ON JORDAN CURVES

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If $f(t)$ and $\varphi(t)$ are two single valued functions of the independent real variable t defined at every point of the interval $t_0 \leqq t = t_\omega$, the pair of equations

$$x = f(t) \qquad y = \varphi(t) \qquad t_0 \leqq t \leqq t_\omega$$

are said to define a *curve*. If $f(t)$ and $\varphi(t)$ are continuous on the interval, we have the representation of a *continuous curve*. If $f(t_0) = f(t_\omega)$ and $\varphi(t_0) = \varphi(t_\omega)$ so that the initial and end points coincide, we have a *closed curve* provided the curve is continuous. If there exist no two values of t on the interval $t_0 \leqq t \leqq t_\omega$ other than t_0 and t_ω for which the corresponding points coincide, the curve will have no multiple points and is called a *simple closed curve*.

These definitions are due to Camille Jordan who was one of the first to study the properties of continuous curves and it is for that reason that simple closed curves are sometimes called *Jordan curves*. A fundamental theorem called the Jordan Theorem concerning simple closed curves is the following: every Jordan curve divides the plane into two regions, an inner region and an outer region such that (1) any two points of the inner region may be connected by a continuous curve which does not cut the given curve and (2) any continuous curve which connects a point of one region with a point of the other must cut the given curve. Most proofs of this theorem are rather difficult because the extreme simplicity of the statement and the lack of facts to work with demand reasoning of the keenest and most logical kind. In this paper however, I shall attempt to show by means of the simplest analytical considerations that for any Jordan curve it is possible to construct a unicursal curve which shall have a loop containing no real multiple points and approximating the Jordan curve as closely as we please. The nature of the approximation will be explained in the following pages.

Let then the equations

$$x = f(t) \qquad y = \varphi(t) \qquad t_0 \leqq t \leqq t_\omega$$

with the conditions placed upon them by definition represent our Jordan curve. The first condition was that $f(t)$ and $\varphi(t)$ should be defined for every t in the interval. The second condition was that both $f(t)$ and $\varphi(t)$ should be continuous on the interval. Continuity on the interval guarantees uniform continuity for both $f(t)$ and $\varphi(t)$ and we may express this fact analytically as follows: given a positive number $\frac{\varepsilon}{\sqrt{2}}$, ε as small as we please, there exists a δ such that when $|t_1 - t_2| < \delta$

$$|f(t_1)-f(t_2)| < \frac{\varepsilon}{\sqrt{2}} \quad \text{and} \quad |\varphi(t_1)-\varphi(t_2)| < \frac{\varepsilon}{\sqrt{2}}$$

where t_1 and t_2 represent any two values of t on the interval $t_0 \leqq t \leqq t_\omega$. Now the geometric distance from the point (x_1, y_1) i.e. the point for which $t = t_1$, to (x_2, y_2) is

$$\Delta = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{[f(t_1) - f(t_2)]^2 + [\varphi(t_1) - \varphi(t_2)]^2}$$

and Δ, always positive, will be smaller than

$$\sqrt{\frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2}} = \varepsilon$$

whenever the points (x_1,y_1) and (x_2,y_2) are such that $|t_1 - t_2| < \delta$.

Suppose now that we divide our interval $t_0 \leqq t \leqq t_\omega$ into a finite number of sub-intervals by the pointd

$$t_0, \ t_1, \ t_2, \ \ldots \ldots t_{n-1}, \ t_\omega$$

such that $t_k < t_{k+1}$ for $k = 0, 1, \quad n-2$, and $t_{n-1} < t_\omega$ and such that the difference between any two successive t's is less in absolute value than δ. Denote by P_i the point on the curve for which $t = t_i$. If we now join the points

$$P_0, \ P_1, \ P_2, \qquad P_{n-1}, \ P_\omega$$

successively by straight line segments, we have a polygon of n sides inscribed on our curve and the length of ecch side will be less than ε.

