HISTORICAL DEVELOPMENT
of the
NUMBER CONCEPT.

by

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CONTENTS

INTRODUCTION.

The purpose of this paper is to give the historical development of the number concept in so far as material is available in the library of the University of Kansas. The development of the different numbers is taken up separately and chronologically. They have been taken up in the following order:

Positive Integers.
Fractions.
Negative Integers.
Irrational Numbers.
Complex Numbers.
Transcendental Numbers.
Transfinite Numbers.

SECTION 1. POSITIVE INTEGERS.

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The history of primitive Aryan numbers has been lost, but those who have made a study of lower races of today have formed a theoretical history. No tribes without a concept of number have been found. In Australia are those that count only as far as 2, and among the low tribes of Brazil are some that do not count beyond 2 or 3.*

The concept of unity seems to be readily grasped by the primitive mind. When one object is distinguished from another they have the idea of duality. Many of the low tribes have not counted beyond two. The word for "one" may be expressed by their word for earth or moon, "two" by ear or wing.** Numbers beyond two are denoted by their word for many. In acquiring this limited concept of number the savage may make no use of a practical method of numeration. If counting is continued practical means are used. That is, the number of things in a group is represented by the number of things in another group. The most natural counters are the fingers, for they are always accessible and familiar to every one. As the counting is continued the word used for 5 is generally their

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*Tylor, page 242.
** Tylor, page 252.

word for "hand", for 10 "two hands", for 15 "foot" and for 20 "feet and hands".* The use of these words for the numerals shows that the fingers and toes were used as counters. A great variety of other practical methods of numeration such as sticks, pebbles, shells, and knots have been used. When practical methods of counting are adopted there is an advance so that complete number systems may be developed. Different systems of finger counting have been developed to such an extent that all numbers from 1 to 100,000 could be expressed. The practice of indicating numbers on the fingers was common among the ancient Egyptians, Babylonians, Greeks and Romans.** The Chinese have a well developed system of finger counting. Finger symbolism is commonly used in bargaining in eastern countries of to-day.

Finger counting has been found in all different parts of the world. This shows that there must have been independent methods of mental development. The child in learning to count makes use of his fingers. He is thus reproducing the mental development of the race.

After the concept of number has been developed begins the formation of numeral words. Preceding the

* Conant, page 55.
** Richardson, American Mathematical Monthly, Vol.23 page 7.

word numerals are the sign words. The words for numerals are often a description of the gesture.

Humboldt says, "When 5 is expressed by 'hand' this is the same as when 2 is expressed by 'wing'. At the root of all numbers are such metaphors as these, though they can not always be traced."*

In the languages of Europe all traces of the origin of number words seems to have disappeared. The words have been made conventional by allowing the descriptive meaning to disappear.**

Among some tribes of Australia fixed names are given to the children in order of their age. This method is also used by the Malays and the Sioux Indians.*** Tylor gives an example of a child using the names of the months for counting. The days of the week could have been used just as well. Any series of names arranged in order can be used in counting. Sometimes distinct numerals are found for various classes of words. This variation in numerals is found in many places, but more commonly among the Indians of British Columbia. It is also used by people as enlightened as the Japanese.

In counting there must be a certain order, and words

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* Tylor, Vol. 1, page 252.
** Tylor, Vol. 1, page 254.
*** Tylor, Vol. 1, page 258.

or symbols for each successive total. If the counting is to be continued to any extent some base must be established, for we can see at once the difficulty there would be in remembering so many words in their proper order, and the symbols for these words.

The numbers used most often as bases are 5, 10, and 20. The direct numeral combination is begun after the base is passed, as 5 and 1 or 10 and 1. Exceptions to this method are noticed in the English where direct method is not used before we reach 21, and in the German before 101. When the smaller number precedes, it generally means multiplication. In the development of a number system the idea of grouping must be understood and the fundamental principles of addition and multiplication. Subtraction is sometimes used in place of addition. This is found among Indians of British Columbia.*

Peacock, writing in 1847, says that the numerical languages of the Indians in the central part of North America are quite complete, and most of them are in the decimal scale. The Algonquins had simple terms for 100 and 1000. The Hurons although speaking a most rude language had numerical scales sufficiently large and simple words for 10, 100, and 1000. The Iroquois also had a sufficiently complete system. The decimal scale was much

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* Conant, page 46.

less common in South America. The systems were not so perfect, rarely extending beyond 100, and frequently limited to small numbers. The names for the numbers that are compounded are often very long and complex. Great complexity of names for numbers is a very frequent occurence among low tribes when their systems are at all extensive. The Peruvians have a very complete and extensive decimal system. Other tribes have borrowed from them the simple words for higher numerals.[*]

The vigesimal system was found in the northern and western parts of North America.[**] The Maya of Yucatan have a pure vigesimal system.[***] The Aztecs who have a well developed system, make use of the quinary to 2o, then the vigesimal.[1] Among the tribes of western South America the vigesimal is used.[2]

The tendency of the quinary system is the establishment of a higher base, the quinary element becoming subordinate. This system has been found in all parts of the world. One of the purest examples of quinary numeration is that from the Betoya dialects of South America. The reckoning is entirely by fives.[3]

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* Encyclopedia of Pure Mathematics, page 379
** Conant, page 200
*** Conant, page 200
1. Conant, page 201.
2. Conant, page 206.
3. Conant, page 57.

The binary system is found among many of the tribes of Australia and South America.* This scale may also be considered natural if we think of the binary combinations of the body. The adoption of this scale would require a rather complete knowledge of the classification of numbers. When this system is found among primitive people there is no extensive development. With a base as small as this it would require too many names and places to express large numbers. Leibnitz (1646-1716) developed a complete binary system and made efforts to have it adopted.**

Quaternary and ternary scales are not as often used as the binary. Indians of British Columbia have used these scales.*** There seems to be no record of systems using 6, 7,8, or 9 as a base, although traces of them have been found in some systems. There would be no natural reason for using such bases.

The primitive people of to-day all make use of the fingers in counting, and some number is used as a base, most generally 5,10, or 20. Where the systems are extensive there are distinct words for 10,100, etc.

The methods used in writing numbers for 10,100, and 1000 in the Egyptian and Phoenician earliest known writings show they used the decimal idea.[1]

* Conant, page 106
** Conant, page 102
*** Conant, page 112
1. Gow,p.43

The Roman notation is based on the scale of 5. The principles of subtraction and addition were used in writing. Simple strokes were probably first used for symbols. Then for 10 they began the use of X . The symbol for 5 would be half of that for 10, V or Λ . Noticing that these symbols represent letters of the alphabet they arranged to use the letters as symbols of the numbers.*

The first letter of the numeral adjective was used to represent the number by the Greeks. These are known as the Herodianic signs. Later the Greeks used the 24 letters of the alphabet and 3 antique letters to represent their numbers. The numbers from 1 to 9 were represented by the first 9 letters, tens were represented by the following 9 letters, and then the hundreds by the other 9.** Although they made use of the decimal notation they did not recognize the importance of place value and the use of zero.

