THE IMPACT OF MEASUREMENT ERROR ON CONTINUOUS TIME PANEL MODELS

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Submitted to the graduate degree program in Psychology and the
Faculty of the Graduate School of the University of Kansas in
partial fulfillment of the requirements for the degree of Master of Arts.

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Date Defended: June 24, 2015

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Abstract

Prior studies have shown that analyzing a continuous time panel model with the Exact Discrete Model (EDM) is less biased and more efficient than approximate methods such as Latent Differential Equations (LDE). Simulation models have included observed variables, latent variables, or a mix of the two types, but prior work has not examined the effects of measurement error on estimation when only a single observation is made at each occasion. This paper compares the performance of the EDM and LDE when measurement error is varied. Data conforming to a first order differential equation was generated for two variables across four time points using a variety of sample sizes, auto-effect values, and cross-effect values. EDM auto-effects were shown to be underestimated and become increasingly biased as measurement error increased while LDE estimates were positively biased, but addition of measurement error had little effect. Estimates for negative cross-effects had smaller absolute bias than positive cross-effects in both models, with LDE estimates closer to the true value than EDM. If expected measurement error is less than 10%, then EDM will produce more accurate estimates than LDE. For measurement error ranging from 10% - 15% each model produced some less biased and more efficient parameters than the other. For measurement error than exceeds 15%, LDE will provide less biased parameters for all but strongly negative cross-effects.

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Introduction

Most researchers being trained today in modern modeling methods are taught latent variable methods, which extol the benefits of separating true variance from error variance. But for a variety of reasons, such as cost of data collection, secondary data use, or parceling for the sake of parsimony, the researcher can be left with single measurements to represent a latent construct. This situation can leave the researcher with the question of what model to use to get the most accurate results given the constraints of the data that have been collected. If the selected model is a discrete time regression, the impact of measurement error on outcomes has been well documented (Fox, 2008). But, less understood is the impact of measurement error within continuous time structural equation models (SEM).

One continuous time model that can be used to estimate a panel model is the Exact Discrete Model (EDM). Oud and Jansen first introduced psychologists to the EDM estimated in SEM (2000). In the 2013 article by Voelkle and Oud introducing the Exact Discrete Model (EDM) with oversampling, the data were created with single indicator constructs and without measurement error in order to simplify the simulation. This choice was understandable because the primary goal was to compare two methods for estimating stochastic differential equations: direct estimation (Voelkle, Oud, Davidov & Schmidt, 2012) and estimation via oversampling (Singer, 2012). Both articles provided data and R syntax files for estimation in OpenMx (Neale et al., 2014) so that substantive researchers can run an example themselves, and the examples were based on single indicator constructs similar to the panel model in Figure 1. Though the code contains a comment that latent variables can be modeled as well, this article led to the question, what impact measurement would error have on estimates from EDM if only single indicator constructs were available for the analysis?

Continuous Time

The main idea of continuous time models is that for every psychological process we are trying to measure, the phenomenon does not start when it is being measured nor stop when we look away. We are capturing snapshots of cognition, behaviors, and emotions in an effort to generalize our findings. In longitudinal research we are taking snapshots over time, with the goal of understanding the underlying psychological process. For example, a researcher conducting a study of attachment observes a mother and child one or more times but assumes that the attachment behavior continues outside of the assessment. But, parameter estimates depend on rate of measurement unless the researcher selects a continuous time model (Gollob & Reichardt, 1987; Deboeck & Preacher, in press).

Imagine there were two longitudinal studies of depression and stress. One study scheduled assessments every week for 1 month, and the second study planned to collect data every three weeks for 12 weeks. Both used the same measures and had comparable sample sizes. A panel model was utilized to look at the relationship between depression and stress variables, but only the study with weekly measures found significant results. Or maybe they both found significant results but the effect sizes were different. What if stress has a fast rate of change and depression a slow rate of change? The different rates of observation could have a negative impact on model estimates resulting in biased or inefficient parameters. This phase problem, as described by Boker and Nesselroade (2002), could be resolved if the data were looked at another way. Rather than obtain parameter estimates tied to the specific intervals selected for study, a continuous time model could be used to identify the parameter estimates that would be relevant for any discrete span of time in the overall length of the study.

After model parameters have been estimated, it can be useful to compare analysis results to other substantive research in order to determine if the statistical results differ. If a continuous time model has been used but all previous research utilized discrete time models, then continuous time parameters can be converted to discrete time parameters or vice-versa. Discrete time parameters for a panel model are called *auto-regressive* and *cross-lags* while continuous time equivalents are called *auto-effects* and *cross-effects*, respectively. The exact relationship between these values is

$$\boldsymbol{A}_D = \frac{ln\big(A(\Delta t_i)\big)}{\Delta t_i}, \tag{1}$$

where $\mathbf{A_D}$ is the *drift* matrix of continuous time values for one interval of time with auto-effects values on the diagonal and cross-effects on off-diagonal; $\mathbf{A}(\Delta t_i)$ is the matrix of auto-regressive and cross-lag discrete time values for some lag, Δt_i. Taking the natural log of the **A** matrix, rather than the individual terms, will result in a corresponding continuous time matrix that is referred to as the drift matrix. The log of a matrix will depend on all values in the matrix because both eigenvalues and eigenvectors of **A** are used in the computation. For example, take two matrices that differ on only one entry. By taking the exponent of the matrix for one unit of time,

$$\begin{bmatrix} -0.12 & 0.45 \\ 0.67 & -0.16 \end{bmatrix} = ln \begin{bmatrix} 0.95 & 0.40 \\ 0.30 & 0.92 \end{bmatrix} /1. \tag{2}$$

Change one value, $\mathbf{A}_{2,1}$, from 0.30 to -0.30, and every entry in the drift matrix changes.

$$\begin{bmatrix} 0.01 & 0.41 \\ -0.31 & -0.02 \end{bmatrix} = ln \begin{bmatrix} 0.95 & 0.40 \\ -0.30 & 0.92 \end{bmatrix} /1. \tag{3}$$

Change $\mathbf{A}_{2,1}$ to another value and every entry will change again. Any value in **A** or the drift matrix can never be considered in isolation.