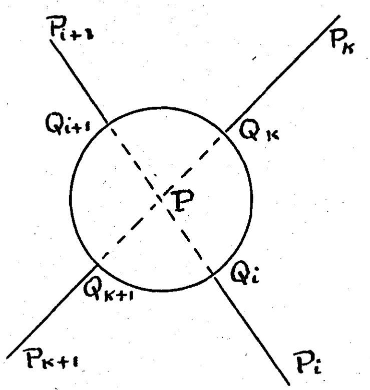
Consider one side of our polygon, the side $P_k P_{k+1}$ for example. As we increase t continuously from t_k to t_{k+1}, the point P will move from P_k to P_{k+1} along the arc joining P_k and P_{k+1}. Let P_α be any point on the arc segment $\widehat{P_k P_{k+1}}$. If P_α does not coincide with P_k or P_{k+1}, we have t_α greater than t_k and less than t_{k+1} and therefore $|t_\alpha - t_k| < \delta$ and $|t_\alpha - t_{k+1}| < \delta$. Consequently $\overline{P_\alpha P_k}$ and $\overline{P_\alpha P_{k+1}}$ are both less than ε, for we have shown that the distance between <u>any</u> two points whose t's differ by less than δ is less than ε. A line joining P_α to any point on the line segment $\overline{P_k P_{k+1}}$ would be shorter than one of the line segments $\overline{P_\alpha P_k}$ or $\overline{P_\alpha P_{k+1}}$ and consequently less than ε in length. Since P_α is any point on the arc $\widehat{P_k P_{k+1}}$, we see that the distance from any point of the arc $\widehat{P_K P_{K+1}}$ to any point of the line segment $\overline{P_K P_{K+1}}$ is less than ε. It will surely be true then that the shortest distance from P_α to the line segment $\overline{P_K P_{K+1}}$ is less than ε. Also, if Q_α is any point on the segment $\overline{P_K P_{K+1}}$, the shortest distance from Q_α to the arc $\widehat{P_K P_{K+1}}$ is less than ε.

The same discussion holds concerning the remaining sides of the polygon and the corresponding arcs. Let now P represent any point of the Jordan curve and Δ_P the shortest distance from P to the polygon. Let Q be any point on the polygon and Δ_Q be the shortest distance from Q to the curve. As P moves along the curve, Δ_P will take on a maximum value which will be smaller than ε. We shall call this the maximum distance from the curve to the polygon. Similarly we shall call the maximum value of Δ_Q the maximum distance from the polygon to the curve. This will also be less than ε. We have shown, then, that it is possible to inscribe a polygon on our Jordan curve such that the maximum distance from the polygon to the curve and the maximum distance from the curve to the polygon will both be less than ε.

The question now arises: can we make the divisions of the interval t_0 to t_ω so small that our polygon will have no multiple points? Intuitively we would answer yes, but the correctness of our guess could not be verified by any simple considerations. We shall therefore admit the possible existence of a finite number of multiple points on our polygon and treat them in the following paragraphs.

Consider first a double point P, formed by the intersection of the two sides $P_i P_{i+1}$ and $P_K P_{K+1}$, assuming first that P does not coincide with P_i, P_{i+1}, P_K, or P_{K+1}. With P as center and radius less than $\frac{\varepsilon'}{2}$, where $0 < \varepsilon' < \varepsilon$, describe a circle cutting the two lines at the points Q_i, Q_{i+1}, Q_K and Q_{K+1}, Q_i being on the line $P_i P_{i+1}$ nearer P etc.

Fig. 1.



Suppose that we start at P and pass thru Q_{i+1} and P_{i+1}. If we continue in the same direction, we shall return to P by way of $P_K Q_K$. If the polygonal line $PP_{i+1} \cdots P_K P$ has no point in common with the remainder of the polygon other than P, it would be completely cut off by joining Q_{i+1} to Q_K and throwing off the lines $Q_i Q_{i+1}$ and $Q_K Q_{K+1}$. To avoid this and at the same time to get rid of our double point we shall join Q_i to Q_K and Q_{i+1} to Q_{K+1}, casting off the lines $Q_i Q_{i+1}$ and $Q_K Q_{K+1}$. If, however, the polygonal line $PP_{i+1} \cdots P_K P$ has one or more points other than P in common with the remainder of the polygon, we may either join Q_i to Q_k and Q_{i+1} to Q_{k+1} or Q_{i+1} to Q_k and Q_i to Q_{k+1} for neither method will divide our polygon into two polygons having no points in common. If P were one of the end points P_i, P_{i+1}, P_k, or P_{k+1}, the process of suppression would be entirely similar.

