} = \operatorname*{Lim}_{n=\infty} \frac{(1-x)(2-x) \cdots (n-x)}{\lfloor n} \left(\frac{x-n}{-n}\right)^{-n-\frac{1}{2}} \left(\frac{x-n}{n}\right)^{x} n^{x} e^{-x} (-1)^{x}$$

As for the function $\Gamma(x)$, eq. (9)

$$\operatorname*{Lim}_{n=\infty} \left(\frac{x-n}{-n}\right)^{-n-\frac{1}{2}} = e^{x}$$

Also $$\lim_{n=\infty} \left(\frac{x-n}{n}\right)^x (-1)^x = \lim_{n=\infty} \left(\frac{n-x}{n}\right)^x = 1.$$

Therefore $$\frac{W(x)}{W(0)} = \lim_{n=\infty} \frac{(1-x)(2-x)\cdots(n-x)\, n^x}{\lfloor n}$$

This infinite product is the same as that occurring in $\Gamma(x)$ and can be evaluated in the same way. Then, choosing the constant s_0 so that $W(0) = 1$ we have

$$W(x) = \prod_{n=1}^{\infty} \left(1 - \frac{x}{n}\right)\left(1 + \frac{1}{n}\right)^x \tag{36}$$

$$W(x) = e^{-\gamma x} \prod_{n=1}^{\infty} \left(1 - \frac{x}{n}\right) e^{\frac{x}{n}} \tag{37}$$

Solution of the Difference Equation by Weierstrass's Theorem.

We can solve the difference equation (33) by use of Weierstrass's Theorem. Since g(x) has zeros at the positive integers, W(x) must have the form

$$W(x) = e^{G(x)} \prod_{n=1}^{\infty} \left(1 - \frac{x}{n}\right) e^{\frac{x}{n}} \qquad (38)$$

The function G(x) must now be determined so that

$$W(x+1) = -x\, W(x)$$

$$W(0) = 1.$$

Then
$$\frac{e^{G(x+1)} \operatorname*{Lim}_{m\to\infty} \prod_{n=1}^{m} \left(1 - \frac{x+1}{n}\right) e^{\frac{x+1}{n}}}{-x\, e^{G(x)} \operatorname*{Lim}_{m\to\infty} \prod_{n=1}^{m} \left(1 - \frac{x}{n}\right) e^{\frac{x}{n}}} = 1.$$

$$e^{G(x+1)-G(x)} = -x \operatorname*{Lim}_{m\to\infty} \prod_{n=1}^{m} \frac{n-x}{n} e^{\frac{x}{n}} \frac{n}{n-1-x} e^{-\frac{x}{n}} e^{-\frac{1}{n}}$$

$$= \operatorname*{Lim}_{m\to\infty} \frac{(-x)(1-x)(2-x)\cdots(m-x)}{(-x)(1-x)\cdots\cdots(m-1-x)} e^{-\sum_{n=1}^{m} \frac{1}{n}}$$

$$= \operatorname*{Lim}_{m\to\infty} (m-x)\, e^{-\sum_{n=1}^{m} \frac{1}{n}}$$

$$e^{G(x+1)-G(x)} = \lim_{m=\infty} e^{\log (m+1) - \sum_{n=1}^{m} \frac{1}{n}} e^{\log \frac{m-x}{m+1}}$$

$$e^{G(x+1)-G(x)} = e^{-\gamma}$$

$$G(x+1) - G(x) = -\gamma$$

Then $$G(x) = -\gamma x + c$$

Substituting this value in eq. (38)

$$W(x) = e^{-\gamma x} e^{c} \prod_{n=1}^{\infty} \left(1 - \frac{x}{n}\right) e^{\frac{x}{n}}$$

To determine the value of c, we have

$$W(0) = e^{c} = 1,$$

since we have chosen to take $W(0) = 1$. Then

$$W(x) = e^{-\gamma x} \prod_{n=1}^{\infty} \left(1 - \frac{x}{n}\right) e^{\frac{x}{n}}$$

which is the same expression for W(x) as that given by eq. (37).

Relation of W(x) and Sin x.

Since W(x) has zeros at all positive integral values of x, W(1-x) has zeros at x = 0 and all negative integral values of x. The product W(x)W(1-x) will then have zeros at all positive and negative integers and at zero. From the difference equation (33)

$$W(1-x) = x\, W(-x)$$

Hence $$W(x)\, W(1-x) = x\, W(-x)\, W(x).$$

From eq. (37) $$W(x) = e^{-\gamma x} \prod_{n=1}^{\infty} \left(1 - \frac{x}{n}\right) e^{\frac{x}{n}}$$

and $$W(-x) = e^{\gamma x} \prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right) e^{-\frac{x}{n}}$$

Therefore $$W(x)\, W(1-x) = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right)$$

The right side of this equation is the infinite product for $\frac{\sin \pi x}{\pi}$. We therefore have

$$W(x)\, W(1-x) = \frac{\sin \pi x}{\pi} \qquad (39)$$

If we set $x = \frac{1}{2}$ in eq. (39) we obtain

$$W(\tfrac{1}{2}) = \frac{1}{\sqrt{\pi}}$$

Relation of W(x) and $\Gamma(x)$.

We will now express W(x) in terms of $\Gamma(x)$. From the equations

$$\bar{\Gamma}(x) = (-2\pi i)(-1)^x \prod_{n=1}^{\infty} \left(1 - \frac{x}{n}\right)\left(1 + \frac{1}{n}\right)^x \qquad (11)$$

$$W(x) = \prod_{n=1}^{\infty} \left(1 - \frac{x}{n}\right)\left(1 + \frac{1}{n}\right)^x \qquad (36)$$

$$\bar{\Gamma}(x) = (1 - e^{2\pi i x})\,\Gamma(x) \qquad (6)$$

we at once obtain $$W(x) = \frac{(1 - e^{2\pi i x})\,\Gamma(x)}{-2\pi i\,(-1)^x}$$

Since $(-1)^x = e^{\pi i x}$, if we take the simplest value for the amplitude of (-1)

$$W(x) = \frac{(e^{-\pi i x} - e^{\pi i x})}{-2\pi i}\,\Gamma(x) = \frac{1}{\pi}\,\frac{e^{\pi i x} - e^{-\pi i x}}{2\,i}\,(x)$$

$$W(x) = \frac{\sin \pi x}{\pi}\,\Gamma(x) \qquad (40)$$

But since[1] $$\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

This reduces to $$W(x) = \frac{1}{\Gamma(1-x)} \qquad (41)$$

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1. Euler, L. "Evolutio Formulae Integralis, etc." Novi Comment. Acad. Petrop. Vol.16. (1771) pp. 91-139.