The Babylonian notation shows that they used the decimal scale,making use of the principles of addition and multiplication. They also used the sexagesimal system. At first the Babylonians reckoned the year as 360 days. This led to the division of the circle into 360 degrees. They were probably familiar with the fact that

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* Brooks,page 141
** Ball,page 127.

the radius can be used as a chord six times, and each of these chords would subtend an arc of 60°. As subdivisions of the degree were made it was divided into 60. This theory is given by Cantor with regard to the adoption of this system.* They made use of place value for 1.4 represents 64. The 1 stands for a unit of the second order, because of its position with respect to 4. Their writings do not show that they used zero for no number is written where they needed to use zero.

Among the Hindoos there is no trace of finger counting like that of the Greeks or Romans.** Not until the third century B.C. do numerals appear in any inscriptions of the Hindoos thus far discovered.*** They appear only in the primitive form of marks as they would have been found in Egypt, Greece or Rome.[4] Evidence tends to show that the complete system was not in common use in India at the beginning of the eighth century.[5] Brahmagupta in the early part of the seventh century gave in his arithmetic a distinct treatment of the properties of zero, which shows that it had acquired a special signifi-

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* Cantor,Vol.I,pp.91-93
** Richardson, American Mathematical Monthly,Vol.23,p.10
*** Smith and Karpinski, p.19.
4.Smith and Karpinski, p.45
5.Smith and Karpinski, p.34.

cance.* The zero is a prerequisite to a place value system. Without this the Hindoo numerals would never have become the computation system of the western world.

The time and place of the introduction of Hindoo numerals into Europe is uncertain. One theory is that they were carried by the Moors into Spain in the eighth and ninth century and then transmitted to Christian Europe. Another is that they were in Spain when the Arabs arrived, having reached the West through the Neo-Pythagoreans.** Considering the trade relations between the East and West it was probably the trader rather than the scholar who carried the Hindoo numerals to various countries. Books explaining the Hindoo art of reckoning appear in the twelfth century.*** Perhaps the most influential in introducing the Hindoo numerals to the scholars of Europe was Leonardo Fibionacci of Pisa.[1] There was a great strife between the abacists and the algorists. The Hindoo numerals were not generally used in school and busihess until the sixteenth century.[2]

The tendency to use 12 as a base is noticed in the use of such measurements as dozen, gross, inch and ounce. The establishment of the duodecimal system has at times been advocated because of its advantages. Were 12 the

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* Smith and Karpinski, p.34.
** Smith and Karpinski, p.63
*** Smith and Karpinski, p.128
1. Smith and Karpinski,p.128
2. Smith and Karpinski,p.137.

base we would have a base divisible by 2, 3, 4, and 6 instead of only 2 and 5 as in the decimal system. In ordinary business the fractions 1/2, 1/3 and 1/4 are used so frequently. There would also be an advantage in writing numbers corresponding to our decimals.

The number systems based on nature have been universal and only those people using the decimal scales have attained any degree of civilization except the Aztecs.

SECTION 2. FRACTION.

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The fraction is found in the oldest numerical records as those of Egypt and Babylonia. Division brought in the difficulty of a remainder, and a fraction, at first came to be understood as a division that was not exact. Because of the primitive symbols it was easier to express the fraction 11/12 as $1/2 + 1/4 + 1/6$. Another method of dealing with the fraction was always to divide the unit into the same number of parts as 1/12 or 1/60. By using either of these two methods the symbols for the fractions would be nearly the same as for integers. The ancient treatment of fractions avoided the handling of the numerator and denominator at the same time.

The Ahmes papyrus written sometime before 1700 B.C. founded on an older writing of about 3400 B.C. deals somewhat with fractions.* The Egyptians understood fractional relations so as to generalize and represent the fraction by a symbol. Fractions were resolved into sums of unit fractions as $2/5 = 1/3 + 1/15$. Ahmes gave a table of answers for all fractions of the form $\frac{2}{2n+1}$ up to 2/99.** No explanation of the work was given, and

* Gow, p.16.
** Eisenlohr, pp.46-48.

only one form was given for each fraction. He gave a formula for multiplying a fraction by 2/3. He says, "2/3 of 1/a is 1/2a + 1/6a." Example, 2/3 x 1/5 was worked, then the statement was made that others are solved similarly.* The unit fraction was expressed by writing the denominator as an integer and placing a dot over it. The only fraction not having unity as a numerator which was called a distinct fraction is 2/3, and it was given a separate symbol .

The unit fraction was used by the Greeks to a late period. They could state fractions easily, but calculations were difficult. They preferred to get rid of numerators and reduce the denominators to numbers, some of which were so low that they could be easily handled, and others so high that they could be omitted without changing the value materially. In writing a fraction the numerator was written first and marked with an accent, then the denominator was written twice and marked with two accents, $\frac{17}{21} = \iota\zeta' \kappa\alpha'' \kappa\alpha''$.** The unit fraction was written by just indicating the denominator. ($\frac{1}{21} = \kappa\alpha''$) The ancient symbol c was used for $\frac{1}{2}$ and ω for 2/3.***

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* Eisenlohr, p.150.
** Cantor, Vol.1, p.65.
*** Tropfke, p.75.

Until 1854 Babylonian mathematics had little or no foundation. At this time two small cylinders, probably written between 2300 B.C. and 1600 B.C. were found at Senekersh. These contain tables of squares and cubes written in sexagesimal system. In about 1889 a number of cylinders were found at Nippur. Prof. Hilprecht has examined many of these and has given an account of his study. He found certain unit fractions where the denominator was 60 or some power of 60.* The Babylonians dealt with fractions by using 60 as the constant denominator. In written form only the denominator was given with a special sign attached. The sexagesimal notation of fractions was introduced into Greece about 200 B.C. From this time until about the sixteenth century the sexagesimal fractions were used in astronomical and mathematical calculations.**

Fractions having a constant denominator were also used by the Romans. The use of duodecimals was due to the division of a mass of copper, weighing one pound, called the as, into twelve equal parts. There were names and symbols for these subdivisions, and also special names for 1/24, 1/48, 1/72, 1/144, and 1/288. Addi-

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* Bulletin of Amer.Math.Soc.,p.392.
** Tropfke,Vol.1, p.77.

tion and subtraction were simple with these fractions. Multiplication was very detailed. Division which did not fit into the duodecimal system could be represented by fractions with extreme difficulty, or only approximately. Tables for multiplication were worked out by Victorius of Aquitanien. These were used by the common people and tradesman.*

Duodecimals were represented on the Roman abacus. The eighth groove from the left was used to represent fractions. On this were five buttons, each representing 1/12. The one button on the upper part of this groove represented 6/12. In the ninth groove, the upper button represented 1/24, the middle 1/48, and each of the two lower ones 1/72.**

Fractions were learned in connection with money, weights, and measures. These fractions were an advantage for the common units were most frequently divided into 2,3,4, or 6 equal parts.

