Stochastic partial differential equations driven by colored noise

By

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Submitted to the graduate degree program in the Department of Mathematics and the Graduate Faculty of the University of Kansas in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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Date Defended: 4-14-15

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Stochastic partial differential equations driven by colored noise

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Acceptance date: 4-14-15

Abstract

This dissertation studies some problems for stochastic partial differential equations, in particular, (nonlinear) stochastic heat and stochastic wave equations, driven by (multiplicative) colored Gaussian noises. These problems considered are existence and uniqueness of the solution, Hölder continuity of the solution, Feynman-Kac formula for the solution, Feynman-Kac formula for the moments of the solution, Smoothness of the density of the solution as a random vectors at different spatial locations, intermittency (asymptotics for the high moments of the solution).

The dissertation is divided into six chapters. Chapter 1 provides a brief summary of this dissertation and Chapter 2 provides some brief background material on Gaussian processes and Malliavin calculus needed in this work.

In Chapter 3, we study the Hölder continuity of the stochastic wave equations in dimension three. This kind of topic has been studied in [91] for one dimensional case and in [30] for two dimensional case. In the three dimensional case, the fundamental solution of wave equation is not a positive function, but a measure supported in a sphere, this poses new difficulties. In [30], this problem is studied for the noise with specific space covariance functions, i.e., the Riesz kernel multiplied by a nice function. Our research deals with more general space covariance function, and give some general criteria to determine the order of the Hölder continuity of the solution. We also give several examples to show the applications of our results.

In Chapter 4, we study the stochastic heat equation with general multiplicative Gaussian noise. For the multiplication of the noise, we consider both Skorohod sense and Stratonovich sense. In the Skorohod case, we obtain the Feynman-Kac formula for the moments of the solution, and in Stratonovich sense, we get the Feynman-Kac formula for the solution. These formulas are used to get some sharp exponential bounds of the moment of the solution. We also study the solution to these equations using rough path theory. The Feynman-Kac representation of the solution to such equations are first studied in [56] for space time fractional noise, our research extends that case to general Gaussian noises.

In Chapter 5, we study the smoothness of the density of the solutions to some stochastic partial differential equations. Using the techniques of Malliavin calculus we derive the smoothness of the density of the solution at a fixed number of points $(t,x_1),\dots,(t,x_n)$, with some suitable regularity and non degeneracy assumptions. We also prove that the density is strictly positive in the interior of the support of the law. In the end of this chapter we will give some examples to show the applications of our results.

In Chapter 6 we study the one-dimensional stochastic heat equation driven by a Gaussian noise which is white in time and which has the covariance of a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{4}, \frac{1}{2})$ in the space variable. When $H < \frac{1}{2}$, there is no positive covariance function for the noise, so the classical theory developed in [24, 31] does not apply. To overcome this difficulty, we use the fractional derivative and some Sobolev space techniques to prove the existence and uniqueness of the solution under some conditions. We also get the moment bound and the Hölder continuity property of the solution. For a specific case, namely, the Anderson model, we also get the moment formula for the solution and the sharp exponential bound for the moment of the solution.

Acknowledgements

I want to take this opportunity to express my gratitude to all the people who have helped me during my Ph.D study, and because of whom my graduate study at the University of Kansas has been a great time in my life.

My deepest gratitude is to my advisors Professor Yaozhong Hu and Professor David Nualart. They introduced me to the area of stochastic analysis which I know almost nothing at the beginning, and then guided and helped me through my study with great patience. Without their help, nothing in this dissertation would come true. Also, I am deeply indebted to Yaozhong and his wife Jun Fu for taking care of me like parents.

I am grateful to Professor Jin Feng, Professor Atanas Stefanov and Professor Jianbo Zhang for their help and suggestions in the past years and for serving in my committee.

I gratefully acknowledge Professor William Fleissner, Professor Estela Gavosto, Professor Craig Huneke, Professor Yasuyuki Kachi, Professor Bozenna Pasik-Duncan, Professor Jack Porter, Professor Albert Sheu, Professor Atanas Stefanov and Professor Erik Van Vleck for their teaching, valuable advice and support. Many thanks are also due to Professor Jeremy Martin, Professor Ingrid Peterson, Professor Judith Roitman for their supervision on my teaching.

I also thank Professor Samy Tindel, with whom we worked on two research projects during his stay here and I benefited a lot from discussing with him.

This dissertation can not be made easily without the efforts of many staffs in the department of mathematics. Especially, I am thankful to Kerrie Brecheisen, Debbie Garcia, Gloria Prothe, Samantha Reinblatt and Lori Springs for their help in many ways.

I would like to thank all my friends who provided so much help and encouragement. Specially, I would like to thank Le Chen, Zheng Han, Jarod Hart, Guannan Hu, Khoa Le, Peter Lewis, Xianping Li, Yanghui Liu, Fei Lu, Billy Sanders, Grant Serio, Jian Song, Xiaoming Song, Guanlin Zhang, Mingji Zhang and Hongjuan Zhou, for numerous mathematical discussions and lectures on related topics that helped me improve my knowledge.

Finally, I would like to thank all the people who have helped me during my study and living at the University of Kansas.

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Chapter 1

Introduction

This dissertation concerns with several topics in stochastic partial differential equations, including (nonlinear) stochastic heat equations, stochastic wave equations, driven by (multiplicative) colored Gaussian noises. The topics contains the existence and uniqueness of the solution, Feynman-Kac formula for the moments of the solution, Feynman-Kac formula for the solution, continuity of the solution, smoothness of the probability density of the solution as a random vector, asymptotic behavior and related properties such as intermittency and chaos.

The dissertation consists of four research articles, jointly with with my advisors Yaozhong Hu and David Nualart, together with my collaborators Khoa Le, Xiaobin Sun, and Samy Tindel. They are listed as follows:

(1) (included in Chapter 3)

Y. Hu, J. Huang, D. Nualart: On Hölder continuity of the solution of stochastic wave equations in dimension three, Stoch. Partial Differ. Equ. Anal. Comput., **2**, (2014), 353-407.

(2) (included in Chapter 4)

Y. Hu, J. Huang, D. Nualart, S. Tindel: Stochastic heat equations with general multiplicative Gaussian noises: Hölder continuity and intermittency. Revised for Electronic Journal of Probability.

(3) (included in Chapter 5)

Y. Hu, J. Huang, D. Nualart, X. Sun: Smoothness of the joint density for spatially homogeneous SPDEs. Accepted by Journal of the Mathematical Society of Japan.

(4) (included in Chapter 6)

Y. Hu, J. Huang, K. Le, D. Nualart, S. Tindel: Stochastic heat equations with rough dependence in space. Preprint.

Next chapter (Chapter 2) will provide some brief background material for Gaussian process and Malliavin calculus. A list of references is provided at the end of the dissertation.

Next we explain the contents of each of the above papers in more detail.

Due to the irregularity of the paths of Gaussian noises, the solution to stochastic partial differential equations (abbreviated as SPDEs) are usually continuous (with some modification) but not smooth, thus the order of Hölder continuity is an important topic for research when it comes to understanding the fine behavior of the process. They can also be used in order to understand the behavior of extreme values of the process, as well as in determining potential theoretical properties of the process. Some research has been done for the Hölder continuity of the solution to heat equations and wave equations in dimensions one and two, see [91, 86, 69]. In the above listed first paper (Chapter 3), the wave equation in dimension three for dimension three is considered.

Namely, we consider

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) u(t,x) = \sigma(t,x,u(t,x))\dot{W}(t,x) + b(t,x,u(t,x))$$

with some initial conditions. Here σ and b are Lipschitz continuous with some growth assumptions. The centered Gaussian noise $\dot{W}$ is assumed to be white in time and with a homogeneous correlation in space and can be informally written as

$$E\left[\dot{W}(t,x)\dot{W}(s,y)\right] = \delta(s-t)f(x-y)\,.$$

Here f is a positive and positive definite function and δ is the Dirac function. The major difficulty is that the fundamental solution to this equation is not a function, but a measure supported on a sphere(see [24]). Our approach to this problem is partly inspired by [30], in which the authors consider a special case where the covariance function f is a nice function multiplied by a Riesz kernel and use fractional derivatives and Sobolev embedding theorem. By computing the convolution of the fundamental solution with itself, using Fourier transform method and the specific structure of the fundamental solution, we are able to obtain some general criteria for the Hölder continuity of the solution, both in space and time, for a large class of covariance functions f. Based on these general criteria, we recover the results proved in [30] and simplify some of their proofs.

Although we are able to give some general results regarding the Hölder continuity of the solution, some further research still remains to be done. For example, although we are able to give some general criterion for the Hölder continuity of the solution, both in the space and time variable, the criterion for the time variable is not as simple as for the space variable. This is due to the fact that the fundamental solution of the wave equation does not enjoy shifting properties for the time variable as for the space variable. So one direction for future research is to obtain a simple criterion for the Hölder continuity of the time variable. As an example, for the heat equation a simple and nice criterion for the Hölder continuity of the solution, both for the time and the space, is obtained in [86]. [91] and [69] give simple criteria for the stochastic wave equations in dimensions one and two, respectively. Our goal is to get such a simple criterion for the wave equation in dimension three.

Another direction for future research is to extend our result to higher dimensions. In [19], the authors studied the stochastic wave equation in dimension $d \geq 4$. Since in this case, the fundamental solution to the wave equation is no longer a positive Schwartz distribution (see [34]), some further generalization of the stochastic integral is required. In that work, they consider such an equation with vanishing initial condition. They are able to prove the moment bound and Hölder continuity of the solution, in the special case when σ is affine, b is zero and the covariance function f is Riesz kernel. One direction for future research is to consider the general Lipschitz functions σ and b, and general covariance function f. Another direction of future research is to consider the equation with non-vanishing initial conditions but a new construction of the stochastic integral may be needed.

The above listed second paper (Chapter 4) is concerned with the d-dimensional ($d \geq 1$) parabolic Anderson model with general multiplicativeGaussian noises (see [53]), i.e.

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u\dot{W} \tag{0.1}$$

with some bounded initial condition $u_0(x)$. Where $\dot{W}$ is a mean zero Gaussian noise with some general time and spatial covariance structure, that is

$$E(\dot{W}(s,x)\dot{W}(t,y)) = \gamma(s-t)\Lambda(x-y),$$

where γ and Λ are some positive and positive definite functions. We also assume that the Fourier transform of Λ is a tempered measure μ. In the recent past there has been a widespread interest in this model since it arises in several important areas, for example, the homogenization problems for PDEs driven by highly oscillating stationary random fields ([41, 47, 57]), the KPZ growth model through the Cole-Hopf's transform ([46, 79]).

We consider the solution in two senses, namely, the Skorohod sense and Stratonovich sense.

Equation in Skorohod sense

This case was studied by [55] when the noise is a fractional Brownian motion with Hurst parameter $H \geq \frac{1}{2}$ in time and a standard Brownian motion in space. Our paper [53] extends their results to the general covariance functions. Using chaos expansion, we are able to prove the existence and uniqueness of the solution to equation (0.1) under the condition that γ is locally integrable and that

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty. \tag{0.2}$$

Condition (0.2) is usually referred as Dalang's condition. This condition guarantees the existence and uniqueness of the solution to a large class of SPDEs when the noise is white in time, see [24]. Using a regularization technique, we are also able to give a formula for the nth moment of the solution $E(u(t,x)^n)$:

$$E(u(t,x)^n) = E_B\left(\prod_{i=1}^{n} u_0(B_t^i + x)\exp\left(\sum_{1\le i<j\le n}\int_0^t\int_0^t \gamma(s-r)\Lambda(B_s^i - B_r^j)dsdr\right)\right),$$

here B^i are independent standard Brownian motions which are independent of W. The chaos expansion and the moment bounds can be used to get some sharp exponential moment bounds for the solution.

Equation in the Stratonovich sense

In this case, if we assume some stronger conditions than (0.2), we are able to give a Feynman-Kac formula for the solution:

$$u(t,x) = E_B\left(\exp\left(u_0(B_t + x)\int_0^t\int_{\mathbb{R}^d} \delta_0(B_{t-r}^x - y)W(dr,dy)\right)\right).$$

We can obtain some Hölder continuity of the solution using this formula. The Feynman-Kac representation is originally studied in [56] when the noise is fractional in both time and space. Our result extends their result to the general covariance. However, the uniqueness of the solution is open. Using the moment bound results from the previous case, i.e., the equation in Skorohod sense, and the results in [18], we are also able to obtain the sharp lower and upper moment bound of the solution, for some specific choices of the covariance functions.

In this project, some questions are still open. For example, one open problem is to prove or disprove the uniqueness of the solution to equation (0.1), interpreted in Stratonovich sense. The difficulty of the problem comes from the construction of Stratonovich integral, which does not have an isometric property as the Itô integral.

In [21], the authors estimated the speed of propagation for the farthest peaks of the solution, when the dimension is 1 and the noise is white in space and time. Using our moment formula for the solution, we hope to get some similar results for the time and space correlated noise case. This topic remains to be a direction for future research.

The above listed third paper (Chapter 5) deals with the smoothness and positivity of the joint probability density at a fixed number of space locations. To be more precisely, consider the SPDE

$$Lu(t,x) = b(u(t,x)) + \sigma(u(t,x))\dot{W}(t,x)\,,$$

$t \geq 0$, $x \in \mathbb{R}^d$, with vanishing initial conditions, where L denotes a second order partial differential operator. The noise $\dot{W}$ is white in time and with some correlation in space and the spatial correlation f satisfies some integrability conditions. This is a generalization of the previous work [69]. Using the techniques of Malliavin calculus as developed in [11, 72], we derived the smoothness of the density of the solution at a fixed number of different points $(t,x_1),\dots,(t,x_n)$ and also the positivity of the density in the support of the law of the random vector $(u(t,x_1),u(t,x_2),\dots,u(t,x_n))$.

In the above listed fourth paper (Chapter 6), we study the one dimensional stochastic heat equation of the following form:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + \sigma(u)\dot{W}\,, \tag{0.3}$$

where W is a centered Gaussian process with covariance given by

$$E[W(s,x)W(t,y)] = \frac{1}{2}(|x|^{2H} + |y|^{2H} - |x-y|^{2H})(s \wedge t)\,,$$

with $\frac{1}{4} < H < \frac{1}{2}$. That is, W is a standard Brownian motion in time and a fractional Brownian motion with Hurst parameter H in the space variable. Since the spatial covariance function for this noise is not locally integrable, the standard methodology used in the classical references [24, 29, 31, 75, 91] does not apply to this case. This research is intended to fill this gap. Since the covariance of two stochastic integrals with respect to $\dot{W}$ is expressed in terms of fractional derivatives, we need to use some Hölder norm to deal with the solution. For the uniqueness of the solution, we apply some factorization and stopping time arguments while for the existence of the solution, we use some tightness arguments. Right now we are only able to deal with the initial condition which is in $L^p(\mathbb{R})$, to consider more general initial conditions needs further research. Some other directions for future research are listed as follows.

1. We want to study some chaotic property of the solution as done in [23, 20, 21, 17], that is, a change of the initial condition may result a totally different behavior of the solution.

2. We may try to study using Malliavin calculus, under which condition the solution has a smooth probability density.

3. We may study the equation in a finite interval, with some boundary conditions. We may study the evolution of the energy of the solution, as done in [59, 35].

4. We may also consider other types of equations, for example, the wave equation or more general spatial differential operators, driven by the same noise.

Chapter 2

Preliminaries

We introduce some basic elements of Gaussian analysis and Malliavin calculus, for which we refer to [74] for further details.

2.1 Isonormal Gaussian process and multiple integrals

Let $\mathscr{H}$ be a real separable Hilbert space (with its inner product and norm denoted by $\langle \cdot, \cdot \rangle_{\mathscr{H}}$ and $\|\cdot\|_{\mathscr{H}}$, respectively). For any integer $q \geq 1$, let $\mathscr{H}^{\otimes q}(\mathscr{H}^{\odot q})$ be the qth tensor product (symmetric tensor product) of $\mathscr{H}$. Let $X = \{X(h), h \in \mathscr{H}\}$ be an isonormal Gaussian process associated with the Hilbert space $\mathscr{H}$, defined on a complete probability space $(\Omega, \mathscr{F}, P)$. That is, X is a centered Gaussian family of random variables such that $E[X(h)X(g)] = \langle h, g \rangle_{\mathscr{H}}$ for all $h, g \in \mathscr{H}$.

For every integer $q \geq 0$, the *qth Wiener chaos* (denoted by $\mathscr{H}_q$) of X is the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\left\{H_q(X(h)) : h \in \mathscr{H}, \|h\|_{\mathscr{H}} = 1\right\}$, where H_q is the qth *Hermite polynomial* recursively defined by $H_0(x) = 1$, $H_1(x) = x$ and

$$H_{q+1}(x) = xH_q(x) - qH_{q-1}(x), \quad q \geq 1. \tag{1.1}$$

For every integer $q \geq 1$, the mapping $I_q(h^{\otimes q}) = H_q(X(h))$, where $\|h\|_{\mathscr{H}} = 1$, can be extended to a linear isometry between $\mathscr{H}^{\odot q}$ (equipped with norm $\sqrt{q!}\,\|\cdot\|_{\mathscr{H}^{\otimes q}}$) and $\mathscr{H}_q$ (equipped with $L^2(\Omega)$ norm). For $q = 0$, $\mathscr{H}_0 = \mathbb{R}$, and I_0 is the identity map.

It is well-known (Wiener chaos expansion) that $L^2(\Omega)$ can be decomposed into the infinite orthogonal sum of the spaces $\mathscr{H}_q$. That is, any random variable $F \in L^2(\Omega)$ has the following chaotic expansion:

$$F = \sum_{q=0}^{\infty} I_q(f_q), \tag{1.2}$$

where $f_0 = E[F]$, and $f_q \in \mathscr{H}^{\odot q}, q \geq 1$, are uniquely determined by F. For every $q \geq 0$ we denote by J_q the orthogonal projection on the qth Wiener chaos $\mathscr{H}_q$, so $I_q(f_q) = J_q F$.

2.2 Malliavin operators

We introduce some basic facts on Malliavin calculus with respect to the Gaussian process X. Let $\mathscr{S}$ denote the class of smooth random variables of the form $F = f(X(h_1),\dots,X(h_n))$, where $h_1,\dots,h_n$ are in $\mathscr{H}$, $n \geq 1$, and $f \in C_p^{\infty}(\mathbb{R}^n)$, the set of smooth functions f such that f itself and all its partial derivatives have at most polynomial growth. Given $F = f(X(h_1),\dots,X(h_n))$ in $\mathscr{S}$, its Malliavin derivative DF is the $\mathscr{H}$–valued random variable given by

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X(h_1),\dots,X(h_n))h_i.$$

The derivative operator D is a closable and unbounded operator on $L^2(\Omega)$ taking values in $L^2(\Omega;\mathscr{H})$. By iteration one can define higher order derivatives $D^k F \in L^2(\Omega;\mathscr{H}^{\odot k})$. For any integer $k \geq 0$ and any $p \geq 1$ and we denote by $\mathbb{D}^{k,p}$ the closure of $\mathscr{S}$ with respect to the norm $\|\cdot\|_{k,p}$ given by:

$$\|F\|_{k,p}^p = \sum_{i=0}^{k} E(\left|\left|D^i F\right|\right|_{\mathscr{H}^{\otimes i}}^p).$$

For $k = 0$ we simply write $\|F\|_{0,p} = \|F\|_p$. For any $p \geq 1$ and $k \geq 0$, we set $\mathbb{D}^{\infty,p} = \cap_{k\geq 0}\mathbb{D}^{k,p}$ and $\mathbb{D}^{k,\infty} = \cap_{p\geq 1}\mathbb{D}^{k,p}$.

We denote by δ (the *divergence* operator) the adjoint operator of D, which is an unbounded operator from a domain in $L^2(\Omega;\mathscr{H})$ to $L^2(\Omega)$. An element $u \in L^2(\Omega;\mathscr{H})$ belongs to the domain of δ if and only if it verifies

$$|E[\langle DF, u\rangle_{\mathscr{H}}]| \leq c_u\sqrt{E[F^2]}$$

for any $F \in \mathbb{D}^{1,2}$, where c_u is a constant depending only on u. In particular, if $u \in$ Dom δ, then $\delta(u)$ is characterized by the following duality relationship

$$E(\delta(u)F) = E(\langle DF, u\rangle_{\mathscr{H}}) \tag{2.1}$$

for any $F \in \mathbb{D}^{1,2}$.

We can factor out a scalar random variable in the divergence in the following sense. Let $F \in \mathbb{D}^{1,2}$ and $u \in$ Dom δ such that $Fu \in L^2(\Omega;\mathscr{H})$. Then $Fu \in$ Dom δ and

$$\delta(Fu) = F\delta(u) - \langle DF, u\rangle_{\mathscr{H}}, \tag{2.2}$$

provided the right hand side is square integrable. The operators δ and D have the following commutation relationship

$$D\delta(u) = u + \delta(Du) \tag{2.3}$$

for any $u \in \mathbb{D}^{2,2}(\mathscr{H})$ (see [74, page 37]).

Chapter 3

Hölder continuity for stochastic wave equations in dimension three

In this chapter, we study the stochastic wave equations in the three spatial dimensions driven by a Gaussian noise which is white in time and correlated in space. Our main concern is the sample path Hölder continuity of the solution both in time variable and in space variables. The conditions are given either in terms of the mean Hölder continuity of the covariance function or in terms of its spectral measure. Some examples of the covariance functions are proved to satisfy our conditions, which include the case of the work [30]. In particular, we obtain the Hölder continuity results for the solution of the stochastic wave equations driven by (space inhomogeneous) fractional Brownian noises. For this particular noise, the optimality of the obtained Hölder exponents is also discussed.

3.1 Introduction

We shall study the following stochastic wave equation in spatial dimension $d = 3$:

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2} - \Delta\right) u(t,x) = \sigma(t,x,u(t,x))\dot{W}(t,x) + b(t,x,u(t,x)), \\ \\ u(0,x) = v_0(x), \qquad \frac{\partial u}{\partial t}(0,x) = \bar{v}_0(x), \end{cases} \tag{1.1}$$

where $t \in (0,T]$ for some fixed $T > 0$, $x \in \mathbb{R}^3$ and $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ denotes the Laplacian on $\mathbb{R}^3$. The coefficients σ and b satisfy some regularity conditions which will be specified later. The Gaussian noise process $\dot{W}$ is assumed to be white in time and with a homogeneous correlation in space. This can be informally written as

$$\mathbb{E}\left[\dot{W}(t,x)\dot{W}(s,y)\right] = \delta(t-s)f(x-y)$$

for a non-negative, non-negative definite and locally integrable function f, where δ is the Dirac delta function. We will explain in Section 2 how this expression can be made formal.

It is known (see, for instance, [31, Theorem 4.3]) that if σ and b are Lipschitz functions with linear growth and f satisfies $\int_{|x|\leq 1} f(x)/|x| dx < \infty$, then there is a unique mild solution to Equation (1.1). Our purpose is to establish the sample path Hölder continuity both in time variable and in space variables of the solution to this equation. When f is given by a Riesz kernel $|x|^{-\beta}$, $\beta \in (0,2)$, the Hölder continuity of the solution has been obtained by Dalang and Sanz-Solé in their monograph [30]. Their approach is based on the fractional Sobolev imbedding theorem and the Fourier transformation technique.

In this chapter, we shall consider more general Gaussian noises, and we introduce a new approach that avoids the Fourier transform. The main idea is to impose conditions on the covariance f itself. To be more precise, let $D_w f = f(\cdot + w)$ be the shift operator. We shall show that if $\|D_w f - f\|_{L^1(\rho)} \leq C|w|^\gamma$ and $\|D_w f + D_{-w} f - 2f\|_{L^1(\rho)} \leq C|w|^{\gamma'}$ for some $\gamma \in (0,1]$ and $\gamma' \in (0,2]$, where ρ is the measure on $\mathbb{R}^3$ defined to be $\rho(dz) = \mathbf{1}_{\{|z| \leq 2T\}} \frac{1}{|z|} dz$, then the solution to (1.1) is locally Hölder continuous of order $\kappa < \min(\gamma, \frac{\gamma'}{2})$ in the space variable (assuming zero initial conditions) (see Theorem 3.5).

The Hölder continuity in the time variable is more involved. Following the methodology used by Dalang and Sanz-Solé in [30], we transform the time increments into space increments, and we impose suitable assumptions on the modulus of continuity of a shift operator which are formulated integrals over $[0,T] \times (S^2)^2$, equipped with the measure $ds\sigma(d\xi)\sigma(d\eta)$, where σ is the uniform measure on the unit sphere S^2 (see Theorem 3.7).

We also obtain a theorem on the Hölder continuity in the space variable using the Fourier transform technique. More precisely, we establish the Hölder continuity of order $\kappa < \gamma$, provided the spectral measure μ satisfies the integrability condition $\int_{\mathbb{R}^3} \frac{\mu(d\zeta)}{1+|\zeta|^{2-2\gamma}} < \infty$ and the Fourier transform of $|\zeta|^{2\gamma}\mu(d\zeta)$ is non-negative. The non-negativity condition on this measure leads to a simple proof of the Hölder continuity in the space variable which avoids the control of the norms of the increments $D_w f - f$ and $D_w f + D_{-w} f - 2f$ (or their respective Fourier transforms). As an application, this method provides a direct proof of the Hölder continuity in the space variable, in the case of the Riesz kernel. However, at this moment we are not able to use this approach to handle the Hölder continuity in the time variable.

To illustrate the scope of our results we provide some examples of covariance functions f which satisfy our conditions. We consider first the Riesz and Bessel kernels.

Then we focus our attention to fractional noises with covariance function of the form

$$f(x) = |x_1|^{2H_1-2}|x_2|^{2H_2-2}|x_3|^{2H_3-2},$$

where $H_1, H_2, H_3 \in (1/2, 1)$ and $\bar{\kappa} := \sum_{i=1}^{3} H_i - 2$. We show (see Theorem 6.1) that, under suitable assumptions on the initial conditions, if $\kappa_i \in (0, \min(H_i - 1/2, \bar{\kappa}))$ and $\kappa_0 = \min(\kappa_1, \kappa_2, \kappa_3)$, then for any bounded rectangle $I \subset \mathbb{R}^3$, there is a finite random variable K, depending on the κ_i's, such that for all $s, t \in [0, T]$ and for all $x, y \in I$

$$|u(t,x) - u(s,y)| \leq K_I(|x_1 - y_1|^{\kappa_1} + |x_2 - y_2|^{\kappa_2} + |x_3 - y_3|^{\kappa_3} + |s - t|^{\kappa_0}).$$

To see if the Hölder exponents κ_i's are optimal or not, we investigate a simple linear stochastic wave equation with additive noise. That means, we consider the equation (1.1) with $v_0 = \bar{v}_0 = 0$, $b = 0$ and $\sigma = 1$. In this situation, we prove (see Theorem 6.2 and a Kolomogorov lemma) that for any bounded rectangle $I \subset \mathbb{R}^3$ and for any $\kappa \in (0, \bar{\kappa})$, there is a random variable $K_{\kappa,I}$ such that for alll $t, s \in [0, T]$ and for all $x, y \in I$

$$|u(t,x) - u(s,y)| \leq K_{\kappa,I}(|x_1 - y_1|^{\kappa} + |x_2 - y_2|^{\kappa} + |x_3 - y_3|^{\kappa} + |s - t|^{\kappa}).$$

On the other hand, we obtain in Theorem 6.2 a lower bound on the variance of the increments of the process u which shows that the exponent $\bar{\kappa}$ is optimal. Notice that in the nonlinear case (see Theorem 6.1), we need the extra conditions $\kappa_i < H_i - 1/2$ for $i = 1, 2, 3$. Also, this extra condition is not necessary if $H_i + H_j \leq 3/2$ for any $i \neq j$ (for instance, if $H_1 = H_2 = H_3 = H \leq 3/4$), and in this case κ_i coincides with the optimal constant $\bar{\kappa}$. It would be interesting to know if the additional conditions $\kappa_i < H_i - 1/2$ are due to the nonlinearity or due to the limitation of our technique.

This chapter is organized as follows. Section 2 contains some preliminary material about the noise process in Equation (1.1). We state our basic assumptions on the covariance function f and prove a general Burkholder inequality. We also give the definition of the mild solution and state the existence and uniqueness theorem of the solution to Equation (1.1). Section 3 contains two main results on the Hölder continuity in the space variables. One is based on the structure of the covariance function f itself and the other one uses the Fourier transform of f. In Section 4 we prove a criterion for the Hölder continuity in the time variable. Section 5 presents some examples of covariance functions f which satisfy the conditions given in our main theorems. In the first example, f is the convolution of a Schwartz function with a Riesz kernel. In the second example, f is the Riesz kernel, which is the case studied in [30]. In the third example, f is the Bessel kernel. Section 6 deals with the case when the noise process is the formal derivative of a fractional Brownian field. The optimality of the Hölder exponents is discussed in this section. Section 7 contains some lemmas which are used in the paper.

3.2 Preliminaries

Consider a non-negative and non-negative definite function f which is a tempered distribution on $\mathbb{R}^3$ (so f is locally integrable). We know that in this case f is the Fourier transform of a non-negative tempered measure μ on $\mathbb{R}^3$ (called the spectral measure of f). That is, for all φ belonging to the space $\mathscr{S}(\mathbb{R}^3)$ of rapidly decreasing C^∞ functions

$$\int_{\mathbb{R}^3} f(x)\varphi(x)dx = \int_{\mathbb{R}^3} \mathscr{F}\varphi(\xi)\mu(d\xi), \tag{2.1}$$

and there is an integer $m \geq 1$ such that

$$\int_{\mathbb{R}^3} (1+|\xi|^2)^{-m} \mu(d\xi) < \infty, \tag{2.2}$$

where we have denoted by $\mathscr{F}\varphi$ the Fourier transform of $\varphi \in \mathscr{S}(\mathbb{R}^3)$, given by

$$\mathscr{F}\varphi(\xi) = \int_{\mathbb{R}^3} \varphi(x) e^{-i\xi \cdot x} dx.$$

Let $G(t)$ be the fundamental solution of the 3-dimensional wave equation $\dfrac{\partial^2 u}{\partial t^2} = \Delta u$. That is

$$G(t) = \frac{1}{4\pi t} \sigma_t \tag{2.3}$$

for any $t > 0$, where σ_t denotes the uniform surface measure (with total mass $4\pi t^2$) on the sphere of radius $t > 0$. Sometimes it is more convenient for us to use the Fourier transform of G given by

$$\mathscr{F}G(t)(\xi) = \frac{\sin(t|\xi|)}{|\xi|}, \quad t > 0. \tag{2.4}$$

Our basic assumption on f is

$$\int_{|x| \leq 1} \frac{f(x)}{|x|} dx < \infty. \tag{2.5}$$

It turns out (see Lemma 3.16 and Equation (7.7) below) that this is equivalent to

$$\int_{\mathbb{R}^3} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty. \tag{2.6}$$

Notice that since we are in $\mathbb{R}^3$, condition (2.5) is satisfied if there is a $\kappa < 2$ such that in a neighborhood of 0, $f(x) \leq C|x|^{-\kappa}$.

The following identities will play an important role,

$$\begin{aligned}\frac{1}{8\pi}\int_{s-t\le|x|\le s+t}\frac{f(x)}{|x|}dx &= \int_{\mathbb{R}^3}[G(s)*G(t)](x)f(x)dx \\ &= \int_{\mathbb{R}^3}\mu(d\xi)\,(\mathscr{F}G(s))\,(\xi)\,(\mathscr{F}G(t))\,(\xi) \qquad (2.7)\end{aligned}$$

for $0<t\le s$. We refer to Lemma 3.16 and Lemma 3.17 for proofs of these two identities.

Fix a time interval $[0,T]$. Let $C_0^\infty([0,T]\times\mathbb{R}^3)$ be the space of infinitely differentiable functions with compact support on $[0,T]\times\mathbb{R}^3$. Consider a zero mean Gaussian family of random variables $W=\{W(\varphi),\varphi\in C_0^\infty([0,T]\times\mathbb{R}^3)\}$, defined in a complete probability space $(\Omega,\mathscr{F},P)$, with covariance

$$\mathbf{E}(W(\varphi)W(\psi))=\int_0^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\varphi(t,x)f(x-y)\psi(t,y)dxdydt. \qquad (2.8)$$

Walsh's classical theory of stochastic integration developed in [91] cannot be applied directly to the mild formulation of Equation (1.1) since G is not a function, but a measure. We shall use the stochastic integral defined in Section 2.3 of [31]. We briefly summarize the construction and properties of this integral.

Let U be the completion of $C_0^\infty(\mathbb{R}^3)$ endowed with the inner product

$$\langle\varphi,\psi\rangle_U=\int_{\mathbb{R}^3}dx\int_{\mathbb{R}^3}dy\varphi(x)f(x-y)\psi(y)=\int_{\mathbb{R}^3}\mathscr{F}(\varphi)(\xi)\overline{\mathscr{F}(\psi)(\xi)}\mu(d\xi), \qquad (2.9)$$

$\varphi,\psi\in C_0^\infty(\mathbb{R}^3)$. Set $U_T=L^2([0,T];U)$.

The Gaussian family W can be extended to the space U_T. We will also denote by $W(g)$ the Gaussian random variable associated with an element $g\in U_T$. Set $W_t(h)=W(\mathbf{1}_{[0,t]}h)$ for any $t\in[0,T]$ and $h\in U$. Then $W=\{W_t,t\in[0,T]\}$ is a cylindrical Wiener process in the Hilbert space U. That is, for any $h \in U$, $\{W_t(h), t \in [0,T]\}$ is a Brownian motion with variance $t\|h\|_U^2$, and

$$\mathbf{E}(W_t(h)W_s(g)) = (s \wedge t)\langle h, g\rangle_U.$$

Let $\mathscr{F}_t$ be the σ-field generated by the random variables $\{W_s(h), h \in U, 0 \leq s \leq t\}$ and the P-null sets. We define the predictable σ-field as the σ-field in $\Omega \times [0,T]$ generated by the sets $\{A \times (s,t], 0 \leq s < t \leq T, A \in \mathscr{F}_s\}$. Then we can define the stochastic integral of a U-valued square-integrable predictable process $g \in L^2(\Omega \times [0,T]; U)$ with respect to the cylindrical Wiener process W, denoted by

$$g \cdot W = \int_0^T \int_{\mathbb{R}^3} g(t,x) W(dt,dx),$$

and we have the isometry property

$$\mathbf{E}|g \cdot W|^2 = \mathbf{E}\int_0^T ||g(t)||_U^2 dt. \tag{2.10}$$

The following lemma provides a sufficient condition for a measure of the form $\varphi(x)G(t,dx)$ to be in the space U.

Lemma 3.1. *Consider a Borel measurable function $\varphi : \mathbb{R}^3 \to \mathbb{R}$, such that for some $t > 0$,*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |\varphi(x)\varphi(y)| G(t,dx)G(t,dy) f(x-y) < \infty. \tag{2.11}$$

Then, $\varphi G(t)$ belongs to U and

$$\|\varphi G(t)\|_U^2 = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \varphi(x)\varphi(y) G(t,dx)G(t,dy) f(x-y). \tag{2.12}$$

Furthermore, when φ is bounded,

$$\|\varphi G(t)\|_U^2 = \int_{\mathbb{R}^3} |\mathscr{F}(\varphi G(t))(\xi)|^2 \mu(d\xi). \tag{2.13}$$

Proof. Suppose first that φ is bounded. Then by Lemma 3.17, the equality (2.13) holds and $\int_{\mathbb{R}^3} |\mathscr{F}(\varphi G(t))(\xi)|^2\mu(d\xi) < \infty$. Let ψ be a nonnegative C^∞ function on $\mathbb{R}^3$ supported in the unit ball such that $\int_{\mathbb{R}^3} \psi(x)dx = 1$. Define $\psi_n(x) = n^3\psi(nx)$, so

$$(\psi_n * (\varphi G(t)))(x) := \int_{\mathbb{R}^3} \psi_n(x-y)\varphi(y)G(t,dy)$$

is in $C_0^\infty(\mathbb{R}^3)$, and we have

$$\begin{aligned} & \int_{\mathbb{R}^3} |\mathscr{F}(\psi_n * (\varphi G(t)) - \varphi G(t))|^2 \mu(d\xi) \\ = & \int_{\mathbb{R}^3} |(\mathscr{F}\psi_n)(\xi) - 1|^2 |\mathscr{F}(\varphi G(t))(\xi)|^2 \mu(d\xi) \to 0 \end{aligned}$$

as $n \to \infty$, by the dominated convergence theorem. This implies that $\varphi G(t)$ is in U, and (2.12) holds.

In the general case, we consider the sequence of functions $\varphi_k(x) = \varphi(x)\mathbf{1}_{\{|\varphi|\le k\}}$. Then $\varphi_k(x)G(t,dx)$ belongs to U, and

$$\begin{aligned} & \|\varphi_k(x)G(t,dx) - \varphi(x)G(t,dx)\|_U^2 \\ \le & \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |\varphi_k(x) - \varphi(x)||\varphi_k(y) - \varphi(y)|G(t,dx)G(t,dy)f(x-y), \end{aligned}$$

which clearly goes to 0 as k goes to infinity, by the dominated convergence theorem.

□

For any $x \in \mathbb{R}^3$ we denote by $G(t,x-dy)$ the shifted measure $A \mapsto G(t,x-A)$. Clearly Lemma 3.1 holds if we replace the kernel $G(t,dy)$ by the shifted kernel $G(t,x-dy)$. Applying Lemma 3.1, we immediately get the following Burkholder inequality.

Lemma 3.2. *Let $Z = \{Z(t,x),(t,x) \in [0,T] \times \mathbb{R}^3\}$ be a predictable process such that for some $p \geq 2$ and $x \in \mathbb{R}^3$,*

$$\mathbf{E}\left(\int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |Z(s,x-y)Z(s,x-z)|G(s,dy)G(s,dz)f(y-z)ds\right)^{\frac{p}{2}} < \infty.$$

Then the measure-valued predictable process $Z(s,y)G(s,x-dy)$ belongs $L^2(\Omega \times [0,T];U)$ and there exists a positive constant C_p, depending only on p, such that

$$\begin{aligned} &\mathbf{E}\left|\int_0^t \int_{\mathbb{R}^3} Z(s,y)G(s,x-dy)W(ds,dy)\right|^p \\ \leq\ & C_p \mathbf{E}\left(\int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} Z(s,x-y)Z(s,x-z)G(s,dy)G(s,dz)f(y-z)ds\right)^{\frac{p}{2}}. \end{aligned}$$

If we have

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^3} \mathbf{E}|Z(t,x)|^p < \infty, \tag{2.14}$$

then an application of Hölder inequality yields

$$\begin{aligned} &\mathbf{E}\left|\int_0^t \int_{\mathbb{R}^3} Z(s,y)G(s,x-dy)W(ds,dy)\right|^p \\ \leq\ & C_p \int_0^t ds \left(\sup_{x\in\mathbb{R}^3} \mathbf{E}|Z(s,x)|^p\right)\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(y-z)G(s,dy)G(s,dz)\right)^{\frac{p}{2}}. \end{aligned}$$

By Lemma 3.16, the above inequality can also be written as

$$\mathbf{E}\left|\int_0^t \int_{\mathbb{R}^3} Z(s,y)G(s,x-dy)W(ds,dy)\right|^p$$

$$\leq C_p \int_0^t ds \left(\sup_{x\in\mathbb{R}^3} \mathbf{E}|Z(s,x)|^p \right) \left(\int_{|x|\leq 2s} \frac{f(x)}{|x|} dx \right)^{\frac{p}{2}}.$$

Using the above notion of stochastic integral one can introduce the following definition:

Definition 3.3. *A real-valued predictable stochastic process* $u = \{u(t,x), 0 \leq t \leq T, x \in \mathbb{R}^3\}$ *is a mild random-field solution of (1.1) if for all* $t \in (0,T]$, $x \in \mathbb{R}^3$,

$$\begin{aligned} u(t,x) &= \frac{d}{dt}\left(G(t) * v_0\right)(x) + \left(G(t) * \bar{v}_0\right)(x) \\ &+ \int_0^t \int_{\mathbb{R}^3} G(t-s, x-dy)\sigma(s,y,u(s,y))W(ds,dy) \\ &+ \int_0^t G(t-s) * \left(b(s,\cdot,u(s,\cdot))\right)(x) ds \quad a.s. \end{aligned}$$

Consider the following condition.

(**H**) The coefficients σ and b satisfy

$$\begin{aligned} |\sigma(t,x,u) - \sigma(t,y,v)| &\leq C(|x-y| + |u-v|) \\ |\sigma(s,x,u)| &\leq C(1+|u|) \end{aligned}$$

and

$$\begin{aligned} |b(t,x,u) - b(t,y,v)| &\leq C(|x-y| + |u-v|) \\ |b(s,x,u)| &\leq C(1+|u|) \end{aligned}$$

for any $x, y \in \mathbb{R}^3$, $s, t \in [0,T]$ and $u, v \in \mathbb{R}$.

Then one can prove the existence and uniqueness of the solution to (1.1) in exactly the same way as in [31, Theorem 4.3].

Theorem 3.4. *Suppose the condition (2.5) holds, and σ, b satisfy the condition (**H**). Let $v_0 \in C^1(\mathbb{R}^3)$ such that v_0 and ∇v_0 are bounded and $\bar{v}_0$ is bounded and continuous. Then there exists a unique mild random-field solution u to (1.1) such that for all $p \geq 1$,*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^3} \mathbf{E}|u(t,x)|^p < \infty. \tag{2.15}$$

Along the paper, C will denote a generic constant which may change from line to line.

3.3 Hölder continuity in the space variable

In this section we will prove Theorems 3.1 and 3.2 which are the main results on the Hölder continuity of the solution of Equation (1.1) in the space variable.

Theorem 3.5. *Let u be the solution to Equation (1.1). Assume the following conditions.*

*(a) The coefficients σ and b satisfy condition (**H**).*

(b) $v_0 \in C^2(\mathbb{R}^3)$, v_0, ∇v_0 and $\bar{v}_0$ are bounded and Δv_0 and $\bar{v}_0$ are Hölder continuous of orders γ_1 and γ_2 respectively, $\gamma_1, \gamma_2 \in (0,1]$.

(c) The function f satisfies condition (2.5) and for some $\gamma \in (0,1]$ and $\gamma' \in (0,2]$ we have for all $w \in \mathbb{R}^3$ such that $|w| \leq 1$

$$\int_{|z|\leq 2T} \frac{|f(z+w)-f(z)|}{|z|} dz \leq C|w|^{\gamma} \tag{3.1}$$

and

$$\int_{|z|\leq 2T} \frac{|f(z+w)+f(z-w)-2f(z)|}{|z|} dz \leq C|w|^{\gamma'}. \tag{3.2}$$

Set $\kappa_1 = \min(\gamma_1, \gamma_2, \gamma, \frac{\gamma'}{2})$. *Then for any* $q \geq 2$, *there exists a constant* C *such that*

$$\sup_{t\in[0,T]} \mathbf{E}|u(t,x)-u(t,y)|^q \leq C|x-y|^{q\kappa_1}$$

for any $x,y \in \mathbb{R}^3$.

Proof. It suffices to assume that $|x-y| \leq 1$. Set $x-y=w$. Fix $q \geq 2$. Then we have

$$\begin{aligned}
\mathbf{E}\,|u(t,x)-u(t,y)|^q \quad &\leq \quad C\Bigg\{\mathbf{E}\Big|\int_0^t\int_{\mathbb{R}^3} G(t-s,x-dz)\sigma\,(s,z,u(s,z))\,W(ds,dz) \\
&\qquad -\int_0^t\int_{\mathbb{R}^3} G(t-s,y-dz)\sigma\,(s,z,u(s,z))\,W(ds,dz)\Big|^q \\
&\qquad +\mathbf{E}\Big|\int_0^t G(t-s)*b\,(s,\cdot,u(s,\cdot))\,(x)ds \\
&\qquad -\int_0^t G(t-s)*b\,(s,\cdot,u(s,\cdot))\,(y)ds\Big|^q \\
&\qquad +\left|\frac{d}{dt}\,(G(t)*v_0)\,(x)-\frac{d}{dt}\,(G(t)*v_0)\,(y)\right|^q \\
&\qquad +|(G(t)*\bar{v}_0)\,(x)-(G(t)*\bar{v}_0)\,(y)|^q\Bigg\} \\
&:= \quad C(I_1+I_2+I_3+I_4).
\end{aligned}$$

For I_4, since $\bar{v}_0$ is Hölder continuous with exponent γ_2 we get

$$\begin{aligned}
|G(t)*\bar{v}_0(x)-G(t)*\bar{v}_0(y)|^q \quad &\leq \quad \left|\int_{\mathbb{R}^3} G(t,dz)\,|\bar{v}_0(x-z)-\bar{v}_0(y-z)|\right|^q \\
&\leq \quad C|w|^{\gamma_2 q}\left|\int_{\mathbb{R}^3} G(t,dz)\right|^q \leq C|w|^{\gamma_2 q}. \qquad (3.3)
\end{aligned}$$

For I_3, we use the identity (see, for instance, [88])

$$\frac{d}{dt}G(t)*v_0(x) = \frac{1}{t}(v_0*G(t))(x)+\frac{1}{4\pi}\int_{|y|<1}(\Delta v_0)(x+ty)dy. \qquad (3.4)$$

Then, since Δv_0 is Hölder continuous with exponent γ_1, we get

$$
\begin{aligned}
& \left| \frac{d}{dt} \left(G(t) * v_0 \right)(x) - \frac{d}{dt} \left(G(t) * v_0 \right)(y) \right|^q \\
\leq\ & \frac{C}{t^q} \left| \int_{\mathbb{R}^3} G(t,dz) \left(v_0(x-z) - v_0(y-z) \right) \right|^q \\
& + C \left| \int_{|z|<1} \left(\Delta v_0(x+tz) - \Delta v_0(y+tz) \right) dz \right|^q \\
\leq\ & C|w|^{\gamma_1 q}. \qquad (3.5)
\end{aligned}
$$

For I_2, we use the Lipschitz condition on b and Hölder's inequality to get

$$
\begin{aligned}
I_2 \ =\ & \mathbf{E} \Big| \int_0^t \int_{\mathbb{R}^3} G(t-s,dz) b\left(s, x-z, u(s,x-z)\right) ds \\
& - \int_0^t \int_{\mathbb{R}^3} G(t-s,dz) b\left(s, y-z, u(s,y-z)\right) ds \Big|^q \\
\leq\ & C\mathbf{E} \left(\int_0^t \int_{\mathbb{R}^3} G(t-s,dz) \left(|w| + |u(s,x-z) - u(s,y-z)| \right) ds \right)^q \\
\leq\ & C\mathbf{E} \left(\int_0^t \int_{\mathbb{R}^3} G(t-s,dz) ds \right)^{q-1} \left(\int_0^t \int_{\mathbb{R}^3} G(t-s,dz) |w|^q ds \right. \\
& \left. + \int_0^t \int_{\mathbb{R}^3} G(t-s,dz) \left| u(s,x-z) - u(s,y-z) \right|^q ds \right) \\
\leq\ & C|w|^q + C \int_0^t ds \sup_{z_1 - z_2 = w} \mathbf{E} \left| u(s,z_1) - u(s,z_2) \right|^q . \qquad (3.6)
\end{aligned}
$$

For I_1, we apply the Burkholder's inequality of Lemma 3.2 to get

$$
\begin{aligned}
I_1 \ \leq\ & C\mathbf{E} \Big| \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sigma\left(s, \xi, u(s,\xi)\right) f(\xi - \eta) \sigma\left(s, \eta, u(s,\eta)\right) \\
& \times \left(G(t-s, x-d\xi) - G(t-s, y-d\xi) \right) \left(G(t-s, x-d\eta) - G(t-s, y-d\eta) \right) ds \Big|^{\frac{q}{2}} \\
:=\ & C\mathbf{E}|Q|^{\frac{q}{2}}.
\end{aligned}
$$

The main idea to estimate the above quantity is to transfer the increments of G to increments of f and σ. We introduce the following notation

$$\Sigma_x(s,\xi) = \sigma(s,x-\xi,u(s,x-\xi)) \tag{3.7}$$

$$\Sigma_{x,y}(s,\xi) = \sigma(s,x-\xi,u(s,x-\xi)) - \sigma(s,y-\xi,u(s,y-\xi)). \tag{3.8}$$

Define

$$h_1 = f(\eta-\xi)\Sigma_{x,y}(s,\xi)\Sigma_{x,y}(s,\eta), \tag{3.9}$$

$$h_2 = (f(\eta-\xi+w)-f(\eta-\xi))\Sigma_x(s,\xi)\Sigma_{x,y}(s,\eta), \tag{3.10}$$

$$h_3 = (f(\eta-\xi-w)-f(\eta-\xi))\Sigma_x(s,\eta)\Sigma_{x,y}(s,\xi), \tag{3.11}$$

$$h_4 = (2f(\eta-\xi)-f(\eta-\xi+w)-f(\eta-\xi-w))\Sigma_x(s,\xi)\Sigma_x(s,\eta). \tag{3.12}$$

and

$$Q_i = \int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G(t-s,d\xi)G(t-s,d\eta)h_i ds, \quad i=1,2,3,4. \tag{3.13}$$

Then by direct calculation, we can verify that $Q=\sum_{i=1}^4 Q_i$. To estimate $\mathbb{E}|Q|^{\frac{q}{2}}$, we need to estimate $\mathbb{E}|Q_i|^{\frac{q}{2}}$ for $i=1,\dots,4$. For $\mathbb{E}|Q_1|^{\frac{q}{2}}$, by the assumptions on σ, using Hölder's inequality and identities (2.7) we have

$$\begin{aligned}
\mathbf{E}|Q_1|^{\frac{q}{2}} &= \mathbf{E}\Big|\int_0^t ds\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G(t-s,d\xi)G(t-s,d\eta)f(\eta-\xi) \\
&\quad \times(\sigma(s,x-\xi,u(s,x-\xi))-\sigma(s,y-\xi,u(s,y-\xi))) \\
&\quad \times(\sigma(s,x-\eta,u(s,x-\eta))-\sigma(s,y-\eta,u(s,y-\eta)))\Big|^{\frac{q}{2}} \\
&\le C\mathbf{E}\Big|\int_0^t ds\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G(t-s,d\xi)G(t-s,d\eta)f(\eta-\xi) \\
&\quad \times(|w|+|u(s,x-\xi)-u(s,y-\xi)|) \\
&\quad \times(|w|+|u(s,x-\eta)-u(s,y-\eta)|)\Big|^{\frac{q}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\le C\int_0^t ds \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G(t-s,d\xi)G(t-s,d\eta)f(\eta-\xi)\right)^{\frac{q}{2}} \\
&\quad \times \left(|w|^q + \sup_{z_1-z_2=w} \mathbf{E}|u(s,z_1)-u(s,z_2)|^q\right) \\
&\le C\int_0^t ds \left(\int_{|z|\le 2T} \frac{f(z)}{|z|}dz\right)^{\frac{q}{2}} \left(|w|^q + \sup_{z_1-z_2=w} \mathbf{E}|u(s,z_1)-u(s,z_2)|^q\right).
\end{aligned}
$$

By the condition (2.5), we get

$$
\mathbb{E}|Q_1|^{\frac{q}{2}} \le C|w|^q + C\int_0^t ds \sup_{z_1-z_2=w} \mathbf{E}|u(s,z_1)-u(s,z_2)|^q. \tag{3.14}
$$

For $\mathbf{E}|Q_2|^{\frac{q}{2}}$, we write $f(\eta-\xi+w)-f(\eta-\xi)=\Phi_1(\eta-\xi,w)$ and using the inequality $ab \le \frac{a^2+b^2}{2}$ we obtain

$$
\begin{aligned}
\mathbb{E}|Q_2|^{\frac{q}{2}} &\le C\mathbf{E}\Big(\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |\Phi_1(\eta-\xi,w)||\Sigma_x(s,\xi)||\Sigma_{x,y}(s,\eta) \\
&\quad \times|G(t-s,d\xi)G(t-s,d\eta)ds\Big)^{\frac{q}{2}} \\
&\le C\mathbf{E}\left(\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |w|^\gamma|\Phi_1(\eta-\xi,w)||\Sigma_x(s,\xi)|^2 G(t-s,d\xi)G(t-s,d\eta)ds\right)^{\frac{q}{2}} \\
&\quad + C\mathbf{E}\left(\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|\Phi_1(\eta-\xi,w)|}{|w|^\gamma}|\Sigma_{x,y}(s,\eta)|^2 G(t-s,d\xi)G(t-s,d\eta)ds\right)^{\frac{q}{2}} \\
&:= C(Q_{2,1}+Q_{2,2}).
\end{aligned}
$$

Applying condition (3.1), identities (2.7) and Hölder's inequality yields

$$
\begin{aligned}
Q_{2,1} &\le |w|^{\frac{q\gamma}{2}}\left(\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|\Phi_1(\eta-\xi,w)|G(t-s,d\xi)G(t-s,d\eta)ds\right)^{\frac{q}{2}} \sup_{s,\xi}\mathbf{E}|\Sigma_x(s,\xi)|^q \\
&\le C|w|^{\frac{q\gamma}{2}}\left(\int_0^t\int_{|z|\le 2T}\frac{|f(z+w)-f(z)|}{|z|}dzds\right)^{\frac{q}{2}} \le C|w|^{q\gamma}.
\end{aligned}
$$

For the second term we obtain

$$
\begin{aligned}
Q_{2,2} &\leq C|w|^{-\frac{q\gamma}{2}}\mathbf{E}\Big(\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|\Phi_1(\eta-\xi,w)|\left(|w|^2+|u(s,x-\xi)-u(s,y-\xi)|^2\right) \\
&\quad \times G(t-s,d\xi)G(t-s,d\eta)ds\Big)^{\frac{q}{2}} \\
&\leq C|w|^{q-\frac{q\gamma}{2}}\left(\int_0^t\int_{|z|\leq 2T}\frac{|f(z+w)-f(z)|}{|z|}dzds\right)^{\frac{q}{2}} \\
&\quad +C|w|^{-\frac{q\gamma}{2}}\left(\int_{|z|\leq 2T}\frac{|f(z+w)-f(z)|}{|z|}dz\right)^{\frac{q}{2}}\int_0^t\sup_{z_2-z_1=w}\mathbf{E}|u(s,z_1)-u(s,z_2)|^q ds \\
&\leq C|w|^q+C\int_0^t\sup_{z_2-z_1=w}\mathbf{E}|u(s,z_1)-u(s,z_2)|^q ds.
\end{aligned}
$$

So we conclude that

$$
\mathbf{E}|Q_2|^{\frac{q}{2}}\leq C|w|^{q\gamma}+C\int_0^t\sup_{z_2-z_1=w}\mathbf{E}|u(s,z_1)-u(s,z_2)|^q ds. \tag{3.15}
$$

The term $\mathbf{E}|Q_3|^{\frac{q}{2}}$ can be treated in the same way and we have

$$
\mathbf{E}|Q_3|^{\frac{q}{2}}\leq C|w|^{q\gamma}+C\int_0^t\sup_{z_2-z_1=w}\mathbf{E}|u(s,z_1)-u(s,z_2)|^q ds. \tag{3.16}
$$

For $\mathbf{E}|Q_4|^{\frac{q}{2}}$, we set $\Phi_2(\eta-\xi,w)=f(\eta-\xi+w)+f(\eta-\xi-w)-2f(\eta-\xi)$, and using the assumption on σ, condition (3.2), Hölder's inequality and the moments estimate (2.5), we have

$$
\begin{aligned}
\mathbf{E}|Q_4|^{\frac{q}{2}} &= \mathbf{E}\left(\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|\Phi_2(\eta-\xi,w)||\Sigma_x(s,\xi)\Sigma_x(s,\eta)|G(t-s,d\xi)G(t-s,d\eta)ds\right)^{\frac{q}{2}} \\
&\leq \left(\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|\Phi_2(\eta-\xi,w)|G(t-s,d\xi)G(t-s,d\eta)ds\right)^{\frac{q}{2}} \\
&\quad \times\sup_{s,\xi,\eta}\mathbf{E}\left(|\Sigma_x(s,\xi)||\Sigma_x(s,\eta)|\right)^{\frac{q}{2}}
\end{aligned}
$$

$$\leq \; C\left(\int_0^t \int_{|z|\leq 2T} \frac{|f(z+w)+f(z-w)-2f(z)|}{|z|} dzds\right)^{\frac{q}{2}} \leq C|w|^{\frac{q\gamma'}{2}}.$$

Combining the above expression with (3.14), (3.15) and (3.16), we can write

$$I_1 \leq C(|w|^{\gamma q} + |w|^{\frac{\gamma' q}{2}}) + C\int_0^t \sup_{z_2 - z_1 = w} \mathbf{E}|u(s,z_1) - u(s,z_2)|^q ds. \tag{3.17}$$

The estimates for I_i, $i = 1,2,3,4$, lead to

$$\begin{aligned} &\sup_{z_1 - z_2 = w} \mathbf{E}|u(t,z_1) - u(t,z_2)|^q \\ \leq \;& C|w|^{q\min(\gamma_1,\gamma_2,\gamma,\frac{\gamma'}{2})} + C\int_0^t ds \sup_{z_1 - z_2 = w} \mathbf{E}|u(s,z_1) - u(s,z_2)|^q. \end{aligned}$$

An application of Gronwall's lemma yields

$$\mathbf{E}|u(t,x) - u(t,y)|^q \leq C|x-y|^{q\min(\gamma_1,\gamma_2,\gamma,\frac{\gamma'}{2})} \tag{3.18}$$

for any x and y in $\mathbb{R}^3$ such that $|x-y| \leq 1$, which completes the proof of the theorem. Notice that, as it can be checked throughout the proof, the generic constant C does not depend on $t \in [0,T]$. □

Next we give a theorem which establishes the Hölder continuity in the space variable using the Fourier transform.

Theorem 3.6. *Let u be the solution to Equation (1.1). Assume conditions (a) and (b) in Theorem 3.5. Suppose the following condition:*

(c′) For some $\gamma \in (0,1]$, the Fourier transform of the tempered measure $|\zeta|^{2\gamma}\mu(d\zeta)$ is a nonnegative locally integrable function and

$$\int_{\mathbb{R}^3} \frac{\mu(d\zeta)}{1+|\zeta|^{2-2\gamma}} < \infty. \tag{3.19}$$

Set $\kappa_1' = \min(\gamma_1, \gamma_2, \gamma)$. Then for any $q \geq 2$, there exists a constant C such that

$$\sup_{t\in[0,T]} \mathbf{E}|u(t,x)-u(t,y)|^q \leq C|x-y|^{q\kappa_1'}$$

for any $x, y \in \mathbb{R}^3$.

Proof. It suffices to assume that $|x-y| \leq 1$. Set $x-y=w$. Fix $q \geq 2$, as in the proof of Theorem 3.5, we still express $\mathbf{E}|u(t,x)-u(t,y)|^q$ as $C(I_1+I_2+I_3+I_4)$, and the estimates for I_2, I_3, I_4 are the same as in the proof of Theorem 3.5. For I_1, use the notation (3.7) -(3.13) and we need to estimate $\mathbf{E}|Q_i|^{\frac{q}{2}}$ for $i = 1, \ldots, 4$.

The estimate for $\mathbf{E}|Q_1|^{\frac{q}{2}}$ is the same as in the proof of Theorem 3.5.