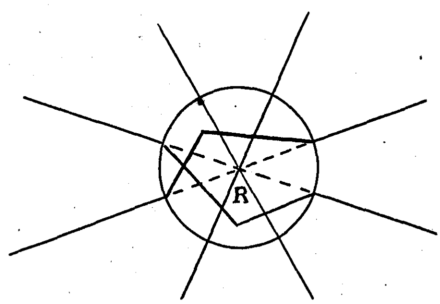
Having suppressed the double point P, we have a polygon with one less double point. Let us proceed to the next double point and suppress it in a similar manner and so on. If our polygon has no points of multiplicity higher than two, it will finally be reduced to a polygon without multiple points.

It must be observed that in each case, the circle used must be enough smaller than $\frac{\varepsilon'}{2}$ in radius that it will cut <u>only</u> segments of the two lines forming the double point. Otherwise new double points may be formen in the act of suppressing the old ones. Also let us notice that our polygon, after any suppression may be reduced to two closed polygons intersecting each other in a finite number of points. These points of intersection may then be treated in a manner the same as that used for double points.

Let us remember that the radii of the circles used in suppression were taken smaller than $\frac{\varepsilon'}{2}$. The distance, then, between any two points in or on one of these circles is less than ε'. It is easily seen therefore, that the maximum distance (as defined above) from the original polygon to the reduced polygon is less than ε', as is also the maximum distance from the reduced polygon to the original polygon.

It still remains to treat points of multiplicity higher than two. If our original polygon contains any such points, each one must be reduced to a number of double points as follows: let such a point be R. With R as center describe a circle of radius less than $\frac{\varepsilon'}{2}$. It must be enough smaller that it will cut no sides of the polygon except those on R. Remove all line segments cut out by the circle and replace *each* segment by a broken line which shall lie wholly within the circle except for its end points. These lines must be such that no two of the points of intersection thus formed by them shall coincide.

Fig. 2.



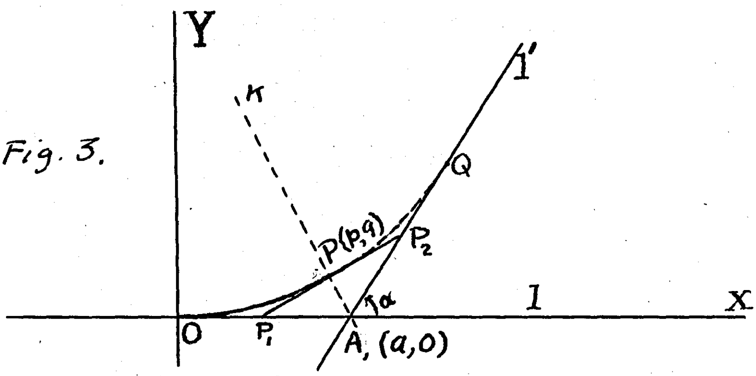
The figure above shows a point of multiplicity four reduced to six double points.

The double points which we thus obtain will all lie inside the circle about R and we shall suppress each one as indicated above. We shall take the circles used in suppression so small that they will lie entirely within the circle about R. We have now a method for suppressing all multiple points in such a way that we shall obtain a polygon of no multiple points and the two maximum distances between the original polygon and the reduced polygon shall both be less than $\mathcal{E}'$.

We shall now round off the corners of our reduced polygon by fitting into them small arcs of certain curves. We desire to obtain in this way a "rounded off" polygon, such that as a point P moves along this rounded off polygon, the curvature at P will vary in a continuous fashion. The problem of rounding off our polygon in this manner reduces itself to this: given two straight lines l and l' intersecting at a given angle. Required to determine the equation of a curve which shall be tangent to l and l' near their point of intersection, and such that if we pass from a point on l to a point on l' by way of this curve, the curvature along our path shall vary continuously. Let us find this curve for a special case. Let l lie along the x-axis and let l' intersect l at the point $A(a,0)$ and at an angle α. Let P_1 be any point on the segment OA and let P_2 be a point on l' such that $P_1A = AP_2$. Call $P(p,q)$ the midpoint of P_1P_2. We wish now to determine uniquely the position of P_1P_2 and

and also the value of a constant m such that the curve $y = mx^3$ shall be tangent to P_1P_2 at the point P.

Fig. 3.



