Hessians and Steinerians

of

Plane Quartic Curves

by

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§1. Introduction.

Among the curves which play an important and interesting role in the theory of plane curves are the Hessian and the Steinerian. The Hessian of a curve may be defined either as the locus of points which are double points on first polar curves of the given curve, or as the locus of points whose polar conics with respect to the given curve have double points, that is, break up. The Steinerian of a curve may be defined either as the locus of points whose first polars with respect to the given curve have double points, or as the locus of points which are double points on polar conics. The Steinerian is clearly the reciprocal of the Hessian.

The order and class of the Hessian of a plane curve are in general $3(n-2)$ and $3(n-2)(3n-7)$; respectively, and of the Steinerian $3(n-2)^2$ and $3(n-1)(n-2)$, respectively. The order and class of the Hessian of a plane quartic are therefore in general 6 and 30, and of the Steinerian of a plane quartic 12 and 18, respectively.*

It is the purpose of this paper to determine the Hessians and the Steinerians of particular quartics and to plot these

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*For the Plücker characteristics of these curves, see Hagen, *Synopsis der höheren Mathematik*, Bd.2, p.203.

curves together with the original quartics.*

The equations of the curves will be given in homogeneous coördinates, where x=0, y=0, z=0 are the y-axis, the x-axis, and the ideal line, respectively.

§2. Quartics having Triple Points.

A k-tuple on a plane curve is a $(k-1)$-tuple point on the first polar of every point in the plane with respect to that curve.** Therefore, if a quartic has a triple point, this triple point is a double point on the first polar of every point in the plane. Since the Steinerian is the locus of points whose first polars have double points, it follows that every point in the plane satisfies the Steinerian of a quartic with a triple point. Therefore a quartic with a triple point has no definite Steinerian.

The maximum number of double points on a proper curve of nth degree is $\frac{1}{2}(n-1)(n-2)$. Since a k-tuple point is equivalent to $\frac{1}{2}k(k-1)$ double points, it follows that a plane quartic having a triple point can have no double points in addition.

A k-tuple point on a plane curve is a $(3k-4)$-tuple point

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*The original quartics will be traced in black, the Hessian in green, and the Steinerian in red.

**cf. Salmon, *A Treatise on Higher Plane Curves*.

on the Hessian of that curve. A triple point on a plane quartic is therefore a quintuple point on its Hessian. Since a quintuple point is equivalent to ten double points, the Hessian can have no double points in addition to the quintuple point.

Every triple point is formed by the *simultaneous* union of three double points. Triple points of plane quartics are of four species according as the tangents at the triple point are

1) all real and distinct,

2) one real and distinct and two real and coincident,

3) all real and coincident,

4) one real and two imaginary;

that is, triple points are composed of

1) three crunodes,

2) two crunodes and one cusp,

3) one crunode and two cusps,

4) one crunode and two acnodes;

respectively*.

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*Basset, *Elementary Treatise on Cubic and Quartic Curves*, pp.102-104.

A.

$$F_1 = (x^2+y^2)^2 + [(a-r)x^2+(a+3r)y^2]xz = 0$$

This curve has a triple point at (0,0,1), its species depending upon the relative values of *r* and *a* , where *r* is the radius of an auxiliary circle* and where *a* is the algebraic distance of the triple point from the center of this auxiliary circle. For all values of *r* and *a* the curve $F_1=0$ is symmetrical with respect to y=0.

The ideal line, z=0, intersects the curve in the points which satisfy

$$(x^2+y^2)^2 = 0$$

The curve therefore passed through the two circular points at infinity and has the ideal line, z=0, as an ordinary tangent at each of these points. The ideal line is therefore a double tangent to the given curve with imaginary points of contact. Such a quartic will be called a *circular* quartic.

We shall first determine the Hessian of the general curve and afterwards note its particular characteristics for different values of *r* and *a*.

The first polar of the point (α,β,γ) with respect to $F_1=0$ is

$$\varphi = \alpha|4x(x^2+y^2)+2(a-r)x^2z+z[(a-r)x^2+(a+3r)y^2]|$$
$$+ \beta[4y(x^2+y^2)+2(a+3r)xyz] + \gamma|x[(a-r)x^2+(a+3r)y^2]|=0$$

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*Wieleitner, *Spezielle Ebene Kurven*, pp.148-156.

If $\varphi=0$ has a double point, such a point must satisfy

$$U = \alpha[4(3x^2+y^2)+6(a-r)xz] + \beta[8xy+2(a+3r)yz] + \gamma[3(a-r)x^2+(a+3r)y^2] = 0$$

$$V = \alpha[8xy+2(a+3r)yz] + \beta[4(x^2+3y^2)+2(a+3r)xz] + \gamma[2(a+3r)xy] = 0$$

$$W = \alpha[3(a-r)x^2+(a+3r)y^2] + \beta[2(a+3r)xy] = 0$$

where U, V, W are the derivatives of φ with respect to x, y, z, respectively. Eliminating (α,β,γ) we have as the locus of the double points of $\varphi=0$

$$H_1 = \begin{vmatrix} 4(3x^2+y^2)+6(a-r)xz & 8xy+2(a+3r)yz & 3(a-r)x^2+(a+3r)y^2 \\ 8xy+2(a+3r)yz & 4(x^2+3y^2)+2(a+3r)xz & 2(a+3r)xy \\ 3(a-r)x^2+(a+3r)y^2 & 2(a+3r)xy & 0 \end{vmatrix} = 0$$

Expanding and reducing

$$H_1 = Ax^6+Bx^5z+Cx^4y^2+Dx^3y^2z+Ex^2y^4+Fxy^4z+Gy^6 = 0$$

where

$$A = 6(a-r)^2$$

$$B = 3(a-r)^2(a+3r)$$

$$C = 18(a-r)^2 - 12(a-r)(a+3r) + 8(a+3r)^2$$

$$D = 2(a-r)(a+3r)^2$$

$$E = 12(a-r)(a+3r) - 2(a+3r)^2$$

$$F = -(a+3r)^3$$

$$G = 2(a+3r)^2$$

This equation shows that the Hessian also is symmetrical to $y=0$.

A_1.

If a=0, the general equation, $F_1=0$, reduces to

$$F_1^1 = (x^2+y^2)^2 - r(x^2-3y^2)xz = 0$$

This is the familiar "three-leaved rose", or, "regular three-leaf"* having a triple point of the first species at (0,0,1) with

$$x(x^2-3y^2) = 0$$

as tangents (Fig.I). The Hessian of $F_1^1=0$ is

$$H_1^1 = 2x^6+3rx^5z+42x^4y^2-6x^3y^2z-18x^2y^4-9rxy^4z+6y^6 = 0$$

It has a quintuple point at the point (0,0,1), and so can have no other double points. The tangents at the quintuple point are given by

$$x(x^2-3y^2)(x^2+y^2) = 0;$$

from which it follows that the quintuple point consists of the triple point of $F_1^1=0$ with the same tangents, and an acnode. The sixth point of intersection of y=0 with the curve is

$$x = -\tfrac{3}{2}rz$$

The ideal line, z=0, intersects the curve in the points for which

(1) $$x^6+21x^4y^2-9x^2y^4+3y^6 = 0$$

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*"regelmässig Dreiblatt", cf. Wieleitner, loc. cit.

