AUTOMORPHISMS

OF

MONOMIAL GROUPS

by

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TABLE OF CONTENTS

PREFACE

This paper was written while the author was a Research Assistant on the project supported by the National Science Foundation through Research Grant NSF-G1126. This paper will be submitted as a report of this project. It will also be submitted as a thesis to the Graduate School of the University of Kansas in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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INTRODUCTION

This thesis is concerned with certain generalized permutation groups called complete monomial groups and some of their subgroups. For the case of finite permutations this group was first studied by Ore [4], and for the case of infinite permutations by Crouch [1]. The most important result obtained is the determination of all automorphisms of a large class of monomial groups. In addition, the derived series is studied.

Let B be a set of n elements and H a group. Then a monomial substitution u is a trnsformation that maps every element x of the set B onto an element of B multiplied by an element h of H in such a manner that it induces a one-to-one mapping of B onto itself. The elements h are called factors of the substitution u. If we consider the set of all such monomial substitutions, and let successive application of the mappings be the defined operation we obtain a group which we call the complete monomial group. Those monomial substitutions which map each x of B onto itself multiplied by some element of H will be called multiplications. The set of all multiplications which we will denote by V_n form a normal subgroup of the complete monomial group $\overline{\sum}_n(H)$. The set of substitutions which map every element of B onto some element of B multiplied by the identity of H form a subgroup S_n of $\overline{\sum}_n(H)$. S_n is the symmetric group on n objects. $\overline{\sum}_n(H)$ is the union of V_n and S_n, and the intersection of V_n and S_n is the identity E of $\overline{\sum}_n(H)$.

In this paper we consider some monomial groups resulting when the restriction that the given set be finite is removed. Such groups we will denote by

$\sum(H; B, C, D)$, $d \leq C$, $D \leq B^{+}$, $d = \aleph_0$, H and B denoting the given group and the order of the given set respectively. C a cardinal such that all monomial substitutions of the group have fewer than C factors different from the identity of H, D a cardinal such that all monomial substitutions of the group have fewer than D elements of the given set mapped into elements distinct from themselves, B^{+} the successor of B. As before the set of all multiplications form a normal subgroup called the basis group, the set of all permutations form a subgroup, and the monomial group is the union of these two groups, and the two groups meet in the identity only of the monomial group.

Ore [4] has determined the derived series and the form of any automorphism of the complete monomial group when the given set has finite order. In this paper we obtain similar results for some of the monomial groups $\sum(H; B, C, D)$, and determine in addition the automorphism groups of some of the monomial groups.

Chapters I, II, and III contain preliminaries for the following chapters.

Chapter IV contains the main results of the paper. For the group $\sum(H; B, d, C)$, $C < B^{+}$, the form of all automorphisms is established and the automorphism group is determined in terms of the automorphism group of H. Chapter V gives the automorphisms of the Alternating Monomial Group when the given set is finite, H contains no subgroup isomorphic to the alternating group on n -1 objects, and n>6. It is also shown that the automorphism group of $\sum(H; B, d, d)$ is isomorphic to the group of sutomorphisms of its subgroup consisting of all alternating substitutions. In The concluding chapter the derived series of $\dot{\overline{\Sigma}}(H; B, C, D)$, $C \leq D$, is determined.

CHAPTER I

PRELIMINARIES

Let H be an arbitrary group, and let S be a set with order B, $B \geq d$, $d = \aleph_0$. We will denote elements of H by h and k, and x will be used to denote elements of S.

A monomial substitution over H is a linear transformation mapping each element x of S in a one-to-one manner onto some element of S multiplied by an element of H. A substitution u will be written,

$$u = \begin{pmatrix} \cdots & x_i & \cdots \\ \cdots & h_j x_j & \cdots \end{pmatrix}.$$

The element h of H will be termed a factor of u. The multiplication hx is a formal one with the associative property $h(kx) = (hk)x$. If a second substitution u' be given by,

$$u' = \begin{pmatrix} \cdots & x_j & \cdots \\ \cdots & h_t x_t & \cdots \end{pmatrix},$$

then the product uu' is defined by,

$$uu' = \begin{pmatrix} \cdots & x_i & \cdots \\ \cdots & h_j h_t x_t & \cdots \end{pmatrix}.$$

With this definition of multiplication the set of monomial substitutions over H form a group, hereafter called the monomial group or symmetry.

A substitution having each of its factors the identity element e of H will be called a permutation. The set of all permutations contained in the monomial group form a subgroup and is the symmetric group on B objects. We will use the cyclic notation commonly used with symmetric groups to represent a substitution which is a permutation. We will use s to denote a substitution which is a permutation.

A substitution which sends each element of B into itself multiplied by an element of H will be called a multiplication. The set of all multiplications contained in the monomial group form a subgroup which is the strong direct sum of groups H_α, each H_α isomorphic to H. We will use v to denote a substitution which is a multiplication and such a substitution will be given by recording only its factors in sequence form.

For the monomial group $\sum(H; B, C, D)$, S(B, D) will denote the subgroup consisting of all permutations, while V(B, C) will denote the subgroup consisting of all multiplications. We may now reinterpt the symbols in the monomial group designation as follows, H a given arbitrary group, B the order of a given set S, C a cardinal number such that for any substitution of the monomial group the number of non-identity factors is less than C, D a cardinal number such that for any substitution of the monomial group the number of elements of S being sent into elements of S distinct from themselves is less than D. It is clear that both C and D must always be less than or equal to B'. In the event $C = D = B'$, the resulting monomial group is refered to as the complete monomial group.

The concept of alternating as associated with permutation groups may be extended in an obvious manner to monomial groups. When considering an alternating monomial group we will indicate this by placing an A as a subscript to $\sum$. In this case the cardinal number D is meaningless unless $D < d$. When all finite even permutations are to be considered the cardinal D will be omitted.

The set of all permutations of the monomial group $\sum(H; B, C, D)$ form a subgroup which will be denoted by $S(B, D)$. This group is well known, and the principal properties of its automorphisms as they relate to this paper will be recorded in the following chapter.

The set of all multiplications of the monomial group $\overline{\sum}(H; B, C, D)$ form a subgroup denoted by $V(B, C)$. This subgroup is moreover a normal subgroup.

Any substitution may be written as the product of a multiplication and a permutation. This shows that any monomial group may be written as the union of the subgroups consisting of all multiplications, and permutations. If we employ E to denote the identity of the group $\sum(H; B, C, D)$, we may write,

$$\sum(H; B, C, D) = V(B, C) \cup S(B, D),$$

$$V(B, C) \cap S(B, D) = E.$$

We say $\sum(H; B, C, D)$ splits over the basis group $V(B, C)$.

A multiplication which has only one distinct factor is called a scalar and will be written $[h]$. The set of all scalars form a subgroup of $\sum(H; B, C, D)$. The scalars are the only elements of the monomial group which commute with all permutations. A scalar $[h]$ will commute will all multiplications if and only if h belongs to the center of H, hence the center of the monomial group is the set of all scalars $[h]$ such that h belongs to the center of H.

CHAPTER II

AUTOMORPHISMS OF THE BASIS AND PERMUTATION GROUPS

In the study of the automorphisms of the various monomial groups we will discuss the automorphisms of the basis group, and isomorphisms of the permutation group with other subgroups of the monomial group, which can be combined in a natural way to form an automorphism of the containing monomial group. For the monomial groups considered, we will be chiefly concerned with the basis group V(B, d), and a variety of permutation groups. We include then a preliminary discussion of the endomorphisms of V(B, d), and automorphisms of some permutation groups.

Theorem 1 All endomorphisms of V_n are obtainable through the possible sets of n^2 endomorphisms T^i_j, i, j = 1, 2, --- , n , of H satisfying the conditions,

$$hT^i_m kT^j_m = kT^j_m hT^i_m, \quad m = 1, \cdots, n, \quad i \neq j,$$

by the correspondence of a general element

$$v = (h_1, h_2, h_3, \cdots, h_n) \text{ to}$$

$(h_1T_1^1h_2T_1^2 \cdots h_nT_1^n,\ h_1T_2^1h_2T_2^2 \cdots h_nT_2^n,\ \cdots)$.

The proof is contained in [4, page 45].

Theorem 2 If T is an endomorphism of V(B, d), then there exists endomorphisms T_j^i of H such that,

(1) $(e, \cdots, e, h_i, e, \cdots)T =$

$(h_iT_1^i, \cdots, h_iT_j^i, \cdots)$, for all $h_i \varepsilon H$.

(2) For all $h \varepsilon H$, and all i, $hT_j^i = e$, for all but a finite number of j.

(3) $h_iT_m^ih_jT_m^j = h_jT_m^jh_iT_m^i$, for all m and all i, j such that $i \neq j$.

Conversely if $\{T_j^i\}$ is a collection of endomorphisms of H, such that (2) and (3) are true, then there exists one and only one endomorphism T of V(B, d) such that (1) is true.

Proof Suppose T is an endomorphism of V(B, d), then,

$(e, \cdots, e, h_j, e, \cdots)T =$

$(k_1, k_2, k_3, \cdots)$.

Let $k_i = h_jT_i^j$, then since T is an endomorphism of V(B, d), each T_j^i maps H onto a subgroup of H, and is moreover an endomorphism of H. Since T is an endomorphism of V(B, d), the image multiplication must be an element of this group, hence for all h ε H, and all i, $hT_j^i = e$, for all but a finite number of j.

The two elements of V(B, d),

$$(e , \cdots , e , h_i, e , \cdots),$$

$$(e , \cdots , e , h_j, e , \cdots), \; i \neq j,$$

commute and hence their endomorphic images commute. That is,

$$h_i T_n^i h_j T_n^j = h_j T_n^j h_i T_n^i, \; i \neq j.$$

Conversely, if $\{T_j^i\}$ be a collection of endomorphisms of H, such that (2) and (3) are true, there exists one and only one endomorphism T of V(B, d) such that (1) is ture. Since the T_j^i are endomorphisms of the group H, and by reason of (2) the correspondence,

$$(e , \cdots , e , h_j, e , \cdots) \text{ to}$$

$$(h_j T_1^j, \; h_j T_2^j, \; h_j T_3^j, \; \cdots),$$

is a correspondence onto a subgroup of V(B, d). It follows from (3) that the correspondence is multiplication preserving. The correspondence T is then an endomorphism of V(B, d). That it is unique follows from the fact that the set of elements of the form

$$(\cdots , e , h_j, e , \cdots)$$

generate the group V(B, d).

We now inquire as to the necessary and sufficient conditions that T be an automorphism of V(B, d). This requirement is that T be one-to-one and onto the group V(B, d), since T is already an endomorphism of V(B, d). That is, given an arbitrary element,

$$v_k = (\cdots , e , k_{j_1} , \cdots , k_{j_m} , e , \cdots)$$

of V(B, d), does there exist an element

$$v_h = (\cdots , e , h_{i_1} , \cdots , h_{i_n} , e , \cdots)$$

such that $v_h T = v_k$. We have that $v_h T =$

$$(h_{i_1} T_1^{i_1} h_{i_2} T_1^{i_2} \cdots h_{i_n} T_1^{i_n}, h_{i_1} T_2^{i_1} h_{i_2} T_2^{i_2} \cdots h_{i_n} T_2^{i_n}, \cdots),$$

where only a finite number of the factors are different from the identity. If equality is to exist between that multiplication and v_k the non-identity factors must occur in the same positions as the non-identity factors of v_k. The equality of factors gives us the following set of equations,

$$h_{i_1} T_{j_w}^{i_1} h_{i_2} T_{j_w}^{i_2} \cdots h_{i_n} T_{j_w}^{i_n} = k_{j_w}, \quad w = 1, \cdots, m.$$

Therefore T is one-to-one and onto if and only if the set of equations have unique solutions h_{i_j}, $j = 1, \cdots, n$, in H.

Thus we may state more precisely, T is an automorphism of V(B, d), if and only if for each finite set M of order m of elements of H, and each finite set of distinct indices A, such that the two sets correspond in a one-to-one manner, there exists a second unique subset N of order n of H, together with an unique set of distinct indices B, where the two sets correspond in a one-to-one manner, such that the set of elements of M, N, A, and B are related in the following manner,

$$h_{i_1} T_{j_w}^{i_1} h_{i_2} T_{j_w}^{i_2} \cdots h_{i_n} T_{j_w}^{i_n} = k_{j_w}, \quad w = 1, \cdots, m,$$

where the $h_{i_t} \in N$, $k_{j_w} \in M$, $i_t \in B$, $j_w \in A$, $t = 1, \cdots, n$, $w = 1, \cdots, m$.

Theorem 3 Every element of $S(B, C)$, $d \leq C \leq B^{\neq}$, may be written as the product of two elements of $S(B, C)$ each having order two.

Theorem 4 Every automorphism of $S(B, B^{\neq})$, $B^{\neq} \geq d$, is an inner automorphism.

Theorem 5 Every automorphism of $S(B, C)$, $d \leq C \leq B^{\neq}$, is the restriction of some automorphism of $S(B, B^{\neq})$ to $S(B, C)$.

Theorem 6 The group of automorphisms of $A(B, B^{\neq})$ is isomorphic to $S(B, B^{\neq})$, $B \geq 5$, $B \neq 6$.

$A(B, B^{\neq})$ is that subgroup of $S(B, B^{\neq})$ consisting of all even permutations contained in $S(B, B^{\neq})$.