Two Types of Error

This paper examines the impact of measurement error on the estimation of continuous time parameters in EDM. But to truly understand how measurement error may differentially impact the estimation process, more needs to be understood about the differing sources of error. In regression models, the theoretical model contains an error term that is associated with the outcome variable while the predictors are assumed to be measured without error. The distribution of errors will have some mean and variance and are usually assumed to be uncorrelated. Within latent variable modeling, a variable's variance is separated into true variance representing the latent variable and unique variance, with measurement error variance being part of the unique variance. The unique variance is a mixture of variance unique to a measure, i.i.d. errors, and systematic error, such as variance related to the type of measurement method (Kline, 2011). Within a longitudinal context, the error structure can be constrained to be equal as in repeated measured analysis of variance (RANOVA), freed at all time points, constrained to decrease over time or take on other patterns (Singer & Willet, 2003; Grimm & Widaman, 2010).

Rather than error variance that is assumed to exist across the whole range of the model, the error in continuous time is stochastic error with variance that has different properties. In EDM, the stochastic error term is assumed to be the Weiner process, a continuous time random walk (Voelkle et al., 2012). A random walk is a discrete time series model with mean of 0 and a variance term that depends on time (Brockwell & Davis, 2010). Likewise, the Weiner process has an initial variance of 0 and then grows proportional to time. If a measurement model with multiple indicator latent variables has been specified, the EDM will be able to estimate unique variances as well as the Weiner process. The stochastic error term is always estimated by EDM, regardless of whether it is also possible to estimate unique variance terms.

For data modeled in SEM the amount of variance present in the data will be estimated in one of the parameters in the model. If it is not possible to define latent variables with multiple indicators in the specification of EDM, it will not be possible to automatically separate the unique variance from the indicator variance. But if those single indicators contain measurement error, how will the estimated parameters be different from their true value? When compared to another SEM-based estimation method that treats errors differently, will EDM still be less biased and more efficient as has been found in previous simulations (Oud, 2007)?

Model Choices

There are several methods available for estimating a continuous time panel model (CTPM). One is EDM, which is a stochastic differential equation that can estimate the stochastic error process separately from the deterministic process of interest. A second approach is Latent Differential Equations (LDE), a type of latent growth curve model (LGC) that generates equivalent parameters for the CTPM as EDM (Oud, 2007). LDE takes a different approach to error in that it can estimate the residual as defined in SEM but cannot estimate the continuous time stochastic error term that is explicitly a part of the EDM.

Exact Discrete Model. The EDM is a stochastic differential equation (SDE) that constrains the nonlinear relationship between discrete time and continuous time parameters in order to estimate the underlying, stationary process. The stochastic error is estimated as part of the equation (Voelkle et al., 2012), and the model is flexible enough to model observed variables through single indicator constructs or multiple indicator latent variables with a measurement model.

Coming from a discrete time and time series perspective, the foundation of EDM is an autoregressive model with lag of 1 (AR1). AR1 is

$$x_i = ax_{i-1} + w_i. \tag{4}$$

The measurement of a variable at any time (x_i) is equivalent to that weighted variable (a) at some previous time (x_{i-1}) plus an error term (w_i). The extension of AR1 into a multivariate form for a cross-lag panel model turns a into matrix $\boldsymbol{A}$ of autoregressive coefficients on the diagonal and cross-lag coefficients on the off-diagonal; x is a vector of outcome variables and w is a vector of uncorrelated error terms:

$$x_i = \boldsymbol{A}x_{i-1} + w_i. \tag{5}$$

The interval of time is still represented from i-1 to i. Another way to represent the interval of time is to write the interval in terms of change in t, Δt_i; x, $\mathbf{A}$ and w are now defined as a function of time (Voelkle, et al., 2012).

$$x(t_i) = \boldsymbol{A}(\Delta t_i)x(t_i - \Delta t_i) + w(\Delta t_i). \tag{6}$$

The interval of time is still discrete and measureable. We could make Δt_i smaller and smaller so that it converges to 0; this is mathematically equivalent to taking the derivative with respect to time. Dropping the error term for the moment, the equation becomes

$$\frac{dx(t)}{dt} = \boldsymbol{A_D}x(t). \tag{7}$$

The derivative is predicted by the vector x and the continuous time drift matrix, $\boldsymbol{A_D}$. Equation 7 is a differential equation because it is a derivative of a function. As shown in the proof provided by Voelkle and his colleagues (2012), the Equation 8 is the solution for the differential equation:

$$x(t) = e^{\boldsymbol{A_D} \times (t - t_0)} x(t_0). \tag{8}$$

Note that this process starts at t_0 and ends at t. A more generic way to write $t\text{-}t0$ is Δt_i for some interval i. Putting the error term aside for the moment, both (6) and (8) are equal to $x(t)$:

$$\boldsymbol{A}(\Delta t_i) = e^{\boldsymbol{A_D} \times \Delta t_i}. \tag{9}$$

The left side of the equation contains in $\boldsymbol{A}(\Delta t_i)$ the autoregressive and cross-lag parameters of discrete time while the right side of the equation holds auto-effect and cross-effects in the drift matrix $\mathbf{A_D}$, which is then multiplied by the change in *t*. This enables the equality to be constrained in the estimation step (Voelkle et al., 2012).

Starting with Equation 7, the error term was deliberately dropped. EDM includes an error term that represents the stochastic error process. The continuous time stochastic error is the Weiner process, a process that is 0 when t = 0 and then grows proportional to time (Voelkle et al., 2012). The Weiner process also has infinite variance (White, 1986) so is not differentiable in the context of Newtonian calculus. New developments in stochastic calculus, a field of calculus devoted to these random processes, can be used to measure the area under the stochastic process.

Returning the formulas and picking up from Equation 8, the error variance added to that formula is

$$x(t) = e^{\boldsymbol{A_D} \times (t - t_0)} x(t_0) + \int_{t_0}^{t} e^{\boldsymbol{A_D} \times (t-s)} G dW(s), \tag{10}$$

Where the $\mathbf{A}_\mathrm{D}$ multiplied by some lag is again included as an exponent, *dW(s)* represents the stochastic process with respect to continuous time and G is the Cholesky triangle for the model. In the case of a multivariate model, the error term changes and the new formula is

$$x(t) = e^{\boldsymbol{A_D} \times (t - t_0)} x(t_0) + \int_{t_0}^{t} e^{\boldsymbol{A_D} \times (t-s)} Q e^{\boldsymbol{A}_D^T \times (t-s)} dW(s). \tag{11}$$

In Equation 11, Q is referred to as the diffusion matrix and is the error covariance matrix; Q = GG^T, and G is its transpose. Q is pre-multiplied by the same term as found in Equation 10, but it is also post-multiplied by a transposed version of that term (Voelkle et al, 2012).