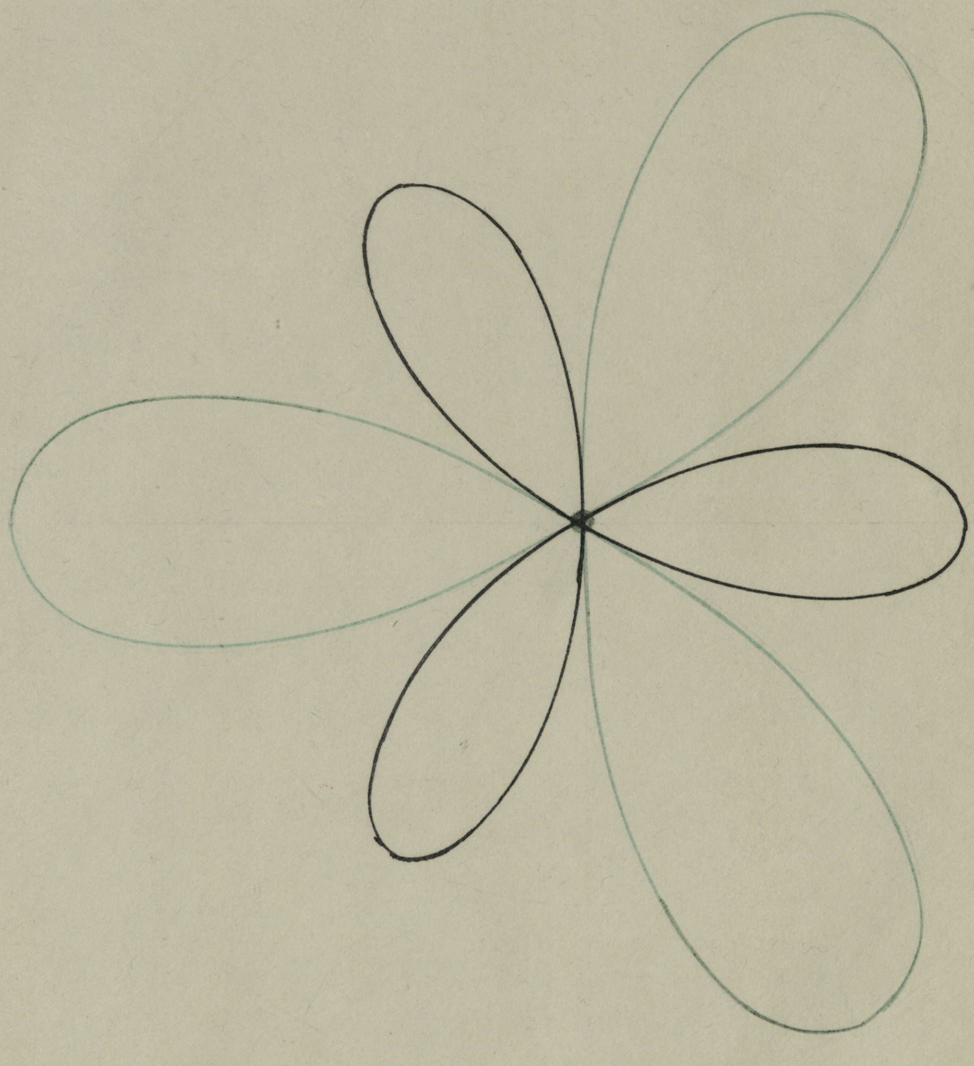

Fig. I

Placing $\frac{x^2}{y^2}=t$, this equation reduces to a cubic in **t** having one negative real root and two imaginary roots. All values of $\frac{x}{y}$ satisfying (1) are therefore imaginary. The curve thus lies entirely in finite space.

A_2.

If $a=r$, the equation $F_1=0$ reduces to

$$F_1^2 = (x^2+y^2)^2 + 4rxy^2z = 0$$

which is the "even two-leaf"* having a triple point at (0,0,1). The tangents at this triple point are

$$xy^2 = 0$$

The triple point therefore belongs to the second species, that is, it is equivalent to two nodes and one cusp (Fig.II). This curve, as was seen in the general case, is a closed curve and therefore lies entirely within a rectangle whose sides are tangent to the curve and parallel to x=0 and y=0. The point of tangency of such a tangent line parallel to y=0 must satisfy

$$(2) \qquad \frac{\partial F_1^2}{\partial x} = 0^{**}$$

Hence eliminating *x* between (2) and $F_1^2=0$, we obtain the ordi-

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*"gerades Zweiblatt", cf. Wieleitner, loc.cit.

**Wieleitner, *Theorie der Ebenen Algebraischen Kurven höh - eren Ordnung*, p.20.

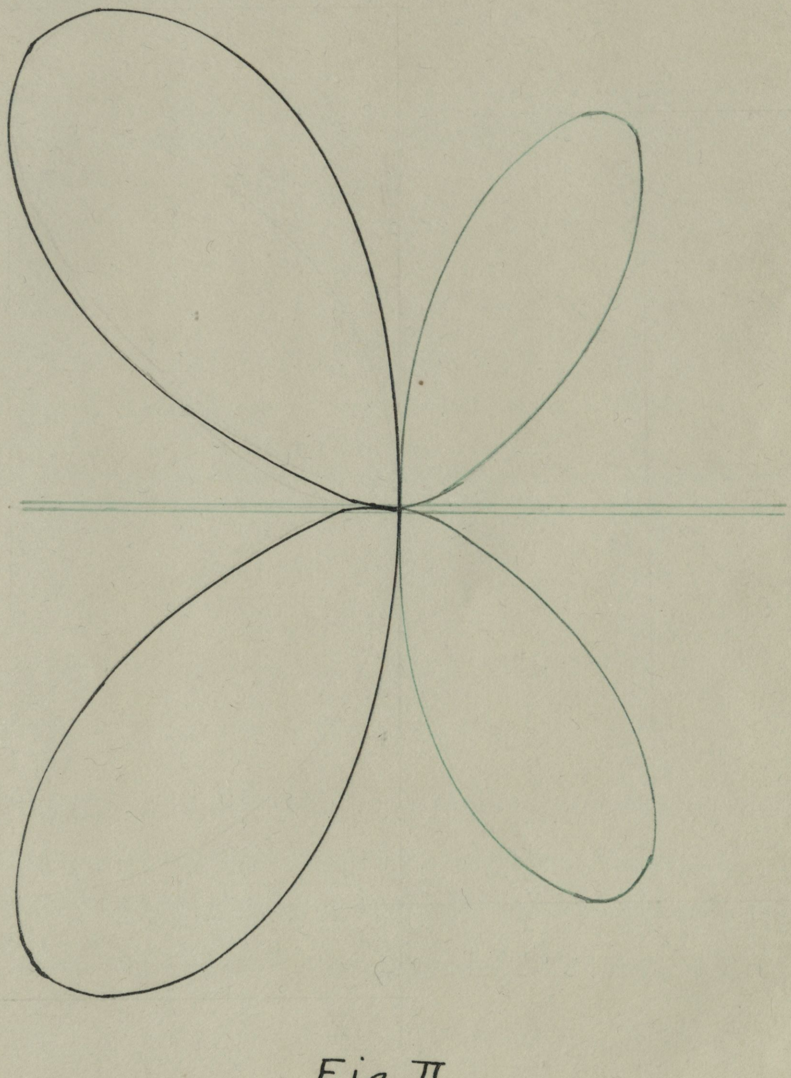

Fig. II

nates of the points of horizontal tangency, and in fact the equations of the horizontal tangents themselves. Similarly, the abscissas of the points of tangency of the vertical tangents may be obtained by eliminating y between $F_1^2=0$ and $\frac{\partial F_1^2}{\partial y}=0$. Thus the vertical tangents are

$$x = 0, \quad x = -rz$$

and the horizontal tangents are

$$y = 0, \quad y = \pm \tfrac{3}{8}\sqrt{3}rz$$

Moreover, $x=-rz$ is a double tangent.

The Hessian of F_1^2 is

$$H_1^2 = y^2(4x^4-x^2y^2-2rxy^2z+y^4) = 0$$

and is therefore composed of $y=0$ counted twice and a quartic having a triple point at (0,0,1), the tangents at the triple point being given by

$$xy^2 = 0$$

This triple point is therefore of the same species as that of the original curve, $F_1^2=0$, having in fact, the same tangents. The points of intersection of $H_1^2=0$ with the ideal line satisfy

$$4x^4-x^2y^2+y^4 = 0$$

The discriminant of this equation considered as a quadratic in $\frac{x^2}{y^2}$ reduces to $\sqrt{-15}$. The curve therefore lies entirely in finite space. Its vertical tangents are

$$x = 0, \quad x = \tfrac{2}{3}rz$$

and its horizontal tangents are

$$y = 0, \quad y = \pm\tfrac{2}{3}\sqrt{3}rz$$

Of these, $x=\frac{2}{3}z$ is a double tangent.

A_3.

If $a=-r$, the equation, $F_1=0$, reduces to

$$F_1^3 = (x^2+y^2)^2 + 2r(y^2-x^2)xz = 0$$

This curve (Fig.III) has a triple point at (0,0,1), the tangents at which are given by

$$x(y^2-x^2) = 0;$$

it is therefore of the first species. The vertical and horizontal tangents are

$$x = 0, \quad x = 2rz, \quad x = -\frac{r}{4}z$$

$$y = \pm\frac{r}{2}z$$

respectively; moreover

$$x = -\frac{r}{4}z, \quad y = \pm\frac{r}{2}z$$

are double tangents. $F_1^3=0$ is called the "even three-leaf"* and can be obtained by a deformation of the "regular three-leaf"

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*"gerade Dreiblatt," Wieleitner, loc. cit.

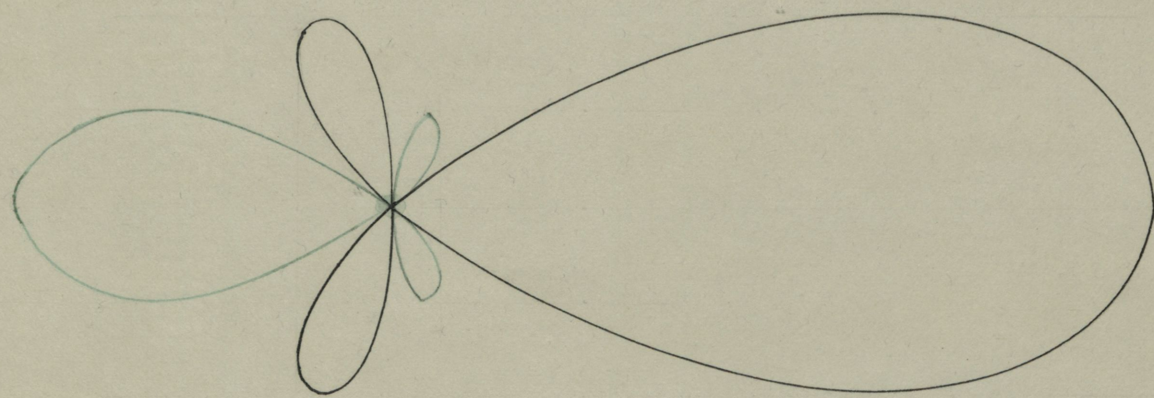

Fig. III

considered above.