For Q_2 we would like to apply Equation (7.3) to $\varphi = \Sigma_{x,y}(s,\eta)$ and $\psi = \Sigma_x(s,\xi)$. Because these functions are not necessarily bounded we we introduce the truncations

$$\begin{aligned} \Sigma_x^k(s,\xi) &= \Sigma_x(s,\xi)\mathbf{1}_{\{|\Sigma_x(s,\xi)|\leq k\}}, & (3.20)\\ \Sigma_{x,y}^k(s,\eta) &= \Sigma_{x,y}(s,\eta)\mathbf{1}_{\{|\Sigma_{x,y}(s,\eta)|\leq k\}}\,, & (3.21) \end{aligned}$$

for any $k > 0$. Clearly, as k tends to infinity, $\Sigma_x^k(s,\xi)$ and $\Sigma_{x,y}^k(s,\eta)$ converge pointwise to $\Sigma_x(s,\xi)$ and $\Sigma_{x,y}(s,\eta)$, respectively. Set

$$Q_2^k = \int_0^t ds \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G(t-s,d\xi)G(t-s,d\eta)\,(f(\eta-\xi+w)-f(\eta-\xi))\,\Sigma_x^k(s,\xi)\Sigma_{x,y}^k(s,\eta).$$

Then Equation (7.3) yields

$$Q_2^k = \int_0^t ds \int_{\mathbb{R}^3} \overline{\mathscr{F}\left(\Sigma_x^k(s,\cdot)G(t-s)\right)(\zeta)}\mathscr{F}\left(\Sigma_{x,y}^k(s,\cdot)G(t-s)\right)(\zeta)(e^{-iw\cdot\zeta}-1)\mu(d\zeta)$$

Using the estimate $|e^{-iw\cdot\zeta}-1| \le C|w|^\gamma|\zeta|^\gamma$ for every $0<\gamma\le 1$, Cauchy-Schwartz's inequality and the inequality $\sqrt{ab}\le \frac{1}{2}(a+b)$ for any $a,b>0$, we can write

$$\begin{aligned}
|Q_2^k| &\le \int_0^t ds \int_{\mathbb{R}^3} \left|\mathscr{F}\left(\Sigma_x^k(s,\cdot)G(t-s)\right)(\zeta)\right|\left|\mathscr{F}\left(\Sigma_{x,y}^k(s,\cdot)G(t-s)\right)(\zeta)\right||w|^\gamma|\zeta|^\gamma\mu(d\zeta)\\
&\le \int_0^t ds|w|^\gamma\left(\int_{\mathbb{R}^3}\left|\mathscr{F}\left(\Sigma_x^k(s,\cdot)G(t-s)\right)(\zeta)\right|^2|\zeta|^{2\gamma}\mu(d\zeta)\right)^{\frac{1}{2}}\\
&\qquad \times\left(\int_{\mathbb{R}^3}\left|\mathscr{F}\left(\Sigma_{x,y}^k(s,\cdot)G(t-s)\right)(\zeta)\right|^2\mu(d\zeta)\right)^{\frac{1}{2}}\\
&\le \frac{1}{2}\int_0^t ds|w|^{2\gamma}\int_{\mathbb{R}^3}\left|\mathscr{F}\left(\Sigma_x^k(s,\cdot)G(t-s)\right)(\zeta)\right|^2|\zeta|^{2\gamma}\mu(d\zeta)\\
&\quad +\frac{1}{2}\int_0^t ds\int_{\mathbb{R}^3}\left|\mathscr{F}\left(\Sigma_{x,y}^k(s,\cdot)G(t-s)\right)(\zeta)\right|^2\mu(d\zeta)\\
&= \frac{1}{2}|w|^{2\gamma}\int_0^t ds\int_{\mathbb{R}^3} d\eta g(\eta)\left[\left(\Sigma_x^k(s,\cdot)G(t-s)\right)*\left(\widetilde{\Sigma_x^k(s,\cdot)G(t-s)}\right)\right](\eta)\\
&\quad +\frac{1}{2}\int_0^t ds\int_{\mathbb{R}^3} d\eta f(\eta)\left[\left(\Sigma_{x,y}^k(s,\cdot)G(t-s)\right)*\left(\widetilde{\Sigma_{x,y}^k(s,\cdot)G(t-s)}\right)\right](\eta)\\
&:= Q_{2,1}^k + Q_{2,2}^k,
\end{aligned}$$

where g is the Fourier transform of the measure $|\cdot|^{2\gamma}\mu$, which by our hypothesis is a nonnegative locally integrable function. In the above formula, for any measure ν, $\widetilde{\nu}$ denotes the measure $\widetilde{\nu}(A)=\nu(-A)$. Treating $g(\eta)G(t-s)*G(t-s)(\eta)d\eta$ as a new measure, and using Minkowski's inequality, we get

$$\begin{aligned}
\mathbf{E}|Q_{2,1}^k|^{\frac{q}{2}} &\le C|w|^{q\gamma}\int_0^t ds\left(\int_{\mathbb{R}^3} d\eta g(\eta)[G(t-s)*G(t-s)](\eta)\right)^{\frac{q}{2}}\\
&\qquad \times \sup_{0\le s\le t,\xi,\eta\in\mathbb{R}^3}\mathbf{E}\left|\Sigma_x(s,\xi)\Sigma_x(s,\xi+\eta)\right|^{\frac{q}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\leq C|w|^{q\gamma}\int_0^t ds\left(\int_{\mathbb{R}^3}|\zeta|^{2\gamma}\mu(d\zeta)\mathscr{F}\left(G(t-s)*G(t-s)\right)(\zeta)\right)^{\frac{q}{2}}\\
&\leq C|w|^{q\gamma}\left(\int_{\mathbb{R}^3}\mu(d\zeta)\frac{|\zeta|^{2\gamma}}{1+|\zeta|^2}\right)^{\frac{q}{2}}\leq C|w|^{q\gamma},
\end{aligned}
$$

where we have used the moments estimate (2.5), Equation (2.4), the fact that $\left(\frac{\sin(s|\xi|)}{|\xi|}\right)^2 \leq \frac{C}{1+|\xi|^2}$, when $s\in[0,T]$ and the inequality $|\Sigma_x^k(s,\xi)|\leq|\Sigma_x(s,\xi)|$. Therefore,

$$
\begin{aligned}
\mathbf{E}\left|Q_2^k\right|^{\frac{q}{2}} \leq\ & C|w|^{q\gamma}\\
&+C\mathbf{E}\int_0^t ds\left(\int_{\mathbb{R}^3}d\eta f(\eta)\left[\left(\Sigma_{x,y}^k(s,\cdot)G(t-s)\right)*\left(\widetilde{\Sigma_{x,y}^k(s,\cdot)G(t-s)}\right)\right](\eta)\right)^{\frac{q}{2}}.
\end{aligned}
$$

Applying the dominated convergence theorem we can show that in the above inequality, as k goes to infinity, the left-hand side converges to $\mathbb{E}\,|Q_2|^{\frac{q}{2}}$ and the expectation on the right-hand side converges to

$$
\mathbf{E}\int_0^t ds\left(\int_{\mathbb{R}^3}f(\eta)d\eta\left[\left(\Sigma_{x,y}(s,\cdot)G(t-s)\right)*\left(\widetilde{\Sigma_{x,y}(s,\cdot)G(t-s)}\right)\right](\eta)\right)^{\frac{q}{2}}.
$$

From the expression for $\Sigma_{x,y}(s,\xi)$ and using Minkowski's inequality, we have

$$
\begin{aligned}
\mathbf{E}\,|Q_2|^{\frac{q}{2}} &\leq C|w|^{q\gamma}+C\int_0^t ds\left(\int_{\mathbb{R}^3}d\eta f(\eta)\left(G(t-s)*G(t-s)\right)(\eta)\right)^{\frac{q}{2}}\\
&\quad\times\sup_{\eta\in\mathbb{R}^3}\mathbf{E}\left[|\sigma(s,x-\eta,u(s,x-\eta))-\sigma(s,y-\eta,u(s,y-\eta))|^q\right]\\
&\leq C|w|^{q\gamma}+C\int_0^t ds\left(\int_{\mathbb{R}^3}\frac{d\mu(\zeta)}{1+|\zeta|^2}\right)^{\frac{q}{2}}\left(|w|^q+\sup_{z_1-z_2=w}\mathbf{E}\,|u(s,z_1)-u(s,z_2)|^q\right)\\
&\leq C\left(|w|^{q\gamma}+|w|^q+\int_0^t ds\sup_{z_1-z_2=w}\mathbf{E}\,|u(s,z_1)-u(s,z_2)|^q\right).
\end{aligned}
$$

The same estimate holds for $\mathbb{E}|Q_3|^{\frac{q}{2}}$.

Consider now the term Q_4. We use the truncation argument as in the estimation for $\mathbb{E}|Q_2|^{\frac{q}{2}}$ and we set

$$\begin{aligned} Q_4^k &= \int_0^t ds \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G(t-s,d\xi)G(t-s,d\eta)\,(2f(\eta-\xi)-f(\eta-\xi+w)-f(\eta-\xi-w)) \\ &\quad \times\Sigma_x^k(s,\xi)\Sigma_x^k(s,\eta). \end{aligned}$$

Then, Equation (7.3) implies

$$\begin{aligned} Q_4^k &= \int_0^t ds \int_{\mathbb{R}^3} \mu(d\zeta)\,(1-\cos(w\cdot\zeta))\,|\left(\mathscr{F}\left(\Sigma_x^k(s,\cdot)G(t-s)\right)\right)(\zeta)|^2 \\ &\leq 2|w|^{2\gamma}\int_0^t ds \int_{\mathbb{R}^3} \mu(d\zeta)|\zeta|^{2\gamma}|\left(\mathscr{F}\left(\Sigma_x^k(s,\cdot)G(t-s)\right)\right)(\zeta)|^2 \\ &= 2|w|^{2\gamma}\int_0^t ds \int_{\mathbb{R}^3} d\eta g(\eta)\left(\left(\Sigma_x^k(s,\cdot)G(t-s)\right)*\left(\Sigma_x^k\widetilde{(s,\cdot)G(t}-s)\right)\right)(\eta). \end{aligned}$$

Then we can use the same argument as before, to conclude that

$$\mathbb{E}\,|Q_4|^{\frac{q}{2}} \leq C|w|^{q\gamma}.$$

Combining the moment estimates for $E\,|Q_i|^{\frac{q}{2}}$, $i=1,2,3,4$, since $|w|\leq 1$ and $0<\gamma\leq 1$, we have

$$I_1 \leq C|w|^{q\gamma}+C\int_0^t ds \sup_{z_1-z_2=w}\mathbf{E}\,|u(s,z_1)-u(s,z_2)|^q\,. \tag{3.22}$$

Finally, the estimates for I_i, $i=1,2,3,4$, allow us to write

$$\begin{aligned} &\sup_{z_1-z_2=x-y}\mathbf{E}|u(t,z_1)-u(t,z_2)|^q \\ &\leq C|x-y|^{q\min(\gamma_1,\gamma_2,\gamma)}+C\int_0^t ds \sup_{z_1-z_2=x-y}\mathbf{E}\,|u(s,z_1)-u(s,z_2)|^q\,. \end{aligned}$$

An application of Gronwall's lemma yields

$$\mathbf{E}|u(t,x)-u(t,y)|^q \le C|x-y|^{q\min(\gamma_1,\gamma_2,\gamma)} \tag{3.23}$$

for any x and y in $\mathbb{R}^3$ such that $|x-y|\le 1$, which completes the proof of the theorem. Notice that, as it can be checked throughout the proof, the generic constant C does not depend on $t\in[0,T]$. □

Under the assumptions of Theorem 3.1 or Theorem 3.2, applying Kolmogorov's continuity criterion, for any fixed $t\in[0,T]$, we deduce the existence of a locally Hölder continuous version for the process $\{u(t,x), x\in\mathbb{R}^3\}$ with exponent $\kappa>0$ where $\kappa<\kappa_1$. Namely, for any $t\in[0,T]$ and any compact rectangle $I\subset\mathbb{R}^3$, there exists a random variable $K_{\kappa,t,I}$ such that

$$|u(t,x)-u(t,y)| \le K_{\kappa,t,I}|x-y|^{\kappa}$$

for and $x,y\in I$.

3.4 Hölder continuity in space and time variables

In this section we obtain a result on the Hölder continuity of the solution of Equation (1.1) in both the space and time variables. Let S^2 denote the unit sphere in $\mathbb{R}^3$ and $\sigma(d\xi)$ the uniform measure on it. We have the following result.

Theorem 3.7. *Let u be the solution to Equation (1.1). Assume conditions (a) and (b) in Theorem 3.5. Suppose the following conditions hold.*

(1) For some $0<\nu\le 1$, $\int_{|z|\le h}\frac{f(z)}{|z|}dz \le Ch^{\nu}$ *for any* $0<h\le 2T$.

(2) For some $0 < \kappa_1 \leq 1$ *and for any* $q \geq 2$ *and* $t \in (0,T]$*, we have*

$$\mathbf{E}|u(t,x) - u(t,y)|^q \leq C|x-y|^{q\kappa_1}.$$

(3) Let ξ *and* η *be unit vectors in* $\mathbb{R}^3$ *and* $0 < h \leq 1$*. We have*

$$\int_0^T \int_{S^2} \int_{S^2} |f(s(\xi+\eta)+h(\xi+\eta)) - f(s(\xi+\eta)+h\eta)| s\sigma(d\xi)\sigma(d\eta)ds \leq Ch^{\rho_1}, \tag{4.1}$$

for some $\rho_1 \in (0,1]$*, and*

$$\int_0^T \int_{S^2} \int_{S^2} \Big| f(s(\xi+\eta)+h(\xi+\eta)) - f(s(\xi+\eta)+h\xi) - f(s(\xi+\eta)+h\eta) + f(s(\xi+\eta)) \Big| s^2\sigma(d\xi)\sigma(d\eta)ds \leq Ch^{\rho_2}, \tag{4.2}$$

for some $\rho_2 \in (0,2]$.

Set $\kappa_2 = \min(\gamma_1, \gamma_2, \kappa_1, \frac{\nu+1}{2}, \frac{\rho_1+\kappa_1}{2}, \frac{\rho_2}{2})$*. Then for any* $q \geq 2$*, there exists a constant* C *such that*

$$\sup_{x\in\mathbb{R}^3} \mathbf{E}|u(\bar{t},x) - u(t,x)|^q \leq C|\bar{t}-t|^{q\kappa_2}$$

for any $t, \bar{t} \in [0,T]$.

Proof. Fix $x \in \mathbb{R}^3$ and $q \in [2,\infty)$. For all $0 \leq t \leq \bar{t} \leq T$ we can write, by Definition 3.3,

$$\mathbf{E}|u(t,x) - u(\bar{t},x)|^q \leq C\sum_{i=1}^4 T_i,$$

where

$$\begin{aligned} T_1 &= \left| \left(\frac{d}{dt}G(t) * v_0\right)(x) - \left(\frac{d}{dt}G(\bar{t}) * v_0\right)(x) \right|^q, \\ T_2 &= |(G(t) * \bar{v}_0)(x) - (G(\bar{t}) * \bar{v}_0)(x)|^q, \end{aligned}$$

$$
\begin{aligned}
T_3 &= \mathbf{E}\Big|\int_0^t ds \int_{\mathbb{R}^3} G(t-s,dy) b(s,x-y,u(s,x-y)) \\
&\quad - \int_0^{\bar t} ds \int_{\mathbb{R}^3} G(\bar t-s,dy) b(s,x-y,u(s,x-y))\Big|^q, \\
T_4 &= \mathbf{E}\Big|\int_0^t \int_{\mathbb{R}^3} G(t-s,x-dy)\sigma(s,y,u(s,y)) W(ds,dy) \\
&\quad - \int_0^{\bar t} \int_{\mathbb{R}^3} G(\bar t-s,x-dy)\sigma(s,y,u(s,y)) W(ds,dy)\Big|^q.
\end{aligned}
$$

Let $\gamma' = \min(\gamma_1, \gamma_2)$. By our assumptions on Δv_0 and $\bar v_0$ and by Lemma 4.9 in [30], we have

$$
T_1 + T_2 \le C|t-\bar t|^{q\gamma'}. \tag{4.3}
$$

Notice that Lemma 4.9 in [30] assumes that x belongs to a bounded set in $\mathbb{R}^3$, but from the proof it is easy to see that the constant C does not depend on x.

The term T_3 is bounded by

$$
T_3 \le C(T_{3,1} + T_{3,2}),
$$

where

$$
\begin{aligned}
T_{3,1} &= \mathbf{E}\left|\int_t^{\bar t} ds \int_{\mathbb{R}^3} G(\bar t-s,dy) b(s,x-y,u(s,x-y))\right|^q, \\
T_{3,2} &= \mathbf{E}\left|\int_0^t ds \int_{\mathbb{R}^3} (G(t-s,dy) - G(\bar t-s,dy))\, b(s,x-y,u(s,x-y))\right|^q.
\end{aligned}
$$

Hölder's inequality, the linear growth of b and the moments estimate (2.5) imply

$$
\begin{aligned}
T_{3,1} &\le C\left(\int_t^{\bar t} ds \int_{\mathbb{R}^3} G(\bar t-s,dy)\right)^{q-1} \\
&\quad \times \left(\int_t^{\bar t} ds \int_{\mathbb{R}^3} G(\bar t-s,dy) \sup_{0\le s\le T} \sup_{x\in\mathbb{R}^3} (1+\mathbf{E}|u(s,x)|^q)\right) \\
&\le C(\bar t - t)^q.
\end{aligned}
$$

For $T_{3,2}$, we split the integral into a difference of two integrals and then we apply the change of variables $\frac{y}{t-s} \to y$ and $\frac{y}{\bar{t}-s} \to y$, respectively. In this way, taking into account that $G(t,dy) = t^{-2}G(1,t^{-1}dy)$, we get

$$\begin{aligned} T_{3,2} &= \mathbf{E}\Big| \int_0^t ds \int_{\mathbb{R}^3} G(1,dy) b\left(s, x-(t-s)y, u\left(s, x-(t-s)y\right)\right)(t-s) \\ &\quad - \int_0^t ds \int_{\mathbb{R}^3} G(1,dy) b\left(s, x-(\bar{t}-s)y, u\left(s, x-(\bar{t}-s)y\right)\right)(\bar{t}-s)\Big|^q. \end{aligned}$$

Hence, $T_{3,2} \le C\left(T_{3,2,1} + T_{3,2,2}\right)$, where

$$T_{3,2,1} = (\bar{t}-t)^q \mathbf{E}\left| \int_0^t ds \int_{\mathbb{R}^3} G(1,dy) b\left(s, x-(\bar{t}-s)y, u\left(s, x-(\bar{t}-s)y\right)\right) \right|^q$$

and

$$\begin{aligned} T_{3,2,2} &= \mathbf{E}\Big(\int_0^t ds(t-s) \int_{\mathbb{R}^3} G(1,dy) \Big| b\left(s, x-(\bar{t}-s)y, u\left(s, x-(\bar{t}-s)y\right)\right) \\ &\quad - b\left(s, x-(t-s)y, u\left(s, x-(t-s)y\right)\right) \Big| \Big)^q. \end{aligned}$$

By the moments estimate (2.5) and the linear growth of b, it follows that

$$T_{3,2,1} \le C|\bar{t}-t|^q.$$

Moreover, by the Lipschitz property of b and Hölder continuity assumption on the space variable (condition (2) in the theorem), we get

$$\begin{aligned} T_{3,2,2} &\le C\mathbf{E}\Big(\int_0^t ds(t-s) \int_{\mathbb{R}^3} G(1,dy) \big((\bar{t}-t)|y| \\ &\quad + |u(s, x-(\bar{t}-s)y) - u(s, x-(t-s)y)| \big) \Big)^q \\ &\le \left(\int_0^t ds(t-s) \int_{\mathbb{R}^3} G(1,dy) \right)^{q-1} \int_0^t ds(t-s) \int_{\mathbb{R}^3} G(1,dy) \Big((\bar{t}-t)^q |y|^q \end{aligned}$$

$$
\begin{aligned}
&\quad + \sup_{x\in\mathbb{R}^3} \mathbf{E}\left|u(s,x-(\bar{t}-s)y)-u(s,x-(t-s)y)\right|^q\Big) \\
&\leq \; C(|\bar{t}-t|^q+|\bar{t}-t|^{q\kappa}) \leq C|\bar{t}-t|^{q\kappa}.
\end{aligned}
$$

Combining the estimates for $T_{3,1}$, $T_{3,2,1}$ and $T_{3,2,2}$ we conclude that

$$
T_3 \leq C|\bar{t}-t|^{q\kappa}. \tag{4.4}
$$

Next we estimate the term T_4 which involves a stochastic integral. Consider the decomposition

$$
T_4 \leq C(T_{4,1}+T_{4,2}),
$$

where

$$
T_{4,1} = \mathbf{E}\left|\int_t^{\bar{t}}\int_{\mathbb{R}^3} G(\bar{t}-s,x-dy)\sigma(s,y,u(s,y))\,W(ds,dy)\right|^q
$$

and

$$
T_{4,2} = \mathbf{E}\left|\int_0^{t}\int_{\mathbb{R}^3} (G(\bar{t}-s,x-dy)-G(t-s,x-dy))\,\sigma(s,y,u(s,y))\,W(ds,dy)\right|^q.
$$

By the linear growth of σ and Burkholder's inequality (Lemma 3.2), we obtain

$$
\begin{aligned}
T_{4,1} &\leq C\mathbf{E}\Big(\int_t^{\bar{t}} ds \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G(\bar{t}-s,x-dy)G(\bar{t}-s,x-dz) \\
&\qquad \times f(y-z)\sigma(s,y,u(s,y))\,\sigma(s,z,u(s,z))\Big)^{\frac{q}{2}} \\
&= C\mathbf{E}\Big(\int_0^{\bar{t}-t} ds \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G(s,x-dy)G(s,x-dz) \\
&\qquad \times f(y-z)\sigma(\bar{t}-s,y,u(\bar{t}-s,y))\,\sigma(\bar{t}-s,z,u(\bar{t}-s,z))\Big)^{\frac{q}{2}} \\
&\leq C\mathbf{E}\Big(\int_0^{\bar{t}-t} ds \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G(s,x-dy)G(s,x-dz) \\
&\qquad \times f(y-z)\,(1+|u(\bar{t}-s,y)|)\,(1+|u(\bar{t}-s,z)|)\Big)^{\frac{q}{2}}.
\end{aligned}
$$

Using Hölder's inequality, the moments estimate (2.5) and condition (1), we can write

$$
\begin{aligned}
T_{4,1} &\leq C(\bar{t}-t)^{\frac{q}{2}-1}\int_0^{\bar{t}-t} ds\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G(s,x-dy)G(s,x-dz)f(y-z)\right)^{\frac{q}{2}} \\
&\quad \times \sup_{s\in[0,T],y,z\in\mathbb{R}^3} \mathbf{E}\Big((1+|u(s,y)|)^{\frac{q}{2}}(1+|u(s,z)|)^{\frac{q}{2}}\Big) \\
&\leq C(\bar{t}-t)^{\frac{q}{2}-1}\int_0^{\bar{t}-t} ds\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G(s,dy)G(s,dz)f(y-z)\right)^{\frac{q}{2}} \\
&\leq C(\bar{t}-t)^{\frac{q}{2}-1}\int_0^{\bar{t}-t} ds\left(\int_{|z|\leq 2s}\frac{f(z)}{|z|}dz\right)^{\frac{q}{2}} \\
&\leq C(\bar{t}-t)^{\frac{q}{2}-1}\int_0^{\bar{t}-t} s^{\nu\frac{q}{2}}ds = C(\bar{t}-t)^{q\frac{\nu+1}{2}}. \qquad (4.5)
\end{aligned}
$$

For $T_{4,2}$, for notational convenience we denote $\bar{t}-t$ by h. Applying Burkholder's inequality (see Lemma 3.2) yields

$$
\begin{aligned}
T_{4,2} &\leq C\mathbf{E}\Big(\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left(G(h+s,dy)-G(s,dy)\right)\left(G(h+s,dz)-G(s,dz)\right) \\
&\quad \times f(y-z)\Theta_{t,x}(s,y)\Theta_{t,x}(s,z)ds\Big)^{\frac{q}{2}},
\end{aligned}
$$

where $\Theta_{t,x}(s,y) = \sigma(t-s,x-y,u(t-s,x-y))$. By making a change of variable, we can transform the integral in the space variable into an integral on the unit sphere S^2. In fact, denote $\xi = \frac{y}{|y|}$ and $\eta = \frac{z}{|z|}$ and we recall that $\sigma(d\xi)$ and $\sigma(d\eta)$ denote the uniform measure on S^2, so

$$
\begin{aligned}
G(s,dy) &= \frac{1}{4\pi}s\sigma(d\xi), \\
G(s,dz) &= \frac{1}{4\pi}s\sigma(d\eta).
\end{aligned}
$$

After some rearrangements similar to those made for Q in the proof of Theorem 3.6 (see also [30] for a similar strategy), we can write

$$T_{4,2} = C\mathbf{E}\Big(R_1 + R_2 + R_3 + R_4\Big)^{\frac{q}{2}} \leq C\sum_{i=1}^{4}\mathbf{E}|R_i|^{\frac{q}{2}},$$

where

$$\begin{aligned}
R_1 &= \int_0^t \int_{S^2\times S^2} (s+h)^2 f\left((s+h)\xi - (s+h)\eta\right) \\
&\quad \times \left(\Theta_{t,x}(s,(s+h)\xi) - \Theta_{t,x}(s,s\xi)\right)\left(\Theta_{t,x}(s,(s+h)\eta) - \Theta_{t,x}(s,s\eta)\right)\sigma(d\xi)\sigma(d\eta)ds, \\
R_2 &= \int_0^t \int_{S^2\times S^2} \left((s+h)^2 f\left((s+h)\xi - (s+h)\eta\right) - s(s+h)f\left(s\xi - (s+h)\eta\right)\right) \\
&\quad \times \left(\Theta_{t,x}(s,(s+h)\eta) - \Theta_{t,x}(s,s\eta)\right)\Theta_{t,x}(s,s\xi)\,\sigma(d\xi)\sigma(d\eta)ds, \\
R_3 &= \int_0^t \int_{S^2\times S^2} \left((s+h)^2 f\left((s+h)\xi - (s+h)\eta\right) - s(s+h)f\left((s+h)\xi - s\eta\right)\right) \\
&\quad \times \left(\Theta_{t,x}(s,(s+h)\xi) - \Theta_{t,x}(s,s\xi)\right)\Theta_{t,x}(s,s\eta)\,\sigma(d\xi)\sigma(d\eta)ds, \\
R_4 &= \int_0^t \int_{S^2\times S^2} \Big((s+h)^2 f\left((s+h)\xi - (s+h)\eta\right) - s(s+h)f\left(s\xi - (s+h)\eta\right) \\
&\quad -s(s+h)f\left((s+h)\xi - s\eta\right) + s^2 f(s\xi - s\eta)\Big)\Theta_{t,x}(s,s\xi)\Theta_{t,x}(s,s\eta)\sigma(d\xi)\sigma(d\eta)ds.
\end{aligned}$$

We estimate each $\mathbf{E}|R_i|^{\frac{q}{2}}$ separately.

For $\mathbf{E}|R_1|^{\frac{q}{2}}$, using Hölder's inequality, the Lipschitz condition on σ, the assumption on the Hölder continuity on the space variable of u (condition (2)), Lemma 3.16 and condition (2.5), we have

$$\begin{aligned}
\mathbf{E}|R_1|^{\frac{q}{2}} &\leq Ch^{q\kappa}\int_0^t \left(\int_{S^2\times S^2} (s+h)^2 f\left((s+h)\xi - (s+h)\eta\right)\sigma(d\xi)\sigma(d\eta)\right)^{\frac{q}{2}} ds \\
&= Ch^{q\kappa}\int_0^t \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(y-z)G(s+h,dy)G(s+h,dz)\right)^{\frac{q}{2}} ds \\
&= Ch^{q\kappa}\int_0^t \left(\int_{|z|\leq 2(s+h)} \frac{f(z)}{|z|}dz\right)^{\frac{q}{2}} ds
\end{aligned}$$

$$\leq \quad Ch^{q\kappa}. \tag{4.6}$$

In order to estimate $\mathbf{E}|R_2|^{\frac{q}{2}}$, we make the decomposition

$$\begin{aligned}
\mathbf{E}|R_2|^{\frac{q}{2}} \quad &\leq \quad C\mathbf{E}\Big(\int_0^t \int_{S^2\times S^2} s(s+h)\,|f\left((s+h)\xi-(s+h)\eta\right)-f\left(s\xi-(s+h)\eta\right)| \\
&\qquad \times|\Theta_{t,x}\left(s,(s+h)\eta\right)-\Theta_{t,x}\left(s,s\eta\right)|\,|\Theta_{t,x}\left(s,s\xi\right)|\sigma(d\xi)\sigma(d\eta)ds\Big)^{\frac{q}{2}} \\
&\quad +C\mathbf{E}\Big(\int_0^t \int_{S^2\times S^2} h(s+h)f\left((s+h)\xi-(s+h)\eta\right) \\
&\qquad \times|\Theta_{t,x}(s,s\xi)|\,|\Theta_{t,x}\left(s,(s+h)\eta\right)-\Theta_{t,x}\left(s,s\eta\right)|\sigma(d\xi)\sigma(d\eta)ds\Big)^{\frac{q}{2}} \\
&:= \quad R_2^1+R_2^2.
\end{aligned}$$

For R_2^1, using the Hölder inequality, the Lipschitz and linear growth conditions on σ, the moments estimate (2.5), the assumption on the Hölder continuity in the space variable of u (condition (2)) and condition (4.1) with the change of variable $\eta\to-\eta$, we have

$$\begin{aligned}
R_2^1 \quad &\leq \quad C\Big(\int_0^t \int_{S^2\times S^2} s(s+h)\Big|f\left((s+h)\xi-(s+h)\eta\right) \\
&\qquad -f\left(s\xi-(s+h)\eta\right)\Big|\sigma(d\xi)\sigma(d\eta)ds\Big)^{\frac{q}{2}} h^{\frac{q\kappa}{2}} \\
&\leq \quad C\left(\int_0^t \int_{S^2\times S^2} s\,|f\left((s+h)\xi+(s+h)\eta\right)-f\left(s\xi+(s+h)\eta\right)|\,\sigma(d\xi)\sigma(d\eta)ds\right)^{\frac{q}{2}} h^{\frac{q\kappa}{2}} \\
&\leq \quad Ch^{\frac{q\rho_1+q\kappa}{2}}.
\end{aligned}$$

For R_2^2, using Hölder's inequality, the Lipschitz condition and linear growth conditions on σ, the moments estimate (2.5), the assumption on the Hölder continuity in the space variable (condition (2)) and condition (1), we have

$$R_2^2 \quad \leq \quad C\left(\int_0^t \int_{S^2\times S^2} (s+h)f\left((s+h)\xi-(s+h)\eta\right)\sigma(d\xi)\sigma(d\eta)ds\right)^{\frac{q}{2}} h^{\frac{q+q\kappa}{2}}$$

$$
\begin{aligned}
&\leq C\left(\int_0^t \frac{1}{s+h}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(y-z)G(s+h,dy)G(s+h,dz)ds\right)^{\frac{q}{2}} h^{\frac{q+q\kappa}{2}}\\
&= C\left(\int_0^t \frac{1}{s+h}\int_{|z|\leq 2(s+h)} \frac{f(z)}{|z|}dzds\right)^{\frac{q}{2}} h^{\frac{q+q\kappa}{2}}\\
&\leq Ch^{\frac{q+q\kappa}{2}}.
\end{aligned}
$$

Combining the estimates for R_2^1 and R_2^2, we have

$$
\mathbf{E}|R_2|^{\frac{q}{2}} \leq Ch^{\frac{q(\rho_1+\kappa)}{2}}. \tag{4.7}
$$

Similarly,

$$
\mathbf{E}|R_3|^{\frac{q}{2}} \leq Ch^{\frac{q(\rho_1+\kappa)}{2}}. \tag{4.8}
$$

For R_4, using the linear growth of σ, the moments estimate (2.5) and the change of variable $\eta \to -\eta$, we have

$$
\begin{aligned}
\mathbf{E}|R_4|^{\frac{q}{2}} \leq\ & C\Big(\int_0^t \int_{S^2\times S^2} \big|(s+h)^2 f\left((s+h)\xi-(s+h)\eta\right) - s(s+h)f\left(s\xi-(s+h)\eta\right)\\
& -s(s+h)f\left((s+h)\xi - s\eta\right) + s^2 f(s\xi - s\eta)\big|\sigma(d\xi)\sigma(d\eta)ds\Big)^{\frac{q}{2}}\\
\leq\ & C\Big(\int_0^t \int_{S^2\times S^2} s^2\big|f\left((s+h)\xi+(s+h)\eta\right) - f\left(s\xi+(s+h)\eta\right)\\
& -f\left((s+h)\xi+s\eta\right) + f\left(s\xi+s\eta\right)\big|\sigma(d\xi)\sigma(d\eta)ds\Big)^{\frac{q}{2}}\\
& + C\Big(\int_0^t \int_{S^2\times S^2} sh\left|f\left((s+h)\xi+(s+h)\eta\right) - f\left(s\xi+(s+h)\eta\right)\right|\\
& \times\sigma(d\xi)\sigma(d\eta)ds\Big)^{\frac{q}{2}}\\
& + C\Big(\int_0^t \int_{S^2\times S^2} sh\left|f\left((s+h)\xi+(s+h)\eta\right) - f\left((s+h)\xi+s\eta\right)\right|\\
& \times\sigma(d\xi)\sigma(d\eta)ds\Big)^{\frac{q}{2}}\\
& + C\left(\int_0^t \int_{S^2\times S^2} h^2 f\left((s+h)\xi-(s+h)\eta\right)\sigma(d\xi)\sigma(d\eta)ds\right)^{\frac{q}{2}}
\end{aligned}
$$

$$:= \quad R_4^1 + R_4^2 + R_4^3 + R_4^4.$$

For R_4^1, condition (4.2) yields

$$R_4^1 \le Ch^{\frac{q\rho_2}{2}}.$$

For R_4^2 and R_4^3, applying condition (4.1) we obtain

$$R_4^2 \le Ch^{\frac{q\rho_1+q}{2}}, \quad R_4^3 \le Ch^{\frac{q\rho_1+q}{2}}.$$

For R_4^4, condition (1) allows us to write

$$\begin{aligned} R_4^4 &= Ch^q \left(\int_0^t \int_{|z| \le 2(s+h)} \frac{1}{(s+h)^2} \frac{f(z)}{|z|} dz ds \right)^{\frac{q}{2}} \\ &\le Ch^q \left(\int_0^t (s+h)^{-2+\nu} ds \right)^{\frac{q}{2}}. \end{aligned}$$

When $\nu < 1$, $R_4^4 \le Ch^{\frac{q(\nu+1)}{2}}$, when $\nu = 1$, $R_4^4 \le Ch^q(\log(t+h) - \log h)^{\frac{q}{2}} \le Ch^q(\log(T + h) - \log h)^{\frac{q}{2}} \le Ch^{q(1-\varepsilon)}$ for any $\varepsilon > 0$.

Combining the estimates for R_4^1, R_4^2, R_4^3, R_4^4, we have

$$\mathbb{E}|R_4|^{\frac{q}{2}} \le C(h^{\frac{q\rho_2}{2}} + h^{\frac{q\rho_1+q}{2}} + h^{q\frac{\nu+1}{2}} + h^{q(1-\varepsilon)}), \tag{4.9}$$

for any $\varepsilon > 0$. By (4.5), (4.6), (4.7), (4.8) and (4.9), we conclude that

$$T_4 \le Ch^{q\rho}, \tag{4.10}$$

where $0 < \rho < \min(\frac{\nu+1}{2}, \frac{\rho_1+\kappa}{2}, \frac{\rho_2}{2}, \kappa)$. From the proof it is easy to see that the constant C in the above expression does not depend on x. Then we combine the estimates of

(4.3), (4.4) and (4.10) to obtain

$$\sup_{x\in\mathbb{R}^3} \mathbf{E}|u(t,x)-u(\bar{t},x)|^q \le C|\bar{t}-t|^{q\kappa'},$$

where $\kappa' \in \left(0, \min(\gamma_1, \gamma_2, \frac{\nu+1}{2}, \frac{\rho_1+\kappa}{2}, \frac{\rho_2}{2}, \kappa)\right)$. □

An application of Kolmogorov's continuity criteria leads to the following Hölder continuity result in the space an time variables.

Corollary 3.8. *Let u be the solution to Equation (1.1). Assume conditions (a) and (b) in Theorem 3.5. Suppose that condition (c) of Theorem 3.1 or condition (a) of Theorem 3.2 hold. Set $\kappa_1 = \min(\gamma_1, \gamma_2, \gamma, \frac{\gamma'}{2})$ in the first case and $\kappa_1 = \min(\gamma_1, \gamma_2, \gamma)$ in the second case. Suppose also that conditions (1), (2) and (3) of Theorem 3.2 hold. Set $\kappa_2 = \min(\gamma_1, \gamma_2, \kappa_1, \frac{\nu+1}{2}, \frac{\rho_1+\kappa_1}{2}, \frac{\rho_2}{2})$. Then, for any $\kappa < \kappa_1$ and $\kappa' < \kappa_2$ there exists a version of the process u which is locally Hölder continuous of order κ in the space variable and of order κ' in the time variable. That is, for any bounded rectangle $I \subset \mathbb{R}^3$ we can find a random variable $K_{\kappa,\kappa',I}$ such that*

$$|u(t,x)-u(s,y)| \le K_{\kappa,\kappa',I}\left(|t-s|^{\kappa'} + |x-y|^{\kappa}\right)$$

for all $s,t \in [0,T]$ and $x,y \in I$.

3.5 Examples

In this section, we give some examples of covariance functions f satisfying the conditions in the previous theorems.

3.5.1 Example 1

Proposition 3.9. *Let f be a non-negative and non-negative definite C^2 function. Then condition (c) of Theorem 3.5 holds with $\gamma = 1$ and $\gamma' = 2$.*

Proof. Using some basic estimate from calculus, we have

$$\int_{|z|\le 2T} \frac{|f(z+w)-f(z)|}{|z|} dz \le C \int_{|z|\le 2T+1} \frac{\sup_{|z|\le 2T+1} |\nabla f(z)|}{|z|} |w| dz \le C|w|\,,$$

and

$$\begin{aligned} &\int_{|z|\le 2T} \frac{|f(z+w)+f(z-w)-2f(z)|}{|z|} dz \\ \le\ & C \int_{|z|\le 2T+1} \max_{1\le i\le 3, 1\le j\le 3} \sup_{|z|\le 2T+1} \left| \frac{\partial^2 f}{\partial z_i \partial z_j} \right| \frac{1}{|z|} |w|^2 dz \le C|w|^2\,. \end{aligned}$$

The claim follows. ☐

Remark 3.10. *Consider the example $f(x) = (\rho * \frac{1}{|\cdot|^\beta})(x)$, where $\rho(x)$ is a nonnegative Schwartz function defined in $\mathbb{R}^3$ such that $(\mathscr{F}^{-1}\rho)(\xi) \ge 0$ (for example, $\rho(x) = e^{-|x|^2}$) and $0 < \beta < 3$. Then it is easy to see that condition (c′) of Theorem 3.6 holds for $0 < \gamma < \min(\frac{3-\beta}{2}, 1)$. The restriction $\gamma < \frac{3-\beta}{2}$ comes from the fact that under the condition $0 < \beta < 3$, the Fourier transform of $\frac{1}{|x|^\beta}$ is $\frac{C_\beta}{|\xi|^{3-\beta}}$ for some constant C_β which only depends on β. We omit the details of the proof. Notice in this example, Theorem 3.6 gives a weaker result than what we would obtain using Theorem 3.5 as we have done in Proposition 3.9.*

3.5.2 The Riesz kernel

Before giving next example, we recall some results from Dalang and Sanz-Solé [30].

Let ξ, η be two unit vectors in $\mathbb{R}^3$ and let u be any point in $\mathbb{R}^3$. Suppose a, b are positive numbers with $a+b \in (0,3)$. Then we have for any $h \in \mathbb{R}$

$$|u+h\xi|^{a+b-3} - |u|^{a+b-3} = |h|^b \int_{\mathbb{R}^3} dw |u-hw|^{a-3} (|w+\xi|^{b-3} - |w|^{b-3}), \qquad (5.1)$$

and

$$\begin{aligned} &\left| |u+h\xi+h\eta|^{a+b-3} - |u+h\xi|^{a+b-3} - |u+h\eta|^{a+b-3} + |u|^{a+b-3} \right| \qquad (5.2) \\ &\le |h|^b \int_{\mathbb{R}^d} dw |u-hw|^{a-3} \left| |w+h\xi+h\eta|^{b-3} - |w+h\xi|^{b-3} - |w+h\eta|^{b-3} + |w|^{b-3} \right|. \end{aligned}$$

Proposition 3.11. *Let $f(x) = |x|^{-\beta}$, $0 < \beta < 2$. Then f satisfies condition (c') in Theorem 3.6 for any $\gamma \in (0, \frac{2-\beta}{2})$ and f also satisfies conditions (1), (4.1) and (4.2) in Theorem 3.7 for $\nu = 2-\beta$, any $0 < \rho_1 < \min(2-\beta, 1)$ and $0 < \rho_2 < 2-\beta$.*

Proof. Let us first check condition (c') in Theorem 3.6. Since $f(x) = |x|^{-\beta}$, we have $\mu(d\xi) = C|\xi|^{-3+\beta} d\xi$. Then it is easy to see that

$$\int_{\mathbb{R}^3} \frac{\mu(d\xi)}{1+|\xi|^{2-2\gamma}} < \infty,$$

since $0 < \gamma < \frac{2-\beta}{2}$, and we have

$$\mathscr{F}\left(|\xi|^{2\gamma}\mu(d\xi)\right)(x) = C\mathscr{F}(|\xi|^{-3+\beta+2\gamma} d\xi)(x) = C|x|^{-(\beta+2\gamma)}$$

for some positive constant C, so the above expression is nonnegative. So, condition (c') in Theorem 3.6 holds.

To verify condition (1) in Theorem 3.7, we notice

$$\int_{|z|\le h} \frac{f(z)}{|z|} dz = \int_{|z|\le h} |z|^{-\beta-1} dz = Ch^{2-\beta}.$$

So condition (1) in Theorem 3.7 is satisfied with $\nu = 2-\beta$.

We turn to condition (4.1). We apply (5.1) with $b = \rho_1 < \min((2-\beta),1)$, $d = 3$, $a = 3-\rho_1-\beta$, $u = s(\xi+\eta)+h\eta$ to get

$$\begin{aligned}
& \int_0^T \int_{S^2}\int_{S^2} s\left|f(s(\xi+\eta)+h(\xi+\eta)) - f(s(\xi+\eta)+h\eta)\right|\sigma(d\xi)\sigma(d\eta)ds \\
\leq\; & h^{\rho_1}\int_0^T \int_{S^2\times S^2} s\int_{\mathbb{R}^3} dw\,|s\xi+(s+h)\eta-hw|^{-\rho_1-\beta}\left||w+\xi|^{\rho_1-3}-|w|^{\rho_1-3}\right| \\
& \times\sigma(d\xi)\sigma(d\eta)ds \\
\leq\; & h^{\rho_1}\int_0^T \int_{S^2\times S^2} s\int_{|w|\leq 3} dw\,|s\xi+(s+h)\eta-hw|^{-\rho_1-\beta}\,|w+\xi|^{\rho_1-3}\sigma(d\xi)\sigma(d\eta)ds \\
& +h^{\rho_1}\int_0^T \int_{S^2\times S^2} s\int_{|w|\leq 3} dw\,|s\xi+(s+h)\eta-hw|^{-\rho_1-\beta}\,|w|^{\rho_1-3}\sigma(d\xi)\sigma(d\eta)ds \\
& +h^{\rho_1}\int_0^T \int_{S^2\times S^2} s\int_{|w|>3} dw\,|s\xi+(s+h)\eta-hw|^{-\rho_1-\beta}\left||w+\xi|^{\rho_1-3}-|w|^{\rho_1-3}\right| \\
& \times\sigma(d\xi)\sigma(d\eta)ds \\
:=\; & h^{\rho_1}(I_1+I_2+I_3).
\end{aligned}$$

For I_1, making the change of variable $w+h\to w$, using the Fourier transform (see Lemma 3.17) and noting that I_1 is real positive, we can write:

$$\begin{aligned}
I_1 \;\leq\; & \int_0^T \int_{S^2\times S^2} s\int_{|w|\leq 4} |(s+h)\xi+(s+h)\eta-hw|^{-\rho_1-\beta}\,|w|^{\rho_1-3}dw\sigma(d\xi)\sigma(d\eta)ds \\
=\; & C\int_0^T \int_{|w|\leq 4}\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{s}{(s+h)^2}|y+z-hw|^{-\rho_1-\beta} \\
& \times G(s+h,dy)G(s+h,dz)|w|^{\rho_1-3}dwds \\
=\; & C\int_0^T \int_{\mathbb{R}^3} \frac{s}{(s+h)^2}\frac{[\sin((s+h)|\xi|)]^2}{|\xi|^2}|\xi|^{-3+\rho_1+\beta}e^{i\xi\cdot hw}d\xi ds\int_{|w|\leq 4}|w|^{\rho_1-3}dw.
\end{aligned}$$

Then using the change of variable $(s+h)\xi = \eta$ and the bound $|e^{i\xi\cdot hw}|\leq 1$, by direct calculation we see that $I_1 < \infty$.

For I_2, we do the same calculation, but we do not need the change of variable for w. Let $2\varepsilon < 2-\beta-\rho_1$, then

$$\begin{aligned} I_2 &\leq \int_0^T \int_{\mathbb{R}^3} \frac{1}{s+h} \frac{|\sin((s+h)|\xi|)\sin(s|\xi|)|}{|\xi|^2} |\xi|^{-3+\rho_1+\beta} d\xi ds \int_{|w|\leq 3} |w|^{\rho_1-3} dw \\ &\leq C\int_0^T \int_{|\xi|\leq 1} s|\xi|^{-3+\rho_1+\beta} d\xi ds + C\int_0^T \int_{|\xi|>1} \frac{1}{s+h} \frac{(s+h)^\varepsilon s^\varepsilon |\xi|^{2\varepsilon}}{|\xi|^2} |\xi|^{-3+\rho_1+\beta} d\xi ds \end{aligned}$$

which is finite by direct calculation.

For I_3 we can write

$$\begin{aligned} I_3 &= \int_0^T \int_{|w|>3} \int_{S^2\times S^2} s\,|s\xi+(s+h)\eta-hw|^{-\rho_1-\beta}\,\sigma(d\xi)\sigma(d\eta)dw \\ &\quad \times \left| \int_0^1 \frac{d}{d\lambda} |w+\lambda\xi|^{\rho_1-3} d\lambda \right| ds \\ &\leq C\int_0^T \int_{|w|>3} \int_{S^2\times S^2} s\,|s\xi+(s+h)\eta-hw|^{-\rho_1-\beta}\,\sigma(d\xi)\sigma(d\eta) \\ &\quad \times (\int_0^1 |w|^{\rho_1-4} d\lambda) dwds, \end{aligned}$$

where the inequality holds because $|w|>3$, $0\leq\lambda\leq 1$ and $|\xi|=1$. We can show that $I_3<\infty$ similarly to the proof for I_2 using the fact that $\int_{|w|>3}|w|^{\rho_1-4}dw<\infty$ since $\rho_1<1$.

It is easy to see that I_1, I_2 and I_3 are finite uniformly for $0<h\leq 1$. Therefore, condition (4.1) is satisfied with $0<\rho_1<\min(2-\beta,1)$.

For condition (4.2), applying (5.3), with $d=3$, $b=\rho_2<2-\beta$, $a=3-\rho_2-\beta$, $u=s(\xi+\eta)$, yields

$$\begin{aligned} &\int_0^T \int_{S^2}\int_{S^2} \big| f(s(\xi+\eta)+h(\xi+\eta)) - f(s(\xi+\eta)+h\xi) \\ &-f(s(\xi+\eta)+h\eta) + f(s(\xi+\eta)) \big| \times s^2 \sigma(d\xi)\sigma(d\eta) \end{aligned}$$

$$\begin{aligned}
&\leq \quad h^{\rho_2}\int_0^T\int_{S^2\times S^2}s^2\int_{\mathbb{R}^3}|s(\xi+\eta)-hw|^{-\rho_2-\beta} \\
&\qquad\times\left||w+\xi+\eta|^{\rho_2-3}-|w+\xi|^{\rho_2-3}-|w+\eta|^{\rho_2-3}+|w|^{\rho_2-3}\right| \\
&\quad\times dw\sigma(d\xi)\sigma(d\eta)ds \\
&\leq \quad h^{\rho_2}\Big(\int_0^T\int_{S^2\times S^2}s^2\int_{|w|\leq 3}|s(\xi+\eta)-hw|^{-\rho_2-\beta}\,|w+\xi+\eta|^{\rho_2-3} \\
&\quad\times dw\sigma(d\xi)\sigma(d\eta)ds \\
&\quad+\int_0^T\int_{S^2\times S^2}s^2\int_{|w|\leq 3}|s(\xi+\eta)-hw|^{-\rho_2-\beta}\,|w+\xi|^{\rho_2-3}dw\sigma(d\xi)\sigma(d\eta)ds \\
&\quad+\int_0^T\int_{S^2\times S^2}s^2\int_{|w|\leq 3}|s(\xi+\eta)-hw|^{-\rho_2-\beta}\,|w+\eta|^{\rho_2-3}dw\sigma(d\xi)\sigma(d\eta)ds \\
&\quad+\int_0^T\int_{S^2\times S^2}s^2\int_{|w|\leq 3}|s(\xi+\eta)-hw|^{-\rho_2-\beta}\,|w|^{\rho_2-3}dw\sigma(d\xi)\sigma(d\eta)ds \\
&\quad+\int_0^T\int_{S^2\times S^2}s^2\int_{|w|>3}|s(\xi+\eta)-hw|^{-\rho_2-\beta} \\
&\qquad\times\left||w+\xi+\eta|^{\rho_2-3}-|w+\xi|^{\rho_2-3}-|w+\eta|^{\rho_2-3}+|w|^{\rho_2-3}\right| \\
&\quad\times dw\sigma(d\xi)\sigma(d\eta)ds\Big) \\
&:= \quad h^{\rho_2}(\sum_{i=1}^{5}L_i).
\end{aligned}$$

For L_i, $i=1,2,3,4$, we can proceed exactly in the same way as for the integrals I_1, I_2 above. For L_5, we can express

$$\begin{aligned}
&|w+\xi+\eta|^{\rho_2-3}-|w+\xi|^{\rho_2-3}-|w+\eta|^{\rho_2-3}+|w|^{\rho_2-3} \\
= &\int_0^1\int_0^1\frac{\partial^2}{\partial\lambda\partial\mu}|w+\lambda\xi+\mu\eta|^{\rho_2-3}d\lambda d\mu\,,
\end{aligned}$$

and since $|w|>3$, $|\eta|=1$, it is easy to see that

$$\left|\frac{\partial^2}{\partial\lambda\partial\mu}|w+\lambda\xi+\mu\eta|^{\rho_2-3}\right|\leq C|w|^{\rho_2-5}.$$

So $\int_{|w|>3}|w|^{\rho_2-5}dw$ is finite, and L_5 is finite, by the same argument as for I_3.

So condition (4.2) is satisfied with $0 < \rho_2 < 2-\beta$. This completes the proof. □

Notice that, with the notation of Corollary 4.2, for the Riesz kernel we can take $\kappa_1 = \kappa_2 < \frac{2-\beta}{2}$, and we deduce the local Hölder continuity of the solution u in space and time variables of order $\kappa < \min(\gamma_1, \gamma_2, \frac{2-\beta}{2})$. In this way we recover the result by Dalang and Sanz-Solé [30].

3.5.3 The Bessel Kernel

In this subsection we consider the Bessel kernel defined by

$$f(x) = \int_0^\infty w^{\frac{\alpha-5}{2}} e^{-w} e^{-\frac{|x|^2}{4w}} dw \tag{5.3}$$

for some $\alpha > 1$. The colored noise with this covariance has received some attention in the literature (see for example, [5, 6]).

Proposition 3.12. *Let f be given by (5.3). Then f satisfies (2.5), (3.1), (3.2), (4.1), (4.2) and condition (1) in Theorem 3.7 for any* $0 < \gamma, \rho_1, \nu < \min(\alpha-1, 1)$ *and* $0 < \gamma', \rho_2 < \min(\alpha-1, 2)$.

Proof. First let us check condition (2.5). We have

$$\int_{|x|\le 1} \frac{f(x)}{|x|} dx = \int_0^\infty w^{\frac{\alpha-5}{2}} e^{-w} \int_{|x|\le 1} \frac{e^{-\frac{|x|^2}{4w}}}{|x|} dx dw.$$

The change of variable $x = 2\sqrt{w} y$ gives

$$\int_{|x|\le 1} \frac{f(x)}{|x|} dx = 4 \int_0^\infty w^{\frac{\alpha-3}{2}} e^{-w} \int_{|y|\le \frac{1}{2\sqrt{w}}} \frac{e^{-|y|^2}}{|y|} dy \le C \int_0^\infty w^{\frac{\alpha-3}{2}} e^{-w} dw < \infty$$

because $\alpha > 1$. To check condition (3.1), we note that for $a, b \geq 0$, we have $|e^{-a} - e^{-b}| \leq |a-b|^\gamma (e^{-a} \vee e^{-b})$, for any $0 \leq \gamma \leq 1$. So

$$\begin{aligned} |e^{-\frac{|z+y|^2}{4w}} - e^{-\frac{|z|^2}{4w}}| &\leq (\frac{1}{4w})^\gamma \big| |z+y|^2 - |z|^2 \big|^\gamma \left(e^{-\frac{|z+y|^2}{4w}} \vee e^{-\frac{|z|^2}{4w}} \right) \\ &\leq C|y|^\gamma (|z+y|^\gamma + |z|^\gamma) \frac{1}{w^\gamma} \left(e^{-\frac{|z+y|^2}{4w}} + e^{-\frac{|z|^2}{4w}} \right). \end{aligned}$$

As a consequence

$$\begin{aligned} & \int_{|z|\leq 2T} \frac{|f(z+y) - f(z)|}{|z|} dz \\ \leq\ & |y|^\gamma \int_0^\infty w^{\frac{\alpha-5}{2} - \gamma} e^{-w} \Big(\int_{|z|\leq 2T} (|z+y|^\gamma + |z|^\gamma) \frac{e^{-\frac{|z+y|^2}{4w}}}{|z|} dz \\ & + \int_{|z|\leq 2T} (|z+y|^\gamma + |z|^\gamma) \frac{e^{-\frac{|z|^2}{4w}}}{|z|} dz \Big) dw \\ :=\ & |y|^\gamma \int_0^\infty w^{\frac{\alpha-5}{2} - \gamma} e^{-w} (I(y) + J(y)) \, dw. \end{aligned}$$

For the integral $I(y)$, with the change of variable $z = \sqrt{w}x - y$, we have

$$\begin{aligned} I(y) &= \int_{|x - \frac{y}{\sqrt{w}}| \leq \frac{2T}{\sqrt{w}}} w^{\frac{\gamma+2}{2}} \left(\frac{|x|^\gamma}{|x - \frac{y}{\sqrt{w}}|} + |x - \frac{y}{\sqrt{w}}|^{\gamma-1} \right) e^{-\frac{|x|^2}{4}} dx \\ &\leq \int_{\mathbb{R}^3} w^{\frac{\gamma+2}{2}} \left(\frac{|x|^\gamma}{|x - \frac{y}{\sqrt{w}}|} + |x - \frac{y}{\sqrt{w}}|^{\gamma-1} \right) e^{-\frac{|x|^2}{4}} dx \\ &\leq C w^{\frac{\gamma+2}{2}}, \end{aligned}$$

where the last inequality follows from the fact that $|x|^\gamma e^{-\frac{|x|^2}{4}} \leq C e^{-\frac{|x|^2}{8}}$ and Lemma 17 in [76]. The term $J(y)$ can be estimated in the same way using the change of variable $z = \sqrt{w} y$, and we have

$$J(y) \leq C w^{\frac{\gamma+2}{2}}.$$

Hence,

$$\int_{|z|\le 2T} \frac{|f(z+y)-f(z)|}{|z|}dz \le C|y|^{\gamma}\int_0^{\infty} w^{\frac{\alpha-\gamma-3}{2}}e^{-w}dw \le C|y|^{\gamma}$$

for any $0<\gamma<\alpha-1$. So condition (3.1) is satisfied with $0<\gamma<\min(\alpha-1,1)$.

To check condition (3.2), note that

$$|f(z+y)+f(z-y)-2f(z)| = \left|\int_0^1\int_0^1 \frac{\partial^2}{\partial\lambda\partial\mu}f(z-(\lambda-\mu)y)d\lambda d\mu\right|.$$

So we have

$$\begin{aligned}
&\left|e^{-\frac{|z+y|^2}{4w}}+e^{-\frac{|z-y|^2}{4w}}-2e^{-\frac{|z|^2}{4w}}\right| \\
\le\ & \int_0^1\int_0^1\left(e^{-\frac{|z-\lambda y+\mu y|^2}{4w}}\frac{1}{4w^2}((z-\lambda y+\mu y)\cdot y)^2+e^{-\frac{|z-\lambda y+\mu y|^2}{4w}}\frac{1}{2w}|y|^2\right)d\lambda d\mu \\
\le\ & \int_0^1\int_0^1 e^{-\frac{|z-\lambda y+\mu y|^2}{4w}}\left(\frac{1}{4w^2}|z-\lambda y+\mu y|^2|y|^2+\frac{1}{2w}|y|^2\right)d\lambda d\mu \\
\le\ & C\int_0^1\int_0^1 \frac{|y|^2}{w}e^{-\frac{|z-(\lambda-\mu)y|^2}{8w}}d\lambda d\mu.
\end{aligned}$$

Here we have used the fact that $x^2e^{-x^2}\le Ce^{-\frac{x^2}{2}}$. By considering the cases $\frac{|y|}{\sqrt{w}}\le 1$ and $\frac{|y|}{\sqrt{w}}>1$, we obtain

$$\begin{aligned}
&\left|e^{-\frac{|z+y|^2}{4w}}+e^{-\frac{|z-y|^2}{4w}}-2e^{-\frac{|z|^2}{4w}}\right| \\
\le\ & C\frac{|y|^{\gamma'}}{w^{\gamma'/2}}\int_0^1\int_0^1\left(e^{-\frac{|z-(\lambda-\mu)y|^2}{8w}}+e^{-\frac{|z+y|^2}{4w}}+e^{-\frac{|z-y|^2}{4w}}+2e^{-\frac{|z|^2}{4w}}\right)d\lambda d\mu
\end{aligned}$$

for any $0\le\gamma'\le 2$. So we have

$$\int_{|z|\le 2T}\frac{|f(z+y)+f(z-y)-2f(z)|}{|z|}dz$$

$$\leq \quad C|y|^{\gamma'} \int_{|z|\leq 2T} \int_0^\infty dw \int_0^1 d\lambda \int_0^1 d\mu\, w^{\frac{\alpha-5-\gamma'}{2}} e^{-w}$$
$$\times \left(e^{-\frac{|z-(\lambda-\mu)y|^2}{8w}} + e^{-\frac{|z+y|^2}{4w}} + e^{-\frac{|z-y|^2}{4w}} + 2e^{-\frac{|z|^2}{4w}} \right) \frac{1}{|z|} dz.$$

By Lemma 17 in [76], we can write

$$\int_{|z|\leq 2T} \left(e^{-\frac{|z-(\lambda-\mu)y|^2}{8w}} + e^{-\frac{|z+y|^2}{4w}} + e^{-\frac{|z-y|^2}{4w}} + 2e^{-\frac{|z|^2}{4w}} \right) \frac{1}{|z|} dz \leq Cw,$$

where the constant C does not depend on y, λ, μ. Therefore,

$$\int_{|z|\leq 2T} \frac{|f(z+y)+f(z-y)-2f(z)|}{|z|} dz \leq C|y|^{\gamma'} \int_0^\infty w^{\frac{\alpha-3-\gamma'}{2}} e^{-w} dw \leq C|y|^{\gamma'}$$

for any $0 < \gamma' < \min(\alpha-1, 2)$. As a consequence, condition (3.2) is satisfied with $0 < \gamma' < \min(\alpha-1, 2)$.