The proof of Theorem 3 is found in [2], Theorem 4 in [6], Theorem 5 in [7], and Theorem 6 in [3] and [7].

Theorem 7 If,

(1) N is a normal subgroup of a group G,

(2) G splits over N, $G = N \cup M$, $M \cap N = e$,

(3) M' and N' are groups isomorphic to M and N respectively, α the isomorphism of M to M', β the isomorphism of N to N', N' normal in G', and $G' = M' \cup N'$, $M' \cap N' = e$,

then the correspondence μ, $(mn)\mu = m\alpha n\beta$ defines an isomorphism between G and G' if and only if

$$m\alpha n\beta m^{-1}\alpha = (mnm^{-1})\beta,$$

for all $m \in M$ and all $n \in N$.

Proof Let μ be an isomorphism of G to G', and let $m \in M$, $n \in N$, then,

$$(nm)\mu = (mm^{-1}nm)\mu,$$

$$n\beta m\alpha = m\alpha(m^{-1}nm)\beta,$$

$$(m^{-1})\alpha(n)\beta(m)\alpha = (m^{-1}nm)\beta.$$

Conversely if $(m)\alpha(n)\beta(m^{-1})\alpha = (mnm^{-1})\beta$, we need only show that multiplication is preserved by μ to know that μ is an isomorphism of G to G'. Consider,

$(m_1n_1)\mu(m_2n_2)\mu = (m_1)\alpha(n_1)\beta(m_2)\alpha(n_2)\beta$, we have

$(n_1)\beta(m_2)\alpha = (m_2)\alpha(m_2^{-1}n_1m_2)\beta$, and hence

$$(m_1n_1)\mu(m_2n_2)\mu = (m_1)\alpha(m_2)\alpha(m_2^{-1}n_1m_2)\beta(n_2)\beta =$$

$$= (m_1m_2)\alpha(m_2^{-1}n_1m_2n_2)\beta = (m_1m_2m_2^{-1}n_1m_2n_2)\mu =$$

$$= (m_1n_1m_2n_2)\mu.$$

CHAPTER III

IMAGES OF SOME SUBGROUPS UNDER AUTOMORPHISMS OF THE CONTAINING MONOMIAL GROUP

Theorem 8 The basis group of $\overline{\sum}(H; B, d, d)$ is a characteristic subgroup of $\overline{\sum}(H; B, d, d)$.

Theorem 9 The basis group of $\overline{\sum}_A(H; B, d)$ is a characteristic subgroup of $\overline{\sum}_A(H; B, d)$.

Theorem 10 The basis group of $\overline{\sum}_{A,n}(H)$ for $n \geq 5$, is a characteristic subgroup of $\overline{\sum}_{A,n}(H)$.

$\overline{\sum}_{A,n}(H)$ is that subgroup of the complete monomial group formed from the given group H, and a set of order n, consisting of all even monomial substitutions contained in the complete monomial group.

The proofs of Theorems 8, 9, and 10 are found in [1]. We will extend the results of Theorem 8 to show that the basis group of $\overline{\sum}(H; B, d, C)$, $d \leq C \leq B^{\neq}$, is a characteristic subgroup of $\overline{\sum}(H; B, d, C)$.

Theorem 11 If $d \leq C \leq B^{\neq}$, $d \leq D \leq B^{\neq}$, and N is a subgroup of V(B, d), then N is normal in $\overline{\sum}(H; B, d, d)$ if and only if N is normal in $\overline{\sum}(H; B, C, D)$.

Proof Suppose N is a normal subgroup of $\overline{\sum}(H; B, d, d)$ and N is contained in the basis group V(B, d).

Let $vs \in \bar{\sum}(H; B, C, D)$ and $v' \in N$, and consider $(vs)(v')(vs)^{-1}$. We may, by reason of Theorem 3, write s as a product of elements s_1 and s_2, where the order of s_1 and s_2 is two, and hence each is the product of disjoint transpositions. Our product of consideration may then be recorded as $(v)(s_1 s_2)(v')(s_1 s_2)^{-1}(v)^{-1}$.

Define $F(v)$, for any multiplication v to be the set of indices i such that the i-th factor of v is different from the identity. The order of $F(v')$ is finite. If (x_i, x_j) is a transposition of s_2 such that neither i nor j belong to $F(v')$, then (x_i, x_j) commutes with the remaining transpositions of s_2 as well as with v', so we may eliminate all such transpositions from s_2. Denote the depressed s_2 by s_2', which will consist of only those transpositions which move some x_i where i belongs to the indexing set $F(v')$. But since the order of $F(v')$ is finite s_2' belongs to $S(B, d)$. But N is normal in $\bar{\sum}(H; B, d, d)$ and hence we have $(s_2')(v')(s_2')^{-1} \in N$. Similarly we may treat s_1 eliminating those transpositions (x_i, x_j) such that neither i nor j belong to $F(s_2'v's_2'^{-1})$, causing s_1 to be depressed to an element s_1' of $\bar{\Sigma}(H; B, d, d)$. We then see that $(s_1's_2')(v')(s_1's_2')^{-1} \varepsilon N$. Finally conjugation by v is equivalent to conjugation by v_1 where v_1 has factors agreeing with v in those positions i such that $i \varepsilon F(s_1's_2'v's_2'^{-1}s_1'^{-1})$ and the remaining factors of v_1 are the identity. Then $v_1 \varepsilon \bar{\Sigma}(H; B, d, d)$, and once more the normality of N in this group insures that $(v_1s_1's_2')(v')(v_1s_1's_2')^{-1} \varepsilon N$, and hence $(vs)(v')(vs)^{-1} \varepsilon N$. We have shown that if N is contained in the basis group of $\bar{\Sigma}(H; B, d, d)$ and is normal in $\bar{\Sigma}(H; B, d, d)$, Then N is normal in $\bar{\Sigma}(H; B, C, D)$, $d \leq C \leq B'$, $d \leq D \leq B'$.

Conversely if N is contained in V(B, d) and if N is normal in $\bar{\Sigma}(H; B, C, D)$, it is clear that N is normal in $\bar{\Sigma}(H; B, d, d)$, which establishes the theorem.

This together with the results of [1, page 77] gives us the following theorem characterizing all normal subgroups of $\bar{\Sigma}(H; B, C, D)$ which are contained in the basis group V(B, d).

Theorem 12 Any normal subgroup N of $\Sigma(H;\ B,\ C,\ D)$, $d \leq C \leq B^{+}$, contained in the subgroup $V(B,\ d)$ is obtained by the following construction. Let subgroups G and G_1 of H be chosen such that,

(1) G and G_1 are normal subgroups of H with G containing G_1,

(2) G/G_1 belongs to the center of H/G_1,

then N is a subgroup of $V(B,\ d)$ consisting of elements of the form,

$$(e , \cdots , e , g_{i_1} , \cdots , g_{i_n} , e , \cdots)$$

where the g_{i_j} belong to G and the product of all non-identity factors belong to G_1.

Theorem 13 If,

(1) M is a normal subgroup of $\Sigma(H;\ B,\ d,\ C)$, $d \leq C \leq B^{+}$,

(2) M is not contained in $V(B,\ d)$,

(3) $N = M \cap V(B,\ d)$,

then,

(1) N is a normal subgroup of $\Sigma(H;\ B,\ d,\ C)$,

(2) the structure of N is as outlined in Theorem 12 such that $G = H$, and H/G_1 is abelian.

Proof Since the intersection of two normal subgroups is a normal subgroup, N is a normal subgroup of $\overline{\sum}(H; B, d, C)$.

Let $u \in M$, $u \notin N$, then there exists i, j such that $i \neq j$ and,

$$u = \begin{pmatrix} \cdots & x_i & \cdots \\ & h_i x_j & \end{pmatrix}.$$

Let $v = (\ \cdots\ , k_i, e\ , \cdots\ , e\ , k_j, e\ , \cdots\ )$ be an element of V(B, d). Then $u^{-1}v^{-1}uv \in N$. The j-th factor of the commutator is $h_i^{-1}k_i^{-1}h_i k_j$, which is an arbitrary element of H since k_i and k_j are arbitrary. Hence $G \neq H$.

Theorem 14 The basis group V(B, d) is a characteristic subgroup of $\overline{\sum}(H; B, d, C)$, $d \leq C \leq B'$.

Proof The proof follows closely the proof that the basis group is a characteristic subgroup of $\overline{\sum}(H; B, d, d)$, as contained in [1].

We deny the theorem, then there exists an automorphism μ such that $V(B, d)\mu$ is not contained in V(B, d). There exists a normal subgroup M such that $M\mu = V(B, d)$. Then $V(B, d)\mu^{-1} = M$, and

V(B, d) is not contained in $V(B, d)\mu^{-1} = M$.

Σ(H; B, d, C)/V(B, d) is isomorphic to S(B, C).

Moreover Σ(H; B, d, C)/M is isomorphic to S(B, C), under the isomorphism α, $(Mu)\alpha = s$, where s is defined by the equalities,

$$(Mu)\mu = M\mu u\mu = V(B, d)(u\mu) = V(B, d)(vs) = V(B, d)s.$$

Let groups K and N be defined by ,

$$K = V(B, d) \cup M, \quad N = V(B, d) \cap M.$$

Both K and N are normal in Σ(H; B, d, C). The quotient group K/M is a normal subgroup of Σ(H; B, d, C)/M, and since V(B, d) is not contained ih M, K/M is not the identity. Then K/M must be isomorphic to a non-identity normal subgroup of S(B, C). But the normal subgroups of S(B, C) are the groups A(B, d) and S(B, D), $D \leq C$, as is shown in [7]. Each of the normal subgroups of S(B, C) are non-abelian and hence K/M is non-abelian.

K/M is isomorphic to V(B, d)N by reason of the second isomorphism law. The form of N was determined in Theoems 12 and 13. We may establish an isomorphism between V(B, d)/N and H/G_1, but H/G_1 is abelian, hence so are

V(B, d)/N and K/M. But this is a contradiction and hence our assumption was false. This establishes the theorem.

Theorem 15 If $G = N \cup M$, $N \cap M = e$, N a characteristic subgroup of G, μ an automorphism of G, $m\mu = n'm'$, $(m)\lambda = m'$, then λ is an automorphism of M.

Proof λ is multiplication preserving.

$(m_1m_2)\mu = (m_1)\mu(m_2)\mu = n_1'm_1'n_2'm_2' =$

$= (n_1'm_1'n_2'm_1'^{-1})(m_1'm_2') = n_3'm_1'm_2'$, and hence

$(m_1)\lambda = m'$, $(m_2)\lambda = m_2'$, $(m_1m_2)\lambda = m_1'm_2'$,

$(m_1)\lambda(m_2)\lambda = (m_1m_2)\lambda$.

The correspondence λ is onto. Let $m \varepsilon M$, then $m\mu^{-1} = n'm'$, $(n'm')\mu = m$, $(n')\mu(m')\mu = m$, $m'\mu = (n'^{-1})\mu m$, hence $(m')\lambda = m$.

Theendomorphism λ of M has kernel e, since N is a characteristic subgroup of G, and hence, $(m)\mu = n'm'$, $m' = e$, if and only if $m = e$. Then λ is an automorphism of M.

Corollary 1 Let μ be an automorphism of $\overline{\sum}(H; B, d, C)$, $d \leq C \leq B^{\not{}}$, and let $s \varepsilon S(B, C)$,

$s\mu = v's'$, $s\lambda = s'$, then λ is an automorphism of $S(B, C)$.

Proof $\Sigma(H; B, d, C)$ splits over the basis group $V(B, d)$. $V(B, d)$ is a characteristic subgroup of $\Sigma(H; B, d, C)$ by reason of Theorems 8 and 14. The Corollary then follows from Theorem 15.

Corollary 2 Let μ be an automorphism of $\Sigma_{A,n}(H)$, $s \in A_n$, $(s)\mu = v's'$, $(s)\lambda = s'$, then λ is an automorphism of A_n.

Proof $\Sigma_{A,n}(H)$ splits over the basis group V_n. V_n is a characteristic subgroup of $\Sigma_{A,n}(H)$ by reason of Theorem 10. The Corollary then follows from Theorem 15.

Corollary 3 Let μ be an automorphism of $\Sigma_A(H; B, d)$, $s \in A(B)$, $(s)\mu = v's'$, then the correspondence λ, $(s)\lambda = s'$ is an automorphism of $A(B)$.

Proof $\Sigma_A(H; B, d)$ splits over the basis group $V(B, d)$, and $V(B, d)$ is a characteristic subgroup of $\Sigma_A(H; B, d)$ by reason of Theorem 9. The Corollary then follows from Theorem 15.

Theorem 16 $\Sigma(H; B, d, d)$ is a characteristic subgroup of $\Sigma(H; B, d, C)$, $d \le C \le B$.