Latent Differential Equations. Latent Differential Equations (LDE) take a different approach in the estimation of differential equations than EDM. Latent in LDE refers to the estimation of latent variables while differential equations refers to the parameterization that results in estimates about a latent variable and its derivatives with respect to time (Boker, Neale, & Rausch, 2004). The foundation of LDE is a growth curve, a model which has a long history that McArdle and Nesselroade (2014) trace back to the seventeenth and eighteenth century work of Newton and Pascal (McArdle & Nesselroade, 2002). The work by Meredith (Meredith & Tisak, 1990) took growth curves into a latent variable framework, enabling the development of hypotheses about change between growth curves via the estimation of structural paths.

Latent growth curves (LGC) typically have a meaningful time point represented in the model. In some applications, the slope factor loading for the first measurement is fixed to 0 indicating a common starting point. The last time point can be fixed to 0 to indicate the slope at the end of the study. A lack of a common time point can result confounded parameters between individual differences in the mean and individual differences in parameter estimates for the latent curve (Boker & Bisconti, 2006). If there is no shared time measurement, LDE can be used to measure the process independently of time because time is a lag (Boker, et al., 2004).

When a LGC has been specified as a LDE, the intercept is the zeroth derivative, and the slope term is the estimate for the first derivative, also referred to as rate of change (Boker, et al., 2004). Two outcomes can be modeled together to create a couple LDE and estimate parameters for the auto- and cross-effects in the CTPM, parameters that are equivalent to the auto- and cross-effects estimated by the EDM (Hu, Boker, Neale, & Klump, 2014; Oud & Singer, 2008). As shown in Figure 2, the parameter from the X-zeroth derivative to the X-first derivative ($\dot{X}$) estimates the auto-effect. The path from the X-zeroth derivative on one variable to the Y-first derivative ($\dot{Y}$) estimates the cross-effect. Because growth curves provide the foundation and may be more familiar to the reader, the rest of this section will describe LDE in terms of a LGC.

The first decision in the estimation of a growth curve is the whether a linear or curved pattern is expected with the model containing the intercept and slope to estimate linear growth. The zeroth and first derivative are sufficient for the estimation of a first order differential equation with regression paths between the latent variables (Boker, et al., 2004). The second decision is how time will be denoted in the factor loadings and whether they will be fixed at each point or freed for estimation. LDE fixes each factor loading equal to the indefinite integral, for example $\int 1 d\tau = \tau$, in order to obtain derivative estimates. Similar to the intercept in a standard latent growth curve, the zeroth derivative is fixed to 1 for every time point in order to set the scale. The factors for the slope, or first derivative, reflect the intervals of time. In LDE the values will be centered on zero; if there are five equally spaced time points, the factor loading values for the first derivative are -2, -1, 0, 1 and 2; four equally spaced time points would be fixed to -1.5, -0.5, 0.5, and 1.5 (Boker, et al., 2004). Because the exact value for the parameters in a LGC can change based on how the factor loadings are specified, forcing the loadings to be centered on 0 results in structural parameters equivalent to the EDM drift matrix and independent of time.

Addition of Measurement Error

Prior simulations using the EDM and LDE have examined the estimation of a damped linear oscillator (DLO), a second order differential equation. Oud and Singer (2008) showed that the two different methods provide the same drift parameter estimates for these models. Oud (2007) compared EDM, LDE and two other models in the estimation of a four time point panel model with a DLO and measurement model. EDM was less biased than LDE on all of the structural paths in the model with the most bias reported for the DLO parameter estimate. LDE latent parameter estimation was also less efficient than the EDM. One of the latent variable parameters that was part of that simulation was the auto-effect. While that parameter was more biased for LDE than the EDM, the difference in bias was small (.006). In the context of a measurement model the linear predictor was estimated equally well by both differential equation estimation methods.

Steele and Ferrer (2011a) used a univariate and coupled LDE to examine how affective processes self-regulate and co-regulate. Residual-based composite scores from observed variables were analyzed instead of latent variables and the data was embedded, a method that attenuates measurement error (von Oertzen & Boker, 2010). In response to a criticism about the use of LDE instead of the EDM (Oud & Folmer, 2011), Steele and Ferrer (2011b) concisely describe the different ways that LDE and EDM approach error. In LDE all error that is not part of the true variance of the derivative is part of the residual. That residual variance will contain both measurement error and random error process with a separate estimate possible for each observed variable while EDM would only estimates the continuous time error process leaving Steele and Ferrer to conclude that LDE was a better methodological choice.

If observed variables are used to represent the process rather than latent variables, then there is no way for EDM to separately estimate measurement error. A simple solution would be to use multiple indicators and build latent variables, but use of observed variables in models are still very common. A search of the journal Child Development for the years 2010-2014 using the terms ‘cross-lag’ and ‘longitudinal’ returned 33 articles. Seven articles used latent variables, 7 articles used a mix of latent and observed variables, and 5 articles estimated LGC parameters from observed variables. The remaining 14 articles used observed variables. The majority of those articles (63.6%) estimated model parameters that would contain measurement error. And while the impact of measurement errors on analysis results are understood in regression (Fox, 2008), the role of error in estimates from differential equation models is less clear.

When measurement error is added to the model, do EDM estimates become biased and less efficient than the condition without added measurement error? How biased and efficient are parameters estimated by LDE in the presence of measurement error? Are there conditions under which one estimation method is preferred over the other based on relative bias and efficiency? To determine the impact of measurement error on the estimation of continuous time parameters, a simulation study was carried out to examine parameter estimation when using LDE and EDM to fit a first order differential equation. Estimation of auto-effect and cross-effect parameters was examined.

Methods

Simulation

A simulation was designed to examine the estimation of a CTPM with two single indicators across four time points and varying A-matrix conditions as measurement error was added to the data. Each data set was analyzed with both LDE and EDM using the OpenMx package (Neale, et al., in press) in R 3.0.2 (R Core Team, 2014). LDE was specified using RAM notation. The oversampling program, CT_SEM.R (Voelkle & Oud, 2013) was utilized for the EDM analysis.

A-matrix values. The A-matrix in discrete time with two constructs is composed of four values for a lag of 1: the X1 to X2 (auto-regressive), X1 to Y2 (cross-lag), Y2 to X1 (cross-lag), and Y1 to Y2 (auto-regressive). The values selected for each element in **A** with the corresponding range of drift matrix values are listed in Table 1. The continuous time auto- and cross-effect values are reported in ranges because there is no one-to-one mapping for each discrete standalone value.