To check condition (1) in Theorem 3.7 holds we compute

$$\int_{|x|\leq h} \frac{e^{-\frac{|x|^2}{4w}}}{|x|} dx = 4\pi \int_0^h e^{-\frac{r^2}{4w}} r dr = 8\pi w \left(1 - e^{-\frac{h^2}{4w}}\right) \leq Ch^\nu w^{1-\frac{\nu}{2}}$$

for any $0 \leq \nu \leq 1$. This implies that

$$\int_{|x|\leq h} \frac{f(x)}{|x|} dx \leq Ch^\nu \int_0^\infty w^{\frac{\alpha-3-\nu}{2}} e^{-w} dw \leq Ch^\nu$$

for any $0 < \nu < \min(\alpha-1, 1)$. So condition (1) in Theorem 3.7 is satisfied with $0 < \nu < \min(\alpha-1, 1)$.

To check the condition (4.1), first we note that

$$\left| e^{-\frac{|x+h\xi|^2}{4w}} - e^{-\frac{|x|^2}{4w}} \right| = \left| \int_0^1 \frac{d}{d\lambda} e^{-\frac{|x+\lambda h\xi|^2}{4w}} d\lambda \right| = \left| \int_0^1 e^{-\frac{|x+\lambda h\xi|^2}{4w}} \frac{\langle x+\lambda h\xi, h\xi \rangle}{2w} d\lambda \right|$$

$$\leq \ C\int_0^1 e^{-\frac{|x+\lambda h\xi|^2}{4w}}\frac{|x+\lambda h\xi|}{\sqrt{w}}\frac{h}{\sqrt{w}}d\lambda \leq C\int_0^1 e^{-\frac{|x+\lambda h\xi|^2}{8w}}\frac{h}{\sqrt{w}}d\lambda\,,$$

where we have used the fact that $|x|e^{-x^2} \leq Ce^{-\frac{x^2}{2}}$. By considering the cases $\frac{h}{\sqrt{w}} \leq 1$ and $\frac{h}{\sqrt{w}} > 1$, we can write

$$\left|e^{-\frac{|x+h\xi|^2}{4w}} - e^{-\frac{|x|^2}{4w}}\right| \leq C\left(\frac{h}{\sqrt{w}}\right)^{\rho_1}\int_0^1\left(e^{-\frac{|x+\lambda h\xi|^2}{8w}} + e^{-\frac{|x+h\xi|^2}{4w}} + e^{-\frac{|x|^2}{4w}}\right)d\lambda$$

for any $\rho_1 \in [0,1]$. So we have

$$|f(x+h\xi)-f(x)| \leq Ch^{\rho_1}\int_0^1\int_0^\infty w^{\frac{\alpha-5-\rho_1}{2}}e^{-w}\left(e^{-\frac{|x+\lambda h\xi|^2}{8w}} + e^{-\frac{|x+h\xi|^2}{4w}} + e^{-\frac{|x|^2}{4w}}\right)dwd\lambda.$$

Therefore, for any $\rho_1 \in [0,1]$.

$$\begin{aligned}
&\int_0^T\int_{S^2}\int_{S^2}|f(s(\xi+\eta)+h(\xi+\eta)) - f(s(\xi+\eta)+h\eta)|s\sigma(d\xi)\sigma(d\eta)ds\\
\leq\ & Ch^{\rho_1}\int_0^1\int_0^\infty w^{\frac{\alpha-5-\rho_1}{2}}e^{-w}\\
&\times\int_0^T\int_{S^2}\int_{S^2}\left(e^{-\frac{|(s+\lambda h)\xi+(s+h)\eta|^2}{8w}} + e^{-\frac{|(s+h)(\xi+\eta)|^2}{4w}} + e^{-\frac{|s\xi+(s+h)\eta|^2}{4w}}\right)\\
&\times s\sigma(d\xi)\sigma(d\eta)dsdwd\lambda.
\end{aligned}$$

We claim that this quantity is bounded by Ch^{ρ_1} if $0 < \rho_1 < \min(\alpha-1,1)$. To show this claim, we first estimate the quantity

$$I_1 := \int_0^T\int_{S^2}\int_{S^2}se^{-\frac{|(s+\lambda h)\xi+(s+h)\eta|^2}{8w}}\sigma(d\xi)\sigma(d\eta)ds.$$

Using the Fourier transform (see Lemma 3.17), the change of variables $\xi\sqrt{w}=\eta$, and taking $0<\varepsilon<1$ we obtain

$$\begin{aligned} I_1 &= C\int_0^T\int_{\mathbb{R}^3}\frac{s}{(s+h)(s+\lambda h)}w^{\frac{3}{2}}e^{-2w|\xi|^2}\frac{\sin(s+h)|\xi|}{|\xi|}\frac{\sin(s+\lambda h)|\xi|}{|\xi|}d\xi ds \\ &\leq Cw^{\frac{3}{2}}\int_0^T s(s+h)^{\varepsilon-1}(s+\lambda h)^{\varepsilon-1}ds\int_{\mathbb{R}^3}e^{-2w|\xi|^2}|\xi|^{2\varepsilon-2}d\xi \\ &\leq Cw^{1-\varepsilon}\int_{\mathbb{R}^3}e^{-2|\eta|^2}|\eta|^{2\varepsilon-2}d\eta\leq Cw^{1-\varepsilon}. \end{aligned}$$

Similarly, we have

$$\int_0^T\int_{S^2}\int_{S^2}se^{-\frac{|(s+h)\xi+(s+h)\eta|^2}{4w}}\sigma(d\xi)\sigma(d\eta)ds\leq Cw^{1-\varepsilon}$$

and

$$\int_0^T\int_{S^2}\int_{S^2}se^{-\frac{|s\xi+(s+h)\eta|^2}{4w}}\sigma(d\xi)\sigma(d\eta)ds\leq Cw^{1-\varepsilon}.$$

Therefore,

$$\begin{aligned} &\sup_{0<h\leq 1}\int_0^1\int_0^\infty w^{\frac{\alpha-5-\rho_1}{2}}e^{-w} \\ &\times\int_0^T\int_{S^2}\int_{S^2}\left(e^{-\frac{|(s+\lambda h)\xi+(s+h)\eta|^2}{8w}}+e^{-\frac{|(s+h)(\xi+\eta)|^2}{4w}}+e^{-\frac{|s\xi+(s+h)\eta|^2}{4w}}\right) \\ &\times s\sigma(d\xi)\sigma(d\eta)dsdwd\lambda \\ \leq\ & C\int_0^1\int_0^\infty w^{\frac{\alpha-5-\rho_1}{2}+1-\varepsilon}e^{-w}dwd\lambda<\infty, \end{aligned}$$

and (4.1) is satisfied with $0<\rho_1<\min(\alpha-1,1))$.

To check condition (4.2), we note that

$$\left|e^{-\frac{|x+h\xi+h\eta|^2}{4w}}-e^{-\frac{|x+h\xi|^2}{4w}}-e^{-\frac{|x+h\eta|^2}{4w}}+e^{-\frac{|x|^2}{4w}}\right|$$

$$
\begin{aligned}
&= \left|\int_0^1\int_0^1 \frac{\partial^2}{\partial\lambda\partial\mu} e^{-\frac{|x+\lambda h\xi+\mu h\eta|^2}{4w}} d\lambda d\mu\right| \\
&= \left|\int_0^1\int_0^1 e^{-\frac{|x+\lambda h\xi+\mu h\eta|^2}{4w}} \Big(\frac{1}{4w^2}\langle h\xi, x+\lambda h\xi+\mu h\eta\rangle\langle h\eta, x+\lambda h\xi+\mu h\eta\rangle \right. \\
&\quad \left. -\frac{1}{2w}\langle h\eta, h\xi\rangle\Big) d\lambda d\mu\right| \\
&\leq \int_0^1\int_0^1 \left(e^{-\frac{|x+\lambda h\xi+\mu h\eta|^2}{4w}} \frac{1}{4w^2}|x+\lambda h\xi+\mu h\eta|^2 h^2 + e^{-\frac{|x+\lambda h\xi+\mu h\eta|^2}{4w}} \frac{1}{2w}h^2\right) d\lambda d\mu \\
&\leq C\int_0^1\int_0^1 e^{-\frac{|x+\lambda h\xi+\mu h|^2}{8w}} \frac{h^2}{w} d\lambda d\mu.
\end{aligned}
$$

By considering the cases $\frac{h^2}{w}\leq 1$ and $\frac{h^2}{w}>1$, we have for any $0\leq\rho_2\leq 2$,

$$
\begin{aligned}
&\left|e^{-\frac{|x+h\xi+h\eta|^2}{4w}} - e^{-\frac{|x+h\xi|^2}{4w}} - e^{-\frac{|x+h\eta|^2}{4w}} + e^{-\frac{|x|^2}{4w}}\right| \\
\leq\ & C(\frac{h^2}{w})^{\frac{\rho_2}{2}}\int_0^1\int_0^1 \left(e^{-\frac{|x+\lambda h\xi+\mu h\eta|^2}{8w}} + e^{-\frac{|x+h\xi+h\eta|^2}{4w}} + e^{-\frac{|x+h\xi|^2}{4w}} + e^{-\frac{|x+h\eta|^2}{4w}} + e^{-\frac{|x|^2}{4w}}\right) d\lambda d\mu \\
:=\ & Ch^{\rho_2} w^{-\frac{\rho 2}{2}} q_{h,\xi,\eta}(x,w).
\end{aligned}
$$

Therefore, we obtain

$$
|f(x+h\xi+h\eta)-f(x+h\xi)-f(x+h\eta)+f(x)| = Ch^{\rho_2}\int_0^\infty w^{\frac{\alpha-5-\rho_2}{2}} e^{-w} q_{h,\xi,\eta}(x,w)dw,
$$

and

$$
\begin{aligned}
&\int_0^T\int_{S^2}\int_{S^2}\Big| f\left(s(\xi+\eta)+h(\xi+\eta)\right) - f\left(s(\xi+\eta)+h\xi\right) \\
&\quad - f\left(s(\xi+\eta)+h\eta\right) + f\left(s(\xi+\eta)\right)\Big| \times s^2\sigma(d\xi)\sigma(d\eta)ds \\
\leq\ & Ch^{\rho_2}\int_0^1\int_0^1\int_0^\infty w^{\frac{\alpha-5-\rho_2}{2}} e^{-w}\int_0^T\int_{S^2}\int_{S^2}\Big(e^{-\frac{|(s+\lambda h)\xi+(s+\mu h)\eta|^2}{8w}} \\
&\quad + e^{-\frac{|(s+h)(\xi+\eta)|^2}{4w}} + e^{-\frac{|(s+h)\xi+s\eta|^2}{4w}} + e^{-\frac{|s\xi+(s+h)\eta|^2}{4w}} + e^{-\frac{|s\xi+s\eta|^2}{4w}}\Big) s^2\sigma(d\xi)\sigma(d\eta)dsdwd\lambda d\mu
\end{aligned}
$$

We claim that when $0 < \rho_2 < \min(\alpha - 1, 2)$, the above expression is bounded by $h^{\frac{\rho_2}{2}}$. To show this claim, we first estimate the integral

$$I_2 := \int_0^T \int_{S^2} \int_{S^2} s^2 e^{-\frac{|(s+\lambda h)\xi + (s+\mu h)\eta|^2}{8w}} \sigma(d\xi)\sigma(d\eta) ds.$$

Using the Fourier transform (see Lemma 3.17) and the change of variable $\sqrt{w}\xi = \eta$, we obtain

$$\begin{aligned} I_2 &= \int_0^T \frac{s^2}{(s+\lambda h)(s+\mu h)} \int_{\mathbb{R}^3} w^{\frac{3}{2}} e^{-2w|\xi|^2} \frac{\sin(s+\lambda h)|\xi|}{|\xi|} \frac{\sin(s+\mu h)|\xi|}{|\xi|} d\xi ds \\ &\leq C \int_{\mathbb{R}^3} e^{-2|\eta|^2} \frac{w}{|\eta|^2} d\eta \leq Cw, \end{aligned}$$

where the constant C does not depend on λ and μ. The same estimation can be done for each of the other integrals and we obtain

$$\begin{aligned} &\int_0^T \int_{S^2} \int_{S^2} s^2 \left(e^{-\frac{|(s+h)(\xi+\eta)|^2}{4w}} + e^{-\frac{|(s+h)\xi + s\eta|^2}{4w}} + e^{-\frac{|s\xi + (s+h)\eta|^2}{4w}} + e^{-\frac{|s\xi + s\eta|^2}{4w}} \right) \\ &\times \sigma(d\xi)\sigma(d\eta) ds \leq Cw. \end{aligned}$$

Thus,

$$\begin{aligned} &\int_0^T \int_{S^2} \int_{S^2} \big| f(s(\xi+\eta) + h(\xi+\eta)) \\ &- f(s(\xi+\eta) + h\xi) - f(s(\xi+\eta) + h\eta) + f(s(\xi+\eta)) \big| \\ &\times s^2 \sigma(d\xi)\sigma(d\eta) ds \\ \leq\ & Ch^{\rho_2} \int_0^\infty w^{\frac{\alpha-3-\rho_2}{2}} e^{-w} dw \leq Ch^{\rho_2}, \end{aligned}$$

and the (4.2) is satisfied for $0 < \rho_2 < \min(\alpha - 1, 2)$. This completes the proof. $\square$

Notice that, with the notation of Corollary 4.2, for the Bessel kernel we can take $\kappa_1 = \kappa_2 < (\alpha - 1) \wedge 1$, and we deduce the local Hölder continuity of the solution u in space and time variables of order $\kappa < \min(\gamma_1, \gamma_2, \frac{\alpha-1}{2} \wedge 1)$.

3.6 The Fractional Noise

In this section we consider the case where $\dot{W}(t,x)$ is fractional Brownian noise in the space variable with Hurst parameters H_1, H_2, H_3 in each direction. That is, suppose that $\{W(t,x), t \geq 0, x \in \mathbb{R}^3\}$ is a centered Gaussian field with the covariance

$$\mathbf{E}[W(s,x)W(t,y)] = (s \wedge t) \prod_{i=1}^{3} R_i(x_i, y_i), \qquad s,t \geq 0, x,y \in \mathbb{R}^3, \tag{6.1}$$

where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ and

$$R_i(u,v) = \frac{1}{2}\left(|u|^{2H_i} + |v|^{2H_i} - |u-v|^{2H_i}\right). \tag{6.2}$$

Then $\dot{W}(t,x)$ is the formal partial derivative $\frac{\partial^4 W}{\partial t \partial x_1 \partial x_2 \partial x_3}(t,x)$. We will require $\frac{1}{2} < H_i < 1$, $i = 1,2,3$. This choice of noise corresponds to the covariance function

$$f(x) = c_H |x_1|^{2H_1 - 2} |x_2|^{2H_2 - 2} |x_3|^{2H_3 - 2}, \tag{6.3}$$

where $H = (H_1, H_2, H_3)$ and $c_H = \prod_{i=1}^{3} H_i(2H_i - 1)$. Here and in what follows for simplicity, we omit the coefficient c_H in the expression of $f(x)$. The corresponding spectral measure is

$$\mu(d\xi) = C_H |\xi_1|^{1-2H_1} |\xi_2|^{1-2H_2} |\xi_3|^{1-2H_3} \tag{6.4}$$

for some constant C_H which depends only on H. We will apply Theorems 3.6 and 3.7 to get the Hölder continuity of the solution to Equation (1.1) in the space and time variables.

Theorem 3.13. *Assume conditions* (a) *and* (b) *in Theorem 3.5 and let* f *be given by (6.3) (without the constant* c_H*) with* $H_1+H_2+H_3>2$*. Set*

$$\bar{\kappa}=H_1+H_2+H_3-2, \tag{6.5}$$

and choose constants $\kappa_i>0$*,* $i=0,1,2,3$ *such that* $\kappa_i<\min(H_i-\frac{1}{2},\bar{\kappa},\gamma_1,\gamma_2)$ *for* $i=1,2,3$*, and* $\kappa_0\le\min(\kappa_1,\kappa_2,\kappa_3)$*. Then the solution to (1.1) is locally Hölder continuous with exponent* κ_0 *in the time variable and with exponent* κ_i *in the ith direction. Namely, for any bounded rectangle* $I\subset\mathbb{R}^3$*, there exists a random variable* K *(depending on* I *and the constants* κ_i*'s), such that*

$$|u(t,x)-u(\bar{t},y)|\le K(|x_1-y_1|^{\kappa_1}+|x_2-y_2|^{\kappa_2}+|x_3-y_3|^{\kappa_3}+|\bar{t}-t|^{\kappa_0})$$

for all $t,\bar{t}\in[0,T]$*,* $x,y\in I$*.*

Proof. First we consider the space variable. Proceeding as in the proof of Theorem 3.6, it is easy to see that if for some number $0<\gamma\le1$, $\mathscr{F}\left(|\xi_1|^{2\gamma}\mu(d\xi)\right)(w)$ is a nonnegative locally integrable function and

$$\int_{\mathbb{R}^3}\frac{|\xi_1|^{2\gamma}\mu(d\xi)}{1+|\xi|^2}<\infty, \tag{6.6}$$

then if $\kappa_1 = \min(\gamma, \gamma_1, \gamma_2)$, for any bounded rectangle $I \subset \mathbb{R}^3$, and for any $q \geq 2$, there exists a constant C such that

$$\mathbf{E}|u(t,x_1,x_2,x_3) - u(t,y_1,x_2,x_3)|^q \leq C|x_1 - y_1|^{q\kappa_1} \tag{6.7}$$

for any $t \in [0,T]$ and $x, y \in I$.

We claim that for $0 < \gamma < \min(H_1 - \frac{1}{2}, \bar{\kappa})$, $\mathscr{F}\left(|\xi_1|^{2\gamma}\mu(d\xi)\right)(w)$ is a nonnegative locally integrable function and (6.6) holds. Indeed, since

$$\mu(d\xi) = |\xi_1|^{1-2H_1}|\xi_2|^{1-2H_2}|\xi_3|^{1-2H_3}d\xi,$$

we have

$$\begin{aligned}
\mathscr{F}\left(|\xi_1|^{2\gamma}\mu(d\xi)\right)(w) &= \mathscr{F}\left(|\xi_1|^{1-2H_1+2\gamma}|\xi_2|^{1-2H_2}|\xi_3|^{1-2H_3}\right)(w) \\
&= C|w_1|^{-2+2H_1-2\gamma}|w_2|^{2H_2-2}|w_3|^{2H_3-2},
\end{aligned}$$

which is well defined because $\gamma < H_1 - \frac{1}{2}$. To show (6.6), we have

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{|\xi_1|^{2\gamma}\mu(d\xi)}{1+|\xi|^2} &= \int_{\mathbb{R}^3} \frac{|\xi_1|^{1-2(H_1-\gamma)}|\xi_2|^{1-2H_2}|\xi_3|^{1-2H_3}}{1+|\xi|^2}d\xi \\
&\leq \int_{\mathbb{R}} \frac{|\xi_1|^{1-2(H_1-\gamma)}}{(1+|\xi_1|^2)^{\alpha_1}}d\xi_1 \int_{\mathbb{R}} \frac{|\xi_2|^{1-2H_2}}{(1+|\xi_2|^2)^{\alpha_2}}d\xi_2 \int_{\mathbb{R}} \frac{|\xi_3|^{1-2H_3}}{(1+|\xi_3|^2)^{\alpha_3}}d\xi_3,
\end{aligned}$$

where the α_i's are positive with $\alpha_1 + \alpha_2 + \alpha_3 = 1$. When $1 - 2(H_1 - \gamma) - 2\alpha_1 < -1$, $1 - 2H_2 - 2\alpha_2 < -1$ and $1 - 2H_3 - 2\alpha_3 < -1$, the above three integrals are finite. It is elementary to see such α_i's exist under the condition $\gamma < H_1 + H_2 + H_3 - 2$. The same argument holds for the other coordinates.

For the time variable, we will check conditions (1) and (3) in Theorem 3.7. To see that condition (1) in Theorem 3.7 is satisfied for some $0 < \nu \leq 1$, take positive numbers ε_i, $i = 1,2,3$ such that $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 1$ and $2H_i - 1 - \varepsilon_i > 0$ for $i = 1,2,3$. Then we have

$$\begin{aligned}\int_{|z|\leq h} \frac{f(z)}{|z|} dz &\leq \int_{|z_1|\leq h} \frac{|z_1|^{2H_1-2}}{|z_1|^{\varepsilon_1}} dz_1 \int_{|z_2|\leq h} \frac{|z_2|^{2H_2-2}}{|z_2|^{\varepsilon_2}} dz_2 \int_{|z_3|\leq h} \frac{|z_3|^{2H_3-2}}{|z_3|^{\varepsilon_3}} dz_3 \\ &\leq Ch^{2H_1-1-\varepsilon_1} h^{2H_2-1-\varepsilon_2} h^{2H_3-1-\varepsilon_3} = h^{2(H_1+H_2+H_3-2)}.\end{aligned}$$

So condition (1) in Theorem 3.7 is satisfied with $\nu = \min(2(H_1+H_2+H_3-2),1)$.

To check (4.1), let $x = a(\xi+\eta)+h\eta$. Then we decompose the difference $f(x+h\xi)-f(x)$ into the sum of three terms, each of them containing an increment in one direction, and we obtain

$$\begin{aligned}&|f(x+h\xi)-f(x)| \\ = &\left| |x_1+h\xi_1|^{2H_1-2}|x_2+h\xi_2|^{2H_2-2}|x_3+h\xi_3|^{2H_3-2} - |x_1|^{2H_1-2}|x_2|^{2H_2-2}|x_3|^{2H_3-2} \right| \\ \leq &\left| |x_1+h\xi_1|^{2H_1-2} - |x_1|^{2H_1-2} \right| |x_2+h\xi_2|^{2H_2-2}|x_3+h\xi_3|^{2H_3-2} \\ &+|x_1|^{2H_1-2} \left| |x_2+h\xi_2|^{2H_2-2} - |x_2|^{2H_2-2} \right| |x_3+h\xi_3|^{2H_3-2} \\ &+|x_1|^{2H_1-2}|x_2|^{2H_2-2} \left| |x_3+h\xi_3|^{2H_3-2} - |x_3|^{2H_3-2} \right|.\end{aligned}$$

We claim that for some $\rho_1 \in (0,1]$, the integral on $[0,T]\times S^2\times S^2$ of each of these three terms with respect to the measure $s\sigma(d\xi)\sigma(d\eta)ds$ is bounded by Ch^{ρ_1}. To show this claim, we apply (5.1) with $d=1$, $b=\rho_1 < \min(2H_1-1, 2H_2-1, 2H_3-1, 2(H_1+H_2+H_3-2))$, $a = 2H_i - \rho_1 - 1$ and $u = x_i$ to the ith summand ($i = 1,2,3$) and we get

$$\begin{aligned}&|f(x+h\xi)-f(x)| \\ \leq\ &h^{\rho_1} \int_{\mathbb{R}} dw |x_1 - hw|^{2H_1-2-\rho_1} \left| |w+\xi_1|^{\rho_1-1} - |w|^{\rho_1-1} \right|\end{aligned}$$

$$
\begin{aligned}
& \times|x_2+h\xi_2|^{2H_2-2}|x_3+h\xi_3|^{2H_3-2} \\
& +h^{\rho_1}\int_{\mathbb{R}} dw|x_2-hw|^{2H_2-2-\rho_1}\left||w+\xi_2|^{\rho_1-1}-|w|^{\rho_1-1}\right||x_1|^{2H_1-2}|x_3+h\xi_3|^{2H_3-2} \\
& +h^{\rho_1}\int_{\mathbb{R}} dw|x_3-hw|^{2H_3-2-\rho_1}\left||w+\xi_3|^{\rho_1-1}-|w|^{\rho_1-1}\right||x_1|^{2H_1-2}|x_2|^{2H_2-2} \\
:= \;& h^{\rho_1}\left(g^1_{h,\xi}(x)+g^2_{h,\xi}(x)+g^3_{h,\xi}(x)\right).
\end{aligned}
$$

We want to show that for $i=1,2,3$

$$
\sup_{0<h\leq 1}\int_0^T\int_{S^2\times S^2} s g^i_{h,\xi}(s\xi+(s+h)\eta)\sigma(d\xi)\sigma(d\eta)ds<\infty. \tag{6.8}
$$

We will consider only the case $i=1$, the other two terms being similar. By splitting the integral with respect to w into two parts, one over $|w|\leq 3$, and another one over $|w|>3$, just as we did for the Riesz kernel, we have

$$
\begin{aligned}
& \int_0^T\int_{S^2\times S^2} s g^1_{h,\xi}(s\xi+(s+h)\eta)\sigma(d\xi)\sigma(d\eta)ds \\
\leq \;& \int_0^T\int_{S^2\times S^2} s\int_{|w|\leq 3}|s\xi_1+(s+h)\eta_1-hw|^{2H_1-2-\rho_1}|(s+h)\xi_2+(s+h)\eta_2|^{2H_2-2} \\
& \quad\times|(s+h)\xi_3+(s+h)\eta_3|^{2H_3-2}|w+\xi_1|^{\rho_1-1}dw\sigma(d\xi)\sigma(d\eta)ds \\
& +\int_0^T\int_{S^2\times S^2} s\int_{|w|\leq 3}|s\xi_1+(s+h)\eta_1-hw|^{2H_1-2-\rho_1}|(s+h)\xi_2+(s+h)\eta_2|^{2H_2-2} \\
& \quad\times|(s+h)\xi_3+(s+h)\eta_3|^{2H_3-2}|w|^{\rho_1-1}dw\sigma(d\xi)\sigma(d\eta)ds \\
& +\int_0^T\int_{S^2\times S^2} s\int_{|w|>3}|s\xi_1+(s+h)\eta_1-hw|^{2H_1-2-\rho_1}|(s+h)\xi_2+(s+h)\eta_2|^{2H_2-2} \\
& \quad\times|(s+h)\xi_3+(s+h)\eta_3|^{2H_3-2}\left||w+\xi_1|^{\rho_1-1}-|w|^{\rho_1-1}\right|dw\sigma(d\xi)\sigma(d\eta)ds \\
:= \;& I_1+I_2+I_3.
\end{aligned}
$$

For integral I_1, using the change of variable $w+\xi_1 \to w$ and the Fourier transform, we can write

$$\begin{aligned}
I_1 &\leq \int_0^T \int_{S^2\times S^2} s \int_{|w|\leq 4} |(s+h)\xi_1+(s+h)\eta_1-hw|^{2H_1-2-\rho_1} \\
&\quad \times |(s+h)\xi_2+(s+h)\eta_2|^{2H_2-2}|(s+h)\xi_3+(s+h)\eta_3|^{2H_3-2}|w|^{\rho_1-1} \\
&\quad \times dw\sigma(d\eta)\sigma(d\xi)ds \\
&\leq \int_0^T \int_{|w|\leq 4} dw|w|^{\rho_1-1} \sup_{w\in\mathbb{R}} \int_{\mathbb{R}^3} \frac{s}{(s+h)^2}|z_1|^{1-2H_1+\rho_1}e^{iz_1hw}|z_2|^{1-2H_2}|z_3|^{1-2H_3} \\
&\quad \times \left(\frac{\sin(s+h)|z|}{|z|}\right)^2 dzds.
\end{aligned}$$

By the change of variable $(s+h)z=x$, the bound $|e^{iz_1hw}|\leq 1$ and direct calculation, we see the integral is finite uniformly in $0<h\leq 1$.

For the integral I_2 we can write

$$\begin{aligned}
I_2 &= \int_{|w|\leq 3}\int_0^T \int_{S^2\times S^2} s|s\xi_1+(s+h)\eta_1-hw|^{2H_1-2-\rho_1}|(s+h)\xi_2+(s+h)\eta_2|^{2H_2-2} \\
&\quad \times |(s+h)\xi_3+(s+h)\eta_3|^{2H_3-2}\sigma(d\xi)\sigma(d\eta)ds|w|^{\rho_1-1}dw \\
&= \int_{|w|\leq 3}\int_0^T \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{s}{(s+h)^2}G(s+h,dy)G(s+h,dz) \\
&\quad \times |\frac{s}{s+h}y_1+z_1-hw|^{2H_1-2-\rho_1}|y_2+z_2|^{2H_2-2}|y_3+z_3|^{2H_3-2}ds|w|^{\rho_1-1}dw \\
&= \int_{|w|\leq 3}\int_0^T \int_{\mathbb{R}^3} \frac{s}{(s+h)^2}G(s+h)*G^{\psi}(s+h)(z) \\
&\quad \times |z_1-hw|^{2H_1-2-\rho_1}|z_2|^{2H_2-2}|z_3|^{2H_3-2}dzds|w|^{\rho_1-1}dw,
\end{aligned}$$

where $\psi(y)=\left(\frac{s}{s+h}y_1,y_2,y_3\right)$, and $G^{\psi}(s+h)$ denotes the image of the measure $G(s+h)$ by the mapping ψ. Then using the Fourier transform we obtain

$$\begin{aligned}
I_2 &= \int_{|w|\leq 3}\int_0^T \int_{\mathbb{R}^3} \frac{s}{(s+h)^2}\left(\mathscr{F}\left(G(s+h)*G^{\psi}(s+h)\right)\right)(\xi) \\
&\quad \times e^{i\xi_1 w}|\xi_1|^{1+\rho_1-2H_1}|\xi_2|^{1-2H_2}|\xi_3|^{1-2H_3}d\xi ds|w|^{\rho_1-1}dw
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{|w|\leq 3}\int_0^T\int_{\mathbb{R}^3}\frac{s}{(s+h)^2}\left|\frac{\sin(s+h)|\xi|}{|\xi|}\right|\left|\frac{\sin(s+h)|(\frac{s}{s+h}\xi_1,\xi_2,\xi_3)|}{|(\frac{s}{s+h}\xi_1,\xi_2,\xi_3)|}\right| \\
&\quad \times|\xi_1|^{1+\rho_1-2H_1}|\xi_2|^{1-2H_2}|\xi_3|^{1-2H_3}d\xi ds|w|^{\rho_1-1}dw \\
&\leq \int_{|w|\leq 3}\int_0^T\int_{|\xi|\leq 1}|\xi_1|^{1+\rho_1-2H_1}|\xi_2|^{1-2H_2}|\xi_3|^{1-2H_3}d\xi ds|w|^{\rho_1-1}dw \\
&\quad +\int_{|w|\leq 3}\int_0^T\int_{|\xi|>1}\frac{s}{(s+h)^2}\left|\frac{\sin(s+h)|\xi|}{|\xi|}\right| \\
&\qquad \times\frac{1}{\frac{s}{s+h}|\xi|}|\xi_1|^{1+\rho_1-2H_1}|\xi_2|^{1-2H_2}|\xi_3|^{1-2H_3}d\xi ds|w|^{\rho_1-1}dw<\infty,
\end{aligned}
$$

where in the first inequality above we used the fact that $|e^{-i\xi_1 w}|\leq 1$, and in the last inequality we used that fact that $|\sin x|\leq |x|^{\varepsilon}$ for any $\varepsilon>0$. The above integral is finite uniformly in w and $0<h\leq 1$.

For the third integral I_3, we can bound $\left||w+\xi_1|^{\rho_1-1}-|w|^{\rho_1-1}\right|$ by $C|w|^{\rho_1-2}$ as in the example of the Riesz kernel, and proceed as in the second integral I_2. Applying the same argument for the other two terms, we get (6.8) with $\rho_1\in(0,\min(2H_1-1,2H_2-1,2H_3-1,2\bar{\kappa})$. Therefore, condition (4.1) is satisfied with $0<\rho_1<\min(2H_1-1,2H_2-1,2H_3-1,2\bar{\kappa})$.

For condition (4.2), we use the inequality

$$
\begin{aligned}
&|f((s+h)\xi+(s+h)\eta)-f((s+h)\xi+s\eta)-f((s+h)\eta+s\xi)+f(s(\xi+\eta))| \\
\leq\ &|f((s+h)\xi+(s+h)\eta)-f((s+h)\xi+s\eta)|+|f((s+h)\eta+s\xi)-f(s\xi+s\eta)|.
\end{aligned}
$$

Then we can apply the previous procedure to both terms on the right-hand side and the argument is the same as in the case of condition (4.1). We conclude that (4.2) is satisfied with $0<\rho_2<\min(2H_1-1,2H_2-1,2H_3-1,2\bar{\kappa})$.

In summary, we can take $\nu=\min(2\bar{\kappa},1)$ and $\rho_1=\rho_2\in(0,\min(2H_1-1,2H_2-1,2H_3-1,2\bar{\kappa}))$, and Theorem 4.2 together with the moment estimate (6.7) leads to the desired Hölder continuity in the space and time variables via an application of Kolmogorov's continuity theorem. □

Consider Equation (1.1) with vanishing initial conditions v_0, $\bar{v}_0$ and coefficients $\sigma \equiv 1$ and $b \equiv 0$. That means, we consider the stochastic wave equation with additive fractional noise

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2} - \Delta u\right)(t,x) &= \dot{W}(t,x), \\ \\ u(0,x) = \frac{\partial u}{\partial t}(0,x) &= 0. \end{cases} \tag{6.9}$$

The covariance function of the noise is given by (6.3) with $H_i > \frac{1}{2}$ for $i = 1,2,3$ and recall that $\bar{\kappa} = H_1 + H_2 + H_3 - 2 > 0$.

For this equation the solution can be written as

$$u(t,x) = \int_0^t \int_{\mathbb{R}^3} G(t-s,x-y) W(ds,dy).$$

In this case we are going to show that $\bar{\kappa}$ is the optimal exponent for the Hölder continuity of the solution u in the space and time variables.

Theorem 3.14. *Let u be the solution to the stochastic partial differential equation (6.9). Then*

(a) There are two positive constants c_1 and c_2 such that

$$c_1|x-y|^{2\bar{\kappa}} \leq \mathbf{E}\left(|u(t,x)-u(t,y)|^2\right) \leq c_2|x-y|^{2\bar{\kappa}} \tag{6.10}$$

for all $x,y \in \mathbb{R}^3$ and $t \in [0,T]$.

(b) For any fixed $t_0 \in (0,T]$ there are two positive constants c_1 and c_2 such that

$$c_1|\bar{t}-t|^{2\bar{\kappa}} \leq \mathbf{E}\left(|u(t,x)-u(\bar{t},x)|^2\right) \leq c_2|\bar{t}-t|^{2\bar{\kappa}}. \tag{6.11}$$

for all $t,\bar{t} \in [t_0,T]$ and $x \in \mathbb{R}^3$.

Proof. For any $x \in \mathbb{R}^3$, set $R(x) = \mathbb{E}\left(u(t,x)u(t,0)\right)$. It is easy to see that

$$\mathbb{E}\left(|u(t,x)-u(t,y)|^2\right) = 2\left(R(0)-R(x-y)\right).$$

Without loss of generality, we may assume that $t=1$ and $y=0$. We have by Lemma 3.17

$$\begin{aligned} R(0)-R(x) &= \int_0^1 ds \int_{\mathbb{R}^3} \mu(d\xi)(1-e^{i\xi\cdot x})|\mathscr{F}G(1-s)(\xi)|^2 \\ &= \frac{1}{2}\int_{\mathbb{R}^3} d\xi |\xi_1|^{1-2H_1}|\xi_2|^{1-2H_2}|\xi_3|^{1-2H_3}\frac{1}{|\xi|^2} \\ &\quad \times (1-\cos(\xi\cdot x))\left(1-\frac{\sin(2|\xi|)}{2|\xi|}\right). \end{aligned} \tag{6.12}$$

The integrand is non-negative. For clarity we may assume that $|x_1| \leq |x_2| \leq |x_3|$. If $|\xi_3| \geq 1$, then $1-\frac{\sin(2|\xi|)}{2|\xi|} \geq \frac{1}{2}$. Thus using the change of variable $\xi x_3 = \eta$, we have

$$\begin{aligned} & R(0)-R(x) \\ \geq\ & \frac{1}{4}\int_{|\xi_3|\geq 1} |\xi_1|^{1-2H_1}|\xi_2|^{1-2H_2}|\xi_3|^{1-2H_3}\left(1-\cos(\xi\cdot x)\right)\frac{1}{|\xi|^2}d\xi \\ =\ & \frac{1}{4}|x_3|^{2\kappa}\int_{|\eta_3|\geq|x_3|} |\eta_1|^{1-2H_1}|\eta_2|^{1-2H_2}|\eta_3|^{1-2H_3} \\ & \times\left[1-\cos\left(\frac{x_1}{x_3}\eta_1+\frac{x_2}{x_3}\eta_2+\eta_3\right)\right]\frac{1}{|\eta|^2}d\eta. \end{aligned}$$

If x is in a bounded interval I, then there is $L > 0$ such that $|x_3| \leq L$. Thus

$$
\begin{aligned}
& R(0) - R(x) \\
\geq\ & \frac{1}{4}|x_3|^{2\kappa} \int_{|\eta_3| \geq L} |\eta_1|^{1-2H_1}|\eta_2|^{1-2H_2}|\eta_3|^{1-2H_3} \\
& \times \left[1 - \cos\left(\frac{x_1}{x_3}\eta_1 + \frac{x_2}{x_3}\eta_2 + \eta_3\right)\right] \frac{1}{|\eta|^2} d\eta \\
\geq\ & \frac{1}{4}|x_3|^{2\kappa} \inf_{|u_1|,|u_2| \leq 1} \int_{|\eta_3| \geq L} |\eta_1|^{1-2H_1}|\eta_2|^{1-2H_2}|\eta_3|^{1-2H_3} \\
& \times [1 - \cos(u_1\eta_1 + u_2\eta_2 + \eta_3)] \frac{1}{|\eta|^2} d\eta.
\end{aligned}
$$

It is easy to see that

$$
g(u_1, u_2) := \int_{|\eta_3| \geq L} |\eta_1|^{1-2H_1}|\eta_2|^{1-2H_2}|\eta_3|^{1-2H_3} [1 - \cos(u_1\eta_1 + u_2\eta_2 + \eta_3)] \frac{1}{|\eta|^2} d\eta
$$

is a continuous function of u_1, u_2 and for any u_1 and u_2, $g(u_1, u_2)$ is positive. Thus

$$
\inf_{|u_1|,|u_2| \leq 1} g(u_1, u_2) > 0
$$

since the infimum is taken on a compact set. This proves the left-hand side inequality in (6.10).

To show the second inequality in (6.10), we can use the triangular inequality, and it suffices to show the inequality for $x = (x_1, 0, 0)$. In this case

$$
\begin{aligned}
R(0) - R(x) &\leq \frac{1}{2} \int_{\mathbb{R}^3} d\xi |\xi_1|^{1-2H_1}|\xi_2|^{1-2H_2}|\xi_3|^{1-2H_3} \frac{1}{|\xi|^2} (1 - \cos(\xi_1 x_1)). \\
&\leq \frac{1}{2}|x_1|^{2\kappa} \int_{\mathbb{R}^3} d\xi |\xi_1|^{1-2H_1}|\xi_2|^{1-2H_2}|\xi_3|^{1-2H_3} \frac{1}{|\xi|^2} (1 - \cos(\xi_1)) \\
&= C|x_1|^{2\kappa}
\end{aligned}
$$

which is the second inequality of (6.10). Hence, (a) is proved.

Now we turn to consider (b). Let $0 \le t < \bar{t} \le T$. Then we have

$$\mathbf{E}\left(|u(t,x) - u(\bar{t},x)|^2\right) = 2Z_1(t,\bar{t},x) + 2Z_2(t,\bar{t},x),$$

where

$$Z_1(t,\bar{t},x) = \mathbf{E}\left(\int_t^{\bar{t}} \int_{\mathbb{R}^3} G(\bar{t}-s, x-y) W(ds,dy)\right)^2$$

and

$$Z_2(t,\bar{t},x) = \mathbf{E}\left(\int_0^t \int_{\mathbb{R}^3} \left(G(\bar{t}-s,x-y) - G(t-s,x-y)\right) W(ds,dy)\right)^2.$$

Integrating with respect to the variable s yields

$$\begin{aligned}
Z_1(t,\bar{t},x) &= \mathbf{E}\left(\int_t^{\bar{t}} \int_{\mathbb{R}^3} G(\bar{t}-s,x-y) W(ds,dy)\right)^2 \\
&= C\int_t^{\bar{t}} ds \int_{\mathbb{R}^3} |\mathscr{F}G(\bar{t}-s)(\xi)|^2 |\xi_1|^{1-2H_1} |\xi_2|^{1-2H_2} |\xi_3|^{1-2H_3} d\xi \\
&= C\int_{\mathbb{R}^3} \left((\bar{t}-t) - \frac{\sin(2(\bar{t}-t)|\xi|)}{2|\xi|}\right) \frac{1}{|\xi|^2} |\xi_1|^{1-2H_1} |\xi_2|^{1-2H_2} |\xi_3|^{1-2H_3} d\xi.
\end{aligned}$$

With the change of variable $(\bar{t}-t)\xi \to \eta$ the last integral becomes

$$C(\bar{t}-t)^{2\sum_{i=1}^3 H_i - 3} \int_{\mathbb{R}^3} \left(1 - \frac{\sin(2|\eta|)}{2|\eta|}\right) \frac{1}{|\eta|^2} |\eta_1|^{1-2H_1} |\eta_2|^{1-2H_2} |\eta_3|^{1-2H_3} d\eta.$$

Therefore, we have

$$c_1 |\bar{t}-t|^{2\bar{\kappa}+1} \le Z_1(t,\bar{t},x) \le c_2 |\bar{t}-t|^{2\bar{\kappa}+1}. \tag{6.13}$$

The term Z_2 is slightly more complicated. A direct integration in the variable s yields

$$\begin{aligned} Z_2(t,\bar{t},x) &= \int_0^t ds \int_{\mathbb{R}^3} d\xi\, |\xi_1|^{1-2H_1}|\xi_2|^{1-2H_2}|\xi_3|^{1-2H_3} \\ &\quad \times \frac{1}{|\xi|^2}\left(\sin((\bar{t}-s)|\xi|) - \sin((t-s)|\xi|)\right)^2 \\ &\geq \int_{|\xi|\geq(\bar{t}-t)^{-1}} d\xi\, |\xi_1|^{1-2H_1}|\xi_2|^{1-2H_2}|\xi_3|^{1-2H_3} \frac{1}{|\xi|^2}\left(A(t,\bar{t},\xi)+B(t,\bar{t},\xi)\right) \\ &= tI_1 + I_2, \end{aligned}$$

where

$$A(t,\bar{t},\xi) = t\left(1-\cos((\bar{t}-t)|\xi|)\right)$$

and

$$\begin{aligned} B(t,\bar{t},\xi) &= \frac{1}{4|\xi|}\sin(2(\bar{t}-t)|\xi|) + \frac{1}{2|\xi|}\sin((\bar{t}-t)|\xi|) - \frac{1}{4|\xi|}\sin(2\bar{t}|\xi|) \\ &\quad -\frac{1}{4|\xi|}\sin(2t|\xi|) - \frac{1}{2|\xi|}\sin((\bar{t}+t)|\xi|). \end{aligned}$$

The change of variable $(\bar{t}-t)\xi = \eta$ yields

$$\begin{aligned} I_1 &\geq \int_{|\xi|\geq(\bar{t}-t)^{-1}} d\xi\, |\xi_1|^{1-2H_1}|\xi_2|^{1-2H_2}|\xi_3|^{1-2H_3} \frac{1}{|\xi|^2}\frac{1}{t}A(t,\bar{t},\xi) \\ &\geq (\bar{t}-t)^{2\bar{\kappa}} \int_{|\eta|\geq 1} d\eta\, |\eta_1|^{1-2H_1}|\eta_2|^{1-2H_2}|\eta_3|^{1-2H_3} \frac{1}{|\eta|^2}(1-\cos|\eta|) \\ &\geq c_1|\bar{t}-t|^{2\bar{\kappa}}. \end{aligned} \tag{6.14}$$

Similarly,

$$I_2 = \int_{|\xi|\geq(\bar{t}-t)^{-1}} d\xi\, |\xi_1|^{1-2H_1}|\xi_2|^{1-2H_2}|\xi_3|^{1-2H_3} \frac{1}{|\xi|^2} B(t,\bar{t},\xi)$$

$$
\begin{aligned}
&\geq \quad -c_1 \int_{|\xi|\geq(\bar{t}-t)^{-1}} d\xi |\xi_1|^{1-2H_1}|\xi_2|^{1-2H_2}|\xi_3|^{1-2H_3}\frac{1}{|\xi|^3} \\
&\geq \quad -c_1(\bar{t}-t)^{2\bar{\kappa}+1}\int_{|\eta|\geq 1} d\eta \frac{1}{|\eta|^3}|\eta_1|^{1-2H_1}|\eta_2|^{1-2H_2}|\eta_3|^{1-2H_3}.
\end{aligned}
$$

Therefore,

$$
Z_2(t,\bar{t},x) \geq c_1 t|\bar{t}-t|^{2\bar{\kappa}} - c_2'|\bar{t}-t|^{2\bar{\kappa}+1} \geq c_1'|\bar{t}-t|^{2\bar{\kappa}}
$$

when $|\bar{t}-t|$ is sufficiently small and $t \geq t_0$. So we conclude that

$$
\mathbb{E}\left(|u(t,x)-u(\bar{t},x)|^2\right) \geq c_1|\bar{t}-t|^{2\bar{\kappa}}.
$$

On the other hand, we have

$$
\begin{aligned}
Z_2(t,\bar{t},x) \quad \leq \quad & c_2\int_0^t ds \int_{|\xi|\leq(\bar{t}-t)^{-1}} d\xi |\xi_1|^{1-2H_1}|\xi_2|^{1-2H_2}|\xi_3|^{1-2H_3}(\bar{t}-t)^2 \\
& +c_2\int_0^t ds \int_{|\xi|\geq(\bar{t}-t)^{-1}} d\xi |\xi_1|^{1-2H_1}|\xi_2|^{1-2H_2}|\xi_3|^{1-2H_3}\frac{1}{|\xi|^2}.
\end{aligned}
$$

Applying the substitution $\xi(\bar{t}-t) = \eta$ to both the above integrals, we see that

$$
Z_2(t,\bar{t},x) \quad \leq \quad c_2|\bar{t}-t|^{2\bar{\kappa}}.
$$

Thus (b) is proved. □

Combining the upper bound in (6.10) and (6.11), taking into account that the process u is Gaussian and applying Kolmogorov continuity criterion, for any $\delta > 0$ and any bounded rectangle $I \subset \mathbb{R}^3$, there is a random variable $c_{\delta,I}$ such that almost surely

$$
|u(s,x)-u(t,y)| \leq c_{\delta,I}\left(|s-t|^{\bar{\kappa}-\delta} + |x-y|^{\bar{\kappa}-\delta}\right).
$$

The first inequalities of (6.10) and (6.11) tell us that the exponent $\bar{\kappa}$ is the optimal.

Remark 3.15. *Theorem 5.1 in [30] shows that the result obtained in Section 5.2 is optimal. The result in Theorem 3.14 suggests that the result in Theorem 3.13 may not be optimal. To prove the result is optimal or to find the optimal result needs further research.*

3.7 Appendix

In this section we prove some lemmas used in this paper.

Lemma 3.16. *For any $s \geq t$*

$$(G(s) * G(t))\,(dx) = \frac{1}{8\pi|x|}\mathbf{1}_{[s-t,s+t]}(|x|)dx. \tag{7.1}$$

Proof. To calculate $(G(t) * G(s))\,(dx)$, let us consider two independent random variables X and Y uniformly distributed on the spheres with radii s and t respectively with $s \geq t$. Note that the distribution of $X+Y$ is rotationally invariant. Consider a bounded continuous function φ on $\mathbb{R}$. We have

$$\mathbf{E}\,(\varphi(|X+Y|)) = \frac{1}{(4\pi)^2}\int_{S^2}\int_{S^2}\varphi(|sx+ty|)\sigma(dy)\sigma(dx).$$

It is easy to see that in the above expression, the integral with respect to $\sigma(dy)$ does not depend on x, so we can take $x = x_0 = (0,0,1)$, and using spherical coordinates $y = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$, we have

$$\begin{aligned} \mathbf{E}\,(\varphi(|X+Y|)) &= \frac{1}{4\pi}\int_{S^2}\varphi\left(|sx_0+ty|\right)\sigma(dy) \\ &= \frac{1}{2}\int_0^{\pi}\varphi\left(\sqrt{s^2+t^2+2ts\cos\phi}\right)\sin\phi\, d\phi. \end{aligned}$$

Making the change of variable $u = \sqrt{s^2+t^2+2ts\cos\phi}$, we obtain

$$\mathbf{E}\left(\varphi(|X+Y|)\right) = \frac{1}{2ts}\int_{s-t}^{s+t}\varphi(u)udu = \frac{1}{2ts}\int_{B(0,s+t)\setminus B(0,s-t)}\varphi(|z|)\frac{1}{4\pi|z|}dz,$$

where $B(0,r)$ is the ball in $\mathbb{R}^3$ with center 0 and radius r. So we conclude that the random variable $X+Y$ has a density given by

$$\rho(z) = \frac{1}{8\pi ts|z|}\mathbf{1}_{[s-t,s+t]}(|z|). \tag{7.2}$$

Taking into account that the distributions of X and Y are given by $\frac{1}{s}G(s,dx)$ and $\frac{1}{t}G(t,dx)$ respectively, we easily get the desired result (7.1). □

Our next result gives an integral identity which is used a lot in this paper. See also Theorem 5.2 in [60] for a similar result.

The following result is related to Theorem 5.2 in [60].

Lemma 3.17. *Let φ and ψ be two bounded Borel measurable functions and assume that (2.5) holds and $s \geq t > 0$. Then for any $w \in \mathbb{R}^3$ we have*

$$\begin{aligned} &\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\varphi(x)G(t,dx)\psi(y)G(s,dy)f(x-y+w) \\ = &\int_{\mathbb{R}^3}\mathscr{F}\left(\varphi G(t)\right)(\xi)\overline{\mathscr{F}\left(\psi G(s)\right)(\xi)}e^{-iw\cdot\xi}\mu(d\xi). \end{aligned} \tag{7.3}$$

Proof. Let $\phi(x) = C\exp(\frac{1}{|x|^2-1})\mathbf{1}_{[0,1)}(|x|)$, where C is a normalization coefficient such that $\int_{\mathbb{R}^3}\phi(x)dx = 1$. Set $\phi_\varepsilon(x) = \frac{1}{\varepsilon^3}\phi(\frac{x}{\varepsilon})$, with $\varepsilon \leq t$. Using the Fourier transform we have

$$\begin{aligned} &\int_{\mathbb{R}^3}\left(\left(\varphi G(t) * \widetilde{\psi G(s)}\right) * \phi_\varepsilon\right)(x)f(x+w)dx \\ = &\int_{\mathbb{R}^3}\mathscr{F}\left(\varphi G(t)\right)(\xi)\overline{\mathscr{F}\left(\psi G(s)\right)(\xi)}(\mathscr{F}\phi)(\varepsilon\xi)e^{-iw\cdot\xi}\mu(d\xi), \end{aligned} \tag{7.4}$$

where $\widetilde{\varphi G(t)}(x) = \varphi(-x)G(t,-dx)$. We are going to show that

$$\left| \left(\left(\varphi G(t) * \widetilde{\psi G(s)} \right) * \phi_\varepsilon \right)(x) \right| \le \frac{C}{|x|} \mathbf{1}_{[0,3s]}(|x|). \tag{7.5}$$

Indeed, since φ and ψ are bounded, we have

$$\begin{aligned} \left| \left(\left(\varphi G(t) * \widetilde{\psi G(s)} \right) * \phi_\varepsilon \right)(x) \right| &\le C\left((G(t) * G(s)) * \phi_\varepsilon \right)(x) \\ &= C\left(\frac{1}{|\cdot|} \mathbf{1}_{[s-t,s+t]}(|\cdot|) * \phi_\varepsilon \right)(x). \end{aligned}$$

Note first that the function

$$\left(\left(\frac{1}{|\cdot|} \mathbf{1}_{[s-t,s+t]}(|\cdot|)\right) * \phi_\varepsilon \right)(x)$$

is supported within a ball centered at the origin with radius $3s$ for every $\varepsilon \le t$ and it converges to $\frac{1}{|z|}\mathbf{1}_{[s-t,s+t]}$ almost everywhere. Next, for $|x| \le 3s$ we have

$$\begin{aligned} \left(\left(\frac{1}{|\cdot|} \mathbf{1}_{[s-t,s+t]}(|\cdot|)\right) * \phi_\varepsilon \right)(x) &= \int_{\mathbb{R}^3} \frac{1}{|x-z|} \mathbf{1}_{[s-t,s+t]}(|x-z|)\phi_\varepsilon(z)dz \\ &\le \int_{|x-z| \ge \frac{|x|}{2}} \frac{1}{|x-z|}\phi_\varepsilon(z)dz + \int_{|x-z| < \frac{|x|}{2}} \frac{1}{|x-z|}\phi_\varepsilon(z)dz \\ &= \frac{2}{|x|} + \int_{|z| < \frac{|x|}{2}} \frac{1}{|z|}\phi_\varepsilon(z+x)dz, \end{aligned}$$

where in the second integral we have used the change of variable $z - x \to z$. Since in the second integral $|z| < \frac{|x|}{2}$, we have $|z+x| \ge |x| - |z| \ge \frac{|x|}{2}$, and

$$\int_{|z| < \frac{|x|}{2}} \frac{1}{|z|}\phi_\varepsilon(z+x) \le \int_{|z| < \frac{|x|}{2}} \frac{1}{|z|}\phi_\varepsilon\left(\frac{x}{2}\right)dz = C|x|^2 \phi_\varepsilon\left(\frac{x}{2}\right) \le C\frac{1}{|x|},$$

where in the last inequality we used the fact that $\sup_{x\in\mathbb{R}^3}|x|^3\phi(x)<\infty$. So (7.5) is proved. Then by an application of the dominated convergence theorem we have

$$\begin{aligned} &\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3}\left(\left(\varphi G(t)*\widetilde{\psi G(s)}\right)*\phi_\varepsilon\right)(x)f(x+w)dx \\ =\;& \int_{\mathbb{R}^3}\left(\varphi G(t)*\widetilde{\psi G(s)}\right)(x)f(x+w)dx \\ =\;& \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\varphi(x)G(t,dx)\varphi(y)G(s,dy)f(x-y+w)\,. \end{aligned}$$

On the other hand, the estimate (7.5) implies that the quantity in (7.4) is uniformly bounded in ε. Hence, by Fatou's lemma $\int_{\mathbb{R}^3}|\mathscr{F}(\varphi G(t))(\xi)||\mathscr{F}(\psi G(s))(\xi)|\mu(d\xi)<\infty$, and by the dominated convergence, the right-hand side of (7.4) converges to

$$\int_{\mathbb{R}^3}\mathscr{F}(\varphi G(t))(\xi)\overline{\mathscr{F}(\psi G(s))(\xi)}e^{-iw\cdot\xi}\mu(d\xi)\,.$$

This completes the proof of the lemma. □

In particular, if in the above lemma, take $\varphi=\psi$, $t=s$ and $w=0$, then for any $t>0$ we have

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\varphi(x)G(t,dx)\varphi(y)G(t,dy)f(x-y)=\int_{\mathbb{R}^3}|\mathscr{F}(\varphi G(t))(\xi)|^2\mu(d\xi)\,. \tag{7.6}$$

More specifically, if in addition, we take $\varphi\equiv 1$, then we obtain

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}G(t,dx)G(t,dy)f(x-y)=\int_{\mathbb{R}^3}\frac{(\sin(t|\xi|))^2}{|\xi|^2}\mu(d\xi)\,. \tag{7.7}$$

Chapter 4

Stochastic heat equations with general multiplicative Gaussian noises: Hölder continuity and intermittency

This chapter studies the stochastic heat equation with multiplicative noises: $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u\dot{W}$, where $\dot{W}$ is a mean zero Gaussian noise and $u\dot{W}$ is interpreted both in the sense of Skorohod and Stratonovich. The existence and uniqueness of the solution are studied for noises with general time and spatial covariance structure. Feynman-Kac formulas for the solutions and for the moments of the solutions are obtained under general and different conditions. These formulas are applied to obtain the Hölder continuity of the solutions. They are also applied to obtain the intermittency bounds for the moments of the solutions.

4.1 Introduction

In this chapter we are interested in the stochastic heat equation in $\mathbb{R}^d$ driven by a general multiplicative centered Gaussian noise. This equation can be written as

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u\dot{W}, \quad t > 0, x \in \mathbb{R}^d, \tag{1.1}$$

with initial condition $u_{0,x} = u_0(x)$, where u_0 is a continuous and bounded function. In this equation, the notation $\dot{W}$ stands for the partial derivative $\frac{\partial^{d+1}W}{\partial t \partial x_1 \cdots \partial x_d}$ (or $\frac{\partial^d W}{\partial x_1 \cdots \partial x_d}$ when the noise does not depend on time), where W is a random field formally defined in the next section. We assume that $\dot{W}$ has a covariance of the form

$$\mathbb{E}\left[\dot{W}_{t,x}\dot{W}_{s,y}\right] = \gamma(s-t)\,\Lambda(x-y),$$

where γ and Λ are general nonnegative and nonnegative definite (generalized) functions satisfying some integrability conditions. The product appearing in the above equation (1.1) can be interpreted as an ordinary product of the solution $u_{t,x}$ times the noise $\dot{W}_{t,x}$ (which is a distribution). In this case the evolution form of the equation will involve a Stratonovich integral (or pathwise Young integral). The product in (1.1) can also be also interpreted as a Wick product (defined in the next section) and in this case the solution satisfies an evolution equation formulated by using the Skorohod integral. We shall consider both of these formulations.

There has been a widespread interest in the model (1.1) in the recent past, with several motivations for its study:

• It is one of the basic stochastic partial differential equations (PDEs) one might wish to solve, either by extending Itô's theory [27, 77] or by pathwise techniques [15, 43]. These developments are also related to Zakai's equation from filtering theory.

• It appears naturally in homogenization problems for PDEs driven by highly oscillating stationary random fields (see [41, 47, 57] and references therein). Notice that in this case limit theorems are often obtained through a Feynman-Kac representation of the solution to the heat equation.