Proof Let μ be an automorphism of $\Sigma(H; B, d, C)$ and $vs \in \Sigma(H; B, d, d)$. Then consider $(vs)\mu = (v)\mu(s)\mu$. Since $V(B, d)$ is a characteristic subgroup of $\Sigma(H; B, d, C)$, $(v)\mu \in V(B, d) \subset \Sigma(H; B, d, d)$. We must then conclude that $(s)\mu \in \Sigma(H; B, d, d)$, for all $s \in S(B, C)$ in order to establish the theorem. $(s)\mu$ is some element $v's' \in \Sigma(H; B, d, C)$. According to Corollary 1 of Theorem 15 the correspondence s to s' induced by μ defines an automorphism of $S(B, C)$. Then according to Theorems 4 and 5,

$(s)\mu = v'(sI_{s^+})$, where $s^+ \in S(B, B^{\neq})$ and I_{s^+} is is the automorphism induced on $S(B, C)$ by μ. If $s \in S(B, d)$, and since $S(B, d)$ is normal in $S(B, C)$, $(sI_{s^+}) \in S(B, d)$. Then $(vs)\mu = (v)\mu(v')(sI_{s^+})$. Each member of this product is an element of $\Sigma(H; B, d, d)$, hence the product is and element of $\Sigma(H; B, d, d)$. Thus any automorphism of $\Sigma(H; B, d, C)$ takes elements of $\Sigma(H; B, d, d)$ into $\Sigma(H; B, d, d)$, and the theorem is established.

Theorem 17 The group $\Sigma_{A,n}(H)$ splits over the basis group, $\overline{\Sigma}_{A,n}(H) = V_n \cup T$, $V_n \cap T = E$.

The group T is conjugate to some group T_o obtained as follows. Let G be a subgroup of H which is the homomorphic image of A_{n-1}. Let $g_4, \cdots, g_n$ be generators of G satisfying the following relations,

(1) $(g_i)^3 = e$, $i = 4, \cdots, n$.

(2) $(g_i g_j)^2 = e$, where $i \neq j$.

Let $s_i = (1, i, 2)$ for $i = 3, \cdots, n$ generate the group A_n. Then the elements of T_o are obtained from the elements of A_n by the isomorphism ϕ defined by

$$s_3 \phi = (e, e, e, g_4, \cdots, g_n)(1, 3, 2)$$

$$s_i \phi = (e, g_i, g_i^2, g_i^2 g_4, \cdots, g_i^2 g_{i-1}, g_i^2, g_i^2 g_{i+1}, \cdots, g_i^2 g_n)(1, i, 2)$$

for $i = 4, \cdots, n$.

The proof of the theorem is contained in [1].

CHAPTER IV

AUTOMORPHISMS OF $\Sigma(H;\ B,\ d,\ C)$, $d \leq C < B^{+}$

We will first find the automorphism group of $\Sigma(H;\ B,\ d,\ d)$ and then the automorphism group of $\Sigma(H;\ B,\ d,\ C)$, $d < C < B^{+}$. By reason of Theorem 16 the problem of finding automorphisms of $\Sigma(H;\ B,\ d,\ C)$ is made easy once the automorphisms of $\Sigma(H;\ B,\ d,\ d)$ are known. It has seemed advisable to treat the problem in the two cases even though some duplication in calculations is involved.

Before proceding to the problem of determining the automorphism group of $\Sigma(H;\ B,\ d,\ d)$ we make the following considerations. If T is any automorphism of the group H, we define an automorphism T' of V(B, C), $d \leq C \leq B^{+}$, by the correspondence,

$$(h_1, h_2, h_3, \cdots)T' =$$

$$(h_1T, h_2T, h_3T, \cdots).$$

Let I denote the identity automorphism of S(B, D), $d \leq D \leq B^{+}$, then according to Theorem 7 the correspondence T^{+}, $(vs)T^{+} = (v)T'(s)I$, for all $v \in V(B, C)$ and all $s \in S(B, D)$ is an automorphism of the group

$\bar{\geq}(H; B, C, D)$ if and only if,

$$(s)I(v)T'(s^{-1})I = (svs^{-1})T'.$$

Since $V(B, C)$ is a normal subgroup of $\bar{\geq}(H; B, C, D)$, this is an equality between multiplications, and it is easy to see that the corresponding factors of the two multiplications are equal. Hence T^+ is an automorphism of $\bar{\geq}(H; B, C, D)$.

In a similar manner we may associate with any endomorphism K of the group H and endomorphism K^+ of $V(B, C)$.

Theorem 18 μ is an automorphism of $\bar{\geq}(H; B, d, d)$ if and only if there exists,

(1) s^+ an element of $S(B, B^+)$,

(2) v^+ an element of $V(B, B^+)$,

(3) T an automorphism of H,

such that,

$$(u)\mu = (u)T^+I_{s^+}I_{v^+}, \text{ for all } u \in \bar{\geq}(H; B, d, d).$$

Proof Suppose μ is an automorphism of $\bar{\geq}(H; B, d, d)$. Then $\bar{\geq}(H; B, d, d) = V(B, d)\mu \cup S(B, d)\mu$. But $V(B, d)$, by reason of Theorem 8, is a characteristic subgroup of $\bar{\geq}(H; B, d, d)$, hence $\bar{\geq}(H; B, d, d) = V(B, d) \cup S(B, d)\mu$, and $V(B, d) \cap S(B, d)\mu = E$.

There exists an isomorphism between S(B, d) and S(B, d)μ, whose form we now seek to discover. Since S(B, d)μ is contained in $\overline{\sum}$(H; B, d, d), the image of any element s ε S(B, d) must have the form v's', where v' ε V(B, d), s' ε S(B, d). We have seen in Corollary 1 of Theorem 15 that the correspondence s to s' is an automorphism of S(B, d), and hence there must exist an element s^+ ε S(B, B^+) such that $s' = (s)I_{s^+}$, since according to Theorems 4 and 5 all automorphisms of S(B, d) have this form. The element s^+ is the element whose existence was asserted in (1) of the theorem.

Any element of S(B, d) may be written as the product of a finite number of elements of the form (1, i). Hence to discover the image of (1, i) under μ, is to know the image of all permutations. We therefore reduce our study of sμ to that of(1, i)μ. $(1, i)\mu = v_i s'$, where $s' = (1, i)I_{s^+}$.

We next proceed to the characterization of v_i and the calculation of the multiplication v^+ of V(B, B^+).

Since the order of any transposition is two, we have,

$$[(1, i)\mu]^2 = [v_i(1s^+, is^+)]^2 = E.$$

This equality can exist if and only if each factor of v_i has order two except possibly the $1s^+$ and is^+ factors, and moreover the $1s^+$ and is^+ factors must be inverses of one another.

We have in Theorem 2 discovered the form which all endomorphisms of V(B, d) must have, and hence the form of all automorphisms of this group. For an arbitrary element v of V(B, d),

$$v = (\cdots , e , h_{i_1}, \cdots , h_{i_n} , e , \cdots)$$

we have,

$$(v)\mu = (h_{i_1}T_1^{i_1}h_{i_2}T_1^{i_2}\cdots h_{i_n}T_1^{i_n}, \quad \cdots$$

$$\cdots, h_{i_1}T_j^{i_1}h_{i_2}T_j^{i_2}\cdots h_{i_n}T_j^{i_n}, \cdots),$$

where the T_j^i are endomorphisms of the group H, and only a finite number of the factors of the multiplication are different from the identity.

In the calculations which follow the subscript of an element h will always indicate the position of h in a multiplication, that is h_j will be the j-th factor of some multiplication v. When ever we require two factors of an element which is a multiplication to be distinct we will indicate this by employing superscripts, distinct superscripts indicate that the two factors are distinct elements of H. Whenever a multiplication has undergone a transofrmation by a permutation we will employ superscripts to indicate, after the shuffling of factors, the equality existing between the factors of the original and resulting multiplication. Like superscripts indicating the same group element.

Let us consider generating elements,

$s = (1, i)$ of $S(B, d)$,

$v = (\cdots , e , h_j , e , \cdots)$ of $V(B, d)$.

Since μ is an automorphism of $\overline{\sum}(H; B, d, d)$ we have,

$(s)\mu(v)\mu(s^{-1})\mu = (svs^{-1})\mu$, where

$(1, i)\mu =$

$= (\cdots , e , k_{i_1} , \cdots , k_{i_n} , e , \cdots)(1s^+, is^+)$,

$(\cdots , e , h_j , e , \cdots)\mu =$

$(h_jT_1^j, h_jT_2^j, h_jT_3^j, \cdots)$,

where only finitely many of the factors are different from the identity. We compute this equality considering two cases.

Case 1 Suppose $j \neq 1$, $j \neq i$. Then since $(svs^{-1}) = v$, the equality reduces to,

$$(s)\mu(v)\mu(s^{-1})\mu = (v)\mu, \text{ or}$$

$$[\![(\cdots , e , k_{i_1} , \cdots , k_{i_n} , e , \cdots)(1s^+, is^+)$$

$$\times (h_jT_1^j, h_jT_2^j, h_jT_3^j, \cdots) \times$$

$$(1s^+, is^+)(\cdots , e , k_{i_1}^{-1}, \cdots , k_{i_n}^{-1}, e , \cdots]\!]$$

$$= (h_jT_1^j, h_jT_2^j, h_jT_3^j, \cdots).$$

Direct computation on the left side of the equality yields the following multiplication,

$$(\cdots , h_jT_m^j, \cdots , k_{i_1} h_jT_{i_1}^j k_{i_1}^{-1}, \cdots$$

$$\cdots, k_{1s^+}h_jT_{is^+}^j k_{1s^+}^{-1}, \cdots , k_{is^+}h_jT_{1s^+}^j k_{is^+}^{-1}, \cdots$$

$$\cdots, k_{i_t} h_jT_{i_t}^j k_{i_t}^{-1}, \cdots).$$

Then the resulting equality between multiplications demands equality between corresponding factors. Hence we have,

(i) $hT_{is^+}^j = k_{is^+}hT_{1s^+}^j k_{is^+}^{-1}$,

(ii) $hT^{j}_{i_m} = k_{i_m} hT^{j}_{i_m} k^{-1}_{i_m}$,

for $m = 1, \cdots, n$, $i_m \neq 1s^{+}$, $i_m \neq is^{+}$, and $j \neq 1s^{+}$, $j \neq is^{+}$. Since in equality (i) and (ii) h represents the same group element we have dropped the subscript.

Case 2 Suppose $j = 1$ or $j = i$. Either equality will yield the same result, and hence both cases are included in one consideration. The calculations recorded are for $j = 1$.

$$v = (h^{1}_{1}, e , \cdots), \quad s = (1, i),$$

$$(svs^{-1}) = (\cdots , e , h^{1}_{i} , e , \cdots), \text{ and}$$

$$(s)\mu(v)\mu(s^{-1})\mu = (\cdots , e , h^{1}_{i} , e , \cdots)\mu, \text{ or}$$

$$\Big[(\cdots , e , k_{i_1} , \cdots , k_{i_n} , e , \cdots)(1s^{+}, is^{+})$$

$$\rtimes (h^{1}_{1}T^{1}_{1}, h^{1}_{1}T^{1}_{2}, h^{1}_{1}T^{1}_{3}, \cdots) \rtimes$$

$$(1s^{+}, is^{+})(\cdots , e , k^{-1}_{i_1} , \cdots , k^{-1}_{i_n} , e , \cdots \Big]$$

$$= (h^{1}_{i}T^{i}_{1}, h^{1}_{i}T^{i}_{2}, h^{1}_{i}T^{i}_{3}, \cdots).$$

Direct computation on the left side of the equality yields the following multiplication,

$$(\cdots , h^{1}_{1}T^{1}_{m}, \cdots , k_{i_1} h^{1}_{1}T^{1}_{i_1} k^{-1}_{i_1} , \cdots$$

$\cdots, k_{1s^+}h_1^1T_{is^+}^1k_{1s^+}^{-1}, \cdots, k_{is^+}h_1^1T_{1s^+}^1k_{is^+}^{-1}, \cdots$

$\cdots, k_{i_t}h_1^1T_{i_t}^1k_{i_t}^{-1}, \cdots).$

Then the resulting equality between multiplications demands the following equality between factors.

(iii) $hT_{i_m}^i = k_{i_m}hT_{i_m}^1k_{i_m}^{-1},$

$m = 1, \cdots, n, i_m \neq 1s^+, i_m \neq is^+.$

(iv) $hT_{is^+}^i = k_{is^+}hT_{1s^+}^1k_{is^+}^{-1},$

(v) $hT_{1s^+}^i = k_{1s^+}hT_{is^+}^1k_{1s^+}^{-1}.$

The equalities (i) through (v) are restrictions on the endomorphisms T_j^i of H. We may now further our study of images of multiplications under μ in view of these restrictions.