Determination of the Constant s_0.

The Asymptotic Series for W(x).

We are now able to determine the constant s_0 in eq. (34)

$$S(x) = (-1)^x\, x^{x-\frac{1}{2}}\, e^{-x}\, (s_0 + \frac{s_1}{x} + - - -) \qquad (34)$$

Since $W(x) \sim S(x)$ in the sector $0 < \text{am } x < 2\pi$, we may write for large values of x

$$W(x) = (-1)^x\, x^{x-\frac{1}{2}}\, e^{-x}\, s_0\, [1 + \varepsilon(x)]$$

where $\underset{x=\infty}{\text{Lim}}\ \varepsilon(x) = 0$, and $x^{x-\frac{1}{2}}$ denotes the branch of the function for which $W(x) \sim S(x)$. We have

$$W(x)\, W(1-x) = x\, W(x)\, W(-x)$$

$$W(1-x) = x\, W(-x) = x\, (-1)^x\, (-x)^{-x-\frac{1}{2}}\, e^x\, s_0\, [1 + \varepsilon_1(x)]$$

Hence $$W(x)\, W(1-x) = x^{x+\frac{1}{2}}\, (-x)^{-x-\frac{1}{2}}\, s_0\, [1 + \varepsilon'(x)]$$

$$W(x)\, W(1-x) = (-1)^{x+\frac{1}{2}}\, s_0\, [1 + \varepsilon'(x)]$$

where $\underset{x=\infty}{\text{Lim}}\ \varepsilon_1(x) = 0$, $\underset{x=\infty}{\text{Lim}}\ \varepsilon'(x) = 0$, and $(-1) = \left(\frac{x}{-x}\right)$.

Then, since am $(-x)$ = am $x + \pi$, if x be taken in the

upper half of the plane, the amplitude of $(-1) = -\pi$, and $(-1) = e^{-\pi i}$. In the upper half plane:

$$(-1)^{x+\frac{1}{2}} = e^{-\pi i(x+\frac{1}{2})} = i\, e^{-\pi i x}$$

and $$W(x)\, W(1-x) = -i\, e^{-\pi i x}\, s_0^2 \left[1 + \varepsilon'(x)\right]$$

But from eq. (39) we have

$$W(x)\, W(1-x) = \frac{\sin \pi x}{\pi} = \frac{(1 - e^{2\pi i x})}{-2\pi i\, e^{\pi i x}}$$

Equating $$-i\, e^{-ix\pi} s_0^2 \left[1 + \varepsilon'(x)\right] = \frac{1 - e^{2\pi i x}}{-2\pi i\, e^{\pi i x}}$$

$$s_0^2 \left[1 + \varepsilon'(x)\right] = \frac{1 - e^{2\pi i x}}{-2\pi}$$

If now we let x become infinite along the positive axis of imaginaries we obtain

$$s_0^2 = \frac{1}{-2\pi}$$

or $$s_0 = \frac{1}{i\sqrt{2\pi}}$$

We here take the positive root since W(x) is positive when x is real and negative.

Substituting the value of s_0 in eq. (34) we obtain

$$S(x) = (-1)^x\, x^{x-\frac{1}{2}}\, e^{-x}\, \frac{1}{i\sqrt{2\pi}} \left[1 + \varepsilon(x)\right]$$

$$S(x) = (-x)^{x-\frac{1}{2}}\, e^{-x}\, \frac{1}{\sqrt{2\pi}} \left[1 + \varepsilon(x)\right] \qquad (42)$$

The Multiplication Theorem for W(x).

From the multiplication theorem for the gamma function

$$\Gamma(nx) = \frac{n^{nx-\frac{1}{2}}}{(2\pi)} \prod_{K=0}^{n-1} \Gamma\left(x + \frac{k}{n}\right) \qquad (15)$$

and the relation of W(x) to the gamma function

$$W(x) = \frac{1}{\Gamma(1-x)} \qquad (41)$$

we can derive a multiplication theorem for W(x).

Substituting (nx) for x

$$W(nx) = \frac{1}{\Gamma(1-nx)} \qquad (43)$$

We have previously derived

$$\Gamma(1-nx) = \frac{-n \times n^{-nx-\frac{1}{2}}}{(2\pi)^{\frac{n-1}{2}}} \prod_{K=0}^{n-1} \Gamma\left(-x + \frac{k}{n}\right) \qquad (17)$$

From eq. (41) $\qquad \Gamma(-x) = \frac{1}{W(1-x)}$

and by replacing x by $\left(x - \frac{k}{n}\right)$

$$\Gamma\left(-x + \frac{k}{n}\right) = \frac{1}{W\left(x+1-\frac{k}{n}\right)} \qquad (44)$$

We also have $$\prod_{k=0}^{n-1} W\left(x + 1 - \frac{k}{n}\right) = \prod_{k=1}^{n} W\left(x + \frac{k}{n}\right) \qquad (45)$$

Combining (17), (43), (44), and (45):

$$W(nx) = \frac{(2\pi)^{\frac{n-1}{2}} n^{nx-\frac{1}{2}}}{-x} \prod_{k=1}^{n} W\left(x + \frac{k}{n}\right) \qquad (46)$$

which is the multiplication theorem for W(x).

Values of W(x) for Real Values of x.

From the fact that $W(0) = 1$, and $W(\frac{1}{2}) = \frac{1}{\sqrt{\pi}}$ and the difference equation

$$W(x+1) = -x\, W(x)$$

we can easily obtain, where n is a positive integer,

$$\begin{aligned} W(+n) &= 0 \\ W(-n) &= \frac{1}{\lfloor n} \\ W\left(\frac{2n+1}{2}\right) &= \frac{2n-1}{2}\,\frac{2n-3}{2} \cdots \frac{3}{2}\,\frac{1}{2}\,\frac{1}{\sqrt{\pi}} \\ W\left(-\frac{2n+1}{2}\right) &= \frac{2}{2n+1}\,\frac{2}{2n-1} \cdots \frac{2}{3}\,\frac{2}{1}\,\frac{1}{\sqrt{\pi}} \end{aligned} \qquad (47)$$

The value of W(x) for any other real value of x may be most readily obtained from the relation

$$W(x) = \frac{1}{\Gamma(1-x)} \qquad (41)$$

Legendre[1] has computed a table of logarithms to twelve places of the gamma function for values of x from 1 to 2 at intervals of one-thousandth. Gauss[2] has also computed a table of the logarithms of $\Pi(x)$ to 20 decimal places where $\Pi(x) = \Gamma(x+1)$ for values of x between 0 and 1 at intervals of one-hundredth. By means of these tables and the difference equation for the gamma function the value of $\Gamma(x)$ and hence of W(x) may be accurately found.