The Hessian of $F_1^3=0$ is

$$H_1^3 = 3x^6+3rx^5z+19x^4y^2-2rx^3y^2z-7x^2y^4-rxy^4z+y^6 = 0$$

This has at (0,0,1) a quintuple point, the tangents at which are given by

$$x(3x^2+y^2)(x^2-y^2) = 0$$

The quintuple point is thus composed of the triple point of $F_1^3=0$ and an acnode. The x-axis meets the Hessian, in addition to the quintuple point, at the point for which $x=-rz$. The points of intersection of the Hessian with the infinite line are given by

$$3x^6+19x^4y^2-7x^2y^4+y^6 = 0$$

In exactly the same manner as in the case of $F_1^1=0$ it is found that this equation can have no real roots. This Hessian, $F_1^3=0$, therefore lies entirely in finite space.

A_4.

If $a=-3r$, the general equation, $F_1=0$, reduces to

$$F_1^4 = (x^2+y^2)^2 - 4rx^3z = 0$$

This curve has a triple point of the third species at (0,0,1), the tangents at this triple point being given by

$$x^3 = 0$$

The curve $F_1^4=0$ (Fig.IV) meets $y=0$ again at $x=4rz$. Its vertical and horizontal tangents are

$$x = 0, \quad x = 4rz$$

$$y = \pm\tfrac{3}{4}\sqrt{3}rz$$

respectively. It is called the "one-leaf".*

Its Hessian is

$$H_1^4 = x^4(x^2+3y^2) = 0;$$

which consists therefore of $x=0$ counted four times and a point ellipse.

B.

$$F_2= x^4+y^4-y(x^2+y^2)z = 0$$

This curve (Fig.V) has a triple point of the fourth species at (0,0,1), the tangents at the triple point being given by

$$y(x^2+y^2) = 0$$

It is symmetrical with respect to $x=0$. Its vertical and horizontal tangents are

$$x = \pm z$$

$$y = 0, \quad y = z, \quad y = \frac{1+\sqrt{2}}{2} z$$

respectively; $y = \frac{1+\sqrt{2}}{2}z$ being, moreover, a double tangent.

The first polar of the point (α,β,γ) with respect to this

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*"Einblatt". It has historical interest in that it was used by Kepler to represent the orbit of Mars.

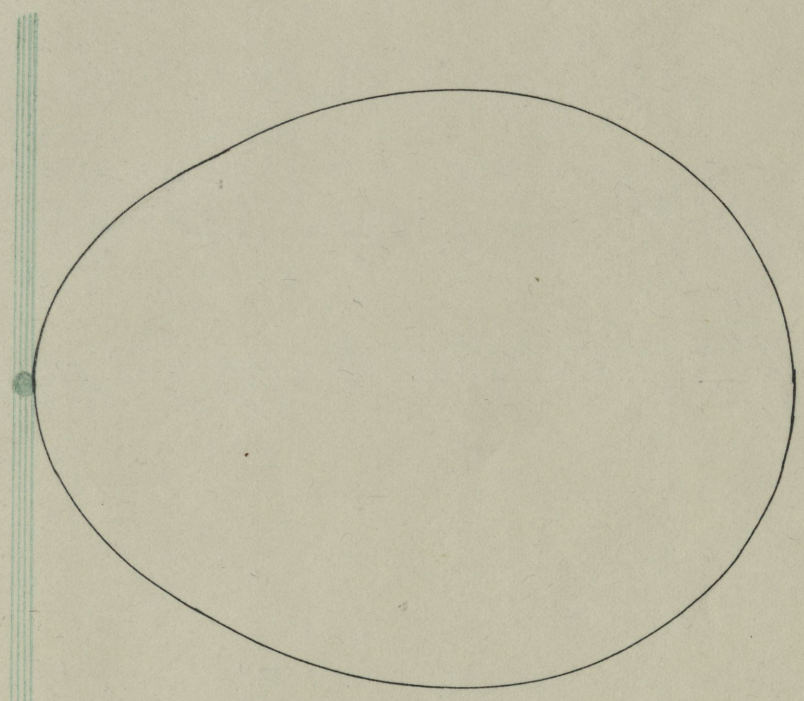

Fig. IV

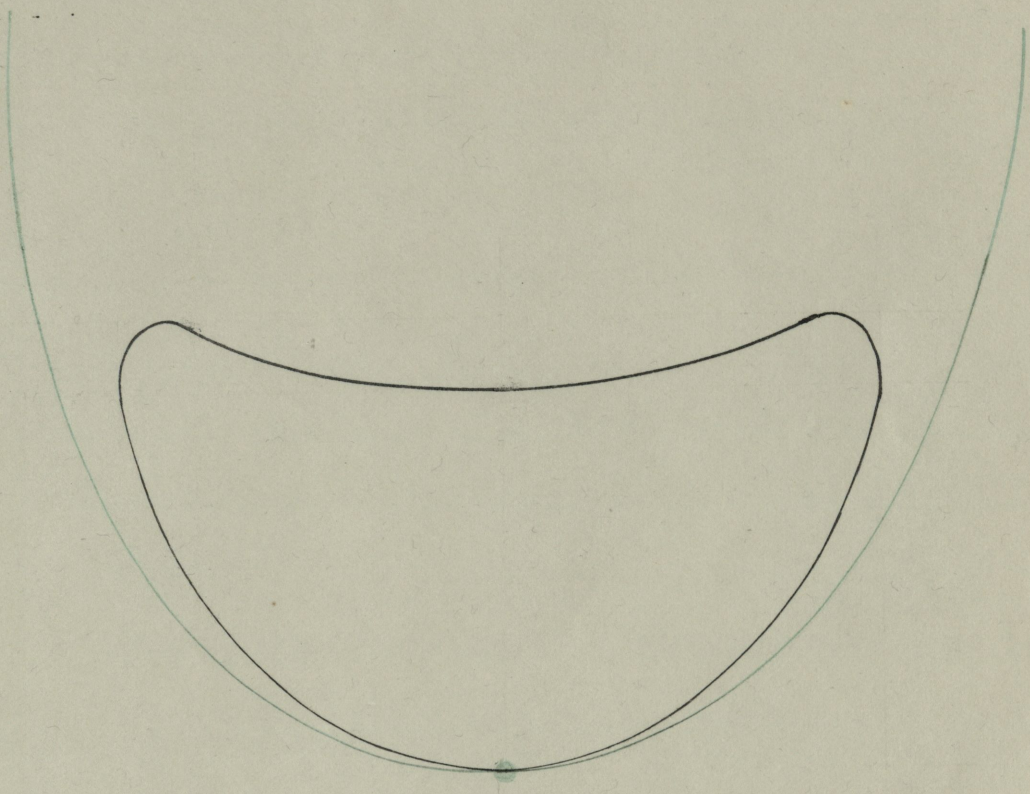

Fig. V

curve is

$$\alpha[2x(2x^2-yz)] + \beta[-x^2z+4y^3-3y^2z] + \gamma[-y(x^2+y^2)] = 0$$

The conditional equations which a point must satisfy for it to be a double point on this first polar are

$$U = \alpha[2(6x^2-yz)] + \beta[-2xz] + \gamma[-2xy] = 0$$

$$V = \alpha[-2xz] + \beta[6y(2y-z)] + \gamma[-x^2-3y^2] = 0$$

$$W = \alpha[-2xy] + \beta[-x^2-3y^2] = 0$$

The equation of the Hessian of $F_2=0$ is therefore

$$H_2 = \begin{vmatrix} 2(6x^2-yz) & -2xz & -2xy \\ -2xz & 6y(2y-z) & -x^2-3y^2 \\ -2xy & -x^2-3y^2 & 0 \end{vmatrix} = 0$$

that is,

$$H_2 = 2x^6+12x^4y^2+x^4yz+26x^2y^4-2x^2y^3z-3y^5z = 0$$

This Hessian has a quintuple point at (0,0,1), with

$$y(x^2-3y^2)(x^2+y^2) = 0$$

as tangents. Hence the quintuple point consists of the triple point of $F_2=0$ together with a crunode. The infinite line touches $H_2=0$ at (0,1,0) but does not meet it in any other real points. Also $H_2=0$ is symmetrical with respect to x=0.

C.

$$F_3 = x^4 + y^3(y-az) = 0$$

This "Pearl curve"* has a triple point of the third species at (0,0,1), the tangents being

$$y^3 = 0$$

It lies entirely in finite space. Its vertical and horizontal tangents are

$$x = \pm \frac{a}{4}\sqrt{3\sqrt{3}}z$$

$$y = 0, \quad y = az$$

respectively. The point (0,a,1) is a point of undulation (Fig.VI).