• Equation (1.1) is also related to the KPZ growth model through the Cole-Hopf's transform. In this context, definitions of the equation by means of renormalization and rough paths techniques have been recently investigated in [42, 46].

• There is a strong connexion between equation (1.1) and the partition function of directed and undirected continuum polymers. This link has been exploited in [64, 82] and is particularly present in [1], where basic properties of an equation of type (1.1) are translated into corresponding properties of the polymer.

• The multiplicative stochastic heat equation exhibits concentration properties of its energy. This interesting phenomenon is referred to as *intermittency* for the process u solution to (1.1) (see e.g [21, 22, 23, 36, 62]), and as a *localization* property for the polymer measure [12]. The intermittency property for our model is one of the main result of the current paper, and will be developed later in the introduction.

• Finally, the large time behavior of equation (1.1) also provides some information on the random operator $Lu = \Delta u + \dot{W} u$. A sample of the related Lyapunov exponent literature is given by [13, 70].

Being so ubiquitous, the model (1.1) has thus obviously been the object of numerous studies.

Indeed, when the noise $\dot{W}$ is white in time and colored in space, that is, when γ is the Dirac delta function $\delta_0(x)$, there is a huge literature devoted to our linear stochastic heat equation. Notice that in this case the stochastic integral involving $\dot{W}$ is interpreted in an extended Itô sense. Starting with the seminal paper by Dalang [24], these equations, even with more general nonlinearities (namely $u\dot{W}$ in (1.1) is replaced by $\sigma(u)\dot{W}$ for a general nonlinear function σ), have received a lot of attention. In this context, the existence and uniqueness of a solution is guaranteed by the integrability condition

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty, \tag{1.2}$$

where μ is the Fourier transform of Λ. This condition is sharp, in the sense that it is also necessary in the case of an additive noise.

Recently, there also has been a growing interest in studying equation (1.1) when the noise is colored in time. Unlike the case where the noise is white in time, one can no longer make use of the martingale structure of the noise, and just making sense of the equation offers new challenges. Recent progresses for some specific Gaussian noises include [6, 55, 56] by means of stochastic analysis methods, and [15, 43, 33] using rough paths arguments.

As mentioned above, we shall focus in this article on intermittency properties for the stochastic heat equation (1.1). There exist several ways to express this phenomenon, heuristically meaning that the process u concentrates into a few very high peaks. However, all the definitions involve two functions $\{a(t); t \geq 0\}$ and $\{\ell(k); k \geq 2\}$ such that $\ell(k) \in (0, \infty)$ and:

$$\ell(k) := \limsup_{t\to\infty} \frac{1}{a(t)} \log\left(\mathbf{E}\left[|u_{t,x}|^k\right]\right), \tag{1.3}$$

where we assume that the limit above is independent of x. In this case, we call $a(t)$ the upper Lyapunov rate and $\ell(k)$ the upper Lyapunov exponent. The process u is then called *weakly intermittent* if

$$\ell(2) > 0, \quad \text{and} \quad \ell(k) < \infty \quad \forall\, k \geq 2.$$

The computation of the exact value of Lyapunov exponents is difficult in general. A related property (which corresponds to the intuitive notion of intermittency) requires that for any $k_1 > k_2$ the moment of order k_1 is significantly greater than the moment of order k_2, or otherwise stated:

$$\limsup_{t\to\infty} \frac{\mathbf{E}^{1/k_1}\left[|u_{t,x}|^{k_1}\right]}{\mathbf{E}^{1/k_2}\left[|u_{t,x}|^{k_2}\right]} = \infty . \tag{1.4}$$

Most of the studies concerning this challenging property involve a white noise in time, and we refer to [4, 13, 36] for an account on the topic. The recent paper [4] tackles the problem for a fractional noise in time, with some special (though important) examples of spatial covariance structures, within the landmark of Skorohod equations. In this case the results are confined to weak intermittency, with an upper bound on L^k moments obtained invoking hypercontractivity arguments and lower bounds computed only for the L^2 norm.

With all those preliminary considerations in mind, the current paper proposes to study existence-uniqueness results, Feynman-Kac representations, chaos expansions and intermittency results for a very wide class of Gaussian noises $\dot{W}$ (including in particular those considered in [4, 24]), for both Skorohod and Stratonovich type equations (1.1). In particular we obtain some lower bounds for $\ell(k)$ defined by (1.3) for all $k \geq 2$, which are sharp in the sense that they have the same exponential order as the upper bounds.

More specifically, here is a brief description of the results obtained in the current paper:

(i) In the Skorohod case, the mild solution has a formal Wiener chaos expansion, which converges in $L^2(\Omega)$ provided γ is locally integrable and the spectral measure μ of the spatial covariance satisfies condition (1.2). Moreover, the solution is unique. This result (proved in Theorem 4.8) is based on Fourier analysis techniques, and covers the particular examples of the Riesz kernel and the Bessel kernel considered by Balan and Tudor in [6]. Our results also encompass the case of the fractional covariance $\Lambda(x) = \prod_{i=1}^d H_i(2H_i-1)|x_i|^{2H_i-2}$, where $H_i > \frac{1}{2}$ and condition (1.2) is satisfied if and only if $\sum_{i=1}^d H_i > d-1$. This particular structure has been examined in [55].

(ii) Under these general hypothesis to ensure the existence and uniqueness of the solution of Skorohod type one cannot expect to have a Feynman-Kac formula for the solution, but one can establish Feynman-Kac-type formulas for the moments of the solution. The formulas we obtain (see (3.21)), generalize those obtained for the Riesz or the Bessel kernels in [6, 55].

(iii) Under more restrictive integrability assumptions on γ and μ (see Hypothesis 4.16) we derive a Feynman-Kac formula for the solution u to (1.1) in the Stratonovich sense. An immediate application of the Feynman-Kac formula is the Hölder continuity of the solution.

(iv) In the Stratonovich case, we give a notion of solution using two different methodologies. One is based on the Stratonovich integral defined as the limit in probability of the integrals with respect to a regularization of the noise, and another one uses a pathwise approach, weighted Besov spaces and a Young integral approach. We show that the two notions coincide and some existence-uniqueness results which are (to the best of our knowledge) the first link between pathwise and Malliavin calculus solutions to equation (1.1).

(v) Under some further restrictions (see hypothesis at the beginning of Section 6), we obtain some sharp lower bounds for the moments of the solution. Namely, we can find explicit numbers κ_1 and κ_2 and constants c_j, C_j for $j = 1,2$ such that

$$C_1 \exp\left(c_1 t^{\kappa_1} k^{\kappa_2}\right) \leq \mathbf{E}\left[|u_{t,x}|^k\right] \leq C_2 \exp\left(c_2 t^{\kappa_1} k^{\kappa_2}\right)$$

for all $t \geq 0$, $x \in \mathbb{R}^d$ and $k \geq 2$.

As it might be clear from the description above, our central object for the study of (1.1) is the Feynman-Kac formula for the solution u or for its moments, which is a very interesting result in its own right. A substantial part of the article is devoted to establish these formulae with optimal conditions on the covariances γ and Λ, including critical cases. Notice that we also handle the case of noises which only depend on the space variable. This situation is usually treated separately in the paper, due to its particular physical relevance.

Here is the organization of this chapter. In Section 6.2, we briefly set up some preliminary material on the Gaussian noises that we are dealing with. We also recall some material from Malliavin calculus. Section 4.3 is devoted to the stochastic heat equation of Skorohod type. Existence and uniqueness of the mild solutions are obtained, and Feynman-Kac formula for the moments of the solution is established. Section 4.4 focuses on the Feynman-Kac formula related to equation (1.1) and studies the regularity of the process u^F defined in that way under some conditions on γ and Λ. In section 4.5 we first prove that the process u^F can really be seen as a solution to the stochastic heat equation interpreted in a mild sense related to Malliavin calculus. However, uniqueness is missing in this general context. Under some slightly more restrictive conditions on the noises, we then study the existence and uniqueness of the mild solution to equation (1.1) using Young integration techniques. Finally, Section 4.6 is concerned with the bounds for the moments and related intermittency results.

Notations. In the remainder of this chapter, all generic constants will be denoted by c, C, and their value may vary from different occurrences. We denote by $p_t(x)$ the d-dimensional heat kernel $p_t(x) = (2\pi t)^{-d/2} e^{-|x|^2/2t}$, for any $t > 0$, $x \in \mathbb{R}^d$.

4.2 Preliminaries

This section is devoted to a further description of the structure of our noise W. We consider first the time dependent case and later the time independent case. We will also provide some basic elements of Malliavin calculus used in the paper.

4.2.1 Time dependent noise

Let us start by introducing some basic notions on Fourier transforms of functions: the space of real valued infinitely differentiable functions with compact support is denoted by $\mathscr{D}(\mathbb{R}^d)$ or $\mathscr{D}$. The space of Schwartz functions is denoted by $\mathscr{S}(\mathbb{R}^d)$ or $\mathscr{S}$. Its dual, the space of tempered distributions, is $\mathscr{S}'(\mathbb{R}^d)$ or $\mathscr{S}'$. If u is a vector of tempered distributions from $\mathbb{R}^d$ to $\mathbb{R}^n$, then we write $u \in \mathscr{S}'(\mathbb{R}^d, \mathbb{R}^n)$. The Fourier transform is defined with the normalization

$$\mathscr{F}u(\xi) = \int_{\mathbb{R}^d} e^{-\imath\langle \xi, x\rangle} u(x) dx,$$

so that the inverse Fourier transform is given by $\mathscr{F}^{-1}u(\xi) = (2\pi)^{-d} \mathscr{F}u(-\xi)$.

Similarly to [24], on a complete probability space $(\Omega, \mathscr{F}, \mathbf{P})$ we consider a Gaussian noise W encoded by a centered Gaussian family $\{W(\varphi);\, \varphi \in \mathscr{D}([0,\infty) \times \mathbb{R}^d)\}$, whose covariance structure is given by

$$\mathbf{E}\left[W(\varphi) W(\psi)\right] = \int_{\mathbb{R}_+^2 \times \mathbb{R}^{2d}} \varphi(s,x) \psi(t,y) \gamma(s-t) \Lambda(x-y) dx dy ds dt, \qquad (2.1)$$

where $\gamma: \mathbb{R} \to \mathbb{R}_+$ and $\Lambda: \mathbb{R}^d \to \mathbb{R}_+$ are non-negative definite functions and the Fourier transform $\mathscr{F}\Lambda = \mu$ is a tempered measure, that is, there is an integer $m \geq 1$ such that $\int_{\mathbb{R}^d} (1+|\xi|^2)^{-m} \mu(d\xi) < \infty$.

Let $\mathscr{H}$ be the completion of $\mathscr{D}([0,\infty) \times \mathbb{R}^d)$ endowed with the inner product

$$\begin{aligned} \langle \varphi, \psi \rangle_{\mathscr{H}} &= \int_{\mathbb{R}_+^2 \times \mathbb{R}^{2d}} \varphi(s,x)\psi(t,y)\gamma(s-t)\Lambda(x-y)\,dxdydsdt \\ &= \int_{\mathbb{R}_+^2 \times \mathbb{R}^d} \mathscr{F}\varphi(s,\xi)\overline{\mathscr{F}\psi(t,\xi)}\gamma(s-t)\mu(d\xi)\,dsdt, \end{aligned} \tag{2.2}$$

where $\mathscr{F}\varphi$ refers to the Fourier transform with respect to the space variable only. The mapping $\varphi \to W(\varphi)$ defined in $\mathscr{D}([0,\infty) \times \mathbb{R}^d)$ extends to a linear isometry between $\mathscr{H}$ and the Gaussian space spanned by W. We will denote this isometry by

$$W(\phi) = \int_0^\infty \int_{\mathbb{R}^d} \phi(t,x) W(dt,dx)$$

for $\phi \in \mathscr{H}$. Notice that if ϕ and ψ are in $\mathscr{H}$, then $\mathbf{E}[W(\phi)W(\psi)] = \langle \phi, \psi \rangle_{\mathscr{H}}$. Furthermore, $\mathscr{H}$ contains the class of measurable functions ϕ on $\mathbb{R}_+ \times \mathbb{R}^d$ such that

$$\int_{\mathbb{R}_+^2 \times \mathbb{R}^{2d}} |\phi(s,x)\phi(t,y)|\,\gamma(s-t)\Lambda(x-y)\,dxdydsdt < \infty. \tag{2.3}$$

We shall make a standard assumption on the spectral measure μ, which will prevail until the end of the paper.

Hypothesis 4.1. *The measure μ satisfies the following integrability condition:*

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty. \tag{2.4}$$

Let us now recall some of the main examples of stationary covariances, which will be our guiding examples in the remainder of the paper.

Example 4.2. *One of the most popular spatial covariances is given by the so-called Riesz kernel, for which* $\Lambda(x) = |x|^{-\eta}$ *and* $\mu(d\xi) = c_{\eta,d}|\xi|^{-(d-\eta)}\,d\xi$. *We refer to this kind of noise as a spatial* η*-Riesz noise. In this case, Hypothesis 4.1 is satisfied whenever* $0 < \eta < 2$*, which will be our standing assumption.*

Example 4.3. *We shall also handle the space white noise case, namely* $\Lambda = \delta_0$ *(notice that in this case* Λ *is not a function but a measure) and* $\mu =$ *Lebesgue. This noise satisfies Hypothesis 4.1 only in dimension 1.*

Example 4.4. *The spatial covariance given by the so-called Bessel kernel is defined by*

$$\Lambda(x) = \int_0^\infty w^{\frac{\eta-d}{2}} e^{-w} e^{-\frac{|x|^2}{4w}}\,dw\,.$$

In this case $\mu(d\xi) = c_{\eta,d}(1+|\xi|^2)^{-\frac{\eta}{2}}d\xi$ *and Hypothesis 4.1 is satisfied if* $\eta > d-2$.

Example 4.5. *An example of time covariance* γ *that has received a lot of attention is the case of a one-dimensional Riesz kernel, which corresponds to the fractional Brownian motion. Suppose that* $\gamma(t) = H(2H-1)|t|^{2H-2}$ *with* $\frac{1}{2} < H < 1$ *and* W *is a noise with this time covariance and a spatial covariance* Λ*. For any* $t \geq 0$ *and any* $\varphi \in C_c^\infty(\mathbb{R}^d)$*, the function* $\mathbf{1}_{[0,t]}\varphi$ *belongs to the space* $\mathscr{H}$*, and we can define* $W_t(\varphi) := W(\mathbf{1}_{[0,t]}\varphi)$*. Then, for any fixed* $\varphi \in \mathscr{D}(\mathbb{R}^d)$*, the stochastic process* $\{c_\varphi^{-1/2}W_t(\varphi); t \geq 0\}$ *is a fractional Brownian motion with Hurst parameter* H*, where*

$$c_\varphi = \int_{\mathbb{R}^d} |\mathscr{F}\varphi(\xi)|^2 \mu(d\xi).$$

That is $\mathbf{E}[W_t(\varphi)W_s(\varphi)] = R_H(s,t)c_\varphi$*, where for each* $H \in (0,1)$ *we have:*

$$R_H(s,t) = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |s-t|^{2H}\right).$$

Example 4.6. *In the same way, the spatial fractional covariance is given by* $\Lambda(x) = \prod_{i=1}^d H_i\,(2H_i-2)|x_i|^{2H_i-2}$*, where* $\frac{1}{2} < H_i < 1$ *for* $i = 1,\dots,d$*. The Fourier transform of* Λ *is* $\mu(d\xi) = C_H \prod_{i=1}^d |\xi_i|^{1-2H_i} d\xi$*, where* C_H *is a constant depending on the parameters* H_i*. Then an easy calculation shows that when* $\sum_{i=1}^d H_i > d-1$*, Hypothesis 4.1 holds.*

If W is a noise with fractional space and time covariances, with Hurst parameters H_0 in time, and $H_1,\dots,H_d$ in space, then we can write formally $W(\varphi)$ as the distributional integral $\int_{\mathbb{R}_+\times\mathbb{R}^d} \varphi(t,x)\dot{W}_H(t,x)dtdx$, where $\dot{W}_H(t,x)$ is the formal partial derivative $\frac{\partial^{d+1}W_H}{\partial t\partial x_1\cdots\partial x_d}(t,x)$ and W_H is centered Gaussian random field which is a fractional Brownian motion on each coordinate, that is,

$$\mathbf{E}[W_H(s,x)W_H(t,y)] = R_{H_0}(s,t)\prod_{i=1}^d R_{H_i}(x_i,y_i), \quad s,t \geq 0, x,y \in \mathbb{R}^d.$$

4.2.2 Time independent noise

In this case we consider a zero mean Gaussian family $W = \{W(\varphi); \varphi \in \mathscr{D}(\mathbb{R}^d)\}$, defined in a complete probability space $(\Omega, \mathscr{F}, \mathbf{P})$, with covariance

$$\mathbf{E}[W(\varphi)W(\psi)] = \int_{\mathbb{R}^{2d}} \varphi(x)\psi(y)\Lambda(x-y)\,dxdy, \tag{2.5}$$

where, as before, $\Lambda: \mathbb{R}^d \to \mathbb{R}_+$ is a non-negative definite function whose Fourier transform μ is a tempered measure. In this case $\mathscr{H}$ is the completion of $\mathscr{D}(\mathbb{R}^d)$ endowed with the inner product

$$\langle \varphi, \psi \rangle_{\mathscr{H}} = \int_{\mathbb{R}^{2d}} \varphi(x)\psi(y)\Lambda(x-y)dxdy = \int_{\mathbb{R}^d} \mathscr{F}\varphi(\xi)\overline{\mathscr{F}\psi(\xi)}\mu(d\xi). \tag{2.6}$$

The mapping $\varphi \to W(\varphi)$ defined in $\mathscr{D}(\mathbb{R}^d)$ extends to a linear isometry between $\mathscr{H}$ and the Gaussian space spanned by W, denoted by

$$W(\phi) = \int_{\mathbb{R}^d} \phi(x)W(dx)$$

for $\phi \in \mathscr{H}$. If ϕ and ψ are in $\mathscr{H}$, then $\mathbf{E}[W(\phi)W(\psi)] = \langle \phi, \psi \rangle_{\mathscr{H}}$ and $\mathscr{H}$ contains the class of measurable functions ϕ on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^{2d}} |\phi(x)\phi(y)|\,\Lambda(x-y)\,dxdy < \infty. \tag{2.7}$$

4.2.3 Elements of Malliavin calculus

Consider first the case of a time dependent noise. We will denote by D the derivative operator in the sense of Malliavin calculus. That is, if F is a smooth and cylindrical random variable of the form

$$F = f(W(\phi_1), \dots, W(\phi_n)),$$

with $\phi_i \in \mathscr{H}$, $f \in C_p^\infty(\mathbb{R}^n)$ (namely f and all its partial derivatives have polynomial growth), then DF is the $\mathscr{H}$-valued random variable defined by

$$DF = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(W(\phi_1), \dots, W(\phi_n))\phi_j.$$

The operator D is closable from $L^2(\Omega)$ into $L^2(\Omega;\mathscr{H})$ and we define the Sobolev space $\mathbb{D}^{1,2}$ as the closure of the space of smooth and cylindrical random variables under the norm

$$\|DF\|_{1,2}=\sqrt{\mathbf{E}[F^2]+\mathbf{E}[\|DF\|_{\mathscr{H}}^2]}\,.$$

We denote by δ the adjoint of the derivative operator given by the duality formula

$$\mathbf{E}\left[\delta(u)F\right]=\mathbf{E}\left[\langle DF,u\rangle_{\mathscr{H}}\right], \tag{2.8}$$

for any $F\in\mathbb{D}^{1,2}$ and any element $u\in L^2(\Omega;\mathscr{H})$ in the domain of δ. The operator δ is also called the *Skorohod integral* because in the case of the Brownian motion, it coincides with an extension of the Itô integral introduced by Skorohod. We refer to Nualart [74] for a detailed account of the Malliavin calculus with respect to a Gaussian process. If DF and u are almost surely measurable functions on $\mathbb{R}_+\times\mathbb{R}^d$ verifying condition (2.3), then the duality formula (2.11) can be written using the expression of the inner product in $\mathscr{H}$ given in (2.2)

$$\mathbf{E}\left[\delta(u)F\right]=\mathbf{E}\left[\int_{\mathbb{R}_+^2\times\mathbb{R}^{2d}}D_{s,x}F\,u_{t,y}\,\gamma(s-t)\,\Lambda(x-y)\,dsdtdxdy\right]. \tag{2.9}$$

Let us recall 3 other classical relations in stochastic analysis, which will be used in the paper:

(i) Divergence type formula. For any $\phi\in\mathscr{H}$ and any random variable F in the Sobolev space $\mathbb{D}^{1,2}$, we have

$$FW(\phi)=\delta(F\phi)+\langle DF,\phi\rangle_{\mathscr{H}}. \tag{2.10}$$

(ii) A duality relationship. Given a random variable $F \in \mathbb{D}^{2,2}$ and two elements $h, g \in \mathscr{H}$, the duality formula (2.11) implies

$$\mathbf{E}[F\,W(h)W(g)] = \mathbf{E}\left[\langle D^2F, h\otimes g\rangle_{\mathscr{H}^{\otimes 2}}\right] + \mathbf{E}[F]\,\langle h, g\rangle_{\mathscr{H}}. \tag{2.11}$$

(iii) Definition of the Wick product of a random and a Gaussian element. If $F \in \mathbb{D}^{1,2}$ and h is an element of $\mathscr{H}$, then Fh is Skorohod integrable and, by definition, the Wick product equals to the Skorohod integral of Fh

$$\delta(Fh) = F \diamond W(h). \tag{2.12}$$

When handling the stochastic heat equation in the Skorohod sense we will make use of chaos expansions, and we should give a small account on this notion. For any integer $n \geq 0$ we denote by $\mathbf{H}_n$ the nth Wiener chaos of W. We recall that $\mathbf{H}_0$ is simply $\mathbb{R}$ and for $n \geq 1$, $\mathbf{H}_n$ is the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_n(W(h)); h \in \mathscr{H}, \|h\|_{\mathscr{H}} = 1\}$, where H_n is the nth Hermite polynomial. For any $n \geq 1$, we denote by $\mathscr{H}^{\otimes n}$ (resp. $\mathscr{H}^{\odot n}$) the nth tensor product (resp. the nth symmetric tensor product) of $\mathscr{H}$. Then, the mapping $I_n(h^{\otimes n}) = H_n(W(h))$ can be extended to a linear isometry between $\mathscr{H}^{\odot n}$ (equipped with the modified norm $\sqrt{n!}\|\cdot\|_{\mathscr{H}^{\otimes n}}$) and $\mathbf{H}_n$.

Consider now a random variable $F \in L^2(\Omega)$ measurable with respect to the σ-field $\mathscr{F}^W$ generated by W. This random variable can be expressed as

$$F = \mathbf{E}[F] + \sum_{n=1}^{\infty} I_n(f_n), \tag{2.13}$$

where the series converges in $L^2(\Omega)$, and the elements $f_n \in \mathscr{H}^{\odot n}$, $n \geq 1$, are determined by F. This identity is called the Wiener-chaos expansion of F.

The Skorohod integral (or divergence) of a random field u can be computed by using the Wiener chaos expansion. More precisely, suppose that $u = \{u_{t,x}; (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ is a random field such that for each (t,x), $u_{t,x}$ is an $\mathscr{F}^W$-measurable and square integrable random variable. Then, for each (t,x) we have a Wiener chaos expansion of the form

$$u_{t,x} = \mathbf{E}[u_{t,x}] + \sum_{n=1}^{\infty} I_n(f_n(\cdot,t,x)). \tag{2.14}$$

Suppose also that

$$\mathbf{E}\left[\int_0^\infty \int_0^\infty \int_{\mathbb{R}^{2d}} |u_{t,x} u_{s,y}| \gamma(s-t)\Lambda(x-y)\, dxdydsdt\right] < \infty.$$

Then, we can interpret u as a square integrable random function with values in $\mathscr{H}$ and the kernels f_n in the expansion (2.13) are functions in $\mathscr{H}^{\otimes(n+1)}$ which are symmetric in the first n variables. In this situation, u belongs to the domain of the divergence (that is, u is Skorohod integrable with respect to W) if and only if the following series converges in $L^2(\Omega)$

$$\delta(u) = \int_0^\infty \int_{\mathbb{R}^d} u_{t,s}\, \delta W_{t,x} = W(\mathbf{E}[u]) + \sum_{n=1}^{\infty} I_{n+1}(\widetilde{f}_n(\cdot,t,x)), \tag{2.15}$$

where $\widetilde{f}_n$ denotes the symmetrization of f_n in all its $n+1$ variables.

The operators D and δ can be introduced in a similar way in the time independent case. If DF and u are almost surely measurable functions on $\mathbb{R}^d$ verifying condition (2.7), then formula (2.11) can be written using the expression of the inner product in $\mathscr{H}$ given in (2.6):

$$\mathbf{E}[\delta(u)F] = \mathbf{E}\left[\int_{\mathbb{R}^{2d}} D_x F\, u(y)\, \Lambda(x-y)\, dxdy\right]. \tag{2.16}$$

4.3 Equation of Skorohod type

The first part of this section is devoted to the study of the following d-dimensional stochastic heat equation with the time dependent multiplicative Gaussian noise W introduced in Section 4.2.1, where the product is understood in the Wick sense (see (2.12)):

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u \diamond \frac{\partial^{d+1} W}{\partial t \partial x_1 \cdots \partial x_d}, \tag{3.1}$$

the initial condition being a continuous and bounded function $u_0(x)$. This equation is formal and below we provide a rigorous definition of a mild solution using the Skorohod integral. The main objective of this section is to show that the mild solution exists and is unique in $L^2(\Omega)$, assuming that the spectral measure μ satisfies Hypothesis 4.1. This is proved by showing that the formal Wiener chaos expansion which defines the solution converges in $L^2(\Omega)$. In a second part of this section we obtain a Feynman-Kac formula for the moments of the solution. In the last part we will extend these results to the case where the noise is time independent.

4.3.1 Existence and uniqueness of a solution via chaos expansions

Recall that we denote by $p_t(x)$ the d-dimensional heat kernel $p_t(x) = (2\pi t)^{-d/2} e^{-|x|^2/2t}$, for any $t > 0$, $x \in \mathbb{R}^d$. For each $t \geq 0$ let $\mathscr{F}_t$ be the σ-field generated by the random variables $W(\varphi)$, where φ has support in $[0,t] \times \mathbb{R}^d$. We say that a random field $u_{t,x}$ is adapted if for each (t,x) the random variable $u_{t,x}$ is $\mathscr{F}_t$-measurable. We define the solution of equation (3.1) as follows.

Definition 4.7. *An adapted random field $u = \{u_{t,x}; t \geq 0, x \in \mathbb{R}^d\}$ such that $\mathbf{E}[u_{t,x}^2] < \infty$ for all (t,x) is a mild solution to equation (3.1) with initial condition $u_0 \in C_b(\mathbb{R}^d)$, if for any $(t,x) \in [0,\infty) \times \mathbb{R}^d$, the process $\{p_{t-s}(x-y) u_{s,y} \mathbf{1}_{[0,t]}(s); s \geq 0, y \in \mathbb{R}^d\}$ is Skorohod* *integrable, and the following equation holds*

$$u_{t,x} = p_t u_0(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) u_{s,y}\, \delta W_{s,y}. \tag{3.2}$$

Suppose now that $u = \{u_{t,x}; t \geq 0, x \in \mathbb{R}^d\}$ is a solution to equation (3.2). Then according to (2.12), for any fixed (t,x) the random variable $u_{t,x}$ admits the following Wiener chaos expansion

$$u_{t,x} = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t,x)), \tag{3.3}$$

where for each (t,x), $f_n(\cdot,t,x)$ is a symmetric element in $\mathscr{H}^{\otimes n}$. Thanks to (2.14) and using an iteration procedure, one can then find an explicit formula for the kernels f_n for $n \geq 1$

$$f_n(s_1,x_1,\dots,s_n,x_n,t,x) = \frac{1}{n!} p_{t-s_{\sigma(n)}}(x - x_{\sigma(n)}) \cdots p_{s_{\sigma(2)}-s_{\sigma(1)}}(x_{\sigma(2)} - x_{\sigma(1)}) p_{s_{\sigma(1)}} u_0(x_{\sigma(1)}),$$

where σ denotes the permutation of $\{1,2,\dots,n\}$ such that $0 < s_{\sigma(1)} < \cdots < s_{\sigma(n)} < t$ (see, for instance, equation (4.4) in [55], where this formula is established in the case of a noise which is white in space). Then, to show the existence and uniqueness of the solution it suffices to show that for all (t,x) we have

$$\sum_{n=0}^{\infty} n! \|f_n(\cdot,t,x)\|^2_{\mathscr{H}^{\otimes n}} < \infty. \tag{3.4}$$

Theorem 4.8. *Suppose that μ satisfies Hypothesis 4.1 and γ is locally integrable. Then relation (5.10) holds for each (t,x). Consequently, equation (3.1) admits a unique mild solution in the sense of Definition 4.7.*

Proof. Fix $t > 0$ and $x \in \mathbb{R}^d$. Set $f_n(s,y,t,x) = f_n(s_1,y_1,\dots,s_n,y_n,t,x)$. We have the following expression

$$\begin{aligned} & n!\|f_n(\cdot,t,x)\|^2_{\mathscr{H}^{\otimes n}} \\ = \; & n!\int_{[0,t]^{2n}}\int_{\mathbb{R}^{2nd}} f_n(s,y,t,x)f_n(r,z,t,x)\prod_{i=1}^n \Lambda(y_i - z_i)\prod_{i=1}^n \gamma(s_i - r_i)\,dydzdsdr \\ \leq \; & n!\|u_0\|^2_\infty \int_{[0,t]^{2n}}\int_{\mathbb{R}^{2nd}} g_n(s,y,t,x)g_n(r,z,t,x)\prod_{i=1}^n \Lambda(y_i - z_i)\prod_{i=1}^n \gamma(s_i - r_i)\,dydzdsdr \end{aligned}$$

where $dx = dx_1 \cdots dx_n$, the differentials dy, ds and dr are defined similarly and

$$g_n(s,y,t,x) = \frac{1}{n!}p_{t-s_{\sigma(n)}}(x - y_{\sigma(n)}) \cdots p_{s_{\sigma(2)}-s_{\sigma(1)}}(y_{\sigma(2)} - y_{\sigma(1)})\,. \tag{3.5}$$

Set now $\mu(d\xi) \equiv \prod_{i=1}^n \mu(d\xi_i)$. Using the Fourier transform and Cauchy-Schwarz, we obtain

$$\begin{aligned} & n!\|f_n(\cdot,t,x)\|^2_{\mathscr{H}^{\otimes n}} \\ \leq \; & n!\|u_0\|^2_\infty \int_{[0,t]^{2n}}\int_{\mathbb{R}^{nd}} \mathscr{F}g_n(s,\cdot,t,x)(\xi)\overline{\mathscr{F}g_n(r,\cdot,t,x)(\xi)}\mu(d\xi)\prod_{i=1}^n \gamma(s_i - r_i)dsdr \\ \leq \; & n!\|u_0\|^2_\infty \int_{[0,t]^{2n}}\left(\int_{\mathbb{R}^{nd}} |\mathscr{F}g_n(s,\cdot,t,x)(\xi)|^2\mu(d\xi)\right)^{\frac{1}{2}} \\ & \times\left(\int_{\mathbb{R}^{nd}} |\mathscr{F}g_n(r,\cdot,t,x)(\xi)|^2\mu(d\xi)\right)^{\frac{1}{2}}\prod_{i=1}^n \gamma(s_i - r_i)dsdr, \end{aligned}$$

and thus, thanks to the basic inequality $ab \leq \frac{1}{2}(a^2 + b^2)$ and the fact that γ is locally integrable, this yields:

$$n!\|f_n(\cdot,t,x)\|^2_{\mathscr{H}^{\otimes n}} \;\leq\; n!\|u_0\|^2_\infty \int_{[0,t]^{2n}}\int_{\mathbb{R}^{nd}} |\mathscr{F}g_n(s,\cdot,t,x)(\xi)|^2\mu(d\xi)\prod_{i=1}^n \gamma(s_i - r_i)dsdr$$

$$\leq \quad C^n n! \|u_0\|_\infty^2 \int_{[0,t]^{2n}} \int_{\mathbb{R}^{nd}} |\mathscr{F} g_n(s,\cdot,t,x)(\xi)|^2 \mu(d\xi) ds, \tag{3.6}$$

where $C = 2\int_0^t \gamma(r)dr$. Furthermore, it is readily checked from expression (3.5) that there exists a constant $C > 0$ such that the Fourier transform of g_n satisfies

$$|\mathscr{F} g_n(s,\cdot,t,x)(\xi)|^2 = \frac{C^n}{(n!)^2} \prod_{i=1}^n e^{-(s_{\sigma(i+1)} - s_{\sigma(i)})|\xi_{\sigma(i)} + \cdots + \xi_{\sigma(1)}|^2},$$

where we have set $s_{\sigma(n+1)} = t$. As a consequence,

$$\begin{aligned}
& (n!)^2 \int_{\mathbb{R}^{nd}} |\mathscr{F} g_n(s,\cdot,t,x)(\xi)|^2 \mu(d\xi) \\
\leq \quad & C^n \prod_{i=1}^n \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} e^{-(s_{\sigma(i+1)} - s_{\sigma(i)})|\xi_{\sigma(i)} + \eta|^2} \mu(d\xi_{\sigma(i)}) \\
= \quad & C^n \prod_{i=1}^n \sup_{\eta \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} \frac{e^{-\frac{|x_{\sigma(i)}|^2}{4(s_{\sigma(i+1)} - s_{\sigma(i)})}}}{(4\pi(s_{\sigma(i+1)} - s_{\sigma(i)}))^{\frac{d}{2}}} e^{\iota x_{\sigma(i)} \cdot \eta} \Lambda(x_{\sigma(i)}) dx_{\sigma(i)} \right| \\
\leq \quad & C^n \prod_{i=1}^n \int_{\mathbb{R}^d} e^{-(s_{\sigma(i+1)} - s_{\sigma(i)})|\xi_{\sigma(i)}|^2} \mu(d\xi_{\sigma(i)}),
\end{aligned} \tag{3.7}$$

where we invoke the fact that $|e^{\iota x_{\sigma(i)} \cdot \eta}| = 1$ to get rid of the supremum in η. Therefore, from relations (3.6) and (3.7) we obtain

$$n! \|f_n(\cdot,t,x)\|_{\mathscr{H}^{\otimes n}}^2 \leq \|u_0\|_\infty^2 C^n \int_{\mathbb{R}^{nd}} \int_{T_n(t)} \prod_{i=1}^n e^{-(s_{i+1} - s_i)|\xi_i|^2} ds\, \mu(d\xi), \tag{3.8}$$

where we denote by $T_n(t)$ the simplex

$$T_n(t) = \{0 < s_1 < \cdots < s_n < t\}. \tag{3.9}$$

Let us now estimate the right hand side of (3.7): making the change of variables $s_{i+1} - s_i = w_i$ for $1 \leq i \leq n-1$, and $t - s_n = w_n$, and denoting $dw = dw_1 dw_2 \cdots dw_n$, we end up with

$$n!\|f_n(\cdot,t,x)\|^2_{\mathscr{H}^{\otimes n}} \le \|u_0\|^2_\infty C^n \int_{\mathbb{R}^{nd}} \int_{S_{t,n}} e^{-\sum_{i=1}^n w_i|\xi_i|^2} dw \prod_{i=1}^n \mu(d\xi_i)\,,$$

where $S_{t,n} = \{(w_1,\dots,w_n) \in [0,\infty)^n : w_1+\cdots+w_n \le t\}$. We also split the contribution of μ in the following way: fix $N \ge 1$ and set

$$C_N = \int_{|\xi|\ge N} \frac{\mu(d\xi)}{|\xi|^2}, \quad \text{and} \quad D_N = \mu\{\xi \in \mathbb{R}^d : |\xi| \le N\}. \tag{3.10}$$

By Lemma 4.9 below, we can write

$$n!\|f_n(\cdot,t,x)\|^2_{\mathscr{H}^{\otimes n}} \le \|u_0\|^2_\infty C^n \sum_{k=0}^n \binom{n}{k} \frac{t^k}{k!} D_N^k (2C_N)^{n-k}. \tag{3.11}$$

Next we choose a sufficiently large N such that $2CC_N < 1$, which is possible because of condition (2.4). Using the inequality $\binom{n}{k} \le 2^n$ for any positive integers n and $0 \le k \le n$, we have

$$\begin{aligned}
&\sum_{n=0}^\infty n!\|f_n(\cdot,t,x)\|^2_{\mathscr{H}^{\otimes n}} \le \|u_0\|^2_\infty \sum_{n=0}^\infty C^n \sum_{k=0}^n \binom{n}{k} \frac{t^k}{k!} D_N^k (2C_N)^{n-k} \\
&\le \|u_0\|^2_\infty \sum_{k=0}^\infty \sum_{n=k}^\infty C^n 2^n \frac{t^k}{k!} D_N^k (2C_N)^{n-k} = \|u_0\|^2_\infty \sum_{k=0}^\infty \frac{t^k}{k!} D_N^k (2C_N)^{-k} \sum_{n=k}^\infty (2CC_N)^n \\
&\le \|u_0\|^2_\infty \frac{1}{1-2CC_N} \sum_{k=0}^\infty \frac{t^k D_N^k (2C_N)^{-k} (2CC_N)^k}{k!} < \infty.
\end{aligned}$$

This proves the theorem. □

Next we establish the lemma that is used in the proof of Theorem 4.8.

Lemma 4.9. *Let μ satisfy the condition (2.4). For any $N > 0$ we let D_N and C_N be given by (3.10). Then we have*

$$\int_{\mathbb{R}^{nd}} \int_{S_{t,n}} e^{-\sum_{i=1}^n w_i|\xi_i|^2} dw \prod_{i=1}^n \mu(d\xi_i) \leq \sum_{k=0}^n \binom{n}{k} \frac{t^k}{k!} D_N^k (2C_N)^{n-k}.$$

Proof. By our assumption (2.4), C_N is finite for all positive N. Let I be a subset of $\{1,2,\dots,n\}$ and $I^c = \{1,2,\dots,n\} \setminus I$. Then we have

$$\begin{aligned}
&\int_{\mathbb{R}^{nd}} \int_{S_{t,n}} \prod_{i=1}^n e^{-w_i|\xi_i|^2} \, dw \, \mu(d\xi) \\
= &\int_{\mathbb{R}^{nd}} \int_{S_{t,n}} \prod_{i=1}^n e^{-w_i|\xi_i|^2} (\mathbf{1}_{\{|\xi_i| \leq N\}} + \mathbf{1}_{\{|\xi_i| > N\}}) \, dw \, \mu(d\xi) \\
= &\sum_{I \subset \{1,2,\dots,n\}} \int_{\mathbb{R}^{nd}} \int_{S_{t,n}} \prod_{i \in I} e^{-w_i|\xi_i|^2} \mathbf{1}_{\{|\xi_i| \leq N\}} \times \prod_{j \in I^c} e^{-w_j|\xi_j|^2} \mathbf{1}_{\{|\xi_j| \geq N\}} \, dw \, \mu(d\xi).
\end{aligned}$$

For the indices i in the set I we estimate $e^{-w_j|\xi_j|^2}$ by 1. Then, using the inclusion

$$S_{t,n} \subset S_t^I \times S_t^{I^c},$$

where $S_t^I = \{(w_i, i \in I) : w_i \geq 0, \sum_{i \in I} w_i \leq t\}$ and $S_t^{I^c} = \{(w_i, i \in I^c) : w_i \geq 0, \sum_{i \in I^c} w_i \leq t\}$ we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{nd}} \int_{S_{t,n}} \prod_{i=1}^n e^{-w_i|\xi_i|^2} \, dw \, \mu(d\xi) \\
\leq &\sum_{I \subset \{1,2,\cdots,n\}} \int_{\mathbb{R}^{nd}} \int_{S_t^I \times S_t^{I^c}} \prod_{i \in I} \mathbf{1}_{\{|\xi_i| \leq N\}} \times \prod_{j \in I^c} e^{-w_j|\xi_j|^2} \mathbf{1}_{\{|\xi_j| \geq N\}} \, dw \, \mu(d\xi).
\end{aligned}$$

Furthermore, one can bound the integral over $S_t^{I^c}$ in the following way

$$\int_{S_t^{I^c}} \prod_{j \in I^c} e^{-w_j|\xi_j|^2} dw \leq \int_{[0,t]^{I^c}} \prod_{j \in I^c} e^{-w_j|\xi_j|^2} dw = \prod_{j \in I^c} \frac{1 - e^{-t|\xi_j|^2}}{|\xi_j|^2} \leq \prod_{j \in I^c} \frac{1}{|\xi_j|^2}.$$

We can thus bound $\int_{\mathbb{R}^{nd}}\int_{S_{t,n}}\prod_{i=1}^n e^{-w_i|\xi_i|^2}\,dw\,\mu(d\xi)$ by:

$$\sum_{I\subset\{1,2,\cdots,n\}}\frac{t^{|I|}}{|I|!}\big(\mu\{\xi\in\mathbb{R}^d:|\xi|\le N\}\big)^{|I|}2^{|I^c|}\int_{|\xi_j|>N,\forall j\in I^c}\prod_{j\in I^c}\frac{\mu(d\xi_j)}{|\xi_j|^2}$$
$$=\sum_{I\subset\{1,2,\cdots,n\}}\frac{t^{|I|}}{|I|!}D_N^{|I|}(2C_N)^{|I^c|}=\sum_{k=0}^n\binom{n}{k}\frac{t^k}{k!}D_N^k(2C_N)^{n-k},$$

which is our claim. □

4.3.2 Feynman-Kac formula for the moments

Our next objective is to find a formula for the moments of the mild solution to equation (3.1). For any $\delta>0$, we define the function $\varphi_\delta(t)=\frac{1}{\delta}\mathbf{1}_{[0,\delta]}(t)$ for $t\in\mathbb{R}$. Then, $\varphi_\delta(t)p_\varepsilon(x)$ provides an approximation of the Dirac delta function $\delta_0(t,x)$ as ε and δ tend to zero.

We set

$$\dot{W}_{t,x}^{\varepsilon,\delta}=\int_0^t\int_{\mathbb{R}^d}\varphi_\delta(t-s)p_\varepsilon(x-y)W(ds,dy)\,. \tag{3.12}$$

Now we consider the approximation of equation (3.1) defined by

$$\frac{\partial u_{t,x}^{\varepsilon,\delta}}{\partial t}=\frac{1}{2}\Delta u_{t,x}^{\varepsilon,\delta}+u_{t,x}^{\varepsilon,\delta}\diamond\dot{W}_{t,x}^{\varepsilon,\delta}\,. \tag{3.13}$$

We recall that the Wick product $u_{t,x}^{\varepsilon,\delta}\diamond\dot{W}_{t,x}^{\varepsilon,\delta}$ is well defined as a square integrable random variable provided the random variable $u_{t,x}^{\varepsilon,\delta}$ belongs to the space $\mathbb{D}^{1,2}$ (see (2.12)), and in this case we have

$$u_{s,y}^{\varepsilon,\delta}\diamond\dot{W}_{s,y}^{\varepsilon,\delta}=\int_0^s\int_{\mathbb{R}^d}\varphi_\delta(s-r)p_\varepsilon(y-z)u_{s,y}^{\varepsilon,\delta}\,\delta W_{r,z}. \tag{3.14}$$

Furthermore, the mild or evolution version of (3.13) is

$$u_{t,x}^{\varepsilon,\delta} = p_t u_0(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) u_{s,y}^{\varepsilon,\delta} \diamond \dot{W}_{s,y}^{\varepsilon,\delta} dsdy. \tag{3.15}$$

Substituting (3.14) into (3.15), and formally applying Fubini's theorem yields

$$u_{t,x}^{\varepsilon,\delta} = p_t u_0(x) + \int_0^t \int_{\mathbb{R}^d} \left(\int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) \varphi_\delta(s-r) p_\varepsilon(y-z) u_{s,y}^{\varepsilon,\delta} dsdy \right) \delta W_{r,z}. \tag{3.16}$$

This leads to the following definition.

Definition 4.10. *An adapted random field $u^{\varepsilon,\delta} = \{u_{t,x}^{\varepsilon,\delta}; t \geq 0, x \in \mathbb{R}^d\}$ is a mild solution to equation (3.13) if for each $(r,z) \in [0,t] \times \mathbb{R}^d$ the integral*

$$Y_{r,z}^{t,x} = \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) \varphi_\delta(s-r) p_\varepsilon(y-z) u_{s,y}^{\varepsilon,\delta} dsdy$$

exists and $Y^{t,x}$ is a Skorohod integrable process such that (3.16) holds for each (t,x).

Notice that according to relation (2.11), the above definition is equivalent to saying that $u_{t,x}^{\varepsilon,\delta} \in L^2(\Omega)$, and for any random variable $F \in \mathbb{D}^{1,2}$, we have

$$\mathbf{E}\left[F u_{t,x}^{\varepsilon,\delta}\right] = \mathbf{E}[F]\, p_t u_0(x) + \mathbf{E}\left[\langle Y^{t,x}, DF \rangle_{\mathscr{H}}\right]. \tag{3.17}$$

In order to derive a Feynman-Kac formula for the moment of order $k \geq 2$ of the solution to equation (3.1) we need to introduce k independent d-dimensional Brownian motions B^j, $j = 1, \ldots, k$, which are independent of the noise W driving the equation. We shall study the probabilistic behavior of some random variables with double randomness, and we thus introduce some additional notation:

Notation 4.11. *We denote by* $\mathbf{P}, \mathbf{E}$ *the probability and expectation with respect to the annealed randomness concerning the couple* (B,W)*, where* $B = (B^1, \dots, B^k)$*, while we set respectively* $\mathbf{E}_B$ *and* $\mathbf{E}_W$ *for the expectation with respect to one randomness only.*

With this notation in mind, define

$$u^{\varepsilon,\delta}_{t,x} = \mathbf{E}_B \left[\exp\left(W(A^{\varepsilon,\delta}_{t,x}) - \frac{1}{2}\alpha^{\varepsilon,\delta}_{t,x} \right) \right], \tag{3.18}$$

where

$$A^{\varepsilon,\delta}_{t,x}(r,y) = \frac{1}{\delta}\left(\int_0^{\delta\wedge(t-r)} p_\varepsilon(B^x_{t-r-s} - y)ds \right) \mathbf{1}_{[0,t]}(r), \quad \text{and} \quad \alpha^{\varepsilon,\delta}_{t,x} = \|A^{\varepsilon,\delta}_{t,x}\|^2_{\mathscr{H}}, \tag{3.19}$$

for a standard d-dimensional Brownian motion B independent of W. Then one can prove that $u^{\varepsilon,\delta}_{t,x}$ is a mild solution to equation (3.13) in the sense of Definition 4.10. The proof is similar to the proof of Proposition 5.2 in [55] and we omit the details.

The next theorem asserts that the random variables $u^{\varepsilon,\delta}_{t,x}$ have moments of all orders, uniformly bounded in ε and δ, and converge to the mild solution of equation (3.1), which is unique by Theorem 4.8, as δ and ε tend to zero. Moreover, it provides an expression for the moments of the mild solution of equation (3.1).

Theorem 4.12. *Suppose* γ *is locally integrable and* μ *satisfies Hypothesis 4.1. Then for any integer* $k \geq 1$ *we have*

$$\sup_{\varepsilon,\delta} \mathbf{E}\left[|u^{\varepsilon,\delta}_{t,x}|^k \right] < \infty, \tag{3.20}$$

the limit $\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0} u^{\varepsilon,\delta}_{t,x}$ *exists in* L^p *for all* $p\geq 1$*, and it coincides with the mild solution u of equation (3.1). Furthermore, we have for any integer* $k\geq 2$

$$\mathbf{E}\left[u^k_{t,x}\right] = \mathbf{E}_B\left[\prod_{i=1}^k u_0(B^i_t+x)\exp\left(\sum_{1\leq i<j\leq k}\int_0^t\int_0^t \gamma(s-r)\Lambda(B^i_s-B^j_r)dsdr\right)\right], \quad (3.21)$$

where $\{B^j; j=1,\dots,k\}$ *is a family of d-dimensional independent standard Brownian motions independent of* W.

Proof. To simplify the proof we assume that u_0 is identically one. Fix an integer $k\geq 2$. Using (3.18) we have

$$\mathbf{E}\left[\left(u^{\varepsilon,\delta}_{t,x}\right)^k\right] = \mathbf{E}_W\left[\prod_{j=1}^k \mathbf{E}_B\left[\exp\left(W(A^{\varepsilon,\delta,B^j}_{t,x}) - \frac{1}{2}\alpha^{\varepsilon,\delta,B^j}_{t,x}\right)\right]\right],$$

where for any $j=1,\dots,k$, $A^{\varepsilon,\delta,B^j}_{t,x}$ and $\alpha^{\varepsilon,\delta,B^j}_{t,x}$ are evaluations of (3.19) using the Brownian motion B^j. Therefore, since $W(A^{\varepsilon,\delta,B^j}_{t,x})$ is a Gaussian random variable conditionally on B, we obtain

$$\begin{aligned}
\mathbf{E}\left[\left(u^{\varepsilon,\delta}_{t,x}\right)^k\right] &= \mathbf{E}_B\left[\exp\left(\frac{1}{2}\|\sum_{j=1}^k A^{\varepsilon,\delta,B^j}_{t,x}\|^2_{\mathscr{H}} - \frac{1}{2}\sum_{j=1}^k \alpha^{\varepsilon,\delta,B^j}_{t,x}\right)\right] \\
&= \mathbf{E}_B\left[\exp\left(\frac{1}{2}\|\sum_{j=1}^k A^{\varepsilon,\delta,B^j}_{t,x}\|^2_{\mathscr{H}} - \frac{1}{2}\sum_{j=1}^k \|A^{\varepsilon,\delta,B^j}_{t,x}\|^2_{\mathscr{H}}\right)\right] \\
&= \mathbf{E}_B\left[\exp\left(\sum_{1\leq i<j\leq k}\langle A^{\varepsilon,\delta,B^i}_{t,x}, A^{\varepsilon,\delta,B^j}_{t,x}\rangle_{\mathscr{H}}\right)\right]. \quad (3.22)
\end{aligned}$$

Let us now evaluate the quantities $\langle A^{\varepsilon,\delta,B^i}_{t,x}, A^{\varepsilon,\delta,B^j}_{t,x}\rangle_{\mathscr{H}}$ above: by the definition of $A^{\varepsilon,\delta,B^i}_{t,x}$, for any $i\neq j$ we have

$$\langle A^{\varepsilon,\delta,B^i}_{t,x}, A^{\varepsilon,\delta,B^j}_{t,x}\rangle_{\mathscr{H}} = \int_0^t\int_0^t\int_{\mathbb{R}^d} \mathscr{F}A^{\varepsilon,\delta,B^i}_{t,x}(u,\cdot)(\xi)\,\overline{\mathscr{F}A^{\varepsilon,\delta,B^j}_{t,x}}(v,\cdot)(\xi)\gamma(u-v)\mu(d\xi)dudv. \quad (3.23)$$

On the other hand, for $u \in [0,t]$ we can write

$$\begin{aligned}\mathscr{F}A_{t,x}^{\varepsilon,\delta,B^i}(u,\cdot)(\xi) &= \frac{1}{\delta}\int_0^{\delta\wedge(t-u)} \mathscr{F}p_\varepsilon(B^i_{t-u-s}+x-\cdot)(\xi)ds \\ &= \frac{1}{\delta}\int_0^{\delta\wedge(t-u)} \exp\left(-\frac{\varepsilon^2|\xi|^2}{2}+\imath\left\langle \xi, B^i_{t-u-s}+x\right\rangle\right)ds.\end{aligned}$$

Thus

$$\begin{aligned}&\langle A_{t,x}^{\varepsilon,\delta,B^i}, A_{t,x}^{\varepsilon,\delta,B^j}\rangle_{\mathscr{H}} \qquad (3.24)\\ &\quad = \int_{\mathbb{R}^d}\left(\int_0^t\int_0^t\left(\frac{1}{\delta^2}\int_0^{\delta\wedge v}\int_0^{\delta\wedge u} e^{\imath\left\langle \xi, B^i_{u-s_1}-B^j_{v-s_2}\right\rangle}ds_1ds_2\right)\gamma(u-v)dudv\right)\\ &\times e^{-\varepsilon^2|\xi|^2}\mu(d\xi),\end{aligned}$$

and we divide the proof in several steps.

Step 1: We claim that,

$$\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0}\langle A_{t,x}^{\varepsilon,\delta,B^i}, A_{t,x}^{\varepsilon,\delta,B^j}\rangle_{\mathscr{H}} = \int_0^t\int_0^t \gamma(u-v)\Lambda(B^i_u-B^j_v)dudv, \qquad (3.25)$$

where the convergence holds in $L^1(\Omega)$. Notice first that the right-hand side of equation (3.25) is finite almost surely because

$$\mathbf{E}_B\left[\int_0^t\int_0^t \gamma(u-v)\Lambda(B^i_u-B^j_v)dudv\right] = \int_0^t\int_0^t\int_{\mathbb{R}^d}\gamma(u-v)e^{-\frac{1}{2}(u+v)|\xi|^2}\mu(d\xi)dudv$$

and we show that this is finite making the change of variables $x = u-v$, $y = u+v$, and using our hypothesis on γ and μ like in the proof of Theorem 4.8.

In order to show the convergence (3.25) we first let δ tend to zero. Then, owing to the continuity of B and applying some dominated convergence arguments to (3.24), we obtain the following limit almost surely and in $L^1(\Omega)$

$$\lim_{\delta\downarrow 0}\langle A_{t,x}^{\varepsilon,\delta,B^i}, A_{t,x}^{\varepsilon,\delta,B^j}\rangle_{\mathscr{H}} = \int_{\mathbb{R}^d}\left(\int_0^t\int_0^t e^{\iota\left\langle \xi, B_u^i - B_v^j\right\rangle}\gamma(u-v)dudv\right)e^{-\varepsilon^2|\xi|^2}\,\mu(d\xi). \tag{3.26}$$

Finally, it is easily checked that the right-hand side of (3.26) converges in $L^1(\Omega)$ to the right-hand side of (3.25) as ε tends to zero, by means of a simple dominated convergence argument again.

Step 2: For notational convenience, we denote by B and $\widetilde{B}$ two independent d-dimensional Brownian motions, and $\mathbf{E}$ will denote here the expectation with respect to both B and $\widetilde{B}$. We claim that for any $\lambda > 0$

$$\sup_{\varepsilon,\delta}\mathbf{E}\left[\exp\left(\lambda\left\langle A_{t,x}^{\varepsilon,\delta,B}, A_{t,x}^{\varepsilon,\delta,\widetilde{B}}\right\rangle_{\mathscr{H}}\right)\right] < \infty. \tag{3.27}$$

Indeed, starting from (3.24), making the change of variables $u - s_1 \to u$, $v - s_2 \to v$, assuming $\delta \le t$, and using Fubini's theorem, we can write

$$\left\langle A_{t,x}^{\varepsilon,\delta,B}, A_{t,x}^{\varepsilon,\delta,\widetilde{B}}\right\rangle_{\mathscr{H}} = \frac{1}{\delta^2}\int_0^\delta\int_0^\delta\int_0^{t-s_1}\int_0^{t-s_2}\int_{\mathbb{R}^d}\exp\left(-\iota(B_u - \widetilde{B}_v)\cdot\xi\right)$$
$$\times\exp(-\varepsilon|\xi|^2)\gamma(u+s_1-v-s_2)\,\mu(d\xi)\,dudvds_1ds_2\,.$$

We now control the moments of $\langle A_{t,x}^{\varepsilon,\delta,B}, A_{t,x}^{\varepsilon,\delta,\tilde{B}}\rangle_{\mathscr{H}}$ in order to reach exponential integrability:

$$\left\langle A_{t,x}^{\varepsilon,\delta,B}, A_{t,x}^{\varepsilon,\delta,\widetilde{B}}\right\rangle_{\mathscr{H}}^n = \frac{1}{\delta^{2n}}\int_{O_{\delta,n}}\int_{\mathbb{R}^{dn}}\exp\left(-\iota\sum_{l=1}^n(B_{u_l} - \widetilde{B}_{v_l})\cdot\xi_l\right)$$
$$\times e^{-\varepsilon\sum_{l=1}^n|\xi_l|^2}\prod_{l=1}^n\gamma(u_l+s_l-v_l-\widetilde{s}_l)\,\mu(d\xi)\,dsd\tilde{s}dudv, \tag{3.28}$$

where $\mu(d\xi)=\prod_{l=1}^{n}\mu(d\xi_l)$, the differentials $ds, d\tilde{s}, du, dv$ are defined similarly, and

$$O_{\delta,n}=\left\{(s,\widetilde{s},u,v);\, 0\le s_l,\widetilde{s}_l\le\delta,\, 0\le u_l\le t-s_l,\, 0\le v_l\le t-\widetilde{s}_l,\ \text{for all } 1\le l\le n\right\}.$$

Moreover, we have:

$$\begin{aligned}\mathbf{E}\left[\exp\left(-\iota\sum_{l=1}^{n}(B_{u_l}-\widetilde{B}_{v_l})\cdot\xi_l\right)\right] &= \exp\left(-\frac{1}{2}\mathbf{Var}\left(\sum_{l=1}^{n}(B_{u_l}-\widetilde{B}_{v_l})\cdot\xi_l\right)\right) \qquad (3.29)\\ &= \exp\left(-\frac{1}{2}\sum_{1\le i,j\le n}(u_i\wedge u_j+v_i\wedge v_j)\xi_i\cdot\xi_j\right).\end{aligned}$$

Taking into account the fact that γ is locally integrable, this yields

$$\begin{aligned}&\mathbf{E}\left[\left\langle A_{t,x}^{\varepsilon,\delta,B},A_{t,x}^{\varepsilon,\delta,\widetilde{B}}\right\rangle_{\mathscr{H}}^{n}\right]\\ \le\ & C^n\int_{[0,t]^{2n}}\int_{\mathbb{R}^{dn}}\exp\left(-\frac{1}{2}\sum_{1\le i,j\le n}(s_i\wedge s_j+\widetilde{s}_i\wedge\widetilde{s}_j)\xi_i\cdot\xi_j\right)\mu(d\xi)ds d\tilde{s}\\ \le\ & C^n\int_{\mathbb{R}^{dn}}\int_{[0,t]^n}\exp\left(-\sum_{1\le i,j\le n}(s_i\wedge s_j)\,\xi_i\cdot\xi_j\right)ds\mu(d\xi).\end{aligned}$$

Since

$$\int_{\mathbb{R}^{dn}}\exp\left(-\sum_{1\le i,j\le n}(s_i\wedge s_j)\xi_i\cdot\xi_j\right)\mu(d\xi)$$

is a symmetric function of $s_1,s_2,\dots,s_n$, we can restrict our integral to $T_n(t)=\{0<s_1<s_2<\cdots<s_n<t\}$. Hence, using the convention $s_0=0$, we have

$$\begin{aligned}&\mathbf{E}\left[\left\langle A_{t,x}^{\varepsilon,\delta,B},A_{t,x}^{\varepsilon,\delta,\widetilde{B}}\right\rangle_{\mathscr{H}}^{n}\right] \qquad (3.30)\\ \le\ & C^n n!\int_{\mathbb{R}^{dn}}\int_{T_n(t)}\exp\left(-\sum_{1\le i,j\le n}(s_i\wedge s_j)\xi_i\cdot\xi_j\right)ds\mu(d\xi)\\ =\ & C^n n!\int_{\mathbb{R}^{dn}}\int_{T_n(t)}\exp\left(-\sum_{i=1}^{n}(s_i-s_{i-1})|\xi_i+\cdots+\xi_n|^2\right)ds\mu(d\xi).\end{aligned}$$

Thus, using the same argument as in the proof of the estimate (3.7), we end up with

$$\begin{aligned}\mathbf{E}\left[\left\langle A_{t,x}^{\varepsilon,\delta,B}, A_{t,x}^{\varepsilon,\delta,\widetilde{B}}\right\rangle_{\mathscr{H}}^{n}\right] &\leq C^n n! \int_{T_n(t)} \prod_{i=1}^{n}\left(\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-(s_i-s_{i-1})|\xi_i+\eta|^2}\mu(d\xi)\right) ds \\ &\leq C^n n! \int_{T_n(t)} \prod_{i=1}^{n}\left(\int_{\mathbb{R}^d} e^{-(s_i-s_{i-1})|\xi_i|^2}\mu(d\xi_i)\right) ds\,.\end{aligned}$$

Making the change of variable $w_i = s_i - s_{i-1}$, the above integral is equal to

$$C^n n! \int_{S_{t,n}} \int_{\mathbb{R}^{dn}} \prod_{i=1}^{n} e^{-w_i|\xi_i|^2}\mu(d\xi)dw \leq C^n n! \sum_{k=0}^{n}\binom{n}{k}\frac{t^k}{k!}D_N^k(2C_N)^{n-k},$$

where we have resorted to Lemma 4.9 for the last inequality. Therefore,

$$\frac{1}{n!}\mathbf{E}\left[\left\langle A_{t,x}^{\varepsilon,\delta,B}, A_{t,x}^{\varepsilon,\delta,\widetilde{B}}\right\rangle_{\mathscr{H}}^{n}\right] \leq C^n \sum_{k=0}^{n}\binom{n}{k}\frac{t^k}{k!}D_N^k(2C_N)^{n-k},$$

which is exactly the right hand side of (3.11). Thus, along the same lines as in the proof of Theorem 4.8, we get

$$\mathbf{E}\left[\exp\left(\lambda\left\langle A_{t,x}^{\varepsilon,\delta,B}, A_{t,x}^{\varepsilon,\delta,\widetilde{B}}\right\rangle_{\mathscr{H}}\right)\right] = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbf{E}\left[\left\langle A_{t,x}^{\varepsilon,\delta,B}, A_{t,x}^{\varepsilon,\delta,\widetilde{B}}\right\rangle_{\mathscr{H}}^{n}\right] < \infty,$$

which completes the proof of (3.27).