Data generation. After an expected covariance matrix was created from discrete-time simulation parameters, time series data conforming to a first-order differential equation model were simulated using the mvrnorm function from the MASS package (Venables & Ripley, 2002) in R 3.0.2 (R Core Team, 2014). Four sample sizes (50, 150, 250, and 500) were crossed with the discrete time A-matrix values listed in Table 1. For the 1280 conditions related to sample size and **A** matrix, the mean was fixed to 0 across four time points for the two variables, X and Y, and 1000 data sets were generated for each condition. After the generation of each data set with 0% measurement error, 10%, 15%, 20%, and 25% measurement error was added. This resulted in a total of 6400 conditions being varied in the data sets. Time interval information was added to the data required by EDM, and each interval was set to 1, indicating equal spacing of the data.

Analysis

The drift matrix parameter estimates, standard errors, convergence status, and data sets were saved for EDM and LDE for each run of the CTPM estimation. The logm function in the R package expm (Goulet et al., 2014) was used to compute the log of the $\mathbf{A}(\Delta t_i)$ matrix for each simulation condition in order to obtain the true values for the drift matrix. Continuous time values for each drift matrix were then used to calculate bias,

$$\widehat{bias} = \left(R^{-1}\sum_{i=1}^{R}\widehat{\theta_i}\right) - \theta \qquad (12)$$

where R is the number of converged replications, $\widehat{\theta_i}$ is the parameter estimate, and θ is the true value. Relative bias was computed as a ratio of bias for LDE divided by bias for EDM; if relative bias was less than 1 then LDE was less biased than EDM. Relative efficiency (RE) of LDE to EDM was computed from the a ratio of mean square error (MSE),

$$RE = \frac{\widehat{bias}^2_{LDE} + var(\hat{\theta}_{LDE})}{\widehat{bias}^2_{EDM} + var(\hat{\theta}_{EDM})} \tag{13}$$

with values less than 1 indicating that LDE is preferred over EDM (Burton, Alton, Royston & Holder, 2006).

Results

The bias of estimates across the four sample sizes, 50, 150, 250, and 500, differed very little in the estimates for LDE and EDM. For example, the difference for LDE's 0.95 XX-auto-effect between N = 50 and N = 150 was 0.00013 with even smaller differences on that parameter between N = 150 and larger sample sizes. More differences were observed in both bias and efficiency for the different A-matrix parameter estimates. Bias will be examined first, starting with values averaged across simulation conditions, including sample size, before looking at auto-effects and cross-effects for a subset based on a cross-effect of 0 and interactions between cross-effects, and then a set of specific drift matrices.

Bias

As seen in Table 2, on average, EDM parameters with 0% measurement error were estimated with little absolute bias (< 0.003), while LDE over-estimated the auto-effects and under-estimated the cross-effects. As measurement error increased, EDM auto-effects became more biased in the negative direction while cross-effects became more biased in the positive direction, though at a slower rate. For LDE bias in auto-effects were larger than bias in cross-effects. Because EDM provided very accurate estimates for models with 0% measurement error, all results for EDM will focus on conditions with added measurement error.

Bias of Cross-effect at 0. A cross-effect of 0 is also 0 for a cross-lag in discrete time. A subset of simulation conditions with N = 150 and YX = 0 was selected as the first step in examining patterns in estimation. Figure 3 shows that EDM auto-effects were more biased as measurement error increased. These estimates were close to each other at 10% measurement error but started to take on a wider range of values as measurement error increased. On the other hand, LDE estimates were nearly consistent in their bias as measurement error increased.

Table 2 and Figure 4 show that, in terms of absolute bias, cross-effect estimates were less biased than the auto-effect estimates. When the XY cross-effect was large and negative, both EDM and LDE produced values with little bias, and LDE was once again consistent at all levels of measurement error. Absolute bias increased for positive XY cross-effects in both models with the most biased estimates for EDM for large, positive values.

Cross-effect interaction. Further examination of the cross-effect focused on positive YX cross-effects (0.2 and 0.4) at all four levels of the XY cross-effect. LDE and EDM estimates were most biased when both cross-effects were positive with EDM parameters more biased than LDE. If a negative cross-effect was paired with a positive cross-effect, the amount of bias between EDM and LDE were approximately equal though LDE estimates were clustered together more than EDM estimates as seen in Figure 5.

Relative bias

The relative bias of LDE to EDM was computed for the 8 drift matrices listed in Table 3 in order to examine how different values would influence the parameter estimates obtained from EDM and LDE. On each row, the diagonal values are the auto-regressive and auto-effects for X and Y. The off-diagonal elements contain the cross-lag and cross-effects between X and Y. Under the 0% measurement error condition, all bias values were greater than 1 indicating that

EDM produced less biased parameters than LDE. At 10% added error, half of the parameter estimates generated by EDM were less biased than LDE; though not shown here, at 15% added error EDM estimates were less biased in only 6 of the 32 parameters. This trend continued at the 25% measurement error condition with only 3 of 32 cross-effect estimates less biased with EDM.

The strength of the auto-effect influenced the EDM parameter estimates with the stronger auto-effects having more bias. With respect to relative bias, LDE produced less biased auto-effects with the largest discrete time simulation conditions (0.95 and 0.92). Cross-effect bias was smallest if one or both of the parameters were negative. EDM estimates were positively biased across all conditions while LDE values were negatively biased with the exception of the XY = -.30 and YX = -.40; in this case bias was close to 0 for stronger auto-effects and positive for the weaker auto-effects. Similar to the auto-effect estimates, LDE parameters were less biased when auto-effects were the largest (0.95 and 0.92).

Relative efficiency

Table 4 shows the relative efficiency of LDE to EDM for the 8 drift matrices with 10% and 25% added measurement error. The patterns of results are very similar to those for relative bias. At 0% measurement error, EDM is more efficient than LDE. With 10% added measurement error, approximately half of the auto-effect and cross-effect parameters estimated by LDE are more efficient. EDM cross-effects are more efficient except for small cross-effects values (0.00 and 0.10) while LDE auto-effects are more efficient if they are strong (0.95 and 0.92) with mixed results with weaker auto-effects (0.80 and 0.77). With 25% added measurement error, EDM is only more efficient for strong, negative cross-effects (-0.30 and -0.40) paired with smaller auto-effects (0.80 and 0.77).