The Hindoos used common fractions as we do to-day. Babylonian and Greek influences were shown in their books. In writing a fraction the numerator was placed above the denominator. The mixed number was written by

* Tropfke,Vol.1, p.77
** Cajori, p.38.

placing the integer above the numerator. $(2\frac{4}{5}$ — $\begin{matrix}2\\4\\5\end{matrix})$ [*] Their rules for the four dundamental operations differed only slightly from ours of to-day. Through the Arabs the knowledge of the Hindoos was transmitted to Western Europe.

Our present method of writing fractions is similar to that of the Hindoos. The first use of the bar, separating the numerator from the denominator is found in Liber abaci (1202) of Leonardo of Pisa.[**] Later the use disappeared, and it was not until the beginning of the sixteenth century that it was spoken of as a necessary part of the fraction.

Our classification of proper and improper fractions was not used in medevial times. Generally only the first was used.[***] Kaufol (1696) considered only proper fractions as true fractions. Kästners in his writings of 1764 divided fractions into proper and improper fractions. The circulation of these words was due to Euler who introduced them in his algebra (1770)[1.]

The word reduction in a true technical sense did not appear before in Rudolff's book of 1532.[2.] The word ex-

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* Tropfke, Vol. 1, p. 78
** Tropfke, Vol. 1, p. 78
*** Tropfke, Vol. 1, p. 81.
1. Tropfke, Vol. 1, p. 81.
2. Tropfke, Vol. 1, p. 81.

pansion was not used until later. The general statement that reduction and expansion does not change the value of the fraction was not given, although problems were given where they were used.

In the addition of fractions Stifel (1545) first used more than two denominators at the same time. He tried to find a least common denominator, but finally contented himself with the product of the given denominators. Tartaglia (1556) first made use of the lowest common denominator. In the beginning of the seventeenth century we have the development of our modern method.

In the multiplication of fractions Leonardo of Pisa multiplied the numerators then divided the product first by one denominator and then the other.* Most of the other authors multiplied numerators, then denominators. The first one to bring up the point generally, that in multiplication of fractions the result is smaller than the multiplicand was Pacinolo (1494). Rudolff, Tartaglia, and Clavius attempted to explain it.** Jordanus Nemorarius carried over to dividion the rule for multiplication. The one used now was first given by Stifel, (1545).

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* Tropfke,Vol.1,p.84
** Tropfke,Vol.1,p.84.

The influence tending toward the development of decimal fractions were (1) methods of extracting square root, (2) division by 10 or its multiples, and (3) the solution of trigonometric functions. The method of extracting square root of a surd by adding 2n ciphers and then writing the result as a fraction whose denominator is one followed by n ciphers was used by John of Seville in the twelfth century. This method was used by Johann of Gumunden during the fourteenth century, Peurbach in the fifteenth, and it continued to be used in the sixteenth century. The bar was sometimes used to separate the fractional part from the integer. Rudolff was the first one who gave evidence of understanding the use of the bar. In his work of 1530 he used the bar as we use the decimal point. As early as 1492 Pellos, in problems of division where the divisor was 10 or a multiple of 10, had used the period to separate the integer from what was really the decimal fraction.

The first discussion of decimal fractions was given by Stevin in La Disme,published 1585. He showed them how all business calculations could be performed as easily as with integers. The subject was clearly explained. He urged governments to establish decimal systems of coinage, weights, and measures. His symbolism was poor. He wrote .37594 in the form 3 ⓪ 7①5②9③4④, and 8.937 in the form

8⓪9①3②7③.

Regiomontanus in solving for the trigonometric functions used 10 as a radius. Although he gave the value of the tangent to five places he did not show any understanding of decimal fractions. Bürgi in making these solutions used one as a radius instead of 10. The function then had the decimal form. ,He used both the period and the comma in his notation and placed a zero under the integer in units place. ($141\underset{0}{4}$ - 141.4). Johann Hartmann Beyer used both the comma and the sexagesimal symbols. In writing 314.1592 it was written in the form 314,1'5''9''' 2'''' . Napier first used the sexagesimal notation, but later the period.* Even at present we have different forms of notation. In England 23 45/100 is written 23·45, in United States 23.45 and on the continent 23,45 or $2\underset{45}{3}$. The decimal fraction became known in England through the publication of The Art of Tens by Henry Lyte (1619).

As early as the twelfth century forms resembling the decimal fraction were used, but it was four hundred years before there appeared any written works which gave an explanation of the subject. The first treatise containing problems that showed an understanding of decimal fractions was written by Rudolff 1530, and this was followed by Stevin's theory of decimal fractions in 1585.

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* Smith, The Invention of Decimal Fractions.

Although fractions were known to the Egypt and Babylonians 3000 years before our time of reckoning, there was not a complete development until in the sixteenth century.

SECTION 3. NEGATIVE INTEGERS.

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Division brought in the difficulty of a remainder, and the fraction came to be understood as a division that was not exact. The solution of equations showed the need of another extension of the number system, that of introducing the negative number. The need of this extension would not appear until the subject of algebra was quite well developed.

In the work of Diophantus (246-330) which represents the greatest achievements of the Greek in Algebra, only those differences were considered where the subtrahend was less than the minuend. He gave the rule for the multiplication of positive and negative numbers. In the solution of the quadratic he gave but one root, the one obtained by using the positive sign of the radical.* In equations having both roots positive he took no account of the smaller roots.** A quantity which is negative was never accepted as a result. This clearly shows that he had no concept of the negative number. His number system was composed only of positive integers, and fractions.