The first polar of (α,β,γ) is

$$\alpha[4x^3] + \beta[y^2(4y-az)] + \gamma[-ay^3] = 0$$

The conditional equations in this case are

$$U = \alpha[12x^2] = 0$$

$$V = \beta[6y(2y-az)] + \gamma[-3ay^2] = 0$$

$$W = \beta[-3ay^2] = 0$$

Therefore

$$H_3 = \begin{vmatrix} 12x^2 & 0 & 0 \\ 0 & 6y(2y-az) & -3ay^2 \\ 0 & -3ay^2 & 0 \end{vmatrix} = 0$$

or,

$$H_3 = [xy^2]^2 = 0$$

Thus $H_3=0$ consists of $x=0$ counted twice and $y=0$, four times.

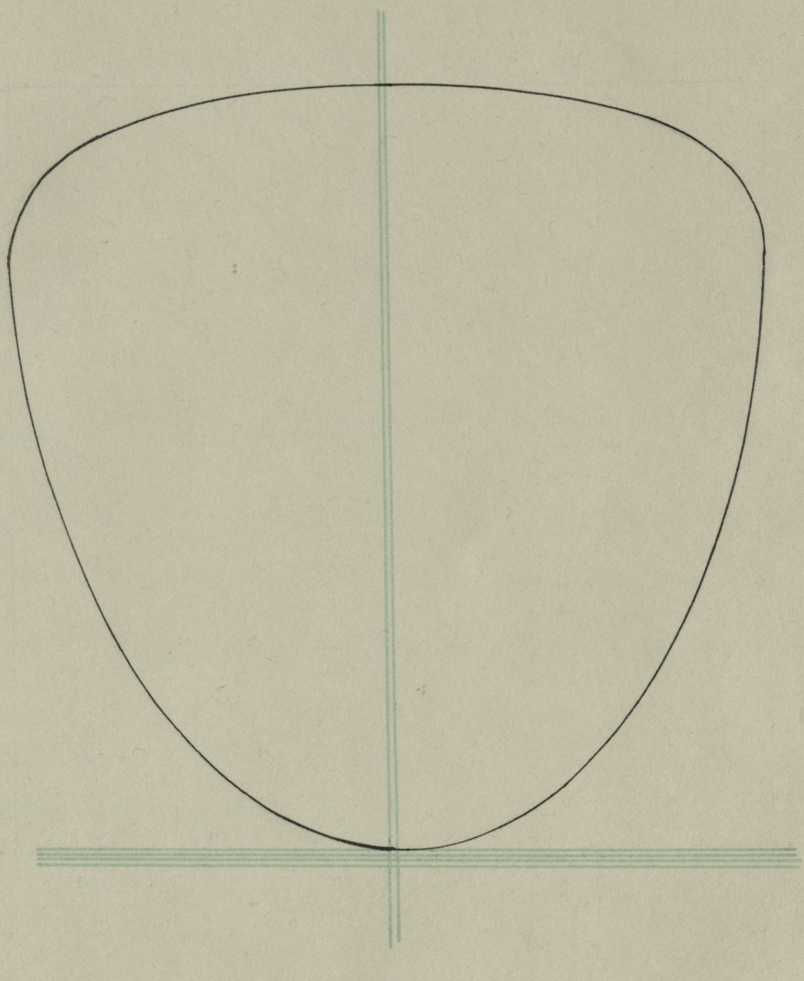

Fig. VI

D.

$$F_4 = (x^2+y^2)^2 - x^2(ax+by)z = 0$$

This curve* has a triple point of the second species at (0,0,1), the tangents at that point being

$$x^2(ax+by) = 0$$

It lies entirely in finite space, but passes through the circular points at infinity, that is, it is a circular quartic(Fig.VII).

The point (α,β,γ) has for its first polar

$$\alpha[4x(x^2+y^2)-x(3ax+2by)z] + \beta[4y(x^2+y^2)-bx^2z] + \gamma[-x^2(ax+by)] = 0$$

The three conditional equations are

$$U = \alpha[4(3x^2+y^2)-2(3ax+by)z] + \beta[2x(4y-bz)] + \gamma[-x(3ax+2by)] = 0$$

$$V = \alpha[2x(4y-bz)] + \beta[4(x^2+3y^2)] + \gamma[-bx^2] = 0$$

$$W = \alpha[-x(3ax+2by)] + \beta[-bx^2] = 0$$

Hence the Hessian is

$$H_4 = \begin{vmatrix} 4(3x^2+y^2)-2(3ax+by)z & 2x(4y-bz) & -x(3ax+2by) \\ 2x(4y-bz) & 4(x^2+3y^2) & -bx^2 \\ -x(3ax+2by) & -bx^2 & 0 \end{vmatrix} = 0$$

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*Loria, *Spezielle Algebraische und Transcendente Ebene Kurven*, zweite Auflage, Bd.I,pp.171-172.

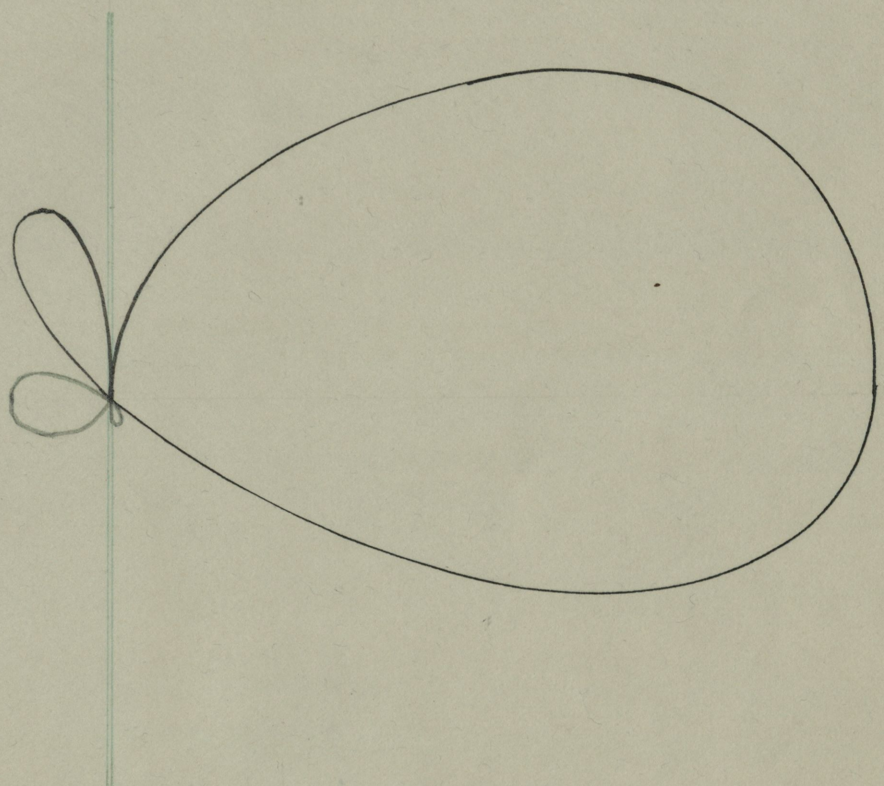

Fig. VII

that is,

$$H_4 = x^2[2(3a^2+b^2)x^4+ab^2x^3z+2(9a^2-b^2)x^2y^2+b^3x^2yz$$
$$+24abxy^3+8b^2y^4] = 0$$

This Hessian is composed of $x=0$ twice, and a quartic having (0,0,1) for a triple point of the same species, and having the same tangents at this triple point as $F_4 = 0$.

If in $F_4=0$, a is 0, the curve reduces to an "even two-leaf"; this was considered in A_2.

§3. Quartics with Singularities other than Triple Points.

E.

$$F_5 = (x^2-a^2z^2)y^2 - bz^4 = 0$$

This curve has a tacnode at (1,0,0) and in addition, two two distinct points of inflexion on $z=0$. At the point (1,0,0), $y=0$ is the tacnodal tangent, while $x=\pm az$ are the tangents at the points of inflexion on $z=0$ (Fig.VIII).