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1. Legendre. (See footnote, page 25.)
2. Gauss, Carl Friedrich, "Werke" (1876) pp. 161-162.

The Logarithmic Derivative of W(x).

To obtain the logarithmic derivative of W(x), we take the logarithm of both sides of the equation

$$W(x) = e^{-\gamma x} \prod_{n=1}^{\infty} \left(1 - \frac{x}{n}\right) e^{\frac{x}{n}} \tag{37}$$

$$\log W(x) = -\gamma x + \sum_{n=1}^{\infty} \left(\log \frac{n-x}{n} + \frac{x}{n}\right)$$

Differentiating and denoting $\frac{W'(x)}{W(x)}$ by $\chi(x)$

$$\chi(x) = \frac{W'(x)}{W(x)} = -\gamma + \sum_{n=1}^{\infty} \left(\frac{1}{x-n} + \frac{1}{n}\right) \tag{48}$$

From the equation

$$W(x+1) = -x\, W(x)$$

we obtain by logarithmic differentiation

$$\frac{W'(x+1)}{W(x+1)} = \frac{1}{x} + \frac{W'(x)}{W(x)}$$

$$\chi(x+1) = \frac{1}{x} + \chi(x) \tag{49}$$

which is the difference equation satisfied by $\chi(x)$. From the form of this equation, we see that $\chi(x)$, $\psi(x)$, and

$\overline{\Psi}(x)$ are all solutions of the equation

$$y(x+1) = \frac{1}{x} + y(x). \tag{22}$$

In fact, both $\chi(x)$ and $\overline{\Psi}(x)$ come from the second principal solution and differ only by the additive constant πi.

From the relation of $W(x)$ and the gamma function

$$W(x) = \frac{1}{\Gamma(1-x)} \tag{41}$$

we obtain $\log W(x) = - \log \Gamma(1-x)$

and differentiating $$\chi(x) = \Psi(1-x) \tag{50}$$

which is the relation of $\chi(x)$ to the logarithmic derivative of the gamma function.

The relation of $\chi(x)$ and the circular functions may be derived from eq. (39):

$$W(x)\, W(1-x) = \frac{\sin \pi x}{\pi} \tag{39}$$

We have $\log W(x) + \log W(1-x) = \log \sin \pi x - \log \pi$

Differentiating $$\chi(x) - \chi(1-x) = \pi \cot \pi x \tag{51}$$

This last result may be obtained directly without use of eq. (39). We have

$$\chi(x) = - \gamma + \sum_{n=1}^{\infty} \left(\frac{1}{x-n} + \frac{1}{n} \right) \tag{48}$$

$$\chi(1-x) = -\gamma + \sum_{n=1}^{\infty}\left(\frac{1}{1-x-n} + \frac{1}{n}\right)$$

Subtracting $$\chi(x) - \chi(1-x) = \sum_{n=1}^{\infty}\left(\frac{1}{x-n} + \frac{1}{x+n-1}\right)$$

$$= \frac{1}{x} + \sum_{n=1}^{\infty}\left(\frac{1}{x-n} + \frac{1}{x+n}\right)$$

$$= \frac{1}{x} + 2\sum_{n=1}^{\infty}\frac{x}{x^2-n^2}$$

We have, however, from Pierpont[1]

$$\cot z = \frac{1}{z} + 2\sum_{n=1}^{\infty}\frac{z}{z^2-n^2\pi^2}$$

or, setting $z = \pi x$, $$\pi\cot\pi x = \frac{1}{x} + 2\sum_{n=1}^{\infty}\frac{x}{x^2-n^2}$$

Therefore $$\chi(x) - \chi(1-x) = \pi\cot\pi x. \qquad (51)$$

From the multiplication theorem

$$W(nx) = \frac{(2\pi)^{\frac{n-1}{2}}\, n^{nx-\frac{1}{2}}}{-x}\prod_{k=1}^{n} W\left(x + \frac{k}{n}\right) \qquad (46)$$

we obtain by logarithmic differentiation the multiplication theorem for $\chi(x)$.

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1. Pierpont, James, "Functions of a Complex Variable." (1914) Page 285.

$$n\,\chi(nx) = n\log n - \frac{1}{x} + \sum_{k=1}^{n}\chi\left(x+\frac{k}{n}\right) \qquad (52)$$

From the difference equation satisfied by $\chi(x)$, eq. (49), we obtain the two expressions:

$$\chi(x+m) = \chi(x) + \frac{1}{x} + \frac{1}{x+1} + - - + \frac{1}{x+m-1} = \chi(x) + \sum_{h=0}^{m-1}\frac{1}{x+n} \qquad (53)$$

$$\chi(x-m) = \chi(x) - \frac{1}{x-1} - \frac{1}{x-2} - - - - \frac{1}{x-m} = \chi(x) - \sum_{n=1}^{m}\frac{1}{x-n} \qquad (54)$$

Equation (48) at once gives us $\chi(0) = -\gamma$ and from eq. (50) we have

$$\chi\left(\tfrac{1}{2}\right) = \psi\left(\tfrac{1}{2}\right) = -\gamma - 2\log 2,$$

where the value of $\psi(\frac{1}{2})$ is that given by Gauss'. Since W(x) has zeros at the positive integers, $\chi(x)$ has poles at these points. From these values and eq. (49) we have, where n is a positive integer,

$$\chi(-n) = -\gamma + \sum_{k=1}^{n}\frac{1}{k}$$

$$\chi\left(\frac{2n+1}{2}\right) = -\gamma - 2\log 2 + 2\sum_{k=1}^{m}\frac{1}{2k-1} \qquad (55)$$

$$\chi\left(-\frac{2n+1}{2}\right) = -\gamma - 2\log 2 + 2\sum_{n=1}^{n+1}\frac{1}{2k-1}$$

$$\chi(+n) = \infty$$

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1. Gauss, Carl Friedrich, "Werke." (1876). Page 155.

Since Gauss'[1] has computed a table of $\psi(x+1)$ for values of $(x+1)$ from 1 to 2 by hundredths, the values being given to eighteen decimal places, the value of $\chi(x)$ for any real value of x may be found from this table and the relations

$$\chi(x) = \psi(1-x) \tag{50}$$

$$\chi(x+1) = \frac{1}{x} + \chi(x) \tag{49}$$

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1. Gauss, Carl Friedrich. "Werke." (1876) pp. 161-162.

The Derivative of the Logarithmic Derivative of W(x).