Suppose $j \neq 1$ and consider,

$(\cdots, e, h_j, e, \cdots)\mu =$

$(h_jT_1^j, h_jT_2^j, h_jT_3^j, h_jT_4^j, \cdots).$

According to restriction (i) each factor in the image multiplication is conjugate to $h_jT_{1s^+}^j$ except the factor $h_jT_{js^+}^j$. But since μ is an automorphism of V(B, d), the image multiplication must be an element of V(B, d), hence only finitely many of the factors may be different from the identity. It then follows that every factor save the factor $h_j T^j_{js^+}$ must be the identity and in this case the factor $h_j T^j_{js^+}$ must be different from the identity. That is for j different from 1,

$$(\cdots , e , h_j , e , \cdots)\mu =$$

$$(\cdots , e , h_j T^j_{js^+} , e , \cdots).$$

We next consider the case where $j = 1$.

$$(h_1 , e , \cdots)\mu = (h_1 T^1_1, h_1 T^1_2, h_1 T^1_3, \cdots).$$

If we rewrite (v) in the form,

$$hT^1_{1s^+} = k_{is^+} hT^i_{1s^+} k^{-1}_{is^+},$$

we see that every factor of the above recorded image multiplication is conjugate to some element $h_i T^i_{1s^+}$. But we have observed in the previous consideration that for $j \neq 1$, $h_j T^j_{1s^+}$ is the identity element, and hence all factors of the image multiplication are the identity except the $1s^+$ factor. That is,

$$(h_1 , e , \cdots)\mu = (\cdots , e , h_1 T^1_{1s^+} , e , \cdots).$$

In the beginning we assumed the most general representation of an automorphism of V(E, d) for μ, and for the correspondence assigned we have only an endomorphism of V(B, d). We must now determine what further restrictions are necessary to insure that the correspondence is an automorphism of V(B, d). Suppose we are given an arbitrary multiplication of V(B, d),

$$(\cdots , e , h'_{i_1} , \cdots , h'_{i_n} , e , \cdots).$$

We ask if this multiplication arose from the image of some other multiplication under μ. This is equivalent to asking under what conditions will the set of equations,

$$h_{i_m} T^{i_m}_{i_m s^+} = h'_{i_m} , m = 1, \cdots , n,$$

have unique solutions h_{i_m}, m = 1, ⋯, n, in H. Such a unique set of solutions can exist if and only if the $T^{i}_{is^+}$ are automorphisms of the group H. With this added restriction we have completed the characterization of the images of multiplications, but will latter employ (iv) to change the representation.

Let us refer to equality (ii) restricting the endomorphisms whose subscripts are different from $1s^+$ and is^+. We have seen that if i_m be different from js^+

then $hT^{j}_{i_m}$ is the identity. In case 1, which produced equality (ii), we have restricted j to be different from 1 and i, so that j may be so chosen that $js^{+} = i_m$, and the following equality results,

$$k_{js^{+}} hT^{j}_{js^{+}} k^{-1}_{js^{+}} = hT^{j}_{js^{+}}.$$

Inasmuch as we have required that $T^{j}_{js^{+}}$ be an automorphism of H, we can only conclude that $k_{js^{+}}$ belongs to the center of the group H. That is the multiplication componnent of the image of (1, i) under μ must have every factor except possibly the $1s^{+}$ and the is^{+} factors belonging to the center of the group H.

We will now show that the factors of this multiplication which do not occupy the $1s^{+}$ and is^{+} positions are the identity element.

Since (1, i)(1, j) has order three, we have,

$$[(1, i)(1, j)\mu]^3 =$$

$$[(k_1, k_2, k_3, \cdots)(1s^{+}, is^{+}) \times$$

$$(h_1, h_2, h_3, \cdots)(1s^{+}, js^{+})]^3 = E.$$

By direct calculation we see that if n be different from $1s^{+}$, is^{+}, and js^{+}, then the n-th factor is,

$$k_n h_n k_n h_n k_n h_n = e.$$

We have prevously seen that both h_n and k_n belong to the center of the group H, and moreover each has order two. It then follows that h_n and k_n are inverses of one another. The $1s^+$ factor of the above product is,

$$k_{1s^+} h_{is^+} k_{is^+} h_{1s^+} k_{js^+} h_{js^+} = e,$$

which, in view of the centrality of the elements k_{js^+} and h_{is^+} together with the equality,

$$k_{1s^+} k_{is^+} = h_{1s^+} h_{js^+} = e,$$

reduces to, $h_{is^+} k_{js^+} = e$. Since h_{is^+} has order two $h_{is^+} = k_{js^+}$. Thus the factors of the image multiplications of (1, i) and (1, j) are the same if we exclude the $1s^+$, is^+, and js^+ factors, and further the js^+ factor of the multiplication componnent of $(1, i)\mu$ is equal to the is^+ factor of the multiplication componnent of $(1, j)\mu$.

In a similar manner by considering $(1, j)\mu$ and $(1, t)\mu$ where $t \neq i$, $t \neq j$, we find that the ts^+ factor of the multiplication component of $(1, j)\mu$ is equal to the js^+ factor of the multiplication component of $(1, t)\mu$.

But the ts^+ factors of the multiplication component of $(1, i)\mu$ and $(1, j)\mu$ are equal, and the js^+ factor of the multiplication component of $(1, i)\mu$ and $(1, t)\mu$ are equal. That is the ts^+ and js^+ factors of the multiplication component of $(1, i)\mu$ are equal, and hence all factors of the multiplication componenet of $(1, i)\mu$ except possibly the $1s^+$ and is^+ factors. But this multiplication is an element of $V(B, d)$ and hence all factors except possibly the $1s^+$ and is^+ factors must be e. Then,

$$(1, i)\mu = (\cdots , e , k_{1s^+}, e , \cdots , e , k_{is^+}, e , \cdots)(1s^+, is^+).$$

Let v^+ be the multiplication of $V(B, B^+)$ whose $1s^+$ factor is e, and whose is^+ factor is the is^+ factor k_{is^+} of the multiplication component of $(1, i)\mu$. This multiplication v^+ is the element of $V(B, B^+)$ whose existance we asserted in (2) of the theorem.

We have seen that,

$$(h_1, e , \dots)\mu = (\dots , e , h_1T^1_{1s^+}, e , \dots)$$

where $T^1_{1s^+}$ is an automorphism of H. Let $T^1_{1s^+}$ generate in a manner described in the discussion preceding this theorem, an automorphism T^+ of $\overline{\Sigma}(H; B, C, D)$, $d \leq C$, $D \leq B^+$, which is moreover an automorphism of $\overline{\Sigma}(H; B, d, d)$ since $\overline{\Sigma}(H; B, d, d)$ is a characteristic subgroup of $\overline{\Sigma}(H; B, D, C)$. This is the automorphism which forms the first component of μ, and $T^1_{1s}+$ is the automorphism of H whose existance we asserted in (3) of the theorem.

If we now refer to restriction (iv) on the automorphisms $T^i_{is}+$, $hT^i_{is}+ = k_{is}+hT^1_{1s}+k^{-1}_{is}+$, we observe that we may write,

$$(\cdots , e , h_j , e , \cdots)\mu =$$

$$(\cdots , e , k_{js}+h_jT^1_{1s}+k^{-1}_{js}+ , e , \cdots),$$

$$(1, i)\mu =$$

$$(\cdots , e , k_{1s}+, e , \cdots , e , k_{is}+ , e , \cdots)$$

$$\times (1s^+, is^+)$$

which we may now record in simplified form as,

$$(\cdots , e , h_J, e , \cdots)\mu =$$

$$(\cdots , e , h_J, e , \cdots)T^+I_s+I_v+,$$

$(1, i)\mu = (1, i)T^+I_s+I_v+$, and hence for an arbitrary element u of $\overline{\sum}(H; B, d, d)$,

$$(u)\mu = (u)T^{+}I_{s^{+}}I_{v^{+}}.$$

Conversely suppose we are given an element s^{+} of $S(B, B^{+})$, $v^{+} \varepsilon V(B, B^{+})$, T an automorphism of H. Then $I_{s^{+}}$, $I_{v^{+}}$, and T^{+} are each automorphisms of $\overline{\sum}(H; B, C, D)$, $d \leq C$, $D \leq B^{+}$, and hence the product $T^{+}I_{s^{+}}I_{v^{+}}$ is an automorphism of the group. Then the groups $\overline{\sum}(H; B, d, d)$ and $\overline{\sum}(H; B, d, d)T^{+}I_{s^{+}}I_{v^{+}}$, are isomorphic. But each of the automorphisms T^{+}, $I_{s^{+}}$, and $I_{v^{+}}$ take elements of $\overline{\sum}(H; B, d, d)$ into $\overline{\sum}(H; B, d, d)$. Hence the restriction of the automorphism $T^{+}I_{s^{+}}I_{v^{+}}$ of $\overline{\sum}(H; B, C, D)$ to $\overline{\sum}(H; B, d, d)$ is an automorphism of the latter group. This is the automorphism μ, and the proof of the theorem is complete.

Corollary 1 $\mu = T^{+}I_{s^{+}}I_{v^{+}}$ is an inner automorphism of $\overline{\sum}(H; B, d, d)$ if and only if T^{+} is generated by an inner automorphism $T = I_{h^{-1}}$ of H, $s^{+} \varepsilon S(B, d)$, v^{+} is the product of an element of $V(B, d)$ and the scalar $[h]$ of $\overline{\sum}(H; B, B^{+}, B^{+})$.

Proof If T^+ is generated by the automorphism $I_{h^{-1}}$, $s^+ \varepsilon S(B, d)$, $v^+ = v_1^+[h]$, $v_1^+ \varepsilon V(B, d)$, $[h] \varepsilon V(B, B^+)$, then,

$$\mu = T^+ I_{s^+} I_{v_1^+[h]} = T^+ I_{s^+} I_{[h]} I_{v_1^+} =$$

$$T^+ I_{[h]} I_{s^+} I_{v_1^+} = I_{s^+} I_{v_1^+} = I_{v_1^+ s^+},$$

and hence μ is an inner automorphism of $\Sigma(H; B, d, d)$.

Conversely suppose μ is an inner automorphism of $\Sigma(H; B, d, d)$, then,

$$\mu = I_u = I_{v's'} = I_{s'} I_{v'}.$$

Hence if $h = e$, and $T = I_{h^{-1}}$, $\mu = T^+ I_{s'} I_{v^+}$, where, $s' \varepsilon S(B, d)$, $v^+ \varepsilon V(B, B^+)$, $v^+ = v'[h] =$ $= v' \varepsilon V(B, d)$.

Theorem 19 The group of three-tuples (T, s^+, v^+), where T is an automorphism of the group H, $s^+ \varepsilon S(B, B^+)$ $v^+ \varepsilon V(B, B^+)$, with the operation,

$$(T_1, s_1^+, v_1^+)(T_2, s_2^+, v_2^+) =$$

$$(T_1 T_2, s_2^+ s_1^+, v_2^+ s_2^+ (v_1^+ T_2^+) s_2^{+-1}),$$

IS homomorphic to the automorphism group of $\Sigma(H; B, d, d)$ under the correspondence λ, $(T, s^+, v^+)\lambda = \mu$, $\mu = T^+ I_{s^+} I_{v^+}$, and the kernel K of λ is the set of all three-tuples (T, s^+, v^+), where s^+ is the identity permutation of $S(B, B^+)$, v^+ is a scalar [h] of $V(B, B^+)$, and T is the inner automorphism $I_{h^{-1}}$ of the group H.

Proof We will first show that the set of three-tuples (T, s^+, v^+) with the above defined operation form a group.

Consider the element (T_o, s_o^+, v_o^+), where T_o is the identity automorphism of the group H, s_o^+ the identity element of the group $S(B, B^+)$, v_o^+ is the identity element of the group $V(B, B^+)$. To demonstrate that this element is the identity element of the set of three-tuples we make the following calculation.

$$(T, s^+, v^+)(T_o, s_o^+, v_o^+) =$$

$$(TT_o, s_o^+s^+, v_o^+s_o^+(v^+T_o^+)s_o^{+-1}) = (T, s^+, v^+).$$

If (T, s^+, v^+) be an arbitrary element of the set we see that, $(T^{-1}, s^{+-1}, s^{+-1}(v^{+-1}T^{-1})s^+)$ is an inverse for this element since,

$$(T, s^+, v^+)(T^{-1}, s^{+-1}, s^{+-1}(v^{+-1}T^{+-1})s^+) =$$

$$(T_o, s_o^+, v_o^+).$$

Then $(T \ , s^{+-1}, s^{+-1}(v^{+-1}T^{+-1})s^+)$ is an element of the set since the first two components clearly belong to the automorphism group of H, and $S(B, B^+)$ respectively and the third component is an element of $V(B, B^+)$ since T^+ restricted to $V(B, B^+)$ is an automorphism of that group, and further since $V(B, B^+)$ is normal in $\overline{\Sigma}(H; B, B^+, B^+)$, conjugation by a permutation of any element of $V(B, B^+)$ produces an element of $V(B, B^+)$.

It follows from the definition of the operation for the set of three-tuples that the set has the closure property. Therefore it remains to demonstrate that the associative law holds.

$$\left[(T_1, s_1^+, v_1^+)(T_2, s_2^+, v_2^+)\right](T_3, s_3^+, v_3^+) =$$

$$\left[T_1T_2, s_2^+s_1^+, v_2^+s_2^+(v_1^+T_2^+)s_2^{+-1}\right](T_3, s_3^+, v_3^+) =$$

$$\left[T_1T_2T_3, s_3^+s_2^+s_1^+, \left\{v_3^+s_3^+ \; v_2^+s_2^+(v_1^+T_2^+)s_2^{+-1}\right\} T_3^+s_3^{+-1}\right] =$$

$$\left[T_1T_2T_3, s_3^+s_2^+s_1^+, v_3^+s_3^+(v_2^+T_3^+)s_2^+(v_1^+T_2^+T_3^+)s_2^{+-1} \; s_3^{+-1}\right].$$

$$(T_1, s_1^+, v_1^+)\left[(T_2, s_2^+, v_2^+)(T_3, s_3^+, v_3^+)\right] =$$

$$(T_1, s_1^+, v_1^+)\left[T_2T_3, s_3^+s_2^+, v_3^+s_3^+(v_2^+T_3^+)s_3^{+-1}\right] =$$

$$\left[T_1T_2T_3, s_3^+s_2^+s_1^+, v_3^+s_3^+(v_2^+T_3^+)s_3^{+-1} s_3^+s_2^+(v_1^+T_2^+T_3^+)s_2^{+-1} s_3^{+-1}\right] =$$

$$\left[T_1T_2T_3, s_3^+s_2^+s_1^+, v_3^+s_3^+(v_2^+T_3^+)s_2^+(v_1^+T_2^+T_3^+)s_2^{+-1}s_3^{+-1}\right].$$

Hence the set of three-tuples with the defined operation form a group.