The three equations, which must be satisfied if the first polar of (α,β,γ) has a double point, are

$$U = \alpha[2y^2] + \beta[4xy] = 0$$

(3) $$V = \alpha[4xy] + \beta[2(x^2-a^2z^2)] + \gamma[-4a^2yz] = 0$$

$$W = \beta[-4a^2yz] + \gamma[-2(a^2y^2+6bz^2)] = 0$$

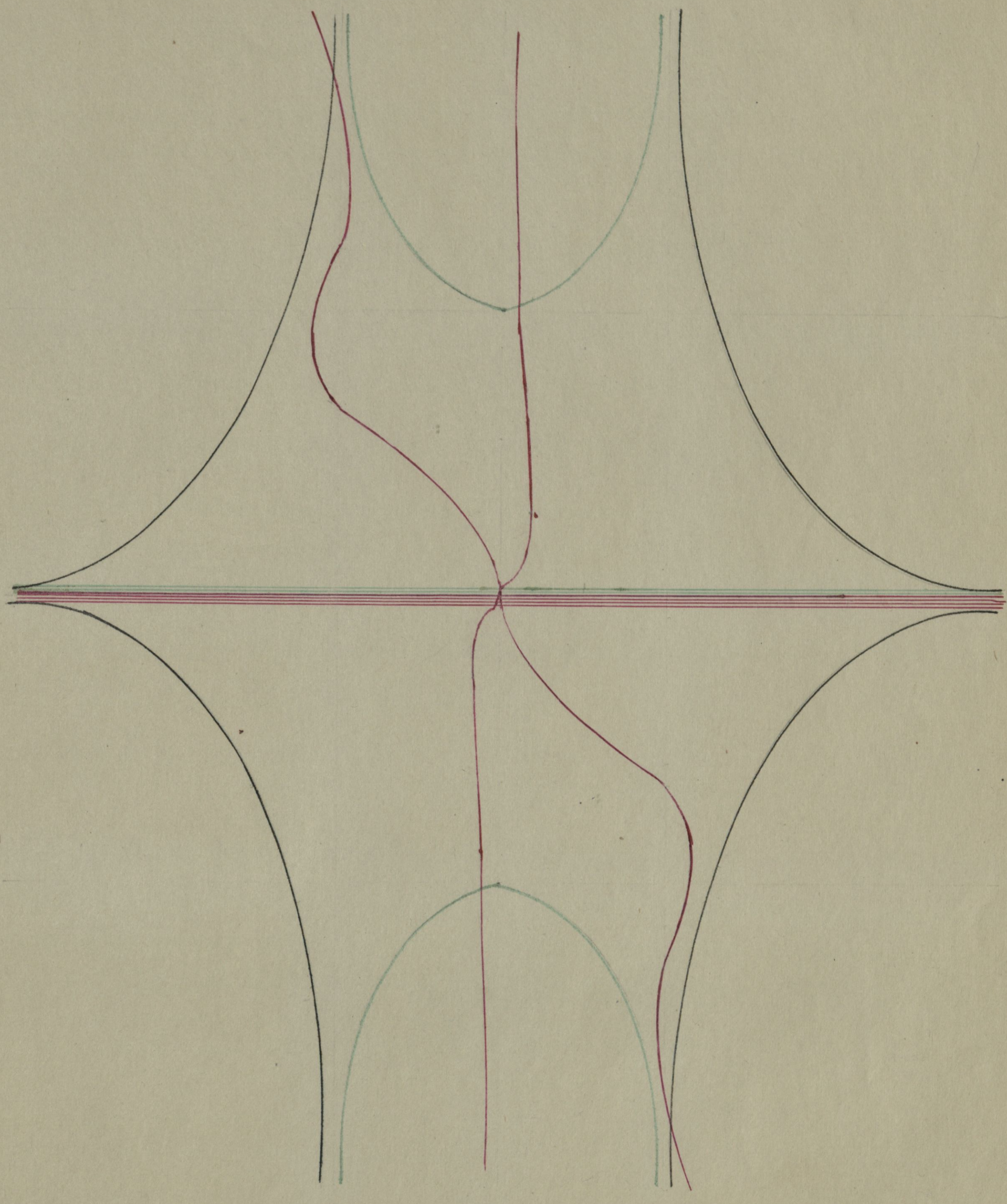

Fig. VIII

Therefore

$$H_5 = \begin{vmatrix} 2y^2 & 4xy & 0 \\ 4xy & 2(x^2-a^2z^2) & -4a^2yz \\ 0 & -4a^2yz & -2(a^2y^2+6bz^2) \end{vmatrix} = 0$$

or

$$H_5 = y^2[a^2y^2(x^2-a^2z^2) + 2bz^2(3x^2+a^2z^2)] = 0$$

$H_5=0$ thus consists of $y=0$ counted twice, and a quartic symmetrical with respect to both $x=0$, $y=0$, having at $(0,1,0)$ the same singularities as $F_5=0$. This quartic also has a double point at $(1,0,0)$ with

$$y^2 + 6z^2 = 0$$

as tangents. The double point is therefore an acnode. The vertical and horizontal tangents are

$$x = \pm\, az \text{ (which are inflexional asymptotes)}$$

$$y = \pm \sqrt{2}\, az$$

respectively.

The equation of the Steinerian is obtained by eliminating x, y, z from equations (3).The equationa (3) are, however, not sufficient for this purpose. It is shown by Salmon* that a system of values which satisfies a set of equations also sat-

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*cf. Salmon, *Modern Higher Algebra*, 4th ed., p.84.

isfies the Jacobian of the set; moreover, that, if the equations are of the same degree, the system of values will also satisfy the derivatives of the Jacobian with respect to each of the variables. The derivatives with respect to x,y,z of the Jacobian of (3) give six equations from which x^2, y^2, z^2, xy, xz, yz can be eliminated.

The Jacobian is

$$J = \begin{vmatrix} \beta y & \beta x+\alpha y & 0 \\ \beta x+\alpha y & \alpha x-a^2\gamma z & -a^2(\gamma y+\beta z) \\ 0 & -a^2(\gamma y+\beta z) & -(a^2\beta y+6b\gamma z) \end{vmatrix} = 0$$

that is,

$$J = a_2x^2y+a_3x^2z+b_2xy^2+d_1xyz+b_1y^3+b_3y^2z+c_3yz^2 = 0$$

where

$$a_2 = a^2\beta(\beta^2-a^2\gamma^2) \qquad d_1 = 2\beta\gamma(3b\alpha-a^4\beta)$$

$$a_3 = 6b\beta^2\gamma \qquad b_1 = a^2\alpha^2\beta$$

$$b_2 = a^2\alpha\beta^2 \qquad b_3 = 6b\alpha^2\gamma + a^4\beta^2\gamma$$

$$c_3 = a^2\beta(-a^2\beta^2+6b\gamma^2)$$

Hence

$$S_5 = \begin{vmatrix} 0 & 2\alpha & 0 & 4\beta & 0 & 0 \\ 2\beta & 0 & -2a^2\beta & 4\alpha & 0 & -4a^2\gamma \\ 0 & -2a^2\gamma & -12b\gamma & 0 & 0 & -4a^2\beta \\ 0 & a^2\alpha\beta^2 & 0 & 2a^2\beta(\beta^2-a^2\gamma^2) & 12b\beta^2\gamma & 2\beta\gamma(3b\alpha-a^4\beta) \\ a^2\beta(\beta^2-a^2\gamma^2) & 3a^2\alpha^2\beta & a^2\beta(-a^2\beta+6b\gamma^2) & 2a^2\alpha\beta^2 & 2\beta\gamma(3b\alpha-a^4\beta) & 2\gamma(6b\alpha^2+a^4\beta^2) \\ 6b\beta^2\gamma & 6b\alpha^2\gamma+a^4\beta^2\gamma & 0 & 2\beta\gamma(3b\alpha-a^4\beta) & 0 & 2a^2\beta(-a^2\beta^2+6b\gamma^2) \end{vmatrix} = 0$$

Expanding and reducing

$$S_5 = \beta^4\gamma^2[243b^3\alpha^4\gamma^2+81a^4b^2\alpha^3\beta\gamma^2+30a^6b\alpha^2\beta^4-$$

$$-6a^4b(a^4+17b)\alpha^2\beta^2\gamma^2+54a^6b^2\alpha^2\gamma^4+2a^{10}\alpha\beta^5$$

$$+6a^{10}b\alpha\beta^3\gamma^2-18a^{10}b\alpha\beta\gamma^4+2a^8(a^4-6b)\beta^4\gamma^2] = 0$$

S_5 therefore breaks up into $\beta=0$ four times, $\gamma=0$ twice, and a sextic having a biflecnode at (0,0,1) with

$$x = 0, \quad 3bx - a^4y = 0$$

as inflexional tangents.The y-axis is an ordinary tangent to the curve at the ideal point (0,1,0). The sextic also has a tacnode at (1,0,0) with z=0 as the tacnodal tangent.

F.

$$F_6 = x^4 \pm a^2(y^2-x^2)z^2 = 0$$

This curve is symmetrical with respect to both x=0, y=0, and has a biflecnode at (0,0,1) with the tangents

$$(x^2-y^2) = 0$$

If the upper sign is used im $F_6=0$, the curve is closed and has a tacnode at the ideal point (0,1,0). This is the familiar "Lemniscate of Gerono" (Fig.IX). If the lower is used, the resulting curve is the lemniscate,rotated 90° with its intercepts on x=0 infinitely large. This curve also has a tacnode at (0,1,0) (Fig.X). We shall call this a "*deformed Lemniscate of Gerono*."