From equation (48)

$$\chi(x) = -\gamma + \sum_{n=1}^{\infty}\left(\frac{1}{x-n} + \frac{1}{n}\right) \tag{48}$$

we obtain by differentiation

$$\frac{d}{dx}\chi(x) = -\sum_{n=1}^{\infty}\frac{1}{(x-n)^2} \tag{56}$$

If we denote $\frac{d}{dx}\chi(x)$ by $\theta(x)$, we have, by comparison with the equation for the derivative of $\overline{\psi}(x)$, eq. (28), and by eq. (30)

$$\theta(x) = \frac{d}{dx}\chi(x) = \frac{d}{dx}\overline{\psi}(x) = -\sum_{n=1}^{\infty}\frac{1}{(x-n)^2} = -\phi(-x,1,2) \tag{57}$$

By differentiating eq. (49) we find for the difference equation satisfied by $\theta(x)$

$$\theta(x+1) = -\frac{1}{x^2} + \theta(x). \tag{58}$$

Similarly, by differentiating eq. (50), we find, where

$$\phi(x) = \frac{d}{dx}\psi(x),^{1} \qquad \theta(x) = -\phi(1-x) \tag{59}$$

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1. The function $\phi(x)$ as here defined is the special case of the function $\phi(x,r,k)$ when $r=1, k=2$; that is, $\phi(x) = \phi(x,1,2)$.

Also, from eq. (51) we find by differentiation,

$$\theta(x) + \theta(1-x) = \frac{-\pi^2}{\sin^2 \pi x} \tag{60}$$

From (56) and (60) we can readily find a series for $\frac{\pi^2}{\sin^2 \pi x}$. We have

$$\theta(x) = -\sum_{n=1}^{\infty} \frac{1}{(x-n)^2} \tag{56}$$

$$\theta(1-x) = -\sum_{n=1}^{\infty} \frac{1}{(1-x-n)^2}$$

Adding $$\theta(x) + \theta(1-x) = -\sum_{n=1}^{\infty} \left[\frac{1}{(x-n)^2} + \frac{1}{(1-x-n)^2}\right]$$

$$\theta(x) + \theta(1-x) = -\sum_{n=1}^{\infty} \left\{\frac{1}{(x-n)^2} + \frac{1}{(x+n-1)^2}\right\}$$

$$\theta(x) + \theta(1-x) = -\sum_{-\infty}^{\infty} \frac{1}{(x+n)^2}$$

Therefore $$\frac{\pi^2}{\sin^2 \pi x} = \sum_{-\infty}^{\infty} \frac{1}{(x+n)^2}$$

From eq. (52) we obtain by differentiation, the multiplication theorem for $\theta(x)$.

$$n^2\, \theta(nx) = \frac{1}{x^2} + \sum_{k=1}^{n} \theta\left(x + \frac{k}{n}\right) \tag{61}$$

The multiplication theorem for $\theta(x)$, eq. (61), may also be obtained from the corresponding equation for $\phi(x)$ given by Godefrey[1]:

$$n^2 \phi(nx) = \sum_{k=0}^{n-1} \phi\left(x + \frac{k}{n}\right) \tag{62}$$

and the relation $\theta(x) = - \phi(1-x)$ (59)

We have, by replacing x by (nx) in this last equation

$$\theta(nx) = - \phi(1-nx)$$

and by replacing x by (-x) in eq. (62)

$$n^2 \phi(-nx) = \sum_{k=0}^{n-1} \phi\left(-x + \frac{k}{n}\right) \tag{63}$$

From the equation $\phi(z+1) = - \frac{1}{z^2} + \phi(z)$

by setting $z = -nx$, $\phi(-nx) = \phi(1-nx) + \frac{1}{n^2 x^2}$

or

$$n^2 \phi(-nx) = n^2 \phi(1-nx) + \frac{1}{x^2} \tag{64}$$

Also from eq. (59), $- \phi\left(-x + \frac{k}{n}\right) = \theta\left(1 + x - \frac{k}{n}\right)$ (65)

Moreover

$$\sum_{k=0}^{n-1} \theta\left(1 + x - \frac{k}{n}\right) = \sum_{k=1}^{n} \theta\left(x + \frac{k}{n}\right) \tag{66}$$

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1. Godefrey, Maurice. "Theorie Elementarie Des Series". (1903) Page 236.

Combining (63), (64), (65), and (66), we finally obtain

$$n^2\,\theta(nx) = \frac{1}{x^2} + \sum_{k=1}^{n} \theta\left(x + \frac{k}{n}\right) \tag{61}$$

From the difference equation (58) we find the two expressions -

$$\theta(x+m) = \theta(x) - \frac{1}{x^2} - \frac{1}{(x+1)^2} - \cdots - \frac{1}{(x+m-1)^2} = \theta(x) - \sum_{n=0}^{m-1} \frac{1}{(x+n)^2}$$

$$\theta(x-m) = \theta(x) + \frac{1}{(x-1)^2} + \frac{1}{(x-2)^2} + \cdots + \frac{1}{(x-m)^2} = \theta(x) + \sum_{n=1}^{m} \frac{1}{(x-n)^2}$$

We easily obtain from eq. (60) that $\theta\left(\frac{1}{2}\right) = -\frac{1}{2}\pi^2$. To obtain the value of $\theta(0)$ we have from eq. (57)

$$\theta(0) = -\sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{\pi^2}{6} \tag{67}$$

From these values and eq. (58) we find, where n is a positive integer

$$\begin{aligned} \theta(+n) &= -\infty \\ \theta(-n) &= -\frac{\pi^2}{6} + \sum_{k=1}^{n} \frac{1}{k^2} \\ \theta\left(\frac{2n+1}{2}\right) &= -\frac{\pi^2}{2} - 4\sum_{k=1}^{n} \frac{1}{(2k-1)^2} \\ \theta\left(\frac{2n+1}{2}\right) &= -\frac{\pi^2}{2} - 4\sum_{k=1}^{n+1} \frac{1}{(2k-1)^2} \end{aligned} \tag{68}$$

To determine the value of $\theta(x)$ for any other real value of x, we can make use of the relation

$$\theta(x) = - \phi(1-x) \tag{59}$$

To do this, however, we must first determine the value of $\phi(x)$. We do this, instead of determining the value of $\theta(x)$ directly so that all our fundamental tables will be for the gamma function or its derivatives.

Calculation of $\phi(x)$ for Real Values of x.