The locus of points P is a fixed line k bisecting the angle (1',1). The equation of k is

$$y = -\frac{1}{\tan\frac{\alpha}{2}}(x - a)$$

and we therefore have the relation

$$q = -\frac{1}{\tan\frac{\alpha}{2}}(p - a) \qquad (1)$$

Now the slope of a tangent at P is

$$y' = 3mp^2 = \tan\frac{\alpha}{2}$$

whence

$$m = \frac{\tan\frac{\alpha}{2}}{3p^2} \qquad (2)$$

If we put this value of m into the relation q H mp we have

$$q = \frac{1}{3}p \cdot \tan\frac{\alpha}{2}$$

Combining this with (1) we find

$$p = \frac{a}{\frac{1}{3}\tan\frac{\alpha}{2} + \frac{1}{\tan\frac{\alpha}{2}}}$$

This value of P immediately determines by (2) a value for m and fixes the position of P and hence of P_1P_2.

Let us now imagine a second arc $\widehat{O'P'}$ coinciding with $\widehat{OP}$, O' falling at O and P' falling at P. Suppose now that we change the position of $\widehat{O'P'}$ in such a way that O' will fall on l' at a point which we shall call Q and P' shall remain at P, and furthermore $\widehat{O'P'}$ shall still remain tangent to P_1P_2 at P. It is obvious that $OP_1 = P_2Q$. It is also obvious that as a movable point approaches P along either arc segment, the curvature at that point will approach a unique value. Moreover the arc $\widehat{OP}$ has a contact of order two with the x-axis at the origin,— a fact which guarantees continuity of curvature as we pass onto $\widehat{PO}$ from the negative half of the x-axis. The same may be said of $\widehat{PQ}$ in reference to line l' and thus we have obtained the arc which fulfills the requirements of the problem which we set for ourselves. We have the equation of the arc $\widehat{OP}$ and the equation of the arc $\widehat{PQ}$ may of course be obtained by the proper transformation of axes.

Let us notice that OA may be taken as small as we please. Suppose that OA is less than β, $\beta > 0$. It is then evident that the maximum distance from any point of the arc $\widehat{OPQ}$ to any point of the broken line OAQ is less than β.

Let T be a vertex of our reduced polygon. On the two adjacent sides respectively take the points T_1 and T_2 equidistant from T and at a distance from it less than ε'', $0<\varepsilon''<\varepsilon'$. Join T_1 and T_2 by an arc such as the one already described and discard the broken line T_1TT_2. If we do this at every vertex, our polygon is transformed into the desired "rounded off" polygon without multiple points and it is evident that the two maximum distances between the reduced polygon and the "rounded off" polygon are both less than ε''.

We shall refer to this "rounded off" polygon as the polygon J. We now wish to represent J analytically in the form of x and y as functions of an independent parameter.

Consider the curve $y = mx^3$. The length of arc from a fixed point P_0 on this curve to a variable point P on the curve is

$$s = \int_{x_0}^{x} \sqrt{(1 + 9m^2x^4)}\, dx$$

Since the integrand is continuous and positive on any finite interval, s is a continuous and monotonic increasing function of x on any finite interval. Since $y = mx^3$, y also is a monotonic continuous function of s on any finite interval. It follows then* that the inverse functions exist and are continuous and we have

*See Veblen and Lennes: "Introduction to Infinitesimal Analysis", page 93.

$$x = \phi_1(s)$$
$$y = \phi_2(s)$$

where ϕ_1 and ϕ_2 are continuous on any finite interval of s. By the linear transformations

$$x = aX + bY + m$$
$$y = cX + dY + n$$

we would have

$$aX + bY + m = \phi_1(s)$$
$$cX + dY + n = \phi_2(s)$$

and X and Y themselves would be continuous functions of s. This shows that the coordinates of a point on our cubical parabola are uniformly continuous functions of the length of arc along the cubical parabola regardless of its position in respect to the coordinate axes. The same statement may obviously be made concerning a straight line. Since J consists of segments of cubical parabolas and straight lines, we may represent it by the pair of equations

$$x = \psi_1(s)$$
$$y = \psi_2(s)$$

s representing the length of arc from a fixed point in a fixed direction. Now s may be allowed to vary from $-\infty$ to $+\infty$ if we put

$$\psi_1(s_0 + S) = \psi_1(s_0)$$
$$\psi_2(s_0 + S) = \psi_2(s_0)$$

where s_0 is any fixed value of s and S is the total length of J, i.e. the total length of arc as we move along J in a fixed direction from a point P_o and finally return to P_o . ψ_1 and ψ_2 are continuous along the arc or line segments of J. At any joining point, x and y take on the same values respectively as we approach the point along either adjacent segment. Therefore ψ_1 and ψ_2 are continuous for all values of s.