Model Convergence

Each simulation conditions was tested with 1000 data sets for a total of 6,400,000 models each for LDE and EDM. With regard to convergence rates, 100% of the EDM models converged. LDE failed to converge in 0.45% of the models, with more than half failing in the 0% measurement error condition. As measurement error increased, more models converged with only 0.004% failing when error was 25%.

Discussion

The goal of longitudinal studies is to understand how behave constructs over time. With panel models, the goal is to understand how two variables are influencing each other over time. But this analysis often utilizes imperfect data, such as data collected at non-optimal intervals or single indicators to represent constructs of interest. Sometimes researchers choose single indicators to represent a construct rather than multiple questions, but other times single indicators are the only option because of the how the data was collected. And depending on how the model is estimated, measurement error can result in biased and inefficient parameters.

Similar to Oud's (2007) simulation results that compared the EDM to LDE, the 0% measurement error condition as estimated by EDM produced accurate and efficient estimates; those estimates became more and more biased as measurement error increased. Across the range of simulation conditions, absolute bias for EDM parameters increased and one would anticipate that the estimates would continue in that linear trajectory in the presence of additional measurement error. LDE was able to effectively separate measurement error and estimates changed little as added measurement error varied from 0% – 25%. Although LDE parameters were biased, they were consistent.

Looking at cross-effects in more detail highlighted how the combination of cross-effects impact the parameters. For both EDM and LDE, cross-effects were the most biased when both values are positive, though LDE was less biased. For the combination of negative and positive cross-effects, LDE parameters are still less biased than EDM, but there is little difference in terms of absolute bias. For the last scenario of two negative cross-effects, absolute bias was smallest for strong values (-0.30, -0.40) for both estimation methods though the EDM was still more biased than LDE expect in the condition with weaker auto-effects.

Recommendations

If a researcher is making a decision between EDM and LDE in order to estimate a CTPM with single variables at each time point, and little is known about the parameter estimates that will be generated by the model, then level of expected measurement error may be the best information to use for model selection. In the case that less than 10% measurement error is expected, then EDM will provide less biased and more efficient results than LDE. For measurement error that ranges from 10-15%, either method can be used because each estimation method will produce better estimates on some of the parameters. If more than 15% error is expected in the data, then LDE will provide less biased estimates than EDM.

If some information is known about strength of parameters and direction of cross-effects, then results from specific drift matrices can be used to select EDM or LDE. When measurement error is expected to be 15% or less, EDM should better estimates for weaker auto-effects (e.g. < 0.80 in discrete time) and stronger, negative cross-effects (e.g. < -0.30 in discrete time). LDE will generate better estimates of auto-effects (e.g. > 0.90 in discrete time) and weaker cross-effects (e.g. -0.20 – 0.20 in discrete time) once added measurement error exceeds 15%.

Limitations and Future Research

This simulation study is only applicable for panel models when only one variable is being used to represent the construct. The situation where latent variables can be constructed for one of the two variables will likely result in different recommended cut-points with respect to measurement error. The impact of measurement error in the estimation of the DLO and other differential equation models is still an open question. It may also be possible to fix the measurement model error term to something other than 0 and correct the standard error (Oberski & Satorra, 2013), but further research needs to be conducted in order to evaluate this possible solution.

Conclusion

As anticipated, the first order differential equation as estimated by the EDM resulted in unbiased and efficient parameters at 0% measurement error; these parameters became increasingly biased and inefficient as more and more measurement error was added to the data. On the other hand, the impact of additional measurement error on LDE estimates was negligible. One surprising finding was the how the presence of negative cross-effects ameliorated the influence of measurement error on the parameters, and that the effect was seen in both the EDM and LDE. EDM is still a good choice if latent variables can be built instead of single indicator constructs, but if it is not possible to build a measurement model, LDE becomes the better model. LDE is more robust to the presence of measurement error, and because it is a LGC, a model familiar to many social scientists, ease of implementation may make this model a better choice for many applied researchers.

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Tables and Figures

Table 1.

A-matrix conditions

A	Discrete	Drift	Continuous
X1 to X2	0.95, 0.90, 0.85, 0.80	XX	-0.331 to 0.019
X1 to Y2	-0.30, -0.10, 0.10, 0.30	XY	-0.410 to 0.410
Y1 to X2	-0.40, -0.20, 0.00, 0.20, 0.40	YX	-0.547 to 0.547
Y1 to Y2	0.92, 0.87, 0.82, 0.77	YY	-0.371 to -0.011

Table 2.

Bias of Drift Matrix Parameters Across Levels of Added Measurement Error

	EDM			LDE		
	Mean	SD	95% CI	Mean	SD	95% CI
XX						
0%	-0.0017	0.0023	[-0.0018, -0.0015]	0.0811	0.0344	[0.0792, 0.0829]
10%	-0.1124	0.0348	[-0.1143, -0.1105]	0.0800	0.0321	[0.0782, 0.0818]
15%	-0.1620	0.0488	[-0.1647, -0.1594]	0.0798	0.0318	[0.0781, 0.0816]
20%	-0.2086	0.0613	[-0.2119, -0.2052]	0.0797	0.0315	[0.0779, 0.0814]
25%	-0.2529	0.0730	[-0.2569, -0.2489]	0.0797	0.0315	[0.0779, 0.0814]
XY						
0%	0.0008	0.0023	[0.0007, 0.0010]	-0.0416	0.0494	[-0.0443, -0.0389]
10%	0.0585	0.0365	[0.0565, 0.0605]	-0.0449	0.0433	[-0.0473, -0.0425]
15%	0.0828	0.0506	[0.0801, 0.0856]	-0.0451	0.0423	[-0.0475, -0.0428]
20%	0.1049	0.0631	[0.1014, 0.1083]	-0.0453	0.0419	[-0.0476, -0.0430]
25%	0.1250	0.0745	[0.1209, 0.1291]	-0.0453	0.0415	[-0.0476, -0.0431]
YX						
0%	0.0009	0.0022	[0.0008, 0.0010]	-0.0424	0.0560	[-0.0455, -0.0393]
10%	0.0584	0.0358	[0.0565, 0.0604]	-0.0457	0.0494	[-0.0484, -0.0430]
15%	0.0827	0.0498	[0.0800, 0.0854]	-0.0460	0.0485	[-0.0486, -0.0433]
20%	0.1048	0.0623	[0.1013, 0.1082]	-0.0461	0.0479	[-0.0487, -0.0434]
25%	0.1252	0.0733	[0.1212, 0.1292]	-0.0461	0.0476	[-0.0487, -0.0435]
YY						
0%	-0.0020	0.0051	[-0.0023, -0.0017]	0.0905	0.0420	[0.0882, 0.0928]
10%	-0.1236	0.0393	[-0.1257, -0.1214]	0.0887	0.0389	[0.0866, 0.0908]
15%	-0.1777	0.0547	[-0.1807, -0.1747]	0.0883	0.0385	[0.0862, 0.0904]
20%	-0.2285	0.0686	[-0.2323, -0.2248]	0.0881	0.0383	[0.0860, 0.0902]
25%	-0.2768	0.0812	[-0.2812, -0.2723]	0.0879	0.0381	[0.0858, 0.0900]