We have, from Batchelder[1], the following equations for $\phi(x)$:

$$\phi(x+1) = -\frac{1}{x^2} + \phi(x) \qquad (69)$$

$$\phi(x) = \sum_{n=0}^{\infty} \frac{1}{(x+n)^2} \qquad (70)$$

$$\phi(x) \sim \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + - - - - - \qquad (71)$$

and from Godefrey[2]:

$$\phi(x-m) = \phi(x) + \frac{1}{(x-1)^2} + \frac{1}{(x-2)^2} + - - - + \frac{1}{(x-m)^2} \qquad (72)$$

$$\phi(x) + \phi(1-x) = \frac{\pi^2}{\sin^2 \pi x} \qquad (73)$$

From eq. (73) by setting $x = \frac{1}{2}$, we immediately obtain the value

$$\phi(\tfrac{1}{2}) = \tfrac{1}{2}\pi^2.$$

By substituting $x=1$ in eq. (70) we obtain, as in eq. (68)

$$\phi(1) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

- -

1. Batchelder, pages 23-29.
2. Godefrey, Maurice. "Theorie Elementarie Des Series." (1903) Page 236.

The series in eq. (70) converges too slowly to be of practical value in the computation of $\phi(x)$ for real values of x. However, for large values of x, $\phi(x)$ may be accurately determined from eq. (71). We must then determine how large x must be taken so that the error will be less than a preassigned constant. By actual substitution of x=10 we find the value of the series in eq. (71) for this value to be, to seven decimal places, 0.1050170. Taking this as the value of $\phi(10)$ we find from eq. (72) that $\phi(1) = 1.644943$. Comparing this with the value $\frac{\pi^2}{6}$ = 1.644934, we see that the error is less than one in the fifth decimal place, which is sufficiently near the correct value for purposes of graphing the function. By calculating $\phi(x)$ at intervals of one tenth between x=10 and x=11 from eq. (71), and reducing these values to the interval between 1 and 2 by eq. (72), we obtain the following fundamental table:

x.	$\phi(x)$.
1.0	1.64493
1.1	1.43330
1.2	1.26738
1.3	1.13425
1.4	1.02536
1.5	0.93480
1.6	0.85862
1.7	0.79323
1.8	0.73698
1.9	0.68797
2.0	0.64493

Graphs.

In the following pages the functions $W(x)$, $\chi(x)$, and $\theta(x)$ are graphed for real values of x.

The function $W(x)$ is given for the interval

$$-3 < x < 6.$$

The values for this graph were obtained for one-tenth intervals from the relation

$$W(x) = \frac{1}{\Gamma(1-x)} \qquad (40)$$

and Legendre's table of the logarithm of the gamma function.

The function $\chi(x)$ is given for the interval

$$-4 < x < 8$$

This function was plotted for values of x differing by .05, the value being found by means of the relation

$$\chi(x) = \psi(1-x) \qquad (50)$$

and Gauss's table for $\psi(x)$.

The function $\theta(x)$ is given for the interval

$$-5 < x < 7$$

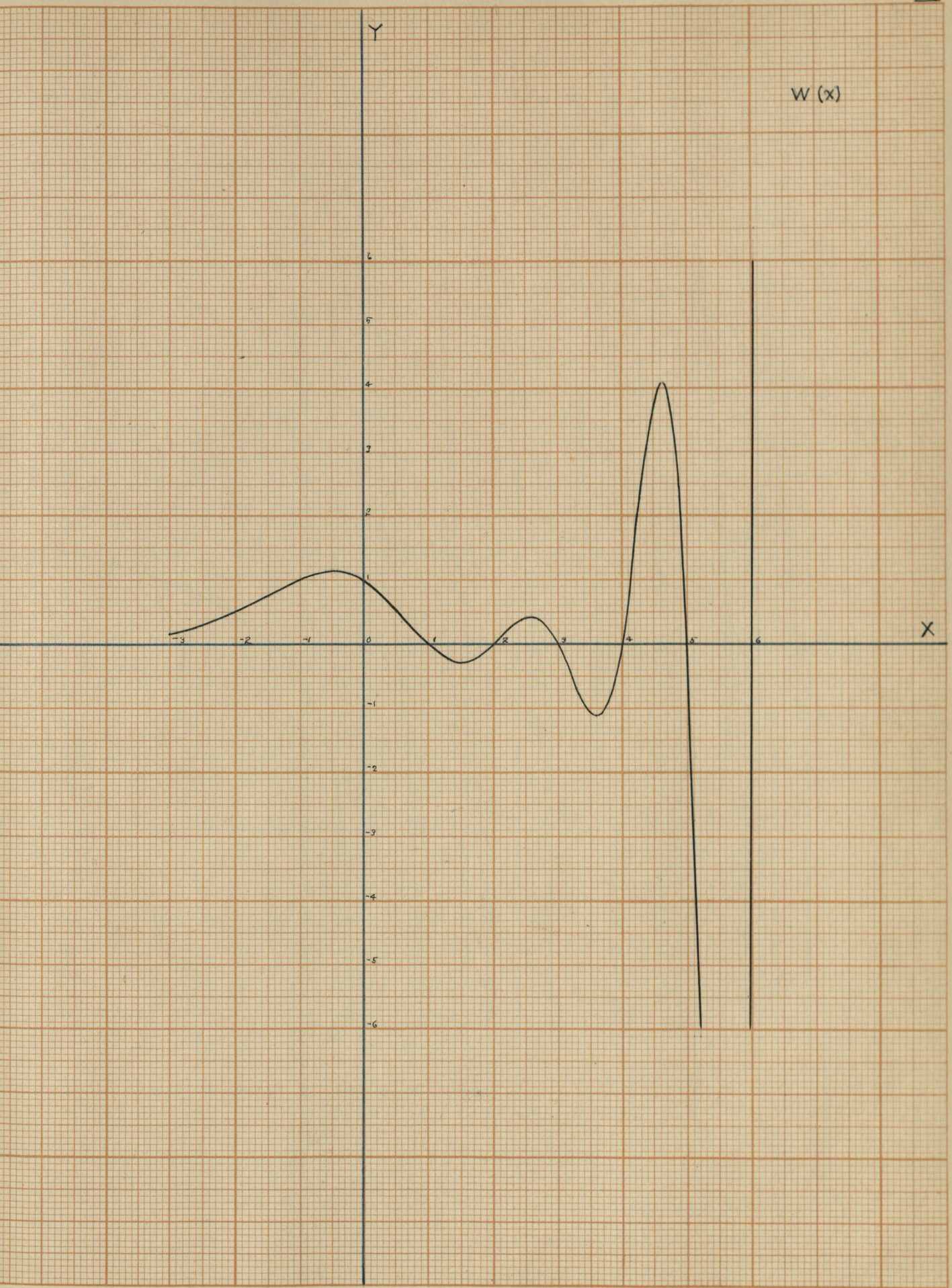
This graph was plotted using the values of $\theta(x)$ derived from the relations

$$\theta(x) = -\phi(1-x) \qquad (59)$$

$$\theta(x+1) = -\frac{1}{x^2} + \theta(x) \qquad (58)$$

and the table for $\phi(x)$ as given on page 51.