It will now be convenient for us to make the change of variable $u = \frac{2\pi}{S} s$. J will then be represented by the pair of equations

$$x = F_1(u)$$
$$y = F_2(u)$$

It is obvious that F_1 and F_2 are continuous for all values of u and since

$$F_1(u_o + 2\pi) = F_1(u_o)$$
$$F_2(u_o + 2\pi) = F_2(u_o)$$

F_1 and F_2 are periodic and of period 2π.

Let α and β represent the angles with the x-axis by the tangent and normal respectively at any point of J. It is clear that as a point moves along J, $\cos\alpha$ and $\cos\beta$ vary in a continuous fashion. In fact the slope of the tangent varies continuously along the line segments and arc segments. At a joining point the tangent assumes the same position as we approach the point along J in either direction. The angles α and β therefore vary in a continuous manner and consequently their cosines.

Now, if to the identities

$$\cos\alpha = \frac{dx}{ds} = \frac{2\pi}{S} F_1'(u)$$

$$\cos\beta = \frac{dy}{ds} = \frac{2\pi}{S} F_2'(u)$$

we apply the formulae of Frenet-Serret:

$$\frac{d\cos\alpha}{ds} = \frac{\cos\beta}{R}$$

$$\frac{d\cos\beta}{ds} = -\frac{\cos\alpha}{R}$$

where R represents the radius of curvature along J,— we obtain the identities

$$F_1''(u) = \frac{\cos\beta}{R} \cdot \frac{4\pi^2}{S^2}$$

$$F_2''(u) = -\frac{\cos\alpha}{R} \cdot \frac{4\pi^2}{S^2}$$

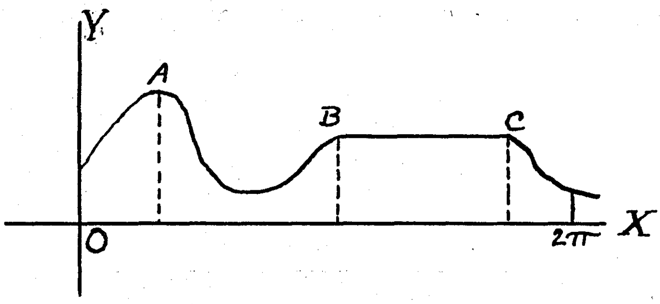
Now as we move along J, $\cos\alpha$, $\cos\beta$, and R vary in a continuous manner, and if we show that R is nowhere zero, we have established the continuity of $F_1''(u)$ and $F_2''(u)$. Since

$$R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

R can only be zero if $\frac{d^2y}{dx^2}$ is infinite. Since $\frac{d^2y}{dx^2}$ is finite along any finite arc of the cubical parabola, R can not be zero along such an arc. Along any straight line R is infinite. Thus R vanishes at no poont of J.

Consider the curve $x = F_1(u)$. This curve may have in the interval 0 to 2π a number of maximum or minimum points or segments such as the point A or the segment BC in Fig. 4. We shall call these points

Fig. 4.



simply maxima or minima and show that they must be finite in number on the interval 0 to 2π. A maximum or minimum may only occur when the corresponding point on J has its tangent parallel to the y-axis or when the corresponding line segment of J is parallel to the y-axis. By the nature of J, such points or segments are finite in number. Furthermore, again bearing in mind that J consists of a finite number of line and arc segments, it is easily seen that $\cos\alpha$, $\cos\beta$, and R have but a finite number of maxima and minima.

Since

$$F_1'(u) = \frac{S}{2\pi}\cos\alpha \qquad F_2'(u) = \frac{S}{2\pi}\cos\beta$$

$$F_1''(u) = \frac{\cos\beta}{R}\,\frac{4\pi^2}{S^2} \qquad F_2''(u) = -\frac{\cos\alpha}{R}\,\frac{4\pi^2}{S^2}$$

it follows that these four derivatives themselves have but a finite number of maxima and minima. Moreover since $\cos\beta$ and $\cos\alpha$ can vary only from +1 to −1 and since R is nowhere zero, it follows that the four derivatives remain always finite.