Note. Averages are reported to the fourth decimal to show the small change in bias for LDE as measurement error was added.

Table 3

Relative Bias for Auto-effects and Cross-effects When Comparing LDE to the EDM

True Value				Relative Bias			
A		Drift		10%		25%	
0.95	0.40	-0.12	0.45	-0.914	-1.043	-0.410	-0.486
0.30	0.92	0.67	-0.16	-0.867	-0.784	-0.404	-0.359
0.95	0.40	0.01	0.41	-0.832	-1.744	-0.330	-0.742
-0.30	0.92	-0.31	-0.02	-1.116	-0.739	-0.555	-0.322
0.95	-0.40	-0.12	-0.45	-0.741	2.523	-0.308	1.141
-0.30	0.92	-0.34	-0.16	1.887	-0.742	0.789	-0.292
0.95	0.00	-0.05	0.00	-0.573	-0.593	-0.253	-0.277
0.10	0.92	0.11	-0.08	-0.679	-0.618	-0.316	-0.274
0.80	0.40	-0.33	0.55	-1.339	-1.540	-0.604	-0.724
0.30	0.77	0.41	-0.37	-1.297	-1.164	-0.612	-0.536
0.80	0.40	-0.14	0.48	-0.375	-2.107	-0.162	-0.972
-0.30	0.77	-0.36	-0.17	-1.233	-0.781	-0.657	-0.357
0.80	-0.40	-0.33	-0.55	-1.241	4.752	-0.519	1.843
-0.30	0.77	-0.41	-0.37	3.886	-1.256	1.433	-0.506
0.80	0.00	-0.22	0.00	-0.689	-0.695	-0.319	-0.342
0.10	0.77	0.13	-0.26	-0.863	-0.752	-0.405	-0.336

Note. Each pair of rows and columns is a 2x2 matrix with the auto-regressive/auto-effects on the diagonal and cross-lag/cross-effects on the diagonal.

Table 4

Relative Efficiency for Auto-effects and Cross-effects When Comparing LDE to the EDM

True Value				Relative Efficiency			
A		Drift		10%		25%	
0.95	0.40	-0.12	0.45	0.829	1.055	0.183	0.254
0.30	0.92	0.67	-0.16	0.743	0.615	0.176	0.140
0.95	0.40	0.01	0.41	0.716	2.112	0.130	0.712
-0.30	0.92	-0.31	-0.02	1.178	0.562	0.350	0.114
0.95	-0.40	-0.12	-0.45	0.563	2.004	0.106	0.857
-0.30	0.92	-0.34	-0.16	1.250	0.563	0.603	0.095
0.95	0.00	-0.05	0.00	0.339	0.375	0.071	0.092
0.10	0.92	0.11	-0.08	0.479	0.392	0.117	0.083
0.80	0.40	-0.33	0.55	1.706	2.172	0.382	0.536
0.30	0.77	0.41	-0.37	1.589	1.309	0.385	0.297
0.80	0.40	-0.14	0.48	0.223	2.748	0.050	1.070
-0.30	0.77	-0.36	-0.17	1.405	0.628	0.491	0.141
0.80	-0.40	-0.33	-0.55	1.495	7.072	0.278	1.969
-0.30	0.77	-0.41	-0.37	4.067	1.517	1.239	0.265
0.80	0.00	-0.22	0.00	0.480	0.498	0.110	0.136
0.10	0.77	0.13	-0.26	0.739	0.570	0.185	0.122

Note. Each pair of rows and columns is a 2x2 matrix with the auto-regressive/auto-effects on the diagonal and cross-lag/cross-effects on the diagonal.