The first trace of the negative is found in the math-

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* Heath, p.60
** Heath, p.61.

ematics of the Hindoos. When one quantity was to be added to another it was placed after it without any particular sign. In subtraction the same method was used with a dot placed over the cofficient of the subtrahend. The dot was used as a symbol to distinguish positive from negative quantities, instead of a sign of operation.* Neg - ative and positive numbers were interpreted as debts and assets. Aryabhatta (476) and Bramhagupta (598) understood the difference between positive and negative numbers. Bramhagupta used the negative in the solution of equations.**

Bhaskara seems to be the first writer who recognized the existence of the negative roots of a quadratic equation. In the solution of the equation $x^2 - 45x = 250$, he gave $x = 50$ and $x = -5$. "But", says he, "the second value in this case is not to be taken, for it is inadequate; people do not approve of negative roots." Commentators speak of this as if negative roots were seen, but not admitted.*** Bhaskara recognized that a square root could be positive or negative and also that $\sqrt{-a}$ was not in the ordinary number system. The concrete idea was attached to the negative and positive. They were represented by lines drawn in opposite directions.[1]

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* Cantor, Vol.1, p.580
** Fink, p.216.
*** Cajori,p.93
1. Fine,p.105.

The Arabs received their learning from Greek and Indian sources. Their geometry was Greek, and their algebra was both Greek and Indian. Their solution of the quadratic was always followed by a geometric demonstration.* The two roots were considered if they were positive. The Arab avoided the use of the negative. Had he adopted the Indian interpretation of the negative, it would have been used earlier in Europe. Leonardo of Pisa (1180-1250) said that the negative solution had a meaning when its explanation was that of a debt.

By the fifteenth century the people of Europe were beginning to devote more attention to learning. Mathematics was at first not considered as a necessary subject and but little time was devoted to it in the schools. Later there came a demand for it, and it finally gained an important place in the curriculum. Many of the teachers were the great mathematicians of that time.

Michel Stifel (1487-1567) wrote his Arithmetica Integra in 1544, which gave him a leading place among mathematicians of his time. Only the positive root value was used in his solution of the quadratic equation. He spoke of the negative number as the "absurd number, smaller than zero". He said that zero was in the middle

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* Ball, p.157.

between the two kinds of number.* Cardan (1501-1576) who taught mathematics in Italy permitted the negative solution of the equation, speaking of the negative number as "fictious." ** Stevin (1548-1620) to whom we owe the invention of decimal fractions knew that the equation with the negative solution was customary.***

The earliest work that gave a symbolical treatment of algebra is In Artem Analyticam Isagoge written by Vieta in 1591. Algebra was applied to geometry, and a careful and extensive study of equations was made.[1] He arrived at a partial knowledge of relations existing between the roots and cofficients of an equation.[2] He did not consider negative roots so did not fully see the relations.

Thomas Harriot (1560-1621) was the earliest writer on algebra in England. He did work on equations similar to that of Vieta and Girard, but failed to recognize the negative roots. He did not allow a negative to stand alone as one member of the equation.[3]

The first to understand negative roots in the solution of geometric problems was Girard (1590-1633).[4]

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* Tropfke,Vol.1,p.165.
** Tropfke,Vol.1,p.165
*** Tropfke,Vol.1,p.166
1. El.Cajori,p.230
2. Tropfke,Vol.1, p.166
3. El.Cajori,p.232
4. El. Cajori, p.231.

He made the statement that every equation has as many roots as was expressed by the number of units in the degree of the equation.* He first showed how to express the sum and products of the roots as cofficients of the dquation.

The representation of a geometric curve by an equation which expressed the relation of its points to two lines of reference was introducted by Descartes in 1637. He saw that mere length of the perpendicular would not show to which side of the line the perpendicular lay. To overcome this difficulty Descartes made the convention that the opposite sides of these lines should have opposite algebraic signs.** With the adoption of this convention the negative gained a new position in mathematics. It has now received a real interpretation, and has just as significant meaning as the positive.

The Hindoos as early as the fifth and sixth century understood the relation of the negative and positive. Bhaskara in the twelfth century made use of the two square roots in his solution of quadratics, and he also used the two directions of the line to represent the negative and positive numbers. In Europe before the seventeenth century the ideas of the negative be-

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* Tropfke Vol.1, p.166.
** Ball, p.272.

gan to occur only occasionally and many of the mathematicians dealt almost entirely with absolute positive quantities. It was only when Descartes hit upon the idea of making a graphical interpretation of the negative that it was generally accepted and came into common use. If for the logical development of arithmetic we are to find suggestions from the historical development, the negative would be taken up much later than the fraction. To give the child an interpretation that will not seem absurd, a concrete explanation by some means as the line or thermometer must be used.

SECTION 4. IRRATIONAL NUMBERS.

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The historical development of the irrational began with the appearance of Greek scientific mathematics and continued to the middle of the nineteenth century. The irrational first appeared through its geometric nature and no number concept was associated with it. Pythagoras (580-500 B.C.?) in his study of the right triangle observed that the side of the right isosceles triangle and its hypotenuse had no common unit of measure. Theodorus of Cyrene (410 B.C.) is said to have proved geometrically that numbers represented by 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, and 17 were incommensurable with unity.* Theaetetus, a pupil of Theodorus relates that after he and the younger Socrates knew the above statement, that it occurred to them, since the squares appeared to be infinite in number, to try to comprise them into one term, by which to designate these squares. He says, "We divided all numbers into two classes. Comparing that number which can be produced by the multiplication of equal numbers to a square in form, we called it quadrangular and equilateral. The numbers which lie between these, such as three and five, and every number which can not be produced by the multiplication of

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* Ball, p.30.

equal numbers, but becomes either a larger number taken a greater number of times, this we likened to an oblong figure, and called it an oblong number.*

The 10th Book of Euclid gives the Greek theory of the irrational. Euclid in conformity to his geometrical conception treated only the irrational which he constructed with the line and circle.** This book has a completeness which none of the others have. Euclid evidently had in mind the classification of incommensurables. He investigated every possible variety of lines which can be represented by $\sqrt{\sqrt{a} \pm \sqrt{b}}$, a and b representing two commensurable lines. He found that there are 25 species of lines that could be represented by this formula. He proved that every one of these species was distinct from every other, and that every line which was commensurable with a line of any one species was itself a line of the same species. He showed that every individual of every species was incommensurable with all the individuals of every other species, and also that no line of any species could belong to that species in two different ways. He also showed how to form other classes of incommensurables.***

Actual approximations to the values of incommensurables among the Greeks were few. Euclid was aware

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* Allman, pp. 208-209.