Since $F_1(u)$ has the period 2π, we have

$$F_1(u) = F_1(u + 2\pi)$$

for all values of u. Since $F_1'(u)$ and $F_1''(u)$ are continuous for all values of u, it follows that

$$F_1'(u) = F_1'(u + 2\pi)$$
$$F_1''(u) = F_1''(u + 2\pi)$$

for all values of u. That is, $F_1'(u)$ and $F_1''(u)$ are both periodic and of period 2π. Likewise $F_2'(u)$ and $F_2''(u)$ are both periodic and of period 2π.

Let us now summarize the properties of the functions $F_1(u)$ and $F_2(u)$ and their derivatives:-

(1) $F_1(u)$ and $F_2(u)$ are finite and continuous for all values of u.

(2) $F_1(u)$ and $F_2(u)$ both admit of period 2π.

(3) $F_1(u)$ and $F_2(u)$ both have maxima and minima (as defined above) finite in number on any interval of length 2π.

(4) Properties (1), (2), and (3) hold for $F_1'(u)$ and $F_2'(u)$.

(5) Properties (1), (2), and (3) hold for $F_1''(u)$ and $F_2''(u)$.

Suppose now that we expand $F_1(u)$ and $F_2(u)$ in the trigonometric series

$$x = F_1(u) = a_0 + \sum_{K=1}^{\infty}(a_K \cos ku + b_K \sin ku)$$
$$y = F_2(u) = c_0 + \sum_{K=1}^{\infty}(c_K \cos ku + d_K \sin ku)$$

As a result of the properties outlined above, these two expansions are absolutely and uniformly convergent. Moreover the expansions

$$F_1'(u) = a_0' + \sum_{K=1}^{\infty}(a_K' \cos ku + b_K' \sin ku)$$

$$F_2'(u) = c_0' + \sum_{K=1}^{\infty}(c_K' \cos ku + d_K' \sin ku)$$

are also absolutely and uniformly convergent and obtainable by differentiating term by term the expansions for $F_1(u)$ and $F_2(u)$ respectively.*

We shall now establish the following proposition:- For every $\varepsilon''' > 0$, ε''' as small as we please, there exists a finite integer N such that the closed curve represented by the pair of equations

$$\left.\begin{aligned} x &= a_0 + \sum_{K=1}^{n}(a_K \cos ku + b_K \sin ku) \\ y &= c_0 + \sum_{K=1}^{n}(c_K \cos ku + d_K \sin ku) \end{aligned}\right\} \quad (3)$$

shall have the following properties whenever n is an integer greater than or equal to N:-

(A) <u>The maximum distance from the curve to J shall be less than ε''', and the maximum distance from J to the curve shall be less than ε'''.</u>

(B) <u>The curve shall have no multiple points.</u>

* See Carslaw "Fourier Series and Integrals", section 62. Also Picard, "Traité d'Analyse", second edition, page 256, Vol. I.

Proof. For convenience let us write the equations (3) as follows

$$x = A_n(u)$$
$$y = B_n(u)$$

and let us designate by C_n the curve thus defined. The distance Δ from a point (x,y) on C_n to the corresponding point on J is

$$\Delta = \sqrt{[F_1(u) - A_n(u)]^2 + [F_2(u) - B_n(u)]^2}$$

Owing to the uniform convergence of the expansions for $F_1(u)$ and $F_2(u)$, we can find an integer N_1, such that

$$|F_1(u) - A_n(u)| < \frac{\varepsilon'''}{\sqrt{2}}$$
$$|F_2(u) - B_n(u)| < \frac{\varepsilon'''}{\sqrt{2}}$$

for all values of u whenever $n \geqq N_1$. Therefore $\Delta < \varepsilon'''$ for $n \geqq N_1$, and for any value of u. That is, the distance between any pair of corresponding points is less than ε''' for $n \geqq N_1$. Thus any point P_c on C_n will be within a distance of ε''' to at least one point P_j (the corresponding point) on J and vica versa. Consequently for $n \geqq N_1$, the two maximum distances between C_n and J are both less than ε'''.