Step 3: Starting from (3.22), (3.25) and (3.27) we deduce that $\mathbf{E}[(u_{t,x}^{\varepsilon,\delta})^k]$ converges as δ and ε tend to zero to the right-hand side of (3.21). On the other hand, we can also write

$$\mathbf{E}\left[u_{t,x}^{\varepsilon,\delta} u_{t,x}^{\varepsilon',\delta'}\right] = \mathbf{E}_B\left[\exp\left(\langle A_{t,x}^{\varepsilon,\delta,B^1}, A_{t,x}^{\varepsilon',\delta',B^2}\rangle_{\mathscr{H}}\right)\right].$$

As before we can show that this converges as $\varepsilon,\delta,\varepsilon',\delta'$ tend to zero. So, $u_{t,x}^{\varepsilon,\delta}$ converges in L^2 to some limit $v_{t,x}$, and the limit is actually in L^p , for all $p \geq 1$. Moreover, $\mathbf{E}[v_{t,x}^k]$ equals to the right hand side of (3.21). Finally, letting δ and ε tend to zero in equation

(3.17) we get

$$\mathbf{E}[Fv_{t,x}] = \mathbf{E}[F] + \mathbf{E}\left[\langle DF, vp_{t-\cdot}(x-\cdot)\rangle_{\mathscr{H}}\right]$$

which implies that the process v is the solution of equation (3.1), and by the uniqueness of the solution we have $v = u$. □

Remark 4.13. *If the space dimension is* 1*, we can consider equation (3.1) assuming that the time covariance function is* $\gamma(t) = H(2H-1)|t|^{2H-2}$*,* $\frac{1}{2} < H < 1$*, and the noise is white in space, which means* $\Lambda(x)$ *is the Dirac delta function* $\delta_0(x)$*. The integral form of this Gaussian noise is a two-parameter process which is a Brownian motion in space and a fractional Brownian motion with Hurst parameter* H *in time. This equation has been studied in [55], where the existence of a unique mild solution has been proved, and the following formula for the moments of the solution has been obtained*

$$\mathbf{E}\left[u_{t,x}^k\right] = \mathbf{E}_B\left[\prod_{i=1}^k u_0(B_t^i + x)\exp\left(\alpha_H \sum_{1\le i<j\le k}\int_0^t\int_0^t |s-r|^{2H-2}\delta_0(B_s^i - B_r^j)dsdr\right)\right], \tag{3.31}$$

where $\alpha_H = H(2H-1)$*. Notice that in the above expression the exponent is a sum of weighted intersection local times.*

4.3.3 Time independent noise

In this section we consider the following stochastic heat equation in the Skorohod sense driven by the multiplicative time independent noise introduced in Section 4.2.2:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u \diamond \frac{\partial^d W}{\partial x_1 \cdots \partial x_d}. \tag{3.32}$$

The notion of mild solution based on the Skorohod integral is similar to Definition 4.7.

Definition 4.14. *An adapted random field $u = \{u_{t,x}; t \geq 0, x \in \mathbb{R}^d\}$ such that $\mathbf{E}[u_{t,x}^2] < \infty$ for all (t,x) is a mild solution to equation (3.32) with initial condition $u_0 \in C_b(\mathbb{R}^d)$, if for any $0 \leq s \leq t, x \in \mathbb{R}^d$, the process $\{p_{t-s}(x-y)u_{s,y}; y \in \mathbb{R}^d\}$ is Skorohod integrable in the sense given by relation* (2.16)*, and the following equation holds:*

$$u_{t,x} = p_t u_0(x) + \int_0^t \left(\int_{\mathbb{R}^d} p_{t-s}(x-y) u_{s,y} \delta W_y \right) ds. \tag{3.33}$$

Suppose that $u = \{u_{t,x}; t \geq 0, x \in \mathbb{R}^d\}$ is a mild solution to equation (3.32). Then for any fixed (t,x), the random variable $u_{t,x}$ admits the following Wiener chaos expansion:

$$u_{t,x} = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t,x)), \tag{3.34}$$

where for each (t,x), $f_n(\cdot,t,x)$ is a symmetric element in $\mathscr{H}^{\otimes n}$. Notice that here the space $\mathscr{H}$ contains functions of the space variable y only. Using an iteration procedure similar to the one described at Section 4.3.1, one can find the explicit formula for the kernels f_n for $n \geq 1$:

$$\begin{aligned} &f_n(x_1, \ldots, x_n, t, x) \\ &= \frac{1}{n!} \int_{[0,t]^n} p_{t-s_{\sigma(n)}}(x - x_{\sigma(n)}) \cdots p_{s_{\sigma(2)}-s_{\sigma(1)}}(x_{\sigma(2)} - x_{\sigma(1)})\, p_{s_{\sigma(1)}} u_0(x_{\sigma(1)})\, ds_1 \cdots ds_n, \end{aligned}$$

where σ denotes the permutation of $\{1, 2, \ldots, n\}$ such that $0 < s_{\sigma(1)} < \cdots < s_{\sigma(n)} < t$. Then, to show the existence and uniqueness of the solution it suffices to show that for all (t,x) we have

$$\sum_{n=0}^{\infty} n! \|f_n(\cdot,t,x)\|_{\mathscr{H}^{\otimes n}}^2 < \infty. \tag{3.35}$$

Theorem 4.15. *Assume that μ satisfies Hypothesis 4.1. Then (3.35) holds for each (t,x) and equation (3.32) has a unique mild solution.*

The proof of this theorem is analogous to the proof of Theorem 4.8 and is omitted for sake of conciseness. As in the previous subsection, we can deduce the following moment formula for the solution to equation (3.32).

$$\mathbf{E}\left[u_{t,x}^k\right] = \mathbf{E}_B\left[\prod_{i=1}^k u_0(B_t^i + x)\exp\left(\sum_{1\le i<j\le k}\int_0^t\int_0^t \Lambda(B_s^i - B_r^j)dsdr\right)\right], \qquad (3.36)$$

where B^i, $i = 1,\dots,k$, are d-dimensional independent Brownian motions.

4.4 Feynman-Kac functional

In this section we construct a candidate solution for equation (1.1) using a suitable version of Feynman-Kac formula. The construction has been inspired by the approach developed in [56] for the case of fractional noises. We will establish the existence and Hölder continuity properties of the Feynman-Kac functional.

4.4.1 Construction of the Feynman-Kac functional

We first consider the time dependent noise introduced in Section 4.2.1, and later we deal with the time independent noise introduced in Section 4.2.2.

Time dependent noise

Suppose first that W is the time dependent noise introduced in Section 4.2.1. If the initial condition of equation (1.1) is a continuous and bounded function u_0, analogously to [56] we define

$$u_{t,x} = \mathbf{E}_B\left[u_0(B_t^x)\exp\left(\int_0^t\int_{\mathbb{R}^d}\delta_0(B_{t-r}^x - y)W(dr,dy)\right)\right], \qquad (4.1)$$

where B^x is a d-dimensional Brownian motion independent of W and starting at $x \in \mathbb{R}^d$. Our first goal is thus to give a meaning to the functional

$$V_{t,x} = \int_0^t \int_{\mathbb{R}^d} \delta_0(B^x_{t-r} - y) W(dr, dy) \tag{4.2}$$

appearing in the exponent of the Feynman-Kac formula (4.1). To this aim, like in the case of the formula for moments (see (3.12)), we will proceed by approximation. Namely, we will approximate V by the process

$$V^{\varepsilon}_{t,x} = \int_0^t \int_{\mathbb{R}^d} p_{\varepsilon}(B^x_{t-r} - y) W(dr, dy), \quad \varepsilon > 0, \tag{4.3}$$

which is well defined as a Wiener integral for a fixed path of the Brownian motion B. The convergence of the approximation V^{ε} is obtained in the next proposition, for which we need to impose the following conditions on the function γ and the measure μ.

Hypothesis 4.16. *There exists a constant* $0 < \beta < 1$ *such that for any* $t \in \mathbb{R}$,

$$0 \le \gamma(t) \le C_{\beta} |t|^{-\beta} \tag{4.4}$$

for some constant $C_{\beta} > 0$, *and the measure* μ *satisfies*

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + |\xi|^{2-2\beta}} < \infty. \tag{4.5}$$

Proposition 4.17. *Let* $V^{\varepsilon}_{t,x}$ *be the functional defined in (4.3) and suppose that Hypothesis 4.16 holds. Then for fixed* $t \ge 0$ *and* $x \in \mathbb{R}^d$, *the random variable* $V^{\varepsilon}_{t,x}$ *converges in* $L^2(\Omega)$ *towards a functional denoted by* $V_{t,x}$. *Moreover, conditioned by* B, $V_{t,x}$ *is a*

Gaussian random variable with mean 0 *and variance*

$$\mathbf{Var}_W(V_{t,x}) = \int_0^t \int_0^t \gamma(r-s)\Lambda(B_r - B_s)drds\,. \tag{4.6}$$

Proof. Our first goal is to find

$$\lim_{\varepsilon_1,\varepsilon_2\to 0} \mathbf{E}\left[V_{t,x}^{\varepsilon_1} V_{t,x}^{\varepsilon_2}\right]. \tag{4.7}$$

To this aim, we set $A_{t,x}^{\varepsilon}(r,y) = p_\varepsilon(B_{t-r}^x - y)\mathbf{1}_{[0,t]}(r)$. Then

$$\begin{aligned} \mathbf{E}\left[V_{t,x}^{\varepsilon_1} V_{t,x}^{\varepsilon_2}\right] &= \mathbf{E}\left[W\left(A_{t,x}^{\varepsilon_1}\right) W\left(A_{t,x}^{\varepsilon_2}\right)\right] = \mathbf{E}_B\left[\left\langle A_{t,x}^{\varepsilon_1}, A_{t,x}^{\varepsilon_2}\right\rangle_{\mathscr{H}}\right] \\ &= \mathbf{E}_B\left[\int_0^t \int_0^t \int_{\mathbb{R}^d} \mathscr{F}A_{t,x}^{\varepsilon_1}(u,\cdot)(\xi)\,\overline{\mathscr{F}A_{t,x}^{\varepsilon_2}}(v,\cdot)(\xi)\gamma(u-v)\mu(d\xi)dudv\right]. \end{aligned}$$

Furthermore, we can write for $u \le t$

$$\mathscr{F}A_{t,x}^{\varepsilon_1}(u,\cdot)(\xi) = \mathscr{F}p_{\varepsilon_1}(B_{t-u}^x - \cdot)(\xi) = e^{-\frac{1}{2}\varepsilon_1^2|\xi|^2 + \iota\langle \xi, B_u^x\rangle},$$

and thus

$$\left\langle A_{t,x}^{\varepsilon_1}, A_{t,x}^{\varepsilon_2}\right\rangle_{\mathscr{H}} = \int_{\mathbb{R}^d}\left(\int_{[0,t]^2} e^{\iota\langle \xi, B_v - B_u\rangle}\gamma(u-v)dudv\right) e^{-\frac{1}{2}(\varepsilon_1^2+\varepsilon_2^2)|\xi|^2}\mu(d\xi). \tag{4.8}$$

This yields

$$\begin{aligned} \mathbf{E}\left[V_{t,x}^{\varepsilon_1} V_{t,x}^{\varepsilon_2}\right] &= \mathbf{E}_B\left[\left\langle A_{t,x}^{\varepsilon_1}, A_{t,x}^{\varepsilon_2}\right\rangle_{\mathscr{H}}\right] \\ &= \int_{\mathbb{R}^d}\left(\int_{[0,t]^2} e^{-\frac{1}{2}|\xi|^2|v-u|}\gamma(u-v)dudv\right) e^{-\frac{1}{2}(\varepsilon_1^2+\varepsilon_2^2)|\xi|^2}\mu(d\xi). \end{aligned} \tag{4.9}$$

Set now

$$\sigma_t^2 := \int_{\mathbb{R}^d} \left(\int_{[0,t]^2} e^{-\frac{1}{2}|\xi|^2|v-u|} \gamma(u-v) du dv \right) \mu(d\xi).$$

Is easily seen by direct integration and by using the hypothesis (4.4) that

$$\int_{[0,t]^2} e^{-\frac{1}{2}|\xi|^2|v-u|} \gamma(u-v) du dv \le c_\beta \int_{[0,t]^2} e^{-\frac{1}{2}|\xi|^2|v-u|} |u-v|^{-\beta} du dv \le \frac{c}{1+|\xi|^{2-2\beta}}.$$

Thus

$$\sigma_t^2 \le c \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^{2-2\beta}},$$

which is a finite quantity by hypothesis (4.5). As a consequence, for every sequence ε_n converging to zero, $V_{t,x}^{\varepsilon_n}$ converges in L^2 to a limit denoted by $V_{t,x}$ which does not depend on the choice of the sequence ε_n. Finally, by a similar argument, we show (4.6). This completes the proof of the proposition. □

Remark 4.18. *We could also regularize the noise in time, and define*

$$V_{t,x}^{\varepsilon,\delta} = W(A_{t,x}^{\varepsilon,\delta}), \tag{4.10}$$

where $A_{t,x}^{\varepsilon,\delta}$ has been introduced in (3.19). Then it is easy to check that $V_{t,x}^{\varepsilon,\delta}$ converges as δ tend to zero in $L^2(\Omega)$ to $V_{t,x}^{\varepsilon}$.

In order to give a meaning to formula (4.1) we need to establish the existence of exponential moments for $V_{t,x}$. To complete this task, we need the following lemma.

Lemma 4.19. *Suppose that Hypothesis 4.16 holds. Then for any $\varepsilon > 0$ there exists a constant C_ε such that for any $\nu > 0$ we have:*

$$\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} e^{-\frac{\nu}{2}|\xi-\eta|^2} \mu(d\xi) \le C_\varepsilon + \frac{\varepsilon}{\nu^{1-\beta}}. \tag{4.11}$$

Proof. The fact that the left hand side of (4.11) is uniformly bounded in η is proven similarly to (3.7), but is included here for sake of readability. Indeed, consider $\eta \in \mathbb{R}^d, v > 0$ and define a function $\varphi_\eta : \mathbb{R}^d \to \mathbb{R}_+$ by $\varphi_\eta(\xi) = e^{-\frac{v}{2}|\xi-\eta|^2}$. Then according to Parseval's identity we have

$$\int_{\mathbb{R}^d} \varphi_\eta(\xi)\,\mu(d\xi) = c\int_{\mathbb{R}^d} \mathscr{F}\varphi_\eta(x)\,\Lambda(x)dx = c\int_{\mathbb{R}^d} v^{-d/2} e^{-\frac{|x|^2}{2v}}\, e^{\imath\langle x,\eta\rangle}\,\Lambda(x)dx.$$

We now use the fact that Λ is assumed to be nonnegative in order to get the following uniform bound in η

$$\int_{\mathbb{R}^d} \varphi_\eta(\xi)\,\mu(d\xi) \le c\int_{\mathbb{R}^d} v^{-d/2} e^{-\frac{|x|^2}{2v}}\,\Lambda(x)dx = \int_{\mathbb{R}^d} \varphi_0(\xi)\,\mu(d\xi) = \int_{\mathbb{R}^d} e^{-\frac{v}{2}|\xi|^2}\,\mu(d\xi).$$

To estimate the right-hand side of the above inequality we introduce a constant $M > 0$, whose exact value is irrelevant for our computations, and let $c_{M,1} = \mu(B(0,M))$, where $B(0,M)$ stands for the ball of radius M centered at 0 in $\mathbb{R}^d$. Then the trivial bound $e^{-\frac{v}{2}|\xi|^2} \le 1$ yields

$$\int_{\mathbb{R}^d} e^{-\frac{v}{2}|\xi|^2}\,\mu(d\xi) \le c_{M,1} + \int_{|\xi|>M} e^{-\frac{v}{2}|\xi|^2}\,\mu(d\xi).$$

Invoking the fact that the function $x \mapsto x^{1-\beta}e^{-x}$ is bounded on $\mathbb{R}_+$, we thus get

$$\int_{|\xi|>M} e^{-\frac{v}{2}|\xi|^2}\,\mu(d\xi) \le \frac{c_2}{v^{1-\beta}}\int_{|\xi|>M} \frac{\mu(d\xi)}{|\xi|^{2-2\beta}} \le \frac{c_2}{v^{1-\beta}}\int_{|\xi|>M} \frac{\mu(d\xi)}{1+|\xi|^{2-2\beta}}.$$

Summarizing the above, we have obtained that

$$\int_{\mathbb{R}^d} e^{-\frac{v}{2}|\xi-\eta|^2}\,\mu(d\xi) \le c_{M,1} + \frac{c_2}{v^{1-\beta}}\int_{|\xi|>M} \frac{\mu(d\xi)}{1+|\xi|^{2-2\beta}},$$

uniformly in $\eta \in \mathbb{R}^d$. Our claim is thus obtained by choosing M large enough so that $c_2 \int_{|\xi|>M} \frac{\mu(d\xi)}{1+|\xi|^{2-2\beta}} \leq \varepsilon$, which is possible by hypothesis (4.5). □

The following elementary integration result will also be crucial for the moment estimates we deduce later.

Lemma 4.20. *Let $\alpha \in (-1+\varepsilon, 1)^m$ with $\varepsilon > 0$ and set $|\alpha| = \sum_{i=1}^m \alpha_i$. Recall (see (3.9)) that $T_m(t) = \{(r_1, r_2, \ldots, r_m) \in \mathbb{R}^m : 0 < r_1 < \cdots < r_m < t\}$. Then there is a constant κ such that*

$$J_m(t,\alpha) := \int_{T_m(t)} \prod_{i=1}^m (r_i - r_{i-1})^{\alpha_i} dr \leq \frac{\kappa^m t^{|\alpha|+m}}{\Gamma(|\alpha|+m+1)},$$

where by convention, $r_0 = 0$.

Proof. Using identities on Beta functions and a recursive algorithm we can snow that

$$J_m(t,\alpha) = \frac{\prod_{i=1}^m \Gamma(\alpha_i+1) t^{|\alpha|+m}}{\Gamma(|\alpha|+m+1)},$$

and the result follows thanks to the fact that the Γ function is bounded on $(\varepsilon, 2)$. □

With these preliminary results in hand, we can now prove the exponential integrability of the random variable $V_{t,x}$ defined in Proposition 4.17.

Theorem 4.21. *Let $V_{t,x}$ be the functional defined in Proposition 4.17, and assume Hypothesis 4.16. Then for any $\lambda \in \mathbb{R}$ and $T > 0$, we have $\sup_{t\in[0,T], x\in\mathbb{R}^d} \mathbf{E}[\exp(\lambda V_{t,x})] < \infty$. In particular, the functional* (4.1) *is well defined.*

Proof. Fix $t > 0$ and $x \in \mathbb{R}^d$. Conditionally to B, the random variable $V_{t,x}$ is Gaussian and centered. From (4.6), we obtain

$$\mathbf{E}[\exp(\lambda V_{t,x})] = \mathbf{E}_B\left[\exp\left(\frac{\lambda^2}{2}\int_0^t\int_0^t \gamma(r-s)\Lambda(B_r - B_s) dr ds\right)\right] = \mathbf{E}_B\left[\exp\left(\frac{\lambda^2}{2} Y\right)\right],$$

where

$$Y = \int_0^t \int_0^t \gamma(r-s)\Lambda(B_r - B_s)drds\,.$$

In order to show that $\mathbf{E}[\exp(\lambda Y)] < \infty$ for any $\lambda \in \mathbb{R}$, we are going to use an elaboration of a method introduced by Le Gall [65] (see also [55, 56]). With respect to those contributions, our case requires a careful handling of the weights Λ and γ. Notice in particular that in our general setting we do not have scaling properties, and some additional work is necessary to overcome this difficulty.

Le Gall's method starts from the following construction: for $n \geq 1$ and $k = 1, \ldots, 2^{n-1}$ we set

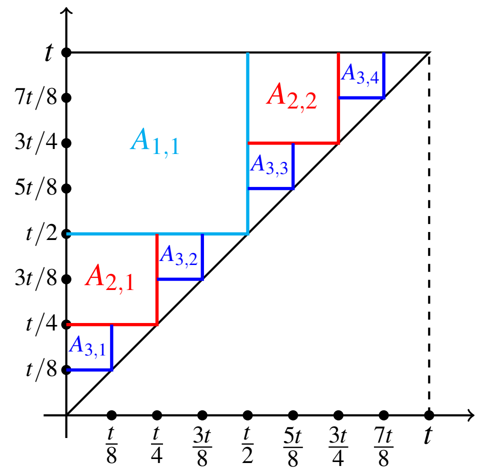
$$J_{n,k} := \left[\frac{(2k-2)t}{2^n}, \frac{(2k-1)t}{2^n}\right), \quad I_{n,k} := \left[\frac{(2k-1)t}{2^n}, \frac{2kt}{2^n}\right), \quad \text{and} \quad A_{n,k} := J_{n,k} \times I_{n,k}.$$

Notice then that $\{A_{n,k}; n \geq 1, k = 1, \ldots, 2^{n-1}\}$ is a partition of the simplex $T_2(t)$, and in addition $I_{n,k-1} \cap I_{n,k} = \varnothing$ and $J_{n,k-1} \cap J_{n,k} = \varnothing$ (see Figure 4.1 for an illustration). We can thus write

$$Y = \sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} a_{n,k}, \quad \text{where} \quad a_{n,k} = \int_{A_{n,k}} \gamma(r-s)\Lambda(B_r - B_s)drds\,.$$

Observe that for fixed n the random variables $\{a_{n,k}; k = 1, \ldots, 2^{n-1}\}$ are independent, owing to the fact that they depend on the increments of B on disjoint sets. Now, thanks to the fact that $J_{n,k} \cap I_{n,k} = \varnothing$, for all $(r,s) \in A_{n,k}$ we can decompose $B_r - B_s$ into $(B_r - B_{\frac{(2k-1)t}{2^n}}) - (B_s - B_{\frac{(2k-1)t}{2^n}})$, where the two pieces of the difference are independent

Figure 4.1: Le Gall's partition of $T_2(t)$ into disjoint rectangles of decreasing area.



Brownian motions. Thus the following identity in law holds true:

$$\left\{B_r - B_s;\, (r,s) \in A_{n,k}\right\} \stackrel{(d)}{=} \left\{B_{r-\frac{(2k-1)t}{2^n}} - \widetilde{B}_{s-\frac{(2k-1)t}{2^n}};\, (r,s) \in A_{n,k}\right\},$$

where B and $\widetilde{B}$ are two independent Brownian motions. With an additional change of variables $r - \frac{(2k-1)t}{2^n} \mapsto r$ and $\frac{(2k-1)t}{2^n} - s \mapsto s$, this easily yields the following identity

$$\begin{aligned} a_{n,k} &\stackrel{(d)}{=} \int_{A_{n,k}} \gamma(r+s)\, \Lambda\left(B_{\frac{(2k-1)t}{2^n}+r} - \widetilde{B}_{\frac{(2k-1)t}{2^n}-s}\right) dsdr \\ &\stackrel{(d)}{=} \int_0^{\frac{t}{2^n}} \int_0^{\frac{t}{2^n}} \gamma(r+s)\, \Lambda(B_r + \widetilde{B}_s)\, dsdr := a_n. \end{aligned}$$

Summarizing the considerations above, we have found that

$$Y = \sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} a_{n,k}, \tag{4.12}$$

where for each $n \geq 1$ the collection $\{a_{n,k}; k = 1, \ldots, 2^{n-1}\}$ is a family of independent random variables such that

$$a_{n,k} \overset{(d)}{=} a_n, \quad \text{with} \quad a_n = \int_0^{\frac{t}{2^n}} \int_0^{\frac{t}{2^n}} \gamma(r+s)\, \Lambda(B_r + \widetilde{B}_s)\, ds dr,$$

where $B, \widetilde{B}$ are two independent Brownian motions. Notice that the transformation of $B_r - B_s$ into $B_r + \widetilde{B}_s$ we have achieved is essential for our future computations. Indeed, it will be translated into some singularities $(r-s)^{-1}$ in a neighborhood of 0 in $\mathbb{R}_+^2$ becoming some more harmless singularities of the form $(r+s)^{-1}$. The proof is now decomposed in several steps.

Step 1. First we need to estimate the moments of the random variable a_n. We claim that for any $\varepsilon > 0$ there exist constants $C_{\varepsilon,1} > 0$ and $C_2 > 0$ (which depend on t) such that

$$\mathbf{E}[a_n^m] \leq C_{\varepsilon,1} m! \left(\frac{C_2 \varepsilon}{2^n} \right)^m. \tag{4.13}$$

In order to show (4.13), we first write

$$\mathbf{E}\left[a_n^m\right] = \int_{[0,\frac{t}{2^n}]^m} \int_{[0,\frac{t}{2^n}]^m} \prod_{i=1}^m \gamma(r_i + s_i) \mathbf{E}\left[\prod_{i=1}^m \Lambda(B_{r_i} + \widetilde{B}_{s_i}) \right] ds dr\,.$$

Let p_B be the joint density of $(B_{r_1} + \widetilde{B}_{s_1}, \ldots, B_{r_m} + \widetilde{B}_{s_m})$, which is a Schwartz function. Hence, using the Fourier transform and the same considerations as for (3.29), we get

$$\mathbf{E}\left[\prod_{i=1}^m \Lambda(B_{r_i} + \widetilde{B}_{s_i}) \right] = \int_{\mathbb{R}^{md}} \prod_{i=1}^m \Lambda(x_i) p_B(x) dx = \int_{\mathbb{R}^{md}} e^{-\frac{1}{2}\sum_{i,j=1}^m \xi_i \cdot \xi_j (r_i \wedge r_j + s_i \wedge s_j)} \prod_{i=1}^m \mu(d\xi_i)\,.$$

We now proceed as in the proof of Theorem 4.12, with an additional care in the computation of terms. Thanks to our assumption (4.4) on γ and the basic inequality

$a+b\geq 2\sqrt{ab}$ for nonnegative a, b, we have

$$\begin{aligned}\mathbf{E}\left[a_n^m\right] &= \int_{[0,\frac{t}{2^n}]^m}\int_{[0,\frac{t}{2^n}]^m}\int_{\mathbb{R}^{md}} e^{-\frac{1}{2}\sum_{i,j=1}^m \xi_i\cdot\xi_j(r_i\wedge r_j+s_i\wedge s_j)}\prod_{i=1}^m\mu(d\xi_i)\prod_{i=1}^m\gamma(r_i+s_i)dsdr\\ &\leq (2^{-\beta}C_\beta)^m\int_{[0,\frac{t}{2^n}]^m}\int_{[0,\frac{t}{2^n}]^m}\int_{\mathbb{R}^{md}} e^{-\frac{1}{2}\sum_{i,j=1}^m \xi_i\cdot\xi_j(r_i\wedge r_j+s_i\wedge s_j)}\prod_{i=1}^m\mu(d\xi_i)\prod_{i=1}^m(r_is_i)^{-\frac{\beta}{2}}dsdr,\end{aligned}$$

and thus, invoking Cauchy-Schwarz inequality with respect to the measure

$$\prod_{i=1}^m(r_is_i)^{-\frac{\beta}{2}}drds,$$

we end up with

$$\begin{aligned}\mathbf{E}\left[a_n^m\right] \leq (2^{-\beta}C_\beta)^m\int_{[0,\frac{t}{2^n}]^m}\int_{[0,\frac{t}{2^n}]^m}&\left(\int_{\mathbb{R}^{md}} e^{-\sum_{i,j=1}^m \xi_i\cdot\xi_j(r_i\wedge r_j)}\prod_{i=1}^m\mu(d\xi_i)\right)^{\frac{1}{2}}\\ &\times\left(\int_{\mathbb{R}^{md}} e^{-\sum_{i,j=1}^m \xi_i\cdot\xi_j(s_i\wedge s_j)}\prod_{i=1}^m\mu(d\xi_i)\right)^{\frac{1}{2}}\prod_{i=1}^m(r_is_i)^{-\frac{\beta}{2}}dsdr.\end{aligned}$$

Since in the above expression, both integrals with respect to the measure $\prod_{i=1}^m\mu(d\xi_i)$ are symmetric functions of the r_i's and s_i's, we can restrict the integral to the region $T_m(\frac{t}{2^n})$, where $T_m(t)$ has been defined in (3.9). Therefore, similarly to (3.30) and with the convention $r_0=0$, we obtain that for any $\varepsilon>0$ the expectation $\mathbf{E}[a_n^m]$ is bounded by

$$\begin{aligned}&(2^{-\beta}C_\beta)^m(m!)^2\left(\int_{T_m(\frac{t}{2^n})}\left(\int_{\mathbb{R}^{md}} e^{-\sum_{i=1}^m(r_i-r_{i-1})|\xi_i+\cdots+\xi_m|^2}\prod_{i=1}^m\mu(d\xi_i)\right)^{\frac{1}{2}}\prod_{i=1}^m|r_i|^{-\frac{\beta}{2}}dr\right)^2\\ &\leq (2^{-\beta}C_\beta)^m(m!)^2\left(\int_{T_m(\frac{t}{2^n})}\prod_{i=1}^m\left(C_\varepsilon+\frac{\varepsilon}{(r_i-r_{i-1})^{1-\beta}}\right)^{\frac{1}{2}}\prod_{i=1}^m(r_i-r_{i-1})^{-\frac{\beta}{2}}dr\right)^2,\end{aligned}$$

where we have used Lemma 4.19 and we have bounded $r_i^{-\frac{\beta}{2}}$ by $(r_i - r_{i-1})^{-\frac{\beta}{2}}$. We now resort to the inequality $(a+b)^{\frac{1}{2}} \le a^{\frac{1}{2}} + b^{\frac{1}{2}}$ in order to get

$$\begin{aligned}\mathbf{E}\left[a_n^m\right] &\le (2^{-\beta} C_\beta)^m (m!)^2 \left(\int_{T_m(\frac{t}{2^n})} \prod_{i=1}^m \left(\sqrt{C_\varepsilon} + \frac{\sqrt{\varepsilon}}{(r_i - r_{i-1})^{\frac{1-\beta}{2}}} \right) \prod_{i=1}^m (r_i - r_{i-1})^{-\frac{\beta}{2}} dr \right)^2 \\ &= (2^{-\beta} C_\beta)^m (m!)^2 \left(\int_{T_m(\frac{t}{2^n})} \sum_{\theta \in \{0,1\}^m} \prod_{i=1}^m C_\varepsilon^{\frac{\theta_i}{2}} \varepsilon^{\frac{1}{2}(1-\theta_i)} (r_i - r_{i-1})^{\frac{\beta-1}{2}(1-\theta_i) - \frac{\beta}{2}} dr \right)^2 .\end{aligned}$$

Hence, a direct application of Lemma 4.20 shows that there exists a positive constant K such that

$$\mathbf{E}\left[a_n^m\right] \le K^m (m!)^2 \left(\sum_{l=0}^m \binom{m}{l} C_\varepsilon^{\frac{l}{2}} \varepsilon^{\frac{m-l}{2}} \frac{\left(\frac{t}{2^n}\right)^{\frac{(1-\beta)l}{2} + \frac{m}{2}}}{\Gamma\left(\frac{(1-\beta)l}{2} + \frac{m}{2} + 1\right)} \right)^2 .$$

We now simply bound $\binom{m}{l}$ by 2^m and recall that $(x/3)^x \le \Gamma(x+1) \le x^x$ for $x \ge 0$. Allowing the constant K to change from line to line, this yields

$$\begin{aligned}\mathbf{E}\left[a_n^m\right] &\le K^m (m!)^2 \left(\sum_{l=0}^m C_\varepsilon^{\frac{l}{2}} \varepsilon^{\frac{m-l}{2}} \frac{\left(\frac{t}{2^n}\right)^{\frac{(1-\beta)l}{2}} \left(\frac{t}{2^n}\right)^{\frac{m}{2}}}{(m!)^{\frac{1}{2}} (l!)^{\frac{1-\beta}{2}}} \right)^2 \\ &\le K^m m! \varepsilon^m \left(\frac{t}{2^n}\right)^m \left(\sum_{l=0}^\infty \frac{C_\varepsilon^{\frac{l}{2}} \varepsilon^{-\frac{l}{2}} t^{\frac{1-\beta}{2} l}}{(l!)^{\frac{1-\beta}{2}}} \right)^2 .\end{aligned}$$

This completes the proof of (4.13) with $C_{\varepsilon,1} = (\sum_{l=0}^\infty C_\varepsilon^{\frac{l}{2}} \varepsilon^{-\frac{l}{2}} t^{\frac{1-\beta}{2} l} (l!)^{\frac{\beta-1}{2}})^2$, which is finite because this series is convergent, and $C_2 = Kt$.

Step 2. We now start from relation (4.13) and prove the finiteness of exponential moments for the random variable Y. It turns out that centering is useful in this context, and we thus define $\overline{a}_{n,k} = a_{n,k} - \mathbf{E}[a_{n,k}]$. Then $\mathbf{E}[\overline{a}_{n,k}] = 0$, and for any integer $m \ge 2$ notice that:

$$\mathbf{E}\left[(\overline{a}_{n,k})^m\right] \leq 2^{m-1}\left(\mathbf{E}\left[a_{n,k}^m\right] + \left(\mathbf{E}[a_{n,k}]\right)^m\right) \leq 2^m \mathbf{E}[a_{n,k}^m].$$

Also recall that $a_{n,k} \overset{(d)}{=} a_n$. Thus, using (4.13)

$$\begin{aligned} \mathbf{E}\left[\exp(\lambda \overline{a}_{n,k})\right] &= 1 + \sum_{m=2}^{\infty} \frac{\lambda^m}{m!} \mathbf{E}\left[(\overline{a}_{n,k})^m\right] \leq 1 + \sum_{m=2}^{\infty} \frac{(2\lambda)^m}{m!} \mathbf{E}\left[(a_{n,k})^m\right] \\ &\leq 1 + \sum_{m=2}^{\infty} C_{\varepsilon,1} \left(\frac{2C_2 \lambda \varepsilon}{2^n}\right)^m. \end{aligned}$$

Now choose and fix ε such that $C_2 \lambda \varepsilon 2^{-n+1} \leq \frac{1}{2}$, and we obtain the bound

$$\mathbf{E}\left[\exp(\lambda \overline{a}_{n,k})\right] \leq 1 + \frac{C_{\varepsilon,2} \lambda^2}{2^{2n}}, \tag{4.14}$$

for some positive constant $C_{\varepsilon,2}$. Next we choose $0 < h < 1$, define $b_N = \prod_{j=2}^{N}\left(1 - 2^{-h(j-1)}\right)$, and notice that $\lim_{N\to\infty} b_N = b_\infty > 0$. Then, by Hölder's inequality, for all $N \geq 2$ we have

$$\begin{aligned} &\mathbf{E}\left[\exp\left(\lambda b_N \sum_{n=1}^{N} \sum_{k=1}^{2^{n-1}} \overline{a}_{n,k}\right)\right] \\ \leq &\left[\mathbf{E}\left[\exp\left(\lambda b_{N-1} \sum_{n=1}^{N-1} \sum_{k=1}^{2^{n-1}} \overline{a}_{n,k}\right)\right]\right]^{1-2^{-h(N-1)}} \\ &\times \left[\mathbf{E}\left[\exp\left(\lambda b_N 2^{h(N-1)} \sum_{k=1}^{2^{N-1}} \overline{a}_{N,k}\right)\right]\right]^{2^{-h(N-1)}}, \end{aligned}$$

and taking into account the independence of the $\{a_{N,k}; k \leq 2^{N-1}\}$ plus the identity $a_{N,k} \stackrel{(d)}{=} a_N$, the above expression is equal to

$$\left(\mathbf{E}\left[\exp\left(\lambda b_{N-1}\sum_{n=1}^{N-1}\sum_{k=1}^{2^{n-1}}\overline{a}_{n,k}\right)\right]\right)^{1-2^{-h(N-1)}}\left(\mathbf{E}\left[\exp\left(\lambda b_N 2^{h(N-1)}\overline{a}_N\right)\right]\right)^{2^{(1-h)(N-1)}}$$
$$:= A_N B_N\,.$$

We now appeal to the estimate (4.14) and the elementary inequality $1+x \leq e^x$, valid for any $x \in \mathbb{R}$. This yields

$$B_N \;\leq\; \left(1+C_{\varepsilon,2}b_N^2 2^{-2N}\lambda^2 2^{2h(N-1)}\right)^{2^{(1-h)(N-1)}} \leq \exp\left(C_{\varepsilon,3}2^{-N(1-h)}\right),$$

for some positive constant $C_{\varepsilon,3}$. Notice that this is where the centering argument on $a_{n,k}$ is crucial. Indeed, without centering we would get a factor 2^{-N} instead of 2^{-2N} in the left hand side of the above expression, and B_N would not be uniformly bounded. Thus, recursively we get

$$\mathbf{E}\left[\exp\left(\lambda b_N\sum_{n=1}^{N}\sum_{k=1}^{2^{n-1}}\overline{a}_{n,k}\right)\right] \leq \exp\left(\sum_{n=2}^{N}C2^{-n(1-h)}\right)\mathbf{E}\left[\exp(\overline{a}_{1,1})\right] < \infty.$$

Recalling now from (4.12) that $Y-\mathbf{E}[Y] = \sum_{n=1}^{\infty}\sum_{k=1}^{2^{n-1}}\bar{a}_{n,k}$ and applying Fatou's lemma, we finally get

$$\mathbf{E}\left[\exp\left(\lambda b_\infty(Y-\mathbf{E}[Y])\right)\right] < \infty,$$

which completes the proof. □

Our next result is an approximation result for the Feynman-Kac functional which will be used in the next section (see Theorem 4.30). Towards this aim, for any $\varepsilon, \delta > 0$

we define

$$u_{t,x}^{\varepsilon,\delta} = \mathbf{E}\left[u_0(B_t^x)\exp(V_{t,x}^{\varepsilon,\delta})\right], \tag{4.15}$$

where $V_{t,x}^{\varepsilon,\delta} = W(A_{t,x}^{\varepsilon,\delta})$ and $A_{t,x}^{\varepsilon,\delta}$ is defined in (3.24).

Proposition 4.22. *For any $p \geq 2$ and $T > 0$ we have*

$$\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0}\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\mathbf{E}\left[|u_{t,x}^{\varepsilon,\delta} - u_{t,x}|^p\right] = 0. \tag{4.16}$$

Proof. First, we recall that (see Proposition 4.17 and Remark 4.18) for any fixed $t \geq 0$ and $x \in \mathbb{R}^d$ the random variable $V_{t,x}^{\varepsilon,\delta}$ converges in $L^2(\Omega)$ to $V_{t,x}$ if we let first δ tend to zero and later ε tend to zero. Then in order to show (4.16) it suffices to check that for any $\lambda \in \mathbb{R}$

$$\sup_{\varepsilon,\delta}\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\mathbf{E}\left[\exp\left(\lambda V_{t,x}^{\varepsilon,\delta}\right)\right] < \infty. \tag{4.17}$$

Taking first the expectation with respect to the noise W yields

$$\mathbf{E}\left[\exp\left(\lambda V_{t,x}^{\varepsilon,\delta}\right)\right] = \mathbf{E}_B\left[\exp\left(\frac{\lambda^2}{2}\|A_{t,x}^{\varepsilon,\delta}\|_{\mathscr{H}}^2\right)\right].$$

Expanding the exponential into a power series, we will need to bound the moments of the random variable $\|A_{t,x}^{\varepsilon,\delta}\|_{\mathscr{H}}^2$. To do this, we use formula (3.28) with $B = \widetilde{B}$ and $\varepsilon = \varepsilon'$, $\delta = \delta'$. Computing the mathematical expectation of this expression, we end up with:

$$\mathbf{E}\left[\left\|A_{t,x}^{\varepsilon,\delta}\right\|_{\mathscr{H}}^{2n}\right] = \frac{1}{\delta^{2n}}\int_{O_{\delta,n}}\int_{\mathbb{R}^{dn}}\exp\left(-\frac{1}{2}\sum_{i,j=1}^{n}\mathbf{E}_B[(B_{u_i}-B_{v_i})(B_{u_j}-B_{v_j})]\langle\xi_i,\xi_j\rangle\right)$$
$$\times\, e^{-\varepsilon\sum_{l=1}^n|\xi_l|^2}\prod_{l=1}^{n}\gamma(u_l+s_l-v_l-\widetilde{s}_l)\,\mu(d\xi)\,ds\,d\widetilde{s}\,du\,dv.$$

Thanks to the estimate

$$\sup_{0\le\delta\le 1}\frac{1}{\delta^2}\int_0^\delta\int_0^\delta |u+s-v-r|^{-\beta}dsdr \le c_{T,\beta}|u-v|^{-\beta}, \tag{4.18}$$

which holds for any $u,v\in[0,T]$, and owing to assumption (4.4), we get

$$\mathbf{E}\left[\left\|A_{t,x}^{\varepsilon,\delta}\right\|_{\mathscr{H}}^{2n}\right] \le c_{T,\beta}^n \mathbf{E}_B\left[\left|\int_0^t\int_0^t |u-v|^{-\beta}\Lambda(B_u-B_v)dudv\right|^n\right]. \tag{4.19}$$

It is now readily checked that (4.17) follows from (4.19) and Theorem 4.21. □

Time independent noise

Suppose that W is the time independent noise introduced in Section 4.2.2. The Feynman-Kac functional is defined as

$$u_{t,x} = \mathbf{E}\left[u_0(B_t^x)\exp\left(\int_0^t\int_{\mathbb{R}^d}\delta_0(B_r^x-y)W(dy)dr\right)\right], \tag{4.20}$$

where $B^x=\{B_t+x, t\ge 0\}$ is a d-dimensional Brownian motion independent of W, starting from xm and $u_0\in C_b(\mathbb{R}^d)$ is the initial condition.

As in the case of a time dependent noise, to give a meaning to this functional for every $t>0$, $x\in\mathbb{R}^d$ and $\varepsilon>0$ we introduce the family of random variables

$$V_{t,x}^\varepsilon = \int_0^t\int_{\mathbb{R}^d} p_\varepsilon(B_r^x-y)W(dy)dr,$$

Then, if the spectral measure of the noise μ satisfies condition (2.4), the family $V_{t,x}^\varepsilon$ converges in L^2 to a limit denoted by

$$V_{t,x} = \int_0^t\int_{\mathbb{R}^d}\delta_0(B_r^x-y)W(dy)dr. \tag{4.21}$$

Conditional on B, $V_{t,x}$ is a Gaussian random variable with mean 0 and variance

$$\mathbf{Var}_W(V_{t,x}) = \int_0^t \int_0^t \Lambda(B_r - B_s) dr ds \,. \tag{4.22}$$

Furthermore, for any $\lambda \in \mathbb{R}$, we have $\mathbf{E}[\exp(\lambda V_{t,x})] < \infty$. These properties can be obtained using the same arguments as in the time dependent case and we omit the details.

4.4.2 Hölder continuity of the Feynman-Kac functional

In this subsection, we establish the Hölder continuity in space and time of the the Feynman-Kac functional given by formulas (4.1) and (4.20). These regularity properties will hold under some additional integrability assumptions on the measure μ. To simplify the presentation we will assume that $u_0 = 1$, and as usual we separate the time dependent and independent cases.

Time dependent noise

For the case of a time dependent noise, we will make use of the following condition in order to ensure Hölder type regularities.

Hypothesis 4.23. *Let W be a space-time stationary Gaussian noise with covariance structure encoded by γ and Λ. We assume that condition (4.4) in Hypothesis 4.16 holds for some $\beta > 0$ and the spectral measure μ satisfies*

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^{2(1-\beta-\alpha)}} < \infty$$

for some $\alpha \in (0, 1-\beta)$.

Theorem 4.24. *Assume Hypothesis 4.23. Let u be the process introduced by relation* (4.1) *with* $u_0 = 1$*, namely:*

$$u_{t,x} = \mathbf{E}_B\left[\exp\left(V_{t,x}\right)\right], \quad \textit{where} \quad V_{t,x} = \int_0^t \int_{\mathbb{R}^d} \delta_0(B^x_{t-r} - y) W(dr, dy). \tag{4.23}$$

Then u admits a version which is (γ_1, γ_2)*-Hölder continuous on any compact set of the form* $[0,T] \times [-M,M]^d$*, with any* $\gamma_1 < \frac{\alpha}{2}$*,* $\gamma_2 < \alpha$ *and* $T, M > 0$*.*

Proof. Owing to standard considerations involving Kolmogorov's criterion, it is sufficient to prove the following bound for all large p and $s,t \in [0,T]$, $x,y \in \mathbb{R}^d$ with $T > 0$:

$$\mathbf{E}\left[|u_{t,x} - u_{s,y}|^p\right] \le c_{p,T}\left(|t-s|^{\frac{\alpha p}{2}} + |x-y|^{\alpha p}\right). \tag{4.24}$$

We now focus on the proof of (4.24). From the elementary relation $|e^x - e^y| \le (e^x + e^y)|x-y|$, valid for $x,y \in \mathbb{R}$ and applying the Cauchy-Schwarz inequality it follows

$$\begin{aligned} \mathbf{E}\left[|u_{t,x} - u_{s,y}|^p\right] &= \mathbf{E}_W\left[\left|\mathbf{E}_B\left[\exp\left(V_{t,x}\right)\right] - \mathbf{E}_B\left[\exp\left(V_{s,y}\right)\right]\right|^p\right] \\ &\le \mathbf{E}_W\left\{\mathbf{E}_B^p\left[\left(\exp(V_{t,x}) + \exp(V_{s,y})\right)|V_{t,x} - V_{s,y}|\right]\right\} \\ &\le \mathbf{E}_W^{1/2}\left\{\mathbf{E}_B^p\left[\left(\exp(V_{t,x}) + \exp(V_{s,y})\right)^2\right]\right\}\mathbf{E}_W^{1/2}\left\{\mathbf{E}_B^p\left[|V_{t,x} - V_{s,y}|^2\right]\right\}. \end{aligned} \tag{4.25}$$

We now resort to our exponential bound of Theorem 4.21 for $V_{t,x}$, Minkowsky inequality and the relation between L^p and L^2 moments for Gaussian random variables in order to obtain:

$$\mathbf{E}\left[|u_{t,x} - u_{s,y}|^p\right] \le c_p\left[\mathbf{E}\left[\left|V_{t,x} - V_{s,y}\right|^2\right]\right]^{p/2}.$$

We now evaluate the right hand side of this inequality.

Let us start by studying a difference of the form $V_{t,x} - V_{t,y}$, for $t \in (0,T]$ and $x, y \in \mathbb{R}$. The variance of $V_{t,x} - V_{t,y}$ conditioned by B can be computed as in (4.6) and we can write

$$
\begin{aligned}
& \mathbf{E}\left[|V_{t,x} - V_{t,y}|^2\right] \\
= \; & 2\mathbf{E}_B\left[\int_0^t \int_0^t \gamma(r-s)\Lambda(B_r - B_s)drds - \int_0^t \int_0^t \gamma(r-s)\Lambda(B_r - B_s + x - y)drds\right] \\
= \; & 2\int_0^t \int_0^t \int_{\mathbb{R}^d} \gamma(r-s)\left(1 - \cos\langle \xi, x-y\rangle\right) e^{-\frac{1}{2}|r-s||\xi|^2} \mu(d\xi)drds.
\end{aligned}
$$

Using condition (4.4) and the estimate $|1 - \cos\langle \xi, x-y\rangle| \le |\xi|^{2\alpha}|x-y|^{2\alpha}$, where $0 < \alpha < 1 - \beta$, yields

$$
\mathbf{E}\left[|V_{t,x} - V_{t,y}|^2\right] \le C|x-y|^{2\alpha} \int_0^t \int_0^t \int_{\mathbb{R}^d} |r-s|^{-\beta} e^{-\frac{1}{2}|r-s||\xi|^2} |\xi|^{2\alpha} \mu(d\xi)drds\,.
$$

Finally, as in the proof of Proposition 4.17, Hypothesis 4.23 implies

$$
\int_0^T \int_0^T \int_{\mathbb{R}^d} |r-s|^{-\beta} e^{-\frac{1}{2}|r-s||\xi|^2} |\xi|^{2\alpha} \mu(d\xi)drds < \infty,
$$

and thus $\mathbf{E}[|V_{t,x} - V_{t,y}|^2] \le C|x-y|^{2\alpha}$.

The evaluation of the variance of $V_{t,x} - V_{s,x}$, with $0 \le s < t \le T$, $x \in \mathbb{R}^d$ goes along the same lines. Indeed, we write $\mathbf{E}[|V_{t,x} - V_{s,x}|^2] \le 2(A_1 + A_2)$, with

$$
\begin{aligned}
A_1 &= \mathbf{E}\left[\left|\int_s^t \int_{\mathbb{R}^d} \delta_0(B_{t-r}^x - y)W(dr,dy)\right|^2\right] \\
A_2 &= \mathbf{E}\left[\left|\int_0^s \int_{\mathbb{R}^d} \left(\delta_0(B_{t-r}^x - y) - \delta_0(B_{s-r}^x - y)\right) W(dr,dy)\right|^2\right].
\end{aligned}
$$

For the term A_1, computing the variance as in (4.6) and using condition (4.4), we obtain

$$
A_1 = \mathbf{E}_B\left[\int_0^{t-s} \int_0^{t-s} \gamma(u-v)\Lambda(B_u - B_v)\,dudv\right]
$$

$$
\begin{aligned}
&\le C_\beta \int_0^{t-s}\int_0^{t-s}\int_{\mathbb{R}^d} |u-v|^{-\beta} e^{-\frac{1}{2}|u-v||\xi|^2}\mu(d\xi)dudv \\
&\le C(t-s)\int_0^{t-s}\int_{\mathbb{R}^d} u^{-\beta} e^{-\frac{1}{2}u|\xi|^2}\mu(d\xi)du.
\end{aligned}
$$

Then, Hypothesis 4.23 allows us to write

$$
\int_{\mathbb{R}^d} e^{-\frac{1}{2}u|\xi|^2}\mu(d\xi) = C_1 + u^{\alpha+\beta-1}\int_{|\xi|>1} |\xi|^{2(\alpha+\beta-1)}\mu(d\xi)
$$

for any $\alpha < 1-\beta$, which leads to the bound $A_1 \le C(t-s)^{1+\alpha}$.

The term A_2 can be handled as follows: as in (4.6) we write:

$$
\begin{aligned}
A_2 = \mathbf{E}_B\Big[\int_0^s\int_0^s \gamma(u-v)\big[\Lambda(B_{t-u}-B_{t-v}) + \Lambda(B_{s-u}-B_{s-v}) \\
- 2\Lambda(B_{t-u}-B_{s-v})\big]\,dudv\Big], \quad (4.26)
\end{aligned}
$$

and changing to Fourier coordinates, this yields:

$$
A_2 \le 2\int_0^s\int_0^s \gamma(u-v)\int_{\mathbb{R}^d}\left|e^{-\frac{1}{2}|u-v||\xi|^2} - e^{-\frac{1}{2}|t-s-u+v||\xi|^2}\right|\mu(d\xi)dudv. \qquad (4.27)
$$

Using the estimate $|e^{-x}-e^{-y}| \le (e^{-x}+e^{-y})|x-y|^\alpha$, for any $0<\alpha<1-\beta$ and $x,y\ge 0$ and condition (4.4), we obtain

$$
A_2 \le C|t-s|^\alpha \int_0^s\int_0^s\int_{\mathbb{R}^d} |u-v|^{-\beta}\left(e^{-\frac{1}{2}|u-v||\xi|^2} + e^{-\frac{1}{2}|t-s-u+v||\xi|^2}\right)|\xi|^{2\alpha}\mu(d\xi)dudv.
$$

Then, in order to achieve the bound $A_2 \le |t-s|^\alpha$, it suffices to prove that

$$
\int_0^s\int_0^s |u-v|^{-\beta}\int_{\mathbb{R}^d} e^{-\frac{1}{2}|t-s-u+v||\xi|^2}|\xi|^{2\alpha}\mu(d\xi)dudv
$$

is uniformly bounded for $0 \leq s < t \leq T$. We decompose the integral with respect to the measure μ into the regions $\{|\xi| \leq 1\}$ and $\{|\xi| > 1\}$. The integral on $\{|\xi| \leq 1\}$ is clearly bounded because μ is finite on compact sets. Taking into account of the hypothesis 4.23, the integral over $\{|\xi| > 1\}$ can be handled using the estimate

$$\sup_{s,t\in[0,T]} \int_0^s \int_0^s |u-v|^{-\beta} e^{-\frac{1}{2}|t-s-u+v||\xi|^2} du dv \leq C|\xi|^{2\beta-2}.$$

Putting together our bounds on A_1 and A_2, we have been able to prove that $\mathbf{E}[|V_{t,x} - V_{s,x}|^2] \leq |t-s|^{\alpha}$. Furthermore, gathering our estimates for $V_{t,x} - V_{t,y}$ and $V_{t,x} - V_{s,x}$, this completes the proof of the theorem. □

Remark 4.25. *The results of Theorem 4.24 do not give the optimal Hölder continuity exponents for the process u defined by* (4.1)*. Another strategy could be implemented, based on the Feynman-Kac representation for the* $(2p)$*-th moments of u. This method is longer than the one presented here, but should lead to some better estimates of the continuity exponents. We stick to the shorter version of Theorem 4.24 for sake of conciseness, and also because optimal exponents will be deduced from the pathwise results of Section 4.5 (in particular Proposition 4.51).*

Time independent noise

In the case of time independent noise, the next result provides a result on the Hölder continuity of the Feynman-Kac functional defined in (4.20). In this case we impose the following additional integrability condition on μ.

Hypothesis 4.26. *Let W be a spatial Gaussian noise with covariance structure encoded by* Λ*. Suppose that the spectral measure* μ *satisfies*

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^{2(1-\alpha)}} < \infty$$

for some $\alpha \in (0,1)$.

Theorem 4.27. *Let* u *be the Feynman-Kac functional defined in (4.20) with* $u_0 \equiv 1$, *namely:*

$$u_{t,x} = \mathbf{E}_B \left[\exp\left(V_{t,x}\right)\right], \quad \textit{where} \quad V_{t,x} = \int_0^t \left(\int_{\mathbb{R}^d} \delta_0(B_r^x - y) W(dy) \right) dr.$$

Then u *admits a version which is* (γ_1, γ_2)*-Hölder continuous on any compact set of the form* $[0,T] \times [-M,M]^d$, *with any* $\gamma_1 < \frac{1+\alpha}{2}$, $\gamma_2 < \alpha$ *and* $T, M > 0$.

Proof. The proof is similar to the proof of Theorem 4.24 and we omit the details. □

4.4.3 Examples

Let us discuss the validity of Hypothesis 4.23 and Hypothesis 4.26 in the examples presented in the introduction. In the case of a time dependent noise we assume that the time covariance has the form $\gamma(x) = |x|^{-\beta}$, $0 < \beta < 1$.

For the Riesz kernel $\Lambda(x) = |x|^{-\eta}$, where $\mu(d\xi) = C_\beta |\xi|^{\eta-d} d\xi$, we already know that Hypothesis 4.1 holds if $\eta < 2$. On the other hand, Hypothesis 4.16, which allows us to define the Feynman-Kac functional in the time dependent case, is satisfied if $\eta < 2 - 2\beta$. For the Hölder continuity, Hypothesis 4.23 holds for any $\alpha \in (0, 1 - \beta - \frac{\eta}{2})$ and Hypothesis 4.26 holds for any $\alpha' \in (0, 1 - \frac{\eta}{2})$. Then, by Theorem 4.24 and 4.27, for any $\alpha \in (0, 1 - \beta - \frac{\eta}{2})$, $\alpha' \in (0, 1 - \frac{\eta}{2})$, assuming $u_0 \equiv 0$, the Feynman-Kac functional (4.1) is Hölder continuous of order α in the space variable and of order $\frac{\alpha}{2}$ in the time variable, and the Feynman-Kac functional (4.20) is Hölder continuous of order α in the space variable and of order $\frac{\alpha'+1}{2}$ in the time variable.

For the Bessel kernel, we know that Hypothesis 4.1 is satisfied when $\eta > d - 2$, and Hypothesis 4.23 holds when $\eta > d + 2\beta - 2$. By Theorem 4.24 and 4.27, for any $\alpha \in (0, \min(\frac{\eta - d}{2} - \beta + 1, 1))$ and $\alpha' \in (0, \min(\frac{\beta - d}{2} + 1, 1))$, assuming $u_0 \equiv 0$, the Feynman-Kac functional (4.1) is Hölder continuous of order α in the space variable and of order $\frac{\alpha}{2}$ in the time variable, the Feynman-Kac functional (4.20) is Hölder continuous of order α' in the space variable and of order $\frac{\alpha'+1}{2}$ in the time variable.