Let λ be the correspondence between the group of three-tuples and the automorphism group of $\sum(H; B, d, d)$ as defined in the theorem. We will show λ is a homomorphism.

The correspondence λ is onto, for given any automorphism $\mu = T^{+}I_{s^+}I_{v^+}$, there exists a three-tuple, namely (T, s^{+}, v^{+}) such that $(T, s^{+}, v^{+})\lambda = \mu$.

λ is a multiplication preserving correspondence. Let, $(T_1, s_1^{+}, v_1^{+})\lambda = \mu_1$, $(T_2, s_2^{+}, v_2^{+})\lambda = \mu_2$, AND then

$$(T_1, s_1, v_1)\lambda(T_2, s_2, v_2)\lambda = \mu_1\mu_2.$$

Consider, $\mu_1\mu_2 = T_1^{+}I_{s_1^+}I_{v_1^+}T_2^{+}I_{s_2^+}I_{v_2^+} =$

$$T_1^{+}T_2^{+}\left\{\left[T_2^{+-1}I_{s_1^+}I_{v_1^+}T_2^{+}\right]I_{s_2^+}I_{v_2^+}\right\}.$$

We desire to change the form of that portion of the product which occurs within the braces, so let us observe its effect upon an element of the group.

$$(vs)\left[T_2^{+-1}I_{s_1^+}I_{v_1^+}T_2^{+}I_{s_2^+}I_{v_2^+}\right] =$$

$$v_2^{+}s_2^{+}\left[v_1^{+}s_1^{+}vT_2^{+-1}ss_1^{+-1}v_1^{+-1}\right]T_2^{+}\ s_2^{+-1}v_2^{+-1} =$$

$$v_2^{+}s_2^{+}(v_1^{+}T_2^{+})s_1^{+}(vs)s_1^{+-1}(v_1^{+}T_2^{+})^{-1}s_2^{+-1}v_2^{+-1} =$$

$$(vs)I_{s_1^+}I_{v_1^+T_2^+}I_{s_2^+}I_{v_2^+} = (vs)I_{v_2^+s_2^+(v_1^+T_2^+)s_1^+} =$$

$$(vs)I_{v_2^+s_2^+(v_1^+T_2^+)\ s_2^{+-1}s_2^+s_1^+} = (vs)I_{s_2^+s_1^+}I_{v_2^+s_2^+(v_1^+T_2^+)s_2^{+-1}}.$$

That is $\mu_1\mu_2 = T_1^+T_2^+I_{s_2^+s_1^+}I_{v_2^+s_2^+(v_1^+T_2^+)s_2^{+-1}}$,

and hence,

$$(T_1T_2,\ s_2^+s_1^+,\ v_2^+s_2^+(v_1^+T_2^+)s_2^{+-1})\lambda = \mu_1\mu_2.$$

But, $(T_1,\ s_1^+,\ v_1^+)(T_2,\ s_2^+,\ v_2^+)\ \lambda =$

$$T_1T_2,\ s_2^+s_1^+,\ v_2^+s_2^+(v_1^+T_2^+)s_2^{+-1}\ \lambda, \text{ hence}$$

$$(T_1,\ s_1^+,\ v_1^+)\lambda(T_2,\ s_2^+,\ v_2^+)\lambda =$$

$$(T_1,\ s_1^+,\ v_1^+)(T_2,\ s_2^+,\ v_2^+)\ \lambda\ .$$

Therefore λ is a homomorphism from the group of three-tuples onto the automorphism group of $\Sigma(H;\ B,\ d,\ d)$.

We compute the kernel K of λ. Let $\mu_o \in K$.

$$(1,\ i)\mu_o = (1,\ i)T^+I_{s^+}I_{v^+} = (1,\ i)$$

But T^+ acts as the identity automorphism on permutations, and therefore the equality reduces to,

$$(1,\ i)I_{s^+}I_{v^+} = (1,\ i),$$

which can exist for all i if and only if s^+ leaves all i fixed and therefore is the identity permutation. Then, $(1,\ i)I_{v^+} = (1,\ i)$, for all i, if and only if v^+ is a scalar [h].

Consider,

$(h, e, \cdots)\mu_o = (h, e, \cdots)T^+I_{s^+}I_{v^+} =$

$(k\bar{h}Tk^{-1}, e, \cdots) = (h, e, \cdots)$.

This equality can exist if and only if $T = I_{k^{-1}}$.

Thus we have shown that the kernel K of λ, is the set of three-tuples (T, s^+, v^+), where s^+ is the identity permutation of $S(B, B^+)$, v^+ a scalar k of $V(B, B^+)$, T the inner automorphism $I_{k^{-1}}$ of H.

Corollary 1 Let A denote the automorphism group of $\sum(H; B, d, d)$, A_s those elements of A which leave $S(B, d)$ fixed elementwise. Then,

(1) A_s is a subgroup of A, such that any automorphism μ in A_s has the form,

$\mu = T^+I_{[h]}$, $[h]$ a scalar of $V(B, B^+)$.

(2) The set of two-tuples (T, h), T an automorphism of H, h an element of H, form a group with the operation, $(T_1, h_1)(T_2, h_2) =$

$(T_1T_2, h_2(h_1T_2))$.

(3) The group of two-tuples are homorphic to A_s under the homomorphism λ,

$$(T, h)\lambda = \mu, \ \mu = T^{+}I_{[h]}.$$

(4) The kernel K of λ is the set of two-tuples

$$(I_{h-1}, h).$$

Proof The assertions (1) through (4) are immediate consequences of the theorem, since the set of two-tuples form a group isomorphic to a subgroup of the group of three-tuples under the correspondence,

$$(T , h) \longleftrightarrow (T , s_o^{+}, [h]).$$

Theorem 20 μ is an automorphism of $\overline{\Sigma}(H; B, d, C)$, $d < C < B^{+}$, if and only if there exists,

(1) $s^{+} \in S(B, B^{+})$,

(2) $v^{+} \in V(B, d)$,

(3) T an automorphism of H,

such that, $(u)\mu = (u)T^{+}I_s + I_{v^{+}}$, for all $u \in \overline{\Sigma}(H; B, d, C)$.

Proof We have seen in Theorem 16 that $\overline{\Sigma}(H; B, d, d)$ is a characteristic subgroup of $\overline{\Sigma}(H; B, d, C)$, hence if μ is an automorphism of $\overline{\Sigma}(H; B, d, C)$ its restriction to $\overline{\Sigma}(H; B, d, d)$ is an automorphism of that group. We have in Theorem 18 determined all automorphisms of $\overline{\Sigma}(H; B, d, d)$, hence we will be concerned with extending the automorphisms of $\bar{\Sigma}(H; B, d, d)$ to automorphisms of $\bar{\Sigma}(H; B, d, C)$. As is evident from the statement of the theorem not all automorphisms of $\bar{\Sigma}(H; B, d, d)$ may be extended to an automorphism of $\bar{\Sigma}(H; B, d, C)$.

There is determined by μ an element s^+ of $S(B, B^+)$ such that,

$$(s)\mu = (v')(sI_{s^+}), \quad s \in S(B, d).$$

If $s \in S(B, C)$ then,

$$(s)\mu = v's', \; v' \in V(B, d), \; s' \in S(B, C).$$

According to Corollary 1 of Theorem 15 the correspondence λ, $s\lambda = s'$, is an automorphism of $S(B, C)$. The automorphism induced on $S(B, d)$ by μ extends to $S(B, C)$ in one and only one way, by reason of Theorems 4 and 5, hence $\lambda = I_{s^+}$, and the elements $s^+ \in S(B, B^+)$ is the element whose existance was assetted in (1) of the theorem.

Any element $s \in S(B, C)$, according to Theorem 3, may be decomposed into the product of two elements $s_1 s_2$ such that the order of each s_1 and s_2 is two. We will therefore reduce our study of $s\mu$ to that of $s_1\mu$.

We then have,

$$(s_1)\mu = v_1(s_1 I_s+),\ v_1 \in V(B, d).$$

and since s_1 has order two, $\left[v_1(s_1 I_s+)\right]^2 = E$.

We observe the factors of v_1 considering two cases.

Suppose n is an index such that x_n does not belong to the set of elements moved by $s_1 I_s+$, then it follows from the above equality that the n-th factor of v_1 has order two. On the other hand if i is an index such that x_i is moved by $(s_1)I_s+$, then there is an index j such that (x_i, x_j) is a transposition of $(s_1)I_s+$. Then the above equality demands that the i-th and j-th factors of v_1 must be inverses of one another.

If n is an index such that x_n does not belong to the set of elements moved by s_1, we will show that $k_{ns}+$ belongs to the center of the group H. Let,

$$v = (\cdots , e , h_n , e , \cdots)$$

and consider,

$$(s_1 v\ s_1^{-1})\mu = (v)\mu = (\cdots , e , h_n T^n_{ns}+, e , \cdots)$$

$$= (s_1)\mu(v)\mu(s_1^{-1})\mu =$$

$$v_1(s_1I_s+)(\ldots , e , h_nT^n_{ns}+, e , \cdots)(s_1^{-1})I_s+(v_1^{-1}) =$$

$$(\cdots , e , k_{ns}\cdot h_nT^n_{ns}+k^{-1}_{ns}+ , e , \cdots).$$

This equality of multiplications demands the following equality of factors,

$$hT^n_{ns}+ = k_{ns}+hT^n_{ns}+k^{-1}_{ns}+.$$

Since $T^n_{ns}+$ is an automorphism of the group H, it follows that $k_{ns}+$ belongs to the center of H. That is all factors of v_1 belong to the center of H except possibly those factors j such that $x_{js}+$ belongs to the set of elements moved by $(s_1)I_s+$.

We next show that each of these factors which belong to the center of H is moreover the identity element of H. Let,

$$s_1 = (x_1, x_2)(x_3, x_4) \cdots$$

and define an element $s_t \varepsilon S(B, C)$ as follows,

$$s_t = (x_1, x_{t_2})(x_3, x_{t_4}) \cdots$$

where the x_{t_i} do not belong to the set of elements moved by s_1, and hence s_t has order two. The existence of such an element s_t is insured since we have required that $c < B^+$, and hence s_1 must move fewer than B elements. Since $s_t s_1$ has order three, we have

$$[(s_1 s_t)\mu]^3 = E,$$

$$(s_1)\mu = v_1(s_1)I_s+, \quad v_1 = (h_1, h_2, h_3, \cdots),$$

$$(s_t)\mu = v_t(s_t)I_s+, \quad v_t = (k_1, k_2, k_3, \cdots).$$

By direct calculation of the above equality we discover that we have in the $1s^+$ position the factor,

$$h_{1s^+}k_{2s^+}h_{2s^+}k_{1s^+}h_{t_2s^+}k_{t_2s^+} = e.$$

But x_2 does not belong to the set of elements moved by s_t and x_{t_2} does not belong to the set of elements moved by s_1, hence k_{2s^+} and $h_{t_2s^+}$ belong to the center of H, and since, $h_{1s^+}h_{2s^+} = k_{1s^+}k_{t_2s^+} = e$, the factor reduces to $k_{2s^+}H_{t_2s^+} = e$. Then $k_{2s^+} = h_{t_2s^+}$ since since each of the elements has order two.

Consider a third permutation of S(B, C),

$$s_w = (x_1, x_{w_2})(x_3, x_{w_4}) \cdots$$

where the x_{w_i} do not belong to the set of elements moved by s_1 or s_t.

$$(s_w)\mu = v_w(s_w)I_s+, \quad v_w = (f_1, f_2, f_3, \cdots).$$

Then calculations similar to those just performed with the elements s_t and s_w yield,

$$K_{w_i s^+} = f_{t_i s^+} \ , \ i = 2, 4 \ , \ 6 \ , \ \cdots$$

but, $h_{w_i s^+} = k_{w_i s^+}$, and hence,

$$h_{w_i s^+} = f_{t_i s^+} = h_{t_i s^+}.$$

Therefore all factors of v_1 are equal except possibly those factors h_j such that $x_j(s_1)I_{s^+} \neq x_j$. But $v_1 \varepsilon V(B, d)$, and hence all factors of v_1 are e except possibly the factors h_j, j an index such that $x_j(s_1)I_{s^+} \neq x_j$.

We have then the following information reagarding v_1, $s_1\mu = v_1(s_1)I_{s^+}$. If (x_i, x_j) is a transposition of s_1 then $h_{is^+}h_{js^+} = e$. If x_m does not belong to the set of elements moved by s_1 then h_{ms^+} is the identity.

Let us consider $(s_1)\mu(x_i, x_j)\mu$, where (x_i, x_j) is a transposition of s_1. Since (x_i, x_j) is an element of $\Sigma(H; B, d, d)$, a characteristic subgroup of $\Sigma(H; B, d, C)$, we have , $(x_i, x_j)\mu =$

$(\ \cdots, e \ , h_{is^+}, e \ , \cdots \ , e \ , h_{js^+} \ , e \ , \cdots)(x_i s^+; x_j s^+)$,

$= v_1'(s_1')I_{s^+}$, where $s_1' = s\ (x_i, x_j)$.