Before going further, let us make this remark. If P_1 and P_2 represent two points on J corresponding respectively to u_1 and u_2, then $|u_1 - u_2|$ will be as small as we please whenever the distance $\overline{P_1 P_2}$ is sufficiently small. Consider the curve $y = mx^3$. If x_1 and x_2 are the abscissas of two points on this curve, the length of arc joining them is

$$s = \int_{x_1}^{x_2} \sqrt{1 + 9\, m^2 x^4}\; dx$$

Since this integral is a uniformly continuous function of x, (see page 11) s may be made as small as we please by taking the points such that $|x_1 - x_2|$ is sufficiently small. Since $|x_1 - x_2|$ is the length of the x-projection of the geometric distance between our points, we can make $|x_1 - x_2|$ as small as we please by taking the points near enough to each other. Thus the length of arc on the curve $y = mx^3$ will be as small as we please whenever the geometric distance between end points is sufficiently small. The same is obviously true for a straight line. Moreover this relation between chord and arc is independent of the position of the curve in respect to the coordinate axes. Let us recall that J consists of segments of straight lines and arcs of cubical parabolas and that $u = (\text{constant})\cdot(\text{length of arc})$. Consequently, from what has been said above, $|u_1 - u_2|$ is as small as we please whenever $\overline{P_1 P_2}$ is sufficiently small.

Let us now continue with our proof. Let $\xi = \frac{\delta}{8\pi}$

Since $F_1'(u)$ is uniformly continuous, there exists an η such that when $|u_i - u_k| < \eta$

$$|F_1'(u_i) - F_1'(u_k)| < \xi \quad . \quad . \quad . \quad . \quad (4)$$

Let P_i and P_k represent points on J corresponding to u_i and u_k. Let ζ be a positive number such that if $\overline{P_i P_k} < \zeta$, $|u_i - u_k| < \eta$. We have proved above the existence of such a ζ.

Let N_2 be a positive integer such that for $n \geqq N_2$

$$\left.\begin{aligned} |F_1(u) - A_n(u)| &< \frac{\zeta}{2\sqrt{2}} {}^{*} \\ |F_2(u) - B_n(u)| &< \frac{\zeta}{2\sqrt{2}} \end{aligned}\right\} \quad (5)$$

and let N_3 be a positive integer such that for $n \geqq N_3$

$$|F_1'(u) - A_n'(u)| < \xi$$
$$|F_2'(u) - B_n'(u)| < \xi$$

The uniform convergence of $A_n(u)$, $B_n(u)$, $A_n'(u)$, and $B_n'(u)$ guarantee the existence of N_2 and N_3.

Now let $n = N_4$ where N_4 is greater than any of the integers N_1, N_2, or N_3. We shall show that such a choice of n will result in a curve C_n satisfying (A) and (B).

Suppose that for $n = N_4$ the curve

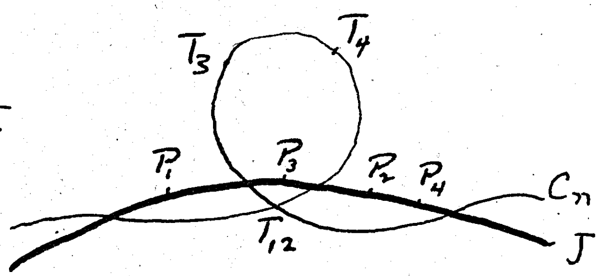
$$\left.\begin{aligned} x &= A_n(u) \\ y &= B_n(u) \end{aligned}\right\} C_n$$

has a loop. We shall denote by P_i and T_i the points on J and C_n respectively corresponding to u_i. Let T_{12}

*This makes the distance Δ between any point on J and the corresponding point on C_n less than $\frac{\zeta}{2}$. See page 17.

be the double point on C_n corresponding to the values u_1 and u_2.

Fig. 5.



From (5) and footnote on preceeding page we have $\overline{T_{12}P_1} < \frac{\zeta}{2}$ and $\overline{T_{12}P_2} < \frac{\zeta}{2}$. $\therefore$ $\overline{P_1 P_2} < \zeta$ and consequently $|u_1 - u_2| < \eta$.