Consider the case of a fractional noise with covariance function $\gamma(t) = H(2H - 1)|t|^{2H-2}$ and $\Lambda(x) = \prod_{i=1}^{d} H_i(2H_i - 2)|x_i|^{2H_i - 2}$. We know that Hypothesis 4.1 holds when $\sum_{i=1}^{d} H_i > d - 1$. Moreover, when $\sum_{i=1}^{d} H_i > d - 2H + 1$, Hypothesis 4.23 is satisfied. By Theorem 4.24 and 4.27, for any $\alpha \in (0, \sum_{i=1}^{d} H_i + 2H - d - 1)$ and $\alpha' \in (0, \sum_{i=1}^{d} H_i - d + 1)$, assuming $u_0 \equiv 0$, Feynman-Kac functional (4.1) is Hölder continuous of order α in the space variable and of order $\frac{\alpha}{2}$ in the time variable, which recovers the result in [56]). On the other hand, Feynman-Kac functional (4.20) is Hölder continuous of order α' in the space variable and of order $\frac{\alpha'+1}{2}$ in the time variable.

4.5 Equation in the Stratonovich sense

In this section we consider the following stochastic heat equation of Stratonovich type with the multiplicative Gaussian noise introduced in Section 4.2.1:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u \frac{\partial^{d+1} W}{\partial t \partial x_1 \cdots \partial x_d}. \tag{5.1}$$

As in the previous sections, the initial condition is a continuous and bounded function u_0. We will discuss two notions of solution. The first one is based on the Stratonovich integral, which is controlled using techniques of Malliavin calculus and a second one is completely pathwise and is based on Besov spaces. We will show that the Feynman-Kac functional (4.1) is a solution in both senses, and in the pathwise formulation it is the unique solution to equation (5.1).

We will also discuss the case of a time independent multiplicative Gaussian noise introduced in Section 4.2.2, that is

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u\frac{\partial^d W}{\partial x_1 \cdots \partial x_d}, \tag{5.2}$$

with an initial condition $u_0 \in C_b(\mathbb{R}^d)$. As in the case of a time dependent noise, we will show that the Feynman-Kac functional (4.20) is both a mild Stratonovich solution and a pathwise solution.

4.5.1 Stratonovich solution

Our aim is to define a notion of solution to equation (5.1) by means of a Russo-Vallois type approach, which happens to be compatible with Malliavin calculus tools. As usual, we divide our study into time dependent and time independent cases.

Time dependent case

Let W be the time dependent noise introduced in Section 4.2.1. In this case, we make use of the following definition of Stratonovich integral.

Definition 4.28. *Given a random field $v = \{v_{t,x}; t \geq 0, x \in \mathbb{R}^d\}$ such that*

$$\int_0^T \int_{\mathbb{R}^d} |v_{t,x}|\, dxdt < \infty$$

almost surely for all $T > 0$, the Stratonovich integral $\int_0^T \int_{\mathbb{R}^d} v_{t,x} W(dt,dx)$ is defined as the following limit in probability, if it exists:

$$\lim_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0} \int_0^T \int_{\mathbb{R}^d} v_{t,x} \dot{W}^{\varepsilon,\delta}_{t,x}\, dxdt\,,$$

where $\dot{W}^{\varepsilon,\delta}_{t,x}$ is the regularization of W defined in (3.12).

With this definition of integral, we have the following notion of solution for equation (5.1).

Definition 4.29. *A random field $u = \{u_{t,x}; t \geq 0, x \in \mathbb{R}^d\}$ is a mild solution of equation (5.1) with initial condition $u_0 \in C_b(\mathbb{R}^d)$ if for any $t \geq 0$ and $x \in \mathbb{R}^d$ the following equation holds*

$$u_{t,x} = p_t u_0(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) u_{s,y} W(ds, dy), \tag{5.3}$$

where the last term is a Stratonovich stochastic integral in the sense of Definition 4.28.

The next result asserts the existence of a solution to equation (5.3) based on the Feynman-Kac representation.

Theorem 4.30. *Assume Hypothesis 4.16 holds true. Then, the process u defined in (4.1) is a mild solution of equation (5.1), in the sense given by Definition 4.29.*

Proof. We proceed similarly to Section 4.3.2. Consider the following approximation to equation (5.1)

$$\frac{\partial u^{\varepsilon,\delta}}{\partial t} = \frac{1}{2}\Delta u^{\varepsilon,\delta} + u^{\varepsilon,\delta} \dot{W}^{\varepsilon,\delta}_{t,x}, \tag{5.4}$$

with initial condition u_0, where $\dot{W}^{\varepsilon,\delta}_{t,x}$ is defined in (3.12). From the classical Feynman-Kac formula, we know that

$$u^{\varepsilon,\delta}_{t,x} = \mathbf{E}_B \left[u_0(B^x_t) \exp\left(\int_0^t \dot{W}^{\varepsilon,\delta}(t-s, B^x_s) ds \right) \right]. \tag{5.5}$$

Moreover, thanks to Fubini's theorem, we can write

$$\int_0^t \dot{W}^{\varepsilon,\delta}(t-s, B^x_s) ds = \frac{1}{\delta} \int_0^t \left(\int_{(t-s-\delta)_+}^{t-s} \int_{\mathbb{R}^d} p_\varepsilon(B^x_s - y) W(dr, dy) \right) ds$$

$$= \quad W(A^{\varepsilon.\delta}_{t,x}) = V^{\varepsilon,\delta}_{t,x},$$

where $A^{\varepsilon,\delta}_{t,x}$ is defined in (3.19) and $V^{\varepsilon,\delta}_{t,x}$ is defined in (4.10). Therefore, the process $u^{\varepsilon,\delta}_{t,x}$ is given by (4.15), and Proposition 4.22 implies that (4.16) holds.

Next we prove that u is a mild solution of equation (5.1) in the sense of Definition 4.29. Taking into account of the definition of the Stratonovich integral, is suffices to show that

$$G^{\varepsilon,\delta} := \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) \left(u^{\varepsilon,\delta}_{s,y} - u_{s,y}\right) \dot{W}^{\varepsilon,\delta}_{s,y}\, dyds$$

converges in $L^2(\Omega)$ to zero when first δ tends to zero and later ε tends to zero. To this aim, we are going to use the following notation:

$$\psi^{\varepsilon,\delta}_{s,y}(r,z) = \frac{1}{\delta}\mathbf{1}_{[(s-\delta)_+,s]}(r)p_\varepsilon(y-z), \quad \text{and} \quad \widetilde{u}^{\varepsilon,\delta}_{s,y} = u^{\varepsilon,\delta}_{s,y} - u_{s,y}.$$

In particular, notice that $\dot{W}^{\varepsilon,\delta}_{s,y} = W\left(\psi^{\varepsilon,\delta}_{s,y}\right)$. Then,

$$\begin{aligned}
&\mathbf{E}\left[\left(G^{\varepsilon,\delta}\right)^2\right] \\
=\ & \int_0^t \int_0^t \int_{\mathbb{R}^{2d}} p_{t-s}(x-y)p_{t-r}(x-z)\mathbf{E}\left[\widetilde{u}^{\varepsilon,\delta}_{s,y}\widetilde{u}^{\varepsilon,\delta}_{r,z} W\left(\psi^{\varepsilon,\delta}_{s,y}\right) W\left(\psi^{\varepsilon,\delta}_{r,z}\right)\right] dydzdsdr,
\end{aligned}$$

and the expected value above can be analyzed by integration by parts. Indeed, according to relation (5.5), it is readily checked that $\widetilde{u}^{\varepsilon,\delta}_{s,y}\widetilde{u}^{\varepsilon,\delta}_{r,z} = \mathbf{E}_{B,\widetilde{B}}[Z^{\varepsilon,\delta}_{s,y,r,z}]$, with

$$Z^{\varepsilon,\delta}_{s,y,r,z} = u_0(B^y_s)\left[\exp\left(V^{\varepsilon,\delta,B}_{s,y}\right) - \exp\left(V^B_{s,y}\right)\right] u_0(\widetilde{B}^z_r)\left[\exp\left(V^{\varepsilon,\delta,\widetilde{B}}_{r,z}\right) - \exp\left(V^{\widetilde{B}}_{r,z}\right)\right],$$

and where $B,\widetilde{B}$ designate two independent d-dimensional Brownian motions. Moreover, a straightforward application of Fubini's theorem yields:

$$\mathbf{E}\left[\widetilde{u}^{\varepsilon,\delta}_{s,y}\widetilde{u}^{\varepsilon,\delta}_{r,z}W\left(\psi^{\varepsilon,\delta}_{s,y}\right)W\left(\psi^{\varepsilon,\delta}_{r,z}\right)\right]=\mathbf{E}_{B,\widetilde{B}}\left\{\mathbf{E}_W\left[Z^{\varepsilon,\delta}_{s,y,r,z}W\left(\psi^{\varepsilon,\delta}_{s,y}\right)W\left(\psi^{\varepsilon,\delta}_{r,z}\right)\right]\right\}.$$

We can now invoke formula (2.11) plus some easy computations of Malliavin derivatives in order to get:

$$\mathbf{E}\left[\left(G^{\varepsilon,\delta}\right)^2\right]=A_1+A_2, \tag{5.6}$$

where

$$A_1=\int_0^t\int_0^t\int_{\mathbb{R}^{2d}}p_{t-s}(x-y)p_{t-r}(x-z)\mathbf{E}\left[\widetilde{u}^{\varepsilon,\delta}_{s,y}\widetilde{u}^{\varepsilon,\delta}_{r,z}\right]\langle\psi^{\varepsilon,\delta}_{s,y},\psi^{\varepsilon,\delta}_{r,z}\rangle_{\mathscr{H}}dydzdsdr$$

and

$$A_2=\int_0^t\int_0^t\int_{\mathbb{R}^{2d}}p_{t-s}(x-y)p_{t-r}(x-z)\mathbf{E}\left[Z^{\varepsilon,\delta}_{s,y,r,z}\Gamma^{\varepsilon,\delta}_{s,y,r,z}\right]dydzdsdr,$$

with the notation

$$\begin{aligned}\Gamma^{\varepsilon,\delta}_{s,y,r,z} \quad = \quad & \langle\psi^{\varepsilon,\delta}_{s,y},A^{\varepsilon,\delta,\widetilde{B}}_{r,z}-\delta_0(\widetilde{B}^z_{r-\cdot}-\cdot)\rangle_{\mathscr{H}}\langle\psi^{\varepsilon,\delta}_{r,z},A^{\varepsilon,\delta,B}_{s,y}-\delta_0(B^y_{s-\cdot}-\cdot)\rangle_{\mathscr{H}}\\
&+\langle\psi^{\varepsilon,\delta}_{s,y},A^{\varepsilon,\delta,B}_{s,y}-\delta_0(B^y_{s-\cdot}-\cdot)\rangle_{\mathscr{H}}\langle\psi^{\varepsilon,\delta}_{r,z},A^{\varepsilon,\delta,\widetilde{B}}_{r,z}-\delta_0(\widetilde{B}^r_{z-\cdot}-\cdot)\rangle_{\mathscr{H}}\\
&+\langle\psi^{\varepsilon,\delta}_{s,y},A^{\varepsilon,\delta,B}_{s,y}-\delta_0(B^s_{y-\cdot}-\cdot)\rangle_{\mathscr{H}}\langle\psi^{\varepsilon,\delta}_{r,z},A^{\varepsilon,\delta,B}_{s,y}-\delta_0(B^y_{s-\cdot}-\cdot)\rangle_{\mathscr{H}}\\
&+\langle\psi^{\varepsilon,\delta}_{s,y},A^{\varepsilon,\delta,\widetilde{B}}_{r,z}-\delta_0(\widetilde{B}^r_{z-\cdot}-\cdot)\rangle_{\mathscr{H}}\langle\psi^{\varepsilon,\delta}_{r,z},A^{\varepsilon,\delta,\widetilde{B}}_{r,z}-\delta_0(\widetilde{B}^r_{z-\cdot}-\cdot)\rangle_{\mathscr{H}}\end{aligned}$$

According to Proposition 4.22, we know that

$$\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0}\sup_{s\in[0,T],y\in\mathbb{R}^d}\mathbf{E}\left[|\widetilde{u}^{\varepsilon,\delta}_{s,y}|^2\right]=0,$$

and with the same arguments as in Proposition 4.22 we can also show that

$$\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0}\sup_{s,r\in[0,T],y,z\in\mathbb{R}^d}\mathbf{E}\left[|Z^{\varepsilon,\delta}_{s,yr,z}|^2\right]=0.$$

Therefore, with formula (5.6) in mind, the convergence to zero of $B^{\varepsilon,\delta}$ will follow, provided we show the following quantities are uniformly bounded in $\varepsilon\in(0,1)$ and $\delta\in(0,1)$

$$\theta_1:=\int_0^t\int_0^t\int_{\mathbb{R}^{2d}}p_{t-s}(x-y)p_{t-r}(x-z)\left|\langle\psi^{\varepsilon,\delta}_{s,y},\psi^{\varepsilon,\delta}_{r,z}\rangle_{\mathscr{H}}\right|dydzdsdr \tag{5.7}$$

and

$$\theta_2:=\int_0^t\int_0^t\int_{\mathbb{R}^{2d}}p_{t-s}(x-y)p_{t-r}(x-z)\left\|\Gamma^{\varepsilon,\delta}_{s,y,r,z}\right\|_2dydzdsdr, \tag{5.8}$$

where $\|\Gamma^{\varepsilon,\delta}_{s,y,r,z}\|_2$ stands for the norm of $\Gamma^{\varepsilon,\delta}_{s,y,r,z}$ in $L^2(\Omega)$. The remainder of the proof is thus just reduced to an estimation of (5.7) and (5.8).

In order to bound θ_1, we apply the estimate (4.18) and the semigroup property of the heat kernel, which yields

$$\begin{aligned}
&\langle\psi^{\varepsilon,\delta}_{s,y},\psi^{\varepsilon,\delta}_{r,z}\rangle_{\mathscr{H}}\\
=&\left(\frac{1}{\delta^2}\int_{(s-\delta)_+}^{s}\int_{(r-\delta)_+}^{r}\gamma(u-v)dudv\right)\int_{\mathbb{R}^{2d}}p_\varepsilon(y-z_1)p_\varepsilon(z-z_2)\Lambda(z_1-z_2)dz_1dz_2\\
\le&\ c_{T,\beta}|r-s|^{-\beta}\int_{\mathbb{R}^d}p_{2\varepsilon}(y-z-w)\Lambda(w)dw.
\end{aligned}$$

Substituting this estimate into (5.7), we obtain

$$\begin{aligned}
\theta_1&\le c_{T,\beta}\int_0^t\int_0^t\int_{\mathbb{R}^d}p_{2t-s-r+2\varepsilon}(w)\Lambda(w)|r-s|^{-\beta}dw\\
&\le c'_{T,\beta}\int_0^{2t}\int_{\mathbb{R}^d}p_{2t-s}(w)\Lambda(w)dwds<\infty,
\end{aligned}$$

where we get rid of ε in Fourier mode, similarly to the proof of (3.27).

We now turn to the control of θ_2: we first write, using the estimate (4.18) and the semigroup property of the heat kernel,

$$\begin{aligned}
&\langle \psi^{\varepsilon,\delta}_{s,y}, A^{\varepsilon,\delta}_{r,z} \rangle_{\mathscr{H}} \\
= &\ \frac{1}{\delta^2} \int_{(s-\delta)_+}^{s} \int_{(r-\sigma-\delta)_+}^{r-\sigma} \int_0^r \int_{\mathbb{R}^{2d}} p_\varepsilon(y-z_1) p_\varepsilon(B^z_\sigma - z_2) \gamma(u-v) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times \Lambda(z_1 - z_2) dz_1 dz_2 d\sigma dv du \\
\le &\ c_{T,\beta} \int_0^r \int_{\mathbb{R}^d} p_{2\varepsilon}(y - B^z_{r-\sigma} - w) \Lambda(w) |s-\sigma|^{-\beta} dw d\sigma
\end{aligned}$$

Invoking again arguments of Fourier analysis, analogous to those in the proof of (3.27), we can show that

$$\begin{aligned}
&\mathbf{E}\left[\left| \int_0^r \int_{\mathbb{R}^d} p_{2\varepsilon}(y - B^z_{r-\sigma} - w) \Lambda(w) |s-\sigma|^{-\beta} dw d\sigma \right|^4 \right] \\
\le &\ \mathbf{E}\left[\left| \int_0^r \Lambda(B_{r-\sigma}) |s-\sigma|^{-\beta} d\sigma \right|^4 \right],
\end{aligned}$$

and

$$\sup_{r,s\in[0,T]} \mathbf{E}\left[\left| \int_0^r \Lambda(B_{r-\sigma}) |s-\sigma|^{-\beta} d\sigma \right|^4 \right] < \infty.$$

This implies that $\left\| \Gamma^{\varepsilon,\delta}_{s,y,r,z} \right\|_2$, and thus, θ_2, are uniformly bounded. The proof of the theorem is complete. □

Remark 4.31. *Consider the case where the space dimension is* 1*,* $\Lambda(x)$ *is the Dirac delta function* $\delta_0(x)$ *corresponding to the white noise, which in our setting means that condition* (4.5) *is satisfied with* $0 < \beta < \frac{1}{2}$*. Then our theorems of Section 4.4 cover assumption* (4.4)*, with* $0 < \beta < \frac{1}{2}$ *too, if we interpret the composition* $\Lambda(B_r - B_s)$ *as a generalized Wiener functional. Notice that in the case of the fractional Brownian* *motion with Hurst parameter H (that is $\gamma(x) = c_H|x|^{2H-2}$) the condition $0 < \beta < \frac{1}{2}$ means that $H > \frac{3}{4}$. In this case it is already known that the process defined by (4.1) is still a solution to equation (5.1) (see [56]).*

Time independent case

Let W be the time independent noise introduced in Section 4.2.2. We claim that as in the time independent case, the Feynman-Kac functional given by (4.20) is a mild solution to equation (5.2) in the Stratonovich sense.

The Stratonovich integral with respect to the noise W is defined as the limit of the integrals with respect to regularization of the noise.

Definition 4.32. *Given a random field $v = \{v_x; x \in \mathbb{R}^d\}$ such that $\int_{\mathbb{R}^d} |v_x| dx < \infty$ almost surely, the Stratonovich integral $\int_{\mathbb{R}^d} v_x W(dx)$ is defined as the following limit in probability, if it exists:*

$$\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^d} v_x \dot{W}_x^{\varepsilon} dx,$$

where $\dot{W}_x^{\varepsilon} = \int_{\mathbb{R}^d} p_{\varepsilon}(x-y) W(dy)$.

With this definition of integral, we have the following notion of solution for equation (5.2).

Definition 4.33. *A random field $u = \{u_{t,x}; t \geq 0, x \in \mathbb{R}^d\}$ is a mild solution of equation (5.2) if we have*

$$u_{t,x} = p_t u_0(x) + \int_0^t \left(\int_{\mathbb{R}^d} p_{t-s}(x-y) u_{s,y} W(dy) \right) ds$$

almost surely for all $t \geq 0$, where the last term is a Stratonovich stochastic integral in the sense of Definition 4.32.

The next result is the existence of a solution based on the Feynman-Kac representation.

Theorem 4.34. *Suppose that μ satisfies (2.4). Then, the process $u_{t,x}$ given by (4.20) is a mild solution of equation (5.2).*

The proof of this theorem is similar to that of Theorem 4.30, and it is omitted.

4.5.2 Existence and uniqueness of a pathwise solution

In this section we define and solve equations (5.1) and (5.2) in a pathwise manner in $\mathbb{R}^d$, when the noise W satisfies some additional hypotheses. Contrarily to the Stratonovich technology invoked at Section 4.5.1, the pathwise method yields uniqueness theorems, which will be used in order to identify Feynman-Kac and pathwise solutions. At a technical level, our results will be achieved in the framework of weighted Besov spaces, that we proceed to recall now.

Besov spaces

The definition of Besov spaces is based on Littlewood-Paley theory, which relies on decompositions of functions into spectrally localized blocks. We thus first introduce the following basic definitions.

Definition 4.35. *We call annulus any set of the form $C = \{x \in \mathbb{R}^d : a \leqslant |x| \leqslant b\}$ for some $0 < a < b$. A ball is a set of the form $B = \{x \in \mathbb{R}^d : |x| \leqslant b\}$.*

The localizing functions for the Fourier domain alluded to above are defined as follows.

Notation 4.36. *In the remaining part of this section, we shall use χ, φ to denote two smooth nonnegative radial functions with compact support such that:*

1. *The support of χ is contained in a ball and the support of φ is contained in an annulus C with $a = 3/4$ and $b = 8/3$;*

2. *We have $\chi(\xi) + \sum_{j \geqslant 0} \varphi(2^{-j}\xi) = 1$ for all $\xi \in \mathbb{R}^d$;*

3. *It holds that $supp(\chi) \cap supp(\varphi(2^{-i}\cdot)) = \varnothing$ for $i \geqslant 1$ and if $|i-j| > 1$, then $supp(\varphi(2^{-i}\cdot)) \cap supp(\varphi(2^{-j}\cdot)) = \varnothing$.*

In the sequel, we set $\varphi_j(\xi) := \varphi(2^{-j}\xi)$.

For the existence of χ and φ see [2, Proposition 2.10]. With this notation in mind, the Littlewood-Paley blocks are now defined as follows.

Definition 4.37. *Let $u \in \mathscr{S}'(\mathbb{R}^d)$. We set*

$$\Delta_{-1}u = \mathscr{F}^{-1}(\chi \mathscr{F}u), \quad \text{and for } j \geqslant 0 \quad \Delta_j u = \mathscr{F}^{-1}(\varphi_j \mathscr{F}u).$$

We also use the notation $S_k u = \sum_{j=-1}^{k-1} \Delta_j u$, valid for all $k \geq 0$.

Observe that one can also write $\Delta_{-1}u = \widetilde{K} * u$ and $\Delta_j u = K_j * u$ for $j \geqslant 0$, where $\widetilde{K} = \mathscr{F}^{-1}\chi$ and $K_j = 2^{jd}\mathscr{F}^{-1}\varphi(2^j\cdot)$. In particular the $\Delta_j u$ are smooth functions for all $u \in \mathscr{S}'(\mathbb{R}^d)$.

In order to handle equations whose space parameter lies in an unbounded domain like $\mathbb{R}^d$, we shall use spaces of weighted Hölder type functions for polynomial or exponential weights, where the weights satisfy some smoothness conditions. In this way we define the following class of weights.

Definition 4.38. *We denote by $\mathscr{W}$ the class of weights $w \in \mathscr{C}_b^\infty(\mathbb{R}^d;\mathbb{R}_+)$ consisting of:*

- *The weights ρ_κ obtained as functions of the form $c\,(1+|x|^\kappa)^{-1}$, with $\kappa \geq 1$, smoothed at 0.*

- *The weights e_λ obtained as functions of the form $c\,e^{-\lambda|x|}$, with $\lambda > 0$, smoothed at 0.*

- *Products of these functions.*

Notice that more general classes of weights are introduced in [90]. We have also tried to stick to the notation given in [47], from which our developments are inspired.

Weighted Besov spaces are sets of functions characterized by their Littlewood-Paley block decomposition. Specifically, their definition is as follows.

Definition 4.39. *Let $w \in \mathscr{W}$ and $\kappa \in \mathbb{R}$. We set*

$$\mathscr{B}_w^\kappa(\mathbb{R}^d) = \left\{ f \in \mathscr{S}'(\mathbb{R}^d);\, \|f\|_{w,\kappa} := \sup_{j \geq -1} 2^{j\kappa} \|w \Delta_j f\|_{L^\infty} < \infty \right\}. \tag{5.9}$$

We call this space a weighted Besov-Hölder space. When $w = 1$, we just denote the space by $\mathscr{B}^\kappa(\mathbb{R}^d)$, and it corresponds to the usual Besov space $\mathscr{B}^\kappa_{\infty,\infty}(\mathbb{R}^d)$.

Notice that we follow here the terminology of [90]. The weighted Besov-Hölder spaces are well understood objects, and let us recall some basic facts about them.

Proposition 4.40. *Let $w, w_1, w_2 \in \mathscr{W}$, $\kappa \in \mathbb{R}$ and $f \in \mathscr{B}_w^\kappa(\mathbb{R}^d)$. Then the following holds true:*

(i) *There exist some positive constants $c^1_{\kappa,w}, c^2_{\kappa,w}$ such that $c^1_{\kappa,w}\|fw\|_\kappa \leq \|f\|_{w,\kappa} \leq c^2_{\kappa,w}\|fw\|_\kappa$.*

(ii) *For $\kappa \in (0,1)$, we have $f \in \mathscr{B}_w^\kappa(\mathbb{R}^d)$ iff fw is a κ-Hölder function.*

(iii) *If $w_1 < w_2$ we have $\|f\|_{w_1,\kappa} \leq \|f\|_{w_2,\kappa}$.*

Proof. Item (i) is borrowed from [90, Chapter 6]. The fact that $\mathscr{B}^\kappa(\mathbb{R}^d)$ coincides with the space of Hölder continuous functions $\mathscr{C}^\kappa(\mathbb{R}^d)$ for $\kappa \in [0,1]$ is shown in [2, Theorem 2.36], and it yields item (ii) thanks to (i). Finally, item (iii) is also taken from [90, Chapter 6]. □

Let us now state a result about products of distributions which turns out to be useful for our existence and uniqueness result.

Proposition 4.41. *Let w_1, w_2 be two weight functions in $\mathscr{W}$, and $\kappa_1, \kappa_2 \in \mathbb{R}$ such that $\kappa_2 < 0 < \kappa_1$ and $\kappa_1 > |\kappa_2|$ and let $w = w_1 w_2$. Then*

$$(f_1, f_2) \in \mathscr{B}_{w_1}^{\kappa_1} \times \mathscr{B}_{w_2}^{\kappa_2} \longmapsto f_1 f_2 \in \mathscr{B}_{w}^{\kappa_2} \quad \textit{is continuous.} \tag{5.10}$$

Furthermore, the following bound holds true:

$$\|f_1 f_2\|_{\mathscr{B}_w^{\kappa_2}} \le \|f_1\|_{\mathscr{B}_{w_1}^{\kappa_1}} \|f_2\|_{\mathscr{B}_{w_2}^{\kappa_2}} . \tag{5.11}$$

Finally we label the action of the heat semigroup on functions in weighted Besov spaces.

Proposition 4.42. *Let $w \in \mathscr{W}$, $\kappa \in \mathbb{R}$ and $f \in \mathscr{B}_w^{\kappa}(\mathbb{R}^d)$. Then for all $t \in [0, \tau]$, $\gamma > 0$ and $\hat{\kappa} > \kappa$ we have*

$$\|p_t f\|_{w,\hat{\kappa}} \le c_{\tau,w,\kappa,\hat{\kappa}} t^{-\frac{\hat{\kappa}-\kappa}{2}} \|f\|_{w,\kappa}, \quad \textit{and} \quad \|[Id - p_t]f\|_{w,\kappa-2\gamma} \le c_{\tau,w,\gamma} t^{\gamma} \|f\|_{w,\kappa}.$$

Notion of solution

In order to give a pathwise definition of solution for equation (5.1), we will replace the noise W by a nonrandom Hölder continuous function in time with values in a Besov space of distributions, denoted by $\mathcal{W}$. We will show later (see Proposition 4.49) that under Hypothesis 4.23, almost surely the mapping $t \to W(\mathbf{1}_{[0,t]}\varphi)$, $\varphi \in \mathscr{D}$, is Hölder continuous with values in this Besov space. We thus label a notation for this kind of space.

Notation 4.43. *Let $\theta \in (0,1)$, $\kappa \in \mathbb{R}$ and $w \in \mathscr{W}$. The space of θ-Hölder continuous functions from $[0,T]$ to a weighted Sobolev space $\mathscr{B}_w^\kappa$ is denoted by $\mathscr{C}_{T,w}^{\theta,\kappa}$. Otherwise stated, we have $\mathscr{C}_{T,w}^{\theta,\kappa} = \mathscr{C}^\theta([0,T];\mathscr{B}_w^\kappa)$. In order to alleviate notations, we shall write $\mathscr{C}_w^{\theta,\kappa}$ only when the value of T is non ambiguous.*

Now we introduce the pathwise type assumption that we shall make on the multiplicative input distribution $\mathcal{W}$.

Hypothesis 4.44. *We assume that there exist two constants $\theta, \kappa \in (0,1)$ satisfying $\frac{1+\kappa}{2} < \theta < 1$, such that $\mathcal{W} \in \mathscr{C}_{T,\rho_\sigma}^{\theta,-\kappa}$, for any $\sigma > 0$ arbitrarily small.*

We also label some more notation for further use:

Notation 4.45. *For a function $f : [0,T] \to \mathscr{B}$, where $\mathscr{B}$ stands for a generic Banach space, we set $\delta f_{st} = f_t - f_s$ for $0 \le s \le t \le T$. Notice that δ has also been used for Skorohod integrals, but this should not lead to ambiguities since Skorohod integrals won't be used in this section.*

With these preliminaries in hand, we shall combine the following ingredients in order to solve equation (5.1):

• Like the input $\mathcal{W}$, the solution u will live in a space of Hölder functions in time, with values in a weighted Sobolev space of the form $\mathscr{B}_{e_\lambda}^{\kappa_u}$. This allows the use of estimates of Young integration type in order to define integrals involving increments of the form $u\, d\mathcal{W}$.

• We have to take into account of the fact that, when one multiplies the function $u_s \in \mathscr{B}_{e_\lambda}^{\kappa_u}$ by the distribution $\delta\mathcal{W}_{st} \in \mathscr{B}_{\rho_\sigma}^{-\kappa}$, the resulting distribution $u_s \delta\mathcal{W}_{st}$ lies (provided

$\kappa_u > \kappa$) into the space $\mathscr{B}^{-\kappa}_{e_\lambda \rho_\sigma}$. This will force us to assume in fact $u_s \in \mathscr{B}^{\kappa_u}_{w_s}$, where the weight $w_s \in \mathscr{W}$ decreases with s.

Let us turn now to the technical part of our task. We first fix positive constants λ, σ and define a weight $w_t = e_{\lambda+\sigma t}$. We shall seek the solution to equation (5.1) in the following space:

$$\mathscr{D}^{\theta_u,\kappa_u}_{\lambda,\sigma} = \Big\{ f \in \mathscr{C}([0,T] \times \mathbb{R}^d);\ \|f_t\|_{\mathscr{B}^{\kappa_u}_{w_t}} \le c_{T,f} \\ \text{and } \|f_t - f_s\|_{\mathscr{B}^{\kappa_u}_{w_t}} \le c_f \, |t-s|^{\theta_u} \ \ \forall 0 \le s < t \le T \Big\}. \quad (5.12)$$

We introduce the Hölder norm in this space by

$$\|f\|_{\mathscr{D}^{\theta_u,\kappa_u}_{\lambda,\sigma}} = \sup_{0 \le s < t \le T} \frac{\|f_t - f_s\|_{\mathscr{B}^{\kappa_u}_{w_t}}}{|t-s|^{\theta_u}}. \quad (5.13)$$

We now introduce a pathwise mild formulation for equation (5.1) in the spaces $\mathscr{D}^{\theta_u,\kappa_u}_{\lambda,\sigma}$.

Definition 4.46. *Suppose that $\mathcal{W}$ satisfies Hypothesis 4.44. Let $u \in \mathscr{D}^{\theta_u,\kappa_u}_{\lambda,\sigma}$ for fixed $\lambda, \sigma > 0$, $\theta_u + \theta > 1$ and $\kappa_u \in (\kappa, 1)$. Consider an initial condition $u_0 \in \mathscr{B}^{\kappa_u}_{e_\lambda}$. We say that u is a mild solution to equation*

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u \frac{\partial \mathcal{W}}{\partial t} \quad (5.14)$$

with initial condition u_0 if it satisfies the following integral equation

$$u_t = p_t u_0 + \int_0^t p_{t-s} \left(u_s \, \mathcal{W}(ds) \right), \quad (5.15)$$

where the product $u\mathcal{W}$ is interpreted in the distributional sense of (5.10) *and the time integral is understood in the Young sense.*

Remark 4.47. *Let us specify what we mean by $J^u_t := \int_0^t p_{t-s}(u_s \mathcal{W}(ds))$ under the conditions of Definition 4.46. First, we should understand J^u_t as*

$$J^u_t = \lim_{\varepsilon \to 0} J^{u,\varepsilon}_t, \quad \textit{where} \quad J^{u,\varepsilon}_t = \int_0^{t-\varepsilon} p_{t-s}(u_s \mathcal{W}(ds)).$$

The integration on $[0, t-\varepsilon]$ avoids any singularity of p_{t-s} as an operator from $\mathscr{B}^{-\kappa}$ to $\mathscr{B}^{\kappa_u}$, so that $J^{u,\varepsilon}_t$ is defined as a Young integral. This integral is in particular limit of Riemann sums along dyadic partitions of $[0,t]$:

$$J^{u,\varepsilon}_t = \lim_{n \to \infty} \sum_{j=0}^{2^n - 1} p_{t-t^n_j} \left(u_{t^n_j} \, \delta\mathcal{W}_{t^n_j t^n_{j+1}} \right) \mathbf{1}_{[0,t-\varepsilon]}(t^n_{j+1}), \quad \textit{where} \quad t^n_j = \frac{jt}{2^n}.$$

We then assume that one can combine the limiting procedures in n and ε (the justification of this step is left to the patient reader), and finally we define:

$$J^u_t = \lim_{n \to \infty} J^{u,n}_t, \quad \textit{where} \quad J^{u,n}_t = \sum_{j=0}^{2^n - 1} p_{t-t^n_j} \left(u_{t^n_j} \, \delta\mathcal{W}_{t^n_j t^n_{j+1}} \right). \tag{5.16}$$

Here again, recall that the product $u_{t^n_j} \, \delta\mathcal{W}_{t^n_j t^n_{j+1}}$ is interpreted according to (5.10)*. This will be our way to understand equation* (5.15)*.*

We can now turn to the resolution of the equation in this context.

Resolution of the equation

Our existence and uniqueness result takes the following form:

Theorem 4.48. *Let $\mathcal{W}$ be a Hölder continuous distribution valued function satisfying Hypothesis 4.44, and let λ, σ be two strictly positive constants. Consider an initial condition $u_0 \in \mathscr{B}_{e_\lambda}^{\kappa_u}$. Then:*

(a) *There exist θ_u, κ_u satisfying $\theta_u + \theta > 1$ and $\kappa_u \in (\kappa, 1)$, such that equation* (5.15) *admits a unique solution in $\mathscr{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}$.*

(b) *The application $(u_0, \mathcal{W}) \mapsto u$ is continuous from $\mathscr{B}_{e_\lambda}^{\kappa_u} \times \mathscr{C}_{T,\rho_\sigma}^{\theta,-\kappa}$ to $\mathscr{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}$.*

Proof. We divide this proof into several steps.

Step 1: Definition of a contracting map. We fix a time interval $[0,\tau]$, where $\tau \leq T$, and along the proof we denote by $\mathscr{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}$ and $\|\cdot\|_{\mathscr{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}}$ the space and the Hölder norm defined in (5.12) and (5.13), respectively, but restricted to the interval $[0,\tau]$.

We consider a map Γ defined on $\mathscr{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}$ by $\Gamma(u) = v$, where v is the function defined by $v := p_t u_0 + J_t^u$ as in Remark 4.47. The proof of our result relies on two steps: (i) Show that Γ defines a map from $\mathscr{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}$ to $\mathscr{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}$, independently of the length of the interval $[0,\tau]$. (ii) Check that Γ is in fact a contraction if τ is made small enough. The two steps hinge on the same type of computations, so that we shall admit point (i) and focus on point (ii) for sake of conciseness.

In order to prove that Γ is a contraction, consider $u^1, u^2 \in \mathscr{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}$, and for $j = 1,2$ set $v^j = \Gamma(u^j)$. For notational sake, we also set $u^{12} = u^1 - u^2$ and $v^{12} = v^1 - v^2$. Consistently with equation (5.15), v^{12} satisfies the relation

$$v_t^{12} = \int_0^t p_{t-r}\left(u_r^{12}\,\mathcal{W}(dr)\right).$$

Notice that the function v^{12} is in fact defined by relation (5.16). We have admitted point (i) above, which means in particular that we assume that the Riemann sums in

(5.16) are converging whenever $u^{12} \in \mathscr{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}$. We now wish to prove that, provided τ is small enough, we have $\|v^{12}\|_{\mathscr{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}} \leq \frac{1}{2}\|u^{12}\|_{\mathscr{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}}$.

Step 2: Study of differences. Let $0 \leq s < t \leq \tau$. We decompose $v_t^{12} - v_s^{12}$ as $L_{st}^1 + L_{st}^2$, with

$$L_{st}^1 = \int_0^s [p_{t-s} - \mathrm{Id}]\, p_{s-v}\left(u_v^{12}\,\mathcal{W}(dv)\right), \quad \text{and} \quad L_{st}^2 = \int_s^t p_{t-v}\left(u_v^{12}\,\mathcal{W}(dv)\right),$$

where the Young integrals with respect to $\mathcal{W}(dv)$ are understood as limit of Riemann sums as in (5.16). We now proceed to the analysis of L_{st}^1 and L_{st}^2.

As in relation (5.16), we write $L_{st}^1 = \lim_{n\to\infty} L_{st}^{1,n}$, where we consider points $s_k^n = 2^{-n}ks$ in the dyadic partition of $[0,s]$ and where we set

$$L_{st}^{1,n} = \sum_{j=0}^{2^n-1} [p_{t-s} - \mathrm{Id}]\, p_{s-s_j^n}\left(u_{s_j^n}^{12}\,\delta\mathcal{W}_{s_j^n s_{j+1}^n}\right). \tag{5.17}$$

In order to estimate L_{st}^1, let us thus first analyze the quantity $L_{st}^{1,n+1} - L_{st}^{1,n}$. Indeed, it is readily checked that $L_{st}^{1,n+1} - L_{st}^{1,n} = \sum_{j=0}^{2^n-1} L_{st}^{1,n,j}$, where $L_{st}^{1,n,j}$ is defined by:

$$L_{st}^{1,n,j} = [p_{t-s} - \mathrm{Id}]\, p_{s-s_{2j+1}^{n+1}}\left(u_{s_{2j+1}^{n+1}}^{12}\,\delta\mathcal{W}_{s_{2j+1}^{n+1} s_{2j+2}^{n+1}}\right) - [p_{t-s} - \mathrm{Id}]\, p_{s-s_{2j}^{n+1}}\left(u_{s_{2j}^{n+1}}^{12}\,\delta\mathcal{W}_{s_{2j+1}^{n+1} s_{2j+2}^{n+1}}\right).$$

We now drop the index $n+1$ in the next computations for sake of readability, and write $L_{st}^{1,n,j} = L_{st}^{11,n,j} - L_{st}^{12,n,j}$ with

$$\begin{aligned} L_{st}^{11,n,j} &= [p_{t-s} - \mathrm{Id}]\, p_{s-s_{2j+1}}\left(\delta u_{s_{2j}s_{2j+1}}^{12}\,\delta\mathcal{W}_{s_{2j+1}s_{2j+2}}\right) := [p_{t-s} - \mathrm{Id}]\,\hat{L}_{st}^{11,n,j} \\ L_{st}^{12,n,j} &= [p_{t-s} - \mathrm{Id}]\left[p_{s_{2j+1}-s_{2j}} - \mathrm{Id}\right] p_{s-s_{2j+1}}\left(u_{s_{2j}}^{12}\,\delta\mathcal{W}_{s_{2j+1}s_{2j+2}}\right) := [p_{t-s} - \mathrm{Id}]\,\hat{L}_{st}^{12,n,j}. \end{aligned}$$

We treat again the two terms $L_{st}^{11,n,j}, L_{st}^{12,n,j}$ separately.

Owing to Proposition 4.42, we have

$$\|L_{st}^{11,n,j}\|_{\mathscr{B}_{w_t}^{\kappa_u}} \leq c\,(t-s)^{\theta_u}\,\|\hat{L}_{st}^{11,n,j}\|_{\mathscr{B}_{w_t}^{\kappa_u+2\theta_u}} \leq \frac{c\,(t-s)^{\theta_u}\,\|\delta u_{s_{2j}s_{2j+1}}^{12}\,\delta\mathcal{W}_{s_{2j+1}s_{2j+2}}\|_{\mathscr{B}_{w_t}^{-\kappa}}}{(s-s_{2j+1})^{\theta_u+\frac{\kappa_u+\kappa}{2}}}$$

Let us now recall the following elementary bound:

$$\varphi_{\alpha,\kappa}(x) := x^{\alpha}\,e^{-\kappa x} \quad \Longrightarrow \quad 0 \leq \varphi_{\alpha,\kappa}(x) \leq \frac{c_\alpha}{\kappa^\alpha}, \quad \text{for} \quad x,\alpha,\kappa \in \mathbb{R}_+. \tag{5.18}$$

This entails $w_t \leq c_\sigma (t-t_{2j+1})^{-\sigma} w_{t_{2j+1}}\rho_\sigma$, and according to (5.11) we obtain

$$\begin{aligned}
\|L_{st}^{11,n,j}\|_{\mathscr{B}_{w_t}^{\kappa_u}} &\leq \frac{c_\sigma\,(t-s)^{\theta_u}\|\delta u_{s_{2j}s_{2j+1}}^{12}\,\delta\mathcal{W}_{s_{2j+1}s_{2j+2}}\|_{\mathscr{B}_{w_{s_{2j+1}}\rho_\sigma}^{-\kappa}}}{(s-s_{2j+1})^{\theta_u+\frac{\kappa_u+\kappa}{2}+\sigma}} \\
&\leq \frac{c_\sigma\,(t-s)^{\theta_u}\|\delta u_{s_{2j}s_{2j+1}}^{12}\|_{\mathscr{B}_{w_{s_{2j+1}}}^{\kappa_u}}\;\|\delta\mathcal{W}_{s_{2j+1}s_{2j+2}}\|_{\mathscr{B}_{\rho_\sigma}^{-\kappa}}}{(s-s_{2j+1})^{\theta_u+\frac{\kappa_u+\kappa}{2}+\sigma}} \\
&\leq \frac{c_\sigma\,(t-s)^{\theta_u}\|u^{12}\|_{\mathscr{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}}\,\|\mathcal{W}\|_{\mathscr{C}_{\rho_\sigma}^{\theta,-\kappa}}}{(s-s_{2j+1})^{\theta_u+\frac{\kappa_u+\kappa}{2}+\sigma}}\,\left(\frac{s}{2^n}\right)^{\theta_u+\theta}.
\end{aligned}$$

As far as $L_{st}^{12,n,j}$ is concerned, we have as above:

$$\|L_{st}^{12,n,j}\|_{\mathscr{B}_{w_t}^{\kappa_u}} \leq c\,(t-s)^{\theta_u}\,\|\hat{L}_{st}^{12,n,j}\|_{\mathscr{B}_{w_t}^{\kappa_u+2\theta_u}}. \tag{5.19}$$

We now take an arbitrarily small and strictly positive constant ε and write:

$$\begin{aligned}
\|\hat{L}_{st}^{12,n,j}\|_{\mathscr{B}_{w_t}^{\kappa_u+2\theta_u}} &\leq (s_{2j+1}-s_{2j})^{1-\theta+\varepsilon}\left\|p_{s-s_{2j+1}}\left(u_{s_{2j}}^{12}\,\delta\mathcal{W}_{s_{2j+1}s_{2j+2}}\right)\right\|_{\mathscr{B}_{w_t}^{\kappa_u+2\theta_u+2(1-\theta+\varepsilon)}} \\
&\leq \frac{(s_{2j+1}-s_{2j})^{1-\theta+\varepsilon}}{(s-s_{2j+1})^{1+\theta_u-\theta+\varepsilon+\frac{\kappa_u+\kappa}{2}}}\,\|u_{s_{2j}}^{12}\,\delta\mathcal{W}_{s_{2j+1}s_{2j+2}}\|_{\mathscr{B}_{w_t}^{-\kappa}},
\end{aligned}$$

and thus relation (5.19) entails:

$$\|L_{st}^{12,n,j}\|_{\mathscr{B}_{w_t}^{\kappa_u}} \leq \frac{c_\sigma\,(t-s)^{\theta_u}\,\|u^{12}\|_{\mathscr{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}}\,\|\mathcal{W}\|_{\mathscr{C}_{\rho\sigma}^{\theta,-\kappa}}}{(s-s_{2j+1})^{1+\theta_u-\theta+\varepsilon+\frac{\kappa_u+\kappa}{2}+\sigma}}\left(\frac{s}{2^n}\right)^{1+\varepsilon}.$$

Putting together the last two estimates on $L_{st}^{11,n,j}$ and $L_{st}^{12,n,j}$ and choosing $\theta_u = 1-\theta+\varepsilon$, we thus end up with:

$$\|L_{st}^{1,n,j}\|_{\mathscr{B}_{w_t}^{\kappa_u}} \leq \frac{c_\sigma\,(t-s)^{\theta_u}\|u^{12}\|_{\mathscr{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}}\,\|\mathcal{W}\|_{\mathscr{C}_{\rho\sigma}^{\theta,-\kappa}}}{(s-s_{2j+1})^{2-2\theta+2\varepsilon+\frac{\kappa_u+\kappa}{2}+\sigma}}\left(\frac{s}{2^n}\right)^{1+\varepsilon}. \tag{5.20}$$

Let us now discuss exponent values: for the convergence of $L_{st}^{1,n}$ we need the condition

$$2-2\theta+2\varepsilon+\frac{\kappa_u+\kappa}{2}+\sigma<1$$

to be fulfilled. If we choose $\kappa_u=\kappa+2\varepsilon$, we can recast this condition into $\theta>\frac{1+\kappa}{2}+\frac{3\varepsilon+\sigma}{2}$. Since ε,σ are chosen to be arbitrarily small, we can satisfy this constraint as soon as $\theta>\frac{1+\kappa}{2}$, which was part of our Hypothesis 4.44. For the remainder of the discussion, we thus assume that

$$2-2\theta+2\varepsilon+\frac{\kappa_u+\kappa}{2}+\sigma=1-\eta, \quad \text{with} \quad \eta>0.$$

Step 3: Bound on L_{st}^1. We express $\lim_{n\to\infty}L_{st}^{1,n}$ as $L_{st}^{1,0}+\sum_{n=0}^{\infty}(L_{st}^{1,n+1}-L_{st}^{1,n})$. Now

$$\sum_{n=0}^{\infty}\|L_{st}^{1,n+1}-L_{st}^{1,n}\|_{\mathscr{B}_{w_t}^{\kappa_u}} \leq \sum_{n=0}^{\infty}\sum_{j=0}^{2^n-1}\|L_{st}^{1,n,j}\|_{\mathscr{B}_{w_t}^{\kappa_u}},$$

and plugging our estimate (5.20), we get that $\sum_{n=0}^{\infty} \|L_{st}^{1,n+1} - L_{st}^{1,n}\|_{\mathscr{B}_{w_t}^{\kappa_u}}$ is bounded by:

$$c_\sigma \|u^{12}\|_{\mathscr{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}} \|\mathcal{W}\|_{\mathscr{C}_{\rho\sigma}^{\theta,-\kappa}} (t-s)^{\theta_u} \sum_{n=0}^{\infty} \left(\frac{s}{2^n}\right)^{\varepsilon} \left(\frac{s}{2^n} \sum_{j=0}^{2^n-1} \frac{1}{(s-s_{2j+1})^{1-\eta}} \right).$$

Furthermore, the following uniform bound holds true:

$$\frac{s}{2^n} \sum_{j=0}^{2^n-1} \frac{1}{(s-s_{2j+1})^{1-\eta}} \le c \int_0^s \frac{dr}{r^{1-\eta}} = c\, s^{\eta},$$

and thus

$$\sum_{n=0}^{\infty} \|L_{st}^{1,n+1} - L_{st}^{1,n}\|_{\mathscr{B}_{w_t}^{\kappa_u}} \le c\, s^{\eta} \,(t-s)^{\theta_u} \sum_{n=0}^{\infty} \left(\frac{s}{2^n}\right)^{\varepsilon} \le c\, s^{\eta+\varepsilon} \,(t-s)^{\theta_u},$$

which ensures the convergence of $L_{st}^{1,n}$. Finally, invoking our definition (5.17) plus the fact that $u_0^{12} = 0$, it is readily checked that $L_{st}^{1,0} = 0$. Thus the relation above transfers into:

$$\begin{aligned} \|L_{st}^{1}\|_{\mathscr{B}_{w_t}^{\kappa_u}} &\le c\, s^{\eta+\varepsilon} \|u^{12}\|_{\mathscr{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}} \|\mathcal{W}\|_{\mathscr{C}_{\rho\sigma}^{\theta,-\kappa}} (t-s)^{\theta_u} \\ &\le c\, \tau^{\eta+\varepsilon} \|u^{12}\|_{\mathscr{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}} \|\mathcal{W}\|_{\mathscr{C}_{\rho\sigma}^{\theta,-\kappa}} (t-s)^{\theta_u}. \end{aligned} \tag{5.21}$$

Step 4: Bound on L_{st}^2. The bound on L_{st}^2 follows along the same lines as for $L_{st}^{1,n}$, and is in fact slightly easier. Let us just mention that we approximate L_{st}^2 by a sequence $L_{st}^{2,n}$ based on the dyadic partition of $[s,t]$, namely $s_j^n = s + j2^{-n}(t-s)$. Like in Step 2, we end up with some terms $L_{st}^{21,n,j}, L_{st}^{22,n,j}$, where

$$L_{st}^{21,n,j} = p_{s-s_{2j+1}}(\delta u_{s_{2j}s_{2j+1}}^{12}\, \delta\mathcal{W}_{s_{2j+1}s_{2j+2}})$$

and

$$L_{st}^{22,n,j} = \left[p_{s_{2j+1}-s_{2j}} - \mathrm{Id}\right] p_{t-s_{2j+1}} \left(u_{s_{2j}}^{12}\, \delta\mathcal{W}_{s_{2j+1}s_{2j+2}}\right).$$

From this decomposition, we leave to the patient reader the task of checking that relation (5.21) also holds true for L_{st}^2.

Step 5: Conclusion. Putting together the last 2 steps, we have been able to prove that for all $0 \le s < t \le \tau$ we have

$$\|v_t^{12} - v_s^{12}\|_{\mathscr{B}_{w_t}^{\kappa_u}} \le c\,\tau^{\eta+\varepsilon}\,\|u^{12}\|_{\mathscr{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}}\,\|\mathcal{W}\|_{\mathscr{C}_{\rho\sigma}^{\theta,-\kappa}}\,(t-s)^{\theta_u}.$$

Thus, choosing $\tau = (c\,\|\mathcal{W}\|_{\mathscr{C}_{\rho\sigma}^{\theta,-\kappa}}/2)^{1/(\varepsilon+\eta)}$, this yields

$$\|v_t^{12} - v_s^{12}\|_{\mathscr{B}_{w_t}^{\kappa_u}} \le \frac{1}{2}\,\|u^{12}\|_{\mathscr{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}}\,(t-s)^{\theta_u},$$

namely the announced contraction property. We have thus obtained existence and uniqueness of the solution to equation (5.15) on $[0,\tau]$. In order to get a global solution on an arbitrary interval, it suffices to observe that all our bounds above do not depend on the initial condition of the solution. One can thus patch solutions on small intervals of constant length τ. The continuity result *(b)* is obtained thanks to the same kind of considerations, and we spare the details to the reader for sake of conciseness.

□

Identification of the Feynman-Kac solution

This section is devoted to the identification of the solution to the stochastic heat equation given by the Feynman-Kac representation formula and the pathwise solution constructed in this section. Calling u^F the Feynman-Kac solution, the global strategy for this identification procedure is the following:

1. Relate the covariance structure (2.1) of the Gaussian noise W to Hypothesis 4.44. We shall see that our Hypothesis 4.23 implies that W satisfies 4.44 almost surely for suitable values of the parameters θ and κ.

2. Prove that u^F coincides with the pathwise solution to (5.15), by means of approximations of the noise W.

We now handle those three problems.

Let us start by establishing the pathwise property of W as a distribution valued function.

Proposition 4.49. *Let W be a centered Gaussian noise defined by μ and γ as in* (2.1)*, satisfying Hypothesis 4.23 for some $0 < \alpha < 1-\beta$. Then the mapping $(t,\varphi) \to W(\mathbf{1}_{[0,t]}\varphi)$ is almost surely Hölder continuous of order θ in time with values in $\mathscr{B}^{-\kappa}_{\rho_\sigma}$ for arbitrarily small σ and for all $\theta, \kappa \in (0,1)$ such that $\theta < 1 - \frac{\beta}{2}$ and $\kappa > 1-\alpha-\beta$. That is, almost surely W satisfies Hypothesis 4.44. Moreover, $\|W\|_{\mathscr{C}^{\theta,-\kappa}_{\rho_\sigma}}$ is a random variable which admits moments of all orders.*

Proof of Proposition 4.49. Fix $\kappa > \kappa' > 1-\alpha-\beta$. For $q \geq 1$, let us denote the Besov space $\mathscr{B}^{-\kappa'}_{2q,2q,\rho_\sigma}$ by $\mathscr{A}_q$, and recall that the norm on $\mathscr{A}_q$ is given by:

$$\|f\|^{2q}_{\mathscr{A}_q} = \sum_{j \geq -1} 2^{-2qj\kappa'} \|\Delta_j f\|^{2q}_{L^{2q}_{\rho_\sigma}}.$$

We will choose q large enough so that $\mathscr{A}_q \hookrightarrow \mathscr{B}^{-\kappa}_{\rho_\sigma}$, a fact which is ensured by Besov embedding theorems. We will show that almost surely:

$$\|\delta W_{st}\|_{\mathscr{A}_q} \leq Z\,(t-s)^{\theta}, \tag{5.22}$$

for any $\theta \in (0, 1-\frac{\beta}{2})$ and the random variable Z admitting moments of all orders. This will complete the proof of the proposition.

To this aim, recall from Section 4.5.2 that $\Delta_j f(x) = [K_j * f](x)$, where $K_j(z) = 2^{jd} K(2^j z)$ and K is the inverse Fourier transform of φ. Otherwise stated, K_j is the inverse Fourier transform of φ_j. With these preliminary considerations in mind, set $K_{j,x}(y) := K_j(x-y)$ and evaluate:

$$\begin{aligned} \mathbf{E}\left[\|\delta W_{st}\|_{\mathscr{A}_q}^{2q}\right] &= \sum_{j\geq -1} 2^{-2qj\kappa'} \int_{\mathbb{R}^d} \mathbf{E}\left[\left|W\left(\mathbf{1}_{[s,t]} \otimes K_{j,x}\right)\right|^{2q}\right] \rho_\sigma^{2q}(x)\, dx \\ &\leq c_q \sum_{j\geq -1} 2^{-2qj\kappa'} \int_{\mathbb{R}^d} \mathbf{E}^q\left[\left|W\left(\mathbf{1}_{[s,t]} \otimes K_{j,x}\right)\right|^{2}\right] \rho_\sigma^{2q}(x)\, dx \quad (5.23) \end{aligned}$$

Moreover, we have

$$\begin{aligned} \mathbf{E}\left[\left|W\left(\mathbf{1}_{[s,t]} \otimes K_{j,x}\right)\right|^{2}\right] &= \int_{[s,t]^2} \left(\int_{\mathbb{R}^d} \left|\mathscr{F} K_{j,x}\right|^2 \mu(d\xi)\right) \gamma(u-v)\, du dv \\ &\leq (t-s)^{2-\beta} \int_{\mathbb{R}^d} \left|\varphi\left(2^{-j}\xi\right)\right|^2 \mu(d\xi). \quad (5.24) \end{aligned}$$

Let us introduce the measure $\nu(d\xi) = \mu(d\xi)/(1+|\xi|^{2(1-\alpha-\beta)})$, which is a finite measure on $\mathbb{R}^d$ according to our standing assumption. Also recall from Notation 4.36 that $\text{Supp}(\varphi) \subset \{x \in \mathbb{R}^d : a \leqslant |x| \leqslant b\}$. Hence

$$\int_{\mathbb{R}^d} \left|\varphi\left(2^{-j}\xi\right)\right|^2 \mu(d\xi) \leq \int_{\mathbb{R}^d} \mathbf{1}_{[0,2^j b]}(|\xi|) \left[1+|\xi|^{2(1-\alpha-\beta)}\right] \nu(d\xi) \leq c_\mu\, 2^{2(1-\alpha-\beta)j}.$$

Plugging this identity into (5.24) and then (5.23) we end up with the relation $\mathbf{E}[\|\delta W_{st}\|_{\mathscr{A}_q}^{2q}] \leq c_q (t-s)^{(2-\beta)q}$, valid for all $0 \leq s < t \leq T$ and any $q \geq 1$. A standard application of Garsia's and Fernique's lemma then yields relation (5.22), and thus Hypothesis 4.44 . □

Remark 4.50. *In particular, equation* (5.14) *driven by* W *admits a unique pathwise solution in* $\mathscr{D}^{\theta_u,\kappa_u}_{\lambda,\sigma}$*, as in Theorem 4.48, for some* $\theta_u > \frac{\beta}{2}$ *and* $\kappa_u > 1-\alpha-\beta$*. Notice here that one obtains (see Theorem 4.30) the existence of a solution to our equation in the Stratonovich sense under Hypothesis 4.16 only. We call this assumption the critical case. In order to get existence and uniqueness of a pathwise solution we have to impose the more restrictive Hypothesis 4.23 with an arbitrarily small constant* α*, which can be seen as a supercritical situation. This is the price to pay in order to get uniqueness of the solution.*

We now turn to the second point of our strategy, namely prove that the Feynman-Kac solution u^F coincides with the unique pathwise solution to equation 5.14 driven by W.

Proposition 4.51. *Let* u^F *be the random field given by equation (4.1). Assume that* W *satisfies Hypothesis 4.23. Then there exist* $\theta_u > \frac{\beta}{2}$ *and* $\kappa_u > 1-\alpha-\beta$ *such that almost surely* u^F *belongs to the space* $\mathscr{D}^{\theta_u,\kappa_u}_{\lambda,\sigma}$*. Moreover,* u^F *is the pathwise solution to equation (5.14) driven by* W*.*

Proof. To show that u^F is the pathwise solution to equation 5.14, we use the fact that $u^F_{t,x}$ is the limit in $L^p(\Omega)$ of the approximating sequence $u^{\varepsilon,\delta}_{t,x}$ introduced in (4.15) (see (4.16)) as ε and δ tend to zero, for any $p \geq 1$. On the other hand, it is clear that $u^{\varepsilon,\delta}$ is the pathwise solution to equation (5.14) driven by the trajectories of $W^{\varepsilon,\delta}$

$$u^{\varepsilon,\delta}_t = p_t u_0 + \int_0^t p_{t-s}\left(u^{\varepsilon,\delta}_s W^{\varepsilon,\delta}(ds)\right).$$

Then, it suffices to take the limit in the above equation to show that u^F is a pathwise solution to equation (5.14) driven by W. In fact, that for two particular sequences $\varepsilon_n \downarrow 0$ and $\delta_n \downarrow 0$ $W^{\varepsilon_n,\delta_n}$ converges to W almost surely in the space $\mathscr{C}^{\theta,-\kappa}_{T,\rho_\sigma}$. This implies (see

Theorem 4.48 item (b)) that $u^{\varepsilon_n,\delta_n}$ converges almost surely to a process u in $\mathscr{D}_{\lambda,\sigma}^{\theta_u,\kappa_u}$, which is the pathwise solution to equation 5.14 driven by W. Therefore, $u = u^F$ and this concludes the proof. □

Time independent case

The case of a time independent noise is obviously easier to handle than the time dependent one. Basically, the Young integration arguments invoked above can be skipped, and they are replaced by Gronwall type lemmas for Lebesgue integration. We won't detail the proofs here, and just mention the main steps for sake of conciseness.