For purposes of comparison the graphs of the functions $\Gamma(x)$, $\psi(x)$, and $\phi(x)$ are given for the corresponding (not the same) intervals.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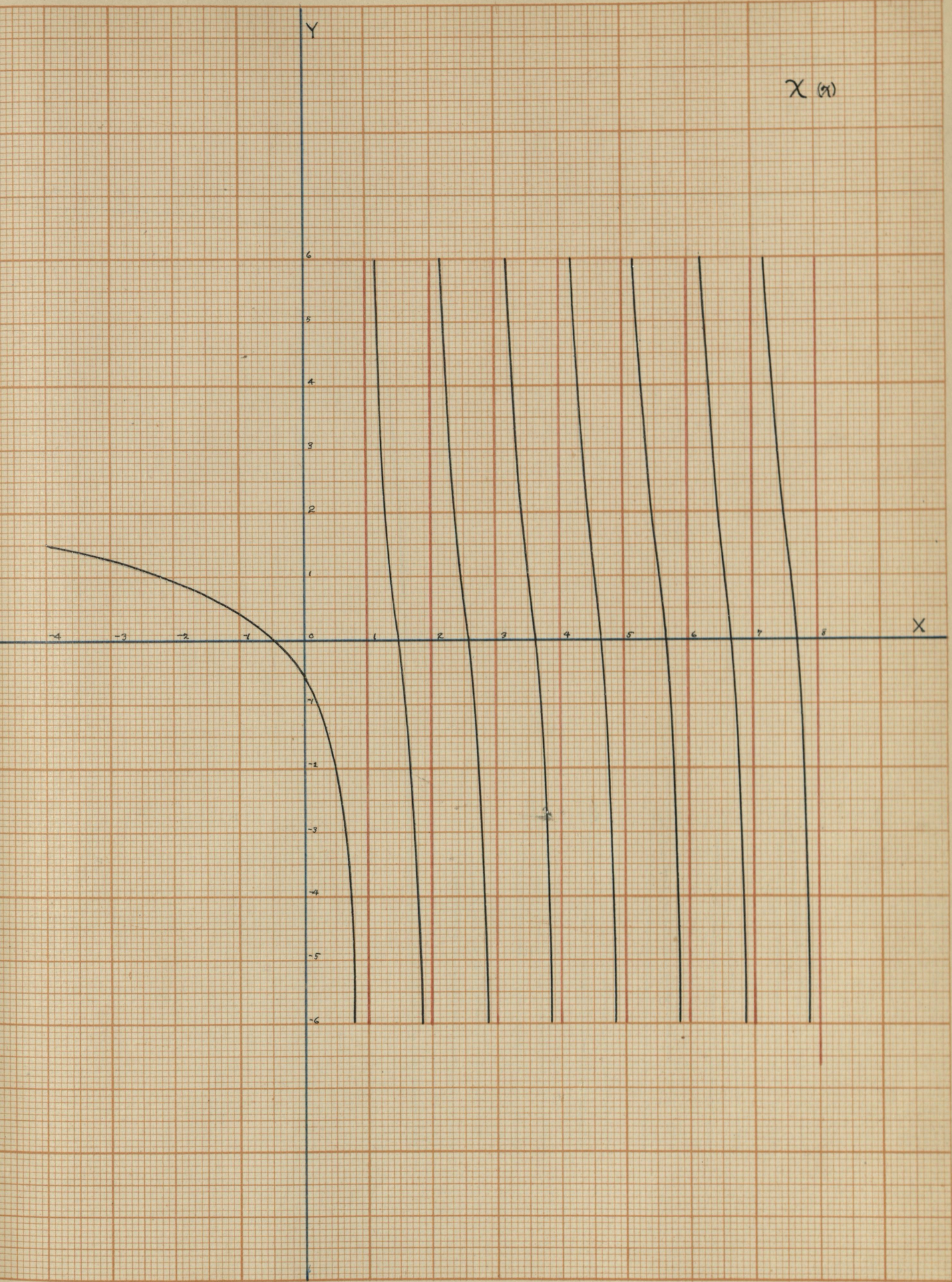