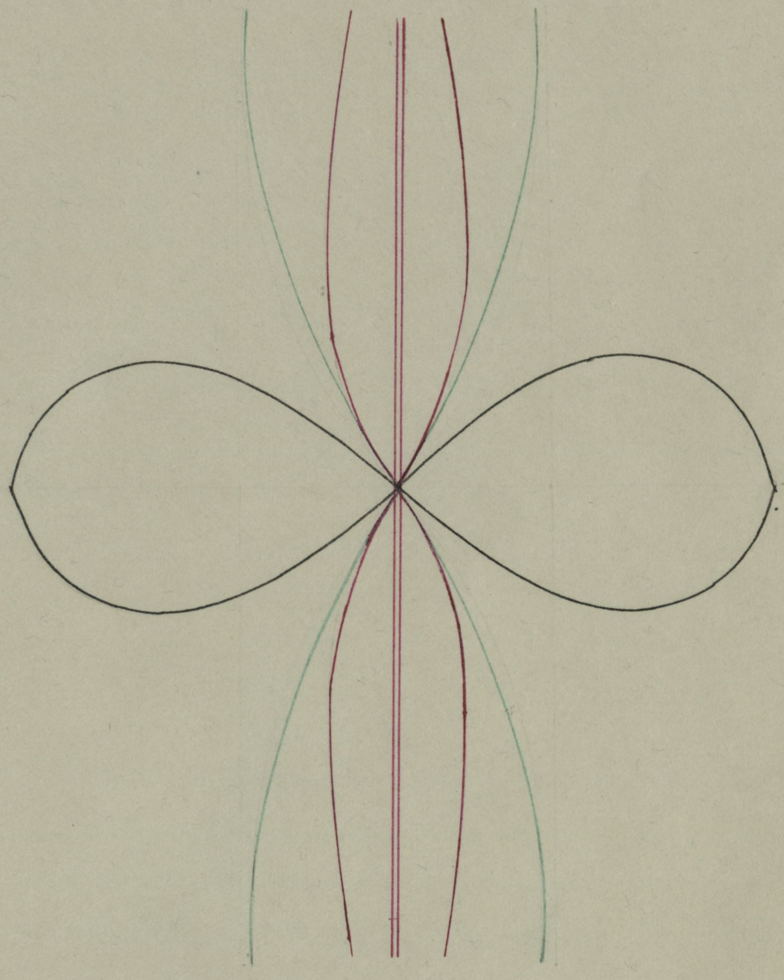

Fig. IX

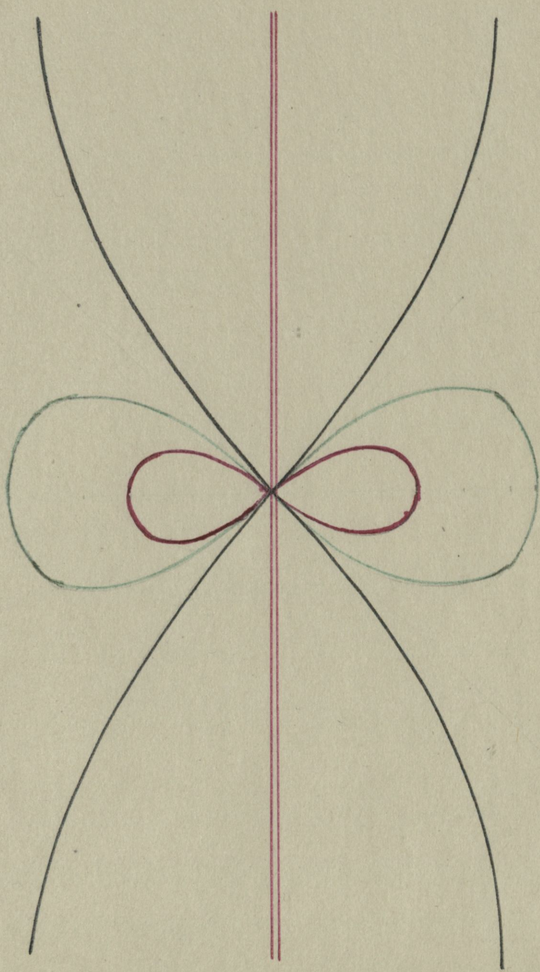

Fig. X

The conditional equations are

$$U = \alpha[6x^2 \mp a^2z^2] + \gamma[\mp 2a^2xz] = 0$$

(3) $$V = \beta[\pm z^2] + \gamma[\pm 2yz] = 0$$

$$W = \alpha[\mp 2xz] + \beta[\pm 2yz] + \gamma[\mp x^2 \pm y^2] = 0$$

The Hessian is

$$H_6 = \begin{vmatrix} 6x^2 \mp a^2z^2 & 0 & \mp 2a^2xz \\ 0 & \pm z^2 & \pm 2yz \\ \mp 2xz & \pm 2yz & \mp x^2 \pm y^2 \end{vmatrix} = 0$$

that is,

$$H_6 = z^2[2x^2(x^2+3y^2) \pm a^2z^2(x^2-y^2)] = 0$$

It is interesting to note that if the *upper* sign is here used, H_6 consists of the infinite twice and a quartic of the same kind as $F_6=0$ using its *lower* sign, that is, a "deformed Lemniscate of Gerono. If the *lower* sign in H_6 is used, H_6 consists of the infinite twice and a quartic of the same kind as $F_6=0$ using its *upper* sign, that is, a "Lemniscate of Gerono".

The Jacobian is

$$J = \gamma(6\alpha^2 \pm a^2\gamma^2)x^2z + 6\alpha\gamma^2xy^2 + 6\alpha\beta\gamma xyz + \alpha(6\beta^2 \pm a^2\gamma^2)xz^2$$
$$\mp a^2\gamma^3y^2z \mp a^2\beta\gamma^2yz^2 \pm a^2\gamma(\alpha^2-\beta^2)z^3 = 0$$

Eliminating x,y,z from this and (4) we obtain

$$S_6 = \begin{vmatrix} 6\alpha & 0 & \mp a^2\alpha & 0 & \mp a^2\gamma & 0 \\ 0 & 0 & \pm\beta & 0 & 0 & \pm\gamma \\ \mp\gamma & \pm\gamma & 0 & 0 & \mp\alpha & \pm\beta \\ 0 & 6\alpha\gamma^2 & \alpha(6\beta^2 \pm a^2\gamma^2) & 0 & \gamma(6\alpha^2 \pm a^2\gamma^2) & 3\alpha\beta\gamma \\ 0 & 0 & \mp a^2\beta\gamma^2 & 2\alpha\gamma^2 & 3\alpha\beta\gamma & \mp a^2\gamma^3 \\ \gamma(6\alpha^2 \pm a^2\gamma^2) & \mp a^2\gamma^3 & \pm 3a^2\gamma(\alpha^2-\beta^2) & \alpha\beta\gamma & \alpha(6\beta^2 \pm a^2\gamma^2) & \mp a^2\beta\gamma^2 \end{vmatrix} = 0$$

which reduces to

$$S_6 = \alpha^2\gamma^4[\alpha^2(32a^2\alpha^2\gamma^2 \mp 27\beta^4 - 36a^2\beta^2\gamma^2) \pm 4a^4\gamma^4(\alpha^2-\beta^2)] = 0$$

Thus S_6 is composed of $\alpha=0$ twice, $\gamma=0$ four times, and a sextic symmetrical with respect to both $\alpha=0$, $\beta=0$. If the upper sign is used, the sextic has a biflecnode at (0,0,1), a tacnode at (0,1,0) with $\alpha=0$ as the tacnodal tangent, and another tacnode at (1,0,0) with $\gamma=0$ as the tacnodal tangent. This sextic, therefore differs from a "deformed Lemniscate of Gerono", only in that at (0,1,0) the tacnodal tangent tangent is $\alpha=0$ instead of $z=0$. The sextic also has an additional tacnode at (1,0,0). We shall also call this a "deformed lemniscate". The lower sign gives a sextic consisting of a lemniscate together with a tacnode (0,1,0) having $x=0$ as the tacnodal tangent, and another tacnode at (1,0,0) having $z=0$ as the tacnodal tangent. These results may be tabulated as follows:

$F_6=0$	$H_6=0$	$S_6=0$
Lemniscate (Fig. IX)	$z^2=0$ and a deformed lemniscate	$\alpha^2\gamma^4=0$ and a def. lem. with tacnode at (100)
Def. lem. (Fig. X)	$z^2=0$ and a lemniscate	$\alpha^2\gamma^4=0$ and a lem. with tacnode at (100)

G.

$$F_7 = y^4 - xz(x^2+z^2) = 0$$

This may be written

$$F_7 = [y^2 + \sqrt{xz(x^2+z^2)}][y^2 - \sqrt{xz(x^2+z^2)}] = 0$$

It is at once seen that this curve consists of two parabolic branches, one of which is entirely imaginary, while the other has points of undulation at (0,0,1) and (1,0,0) with x=0 and z=0 as tangents, respectively (Fig.XI).