First, the pathwise type assumption we make on the noise W, considered as a distribution on $\mathbb{R}^d$, is the following counterpart of Hypothesis 4.44:

Hypothesis 4.52. *Suppose that $\mathcal{W}$ is a distribution on $\mathbb{R}^d$ such that $\mathcal{W} \in \mathscr{B}_{\rho_\sigma}^{-\kappa}$ with $\kappa \in (0,1)$ and an arbitrarily small constant $\sigma > 0$.*

Another simplification of the time independent case is that one can solve the equation in a space of continuous functions in time (compared to the Hölder regularity we had to consider before), with values in weighted Besov spaces. We thus define the following sets of functions

$$\mathscr{C}_{\lambda,\sigma}^{\kappa_u} = \left\{ f \in \mathscr{C}([0,T] \times \mathbb{R}^d); \|f_t\|_{\mathscr{B}_{w_t}^{\kappa_u}} \leq c_f \right\}, \quad \text{where} \quad w_t := e_{\lambda+\sigma t}.$$

With these conventions in hand, we interpret equation (5.2) as a mild equation in the spaces $\mathscr{C}_{\lambda,\sigma}^{\kappa_u}$.

Definition 4.53. *Let $u \in \mathscr{C}^{\kappa_u}_{\lambda,\sigma}$ for $\lambda, \sigma > 0$ and $\kappa_u \in (\kappa, 1)$. Consider an initial condition $u_0 \in \mathscr{B}^{\kappa_u}_{e_\lambda}$. We say that u is a mild solution to equation*

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u\mathcal{W} \tag{5.25}$$

with initial condition u_0 if it satisfies the following integral equation

$$u_t = p_t u_0 + \int_0^t p_{t-s}\left(u_s \mathcal{W}\right) ds, \tag{5.26}$$

where the product $u\mathcal{W}$ is interpreted in the distributional sense.

We can now turn to the resolution of the equation in this context, and the main theorem in this direction is the following.

Theorem 4.54. *Let $\mathcal{W}$ be a distribution satisfying Hypothesis 4.52 and let λ be a strictly positive constant. Then equation* (5.26) *admits a unique solution in $\mathscr{C}^{\kappa_u}_{\lambda,\sigma}$, in the sense given by Definition 4.53, with $\kappa < \kappa_u < 1$.*

Proof. As in the proof of Theorem 4.48, we focus on the proof of uniqueness, and fix a small time interval $[0,\tau]$. Consider u^1, u^2 two solutions in $\mathscr{C}^{\kappa_u}_{\lambda,\sigma}$ and we set $u^{12} = u^1 - u^2$. Consistently with Definition 4.53, the equation for u^{12} is given by:

$$u^{12}_t = \int_0^t p_{t-s}\left(u^{12}_s \mathcal{W}\right) ds, \tag{5.27}$$

and we wish to prove that $u^{12} \equiv 0$.

Towards this aim, let us bound the Besov norm of u starting from equation (5.27). Owing to Proposition 4.42, we get

$$\|u^{12}_t\|_{\mathscr{B}^{\kappa_u}_{w_t}} \le \int_0^t \|p_{t-s}\left(u^{12}_s \mathcal{W}\right)\|_{\mathscr{B}^{\kappa_u}_{w_t}} ds \le c_{\tau,\lambda,\sigma} \int_0^t (t-s)^{-\frac{(\kappa_u+\kappa)}{2}} \|u^{12}_s \mathcal{W}\|_{\mathscr{B}^{-\kappa}_{w_t}} ds.$$

Along the same lines as in the proof of Theorem 4.48, we now invoke the bound (5.18), which yields $w_t \leq c_{\tau,\lambda,\sigma}(t-s)^{-\sigma} w_s \rho_\sigma$. Hence, according to Proposition 4.40 item (iii), we have

$$\|u_t^{12}\|_{\mathscr{B}_{w_t}^{\kappa_u}} \leq c_{\lambda,\sigma} \int_0^t (t-s)^{-\frac{(\kappa_u+\kappa)}{2}-\sigma} \|u_s^{12}\mathcal{W}\|_{\mathscr{B}_{w_s\rho_\sigma}^{-\kappa}}\, ds.$$

Since $\kappa_u > \kappa$, we now apply relation (5.11) with $w_1 = w_s$, $\kappa_1 = \kappa_u$, $w_2 = \rho_\sigma$ and $\kappa_2 = \kappa$. We end up with

$$\|u_t^{12}\|_{\mathscr{B}_{w_t}^{\kappa_u}} \leq c_{\tau,\lambda,\sigma} \|\mathcal{W}\|_{\mathscr{B}_{\rho_\sigma}^{-\kappa}} \int_0^t \frac{\|u_s^{12}\|_{\mathscr{B}_{w_s}^{\kappa_u}}}{(t-s)^{\frac{(\kappa_u+\kappa)}{2}+\sigma}}\, ds\,.$$

Taking into account that $\kappa_u + \kappa < 2$ and σ can be arbitrarily small, our conclusion $u^{12} \equiv 0$ follows easily from a Gronwall type argument. □

We now state a result which allows to identify the Feynman-Kac and the pathwise solution to our spatial equation. Its proof is omitted for sake of conciseness, since it is easier than in the time dependent case.

Proposition 4.55. *Let W be a spatial Gaussian noise defined by the covariance structure* (2.5) *and* (2.6). *Assume that the measure μ satisfies the condition*

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^{2(1-\alpha)}} < \infty, \tag{5.28}$$

for a constant $\alpha \in (0,1)$. Then:

(i) *There exists $\kappa \in (0,1)$ such that for any arbitrarily $\sigma > 0$, W has a version in $\mathscr{B}_{\rho_\sigma}^{-\kappa}$ and the random variable $\|W\|_{\mathscr{B}_{\rho_\sigma}^{-\kappa}}$ has moments of all orders, that is the trajectories of W satisfy Hypothesis 4.52. As a consequence, equation* (5.26) *driven by the trajectories of W admits a unique pathwise solution in $\mathscr{C}_{\lambda,\sigma}^{\kappa_u}$.*

(ii) *Let* u^F *be the Feynman-Kac solution to the heat equation given by* (4.20). *Then almost surely the process* u^F *lies into* $\mathscr{C}^{\kappa_u}_{\lambda,\sigma}$*, and it coincides with the unique pathwise solution to equation* (5.26).

Remark 4.56. *Here again, we see that the Feynman-Kac solution* u^F *exists under the critical condition* $\int_{\mathbb{R}^d}(1+|\xi|^2)^{-1}\mu(d\xi)<\infty$*, while the pathwise solution requires the more stringent condition* (5.28).

4.6 Moment estimates

As mentioned in the introduction, intermittency properties for u are characterized by the family of Lyapounov type coefficients $\ell(k)$ defined by (1.3) or by the limiting behavior (1.4). In any case, the intermittency phenomenon stems from an asymptotic study of the moments of u, for large values of k and t. We propose to lead this study in the context of the general Gaussian noises considered in the current paper.

Notice that delicate results such as limiting behaviors for moments will rely on more specific conditions on the noise W. We are thus going to make use of the following conditions.

Hypothesis 4.57. *There exist constants* c_0, C_0 *and* $0<\beta<1$*, such that*

$$c_0|x|^{-\beta} \leq \gamma(x) \leq C_0|x|^{-\beta}.$$

Hypothesis 4.58. *There exist constants* c_1, C_1 *and* $0<\eta<2$*, such that*

$$c_1|x|^{-\eta} \leq \Lambda(x) \leq C_1|x|^{-\eta}.$$

Hypothesis 4.59. *There exist constants c_2, C_2 and $0 < \eta_i < 1$, with $\sum_{i=1}^d \eta_i < 2$, such that*

$$c_2 \prod_{i=1}^d |x_i|^{-\eta_i} \le \Lambda(x) \le C_2 \prod_{i=1}^d |x_i|^{-\eta_i}.$$

Clearly, Hypothesis 4.57 and Hypothesis 4.58 generalize the case of Riesz kernels and Hypothesis 4.59 generalizes the case of fractional noises. Notice that under Hypotheses 4.58 or 4.59 the spectral measure μ satisfies the integrability condition (2.4).

Theorem 4.60. *Suppose that γ satisfies Hypothesis 4.57 and Λ satisfies Hypothesis 4.58 or Hypothesis 4.59. Denote*

$$a = \begin{cases} \eta & \text{if Hypothesis 4.58 holds} \\ \sum_{i=1}^d \eta_i & \text{if Hypothesis 4.59 holds.} \end{cases}$$

Consider the following two cases:

(i) u is the solution to the Skorohod equation (3.1) driven by a time dependent noise with time covariance γ and space covariance Λ.

(ii) u is the solution to the Stratonovich equation (5.1) driven by a time dependent noise with time covariance γ and space covariance Λ, and we assume that $a < 2 - 2\beta$.

Then in both of these two cases we have

$$\exp\left(Ct^{\frac{4-2\beta-a}{2-a}} k^{\frac{4-a}{2-a}}\right) \le \mathbf{E}\left[u_{t,x}^k\right] \le \exp\left(C' t^{\frac{4-2\beta-a}{2-a}} k^{\frac{4-a}{2-a}}\right) \tag{6.1}$$

for all $t \ge 0, x \in \mathbb{R}^d, k \ge 2$, where C, C' are constants independent of t and k.

Proof. Let us first discuss the upper bound. For the Skorohod equation, using the chaos expansion and the hypercontractivity property we can derive the upper bound as it has been done in [4]. For the Stratonovich equation, notice first that Hypothesis 4.23 holds because $a<2-2\beta$. Using the Feynmann-Kac formula (4.1) for the solution to equation (5.1), and applying Cauchy-Schwartz inequality yields

$$\begin{aligned}\mathbf{E}\left[u_{t,x}^k\right] &= \mathbf{E}_B\left[\exp\left(\sum_{1\le i,j\le k}\int_0^t\int_0^t \gamma(r-s)\Lambda(B_r^i-B_s^j)drds\right)\right]\\ &\le \left[\mathbf{E}_B\left[\exp\left(2\sum_{1\le i<j\le k}\int_0^t\int_0^t \gamma(r-s)\Lambda(B_r^i-B_s^j)drds\right)\right]\right]^{\frac12}\\ &\quad\times\left[\mathbf{E}_B\left[\exp\left(2\sum_{i=1}^k\int_0^t\int_0^t \gamma(r-s)\Lambda(B_r^i-B_s^i)drds\right)\right]\right]^{\frac12}.\end{aligned}$$

In the above expression, the first term is just the square root of the Feynman-Kac formula (3.21) for the moment of order k of the solution of a Skorohod equation with multiplicative noise, with covariances 2γ and 2Λ. For this term we know that we can derive the upper bound (6.1) using the chaos expansion and the hypercontractivity property as it has been done in [4]. For the second factor, using the asymptotic result proved in Proposition 2.1 in [18], we derive the estimate

$$\mathbf{E}^{\frac{k}{2}}\left[\exp\left(2\int_0^t\int_0^t \gamma(r-s)\Lambda(B_r^1-B_s^1)drds\right)\right]\le C^k\exp\left(Ct^{\frac{4-2\beta-a}{2-a}}k\right).$$

Therefore, in this way we can obtain the desired upper bound of $\mathbf{E}\left[u_{t,x}^k\right]$.

Let us now discuss the lower bound. Taking into account again the Feynman-Kac formula (3.21) for the moments of u, it suffices to consider the case of the Skorohod equation (it is readily checked from (3.21) that the moments of u for the Stratonovich equation are greater than those of the Skorohod equation). The argument of the proof is then based in the small ball probability estimates for Brownian motion. We consider only the case when Λ satisfies the lower bound given in hypothesis Hypothesis 4.58

(Riesz kernel case), since the case Hypothesis 4.59 (fractional noise) is analogous. In this case, owing to formula (3.21) and the scaling property of the Brownian motion, it is easy to see that

$$\mathbf{E}\left[u_{t,x}^k\right] \geq \mathbf{E}\left[\exp\left(c_0 c_1 t^{2-\beta-\frac{\eta}{2}} \sum_{1\leq i<j\leq k} \int_0^1\int_0^1 |s-r|^{-\beta}|B_s^i - B_r^j|^{-\eta} ds dr\right)\right].$$

Denote $B_s^{i,l}, l=1,2,\cdots,d$ the l-th component of the d-dimensional Brownian motion B_s^i. Consider the set

$$A_\varepsilon = \left\{ \sup_{1\leq i<j\leq k} \sup_{1\leq l\leq d} \sup_{0\leq s,r\leq 1} |B_s^{i,l} - B_r^{j,l}| \leq \varepsilon \right\}.$$

Restricting the above expectation to this event and recalling that the value of a generic constant c might change from line to line, we obtain:

$$\begin{aligned} \mathbf{E}\left[u_{t,x}\right]^k &\geq \mathbf{E}\left[\exp\left(c t^{2-\beta-\frac{\eta}{2}} \sum_{1\leq i<j\leq k} \int_0^1\int_0^1 |s-r|^{-\beta}|B_s^i - B_r^j|^{-\eta} ds dr\right) \mathbf{1}_{A_\varepsilon}\right] \qquad (6.2) \\ &\geq \exp\left(\frac{c k(k-1)}{(2-\beta)(1-\beta)} t^{2-\beta-\frac{\eta}{2}} \varepsilon^{-\eta}\right) \mathbf{P}\big(A_\varepsilon\big) \geq \exp\left(c t^{2-\beta-\frac{a}{2}} k^2 \varepsilon^{-\eta}\right) \mathbf{P}\big(A_\varepsilon\big). \end{aligned}$$

Moreover, notice that

$$\cap_{i=1}^k \cap_{l=1}^d F_{i,l} \subset A_\varepsilon, \quad \text{with} \quad F_{i,l} = \left(\sup_{0\leq s\leq 1} |B_s^{i,l}| \leq \frac{\varepsilon}{2}\right).$$

The events $F_{i,l}$ being i.id, we get:

$$\mathbf{P}\big(A_\varepsilon\big) \geq \mathbf{P}^{kd}\left(F_\varepsilon\right), \quad \text{with} \quad F_\varepsilon = \left(\sup_{0\leq s\leq 1} |b_s| \leq \frac{\varepsilon}{2}\right),$$

where b stands for a one dimensional standard Brownian motion. In addition, it is a well known fact (see e.g (1.3) in [66]) that $\lim_{\varepsilon\to 0}\mathbf{P}(F_\varepsilon)/\exp(-\frac{\pi^2}{2\varepsilon^2})=1$. Hence, there exists an $\varepsilon_0>0$ such that for $\varepsilon\le\varepsilon_0$, we have $\mathbf{P}(F_\varepsilon)\ge\exp(-C\varepsilon^{-2})$, for some constant $C>0$. Under the condition $\varepsilon\le\varepsilon_0$, this entails:

$$\mathbf{E}\left[u_{t,x}\right]^k \ge \exp\left(c\,t^{2-\beta-\frac{\eta}{2}}k^2\varepsilon^{-\eta}-\frac{Cdk}{\varepsilon^2}\right).$$

In order to optimize this expression, we try to equate the two terms inside the exponential above. To this aim, we set

$$\varepsilon=\frac{t^{\frac{2-\beta-\frac{\eta}{2}}{\eta-2}}(ck)^{\frac{1}{\eta-2}}}{(2dC)^{\frac{1}{\eta-2}}},$$

and notice that for $k\ge 2$ and t sufficiently large, the condition $\varepsilon\le\varepsilon_0$ is fulfilled. Therefore, we conclude that for t and k large enough

$$\mathbf{E}\left[u_{t,x}^k\right]\ge\exp\left(\frac{c^{\frac{\eta}{2-\eta}}\,t^{\frac{4-2\beta-\eta}{2-\eta}}\,k^{\frac{4-\eta}{2-\eta}}}{8(2dC)^{\frac{a}{2-a}}}\right), \tag{6.3}$$

which finishes the proof of (6.1). □

We now give two extensions of the theorem above. The first one concerns the moment estimates in the time independent case. Its proof is very similar to the proof of Theorem 4.60, and is thus omitted for sake of conciseness.

Theorem 4.61. *Suppose that Λ satisfies Hypothesis 4.58 or Hypothesis 4.59. Set $a=\eta$ if Hypothesis 4.58 holds, and $a=\sum_{i=1}^d\eta_i$ if Hypothesis 4.59 holds. Suppose that u is the solution to the Skorohod equation (3.32) or the Stratonovich equation (5.2) driven by a multiplicative time independent noise with covariance Λ. Then, for any $x\in\mathbb{R}^d$,*

$k \geq 2$, we have

$$\exp\left(Ct^{\frac{4-a}{2-a}}k^{\frac{4-a}{2-a}}\right) \leq \mathbf{E}\left[u_{t,x}^k\right] \leq \exp\left(C't^{\frac{4-a}{2-a}}k^{\frac{4-a}{2-a}}\right), \tag{6.4}$$

where $C, C' > 0$ are constants independent of t and k.

Finally, when $d = 1$ we can also obtain moment estimates in the case where the space covariance is a Dirac delta function, that is, the noise is white in space.

Theorem 4.62. *Suppose that γ satisfies condition Hypothesis 4.57 and the spatial dimension is 1. Consider two cases:*

(i) Suppose that u satisfies either the Skorohod equation (3.1) or the Stratonovich equation (5.1) driven by a multiplicative noise with time covariance γ and spatial covariance $\Lambda(x) = \delta_0(x)$. Then, for any $x \in \mathbb{R}^d$, $k \geq 2$ and $t > 0$, we have

$$\exp\left(Ct^{3-2\beta}k^3\right) \leq \mathbf{E}\left[u_{t,x}^k\right] \leq \exp\left(C't^{3-2\beta}k^3\right), \tag{6.5}$$

where $C, C' > 0$ are constants independent of t and k.

(ii) Suppose that u satisfies either the Skorohod equation (3.32) or the Stratonovich equation (5.2) driven by a time independent multiplicative noise with spatial covariance $\Lambda(x) = \delta_0(x)$. Then, for any $x \in \mathbb{R}^d$, $k \geq 2$ and $t > 0$, we have

$$\exp\left(Ct^3k^3\right) \leq \mathbf{E}\left[u_{t,x}^k\right] \leq \exp\left(C't^3k^3\right), \tag{6.6}$$

where $C, C' > 0$ are constants independent of t and k.

Proof. In the Skorohod case with time dependent noise, the moments of $u_{t,x}$ are given by equation (3.31). We will only discuss the lower bound because the upper bound can be obtained by using chaos expansions as in [4]. We consider the approximation of the Dirac delta function by the heat kernel p_ε, and define

$$I_{t,k,\varepsilon} = \mathbf{E}_B \left[\exp\left(\sum_{1\le i<j\le k} \int_0^t \int_0^t \gamma(s-r) p_\varepsilon(B_s^i - B_r^j) ds dr \right) \right]. \tag{6.7}$$

Expanding the exponential and using Fourier analysis as in [55], one can show that $\mathbf{E}\left[u_{t,x}^k\right] \ge I_{t,k,\varepsilon}$, for any $\varepsilon > 0$. For any positive ε, denote

$$A_{k,\varepsilon,t} = \left\{ \max_{1\le i\le k} \sup_{0\le s\le t} |B_s^i| \le \sqrt{\varepsilon} \right\}.$$

On the event $A_{k,\varepsilon,t}$ we have $p_\varepsilon(B_s^i - B_r^j) \ge \frac{C}{\sqrt{\varepsilon}}$ for some positive constant C. Therefore, using the lower bound in Hypothesis 4.57, we can write similarly to (6.2):

$$I_{t,k,\varepsilon} \ \ \ge \ \ \exp\left(c k^2 \int_0^t \int_0^t |s-r|^{-\beta} \frac{C}{\sqrt{\varepsilon}} ds dr \right) \mathbf{P}\left(A_{k,\varepsilon,t}\right).$$

Furthermore, by the scaling property of Brownian motion, $\mathbf{P}\left(A_{k,\varepsilon,t}\right)$ can be written as:

$$\mathbf{P}\left(A_{k,\varepsilon,t}\right) = \mathbf{P}\left(\max_{1\le i\le k} \sup_{0\le s\le 1} |B_s^i| \le \sqrt{\varepsilon/t} \right) = \left(\mathbf{P}\left(\max_{0\le s\le 1} |b_s| \le \sqrt{\varepsilon/t} \right) \right)^k,$$

where b stands for a one-dimensional standard Brownian motion. We now invoke again (1.3) in [66], which yields $\lim_{\varepsilon\to 0} \mathbf{P}(\sup_{0\le s\le 1} |B_s| \le \sqrt{\frac{\varepsilon}{t}}) / \exp(-\frac{\pi^2}{8}\frac{t}{\varepsilon}) = 1$. Thus, when ε is sufficiently small,

$$\mathbf{P}\left(\sup_{0\le s\le 1} |B_s| \le \sqrt{\frac{\varepsilon}{t}} \right) \ge \exp\left(-C\frac{t}{\varepsilon} \right),$$

for some positive constant C which does not depend on t. Hence, we end up with the following lower bound:

$$I_{t,k,\varepsilon} \geq \exp\left(C_1 k^2 t^{2-\beta} \frac{1}{\sqrt{\varepsilon}} - C_2 \frac{t}{\varepsilon}\right).$$

As in the proof of Theorem 4.60, we optimize this expression by choosing $\varepsilon = \frac{4C_2^2}{C_1^2 k^3 t^{2-2\beta}}$, and we obtain that

$$I_{t,k,\varepsilon} \geq \exp(C_3 t^{3-2\beta} k^3) \tag{6.8}$$

when t is sufficiently large, where the positive constant C_3 does not depend on t or k.

For the Stratonovich case, the lower bound is obvious and for the upper bound we use the Cauchy-Schwartz inequality and Lemma 2.2 in [18]. The estimate (6.6) is proved similarly, which completes the proof. □

Remark 4.63. *As a consequence of Theorems 6.4, 6.5 and 6.6, the solution u of both the Skorohod and Stratonivich equations is intermittent in the sense of condition (1.4).*

Chapter 5

Smoothness of the joint density for spatially homogeneous SPDEs

In this chapter we consider a general class of second order stochastic partial differential equations on $\mathbb{R}^d$ driven by a Gaussian noise which is white in time and has a homogeneous spatial covariance. Using the techniques of Malliavin calculus we derive the smoothness of the density of the solution at a fixed number of points $(t,x_1),\ldots,(t,x_n)$, $t>0$, with some suitable regularity and non degeneracy assumptions. We also prove that the density is strictly positive in the interior of the support of the law.

5.1 Introduction

Consider the stochastic partial differential equation

$$Lu(t,x) = b(u(t,x)) + \sigma(u(t,x))\dot{W}(t,x), \tag{1.1}$$

$t \geq 0$, $x \in \mathbb{R}^d$, with vanishing initial conditions, where L denotes a second order partial differential operator. The coefficients b and σ are real-valued functions and the noise $\dot{W}(t,x)$ is a Gaussian field which is white in time and has a spatially homogeneous covariance in the space variable. A mild solution to this equation can be formulated using the Green kernel $\Gamma(t,dx)$ associated with the operator L (see Definition 5.1). This requires the notion of stochastic integral introduced by Walsh in [91] if $\Gamma(t,x)$ is a real-valued function or Dalang's extension of Walsh integral (see [24]) when Γ is a measure.

In [75], Nualart and Quer-Sardanyons have studied the existence and smoothness of the density of the solution $u(t,x)$ at a fixed point $(t,x)\in(0,\infty)\times\mathbb{R}^d$ using techniques of Malliavin calculus. The smoothness of the density follows from the fact that the norm of the Malliavin derivative of $u(t,x)$ has inverse moments of all orders, assuming some suitable non degeneracy and regularity conditions. The basic assumptions there are that b and σ are smooth with bounded partial derivatives of all orders, $|\sigma(z)|\geq c>0$ for all z (In this chapter, we shall assume a weaker condition, see Theorem 5.3) and

$$C\varepsilon^{\eta}\leq\int_0^{\varepsilon}\int_{\mathbb{R}^d}|\mathscr{F}\Gamma(r)(\xi)|^2\mu(d\xi)dr<\infty \tag{1.2}$$

for some $\eta>0$ and all ε small enough, where μ is the spectral measure of the noise and $\mathscr{F}$ denotes the Fourier transform. This general result extends previous work of Quer-Sardanyons and Sanz-Solé [80] for the case when L corresponds to the three dimensional wave equation.

The purpose of this chapter is to establish the smoothness of the joint density of the solution to equation (1.1) at a fixed number of points $(t,x_1),\dots,(t,x_n)$, where $t>0$ and $x_i\in\mathbb{R}^d$. This kind of problem was studied by Bally and Pardoux in [11] for the one-dimensional stochastic heat equation driven by a space-time white noise. The extension of this result to equation (1.1) presents new difficulties and requires additional non degeneracy conditions, in addition to (1.2), because we need to handle the determinant of the Malliavin matrix of the random vector $u(t,x_1),\dots,u(t,x_n)$. The basic ingredient is to impose that leading terms as $\to 0$ in the matrix

$$\left(\int_0^\varepsilon \int_{\mathbb{R}^d} \langle \Gamma(r, * + x_j), \Gamma(r, * + x_i)\rangle_{\mathscr{H}} dr\right)_{1\le i,j\le n}$$

are the diagonal ones given by (1.2) (see the hypotheses (**H3**) and (**H4**) below). These hypotheses are related, although different, to the ones imposed by Nualart in [72] to establish the smoothness of the density for the solution of a system of SPDEs.

The chapter is organized as follows. After some preliminaries, Section 3 is devoted to the proof of the smoothness of the density of the vector $u(t,x_1),\dots,u(t,x_n)$. In Section 4 we derive the positivity of the density in the interior of the support following the general criterion established by Nualart in [74]. Finally, in Section 5, we apply these results to the basic examples of the stochastic heat and wave equations and to the spatial covariances given by the Riesz, Bessel and fractional kernels.

5.2 Preliminaries

The noise we are considering in this chapter is almost of the same type as in Chapter 3. We will still describe the noise in detail, for the sake of completeness. Consider a non-negative and non-negative definite function f which is continuous on $\mathbb{R}^d \setminus \{0\}$. We assume that f is the Fourier transform of a non-negative tempered measure μ on $\mathbb{R}^d$ (called the spectral measure of f). That is, for all φ belonging to the space $\mathscr{S}(\mathbb{R}^d)$ of rapidly decreasing $\mathscr{C}^\infty$ functions on $\mathbb{R}^d$

$$\int_{\mathbb{R}^d} f(x)\varphi(x)dx = \int_{\mathbb{R}^d} \mathscr{F}\varphi(\xi)\mu(d\xi), \tag{2.1}$$

and there is an integer $m \geq 1$ such that

$$\int_{\mathbb{R}^d} (1+|\xi|^2)^{-m} \mu(d\xi) < \infty.$$

Here we have denoted by $\mathscr{F}\varphi$ the Fourier transform of $\varphi \in \mathscr{S}(\mathbb{R}^d)$, given by $\mathscr{F}\varphi(\xi) = \int_{\mathbb{R}^d} \varphi(x) e^{-i\xi\cdot x} dx$.

Let $\mathscr{C}_0^\infty([0,\infty)\times\mathbb{R}^d)$ be the space of smooth functions with compact support on $[0,\infty)\times\mathbb{R}^d$. Consider a family of zero mean Gaussian random variables $W = \{W(\varphi), \varphi \in \mathscr{C}_0^\infty([0,\infty)\times\mathbb{R}^d)\}$, defined on a complete probability space $(\Omega, \mathscr{F}, \mathbf{P})$, with covariance

$$\mathbf{E}(W(\varphi)W(\psi)) = \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(t,x) f(x-y) \psi(t,y) dx dy dt\,. \tag{2.2}$$

The covariance (1.2) can also be written, using Fourier transform, as

$$\mathbf{E}(W(\varphi)W(\psi)) = \int_0^\infty \int_{\mathbb{R}^d} \mathscr{F}\varphi(t)(\xi)\overline{\mathscr{F}\psi(t)(\xi)} \mu(d\xi) dt\,.$$

The main assumptions on the differential operator L in (1.1) can be stated as follows:

(**H1**) The fundamental solution to $Lu = 0$, denoted by Γ, satisfies that for all $t > 0$, $\Gamma(t)$ is a nonnegative measure with rapid decrease, such that for all $T > 0$

$$\int_0^T \int_{\mathbb{R}^d} |\mathscr{F}\Gamma(t)(\xi)|^2 \mu(d\xi) dt < \infty,$$

and

$$\sup_{t\in[0,T]} \Gamma(t, \mathbb{R}^d) \leq C_T < \infty.$$

The basic examples we are interested in are the stochastic heat and wave equations. More precisely, it is well-known that if L is the heat operator in $\mathbb{R}^d$, that is, $L = \frac{\partial}{\partial t} - \frac{1}{2}\Delta$, where Δ denotes the Laplacian operator in $\mathbb{R}^d$, or if L is the wave operator in $\mathbb{R}^d$,

$d \in \{1,2,3\}$, i.e., $L = \frac{\partial^2}{\partial t^2} - \Delta$, hypothesis (**H1**) is satisfied if and only if

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty.$$

Let $\mathscr{H}$ be the Hilbert space obtained by the completion of $\mathscr{C}_0^\infty(\mathbb{R}^d)$ endowed with the inner product

$$\langle \varphi, \psi \rangle_{\mathscr{H}} = \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \varphi(x) f(x-y) \psi(y) = \int_{\mathbb{R}^d} \mathscr{F}\varphi(\xi) \overline{\mathscr{F}\psi(\xi)} \mu(d\xi), \qquad (2.3)$$

$\varphi, \psi \in \mathscr{C}_0^\infty(\mathbb{R}^d)$. Notice that $\mathscr{H}$ may contain distributions. Set $\mathscr{H}_0 = L^2([0,\infty); \mathscr{H})$.

Walsh's classical theory of stochastic integration developed in [91] cannot be applied directly to the mild formulation of equation (1.1) since Γ may not be absolutely continuous with respect to the Lebesgue measure. We shall use the stochastic integral defined in [31, Section 2.3] (see also [75, Section 3]). We briefly review the construction and properties of this integral.

The Gaussian family W can be extended to the space $\mathscr{H}_0$ and we denote by $W(g)$ the Gaussian random variable associated with an element $g \in \mathscr{H}_0$. It is obvious that $\mathbf{1}_{[0,t]}h$ is in $\mathscr{H}_0$ and we set $W_t(h) = W(\mathbf{1}_{[0,t]}h)$ for any $t \geq 0$ and $h \in \mathscr{H}$. Then $W = \{W_t, t \geq 0\}$ is a cylindrical Wiener process in the Hilbert space $\mathscr{H}$. That is, for any $h \in \mathscr{H}$, $\{W_t(h), t \geq 0\}$ is a Brownian motion with variance $t\|h\|_{\mathscr{H}}^2$, and

$$\mathbf{E}(W_t(h)W_s(g)) = (s \wedge t)\langle h, g \rangle_{\mathscr{H}}.$$

Let $\mathscr{F}_t$ be the σ-field generated by the random variables $\{W_s(h), h \in \mathscr{H}, 0 \leq s \leq t\}$ and the $\mathbf{P}$-null sets. We define the predictable σ-field as the σ-field in $\Omega \times [0,\infty)$ generated by the sets $\{A \times (s,t], 0 \leq s < t, A \in \mathscr{F}_s\}$. Then we can define the stochastic integral of an $\mathscr{H}$-valued square-integrable predictable process $g \in L^2(\Omega \times [0,\infty); \mathscr{H})$ with respect to the cylindrical Wiener process W, denoted by

$$g \cdot W = \int_0^\infty \int_{\mathbb{R}^d} g(t,x) W(dt,dx),$$

and we have the isometry property

$$\mathbf{E}|g \cdot W|^2 = \mathbf{E} \int_0^\infty \|g(t)\|_{\mathscr{H}}^2 dt. \tag{2.4}$$

Using the above notion of stochastic integral one can introduce the following definition:

Definition 5.1. *A real-valued predictable stochastic process $u = \{u(t,x), t \geq 0, x \in \mathbb{R}^d\}$ is a mild solution of equation (1.1) if for all $t \geq 0$, $x \in \mathbb{R}^d$,*

$$\begin{aligned} u(t,x) &= \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y) \sigma(u(s,y)) W(ds,dy) \\ &\quad + \int_0^t \int_{\mathbb{R}^d} b(u(s,x-y)) \Gamma(t-s,dy) ds \quad a.s. \end{aligned}$$

Now we state the existence and uniqueness result of the solution to equation (1.1). For a proof of this result, see, for example, [31, Theorem 4.3].

Theorem 5.2. *Suppose hypothesis (**H1**) holds, and σ, b are Lipschitz continuous. Then there exists a unique mild solution u to equation (1.1) such that for all $p \geq 1$ and $T > 0$,*

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \mathbf{E}|u(t,x)|^p < \infty. \tag{2.5}$$

Next we recall some elements of Malliavin calculus which will be used to prove the main results of this paper. We consider the Hilbert space $\mathscr{H}_0$ and the Gaussian family of random variables $\{W(h), h \in \mathscr{H}_0\}$ defined above. Then $\{W(h), h \in \mathscr{H}_0\}$ is a centered Gaussian process such that $\mathbb{E}(W(h_1)W(h_2)) = \langle h_1, h_2\rangle_{\mathscr{H}_0}, h_1, h_2 \in \mathscr{H}_0$. In this framework we can develop a Malliavin calculus (see, for instance, [74]). The Malliavin derivative is denoted by D and for any $N \geq 1$ and any real number $p \geq 2$, the domain of the iterated derivative D^N in $L^p(\Omega; \mathscr{H}_0^{\otimes N})$ is denoted by $\mathbb{D}^{N,p}$. We shall also use the notation

$$\mathbb{D}^\infty = \cap_{p\geq 1} \cap_{k\geq 1} \mathbb{D}^{k,p} .$$

Note that for any random variable X in the domain of the derivative operator D, DX defines an $\mathscr{H}_0$-valued random variable. In particular, for some fixed $r \geq 0$, $DX(r,*)$ is an element of $\mathscr{H}$, which will be denoted by $D_{r,*}X$.

If $x_1, \dots, x_n$ are points in $\mathbb{R}^d$ we will make use of the notation

$$u(t, \underline{x}) = (u(t,x_1), \dots, u(t,x_n)).$$

In order to study the smoothness and strict positivity of the (joint) density of a random vector of the form $u(t,\underline{x})$, we need to assume some moment estimates for the increments of the solution. We will also need to assume some integral bounds of the fundamental solution Γ. We list these assumptions below.

(**H2**) There exist positive constants κ_1 and κ_2 such that for all $s,t \in [0,T]$, $x,y \in \mathbb{R}^d$, $T > 0$ and $p \geq 1$,

$$\mathbf{E}|u(s,x) - u(t,x)|^p \leq C_{p,T}|t-s|^{\kappa_1 p}, \tag{2.6}$$

$$\mathbf{E}|u(t,x) - u(t,y)|^p \leq C_{p,T}|x-y|^{\kappa_2 p} \tag{2.7}$$

for some constant $C_{p,T}$ which only depends on p, T.

(**H3**) There exist $\eta > 0$ and $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$,

$$C\varepsilon^{\eta} \leq \int_0^{\varepsilon} \|\Gamma(r)\|_{\mathscr{H}}^2 dr$$

for some constant $C > 0$.

(**H4**) Let η be as defined in (**H3**) and let κ_1 and κ_2 be defined in (**H2**).

(i) There exists $\eta_1 > \eta$ and $\varepsilon_1 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_1$,

$$\int_0^{\varepsilon} r^{\kappa_1} \|\Gamma(r)\|_{\mathscr{H}}^2 dr \leq C\varepsilon^{\eta_1}. \tag{2.8}$$

(ii) There exists $\eta_2 > \eta$ such that for each fixed non zero $w \in \mathbb{R}^d$, there exists a positive constant C_w and $\varepsilon_2 > 0$ satisfying

$$\int_0^{\varepsilon} \langle \Gamma(r,*), \Gamma(r,w+*)\rangle_{\mathscr{H}} dr \leq C_w \varepsilon^{\eta_2} \tag{2.9}$$

for all $0 < \varepsilon \leq \varepsilon_2$.

(iii) The measure $\Psi(t)$ defined by $|x|^{\kappa_2}\Gamma(t,dx)$ satisfies $\int_0^T \int_{\mathbb{R}^d} |\mathscr{F}\Psi(t)(\xi)|^2 \mu(d\xi) dt < \infty$ and there exists $\eta_3 > \eta$ such that for each fixed $w \in \mathbb{R}^d$, there exist a positive constant C_w and $\varepsilon_3 > 0$ satisfying

$$\int_0^{\varepsilon} \langle |*|^{\kappa_2}\Gamma(r,*), \Gamma(r,w+*)\rangle_{\mathscr{H}} dr \leq C_w \varepsilon^{\eta_3} \tag{2.10}$$

for all $0 < \varepsilon \leq \varepsilon_3$.

Along the paper, C_p and C will denote generic constants which may change from line to line and C_p depends on $p \geq 2$.

5.3 Existence and smoothness of the density

Fix $t > 0$ and fix distinct points $x_1, \ldots, x_n$ of $\mathbb{R}^d$. Let $u(t,x)$ denote the solution of equation (1.1). Recall that $u(t,\underline{x}) = (u(t,x_1), \ldots, u(t,x_n))$. In this section we give sufficient conditions for the existence and smoothness of the density of the law of the random vector $u(t,\underline{x})$, using Malliavin calculus. The main result is the following theorem.

Theorem 5.3. *Assume that conditions (**H1**)-(**H4**) hold, and the coefficients σ, b are $\mathscr{C}^\infty$ functions with bounded derivatives of all orders. Assume that there exists a positive constant C_1 such that $|\sigma(u(t,x_i))| \geq C_1$ $\mathbb{P}$-a.s. for any $i = 1, \ldots, n$. Then the law of the random vector $u(t,\underline{x})$ has a $\mathscr{C}^\infty$ density with respect to the Lebesgue measure on $\mathbb{R}^n$.*

Remark 5.4. *(1)Our assumption on σ in Theorem 5.3 is implied by $|\sigma(z)| \geq c > 0$.*

(2)Using a localization procedure developed in [11, Theorem 3.1], we can prove a version of Theorem 5.3 without assuming that $|\sigma(u(t,x_i))| \geq C_1$ $\mathbb{P}$-a.s., for any $i = 1, \ldots, n$. In this case, we conclude that the law of $u(t,\underline{x})$ has a smooth density on $\{y \in \mathbb{R} : \sigma(y) \neq 0\}^n$.

Proof. We begin by noting that according to Proposition 6.1 in [75], for each fixed $(t,x) \in [0,\infty) \times \mathbb{R}^d$, $u(t,x) \in \mathbb{D}^\infty$. If we denote by $M_t(\underline{x})$ the Malliavin covariance matrix $(\langle Du(t,x_i), Du(t,x_j)\rangle_{\mathscr{H}_0})_{1 \leq i,j \leq n}$, then, taking into account Theorem 2.1.4 in [74], we only need to show that the determinant of the Malliavin covariance matrix of $u(t,\underline{x})$ has negative moments of all orders, that is

$$\mathbf{E}\left(\det M_t(\underline{x})\right)^{-p} < \infty$$

for all $p \geq 2$. It suffices to check that for any $p \geq 2$, there exists an $\delta_0(p) > 0$ such that for all $0 < \delta \leq \delta_0(p)$

$$\mathbf{P}\{\det M_t(\underline{x}) \leq \delta\} \leq C\delta^p,$$

for some constant C independent on δ.

We begin by noting that

$$\det M_t(\underline{x}) \geq \left(\inf_{\|\xi\|=1} \xi^T M_t(\underline{x}) \xi \right)^n. \tag{3.1}$$

The derivative of the solution satisfies the following equation (see Proposition 5.1 in [75])

$$\begin{aligned} & D_{r,*}u(t,x) \\ = \; & \Gamma(t-r, x-*)\sigma(u(r,*)) + \int_r^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y)\sigma'(u(s,y)) D_{r,*}u(s,y) W(ds,dy) \\ & + \int_r^t \int_{\mathbb{R}^d} b'(u(s,x-y)) D_{r,*}u(s,x-y)\Gamma(t-s,dy)ds. \end{aligned}$$

Therefore, we can write

$$\xi^T M_t(\underline{x})\xi \geq \int_{t-\varepsilon}^t \| \sum_{i=1}^n D_{r,*}u(t,x_i)\xi_i \|_{\mathscr{H}}^2 dr \geq \frac{1}{2}\mathscr{A}_1 - \mathscr{A}_2,$$

where

$$\begin{aligned} \mathscr{A}_1 &= \int_{t-\varepsilon}^t \| \sum_{i=1}^n \Gamma(t-r, x_i - *)\sigma(u(r,*))\xi_i \|_{\mathscr{H}}^2 dr, \\ \mathscr{A}_2 &= \int_{t-\varepsilon}^t \| a(r,t,\underline{x},*) \|_{\mathscr{H}}^2 dr, \end{aligned}$$

and

$$\begin{aligned} a(r,t,\underline{x},*) &= \sum_{i=1}^n \int_r^t \int_{\mathbb{R}^d} \Gamma(t-s, x_i - y)\sigma'(u(s,y)) D_{r,*}u(s,y) W(ds,dy)\xi_i \\ & \quad + \sum_{i=1}^n \int_r^t \int_{\mathbb{R}^d} b'(u(s,x_i - y)) D_{r,*}u(s,x_i - y)\Gamma(t-s,dy)ds\xi_i. \end{aligned}$$

The term $\mathscr{A}_1$ can be estimated as follows

$$\begin{aligned}
\mathscr{A}_1 &= \int_{t-\varepsilon}^{t} \langle \sum_{i=1}^{n} \Gamma(t-r,x_i-*)\sigma(u(r,*))\xi_i, \sum_{j=1}^{n} \Gamma(t-r,x_j-*)\sigma(u(r,*))\xi_j \rangle_{\mathscr{H}} dr \\
&= \int_{t-\varepsilon}^{t} \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i\xi_j \langle \Gamma(t-r,x_i-*)\sigma(u(t,x_i)), \Gamma(t-r,x_j-*)\sigma(u(t,x_j))\rangle_{\mathscr{H}} dr \\
&\quad + \int_{t-\varepsilon}^{t} \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i\xi_j \Big[\langle \Gamma(t-r,x_i-*)\sigma(u(r,*)), \Gamma(t-r,x_j-*)\sigma(u(r,*))\rangle_{\mathscr{H}} \\
&\quad - \langle \Gamma(t-r,x_i-*)\sigma(u(t,x_i)), \Gamma(t-r,x_j-*)\sigma(u(t,x_j))\rangle_{\mathscr{H}} \Big] dr \\
&= \int_{t-\varepsilon}^{t} \sum_{i=1}^{n} \|\xi_i\|^2 \|\Gamma(t-r,x_i-*)\sigma(u(t,x_i))\|_{\mathscr{H}}^2 dr \\
&\quad + \int_{t-\varepsilon}^{t} \sum_{i,j=1, i\neq j}^{n} \xi_i\xi_j \langle \Gamma(t-r,x_i-*)\sigma(u(t,x_i)), \Gamma(t-r,x_j-*)\sigma(u(t,x_j))\rangle_{\mathscr{H}} dr \\
&\quad + \int_{t-\varepsilon}^{t} \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i\xi_j \left[\langle \Gamma(t-r,x_i-*)\sigma(u(r,*)), \Gamma(t-r,x_j-*)\sigma(u(r,*))\rangle_{\mathscr{H}} \right. \\
&\quad \left. - \langle \Gamma(t-r,x_i-*)\sigma(u(t,x_i)), \Gamma(t-r,x_j-*)\sigma(u(t,x_j))\rangle_{\mathscr{H}} \right] dr \\
&\geq \mathscr{A}_{11} - |\mathscr{A}_{12}| - |\mathscr{A}_{13}|,
\end{aligned}$$

where

$$\begin{aligned}
\mathscr{A}_{11} &= \int_{t-\varepsilon}^{t} \sum_{i=1}^{n} \|\xi_i\|^2 \|\Gamma(t-r,x_i-*)\sigma(u(t,x_i))\|_{\mathscr{H}}^2 dr, \\
\mathscr{A}_{12} &= \int_{t-\varepsilon}^{t} \sum_{i,j=1, i\neq j}^{n} \xi_i\xi_j \langle \Gamma(t-r,x_i-*)\sigma(u(t,x_i)), \Gamma(t-r,x_j-*)\sigma(u(t,x_j))\rangle_{\mathscr{H}} dr, \\
\mathscr{A}_{13} &= \int_{t-\varepsilon}^{t} \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i\xi_j \Big[\langle \Gamma(t-r,x_i-*)\sigma(u(r,*)), \Gamma(t-r,x_j-*)\sigma(u(r,*))\rangle_{\mathscr{H}} \\
&\quad - \langle \Gamma(t-r,x_i-*)\sigma(u(t,x_i)), \Gamma(t-r,x_j-*)\sigma(u(t,x_j))\rangle_{\mathscr{H}} \Big] dr.
\end{aligned}$$

Then, using the fact that $|\sigma(u(t,x_i))| \geq C_1$, for all $i = 1, \dots, n$, we have

$$\xi^T M_t(\underline{x})\xi \quad \geq \quad \frac{1}{2}\mathscr{A}_{11} - \frac{1}{2}|\mathscr{A}_{12}| - \frac{1}{2}|\mathscr{A}_{13}| - \mathscr{A}_2$$

$$\geq \frac{1}{2}C_1^2 g(\varepsilon) - \frac{1}{2}|\mathscr{A}_{12}| - \frac{1}{2}|\mathscr{A}_{13}| - \mathscr{A}_2,$$

where

$$g(\varepsilon) = \int_0^\varepsilon \int_{\mathbb{R}^d} |\mathscr{F}\Gamma(s)(\xi)|^2 \mu(d\xi) ds.$$

Taking ε such that $\frac{1}{4}C_1 g(\varepsilon) = \delta^{1/n}$, we obtain

$$\begin{aligned}
& \mathbf{P}\left\{ \inf_{\|\xi\|=1} \xi^T M_t(\underline{x})\xi \leq \delta^{1/n} \right\} \\
\leq\ & \mathbf{P}\left\{ \sup_{\|\xi\|=1} (|\mathscr{A}_{12}| + |\mathscr{A}_{13}| + 2\mathscr{A}_2) \geq \frac{1}{2}C_1 g(\varepsilon) \right\} \\
\leq\ & C_p g(\varepsilon)^{-p} \left[\mathbf{E}\Big(\sup_{\|\xi\|=1} |\mathscr{A}_{12}|^p \Big) + \mathbf{E}\Big(\sup_{\|\xi\|=1} |\mathscr{A}_{13}|^p \Big) + \mathbf{E}\Big(\sup_{\|\xi\|=1} |\mathscr{A}_2|^p \Big) \right]. \quad (3.2)
\end{aligned}$$

Next, we treat each of the above expectations separately. For the first expectation of (3.2), using (2.5) and property (ii) in condition (**H4**), we can write

$$\begin{aligned}
& \mathbf{E}\left(\sup_{\|\xi\|=1} |\mathscr{A}_{12}|^p \right) \\
=\ & \mathbf{E}\left(\sup_{\|\xi\|=1} \left| \int_0^\varepsilon \sum_{i,j=1, i\neq j}^n \xi_i \xi_j \langle \Gamma(r, x_i - *)\sigma(u(t,x_i)), \Gamma(r, x_j - *)\sigma(u(t,x_j))\rangle_{\mathscr{H}} dr \right|^p \right) \\
\leq\ & C_p \sum_{i,j=1, i\neq j}^n \left[\mathbb{E}\left(|\sigma(u(t,x_i))\sigma(u(t,x_j))|^p \right) \left| \int_0^\varepsilon \langle \Gamma(r, x_i - *), \Gamma(r, x_j - *)\rangle_{\mathscr{H}} dr \right|^p \right] \\
\leq\ & C_p \varepsilon^{\eta_2 p}. \quad (3.3)
\end{aligned}$$

For the second expectation of (3.2), using Minkowski's inequality and property (i) and (iii) in condition (**H4**), we get

$$\mathbf{E}\left(\sup_{\|\xi\|=1} |\mathscr{A}_{13}|^p \right)$$

$$
\begin{aligned}
&\leq C_p \sum_{i,j=1}^{n} \mathbf{E}\Big| \int_{t-\varepsilon}^{t} dr \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} [\sigma(u(r,z))\sigma(u(r,y)) - \sigma(u(t,x_i))\sigma(u(t,x_j))] \\
&\quad \times \Gamma(t-r, x_i - dz)\Gamma(t-r, x_j - dy) f(z-y)\Big|^p \\
&\leq C_p \sum_{i,j=1}^{n} \Big(\int_{t-\varepsilon}^{t} dr \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \|\sigma(u(r,z))\sigma(u(r,y)) - \sigma(u(t,x_i))\sigma(u(t,x_j))\|_{L^p(\Omega)} \\
&\quad \times \Gamma(t-r, x_i - dz)\Gamma(t-r, x_j - dy) f(z-y)\Big)^p \\
&\leq C_p \sum_{i,j=1}^{n} \Big(\int_{0}^{\varepsilon} dr \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (r^{\kappa_1} + |x_i - z|^{\kappa_2} + |x_j - y|^{\kappa_2}) \\
&\quad \times \Gamma(r, x_i - dz)\Gamma(r, x_j - dy) f(z-y)\Big)^p \\
&\leq C_p \left| \int_0^{\varepsilon} r^{\kappa_1} \|\Gamma(r,*)\|^2_{\mathscr{H}} dr \right|^p + C_p \sum_{i,j=1}^{n} \left| \int_0^{\varepsilon} \langle |*|^{\kappa_2}\Gamma(r,*), \Gamma(r, x_j - x_i + *)\rangle_{\mathscr{H}} dr \right|^p \\
&\leq C_p \varepsilon^{\eta_1 p} + C_p \varepsilon^{\eta_3 p}. \qquad (3.4)
\end{aligned}
$$

Finally, we treat the last expectation of (3.2) and we obtain the following inequalities

$$
\begin{aligned}
&\mathbf{E}\left(\sup_{\|\xi\|=1} |\mathscr{A}_2|^p \right) \\
&\leq C_p \mathbf{E}\left(\sup_{\|\xi\|=1} \int_{t-\varepsilon}^{t} \left\| \sum_{i=1}^{n} \int_r^t \int_{\mathbb{R}^d} \Gamma(t-s, x_i - y)\sigma'(u(s,y)) D_{r,*}u(s,y) W(ds,dy)\xi_i \right\|^2_{\mathscr{H}} dr \right)^p \\
&+ C_p \mathbf{E}\left(\sup_{\|\xi\|=1} \int_{t-\varepsilon}^{t} \left\| \sum_{i=1}^{n} \int_r^t \int_{\mathbb{R}^d} b'(u(s, x_i - y)) D_{r,*}u(s, x-y)\Gamma(t-s, dy) ds \xi_i \right\|^2_{\mathscr{H}} dr \right)^p \\
&:= \quad T_1 + T_2.
\end{aligned}
$$

For any φ, ψ in $\mathscr{H}_0$ we use the notation

$$
\langle \varphi, \psi \rangle_{\mathscr{H}_{t-\varepsilon,t}} := \int_{t-\varepsilon}^{t} \langle \varphi(s,*), \psi(s,*)\rangle_{\mathscr{H}} ds.
$$

Using equation (3.13) and the inequality (5.26) in [75], we obtain

$$\begin{aligned} T_1 &\leq C_p \sum_{i=1}^{n} \mathbb{E} \left\| \int_{t-\varepsilon}^{t} \int_{\mathbb{R}^d} \Gamma(t-s, x_i - y) \sigma'(u(s, x_i - y)) Du(s, x-y) W(ds, dy) \right\|_{\mathscr{H}_{t-\varepsilon,t}}^{2p} \\ &\leq g(\varepsilon)^p \sup_{t-\varepsilon \leq s \leq t, x \in \mathbb{R}^d} \mathbf{E} \|Du(s,x)\|_{\mathscr{H}_{t-\varepsilon,t}}^{2p} \\ &\leq C_p g(\varepsilon)^{2p}. \end{aligned} \tag{3.5}$$

For T_2, using Cauchy-Schwartz inequality, our assumption on b', Minkowski's inequality and the estimate (5.26) in [75], we obtain the bound

$$\begin{aligned} T_2 &\leq C_p \sum_{i=1}^{n} \mathbf{E} \left\| \int_{t-\varepsilon}^{t} \int_{\mathbb{R}^d} b'(u(s, x_i - y)) Du(s, x_i - y) \Gamma(t-s, dy) ds \right\|_{\mathscr{H}_{t-\varepsilon,t}}^{2p} \\ &\leq C_p \left(\int_{t-\varepsilon}^{t} \int_{\mathbb{R}^d} \Gamma(t-s, dy) ds \right)^{2p} \sup_{t-\varepsilon \leq s \leq t, x \in \mathbb{R}^d} \mathbf{E} \|Du(s,x)\|_{\mathscr{H}_{t-\varepsilon,t}}^{2p} \\ &\leq C_p g(\varepsilon)^p \varepsilon^{2p}. \end{aligned} \tag{3.6}$$

The estimates (3.5) and (3.6) imply that

$$\mathbf{E} \left(\sup_{\|\xi\|=1} |\mathscr{A}_2|^p \right) \leq C_p g(\varepsilon)^{2p} + C_p g(\varepsilon)^p \varepsilon^{2p}. \tag{3.7}$$

Then by (3.1),(3.2),(3.3),(3.4) and (3.7), for $\delta < 1$, we obtain

$$\begin{aligned} \mathbf{P}\{\det M_t(\underline{x}) \leq \delta\} &\leq C_p g(\varepsilon)^{-p} (\varepsilon^{\eta_1 p} + \varepsilon^{\eta_2 p} + \varepsilon^{\eta_3 p} + g(\varepsilon)^{2p} + g(\varepsilon)^p \varepsilon^{2p}) \\ &\leq C_p \delta^{\lambda p}, \end{aligned}$$

where $\lambda = \min\{\frac{\eta_1 - \eta}{n\eta}, \frac{\eta_2 - \eta}{n\eta}, \frac{\eta_3 - \eta}{n\eta}, \frac{1}{n}, \frac{2}{n\eta}\}$. The proof is completed. □

5.4 Strict positivity of the density

In this section, we proceed to the study of the positivity of the density $p_{t,\underline{x}}(\cdot)$ of the law of $u(t,\underline{x})$, where $t>0$, $\underline{x}=(x_1,\dots,x_n)$ are distinct points of $\mathbb{R}^d$. The main theorem of this section is:

Theorem 5.5. *Assume that conditions (***H1***)-(***H4***) hold, and the coefficients σ, b are $\mathscr{C}^\infty$ functions with bounded derivatives of all orders and σ is bounded. We also assume $\sigma \neq 0$ on $\mathbb{R}$. Then the law of the random vector $u(t,\underline{x})$ has a $\mathscr{C}^\infty$ density $p_{t,\underline{x}}(y)$, and $p_{t,\underline{x}}(y)>0$ if y belongs to the interior of the support of the law of $u(t,\underline{x})$.*

To prove this theorem we will use the criterion given by Theorem 3.3 in [11]. To state this criterion in the context of framework, first we introduce some notation and concepts.

Given predictable processes $(g^1,\dots,g^n)\in\mathscr{H}_0^n$ and $z=(z_1,\cdots,z_n)\in\mathbb{R}^n$, for any $h\in\mathscr{H}$ and $t\geq 0$ we define a translation of $W_t(h)$:

$$\widehat{W}_t(h):=\widehat{W}(\mathbf{1}_{[0,t]}h)=W(\mathbf{1}_{[0,t]}h)+\sum_{k=1}^n z_k\langle \mathbf{1}_{[0,t]}h,g^k\rangle_{\mathscr{H}_0}.$$

Then $\{\widehat{W}_t,t\geq 0\}$ is a cylindrical Wiener process in $\mathscr{H}$ on the probability space $(\Omega,\mathscr{F},\widehat{\mathbf{P}})$, where

$$\frac{d\widehat{\mathbf{P}}}{d\mathbf{P}}=\exp\left(-\sum_{k=1}^n z_k\int_0^\infty\int_{\mathbb{R}^d}g^k(s,y)W(ds,dy)-\frac{1}{2}\sum_{k=1}^n z_k^2\int_0^\infty\|g^k(s,*)\|_{\mathscr{H}}^2ds\right).$$

Then, for any predicable process $Z\in L^2(\Omega\times[0,\infty);\mathscr{H})$, we can write

$$\int_0^\infty\int_{\mathbb{R}^d}Z(s,y)\widehat{W}(ds,dy)=\int_0^\infty\int_{\mathbb{R}^d}Z(s,y)W(ds,dy)+\sum_{k=1}^n z_k\int_0^\infty\langle Z(s,*),g^k(s,*)\rangle_{\mathscr{H}}ds.$$

For any $(t,x) \in [0,\infty) \times \mathbb{R}^d$, let $\widehat{u}^z(t,x)$ be the solution to equation (1.1) with respect to the cylindrical Wiener process $\widehat{W}$, that is,

$$\begin{aligned}
\widehat{u}^z(t,x) &= \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y)\sigma\left(\widehat{u}^z(s,y)\right) W(ds,dy) \\
&\quad + \sum_{k=1}^n z_k \int_0^t \langle \Gamma(t-s,x-*)\sigma\left(\widehat{u}^z(s,*)\right), g^k(s,*)\rangle_{\mathscr{H}} ds \\
&\quad + \int_0^t \int_{\mathbb{R}^d} b\left(\widehat{u}^z(t-s,x-y)\right)\Gamma(s,dy)ds. \qquad (4.1)
\end{aligned}$$

Then, the law of u under $\mathbf{P}$ coincides with the law of $\widehat{u}^z$ under $\widehat{\mathbf{P}}$.

Now we consider a sequence $\{g_m\}_{m\geq 1}$ of predictable processes in $\mathscr{H}_0^n$ and $z \in \mathbb{R}^n$. Let $\widehat{u}^z_m(t,x)$ be the solution to equation (1.1) with respect to the cylindrical Wiener process $\{\widehat{W}^m_t, t \geq 0\}$, where $\widehat{W}^m_t(h) = \widehat{W}^m(\mathbf{1}_{[0,t]}h)$ for any $h \in \mathscr{H}$, and

$$\widehat{W}^m(\mathbf{1}_{[0,t]}h) = W(\mathbf{1}_{[0,t]}h) + \sum_{k=1}^n z_k \langle \mathbf{1}_{[0,t]}h, g^k_m \rangle_{\mathscr{H}_0}.$$

Set $\varphi^z_{m,j}(t,x) := \partial_{z_j}\widehat{u}^z_m(t,x)$ and denote by $\varphi^z_m(t,\underline{x})$ the $n \times n$ matrix $\{\varphi^z_{m,j}(t,x_i)\}_{1\leq i,j\leq n}$. Also, denote the Hessian matrix of $\widehat{u}^z_m(t,x)$ by $\psi^z_m(t,x) := \partial^2_z \widehat{u}^z_m(t,x)$, and let $\psi^z_m(t,\underline{x}) := (\psi^z_m(t,x_1),\ldots,\psi^z_m(t,x_n))$. In fact, it can be shown that

$$\partial_{z_j}\widehat{u}^z_m(t,x) = \int_0^t \langle D_{r,*}\widehat{u}^z_m(t,x), g^j_m(r,*)\rangle_{\mathscr{H}} dr.$$

We denote the operator norms of these matrices by $\|\varphi^z_m(t,x)\|$ and $\|\psi^z_m(t,x)\|$, respectively.