** Heath, Euclid, Vol. 3, pp. 15-260

*** De Morgan Euclides Smith's Dictionary, Vol. 2, p. 67.

of these as were others before him. Plato and the Pythagoreans were familiar with $\frac{7}{5}$ as an approximation to $\sqrt{2}$. Archimedes (287 B.C.) gave $\frac{1351}{780} > \sqrt{3} > \frac{265}{153}$ and proved that the ratio of the circumference of a circle to its diameter was less than 3 1/7 but greater than 3 10/71. The use of sexagesimal fractions to approximate surds appeared fully developed by Ptolemy. The approximation given to $\sqrt{3}$ is $\frac{103}{60} + \frac{55}{60^2} + \frac{23}{60^3}$ which is equivalent to 1.7326509.*

Diophantus did not give the irrational as a result in his solution of the indeterminate quadratics.**

The idea of incommensurable magnitude can be traced to Pythagoras, the geometry of the Greeks who followed him show a study of incommensurable quantities obtained by geometrical construction, as well as some approximations, yet no conception of extending the notion of natural numbers so as to give them, and the notion of incommensurable magnitudes the same character of generality is shown.

The Hindus saw the relations of commensurable and incommensurable quantities as if they depended on number. *** The works of Brahmagupta (born 598) and Bhaskara

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* Heath Euclid, Vol.3, p.119
** Tropfke, Vol.1, p.159.
*** Encylopedie des Sciences Mathematique, p.137

(born 1114) show the use of the irrational. Bhaskara shows how to find the square root of a binomial surd.* The irrationals were used as ordinary numbers. This method of proceeding was not justified for they did not seem to realize that there was such a need.

The Arabs who were familiar with the mathematics of the Hindus and the Greeks did not show any different conception of the irrational.

The name "surd number" is found in the writings of mathematicians of the middle ages; but this name in no way implies a generalization of the notion of number. Their methods of demonstration rested entirely upon the Euclidean. This manner of proceeding tended to remove all idea of speculating upon the surds themselves as if they were like the natural numbers.** The irrational numbers were looked upon as not real but improper numbers which would only be endured as a necessary evil.[1.]

Michel Stifel (1487-1576) is probably the first to present any analogy between the character of surd numbers and that of rational numbers. In his Arithmetica Integra he divided numbers into classes; class of rational numbers included in two natural consecutive numbers,

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* Cantor,Vol.1, p.586
** Encyclopedie des Science Mathematique p.138.
1. Gropfke, Vol. 1,p.150.

class of roots of any natural numbers, included in two natural consecutive number. He noticed that the same number could not belong to the two classes. Yet with his contemporaries he believed that surd numbers were not true numbers. His demonstrations show that in his foundations he was attached to the idea of Euclid.*

The construction of trigonometric tables and the invention of logarithms contributed to prepare the way for the generalization of rational numbers to extend to irrational numbers which were the roots of natural numbers. The development of analytic geometry made a need of the generalization of the notion of numbers. This need was emphasized through the introduction of infinitesimal calculus by Leibnitz and Newton the last part of the seventeenth century. No scientific study of the irrational was made until the nineteenth century.

Dedekind relates how he felt the lack of a really scientific foundation for arithmetic as he was preparing a lecture on the elements of differental calculus. He says," In discussing the notion of the approach of a variable magnitude to a fixed limiting value, and especially in proving the theorem that every magnitude

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* Encyclopedie des Science Mathematique, pp.139-140.

which grows continually, but not beyond all limits must certainly approach a limiting value, I had recourse to geometric evidences. But that this form of introduction into differential calculus can make no claim to being scientific no one will deny. The statement is so frequently made that the differential calculus deals with continuous magnitude, and yet an explanation of this continuity is nowhere given. Even the most rigorous expositions of differential calculus do not base their proofs upon continuity but, with more or less consciousness of the fact, they either appeal to geometric notions or those suggested by geometry, or depend upon theorems which are never established in a purely arithmetic manner. It then only remained to discover its true origin in the elements of arithmetic, and thus at the same time secure a real definition of the essence of continuity."*

In 1858 Dedekind presented his theory of the irrational, which he based upon the ideas of rational number. This theory was founded on the idea of a cut (Schnitt) in the system of real numbers. He separated all the real numbers into two classes A_1 and A_2 such that each number in A_{11} preceded every number in A_2 and there was no last number in A_1 and no first in A_2. The irrat-

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* Dedekind, pp. 1-2.

ional a being that one number which lay between all numbers in A_1 and all numbers in A_2.*

Weierstrass in developing his theory of the irrational proceeded from the concept of the whole number. Division brought in the need of the fraction. Certain fractions, if referred to the decimal system, consist of an infinite number of elements, but they also arise from a combination of a finite number of elements. He showed that every number formed from an infinite number of elements of a known species and which contained a known finite number of those elements possessed a definite meaning, whether it was capable of actual expression or not. When a number could only be expressed by an infinite number of elements he called it irrational. His theory was made public in 1872 through E. Kossak who had followed his course of lectures at the University of Berlin.**

G. Cantor introduced the irrational number through a fundamental series. He showed that a number defined by a series is either identical with a rational number or not identical. In the latter case it is irrational.*** This theory appeared in Math. Annalen in 1872.

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* Dedekind,p.11.
** Encyclopedie des Sciences Mathematique, p.149.
*** Hobson, pp.25-30.

SECTION 5. COMPLEX NUMBERS.

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Division can not be performed exactly in arithmetic if we are restricted to integers, but if fractions are admitted division is always possible, so with their introduction we have a new class of numbers. The solution of equations brought in the need of another extension of the number system. With this extension both addition and subtraction are always possible with results as real as those of purely arithmetical addition. The form $\sqrt{-b^2}$ indicates an impossible operation if we consider it as the square root of a negative quantity. If we resort to a geometrical representation and consider a system of two numbers, a and b which are combined with each other as the co-ordinates of a point in a plane, quantities of the form $\sqrt{-b}$, when reduced to the form $a+b\sqrt{-1}$ are no longer impossible. They are just as real as the fraction or the negative.

The history of the complex number began when it was noticed that a negative number does not have a square root. Bhaskara (1114) noticed that the square root of a negative quantity was not possible.*

Cardan (1501-1576) in his study of the cubic

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* Tropfke, Vol.1, p.169.

met quantities of the form $\sqrt{-b^2}$. He used the word "impossible" to designate a solution of an equation of the second degree in real cofficients having been carried on with these imaginary roots. He observed that $(5+\sqrt{-15})(5-\sqrt{-15}) = [25-(-15)] = 40.$*

Bombelli (born 1530) gave the roots of the equation $x^3 = 15x+4$, as 4, $2\pm\sqrt{-1}$. His work showed that $\sqrt{2\pm\sqrt{-121}} = 2\pm\sqrt{-1}$**. He probably found these roots by trial.

Girard (1590-1633) justified the introduction of imaginary roots. He gave as roots of the equation $x^4 = 4x-3$, 1, 1, $1+\sqrt{-2}$, $1-\sqrt{-2}$. He also stated that $(-1+\sqrt{-2})(-1-\sqrt{-2}) = 3.$*** This product was probably considered as the product of the sum and difference of two quantities. This product then checked by the fact that he knew the product of the roots should be 3.