The conditional equations are

(5)
$$\begin{aligned} U &= \alpha[-6xz] + \gamma[-3(x^2+z^2)] = 0 \\ V &= \beta[12y^2] = 0 \\ W &= \alpha[-3(x^2+z^2)] + \gamma[-6xz] = 0 \end{aligned}$$

The Hessian reduces to

$$H_7 = [y(x^2-z^2)]^2 = 0$$

The Jacobian is

$$J = \beta y(\alpha^2-\gamma^2)(x^2-z^2) = 0$$

Eliminating x,y,z between the Jacobian and (5)

$$S_7 = \begin{vmatrix} -3\gamma & 0 & -3\gamma & 0 & -6\alpha & 0 \\ 0 & 12\beta & 0 & 0 & 0 & 0 \\ -3\alpha & 0 & -3\alpha & 0 & -6\gamma & 0 \\ 0 & 0 & 0 & 2\beta(\gamma^2-\alpha^2) & 0 & 0 \\ \beta(\gamma^2-\alpha^2) & 0 & -\beta(\gamma^2-\alpha^2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2\beta(\gamma^2-\alpha^2) \end{vmatrix} = 0$$

or,

$$S_7 = [\beta(\alpha^2-\gamma^2)]^4 = 0$$

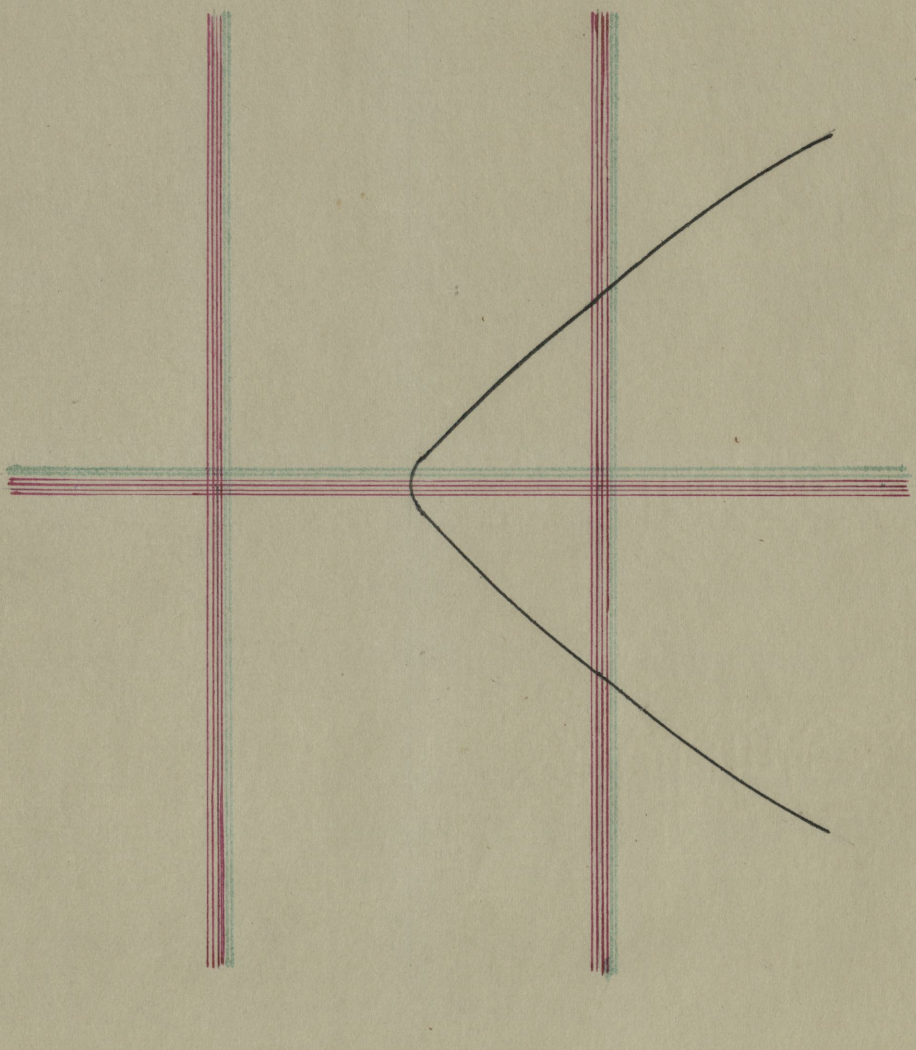

Fig. XI

H.

$$F_8 = y^3(az-y) - a^2x^2z^2 = 0$$

This curve is a unifolium having its points of inflexion at $(\pm\frac{a}{4}, \frac{a}{2}, 1)$. It has a cusp at (0,0,1) with $x=0$ as a cuspidal tangent. The vertical and horizontal tangents are

$$x=0, \quad x = \pm \frac{3}{16}\sqrt{3}az$$

$$y = az$$

respectively. This curve also has a tacnode at (1,0,0) with the ideal line, $z=0$, as the tacnodal tangent (Fig.XII).

The Hessian reduces to

$$H_8 = yz^2[4x^2(az-2y) - y^3] = 0$$

which breaks up into $y=0$, the ideal line twice, and a cubic curve. The cubic factor is a "Cissoid of Diocles" symmetrical with respect to $x=0$ and having $y=\frac{a}{2}z$ as its asymptote.

The Jacobian is

$$J = 2[(4\beta-a\gamma)y-a\beta z](\gamma^2x^2+\alpha\gamma xz+\alpha^2z^2) + 3\beta^2\gamma y^2z = 0$$

The Steinerian reduces to

$$S_8 = \alpha^2\gamma^4(4\beta-a\gamma)^2[\alpha^2(4\beta-a\gamma)^2 + 12a\beta^3\gamma] = 0$$

The quartic factor is a curve having a cusp at (0,0,1) and a parabolic branch symmetrical to $x=0$. This cusp and parabolic branch lie entirely in the lower half of the plane. The curve also has a tacnode at infinity with $y = \frac{a}{4}z$ as tacnodal tangent.

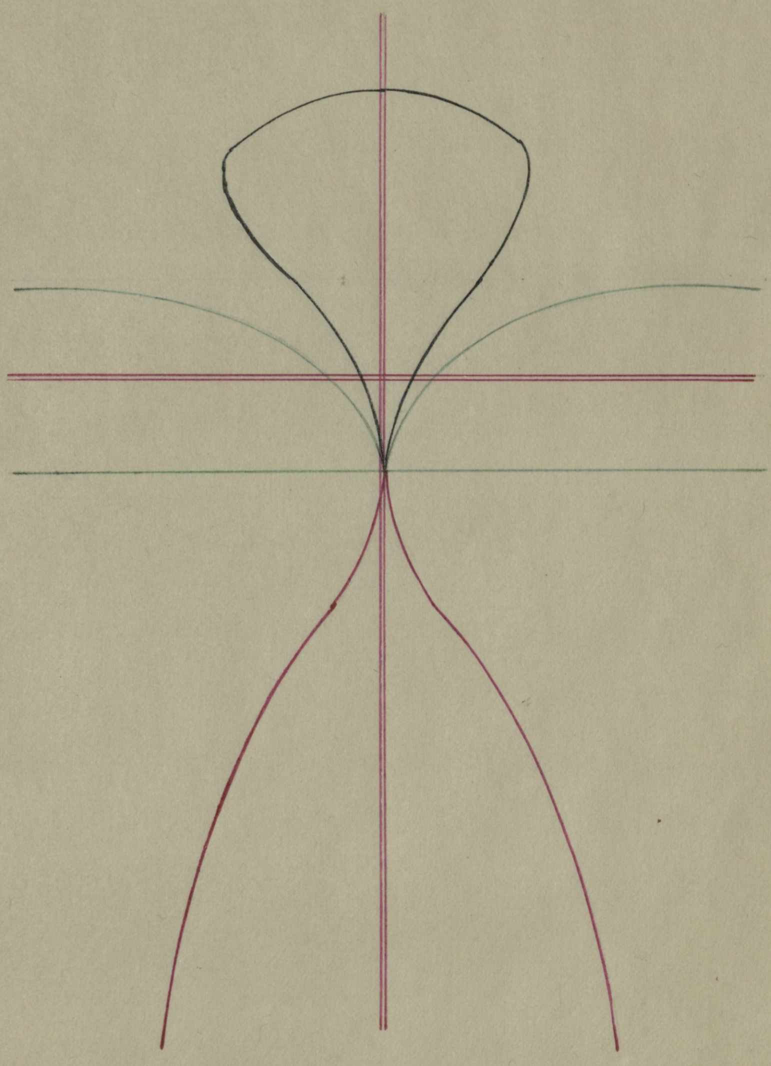

Fig. XII

35

I.

$$F_9 = x^4 + yz(x^2 - yz) = 0$$

Solving for y, we have

$$y = \frac{x^2}{2z}(1 \pm \sqrt{5})$$

Hence the curve has one tacnode at (0,0,1) and another at (0,1,0) (Fig.XIII).

The conditional equations are

$$U = \alpha[2(6x^2+yz)] + \beta[2xz] + \gamma[2xy] = 0$$

$$V = \alpha[2xz] + \beta[-2z^2] + \gamma[x^2-4yz] = 0$$

$$W = \alpha[2xy] + \beta[x^2-4yz] + \gamma[-2y^2] = 0$$

The Hessian therefore reduces to

$$H_9 = 3x^6 - 17x^4yz + 24x^2y^2z^2 + 4y^3z^3 = 0$$

The Hessian therefore has a parabolic branch lying entirely in the lower half of the plane; is symmetrical with respect to x=0; and has two triple points of the third species, one at (0,0,1), and one at (0,1,0), with y=0 and z=0 as tangents, respectively. Each of these tangents has six-point contact with the curve.