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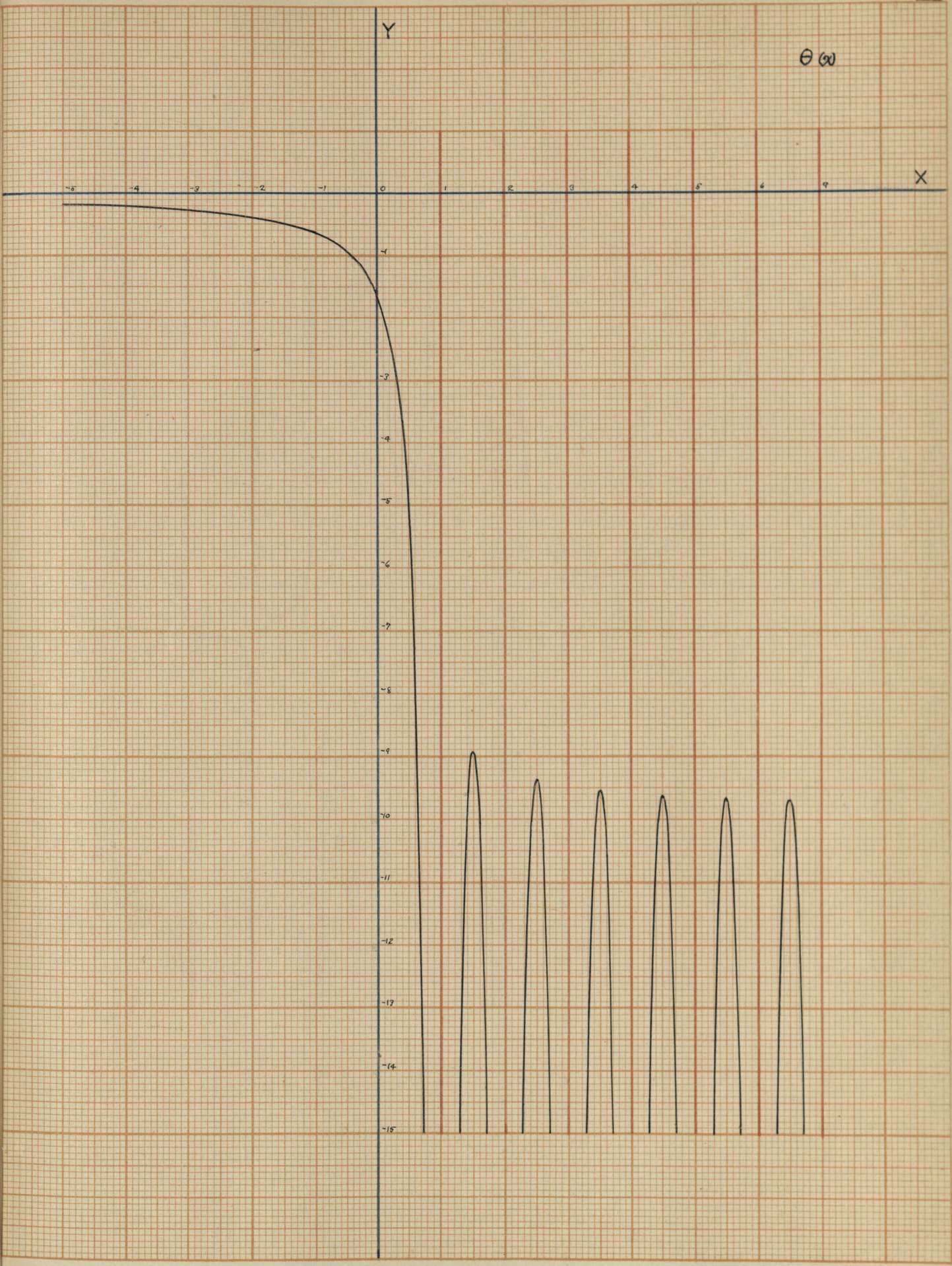
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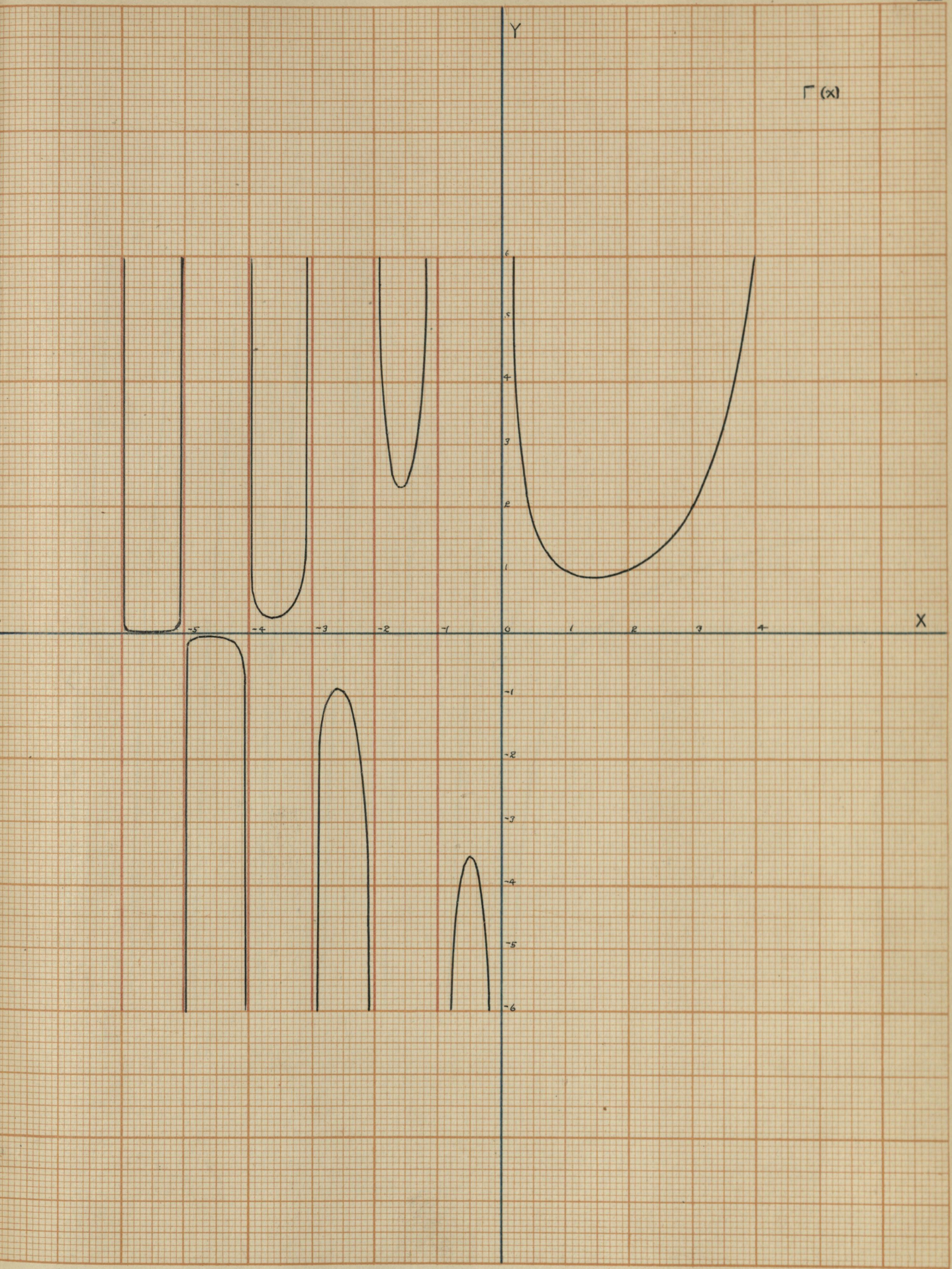