Descartes (1596-1650) made a distinction between real and imaginary roots. He was the first to use the word imaginary.[1]

The imaginary was used by mathematicians a long time without accounting for its true nature. They used them as signs of operation which had no meaning in them-

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* Encyclopedie des Sciences Mathematiques, p.330
** " " " " " p.331
*** Girard (sign F. verso)
1. Encyclopedie des Sciences Mathematiques, p.330

selves. They were unable to avoid the imaginary in their results in analysis. Wallis (1616-1703) appears to be the first who used the idea of graphical representation to express the imaginary number. In his de Algebra tractus, which appeared in 1685 the imaginary number was represented by the geometric sum of two vectos, the one representing a pure imaginary number, the other a real number. The vector representing a pure imaginary number ib, where $b > 0$ formed with the axis of the real positive quantities an angle which was some-times fixed arbitrarily and sometimes depended upon the arc tan $\frac{b}{a}$.*

In a publication of 1750 Kühn represented $a\sqrt{-1}$ by a line ⊥ to the line a and equal in length to a. The $\sqrt{-1}$ was constructed as a mean proportional between 1 and -1.**

Wessel's work presented to the Academy of Copenhagen (1795) showed an advance toward our present theory.*** His work is clear and complete.***

Argand's theory of the imaginary appeared in a pamphlet published in Paris in 1806. This showed how to represent the imaginary forms $a+bi$ by points in a plane by use of real and imaginary axes, and it gave

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* Encyclopedie des Sciences Mathematiques, p.339
** Cajori, p.317
*** Merriman & Woodard, p.515.

rules for their geometric multiplication and addition. Seven years later J.F.Francais sent to the editor of the Annales, the outline of a theory whose origin he had found in a letter written to his brother by Legendre who had obtained it from an author whose name he did not give. When this came to the notice of Argand he made himself known as the author referred to in the work given by Legendre. This gave rise to a discussion in the Annales. Through these articles Argand explained more satisfactorily his theory. This geometrical interpretation of the imaginary made it a real quantity just as Descarte's geometrical interpretation of the negative had made it a real quantity. Neither the negative or the imaginary were accepted until there was a correspondence to some actual thing.*

The imaginary was of much interest to the mathematicians of the seventeenth and eighteenth centuries. Important relations were worked out, although it was not until the beginning of the nineteenth century that there was a generally accepted geometrical interpretaion.

In 1675 Leibnitz had commenced to study the imaginary as is shown by his letters. His work shows a study

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* Argand, pp.3-16.

of the general formula $\sqrt[n]{a+\sqrt{-b}}+\sqrt[n]{a-\sqrt{-b}}$. Leibnitz and Bernoulli in their integration of rational functions by a separation into simple elements had noticed the logarithms of imaginary quantities. Leibnitz and Bernoulli in their integration of rational functions by a separation into simple elements had noticed the logarithms of imaginary quantities. Leibnitz first spoke of these in his letter to Bernoulli in 1702.* At this time Bernoulli showed the relation between the arc of the circle and the logarithms of the imaginary. In 1712 he deduced the expression for the tangent of a multiple arc by means of the tangent of this arc.** It would seem that having advanced thus far they would have been able to construct the theory of the logarithm of the imaginary.

It was Euler (1707-1783) who was able to solve the difficulties of this question. He saw that he needed to abandon the uniform character of the logarithm and admit for all numbers an infinity of logarithms. In 1749 he explained the complete and definite theory of logarithms which is like that universally adopted today. In 1740 he announced the formula $\cos x = e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}$ and in 1748 the formula $e^{x\sqrt{-1}} = \cos x + \sqrt{-1}\sin$ which has kept his name. He placed thus in evidence the identity, in a certain sense, of the exponential functions and the

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* Encyclopedie des Sciences Mathematiques, p.334.

** " " " p.335.

trigonometric functions. It is upon this formula that he founded in the following year the theory of logarithms, in regarding the logarithm as the inverse function of the exponential.*

The theorem that all functions of one or more imaginary numbers could always be placed in the form of $p+q\sqrt{-1}$ had been given by D'Alembert (1717-1783) in 1746. He had considered for the first time the power of which the base and the exponent are imaginary.**

Euler took up the theory of the same expression in 1749 and founded it upon his theory of logarithms. The same year he gave expressions for the $\sin(a+b\sqrt{-1})$, $\cos(a+b\sqrt{-1})$ and $\tan(a+b\sqrt{-1})$ in the form of $(p+q\sqrt{-1})$ and inversely the expressions of arc sin $(p+q\sqrt{-1})$, arc cos $(p+q\sqrt{-1})$ and arc tan $(p+q\sqrt{-1})$ in the form $(a+b\sqrt{-1})$.***

In 1730 De Moivre (1667-1754) published the formula $\cos B = \frac{1}{2}\sqrt{\cos nB + \sqrt{-1}\sin nB} + \frac{1}{2}\sqrt{\cos nB - \sqrt{-1}\sin nB}$ Later in 1783 he considered directly the expression $\sqrt{a+\sqrt{-b}}$. He showed that $\sqrt[n]{\cos a - \sqrt{-1}\sin a}$ admitted n values all of the form $p+q\sqrt{-1}$, which he obtained in dividing the arc a, and the arcs differing by a multiple of the circumference into n equal parts. He had thus established the re-

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* Encyclopedie des Sciences Mathematiques, p.335.
** " " " " p.336.
*** " " " " pp.336,337.

lation between the extraction of roots of imaginary numbers and the division of arcs.*

By the middle of the eighteenth century the theory of the fundamental operations, trigonometric functions and logarithms had been worked out and some attempts had been made to represent these numbers graphically. Argand had developed his theory of graphical representation the early part of the nineteenth century, but it had not been generally brought to the knowledge of mathematicians. The imaginary was still looked upon with doubt.

Much of the opposition to the imaginary was removed by the influence of Gauss (1777-1855). In 1831 he published his work on theory of imaginaries which showed their development in the same manner as that presented by Argand 25 years earlier. Unfortunately he never published the demonstration which he had promised for the justification of imaginaries.** He was the first to use the word complex to designate these numbers together with the real numbers.***

Euler was the first to introduce the symbol i . It appeared in his work presented to the Academy of St. Petersburg in 1777.[1]

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* Encyclopedie des Sciences Mathematiques, p.332.
** " " " " p.339
*** " " " " p.337
1. " " " " p.343

In France Cauchy (1789-1857) contributed much to the spread of the theory of Argand. His authority and the importance of his works upon the theory of the functions of a complex variable and finally the importance of the works by Abel (1802-1829) and Jacobi (1804-1851) upon elliptic functions removed definitely all doubt.