The Jacobian is

$$J = a_1x^3 + a_2x^2y + a_3x^2z + b_2xy^2 + d_1xyz + c_2xz^2 + b_1y^3 + b_3y^2z + c_3yz^2 + c_1z^3 = 0$$

where

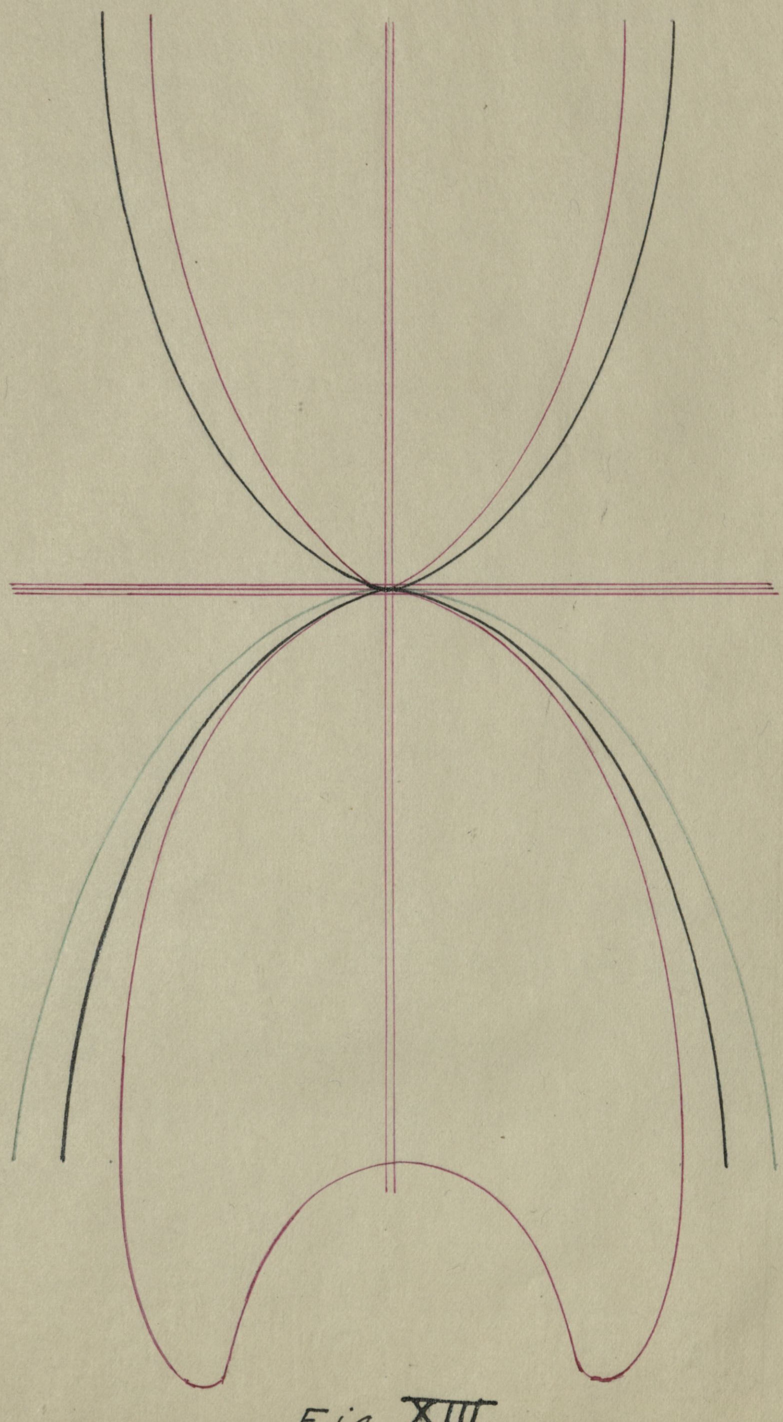

Fig. XIII

$a_1 = 2\alpha(\beta\gamma-6\alpha^2)$ $\qquad$ $c_2 = -48\alpha\beta^2$

$a_2 = \gamma(49\alpha^2-2\beta\gamma)$ $\qquad$ $b_1 = -4\gamma^3$

$a_3 = \beta(49\alpha^2-2\beta\gamma)$ $\qquad$ $b_3 = 2\gamma(-\alpha^2-4\beta\gamma)$

$b_2 = -48\alpha\gamma^2$ $\qquad$ $c_3 = 2\beta(-\alpha^2-4\beta\gamma)$

$d_1 = 2\alpha(\alpha^2-20\beta\gamma)$ $\qquad$ $c_1 = -4\beta^3$

The Steinerian is

$$S_9 = \begin{vmatrix} 12\alpha & 0 & 0 & 2\gamma & 2\beta & 2\alpha \\ \gamma & 0 & -2\beta & 0 & 2\alpha & -4\gamma \\ \beta & -2\gamma & 0 & 2\alpha & 0 & -4\beta \\ 3a_1 & b_2 & c_2 & 2a_2 & 2a_3 & d_1 \\ a_2 & 3b_1 & c_3 & 2b_2 & d_1 & 2b_3 \\ a_3 & b_3 & 3c_1 & d_1 & 2c_2 & 2c_3 \end{vmatrix} = 0$$

which reduces to

$$S_9 = \alpha^2\beta^3\gamma^3(2472\alpha^4+5760\alpha^2\beta^2+221248\alpha^2\beta\gamma-720\beta^3\gamma-2881\beta^2\gamma^2)=0$$

It thus consists of the linear factors, $\alpha=0$ twice, $\beta=0$ three times, $\gamma=0$ three times and a quartic symmetrical with respect to $\alpha=0$, having, in the upper half of the plane, a parabolic branch, and in the lower half, a peculiar finite branch some what similar to the "Cocked Hat."* The Steinerian, therefore, has a tacnode at (0,0,1).

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*For a discussion of the Cocked Hat, see Loria, loc.cit., p.151.

Addenda.

The cases A_1 to A_4 inclusive - §2 - are, as has been seen, special cases of $F_1=0$. The relationship between the curves which arise in the different cases, and their connection with the auxiliary circle is shown in Figure XIV. below.

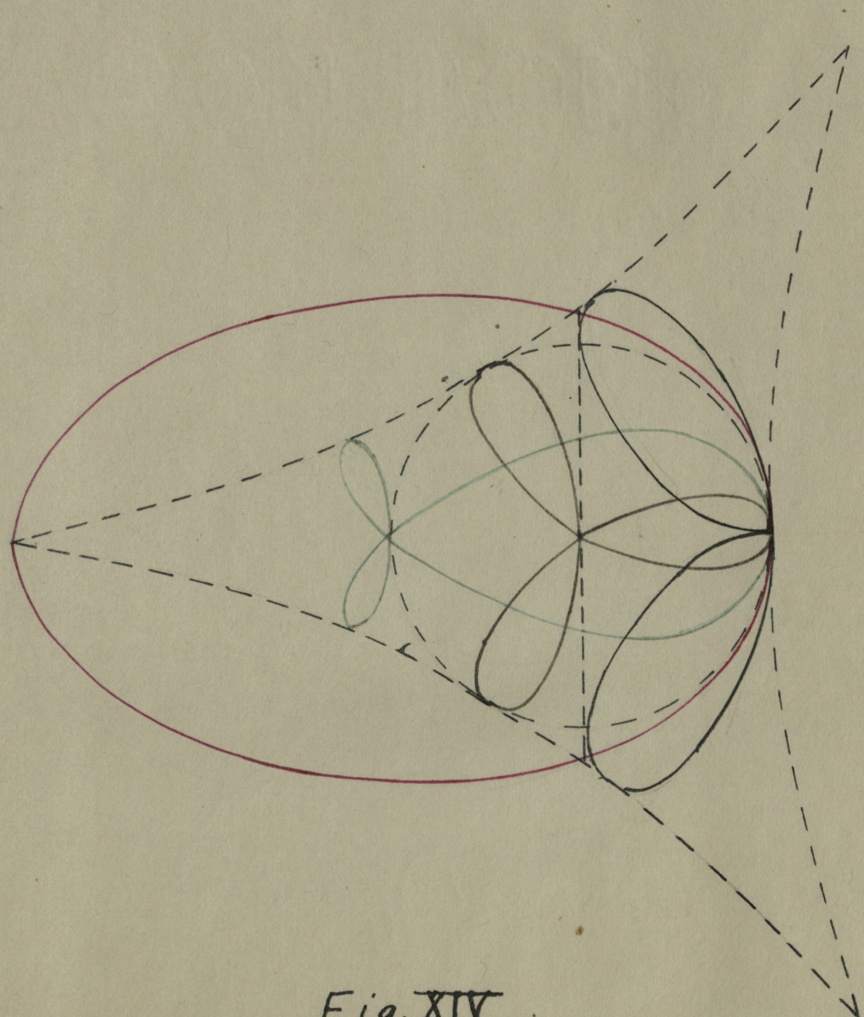

Fig. XIV.