Since (x_i, x_j) is not a transposition of s_1' the is^+-th and js^+-th factors of v_1' are e, but the is^+-th factor of v_1' is the product of the is^+-th factor of v_1 and the js^+-th factor of the multiplication component of $(x_i, x_j)\mu$. Hence the is^+ and js^+ factors of v_1 are identical with the factors in the corresponding positions of the multiplication component of $(x_i, x_j)\mu$. The multiplication component of $(x_i, x_j)\mu$ was formed by conjugating $(x_i, x_j)I_{s^+}$ with an element $v_1^+ \ \varepsilon\ V(B, B^+)$. It is evident that the is^+ and js^+ factors of v_1 can be formed in the same manner.

In the event that s_1 moves an infinite number of elements, it is not possible that all h_{js^+} be different from the identity, yet we have seen that all h_{js^+} are formed by conjugation by the element y_1^+ determined by the restricition of μ to $\sum(H;\ B,\ d,\ d)$. If (x_i, x_j) be a transposition of s_1, and if the is^+ and js^+ factors of v_1^+ are distinct, then the is^+ and js^+ factors of v_1 will be distinct. We must then restrict v_1^+ in such a manner that this situation can happen only a finite number of times. Hence we must require that no two factors of v_1^+ be repeated infinitely often, and there must not occur in v_1^+ an infinite number of distinct factors. Under these restrictions v_1 will always be an element of $V(B, d)$.

v_1^+ so restricted may then be written as a product $v^+[k]$, where $v^+ \varepsilon V(B, d)$, and $[k] \varepsilon V(B, B^+)$ k being that factor of v_1^+ which was repeated infinitely often. Then,

$$\mu = T_1^+ I_{s^+} I_{v^+} = T_1^+ I_{s^+} I_{v^+[k]} =$$

$$T_1^+ I_{s^+} I_{[k]} I_{v^+} = T_1^+ I_{[k]} I_{s^+} I_{v^+} = T^+ I_{s^+} I_{v^+},$$

where T^+ is generated by the automorphism TI_k of H.

Conversely given an element $s^+ \varepsilon S(B, B^+)$, $v^+ \varepsilon V(B, B^+)$, and T an automorphism of H, then I_{s^+}, I_{v^+}, and T^+ are automorphisms of $\overline{\sum}(H; B, B^+, B^+)$. Hence the groups $\overline{\sum}(H; B, d, C)$ and $\overline{\sum}(H; B, d, C)T^+ I_{s^+} I_{v^+}$

are isomorphic. But each of the automorphisms T^+, I_{s^+}, and I_{v^+} of $\overline{\Sigma}(H; B, B^+, B^+)$ takes elements of $\overline{\Sigma}(H; B, d, C)$ into elements of $\overline{\Sigma}(H; B, d, C)$. Hence the restriction of the automorphism $T^+I_{s^+}I_{v^+}$ of $\overline{\Sigma}(H; B, B^+, B^+)$ to $\overline{\Sigma}(H; B, d, C)$ is an automorphism of the latter group. This is the automorphism μ, and this completes the proof of the theorem.

Corollary 1 μ is an inner automorphism of $\overline{\Sigma}(H; B, d, C)$, $d < C < B^+$, if and only if T^+ is generated by the identity automorphism of H, and s^+ is an element of S(B, C).

Proof If T^+ is generated by the identity automorphism of H, and $s^+ \varepsilon S(B, C)$, then,

$$\mu = T^+I_{s^+}I_{v^+} = I_{v^+s^+}, \quad v^+s^+ \varepsilon \overline{\Sigma}(H; B, d, C),$$

and hence μ is an inner automorphism.

Conversely suppose μ is inner,

$$\mu = I_u, \quad u \varepsilon \overline{\Sigma}(H; B, d, C), \text{ then}$$

$$\mu = I_u = T^+I_{s^+}I_{v^+}, \text{ and } T^+I_{s^+} = I_uI_{v^+}{}^{-1},$$

Moreover, $(s)I_{s^+} = (s)I_uI_{v^+}$ for all $s \varepsilon S(B, C)$, therefore $s^+ \varepsilon S(B, C)$. Then finally $T^+ = I_uI_{(v^+s^+)^{-1}}$ is an inner automorphism. Since T^+ leaves S(B, C) fixed elementwise, $T^+ = I_{[h]}$, but $[e]$ is the only scalar of $\Sigma(H; B, d, C)$, hence T^+ is generated by the identity automorphism.

Theorem 21 The group of three-tuples (T, s^+, v^+), T an automorphism of H, $s^+ \varepsilon S(B, B^+)$, $v^+ \varepsilon V(B, d)$, with the operation,

$$(T_1, s_1^+, v_1^+)(T_1, s_1^+, v_1^+) =$$

$$(T_1T_2, s_2^+s_1^+, v_2^+s_2^+(v_1^+T_2^+)s_2^{+-1}),$$

is isomorphic to the automorphism group of

$$\Sigma(H; B, d, C), \ d < C < B^+.$$

Proof The set of three-tuples form a subgroup of the set of three-tuples of Theorem 19, and hence the mapping defined there is a homomorphic mapping of the set of three-tuples named above onto the automorphism group of $\Sigma(H; B, d, C)$. Call this restriction of the homomorphism λ of Theorem 19, λ'. Then the kernel K' of λ' is contained in the kernel K of λ. But the only scalar contained in $V(B, d)$ is the identity multiplication, hence K' has order one, and λ' is the desired isomorphism.

CHAPTER V

AUTOMORPHISMS OF THE ALTERNATING MONOMIAL GROUP

Suppose μ' is an automorphism of $\overline{\sum}_{A,n}(H)$. According to Theorem 10, V_n is a characteristic subgroup of $\overline{\sum}_{A,n}(H)$, and hence,

$$\overline{\sum}_{A,n}(H) = V_n \cup A_n\mu', \text{ and } V_n \cap A_n\mu' = E.$$

Then $A_n\mu'$ is conjugate to a group T_o which is isomorphic to A_n under an isomorphism ϕ as determined in Theorem 17.

$$A_n\mu' = u^{-1}T_o u, \ u \in \overline{\sum}_{A,n}(H), \text{ and we may write,}$$

$$A_n\mu' I_u = T_o, \ A_n\phi = T_o, \text{ and } A_n\mu' I_u \phi^{-1} = A_n.$$

Then $\mu' I_u \phi^{-1}$ is an automorphism of A_n, and by reason of Theorems 4 and 6 there exists an $s^+ \in S_n$ such that $I_{s^+} = \mu' I_u \phi^{-1}$ on A_n. Then $\phi = I_{s^+}{-1}\mu' I_u$, and we have extended ϕ to an automorphism μ of the group $\overline{\sum}_{A,n}(H)$. $\mu = I_{s^+}{-1}\mu' I_u$, and $\mu = \phi$ on A_n.

Then we may write,

$$(1, 3, 2)\mu = (e, e, e, g_4, \cdots, g_n)(1, 3, 2)$$

$$(1, i, 2)\mu = (e, g_i, g_i^2, g_i^2 g_4, \cdots,$$

$$g_i^2 g_{i-1}, g_i^2, g_i^2 g_{i+1}, \cdots, g_i^2 g_n)(1, i, 2),$$

$i = 4, \cdots, n$, where the g_i are as defined in Theorem 17.

Consider generating elements (1, i, 2) of A_n and $(\cdots , e , h_j, e , \cdots)$ of V_n. Then since μ is an automorphism of $\overline{\sum}_{A,n}(H)$,

$(s)\mu(v)\mu(s^{-1})\mu = (svs^{-1})\mu$, for all $s \in A_n$ and all $v \in V_n$, and in particular for the generating elements named above. The form of an endomorphism of V_n was determined in Theorem 1 as a function of n^2 endomorphisms of H. The above equality will serve to restrict these endomorphisms of H in such a manner as to have an automorphism of V_n which extends to an automorphism of $\overline{\sum}_{A,n}(H)$. It will be necessary to consider a number of cases. In each case the above equality has been calculated for the generating elements, but such calculations have not been recorded, only the resulting restrictions of the endomorphisms T^i_j of H.

Case 1 $j \neq 1, 2, 3$ and $i = 3$

(1) $hT^j_1 = hT^j_2 = hT^j_3$

(2) $g_i hT^j_i g_i = h\, T^j_i$, $i = 4, \cdots, n$.

Case 2 $j = 1$ and $i = 3$

(1) $hT_3^1 = hT_1^2$

(2) $hT_1^1 = hT_2^2$

(3) $hT_2^1 = hT_3^2$

(4) $g_m hT_m^1 g_m^{-1} = hT_m^2$, $m = 4, \cdots, n$.

Case 3 $j = 2$ and $i = 3$

(1) $hT_3^2 = hT_1^3$

(2) $hT_1^2 = hT_2^3$

(3) $hT_2^2 = hT_3^3$

(4) $g_m hT_m^2 g_m^{-1} = hT_m^3$, $m = 4, \cdots, n$.

Case 4 $j=3$ and $i = 3$

(1) $g_m hT_m^3 g_m^{-1} = hT_m^1$, $m = 4, \cdots, n$.

Case 5 $i \neq 3$ and $j \neq i, 1, 2$.

(1) $hT_i^j = hT_1^j$

(2) $g_i hT_1^j g_i^{-1} = hT_2^j$

(3) $g_i^2 hT_2^j {g_i^{-1}}^2 = hT_i^j$

(4) $g_i^2 hT_e^u {g_i^{-1}}^2 = hT_3^j$

(5) $g_i^2 g_m hT^j g_m^{-1} g_i^{2-1} = hT_m^j$, $m = 4, \cdots, n$. $m \neq i$.

Case 6 $i \neq 3$ and $i = j$

(1) $hT_i^i = hT_1^1$

(2) $g_i hT_1^i g_i^{-1} = hT_2^1$

(3) $g_i^2 hT_2^i g_i^{2^{-1}} = hT_i^1$

(4) $g_i^2 hT_3^i g_i^{2^{-1}} = hT_3^1$

(5) $g_i^2 g_m hT_m^i g_m^{-1} g_i^{2^{-1}} = hT_m^1$, $m = 4, \cdots, n$. $m \neq i$.

Case 7 $i \neq 3$ and $j = 1$

(1) $hT_i^1 = hT_1^2$

(2) $g_i hT_1^1 g_i^{-1} = hT_2^2$

(3) $g_i^2 hT_2^1 g_i^{2^{-1}} = hT_i^2$

(4) $g_i^2 hT_3^1 g_i^{2^{-1}} = hT_3^2$

(5) $g_i^2 g_m hT_m^1 g_m^{-1} g_i^{2^{-1}} = hT_m^2$, $m = 4, \cdots, n$. $m \neq i$.

Case 8 $i \neq 3$ and $j = 2$

(1) $hT_i^2 = hT_1^i$

(2) $g_i hT_1^2 g_i^{-1} = hT_2^i$

(3) $g_i^2 hT_2^2 g_i^{2^{-1}} = hT_i^i$

(4) $g_i^2 hT_3^2 g_i^{2^{-1}} = hT_3^i$

(5) $g_i^2 g_m hT_m^2 g_m^{-1} g_m^{2^{-1}} = hT_m^i$, $m = 4, \cdots, n$. $m \neq i$.

Theorem 22 If $n > 6$ and H contains no subgroup isomorphic to A_{n-1}, then any automorphism of $\overline{\sum}_{A,n}(H)$ differs from an automorphism μ of $\overline{\sum}_{A,n}(H)$ by an automorphism I_{u^+}, $u^+ \varepsilon \overline{\sum}_n(H)$, where μ is constructed in the following manner,

$(s)\mu = s$, for all $s \varepsilon A_n$,

$(h_1, h_2, h_3, \cdots , h_n)\mu =$

$\{(h_1, h_2, h_3, \cdots , h_n)[pK]\}T^{+}, \quad p = \prod_{i=1}^{n} h_i$

where T^{+} is generated by an automorphism of H, and K is an endomorphism of H mapping H upon a subgroup of its center in such a manner that 1 + nK is a central automorphism of H.

Proof For n > 6 the only homomorphic image of A_{n-1} in the group H is the identity subgroup, since we have required that H contain no subgroup isomorphic to A_{n-1}. That is in the preceding calculations, $g_i = e$, $i = 4, \cdots , n$. From these calculations we pick the following restrictions on the endomorphisms T^i_j.

(i) $hT^i_1 = hT^1_i$

(ii) $hT^1_1 = hT^i_i$

(iii) $hT^j_i = hT^j_1, \ i \neq j, \ j \neq 1$

(iv) $hT^2_1 = hT^i_1, \ i \geq 2$

(i) follows from 3 case 6, 3 case 5, 1 case 5 for $i \neq 2, 3$. Then from 1 case 7, 1 case 2, and from 1 case 8, 3 case 7, 3 case 2, 1 case 3, combined with the equality just established we have (i), and moreover have shown that $hT_1^2 = hT_1^3$.

(ii) follows from 1 case 6, 2 case 2, and 3 case 3.

(iii) follows from 1 case 5, and 1 case 1 for $j \neq 2$, and for $j = 2$, from 1 case 7, 3 case 6, 1 case 1, and 1 case 8, 1 case 2, (i) and 1 case 3.

Finally (iv) follows from (i), 1 case 7, and 1 case 2.