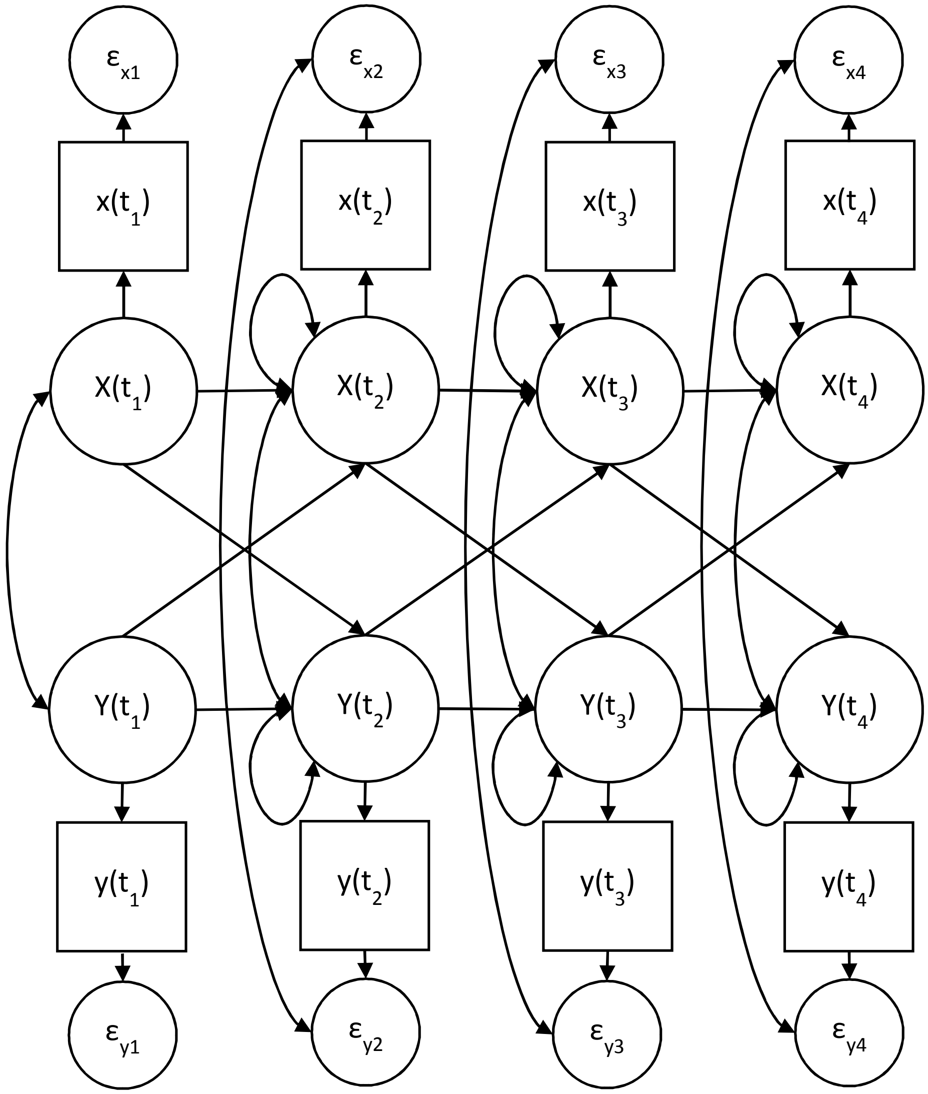

Figure 1. Panel model with four time points and correlated error terms.

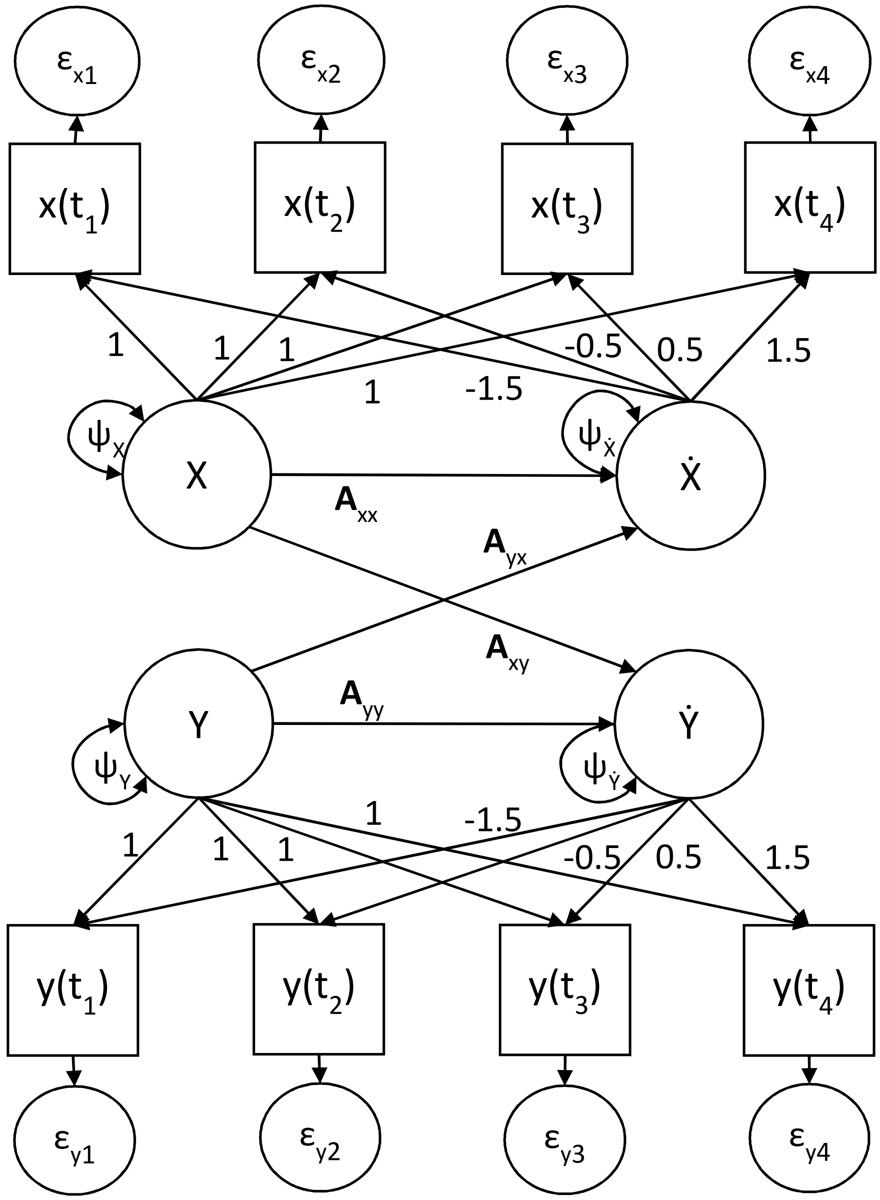

Figure 2. Coupled latent differential equation.