The connection between points on a plane and complex numbers was a most powerful aid in the study of symbolic algebra. In attempting to find a second imaginary unit that would correspond to the perpendicular which may be drawn in space to the lines representing 1 and i in the plane, Hamilton (1805-1865) discovered quaternions.* In the following year 1844 an account of his discovery appeared in the Philosophical Magazine. He discovered that the commutative law for multiplication did not hold. If the factors were interchanged the sign of the product had to be changed. These quantities were represented by the use of rectangular co-ordinate in space. A line drawn from the origin to a point in space was called a vector. The application of quaternions to physics has not been as wide as was expected. Kummer (1844), Kronecker (1845), Scheffer (1845), Bellaviti (1835), Peacock (1845) and De Morgan (1849) have

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* Fine, p.128.

made important contributions. Möbius can be mentioned because of his numerous geometric applications of complex numbers. Contributors of the latter part of the nineteenth century are Wierstrass, Schwarz, Dedekind and Poincare.*

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* Merriman & Woodard, p.516.

SECTION 6. TRANSCENDENTAL NUMBERS.

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The history of the number commonly known as "π" can be traced from the earliest times, but it was not until 1882 that it was proved transcendental. From earliest times to the middle of the seventeenth century attempts were made to find a square whose area was equal to that of a given circle; that is, to find the approximate value of π by purely geometric methods. During this period π was regarded as equivalent to a geometrical ratio. The earliest approximation is probably the one found in the Bible (I.Kings VII.,23) where the circumference is given as 3 times the diameter. The Babylonians also used the value 3 for π .*

Ahmes showed how to find the area of a square equivalent to a circle. The diameter diminished by 1/9 of itself was used as the side of the square. If the radius of the circle was unity, 16/9 was used for the side of the square and its area was $\left(\frac{16}{9}\right)^2 = 3.1604$. This gave a close approximation.**

Anaxagoras (499-428 B.C.) attempted to square the circle while he was in prison. This is the first record of any attempt to find the exact ratio. No

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* Tropfke, Vol.2, p.109
** Eisenlohr, p.124.

solution was offered.*

The process of exhaustion was introduced by Antiphon about the fifth century B.C. He inscribed in a circle a square and continued to double the number of sides of the polygon, thus obtaining polygons whose perimeters approached the circumference of the circle. He concluded that if a polygon was inscribed in this manner its perimeter would coincide with the circumference, a square could be found equal in area to the polygon, and hence a square could be constructed equal in area to a given circle.** Bryson, a contemporary of Antiphon circumscribed polygons about the circle, as well as inscribed them. He made the error that the area of the circle was equivalent to the arithmetical mean of the areas of the two polygons.***

Antiphon believed the circumference of the circle and the perimeter of the polygon could be made to coincide. Bryson and many of the other Greek mathematicians did not believe this was possible. This question gave rise to many lively discussions.

Euclid found no approximations to the ratio between the circumference and the diameter.[1] This was probably due to the fact that the Greek geometers ex-

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* Cantor, Vol.1, p.177
** Cantor, Vol.1, pp.189-190
*** Cantor, Vol.I, pp. 190-191
1. El.Cajori,p.173.

cluded calculations from their work. It was not evident to them that a straight line could be equal in length to a curved line. In Euclid the equality of lines was based on area and congruence. Nowhere in Euclid is there given the equality of a straight line and a curved line.

Archimedes (287 B.C.?) in his book on the measurement of the circle proved that a circular area is equal to that of a right triangle whose base is the circumference of the circle and the altitude, the radius. To find the upper limit for the ratio of the circumference to the diameter he constructed an equilateral triangle whose vertex was the center of the circle and whose sides are tangent to the circle. The angle at the center was bisected, and the ratio of the legs of the triangle was determined. This led him to conclude that $\pi < 3\frac{1}{7}$. To find the upper limit he inscribed a hexagon and continued to double the number of sides until he had a polygon of 96 sides. He found the perimeter of this, and then concluded that the lower limit was $3\frac{10}{71}$. The value 3 1/7 has continued to be used where approximate results are satisfactory. Archimedes assumed that a straight line existed which in length equaled a curved line. On this basis he made many valuable contributions to geometry.*

Ptolemy (87-165) in the sixth book of the Almagest which is devoted to the theory of eclipses, gave as the value of π, $3^\circ 8' 30''$ $(3+\frac{8}{60}+\frac{30}{3600})$ which in our notations is equivalent to $3\frac{17}{120}=3.1416$.**

The work of the Hindoos shows approximations for the value π. Aryabhatta (478) gave the value $\frac{62832}{20000}$ which equals 3.1416. "He showed that, if a is the side of a regular polygon of n sides inscribed in a circle of unit diameter, and if b is the side of a regular inscribed polygon of 2n sides, then $b^2=\frac{1}{2}-\frac{1}{2}(1-a^2)^{\frac{1}{2}}$"*** He begun with the side of an inscribed hexagon and found successively the sides of polygons of twice the number of sides, and finally the side of a polygon of 384 sides.

Brahmagupta (598) gave $\sqrt{10}$ as the value of π. He obtained his value by inscribing in a circle of unit diameter, regular polygons of 12, 24, 48, and 96 sides, and calculating their perimeters.[1]

Bhaskara (1114) gave two values, 'accurate' $\frac{3927}{1250}$ and 'inaccurate' $\frac{22}{7}$.[2]

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* Cantor, Vol.1, pp.285-288
** Ball Math.Recreations, p.252
*** Ball, Math.Recreations, p.253
1. Cantor, p.607
2. Cantor, p.612

The ancient Chinese used the value 3 for π. This value was used as early as the twelfth century B. C.[*] Chang Heng who lived during the first century A.D. gave $\sqrt{10}$ as value of π.[**] Liu Hiu in the third century calculated the value of π by inscribing a hexagon and then doubling the number of sides. He obtained the ratio $\frac{157}{50} = 3.14$.[***] Tsu Ch'ung chih in the fifth century showed that the value of π was between $\frac{22}{7}$ and $\frac{355}{133}$.[1]

The Arabs used the value $3\frac{1}{7}$ and the two Indian values $\sqrt{10}$ and $\frac{62832}{20000}$.[2]

Leonardo of Pisa (1202) showed that the limits of π were $\frac{1440}{458\frac{1}{8}} = 3.1427$ and $\frac{1440}{458\frac{4}{9}} = 3.1410$. As a mean he gave $\frac{1440}{458\frac{1}{3}} = 3.1418$.[3]

Viete (1540-1635) by using the polygon, and considering those of $6 \cdot 2^{16}$ sides found correct to nine decimal places the value of π.[4] Adrianus Romanus (1561-1613) worked out the value to seventeen places.[5] Ludolph van Ceulen (1540-1610) spent much time in finding the value of π. He gave the value to thirty-five

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* Mikami, p.46.
** Mikami, p.47.
*** Mikami, p.49
1. Mikami, p.80
2. Cantor, Vol.1, p.685.
3. Young, p.395.
4. Young, p.395.
5. Young, p.395.

places (1610).* The value of π is sometimes called Ludolph's number.