If we set, $T_1^1 = T'$, and $T_1^2 = \overline{T}$

we may write by reason of Theorem 1, and (i) through (iv) above,

$$(\cdots , e , h_j , e , \cdots)\mu =$$

$$(h_j\overline{T}, \cdots , h_j\overline{T} , h_jT', h_j\overline{T}, \cdots , h_j\overline{T}).$$

where h_jT' is the j-th factor of the image multiplication.

The permutability conditions of the endomorphisms of H now become, $hT'k\overline{T} = k\overline{T}hT'$.

Since the elements $(h , e , \cdots)$ and $(e , k , e , \cdots)$ commute, we have $h\overline{T}k\overline{T} = k\overline{T}h\overline{T}$, for all $h \varepsilon H$, $k \varepsilon H$. That is $H\overline{T}$ is an abelian subgroup, and is moreover contained in the center of H, since $H = HT' \cup H\overline{T}$.

Let $(k_1, k_2, k_3, \cdots, k_n) \in V_n$, then since μ is an automorphism of $\sum_{A,n}(H)$, the following set of equations must have a unique set of solutions $h_i \in H$, $i = 1, \cdots, n$.

$$h_1T'h_2\overline{T}h_3\overline{T}\cdots h_n\overline{T} = k_1$$

$$h_1\overline{T}h_2T'h_3\overline{T}\cdots h_n\overline{T} = k_2$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$h_1\overline{T}h_2\overline{T}h_3\overline{T}\cdots h_nT' = K_n.$$

If we agree to let T be the correspondence

$$hT = hT'h^{-1}\overline{T},$$

we may rewrite the above set of equations in the following simplified form, with the aid of the permutatibility conditions,

$$h_1Tp\overline{T} = k_1, \; h_2Tp\overline{T} = k_2, \; \cdots, \; h_nTp\overline{T} = k_n,$$

where $p = \prod_{i=1}^{n} h_i$. If we further alter the equations by employing the relations,

$$h_m = y_mh_1, \quad k_m = x_mk_1, \; m = 2, \cdots, n,$$

and multiply the first equation by the equation resulting from taking the inverse of both sides of the m-th equation we obtain,

$$y_m T = x_m, \; m = 2, \cdots , n.$$

Since these conditions must be satisfied by some y_i for every set of elements $x_i \varepsilon H$, it follows that T must be a correspondence of H onto itself. The correspondence T is moreover an automorphism of H. That T is multiplication preserving follows from the fact that $H\overline{T}$ is contained in the center of H. T is onto, and has kernel e. Suppose the kernel is different from e, then there exists an element $h \varepsilon H$, $h \neq e$, such that,

$$hT = hT'h^{-1}\overline{T} = e, \text{ hence, } hT' = h\overline{T}, \text{ and}$$

$$(e , h , e , \cdots)\mu = \left[h\overline{T}\right] = (h , e , \cdots)\mu.$$

But μ is an automorphism of V_n, and hence we have reached a contridiction. This shows that T is an automorphism of the group H.

Then by reason of Theorem 1 and the permutatibility conditions we may write,

$$(h_1, h_2, h_3, \cdots , h_n)\mu =$$

$$(h_1 Tp\overline{T}, h_2 Tp\overline{T}, h_3 Tp\overline{T}, \cdots ,h_n Tp\overline{T}) =$$

$= (h_1, h_2, h_3, \cdots , h_n)\Gamma^+[p\bar{T}]$.

Define correspondences α and β of V_n as follows,

$(h_1, h_2, h_3, \cdots , h_n)\alpha =$

$(h_1, h_2, h_3, \cdots , h_n)[pK]$, where $K = \bar{T}T^{-1}$,

$(h_1, h_2, h_3, \cdots , h_n)\beta =$

$(h_1, h_2, h_3, \cdots , h_n)T^+$.

Then $(v)\mu = (v)\alpha\beta$, for all $v \varepsilon V_n$. But β is an automorphism of V_n and hence α must be an automorphism of V_n. That is the set of equations,

$h_i(pK) = k_i, \ i = 1, \cdots , n,$

where the k_i are an arbitrary given set of elements of H, must have unique solutions h_i in H. Since the center of any group is a characteristic subgroup, $K = \bar{T}T^{-1}$, maps H onto a subgroup of its center.

If as before we set,

$h_m = y_m h_1, \quad k_m = x_m k_1, \ m = 2, \cdots , n,$

we see from the former equality that $y_m = x_m$, and moreover the set of equations reduce to a single equation, $h(h^n)K = h\ (1 + nK)$ be a defining relation for $(1 + nK)$, the single equation of consideration being $h(h^n)K(x_2 x_3 \cdots x_n)K = k$.

We see that the correspondence 1 + nK must be onto the group H, since this equation must be satisfied for all $k \in H$. (1 + nK) is a multiplication preserving correspondence since K maps H onto a subgroup of its center. Thus 1 + nK is a homomorphic mapping, with kernel e. To establish this we suppose that there exists an $h \in H$, $h \neq e$, such that,

$$h(1 + nK) = h(h^n)K = e, \text{ then}$$

$$[h]\alpha = [h(h^n)K] = [e].$$

But α is an automorphism of V_n and hence this cannot be, and therefore 1 + nK has kernel e and is an automorphism of the group H. Then,

$$(h_1, h_2, \cdots, h_n)\mu =$$

$$\{(h_1, h_2, h_3, \cdots, h_n)[pK]\}T^+ ,$$

where T and 1 + nK are automorphisms of H, 1 + nK being a central automorphism of H.

In the beginning we started with an arbitrary automorphism μ' of $\Sigma_{A,n}(H)$, but multiplied μ' by another automorphism of $\Sigma_{A,n}(H)$ to form another automorphism μ of $\Sigma_{A,n}(H)$, which took generating three cycles of A_n onto the product of an element of V_n and the initial three cycle. The further requirement that $n > 6$ caused the multiplication component of the image element to be E. Then $(s)\mu = s$, for all $s \varepsilon A_n$. This completes the proof of the theorem.

Theorem 23 Given an automorphism T of H, an endomorphism K of H, such that K maps H onto a subgroup of its center and $(1 + nK)$ is a (central) automorphism of H, the the correspondence μ,

$$(s)\mu = s, \quad s \varepsilon A_n ,$$

$$(v)\mu = \left\{v\,[pK]\right\}T^{+}, \quad v \varepsilon V_n,$$

where p is the product of the factors of v, is an automorphism of $\bar{\sum}_{A,n}(H)$.

Proof V_n and $V_n\mu$ are isomorphic, under the correspondence μ. Let,

$$(v_1)\mu = \left\{v_1\,[p_1K]\right\}T^{+}, \quad (v_2)\mu = \left\{v_2\,[p_2K]\right\}T^{+}, \text{ then}$$

$$(v_1)\mu(v_2)\mu = \left\{v_1\,[p_1K]\right\}T^{+}\left\{v_2\,[p_2K]\right\}T^{+} =$$

$$\left\{v_1\,[p_1K]\,v_2[p_2K]\right\}T^{+} = \left\{v_1v_2\,[p_1p_2K]\right\}T^{+} = (v_1v_2)\mu.$$

Hence μ preserves multiplication, and μ is clearly onto. The homomorphism μ has kernel e, which will be established by denying this statement. Then there exists $v \in V_n$, $v \neq E$, such that

$$(v)\mu = \{v[pK]\} T^+ = E.$$

Since T^+ is an automorphism of V_n, $v[pK] = E$,

and hence v must be a scalar h, and

$$h(h^n)K = h(1 + nK) = e,$$

but $(1 + nK)$ is an automorphism of H and hence $h = e$,

and $v \neq E$, is a contridiction.

Let $G = (V_n)\mu \cup A_n$. $(V_n)\mu$ is a normal subgroup of G, and $(V_n)\mu \cap A_n = E$. Then according to Theorem 7, G and $\overline{\sum}_{A,n}(H)$ are isomorphic if and only if,

$$(s)\mu(v)\mu(s^{-1})\mu = (svs^{-1})\mu, \text{ for all } s \in A_n,$$

and all $v \in V_n$.

$$(s)\mu(v)\mu(s^{-1})\mu = s(v)\mu s^{-1} = s\{v[pK]\} T^+ s^{-1} =$$

$$= s(vT^+)[pKT]s^{-1} = s(vT^+)s^{-1}[pKT].$$

$(svs^{-1})\mu = \{(svs^{-1})[p'K]\}T^+$, where p' is the product of the factors of (svs^{-1}). But since K maps H onto a subgroup of its center $pK = p'K$, and

$$\{(svs^{-1})[p'K]\}T^+ = s(vT^+)s^{-1}[pKT].$$

Therefore G and $\overline{\sum}_{A,n}(H)$ are isomorphic under μ.

But $G = \sum_{A,n}(H)$. It is clear that $\sum_{A,n}(H)$ contains G. To show the inclusion in the reverse sense we need only show that if $v \varepsilon \sum_{A,n}(H)$ then $v \varepsilon G$. Let

$v = (h_1, h_2, h_3, \cdots, h_n)$, then we ask if there exists an element,

$v' = (k_1, k_2, k_3, \cdots, k_n) \varepsilon V_n$, such that

$(v')\mu = \{v' [p'K]\} T^+ = v$, that is does the set of equations,

$$\{k_i(p'K)\} T = h_i, \; i = 1, \cdots, n,$$

have solutions $k_i \varepsilon H$. The construction employed in the previous theorem when viewed in reverse order shows that if we set,

$$h_m = x_m h_1, \; m = 2, \cdots, n,$$

and define y_m, $m = 2, \cdots, n$, to be $y_m = x_m T^{-1}$, the set of equations are seen to have solutions,

$$k_1 = \{h_1 T^{-1} (\prod_{i=2}^{n} y_i)^{-1} K\}(1 + nK)^{-1}$$

$$k_m = y_m k_1, \; m = 2, \cdots, n.$$

We demonstrate that the k_i, $i = 1, \cdots, n$, are factors of a multiplication v' such that $(v')\mu = v$, by showing that they satisfy the above set of equations. We show that the first equation is satisfied.

$$k_1(k_1^n)K(\prod_{i=2}^{n} y_i)K = k_1T^{-1},$$

$$k_1(p')K = h_1T^{-1}, \quad [k_1(p')K]T = h_1.$$

We now show that the m-th equation is satisfied.

$$k_m = (y_m)\left[\left\{h_1T^{-1}(\prod_{i=2}^{n} y_i)^{-1}K\right\}(1 + nK)^{-1}\right],$$

$$k_m(k_m^n)K = y_m(y_m^n)Kh_1T^{-1}(\prod_{i=2}^{n} y_i)^{-1}K,$$

$$k_m(k_m^n)K\ (\prod_{i=2}^{n} yi)K(y_m^n)^{-1}K = y_m(h_1)T^{-1} = (x_mh_1)T^{-1},$$

$$k_m(k_1^n)K(\prod_{i=2}^{n} yi)K = (x_mh_1)T^{-1} = h_mT^{-1},$$

$$k_m(p')K = h_mT^{-1}, \quad [k_m(p')K]\ T = h_m.$$

$G = \overline{\Sigma}_{A,n}(H)$, and hence μ is an automorphism of $\overline{\Sigma}_{A,n}(H)$.

Theorem 24 The automorphism group of $\overline{\Sigma}_{A,n}(H, B, d)$ is isomorphic to the automorphism group of $\overline{\Sigma}(H; B, d, d)$.

Proof Let μ be an automorphism of $\overline{\Sigma}_A(H; B, d)$,

$s = (1, i)(m, n) \ \varepsilon \ A(B)$

$v = (\cdots , e , h_j, e , \cdots) \ \varepsilon \ V(B, d).$

Then by Theorem 6,

$$(s)\mu = (k_1, k_2, \cdots, k_n)(1, i)(m, n) I_s{}^{+},$$

$s^{+} \varepsilon S(B, B^{+})$, and only finitely many of the k's are different from the identity. From Theorem 2 we have,

$$(v)\mu = (h_j T_1^j, h_j T_2^j, h_j T_3^j, \cdots)$$

T_j^i endomorphisms of H. If we then compute the equality

$$(s)\mu(v)\mu(s^{-1})\mu = (svs^{-1})\mu,$$

the restrictions placed on the endomorphisms T_j^i are such that the images of multiplications under and automorphism of $\overline{\sum}_A(H; B, d)$ are determined in the same manner as under an automorphism of $\overline{\sum}(H; B, d, d)$. We may determine the image of permutations by reproducing the calculations of Theorem 20, for the images of permutations there were determined irrespective oftheir being and even or an odd permutation. Thus all automorphisms of $\overline{\sum}_A(H; B, d)$ are restrictions of automorphisms of $\overline{\sum}(H; B, d, d)$.

CHAPTER VI

COMMUTATOR SUBGROUPS

OF THE MONOMIAL GROUPS

We will use G' to denote the commutator subgroup of the group G.

Theorem 25 The commutator subgroup V'(B, C), $d \leq C \leq B^+$, of V(B, C) is the set of all elements

$$v' = (h_1', h_2', h_3', \ldots \ ), \ h_i' \in H',$$

where there exists an integer N such that each h_i' is the product of N or fewer commutators of H.

Proof Suppose $v \in V'(B, C)$ and v is a commutator, then there exists a v_1 and v_2 of V(B, C) such that $v = v_1 v_2 v_1^{-1} v_2^{-1}$. It then follows that every factor of v must be a commutator of H and hence an element of H'.