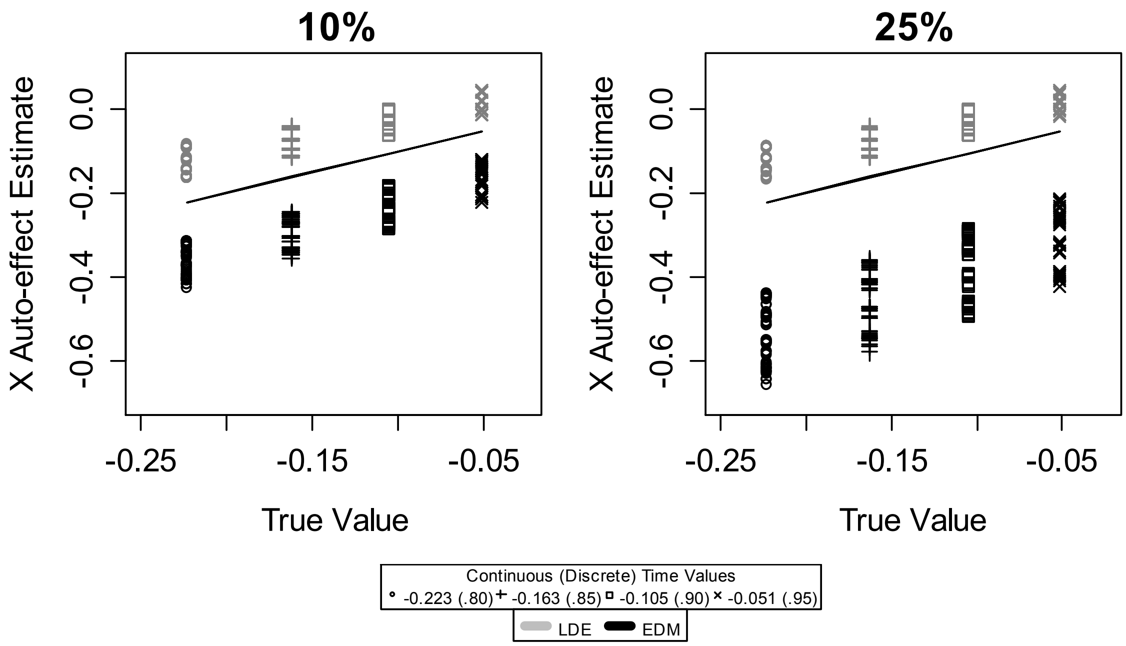

Figure 3. True Value Versus Estimated Auto-effects for EDM and LDE (YX = 0 and N = 150). When the YX cross-effect equals 0 then X auto-effect has a one-to-one mapping between discrete and continuous time.

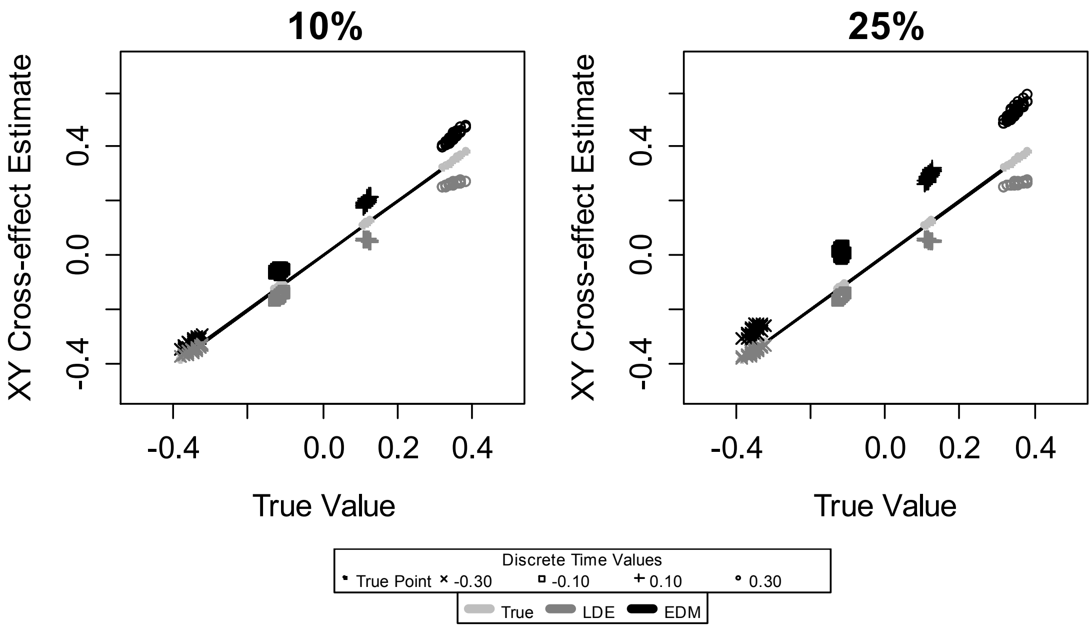

Figure 4. True Versus Estimated XY Cross-effects for EDM (black) and LDE (gray) for YX = 0. Each true value for a discrete time parameter represents a range of continuous time values. For a discrete time value of -0.30, continuous time values range from -0.382 to -0.321; -0.10 in discrete time ranges from -0.127 to -0.107; 0.10 in discrete time ranges from 0.107 to 0.127; and 0.30 in discrete time ranges from 0.321 to 0.382.

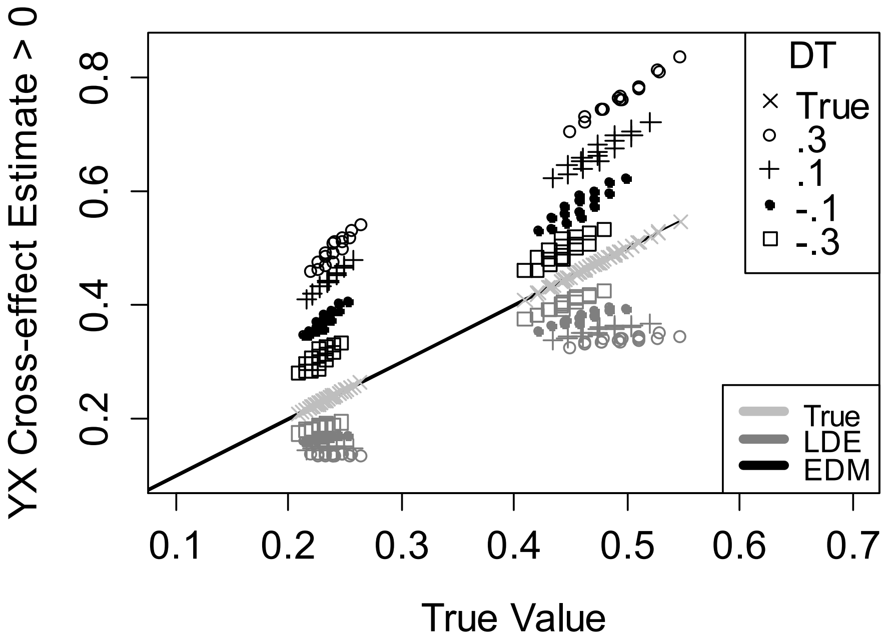

Figure 5. Positive YX Cross-effects for All Values of XY as Estimated by EDM and LDE. The simulation sample size was N = 150 and 25% added measurement error. Each true value for a discrete time parameter represents a range of continuous time values. For a discrete YX = 0.2, continuous time values range from 0.209 to 0.264; and YX = 0.40 in discrete time ranges from 0.450 to 0.547. For a discrete time XY = 0.30, continuous time values range from 0.328 to -0.410; XY = 0.10 in discrete time ranges from 0.107 to 0.130; XY = -0.10 in discrete time ranges from -0.126 to -0.105; and XY = -0.30 in discrete time ranges from -0.370 to -0.307.