From the middle of the seventeenth century to the middle of the eighteenth mathematicians attempted to develop as an infinite series. John Wallis (1616-1703) made the first attempt by this method. He proved

that $\frac{\pi}{2}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdot\frac{8}{9}\cdots,$

and that $\frac{4}{\pi}=1-\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+\cfrac{81}{2+\cdots}}}}}$

The second form, that of the continued fractions had been given to him by Lord Brouncker (1620-1684)**

James Gregory (1638-1675) and Leibnitz (1646-1716) developed independently the series

$\tan^{-1}x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\cdots\cdots$

Gregory recognized the need of considering the convergency of such a series. If $x=1$ we have the series

$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\cdots$

This was developed by Leibnitz in 1674, and published 1682. If $x=\sqrt{\frac{1}{3}}$ the series of the arc tan becomes

$\frac{\pi}{6}=\sqrt{\frac{1}{3}}\cdot\left[1-\frac{1}{3\cdot3}+\frac{1}{3\cdot5}-\frac{1}{3^3\cdot7}+\frac{1}{3^4\cdot9}-\frac{1}{3^5\cdot11}+\cdots\right]$

which is more convenient as it does not converge so slowly.***

The addition formula $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}$ has also been used.[1]

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* Sherwin's Tables, p.108 ** Young, p.396

1 Young, p.397 *** Young, p.397

Using this method the value has been computed to 100 decimal places. Many other attempts at computing the value of π by means of series has been made. The value was computed to 700 places by Shanks. *

The results of these extensive computations show nothing about the nature of π , that is, whether it is rational or irrational, or its transcendental character. From the middle of the eighteenth century to the present time efforts have been made to determine the nature of π. The foundations for the solution of this problem were given by Euler in connection with his formulae involving e.

The number e is the limit $\lim_{x=0} (1+x)^{\frac{1}{x}}$ which equals 2.71828... In working out the derivative of the logarithm the form $(1+\frac{\Delta v}{v})^{\frac{v}{\Delta v}}$ appears in the solution. Mathematicians found that no number in the number system corresponded to the limit of this form. The first use of a single symbol to represent this limit seems to be due to Cotes, who represented it by M. Euler used the symbol e for this number.**

By means of Maclaurin's formula Euler expanded e^x, cos x, and sin x. Using these series he showed that $e^{ix} = \cos x + i \sin x$ and if $x = \pi$, $e^{i\pi} = -1$. He gave many

* Young, p.398

** Ball, p.394.

other relations between e and π.*

Making use of Euler's work Lambert proved in 1761 that π could not be rational.** Legendre in his geometry proved the irrationality of π and also that of π^2. In 1840 Liouville showed that e can not be the root of a quadratic equation with rational cofficients.***

The questions after this contribution were, of what if any algebraic equation with a finite number of terms with rational cofficients can e and π be roots? Is it not possible to find numbers that are not roots of an algebraic equation? [1] Legendre was the first to express this latter idea. Liouville in 1884 proved the existence of non algebraic numbers and justified the division of numbers into algebraic and transcendental numbers.

In 1873 Hermite provedd that the number e is transcendental. In 1882 Lindemann by basing his work on that of Hermite proved that π was transcendental.[2] This proof lead to the conclusion that if x is a root of a rational integral algebraic equation, then e^x cannot be rational; hence, if πi was the root of such an equation, $e^{\pi i}$ could not be rational; but $e^{\pi i}$ equals -1, and therefore is rational; hence πi cannot be the root of such an

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* Young, p.398.
** Merriman & Woodard, p.514.
*** Young, p.401
1. Young, p.401
2. Mathmatische Annalen, Vol.20, pp.213-225.

algebraic equation, and therefore neither can π.

Ball says,"If π represented merely the ratio of the circumference of a circle to its diameter the determination of its numerical value would have but slight interest. It is mere accident that it was defined in this way. It really represents a certain number which would enter analysis from whatever side it was approached.* The approximate value of π has also been obtained experimentally by theory of probability. **

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* Ball Recreations,p.249.

** Ball Recreations,p.260.

SECTION 7. TRANSFINITE NUMBERS.

Bibliography

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Hobson, E.W. The Theory of Functions of a Real Variable.
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Transfinite numbers first appeared in connection with point aggregates. In 1882 Georg Cantor defined "transfinite numbers" independently of the aggregates.* The theory of irrational numbers on which the founding of the theory of functions depends were never investigated with important results until the time of Dedekind and Weierstrass. However, they made no valuable contributions as to the nature of the whole number.** After the appearance of Cantor's articles, arithmetic received a development into a theory of cardinal and ordinal numbers, both finite and transfinite.

Aggregates with finite cardinal numbers are called "transfinite aggregates , and their cardinal numbers "transfinite cardinal numbers." The first example of a transfinite aggregate is given by the totality of finite cardinal numbers . Its cardinal number is called "Aleph-zero", and it is denoted by $\aleph_0$ (Aleph, the first letter of the Hebrew alphabet written with subscript zero).*** By a definite law, out of $\aleph_0$ proceeds the next greater cardinal number $\aleph_1$, and out of this by the same law the next greater $\aleph_2$ and so on.[1] Corresponding to a single

*Jourdain, p.4
**Jourdain, p.23
*** " p.103, Mathematische Annalen, Vol.46, p.488
1. " p.109, " " .p.495

transfinite cardinal number there is an infinity of transfinite ordinal numbers.*

In general the commutative law does not hold with transfinite numbers. The associative law holds, but the distributive law is only generally valid.**

Transfinite numbers are in a sense new irrationalities. Transfinite numbers and finite irrational numbers are both definitely marked off modifications of the actually infinite.***

Hobson in his discussion of the transfinite number says, that it has already become of great value for purposes of exact formulations in Analysis and in Geometry. It is constantly receiving new applications because of its power of providing the language necessary for expressing results in the theory of functions with the highest degree of rigour and generality.[1]

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* Hobson, p.177
** Jourdain, p.66
*** Jourdain, p.77
1. Hobson, p.211.

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