If $v \in V'(B, C)$ but is not a commutator, it is a product of a finite number of commutators, $v = v_1 v_2 v_3 \cdots v_N$. Since each v_i, $i = 1, \cdots, N$, is a commutator of V(B, C), each factor of v_i, $i = 1, \cdots, N$, is a commutator of the group H. Therefore every factor of v is the product of N or fewer commutators of H, and is then an element of H'.

Conversely if $v \in V(B, C)$ and has the form

$$v = (h'_1, h'_2, h'_3, \cdots), \ h'_i \in H', \text{ and}$$

there exists an integer N such that each h'_i is the product of N or fewer commutators of H, we see that v can be decomposed into a product of N or fewer commutators from V(B, C). It then follows that $v \in V'(B, C)$.

Theorem 26 The commutator subgroup S'(B, C), $d < C \leq B^+$, of S(B, C) is S(B, C). The commutator subgroup S'(B, d) of S(B, d) is A(B, d).

The proof is contained in [5].

Theorem 27 The commutator subgroup $\Sigma'(H; B, d, d)$ of $\Sigma(H; B, d, d)$ is $A(B, d) \cup V^+(B, d)$ where $V^+(B, d)$ is the set of all elements of V(B, d) whose product of factors is a member of H'.

Proof By reason of Theorem 26 we have,

$$\Sigma'(H; B, d, d) \supset A(B, d),$$

and we now show that,

$$\Sigma'(H; B, d, d) \supset V^+(B, d).$$

Suppose $(h_1, h_2, h_3, \cdots) \varepsilon V^+(B, d)$, and let i_j, $j = 1, \cdots , n$, denote the subscripts of its non-identity factors. Then consider the elements,

$$v = (\cdots , e , h_{i_j} , e , \cdots)$$

$$s = (x_{i_j}, x_k), \text{ then}$$

$$v\, s\, v^{-1} s^{-1} = (\cdots , e , h_{i_j} , e , \cdots , e , h_{i_j}^{-1} , e , \cdots) \text{ belongs to } \Sigma'(H; B, d, d).$$

Moreover since h and i_j are arbitrary, any element of the above form belongs to $\Sigma'(H; B, d, d)$. This being the case, the element,

$$(\cdots , e , h_{i_1} , \cdots , h_{i_n} , e , \cdots , e , h_{i_1}^{-1} h_{i_2}^{-1} h_{i_3}^{-1} \cdots h_{i_n}^{-1} , e , \cdots),$$

belongs to $\Sigma'(H; B, d, d)$, and further since $h_{i_1} h_{i_2} h_{i_3} \cdots h_{i_n} \varepsilon H'$, we have,

$$(\cdots , e , h_{i_1} h_{i_2} h_{i_3} \cdots h_{i_n} , e , \cdots)$$

belongs to $\Sigma'(H; B, d, d)$. Finally the product of the two multiplications must belong to $\Sigma'(H; B, d, d)$, but this product is the element of $V^+(B, d)$ selected earlier. That is,

$\Sigma'(H;\ B,\ d,\ d) \supset V^{+}(B,\ d)$, and

$\Sigma'(H;\ B,\ d,\ d) \supset V^{+}(B,\ d) \cup A(B,\ d)$.

Since G/G' is abelian for any group G, and G' is the smallest group for which this is true, we will have $\Sigma'(H;\ B,\ d,\ d) \subset V^{+}(B,\ d) \cup A(B,\ d)$, if we can show that $\Sigma(H;\ B,\ d,\ d)/V^{+}(B,\ d) \cup A(B,\ d)$ is abelian.

It follows from the definition of $V^{+}(B,\ d)$ that $V^{+}(B,\ d)$ contains $V'(B,\ d)$, and hence $V(B,\ d)/V^{+}(B,\ d)$ is an abelian group. Therefore any two multiplications commute mod $\left[V^{+}(B,\ d) \cup A(B,\ d)\right]$. Since $A(B,\ d)$ consists of all even permutations there are but two cosets of $A(B,\ d)$ in $S(B,\ d)$, namely $A(B,\ d)$ and $(x_1,\ x_2)A(B,\ d)$. Thus any element of the factor group $\Sigma(H;\ B,\ d,\ d)/V^{+}(B,\ d) \cup A(B,\ d)$ has one of the forms,

$$v\left[V^{+}(B,\ d) \cup A(B,\ d)\right], \text{ or}$$

$$v(x_1,\ x_2)\left[V^{+}(B,\ d) \cup A(B,\ d)\right], \quad v \in V(B,\ d).$$

Now if, $v = (\ k_1,\ k_2,\ k_3,\ \cdots\ )$,

$$v(x_1,\ x_2)v^{-1}(x_1,\ x_2)^{-1} =$$

$$(k_1k_2^{-1},\ k_2k_1^{-1},\ \cdots\ ) \in V^{+}(B,\ d).$$

That is (x_1, x_2) and v commute mod $[V^+(B, d) \cup A(B, d)]$.

It then follows that $\overline{\Sigma}(H; B, d, d)/V^+(B, d) \cup A(B, d)$, is abelian, and hence,

$$\overline{\Sigma}'(H; B, d, d) \subset V^+(B, d) \cup A(B, d),$$

which together with the inclusion in the reverse sense which was previously established, yields,

$$\overline{\Sigma}'(H; B, d, d) = V^+(B, d) \cup A(B, d).$$

This completes the proof of the theorem. The next theorem assets that the derived series for $\overline{\Sigma}(H; B, d, d)$ consists of but two distinct terms.

Theorem 28 The commutator subgroup $\overline{\Sigma}''(H; B, d, d)$ of $\overline{\Sigma}'(H; B, d, d)$ is $\overline{\Sigma}'(H; B, d, d)$.

Proof We show that $\overline{\Sigma}''(H; B, d, d)$ contains both $V^+(B, d)$ and $A(B, d)$, and then the conclusion will follow.

A(B, d) is simple [7], while A'(B, d) is a characteristic subgroup of A(B, d). But A'(B, d) is different from the identity, hence A(B, d) = A'(B, d), and hence $\overline{\Sigma}''(H; B, d, d) \supset A(B, d)$.

We next show $\overline{\Sigma}''(H; B, d, d) \supset V'(B, d)$. According to Theorem 27, $\overline{\Sigma}'(H: B, d, d)$ contains the elements, $v = (h , h^{-1}, e , \cdots)$,

$v' = (k , e , k^{-1}, e , \cdots)$. It then follows that,

$$v\, v'\, v^{-1} v'^{-1} = (hkh^{-1}k^{-1}, e , \cdots) \varepsilon \sum{}''(H; B, d, d).$$

Therefore any element of $V'(B, d)$ is the product of elements of $\sum''(H; B, d, d)$ and hence an element of $\sum''(H; B, d, d)$. That is $\sum''(H; B, d, d)$ contains $V'(B, d)$.

Let $v = (h , e , e , h^{-1}, e , \cdots)$

$s = (1, 3, 2)$, then

$$v\, s\, v^{-1} s^{-1} = (h , h^{-1}, e , \cdots) \varepsilon \sum{}''(H, B, d, d)$$

since s and v belong to $\sum'(H; B, d, d)$. Then conjugation by appropriate elements of $A(B, d)$ will move the non-identity factors into any desired position, and the resulting multiplication is again an element of $\sum''(H; B, d, d)$ since the commutator subgroup is a characteristic subgroup.

Let $v' = (h_1, h_2, h_3, \cdots , h_n, e , \cdots)$

be an element of $V^+(B, d)$, then,

$$v' = (h_1, h_1^{-1}, e , \cdots)(e, h_1h_2, h_2^{-1}h_1^{-1}, e ,$$

$$\cdots)(e , e , h_1h_2h_3, h_3^{-1}h_2^{-1}h_1^{-1}, e , \cdots)\cdots$$

$$\cdots(\cdots , e , h_1h_2\cdots h_n, h_n^{-1}h_{n-1}^{-1}\cdots h_1^{-1}, e , \cdots)$$

$(\cdots , e , h_1 h_2 h_3 \cdots h_n , e , \cdots) \in \bar{\geq}''(H; B, d, d)$,

since the last multiplication in the product is an element of $V'(B, d)$ which is contained in $\bar{\geq}''(H; B, d, d)$. Therefore $\bar{\geq}''(H; B, d, d)$ contains $V^+(B, d)$, and hence $\bar{\geq}''(H; B, d, d)$ contains $V^+(B, d) \cup A(B, d) = \bar{\geq}'(H; B, d, d)$.

Theorem 29. The commutator subgroup $\bar{\geq}'_A(H; B, d)$ of $\bar{\geq}_A(H; B, d)$ is $V^+(B, d) \cup A(B, d)$.

Proof We have,

$\bar{\geq}'(H; B, d, d) \subset \bar{\geq}_A(H; B, d) \subset \bar{\geq}(H; B, d, d)$,

hence,

$\bar{\geq}''(H; B, d, d) \subset \bar{\geq}'_A(H; B, d) \subset \bar{\geq}'(H; B, d, d)$.

Then by reason of Theorem 28,

$\bar{\geq}'(H; B, d, d) = \bar{\geq}''(H; B, d, d) = V^+(B, d) \cup A(B, d)$.

Hence $\bar{\geq}'_A(H; B, d) = V^+(B, d) \cup A(B, d)$, as was to be shown.

Theorem 30 The commutator subgroup $\bar{\geq}'(H; B, C, D)$, $d < C \leq D \leq B^+$, of $\bar{\geq}(H; B, C, D)$ is $\bar{\geq}(H; B, C, D)$.

Proof It is shown in [5] that the commutator subgroup of S'(B, D) of S(B, D) is S(B, D). Hence $\bar{\Sigma}$'(H; B, C, D) contains S(B, D). We next show that $\bar{\Sigma}$'(H; B, C, D) contains V(B, C), and having established this the conclusion of the theorem will follow.

Let, $s = (\cdots , x_{-1}, x_0, x_1, x_2, \cdots ; \cdots x_\varepsilon,$

$x_{\varepsilon+1}, x_{\varepsilon+2}, x_{\varepsilon+3}, \cdots),$

$v = (\cdots , h_{-1}, h_0, h_1, h_2, \cdots ; \cdots h_\varepsilon,$

$h_{\varepsilon+1}, h_{\varepsilon+2}, h_{\varepsilon+3}, \cdots).$

Then, $s\, v\, s^{-1} v^{-1} = (\cdots , h_0 h_{-1}^{-1}, h_1 h_0^{-1}, h_2 h_1^{-1}, \cdots ;$

$\cdots, h_{\varepsilon+1} h_\varepsilon^{-1}, h_{\varepsilon+2} h_{\varepsilon+1}^{-1}, h_{\varepsilon+3} h_{\varepsilon+2}^{-1}, \cdots)$, belongs to $\bar{\Sigma}$'(H; B, C, D).

Let $v_c = (\cdots , c_{-1}, c_0, c_1, \cdots ; \cdots , c_\varepsilon,$ $c_{\varepsilon+1}, c_{\varepsilon+2}, \cdots)$, be an arbitrary element of V(B, C), and consider the following set of equations.

$$
\begin{array}{llll}
\vdots & & \vdots & \\
h_0 h_{-1}^{-1} = c_{-1}, & h_{\varepsilon+1} h_\varepsilon^{-1} = c_\varepsilon, & \cdots \\
h_1 h_0^{-1} = c_0, & h_{\varepsilon+2} h_{\varepsilon+1}^{-1} = c_{\varepsilon+1}, & \cdots \\
h_2 h_1^{-1} = c_1, & h_{\varepsilon+3} h_{\varepsilon+2}^{-1} = c_{\varepsilon+2}, & \cdots \\
\vdots & & \vdots &
\end{array}
$$

This set of equations has solutions,

$$\begin{array}{llll} \vdots & \vdots & \vdots & \vdots \\ h_{-1} = e & , & h_{\varepsilon} = e & , \cdots \\ h_0 = c_{-1} & , & h_{\varepsilon+1} = c_{\varepsilon} & , \cdots \\ h_1 = c_0 c_{-1} & , & h_{\varepsilon+2} = c_{\varepsilon+1} c_{\varepsilon} & , \cdots \\ h_2 = c_1 c_0 c_{-1} & , & h_{\varepsilon+3} = c_{\varepsilon+2} c_{\varepsilon+1} c_{\varepsilon} & , \cdots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

The factors of v are completely arbitrary. If we take the factors of v to be as indicated above we see that,

$s\,v\,s^{-1}v^{-1} = v_c \in \overline{\Sigma}'(H; B, C, D)$, and hence $\overline{\Sigma}'(H, B, D, C)$ contains $V(B, C)$, and therefore $\overline{\Sigma}(H; B, D, C) = \overline{\Sigma}'(H; B, D, C)$, as was to be shown.

Corollary 1 Any element $u \in \overline{\Sigma}(H; B, C, D)$, $d < C \leq D \leq B^{+}$, is the product of at most two commutators.

Proof Every element of $S(B, D)$ is a commutator of $S(B, D)$, which is demonstrated in [5]. Every element of $V(B, C)$ is a commutator of $\overline{\Sigma}(H; B, C, D)$ as was shown in Theorem 30. Therefore any element of $\Sigma(H; B, C, D)$ which is either a multiplication or a permutation is a commutator. Since every element of $\Sigma(H; B, C, D)$ has the form vs, $v \in V(B, C)$, $s \in S(B, D)$, other elements of $\Sigma(H; B, C, D)$ are the product of at most two commutators of $\Sigma(H; B, C, D)$.

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