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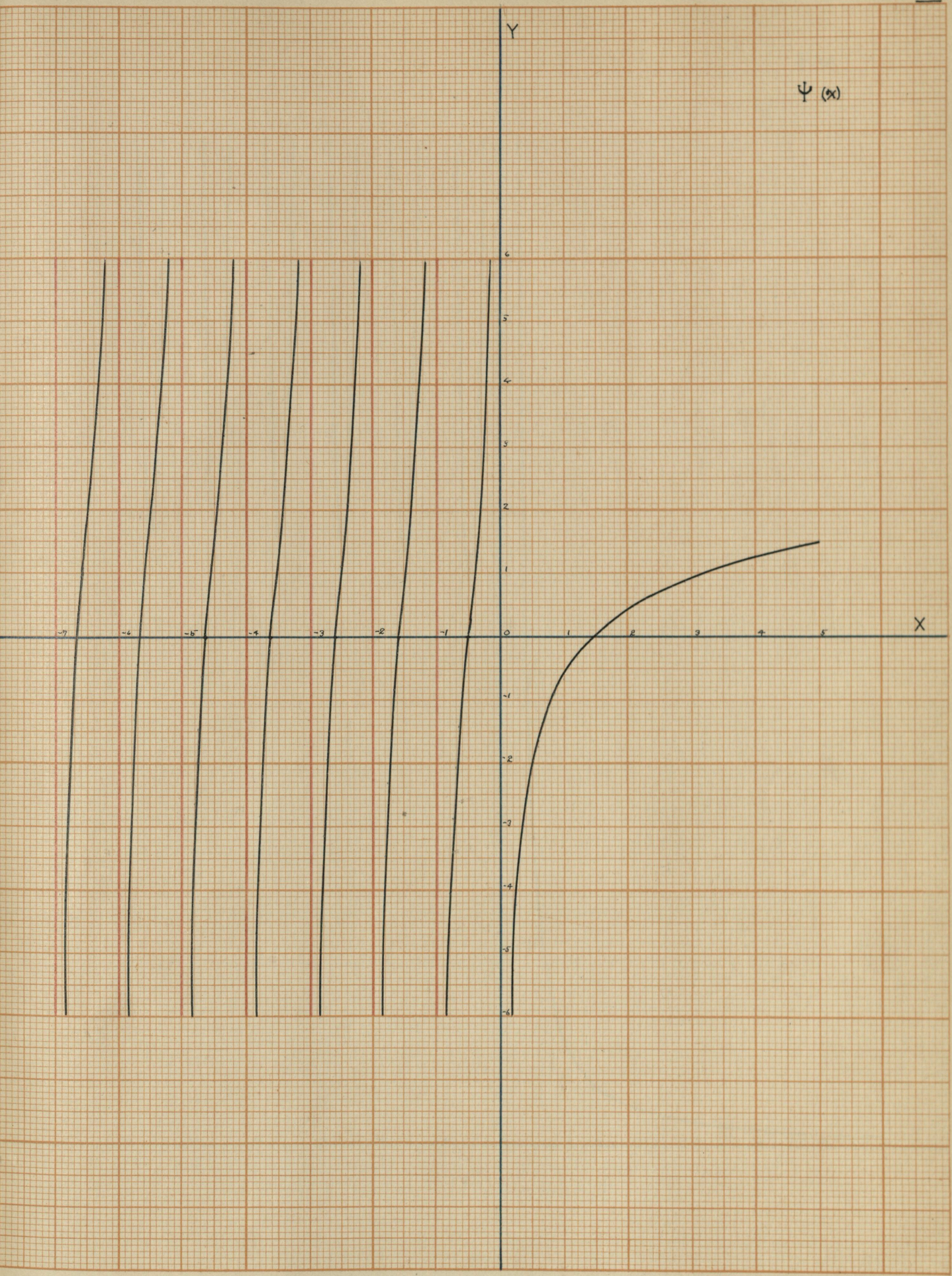
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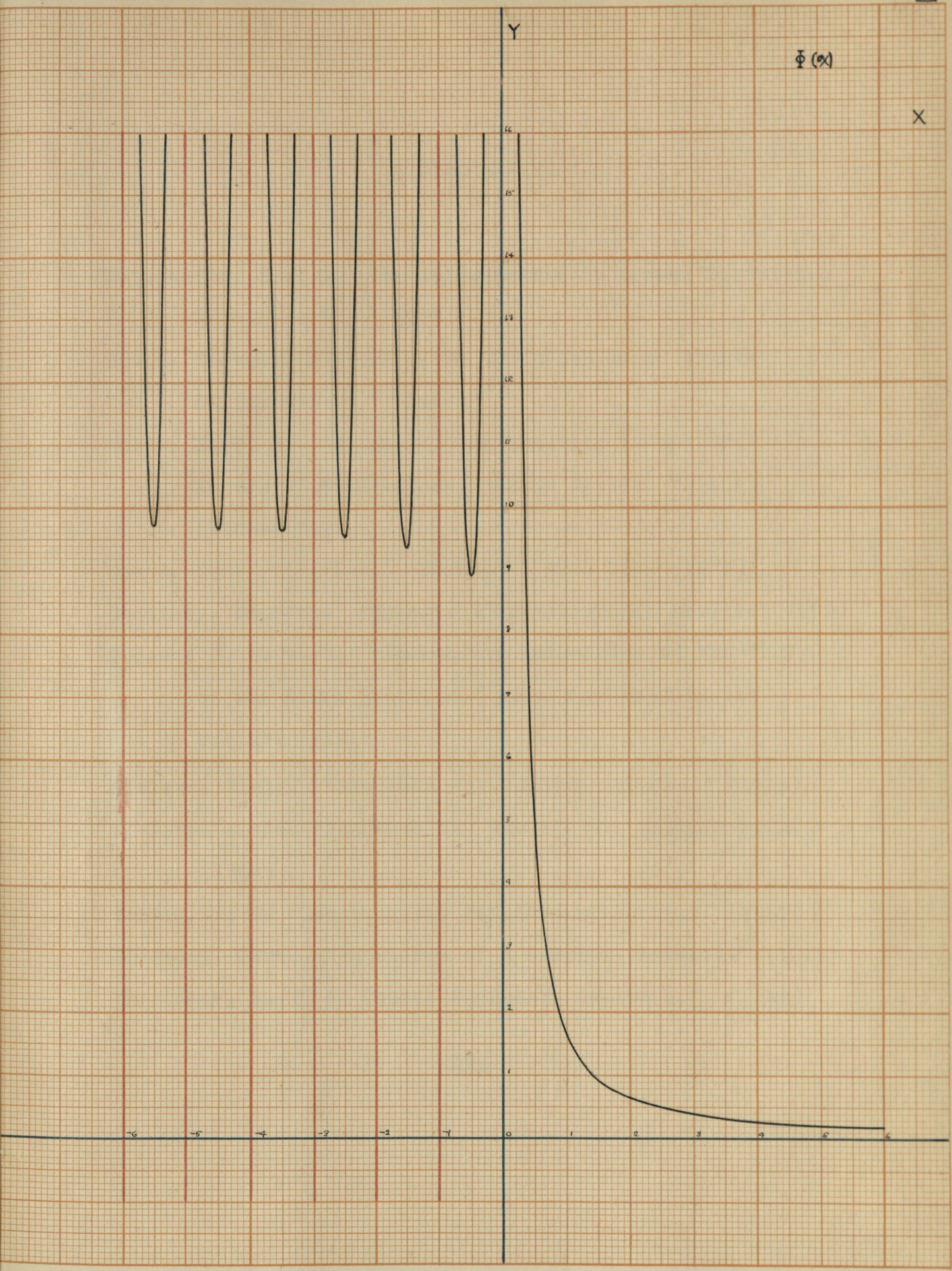
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