At our double point we have

$$A_n(u_1) = A_n(u_2)$$
$$B_n(u_1) = B_n(u_2)$$

By Rolle's Theorem there exists a u_3 and a u_3 both lying between u_1 and u_2 such that

$$A'_n(u_3) = 0$$
$$B'_n(u_4) = 0$$

Since $|u_1 - u_2| < \eta$ and since u_3 and u_4 both lie between u_1 and u_2, we have $|u_3 - u_4| < \eta$, making from (4)

$$\left| F'_1(u_3) - F'_1(u_4) \right| < \xi \quad . \quad . \quad . \quad (6)$$

Owing to our choice of n we have

$$\left| F'_1(u_3) - A'_n(u_3) \right| < \xi$$
$$\left| F'_2(u_4) - B'_n(u_4) \right| < \xi$$

But since $A_n'(u_3) = 0$ and $B_n'(u_4) = 0$, we have

$$\left.\begin{array}{l} |F_1'(u_3)| < \xi \\ |F_2'(u_4)| < \xi \end{array}\right\} \quad . \; . \; . \quad (7)$$

From (6) and (7) it follows that

$$|F_1'(u_4)| < 2\xi$$

We shall make use of the pair of inequauities:

$$|F_1'(u_4)| < 2\xi$$
$$|F_2'(u_4)| < \xi$$

If θ represents the angle which the tangent to J at P_4 makes with the x-axis, we have

$$|F_1'(u_4)| = \left|\frac{S}{2\pi}\cos\theta\right| < 2\xi$$
$$|F_2'(u_4)| = \left|\frac{S}{2\pi}\sin\theta\right| < \xi$$

from which

$$|\cos\theta| < \frac{4\pi}{S} \cdot \xi$$
$$|\sin\theta| < \frac{2\pi}{S} \cdot \xi$$

But we have chosen $\xi = \dfrac{S}{8\pi}$ and we have therefore

$$|\cos\theta| < \frac{1}{2}$$
$$|\sin\theta| < \frac{1}{4}$$

which is absurd. Consequently for $n = N_4$, a double point on C_n is impossible.

The same arguments would prove the impossibility of the existence of multiple points of any order. Since $N_4 \geqq N_1$, C_n satisfies condition (A) as well as (B) for $n = N_4$. Thus our proposition is proved for we may take $N = N_4$.

Suppose that in the identities

$$\cos ku = \frac{e^{iku} + e^{-iku}}{2}$$

$$\sin ku = \frac{e^{iku} - e^{-iku}}{2i}$$

we put $e^{iu} = v$. We have then

$$\left.\begin{aligned} \cos ku &= \frac{v^{2k}+1}{2v^k} \\ \sin ku &= \frac{v^{2k}-1}{2iv^k} \end{aligned}\right\} \qquad (8)$$

If we make the substitutions (8) in the pair of equations

$$x = a_0 + \sum_{k=1}^{N} (a_k \cos ku + b_k \sin ku)$$

$$y = c_0 + \sum_{k=1}^{N} (c_k \cos ku + d_k \sin ku)$$

we obtain

$$x = \frac{P(v)}{2v^N} \qquad y = \frac{R(v)}{2iv^N} \qquad (9)$$

where P(v) and R(v) are polynomials in v and both of degree 2N. That is, we have represented x and y as rational functions of the independent variable v.

Let δ be a positive number as small as we please, and suppose that we choose

$$\epsilon < \frac{\delta}{2} \qquad \epsilon' < \frac{\delta}{4} \qquad \epsilon'' < \frac{\delta}{8} \qquad \epsilon''' < \frac{\delta}{16}$$

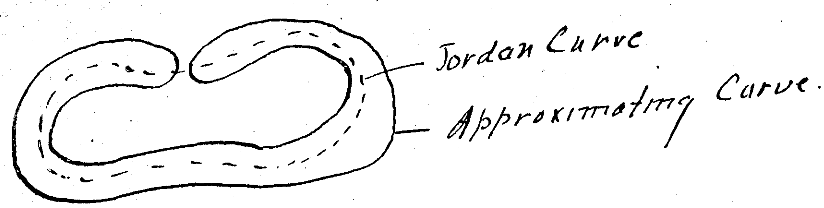
We have then

$$\epsilon + \epsilon' + \epsilon'' + \epsilon''' < \delta$$

The equations (9) therefore represent a unicursal curve which approximates our Jordan curve so that the maximum distance from the Jordan curve to the curve (9) is less than and the maximum distance from the curve (9) to the Jordan curve is less than δ.

We have thus shown that any Jordan curve may be approximated by a real loop of a unicursal curve, the loop containing no multiple points.

The nature of the approximation is somewhat weak. It does not prevent, for example, an approximation by a double loop as shown in the figure below.



It may be possible that by treating our multiple points of the first polygon in a different manner, we could strengthen the nature of our approximation to an extent which would enable us to develope a new proof for the Jordan Theorem.

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