We say that $y \in \mathbb{R}^d$ satisfies $\mathbf{H}_{t,\underline{x}}(y)$ if there exist a sequence of predictable processes $\{g_m\}_{m\geq 1}$ in $\mathscr{H}_0^n$, and positive constants c_1, c_2, r_0 and δ such that

(i) $\limsup_{m\to\infty} \mathbf{P}\left\{(\|u(t,\underline{x}) - y\| \leq r) \cap (|\det\varphi^0_m(t,\underline{x})| \geq c_1)\right\} > 0, \forall r \in (0,r_0]$.

(ii) $\lim_{m\to\infty} \mathbf{P}\left\{\sup_{|z|\leq\delta}\left(\|\varphi^z_m(t,\underline{x})\| + \|\psi^z_m(t,\underline{x})\|\right) \leq c_2\right\} = 1$.

Now we can state the criterion in [11] (Theorem 3.3) that we are going to use: Suppose that $y \in \mathbb{R}^d$ belongs to the interior of the support of the law of $u(t,\underline{x})$. If y satisfies $\mathbf{H}_{t,\underline{x}}(y)$, then $p_{t,\underline{x}}(y) > 0$.

Proof of Theorem 5.5 From the above criterion it suffices to check that y satisfies the two conditions in $\mathbf{H}_{t,\underline{x}}(y)$. We will do this in several steps.

Step 1. Consider the sequence of predictable processes $\{g_m\}_{m\geq 1}$ in $\mathscr{H}_0^n$, defined by

$$g_m^k(s,*) = v_m^{-1}\mathbf{1}_{[t-2^{-m},t]}(s)\Gamma(t-s,x_k-*) \text{ for } 1 \leq k \leq n\,,$$

where

$$v_m = \int_0^{2^{-m}} \int_{\mathbb{R}^d} |\mathscr{F}\Gamma(r)(\xi)|^2 \mu(d\xi)dr\,.$$

Taking the partial derivatives on both sides of (4.1) with g replaced by g_m, we obtain that

$$\begin{aligned}
\partial_{z_j}\widehat{u}_m^z(t,x) \quad = \quad & \int_{t-2^{-m}}^t \langle \Gamma(t-s,x-*)\sigma\left(\widehat{u}_m^z(s,*)\right), g_m^j(s,*)\rangle_{\mathscr{H}} ds \\
&+ \sum_{k=1}^m z_k \int_{t-2^{-m}}^t \langle \Gamma(t-s,x-*)\sigma'\left(\widehat{u}_m^z(s,*)\right)\partial_{z_j}\widehat{u}_m^z(s,*), g_m^k(s,*)\rangle_{\mathscr{H}} ds \\
&+ \int_{t-2^{-m}}^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y)\sigma'\left(\widehat{u}_m^z(s,y)\right)\partial_{z_j}\widehat{u}_m^z(s,y) W(ds,dy) \\
&+ \int_{t-2^{-m}}^t \int_{\mathbb{R}^d} b'\left(\widehat{u}_m^z(t-s,x-y)\right)\partial_{z_j}\widehat{u}_m^z(t-s,x-y)\Gamma(s,dy)ds \\
:= \quad & A_{m,j}^z(t,x) + B_{m,j}^z(t,x) + C_{m,j}^z(t,x) + D_{m,j}^z(t,x)\,. \qquad (4.2)
\end{aligned}$$

Step 2. We are going to bound the moments of the four terms on the right hand side of (4.2). We assume that $\|z\| \leq \delta$ for some $\delta > 0$. Since σ is bounded, there is a positive constant K such that

$$|A^z_{m,j}(t,x)| \leq K\,. \tag{4.3}$$

Using Minkowski's inequality and the fact that the partial derivatives of σ are bounded, we get that for all $p \geq 1$, $t \leq T$,

$$\mathbf{E}\left|B^z_{m,j}(t,x)\right|^p \leq C\delta^p \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \mathbf{E}\left|\partial_{z_j}\widehat{u}^z_m(s,y)\right|^p\,. \tag{4.4}$$

From the Burkholder-Davis-Gundy inequality and from the definition of v_m, we have

$$\begin{aligned} & \mathbf{E}\left|C^z_{m,j}(t,x)\right|^p && (4.5)\\ \leq\ & C \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \mathbf{E}\left|\partial_{z_j}\widehat{u}^z_m(s,y)\right|^p \left(\int_{t-2^{-m}}^t \int_{\mathbb{R}^d} |\mathscr{F}\Gamma(t-s)(\xi)|^2 \mu(d\xi)ds\right)^{\frac{p}{2}} \\ \leq\ & C v_m^{\frac{p}{2}} \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \mathbf{E}\left|\partial_{z_j}\widehat{u}^z_m(s,y)\right|^p\,. && (4.6)\end{aligned}$$

Since b' is bounded and by condition (**H1**),

$$\mathbf{E}\left|D^z_{m,j}(t,x)\right|^p \leq C2^{-mp} \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \mathbf{E}\left|\partial_{z_j}\widehat{u}^z_m(s,y)\right|^p\,. \tag{4.7}$$

Combing (4.3), (4.4), (4.6) and (4.7) we obtain

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbf{E}\left|\partial_{z_j}\widehat{u}^z_m(t,x)\right|^p \leq K + C(\delta^p + v_m^{\frac{p}{2}} + 2^{-mp}) \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \mathbf{E}\left|\partial_{z_j}\widehat{u}^z_m(s,y)\right|^p\,. \tag{4.8}$$

Proceeding as in the proof of Proposition 6.1 in [75], we can show

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d, |z|\leq\delta} \mathbf{E}\left|\partial_{z_j}\widehat{u}^z_m(t,x)\right|^p < \infty. \tag{4.9}$$

Then, when m is large enough and δ is small enough, $C(\delta^p + v_m^{\frac{p}{2}} + 2^{-mp})$ on the right hand side of equation (4.8) is less than $\frac{1}{2}$ and we obtain

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d, |z|\leq\delta} \mathbf{E}\left|\partial_{z_j}\widehat{u}_m^z(t,x)\right|^p \leq C \tag{4.10}$$

for some constant C.

Recall that $\varphi_{m,j}^z(t,x_i) = \partial_{z_j}\widehat{u}_m^z(t,x_i)$. Take $z=0$ and decompose $\varphi_{m,j}^0(t,x_i)$ as follows

$$\varphi_{m,j}^0(t,x_i) = A_{m,j}^0(t,x_i) + C_{m,j}^0(t,x_i) + D_{m,j}^0(t,x_i)\,. \tag{4.11}$$

From (4.6) and (4.7) it follows that

$$\mathbf{E}\left|C_{m,j}^0(t,x_i) + D_{m,j}^0(t,x_i)\right|^p \leq C(v_m^{\frac{p}{2}} + 2^{-mp})\,. \tag{4.12}$$

For $A_{m,j}^0(t,x_i)$,

$$\begin{aligned} A_{m,j}^0(t,x_i) &= \int_{t-2^{-m}}^t \langle \Gamma(t-s,x_i-*)\sigma(u(s,*)), g_m^j(s,*)\rangle_{\mathscr{H}} ds \\ &= \int_{t-2^{-m}}^t \langle \Gamma(t-s,x_i-*)\left[\sigma(u(s,*)) - \sigma(u(t,x_i))\right], g_m^j(s,*)\rangle_{\mathscr{H}} ds \\ &\quad + \sigma(u(t,x_i)) \int_{t-2^{-m}}^t \langle \Gamma(t-s,x_i-*), g_m^j(s,*)\rangle_{\mathscr{H}} ds \\ &:= O_{m,i,j} + \widetilde{O}_{m,i,j}\,. \end{aligned} \tag{4.13}$$

By the assumption (**H2**) and Minkowski's inequality, we have

$$\begin{aligned} &\mathbf{E}\left|O_{m,i,j}\right|^p \\ = &\left\| \frac{1}{v_m} \int_{t-2^{-m}}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Gamma(t-s,x_i-dy)\left[\sigma(u(s,y)) - \sigma(u(t,x_i))\right] \right. \\ &\left. \times \Gamma(t-s,x_j-dz) f(y-z) ds \right\|_{L^p(\Omega)}^p \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{v_m^p}\left(\int_{t-2^{-m}}^{t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\|\sigma\left(u(s,y)\right)-\sigma\left(u(t,x_i)\right)\|_{L^p(\Omega)}\right. \\
&\quad \left.\times\Gamma(t-s,x_j-dz)f(y-z)\Gamma(t-s,x_i-dy)ds\right)^p \\
&\leq \frac{C}{v_m^p}\left(\int_{t-2^{-m}}^{t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\right. \\
&\quad \left.\times\Gamma(t-s,x_i-dy)\left(|x_i-y|^{\kappa_2}+|s-t|^{\kappa_1}\right)\Gamma(t-s,x_j-dz)f(y-z)ds\right)^p \\
&\leq \frac{C}{v_m^p}\left(2^{-m\eta_1}+2^{-m\eta_3}\right)^p \to 0 \text{ as } m\to\infty.
\end{aligned}
$$

For $\widetilde{O}_{m,i,j}$, when $i=j$, it is easy to see that

$$
\widetilde{O}_{m,i,i}=\sigma(u(t,x_i))\,, \tag{4.14}
$$

while when $i\neq j$, we have the pth moment bound

$$
\begin{aligned}
\mathbf{E}\left|\widetilde{O}_{m,i,j}\right|^p &\leq \mathbf{E}\left|\sigma(u(t,x_i))\right|^p\left(\int_{t-2^{-m}}^{t}\langle\Gamma(t-s,x_i-*),g_m^j(s,*)\rangle_{\mathscr{H}}ds\right)^p \\
&\leq C_p\left(\frac{1}{v_m}\int_0^{2^{-m}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Gamma(s,x_i-dy)f(y-z)\Gamma(s,x_j-dz)ds\right)^p \\
&\leq C_p\left(\frac{2^{-m\eta_2}}{v_m}\right)^p,
\end{aligned} \tag{4.15}
$$

which goes to 0 as $m\to\infty$.

Step 3. We check condition (i) in hypothesis $\mathbf{H}_{t,\underline{x}}(y)$. Recall that $y\in\operatorname{Supp}\left(P_{u(t,\underline{x})}\right)$, there exists $r_0>0$ such that for all $0<r\leq r_0$,

$$
\mathbf{P}\left\{u(t,\underline{x})\in B(y;r)\right\}>0\,.
$$

By the assumption on σ, there is a $c_1 > 0$ such that

$$\mathbf{P}\left\{(\|u(t,\underline{x})-y\| \le r) \cap \left(\left|\prod_{i=1}^{n}\sigma(u(t,x_i))\right| \ge 2c_1\right)\right\} > 0 \tag{4.16}$$

where

$$c_1 = \frac{1}{2}\left(\inf_{z\in B(y;r)}\prod_{i=1}^{n}|\sigma(z_i)|\right),$$

here $z = (z_1,\dots,z_n)$. Recall that $\varphi_m^0(t,\underline{x})$ is the matrix $\left(\varphi_{m,j}^0(t,x_i)\right)_{1\le i,j\le n}$. By (4.11), (4.12), (4.13), (4.14) and (4.15), we obtain

$$\mathbf{E}\left|\det\varphi_m^0(t,\underline{x}) - \prod_{i=1}^{n}\sigma(u(t,x_i))\right|^p \to 0 \text{ as } m\to\infty. \tag{4.17}$$

Combining (4.16) and (4.17) yields

$$\limsup_{m\to\infty}\mathbf{P}\left\{(\|u(t,\underline{x})-y\| \le r) \cap \left(\left|\det\varphi_m^0(t,\underline{x})\right| \ge c_1\right)\right\} > 0.$$

Step 4. We check condition (ii) in the hypothesis $\mathbf{H}_{t,\underline{x}}(y)$.

We first show that there exist $c_2 > 0$ and $\delta > 0$ such that

$$\lim_{m\to\infty}\mathbf{P}\left\{\sup_{|z|\le\delta}\|\varphi_m^z(t,\underline{x})\| \le c_2\right\} = 1.$$

Consider the following equation

$$v_{m,j}^z(t,x) = A_{m,j}^z(t,x) + \sum_{k=1}^{n} z_k\int_{t-2^{-m}}^{t}\langle\Gamma(t-s,x-*)\sigma'\left(\widehat{u}_m^z(s,*)\right)v_{m,j}^z(s,*),g^k(s,*)\rangle_{\mathscr{H}}\,ds. \tag{4.18}$$

By the contraction mapping theorem we can prove that this equation has a unique solution $v^z_{m,j}(t,x)$ and there exists a constant C such that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d,|z|\le\delta} |v^z_{m,j}(t,x)| \le C \quad \forall 1 \le j \le n\,, \tag{4.19}$$

when δ is small.

Then we claim that for each j, $v^z_{m,j}(t,x) - \partial_{z_j}\widehat{u}^z_m(t,x)$ converges to 0 in $L^p(\Omega)$ norm, uniformly in $(t,x) \in [0,T]\times\mathbb{R}^d$, and $|z| \le \delta$ when δ is small. Indeed, we have

$$\begin{aligned}
& \mathbf{E}\left|\partial_{z_j}\widehat{u}^z_m(t,x) - v^z_{m,j}(t,x)\right|^p \\
\le\ & C_p \sum_{k=1}^n |z_k|^p \left(\int_{t-2^{-m}}^t \langle \Gamma(t-s,x-*)\sigma'(\widehat{u}^z_m(s,*)), g^k(s,*)\rangle_{\mathscr{H}} ds\right)^p \\
& \times \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \mathbf{E}\left|\partial_{z_j}\widehat{u}^z_m(s,y) - v^z_{m,j}(s,y)\right|^p \\
& + C_p \left\|\int_{t-2^{-m}}^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y)\sigma'\left(\widehat{u}^z_m(s,y)\right)\partial_{z_j}\widehat{u}^z_m(s,y) W(ds,dy)\right\|^p_{L^p(\Omega)} \\
& + C_p \left\|\int_{t-2^{-m}}^t \int_{\mathbb{R}^d} b'\left(\widehat{u}^z_m(t-s,x-y)\right)\partial_{z_j}\widehat{u}^z_m(t-s,x-y)\Gamma(s,dy)ds\right\|^p_{L^p(\Omega)} \\
\le\ & C_p\delta^p \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbf{E}\left|\partial_{z_j}\widehat{u}^z_m(t,x) - v^z_{m,j}(t,x)\right|^p \\
& + C_p\left(\int_{t-2^{-m}}^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Gamma(t-s,x-dy)f(y-\tilde{y})\Gamma(t-s,x-d\tilde{y})ds\right)^{\frac{p}{2}} \\
& \times \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \mathbf{E}\left|\partial_{z_j}\widehat{u}^z_m(s,y)\right|^p \\
& + C_p\left(\int_{t-2^{-m}}^t \int_{\mathbb{R}^d}\Gamma(s,dy)ds\right)^p \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \mathbf{E}\left|\partial_{z_j}\widehat{u}^z_m(s,y)\right|^p \\
\le\ & C_p\delta^p \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbf{E}\left|\partial_{z_j}\widehat{u}^z_m(t,x) - v^z_{m,j}(t,x)\right|^p \\
& + C_p\left(\int_0^{2^{-m}} \int_{\mathbb{R}^d} |\Gamma(s)(\xi)|^2\mu(d\xi)ds\right)^{\frac{p}{2}} \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \mathbf{E}\left|\partial_{z_j}\widehat{u}^z_m(s,y)\right|^p \\
& + C_p\left(\int_{t-2^{-m}}^t \int_{\mathbb{R}^d}\Gamma(s,dy)ds\right)^p \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \mathbf{E}\left|\partial_{z_j}\widehat{u}^z_m(s,y)\right|^p .
\end{aligned}$$

We can choose δ small enough such that $C_p\delta^p \leq \frac{1}{2}$. Then, using condition (**H1**) and (4.10) to conclude that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d, |z|\leq\delta} \mathbf{E}\left|\partial_{z_j}\widehat{u}_m^z(t,x) - v_{m,j}^z(t,x)\right|^p \tag{4.20}$$

goes to 0 as m tends to ∞.

Next, we will calculate the pth moment of the increments with respect to z of $\partial_{z_j}\widehat{u}_m^z(t,x)$ and $v_{m,j}^z(t,x)$.

$$\begin{aligned}
& \mathbf{E}\left|\partial_{z_j}\widehat{u}_m^z(t,x) - \partial_{z_j}\widehat{u}_m^{z'}(t,x)\right|^p \\
\leq\ & \mathbf{E}\left|\int_{t-2^{-m}}^t \left\langle \Gamma(t-s,x-*)\left[\sigma\left(\widehat{u}_m^z(s,*)\right) - \sigma(\widehat{u}_m^{z'}(s,*))\right], g_m^j(s,*)\right\rangle_{\mathscr{H}} ds\right|^p \\
& +\mathbf{E}\Big|\sum_{k=1}^n \int_{t-2^{-m}}^t \langle\Gamma(t-s,x-*)[z_k\sigma'\left(\widehat{u}_m^z(s,*)\right)\partial_{z_j}\widehat{u}_m^z(s,*) \\
& \quad -z'_k\sigma'(\widehat{u}_m^{z'}(s,*))\partial_{z_j}\widehat{u}_m^{z'}(s,*)], g_m^k(s,*)\rangle_{\mathscr{H}} ds\Big|^p \\
& +\mathbf{E}\Big|\int_{t-2^{-m}}^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y)[\sigma'\left(\widehat{u}_m^z(s,y)\right)\partial_{z_j}\widehat{u}_m^z(s,y) \\
& -\sigma'(\widehat{u}_m^{z'}(s,y))\partial_{z_j}\widehat{u}_m^{z'}(s,y)]W(ds,dy)\Big|^p \\
& +\mathbf{E}\Big|\int_{t-2^{-m}}^t \int_{\mathbb{R}^d} \Big[b'\left(\widehat{u}_m^z(t-s,x-y)\right)\partial_{z_j}\widehat{u}_m^z(t-s,x-y) \\
& \quad -b'(\widehat{u}_m^{z'}(t-s,x-y))\partial_{z_j}\widehat{u}_m^{z'}(t-s,x-y)\Big]\Gamma(s,dy)ds\Big|^p.
\end{aligned}$$

Proceeding as before, we obtain that

$$\mathbf{E}\left|\partial_{z_j}\widehat{u}_m^z(t,x) - \partial_{z_j}\widehat{u}_m^{z'}(t,x)\right|^p \leq C|z-z'|^p$$

uniformly in $(t,x)\in[0,T]\times\mathbb{R}^d$, $|z|\leq\delta$ and m. Similarly, we have

$$\mathbf{E}\left|v_{m,j}^z(t,x) - v_{m,j}^{z'}(t,x)\right|^p \leq C|z-z'|^p$$

uniformly in $(t,x) \in [0,T] \times \mathbb{R}^d$, $|z| \le \delta$ and m. Using Kolmogorov's continuity theorem and (4.19), (4.20) we obtain

$$\lim_{m\to\infty} \mathbf{P}\left\{ \sup_{|z|\le\delta} \|\varphi_m^z(t,\underline{x})\| \le C \right\} = 1$$

for some positive constant C.

Next we will show that there exists a positive constant C such that

$$\lim_{m\to\infty} \mathbf{P}\left\{ \sup_{|z|\le\delta} \|\psi_m^z(t,\underline{x})\| \le C \right\} = 1 \,.$$

This proof is analogous to that for $\varphi_m^z(t,\underline{x})$, but the computations are more involved. Let us just write the equation for the quantity of interest and the main steps. Taking the partial derivative on both sides of (4.2), we obtain

$$\begin{aligned}
&\partial_{z_l}\partial_{z_j}\widehat{u}_m^z(t,x)\\
= \;& \int_{t-2^{-m}}^t \langle \Gamma(t-s,x-*)\sigma'\left(\widehat{u}_m^z(s,*)\right)\partial_{z_l}\widehat{u}_m^z(s,*), g_m^j(s,*)\rangle_{\mathscr{H}} ds\\
&+\int_{t-2^{-m}}^t \langle \Gamma(t-s,x-*)\sigma'\left(\widehat{u}_m^z(s,*)\right)\partial_{z_j}\widehat{u}_m^z(s,*), g_m^l(s,*)\rangle_{\mathscr{H}} ds\\
&+\sum_{k=1}^m z_k \int_{t-2^{-m}}^t \langle \Gamma(t-s,x-*)\Big(\sigma''\left(\widehat{u}_m^z(s,*)\right)\partial_{z_l}\widehat{u}_m^z(s,*)\partial_{z_j}\widehat{u}_m^z(s,*)\\
&\quad +\sigma'\left(\widehat{u}_m^z(s,*)\right)\partial_{z_l}\partial_{z_j}\widehat{u}_m^z(s,*)\Big), g_m^k(s,*)\rangle_{\mathscr{H}} ds\\
&+\int_{t-2^{-m}}^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y)\Big(\sigma''\left(\widehat{u}_m^z(s,y)\right)\partial_{z_l}\widehat{u}_m^z(s,y)\partial_{z_j}\widehat{u}_m^z(s,y)\\
&\quad +\sigma'\left(\widehat{u}_m^z(s,y)\right)\partial_{z_l}\partial_{z_j}\widehat{u}_m^z(s,y)\Big)W(ds,dy)\\
&+\int_{t-2^{-m}}^t \int_{\mathbb{R}^d} \Big(b''\left(\widehat{u}_m^z(t-s,x-y)\right)\partial_{z_l}\widehat{u}_m^z(t-s,x-y)\partial_{z_j}\widehat{u}_m^z(t-s,x-y)\\
&\quad +b'\left(\widehat{u}_m^z(t-s,x-y)\right)\partial_{z_l}\partial_{z_j}\widehat{u}_m^z(t-s,x-y)\Big)\Gamma(s,dy)ds
\end{aligned}$$

and a similar equation for $\partial_{z_l} v_{m,j}^z(t,x)$.

We can show that for every $1 \leq l, j \leq n$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d, |z|\leq\delta} \mathbf{E}\left|\partial_{z_l}\partial_{z_j}\widehat{u}^z_m(t,x) - \partial_{z_l} v^z_{m,j}(t,x)\right|^p \to 0\,,$$

as m goes to ∞. Bound $\partial_{z_l} v^z_{m,j}(t,x)$ and calculate the pth moment of the increments with respect to z of $\partial_{z_l}\partial_{z_j}\widehat{u}^z_m(t,x)$ and $\partial_{z_l} v^z_{m,j}(t,x)$. The result follows as in the previous steps.

Step 5. By combining the results in step 3 and step 4, together with the criterion developed by Theorem 3.3 in [11] that we cited just before the proof, we complete the proof. ■

5.5 Examples

In this section we will give some examples of fundamental solutions Γ and covariance functions f satisfying hypotheses (**H1**) to (**H4**). This implies that Theorem 5.3 and Theorem 5.5 can be applied to these examples. We consider the fundamental solution to the heat equation in any dimension and the wave equation in dimensions up to three and the covariance functions given by the Riesz, Bessel, and fractional kernels.

5.5.1 Heat equation

Let $\Gamma(r,dx)$ be the fundamental solution to the heat equation on $\mathbb{R}^d$, i.e., $\Gamma(r,dx) = p_r(x)dx$, where $p_r(x) = (2\pi r)^{-d/2} e^{-\frac{|x|^2}{2r}}$ is the d-dimensional heat kernel. Then, hypothesis (**H1**) to (**H4**) are satisfied for the following covariance functions:

(A) *Riesz kernel.* Let $f(x) = |x|^{-\beta}$ with $0 < \beta < 2 \wedge d$. It is well-known that (**H1**) holds. According to [86], (**H2**) is satisfied with $0 < \kappa_1 < \frac{2-\beta}{4}$ and $0 < \kappa_2 < \frac{2-\beta}{2}$. In

[72] it is proved that (**H3**) holds with $\eta = \frac{2-\beta}{2}$, and property (i) in (**H4**) holds with $\eta_1 = \frac{2-\beta}{2} + \kappa_1$.

Next we check conditions (ii) and (iii) in (**H4**). To show (2.9) we use the fact that there exists a constant $C > 0$ such that for any non zero $y \in \mathbb{R}^d$ and $r \geq 0$

$$\int_{\mathbb{R}^d} p_r(x)|x-y|^{-\beta} dx \leq C|y|^{-\beta}. \tag{5.1}$$

For a non zero $w \in \mathbb{R}^d$, using (5.1) we can write

$$\begin{aligned}
\int_0^\varepsilon \langle p_r(*), p_r(w+*)\rangle_{\mathscr{H}} dr &= \int_0^\varepsilon \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_r(x) p_r(y+w)|x-y|^{-\beta} dxdydr \\
&\leq C \int_0^\varepsilon \int_{\mathbb{R}^d} p_r(y+w)|y|^{-\beta} dydr \\
&\leq C\varepsilon |w|^{-\beta}.
\end{aligned}$$

So (2.9) is satisfied with $\eta_2 = 1 > \eta$. For (2.10), using the fact that $\sup_{x\in\mathbb{R}^d} |x|^\alpha e^{-x^2} < \infty$ for any positive α, we have

$$\begin{aligned}
\int_0^\varepsilon \langle |*|^{\kappa_2} p_r(*), p_r(w+*)\rangle_{\mathscr{H}} dr &= \int_0^\varepsilon \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x|^{\kappa_2} p_r(x) p_r(y+w)|x-y|^{-\beta} dxdydr \\
&\leq C \int_0^\varepsilon \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} r^{\frac{\kappa_2}{2}} p_{2r}(x) p_r(y+w)|x-y|^{-\beta} dxdydr \\
&\leq C \int_0^\varepsilon r^{\frac{\kappa_2}{2}} \int_{\mathbb{R}^d} e^{-r|\xi|^2} e^{-\frac{1}{2}r|\xi|^2} |\xi|^{\beta-d} d\xi dr \\
&= C \int_0^\varepsilon r^{\frac{\kappa_2-\beta}{2}} dr = C\varepsilon^{\frac{\kappa_2-\beta}{2}+1}.
\end{aligned}$$

Therefore, (2.10) is satisfied with $\eta_3 = \frac{\kappa_2-\beta}{2} + 1 > \eta$.

(B) *Bessel kernel*. Let $f(x) = \int_0^\infty u^{\frac{\alpha-d-2}{2}} e^{-u} e^{-\frac{|x|^2}{4u}} du$, $d-2 < \alpha < d$. In this case $\mu(d\xi) = c_{\alpha,d}(1+|\xi|^2)^{-\frac{\alpha}{2}} d\xi$. Hypothesis (**H1**) can be easily verified by direct computation. According to [86], (**H2**) is satisfied with $0 < \kappa_1 < \frac{2-d+\alpha}{4}$ and $0 < \kappa_2 < \frac{2-d+\alpha}{2}$.

For (**H3**), we note that, assuming $\varepsilon < 1$,

$$\begin{aligned}\int_0^\varepsilon \int_{\mathbb{R}^d} |\mathscr{F}\Gamma(r)(\xi)|^2 \mu(d\xi)dr &= C\int_0^\varepsilon \int_{\mathbb{R}^d} e^{-r|\xi|^2}(1+|\xi|^2)^{-\frac{\alpha}{2}} d\xi dr \\ &= C\int_0^\varepsilon \int_{\mathbb{R}^d} e^{-|\theta|^2} \frac{r^{\frac{\alpha-d}{2}}}{(|\theta|^2+r)^{\frac{\alpha}{2}}} d\theta dr \\ &\geq C\int_0^\varepsilon r^{\frac{\alpha-d}{2}} dr \int_{\mathbb{R}^d} e^{-|\theta|^2} \frac{1}{(|\theta|^2+1)^{\frac{\alpha}{2}}} d\theta \\ &= C\varepsilon^{\frac{\alpha-d}{2}+1}.\end{aligned}$$

Thus, (**H3**) is satisfied with $\eta = \frac{\alpha-d}{2}+1$. To show (**H4**) we use the fact that for any $x \in \mathbb{R}^d$, $f(x) \leq C|x|^{-d+\alpha}$ (see Proposition 6.1.5 in [40]). Therefore, proceeding as in the case of the Riesz kernel with $\beta = d-\alpha$ we obtain that conditions (2.8), (2.9) and (2.10) in (**H4**) hold, with $\eta_1 = \frac{\alpha-d}{2}+1+\kappa_1$, $\eta_2 = 1$ and $\eta_3 = \frac{\alpha-d}{2}+1+\frac{\kappa_2}{2}$, respectively.

(C) *Fractional kernel*. Let $f(x) = \prod_{j=1}^d |x_j|^{2H_j-2}$, $\frac{1}{2} < H_j < 1$ for $1 \leq j \leq d$ such that $\sum_{j=1}^d H_j > d-1$. First notice that although we have assumed $f(x)$ to be a continuous function on $\mathbb{R}^d \setminus \{0\}$, it is clear that all of our theory still works for this case. Then we note that since $f(x) = \prod_{j=1}^d |x_j|^{2H_j-2}$, we have $\mu(d\xi) = C_H \prod_{j=1}^d |\xi_j|^{1-2H_j} d\xi$, where C_H only depends on $H := (H_1, H_2, \ldots, H_d)$. According to [86], (**H1**) holds and (**H2**) is satisfied for $0 < \kappa_1 < \frac{1}{2}(\sum_{j=1}^d H_j - d + 1)$ and $0 < \kappa_2 < \sum_{j=1}^d H_j - d + 1$. For (**H3**), using the change of variable $\sqrt{t}\xi \to \xi$, we obtain

$$\int_0^\varepsilon \int_{\mathbb{R}^d} |\mathscr{F}\Gamma(t)(\xi)|^2 \mu(d\xi)dt = \int_0^\varepsilon \int_{\mathbb{R}^d} e^{-t|\xi|^2} \prod_{j=1}^d |\xi_j|^{1-2H_j} d\xi dt = C\varepsilon^{\sum_{j=1}^d H_j - d + 1}.$$

Therefore, (**H3**) is verified with $\eta = \sum_{j=1}^d H_j - d + 1$. For (2.8), we can proceed as in checking (**H3**) to get

$$\int_0^\varepsilon r^{\kappa_1} \|\Gamma(r)\|_{\mathscr{H}}^2 dr = C\int_0^\varepsilon r^{\kappa_1 + \sum_{j=1}^d H_j - d} dr = C\varepsilon^{\sum_{j=1}^d H_j - d + 1 + \kappa_1}.$$

So (2.8) is satisfied with $\eta_2 = \sum_{j=1}^d H_j - d + 1 + \kappa_1$ which is strictly greater than η.

To check (2.9), fix a nonzero point $w = (w_1, w_2, \ldots, w_d) \in \mathbb{R}^d$, without loss of generality, we may assume that $w_1 \neq 0$. Then using Fourier transform and (5.1) we have

$$\int_0^\varepsilon \langle \Gamma(r,*), \Gamma(r, w+*) \rangle_{\mathscr{H}} dr = \int_0^\varepsilon \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_r(x) p_r(w+y) \prod_{j=1}^d |x_j - y_j|^{2H_j - 2} dy dx dr$$

$$\begin{aligned}
&= \int_0^\varepsilon \prod_{j=1}^d \left(\int_{\mathbb{R}} \frac{1}{(2\pi r)^{\frac{1}{2}}} e^{-\frac{|x_j|^2}{2r}} \frac{1}{(2\pi r)^{\frac{1}{2}}} e^{-\frac{|w_j + y_j|^2}{2r}} |x_j - y_j|^{2H_j - 2} dy_j dx_j \right) dr \\
&= C \int_0^\varepsilon \left(\int_{\mathbb{R}} \frac{1}{(2\pi r)^{\frac{1}{2}}} e^{-\frac{|x_1|^2}{2r}} \frac{1}{(2\pi r)^{\frac{1}{2}}} e^{-\frac{|w_1 + y_1|^2}{2r}} |x_1 - y_1|^{2H_1 - 2} dy_1 dx_1 \right) \\
&\quad \times \prod_{j=2}^d \left(\int_{\mathbb{R}} e^{-r|\xi_j|^2} e^{-iw_j \xi_j} |\xi_j|^{1-2H_j} d\xi_j \right) dr \\
&\leq C \int_0^\varepsilon \left(\int_{\mathbb{R}} \frac{1}{(2\pi r)^{\frac{1}{2}}} e^{-\frac{|x_1|^2}{2r}} \frac{1}{(2\pi r)^{\frac{1}{2}}} e^{-\frac{|w_1 + y_1|^2}{2r}} |x_1 - y_1|^{2H_1 - 2} dy_1 dx_1 \right) \\
&\quad \times \prod_{j=2}^d \left(\int_{\mathbb{R}} e^{-r|\xi_j|^2} |\xi_j|^{1-2H_j} d\xi_j \right) dr \\
&\leq C |w_1|^{2H_1 - 2} \int_0^\varepsilon r^{\sum_{j=2}^d H_j - d + 1} dr = C \varepsilon^{\sum_{j=2}^d H_j - d + 2},
\end{aligned}$$

where in the last inequality we have used the change of variable $\sqrt{r}\xi \to \xi$. So (2.9) is satisfied with $\eta_1 = \min_{1 \leq k \leq d} (\sum_{j \neq k}^d H_j - d + 2)$, which is strictly greater than η. For (2.10), fixing again a non zero element $w \in \mathbb{R}^d$ and using the bound $|x|^\alpha p_r(x) \leq C r^{\frac{\alpha}{2}} p_{2r}(x)$, for all $x \in \mathbb{R}^d$, we have

$$\begin{aligned}
&\int_0^\varepsilon \langle |*|^{\kappa_2} \Gamma(r,*), \Gamma(r, w+*) \rangle_{\mathscr{H}} dr \\
&= \int_0^\varepsilon \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x|^{\kappa_2} p_r(x) p_r(y+w) \prod_{j=1}^d |x_j - y_j|^{2H_j - 2} dx dy dr \\
&\leq C \int_0^\varepsilon \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} r^{\frac{\kappa_2}{2}} p_{2r}(x) p_r(y+w) \prod_{j=1}^d |x_j - y_j|^{2H_j - 2} dx dy dr
\end{aligned}$$

$$
\begin{aligned}
&\leq C\int_0^{\varepsilon}\int_{\mathbb{R}^d} r^{\frac{\kappa_2}{2}} e^{-\frac{3r}{2}|\xi|^2}\prod_{j=1}^d |\xi_j|^{1-2H_j} d\xi dr \\
&= C\int_0^{\varepsilon} r^{\frac{\kappa_2}{2}+\sum_{j=1}^d H_j - d} dr = C\varepsilon^{\frac{\kappa_2}{2}+\sum_{j=1}^d H_j-d+1}.
\end{aligned}
$$

So (2.10) is satisfied with $\eta_3 = \frac{\kappa_2}{2} + \sum_{j=1}^d H_j - d + 1$, which is strictly greater than η.

5.5.2 Wave equation

Let $\Gamma_d(t,dx)$ be the fundamental solution to the wave equation on $\mathbb{R}^d$, for $d = 1,2,3$, i.e., $\Gamma_1(t,dx) = \frac{1}{2}\mathbf{1}_{\{|x|<t\}}dx$, $\Gamma_2(t,dx) = \frac{1}{2\pi}(t^2 - |x|^2)_+^{-1/2}dx$, $\Gamma_3(t,dx) = \frac{1}{4\pi t}\sigma_t(dx)$, where σ_t denotes the surface measure on the two-dimensional sphere of radius t. We recall that the Fourier transform of $\Gamma_d(t,dx)$ is given by

$$
\mathscr{F}\Gamma_d(t)(\xi) = \frac{\sin(t|\xi|)}{|\xi|}.
$$

(A) *Riesz kernel*. Let $f(x) = |x|^{-\beta}$ with $0 < \beta < 2 \wedge d$. It is known that hypothesis (**H1**)is satisfied. According to [51], (**H2**) is satisfied with $0 < \kappa_1 = \kappa_2 < \frac{2-\beta}{2}$. In [72] it is proved that condition (**H3**) is satisfied for $\eta = 3 - \beta$ and (2.8) holds with $\eta_1 = \kappa_1 + 3 - \beta > \eta$. To show (2.9), we fix $w \neq 0$, and taking ε such that $4\varepsilon < |w|$ we get $\frac{|w|}{2} \leq |x-y| \leq \frac{3|w|}{2}$ if $|x| \leq \varepsilon$ and $|w+y| \leq \varepsilon$. Then, $|x-y|^{-\beta}$ is bounded by some constant C depending on $|w|$. Hence we have

$$
\begin{aligned}
\int_0^{\varepsilon}\langle \Gamma_d(r,*), \Gamma_d(r,w+*)\rangle_{\mathscr{H}} dr &= \int_0^{\varepsilon}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Gamma_d(r,dx)\Gamma_d(r,w+dy)|x-y|^{-\beta}dr \\
&\leq C_w \int_0^{\varepsilon}\int_{\mathbb{R}^d}\Gamma_d(r,dx)\int_{\mathbb{R}^d}\Gamma_d(r,w+dy)dr \\
&\leq C_w\int_0^{\varepsilon} r^2 dr \leq C_w\varepsilon^3.
\end{aligned}
$$

So (2.9) is satisfied with $\eta_2 = 3 > \eta$. For (2.10), any fixed $w \in \mathbb{R}^d$, using again the same arguments, we have

$$
\begin{aligned}
& \int_0^{\varepsilon} \langle |*|^{\kappa_2}\Gamma_d(r,*), \Gamma_d(r,w+*)\rangle_{\mathscr{H}} dr \\
= & \int_0^{\varepsilon} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |x|^{\kappa_2}\Gamma_d(r,dx)\Gamma_d(r,w+dy)|x-y|^{-\beta} dr \\
\leq & \int_0^{\varepsilon} |r|^{\kappa_2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Gamma_d(r,dx)\Gamma_d(r,w+dy)|x-y|^{-\beta} dr \\
\leq & \; C\varepsilon^{\kappa_2+3-\beta}.
\end{aligned}
$$

Therefore, (2.10) is satisfied with $\eta_3 = \kappa_2 + 3 - \beta > \eta$.

(B) *Bessel kernel*. Let $f(x) = \int_0^\infty u^{\frac{\alpha-d-2}{2}} e^{-u} e^{-\frac{|x|^2}{4u}} du$, $\max(d-2,0) < \alpha < d$. According to section 3 in [72] and [51], (**H1**) holds and (**H2**) is satisfied with $0 < \kappa_1 = \kappa_2 < \frac{\alpha-d+2}{2}$. Making the change of variable $r\xi \to \xi$ and assuming $\varepsilon < 1$, we get that

$$
\begin{aligned}
\int_0^{\varepsilon}\int_{\mathbb{R}^d} |\mathscr{F}\Gamma_d(r)(\xi)|^2 \mu(d\xi) dr &= C\int_0^{\varepsilon}\int_{\mathbb{R}^d} \frac{\sin^2(r|\xi|)}{|\xi|^2}(|\xi|^2+1)^{-\frac{\alpha}{2}} d\xi dr \\
&= C\int_0^{\varepsilon}\int_{\mathbb{R}^d} \frac{\sin^2|\xi|}{|\xi|^2}\frac{r^{\alpha+2-d}}{(|\xi|^2+r^2)^{\frac{\alpha}{2}}} d\xi dr \\
&\geq C\int_0^{\varepsilon} r^{\alpha+2-d} dr \int_{\mathbb{R}^d} \frac{\sin^2|\xi|}{|\xi|^2}\frac{1}{(|\xi|^2+1)^{\frac{\alpha}{2}}} d\xi \\
&= C\varepsilon^{\alpha+3-d}.
\end{aligned}
$$

Therefore, condition (**H3**) is satisfied for $\eta = \alpha + 3 - d$. To show (**H4**) as in the case of the heat equation we use the fact that for any $x \in \mathbb{R}^d$, $f(x) \leq C|x|^{-d+\alpha}$. Therefore, proceeding as in the case of the Riesz kernel with $\beta = d - \alpha$ we obtain that conditions (2.9), (2.8) and (2.10) in (**H4**) hold, with $\eta_1 = \alpha + 3 - d + \kappa_1$, $\eta_2 = 3$ and $\eta_3 = \alpha + 3 - d + \kappa_2$, respectively.

(C) *Fractional kernel*. Let $f(x) = \prod_{j=1}^{d} |x_j|^{2H_j-2}$, $\frac{1}{2} < H_j < 1$ for $1 \leq j \leq d$ such that $\sum_{j=1}^{d} H_j > d-1$. Hypothesis (**H1**) is verified by direct calculation. By Section 3 in [72], (**H2**) holds when $d = 1$ with $\kappa_1, \kappa_2 \in (0, H_1)$ and when $d = 2$, it is satisfied for $\kappa_1, \kappa_2 \in (0, H_1 + H_2 - 1)$. By Theorem 6.1 in [51], when $d = 3$ (**H2**) is satisfied with $\kappa_1, \kappa_2 \in (0, \min(H_1 + H_2 + H_3 - 2, H_1 - \frac{1}{2}, H_2 - \frac{1}{2}, H_3 - \frac{1}{2}))$. For (**H3**), direct calculation and the change of variable $t\xi \to \xi$ yields

$$\begin{aligned}
& \int_0^\varepsilon \int_{\mathbb{R}^d} |\mathscr{F}\Gamma_d(t)(\xi)|^2 \mu(d\xi) dt \\
= \;& C \int_0^\varepsilon \int_{\mathbb{R}^d} \frac{(\sin(t|\xi|))^2}{|\xi|^2} \prod_{j=1}^{d} |\xi_j|^{1-2H_j} d\xi dt \\
= \;& C \int_0^\varepsilon t^{2\sum_{j=1}^{d} H_j - 2d + 2} dt \int_{\mathbb{R}^d} \frac{(\sin(|\xi|))^2}{|\xi|^2} \prod_{j=1}^{d} |\xi_j|^{1-2H_j} d\xi \\
= \;& C\varepsilon^{2\sum_{j=1}^{d} H_j - 2d + 3}.
\end{aligned}$$

So (**H3**) is satisfied with $\eta = 2\sum_{j=1}^{d} H_j - 2d + 3$. For (**H4**), we will check (2.8) and (2.10) first. For (2.8), proceeding as before,

$$\begin{aligned}
\int_0^\varepsilon r^{\kappa_1} \|\Gamma_d(r)\|_{\mathscr{H}}^2 dr &= \int_0^\varepsilon \int_{\mathbb{R}^d} r^{\kappa_1} \frac{(\sin(r|\xi|))^2}{|\xi|^2} \prod_{j=1}^{d} |\xi_j|^{1-2H_j} d\xi dr \\
&= C \int_0^\varepsilon r^{\kappa_1 + 2\sum_{j=1}^{d} H_j - 2d + 2} dr = C\varepsilon^{\kappa_1 + 2\sum_{j=1}^{d} H_j - 2d + 3}.
\end{aligned}$$

So (2.8) is satisfied with $\eta_1 = \kappa_1 + 2\sum_{j=1}^{d} H_j - 2d + 3$, which is strictly greater than η. For (2.10), noting that the support of $\Gamma_d(r, *)$ is contained in the ball centered at the origin with radius r, we get

$$\begin{aligned}
\int_0^\varepsilon \langle |*|^{\kappa_2} \Gamma_d(r,*), \Gamma_d(r, \tilde{w} + *) \rangle_{\mathscr{H}} dr &\leq \int_0^\varepsilon r^{\kappa_2} \langle \Gamma_d(r,*), \Gamma_d(r, \tilde{w} + *) \rangle_{\mathscr{H}} dr \\
&\leq \int_0^\varepsilon r^{\kappa_2} \int_{\mathbb{R}^d} \frac{(\sin(r|\xi|))^2}{|\xi|^2} \prod_{j=1}^{d} |\xi_j|^{1-2H_j} d\xi dr
\end{aligned}$$

$$= C\varepsilon^{\kappa_2+2\sum_{j=1}^d H_j-2d+3}.$$

So (2.10) is satisfied with $\eta_3 = \kappa_2 + 2\sum_{j=1}^d H_j - 2d + 3$, which is strictly greater than η. For (2.9), we need to treat the cases $d = 1, 2, 3$ separately. When $d = 1$, fix $w \neq 0$. We have

$$\int_0^\varepsilon \langle \Gamma_1(r,*), \Gamma_1(r, w+*)\rangle_{\mathscr{H}} dr = \frac{1}{4}\int_0^\varepsilon \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{1}_{\{|x|<r\}} |x-y|^{2H_1-2} \mathbf{1}_{\{|y+w|<r\}} dydxdr.$$

When ε is small enough, we need to have $|x-y| \geq C$ for some positive constant C for the above integrand to be non zero. Hence,

$$\begin{aligned}
\int_0^\varepsilon \langle \Gamma_1(r,*), \Gamma_1(r, w+*)\rangle_{\mathscr{H}} dr &\leq C\int_0^\varepsilon \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{1}_{\{|x|<r\}} \mathbf{1}_{\{|y+w|<r\}} dydxdr \\
&= C\int_0^\varepsilon r^2 dr = C\varepsilon^3,
\end{aligned}$$

and when $d = 1$ (2.9) is satisfied with $\eta_2 = 3$, which is strictly greater than η.

When $d = 2$, fix a nonzero point $w = (w_1, w_2)$. Without loss of generality, we may assume w_1 is not zero. We have

$$\begin{aligned}
&\int_0^\varepsilon \langle \Gamma_2(r,*), \Gamma_2(r, w+*)\rangle_{\mathscr{H}} dr \\
= \; &\frac{1}{4\pi^2}\int_0^\varepsilon \int_{|x|<r}\int_{|y+w|<r} \frac{1}{\sqrt{r^2-|x|^2}} |x_1-y_1|^{2H_1-2} |x_2-y_2|^{2H_2-2} \\
&\times \frac{1}{\sqrt{r^2-|y+w|^2}} dxdydr.
\end{aligned}$$

Again, if ε is small enough, we must have $|x_1 - y_1| > C$ for some positive constant C for the above integral to be non zero. Hence, using the Fourier transform we obtain

$$\int_0^\varepsilon \langle \Gamma_2(r,*), \Gamma_2(r, w+*)\rangle_{\mathscr{H}} dr$$

$$
\begin{aligned}
&\le C\int_0^\varepsilon\int_{|x|<r}\int_{|y+w|<r}\frac{1}{\sqrt{r^2-|x|^2}}|x_2-y_2|^{2H_2-2}\frac{1}{\sqrt{r^2-|y+w|^2}}dxdydr\\
&= C\lim_{\delta\to0}\int_0^\varepsilon\int_{|x|<r}\int_{|y+w|<r}\frac{1}{\sqrt{r^2-|x|^2}}e^{-\frac{\delta}{2}|x_1-y_1|^2}|x_2-y_2|^{2H_2-2}\frac{1}{\sqrt{r^2-|y+w|^2}}dxdydr\\
&= C\lim_{\delta\to0}\int_0^\varepsilon\int_{\mathbb{R}^2}\frac{(\sin(r|\xi|))^2}{|\xi|^2}p_\delta(\xi_1)|\xi_2|^{1-2H_2}e^{-iw\cdot\xi}d\xi dr\\
&\le C\lim_{\delta\to0}\int_0^\varepsilon\int_{\mathbb{R}^2}\frac{(\sin(r|\xi|))^2}{|\xi|^2}p_\delta(\xi_1)|\xi_2|^{1-2H_2}d\xi dr\\
&= C\int_0^\varepsilon\int_{\mathbb{R}}\frac{(\sin(r|\xi_2|))^2}{|\xi_2|^2}|\xi_2|^{1-2H_2}d\xi_2dr = C\varepsilon^{2H_2+1}.
\end{aligned}
$$

Therefore, (2.9) is satisfied with $\eta_2=\min(2H_1+1,2H_2+1)$, which is strictly greater than η.

When $d=3$, fix a nonzero $w=(w_1,w_2,w_3)\in\mathbb{R}^3$, without loss of generality, we may assume that $w_1\neq0$. We have

$$
\begin{aligned}
&\int_0^\varepsilon\langle\Gamma_3(r,*),\Gamma_3(r,w+*)\rangle_{\mathscr{H}}dr\\
=\ &\int_0^\varepsilon\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Gamma_3(r,dx)\Gamma_3(r,w+dy)\prod_{j=1}^3|x_j-y_j|^{2H_j-2}dr.
\end{aligned}
$$

Again, when ε is small enough, to make x and $w+y$ in the support of the measure $\Gamma_3(r)$, we must have $|x_1-y_1|>C$ for some positive constant C. So

$$
\begin{aligned}
&\int_0^\varepsilon\langle\Gamma_3(r,*),\Gamma_3(r,w+*)\rangle_{\mathscr{H}}dr\\
&\le C\int_0^\varepsilon\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Gamma_3(r,dx)\Gamma_3(r,w+dy)\prod_{j=2}^3|x_j-y_j|^{2H_j-2}dr\\
&= C\lim_{\delta\to0}\int_0^\varepsilon\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Gamma_3(r,dx)\Gamma_3(r,w+dy)e^{-\frac{\delta}{2}|x_1-y_1|^2}\prod_{j=2}^3|x_j-y_j|^{2H_j-2}dr\\
&\le C\lim_{\delta\to0}\int_0^\varepsilon\int_{\mathbb{R}^3}\frac{(\sin r|\xi|)^2}{|\xi|^2}p_\delta(\xi_1)\prod_{j=2}^3|\xi_j|^{1-2H_j}d\xi dr
\end{aligned}
$$

$$
\begin{aligned}
&= C\int_0^{\varepsilon}\int_{\mathbb{R}^2}\frac{(\sin(r|(\xi_2,\xi_3)|))^2}{(\xi_2^2+\xi_3^2)}\prod_{j=2}^{3}|\xi_j|^{1-2H_j}d\xi_2 d\xi_3 dr \\
&= C\int_0^{\varepsilon} r^{2(H_2+H_3)-2}dr = C\varepsilon^{2(H_2+H_3)-1},
\end{aligned}
$$

and (2.9) is satisfied with $\eta_2 = \min(2(H_2+H_3)-1, 2(H_1+H_3)-1, 2(H_1+H_2)-1)$, which is strictly greater than η.

Chapter 6

Stochastic heat equation with rough dependence in space

This chapter studies the one-dimensional stochastic heat equation driven by a Gaussian noise which is white in time and which has the covariance of a fractional Brownian motion with Hurst parameter $H \in \left(\frac{1}{4}, \frac{1}{2}\right)$ in the space variable. The existence and uniqueness of the solution u are proved assuming the nonlinear coefficient $\sigma(u)$ is Lipschitz continuous. In the case of a multiplicative noise, that is, $\sigma(u) = u$, we derive the Wiener chaos expansion of the solution and a Feynman-Kac formula for the moments of u. These results allow us to establish sharp lower and upper asymptotic bounds for $\mathbf{E}[|u(t,x)|^n]$.

6.1 Introduction

In this chapter we are interested in the one-dimensional stochastic partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\kappa}{2}\frac{\partial^2 u}{\partial x^2} + \sigma(u)\dot{W}\,, \quad t \geq 0, \quad x \in \mathbb{R}\,, \tag{1.1}$$

where W is a centered Gaussian process with covariance given by

$$\mathbf{E}\left[W(s,x)W(t,y)\right] = \frac{1}{2}\left(|x|^{2H} + |y|^{2H} - |x-y|^{2H}\right)(s \wedge t), \tag{1.2}$$

with $\frac{1}{4} < H < \frac{1}{2}$. That is, W is a standard Brownian motion in time and a fractional Brownian motion with Hurst parameter H in the space variable. For this stochastic heat equation with a rough noise in space, understood in the Itô sense, our aim is twofold: on one hand, for a general coefficient σ with $\sigma(0) = 0$ and satisfying some regularity assumptions, we will obtain the existence and uniqueness of the solution. On the other hand, we shall further investigate the special relevant case $\sigma(u) = u$. We now detail those two main points.

(1) Since the pioneering work by Peszat-Zabczyk [77] and Dalang (see [24]), there has been a lot of interest in stochastic partial differential equations driven by a Brownian motion in time with spatial homogeneous covariance. After more than a decade of investigations, the standard assumptions on W under which existence and uniqueness hold take the following form:

(i) $\mathbf{E}[W(s,x)W(t,y)] = \Lambda(x-y)(s \wedge t)$, where Λ is a positive distribution of positive type.

(ii) The Fourier transform of the spatial covariance Λ is a tempered measure μ that satisfies the integrability condition $\int_{\mathbb{R}} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty$.

In case of the covariance (1.2) under consideration, one can easily compute the measure μ, whose explicit expression is $\mu(d\xi) = c_1(H)|\xi|^{1-2H}d\xi$, where $c_1(H)$ is a constant depending on H (see expression (2.2) below). In addition, it is readily checked that μ fulfills the condition $\int_{\mathbb{R}} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty$ for all $H \in (0,1)$. However, the corresponding covariance Λ is a distribution which fails to be positive when $H < 1/2$, and the covariance of two stochastic integrals with respect to $\dot{W}$ is expressed in terms of fractional derivatives. For this reason, the standard methodology used in the classical references [24, 31, 77] to handle homogeneous spatial covariances does not apply to our case of interest.

In a recent paper, Balan, Jolis and Quer-Sardanyons [3] proved the existence of a unique mild solution for equation (1.1) in the case $\sigma(u) = au + b$, using techniques of Fourier analysis. The current paper can be seen as another step forward in this direction. Indeed, we first give some a priori estimates for the moments and Hölder continuity, both in space and time. Then following Gyöngy [44] Summarizing, we get a complete basic picture of the solution to equation (1.1) as long as σ is a Lipschitz coefficient and $H > 1/4$. This boundary $H = 1/4$ is worthwhile noting, since it is also the threshold under which rough differential equations driven by a fractional Brownian motion are ill-defined.

(2) The particular case $\sigma(u) = u$ in equation (1.1) deserves a special interest. Indeed, this linear equation turns out to be a continuous version of the parabolic Anderson model, and is related to challenging systems in random environment like KPZ equation [46, 4] or polymers [1, 9]. The localization and intermittency properties of the linear version of (1.1) have thus been thoroughly studied for equations driven by a Brownian motion (see [58] for a nice survey), while a recent trend consists in extending this kind of result to equations driven by very general Gaussian noises [17, 53, 55, 56].

Nevertheless, the rough noise W with covariance (1.2) presented here is uncovered by the aforementioned references, and we wish to fill this gap. We will thus particularize our setting to $\sigma = \mathrm{Id}$, and first go back to the existence and uniqueness problem. Indeed, in this linear case, one can implement a rather simple procedure involving Fourier transform, as well as a chaos expansion technique, in order to achieve existence and uniqueness of the solution to (1.1). Since this point of view is interesting in its own right and short enough, we develop it at Section 6.5. Moreover in this case we can consider more general initial conditions.

We then move to a Feynman-Kac type representation for the equation: following the approach introduced in [55, 53], we obtain an explicit formula for the kernels of the Wiener chaos expansion and we show its convergence. In fact we cannot expect a Feynman-Kac formula for the solution, because the covariance is rougher that the space-time white noise case, and this type of formula requires smoother covariances (see, for instance, [56]). However, by means of Fourier analysis techniques as in [55, 53], we have been able to obtain a Feynman-Kac formula for the moments that involves a fractional derivative of the Brownian local time.

Finally, the previous considerations allow to handle, in the last section of the paper, the intermittency properties of the solution. More precisely, we show sharp lower bounds for the moments of the solution of the form $\mathbf{E}[|u(t,x)|^n] \leq \exp(Cn^{1+\frac{1}{H}}t)$, for all $t \geq 0$, $x \in \mathbb{R}$ and $n \geq 2$. This bounds entail the intermittency phenomenon and match the corresponding estimates for the case $H > \frac{1}{2}$ obtained in [53].

6.2 Preliminaries

6.2.1 Noise structure and stochastic integration

Our noise W can be seen as a Brownian motion with values in an infinite dimensional Hilbert space. One might thus think that the stochastic integration theory with respect to W can be handled by classical theories (see e.g [11, 24, 32]). However, the spatial covariance function of W is not positive whenever $H < 1/2$ (as mentioned in the introduction), and W thus lies outside the scope of application of these classical references.

Due to this fact, we provide some details about the construction of a stochastic integral with respect to our noise.

Let us start by introducing our basic notation on Fourier transforms of functions. The space of real valued infinitely differentiable functions with compact support on $\mathbb{R}$ is denoted by $\mathscr{D}$. The space of Schwartz functions is denoted by $\mathscr{S}$. Its dual, the space of tempered distributions, is $\mathscr{S}'$. The Fourier transform is defined with the normalization

$$\mathscr{F}u(\xi) = \int_{\mathbb{R}} e^{-i\langle \xi, x\rangle} u(x)dx,$$

so that the inverse Fourier transform is given by $\mathscr{F}^{-1}u(\xi) = (2\pi)^{-1}\mathscr{F}u(-\xi)$.

Taking into account the spectral representation of the covariance function of the fractional Brownian motion in the case $H < \frac{1}{2}$ proved in [78, Theorem 3.1], we represent our noise W by a Gaussian family $\{W(\varphi); \varphi \in \mathscr{D}([0,\infty) \times \mathbb{R})\}$ defined on a complete probability space $(\Omega, \mathscr{F}, \mathbf{P})$, whose covariance structure is given by

$$E\left[W(\varphi)W(\psi)\right] = c_1(H)\int_{\mathbb{R}_+ \times \mathbb{R}} \mathscr{F}\varphi(s,\xi)\,\overline{\mathscr{F}\psi(s,\xi)}\,|\xi|^{1-2H}\,dsd\xi, \qquad (2.1)$$

where the Fourier transforms $\mathscr{F}\varphi, \mathscr{F}\psi$ are understood as Fourier transforms in space only and

$$c_1(H) = \frac{1}{2\pi}\Gamma(2H+1)\sin(\pi H). \qquad (2.2)$$

The inner product appearing in (2.1) can be expressed in terms of fractional derivatives. Let $\beta \in (0,1)$. Define (see [78]) the Marchaud fractional derivative D_-^{β} of order β of a function $\varphi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ as

$$D_-^{\beta}\varphi(s,x) = \lim_{\varepsilon \to 0} D_{-,\varepsilon}^{\beta}\varphi(s,x), \qquad (2.3)$$

where

$$D^{\beta}_{-,\varepsilon}\varphi(s,x) = \frac{\beta}{\Gamma(1-\beta)}\int_{\varepsilon}^{\infty}\frac{\varphi(s,x)-\varphi(s,x+y)}{y^{1+\beta}}dy,$$

and define the fractional integral of order α of a function $\psi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ by

$$I^{\beta}_{-}\psi(s,x) := \frac{1}{\Gamma(\beta)}\int_{\mathbb{R}}\psi(s,u)(x-u)^{\beta-1}_{-}du,$$

where $(x-u)_-$ means $\max(0,u-x)$. Note that here the fractional differentiation and integration are only with respect to space variables. Then for our noise it is known (cf. [78] for further details) that:

$$E[W(\varphi)W(\psi)] = c_2(H)\int_{\mathbb{R}_+\times\mathbb{R}} D^{\frac{1}{2}-H}_{-}\varphi(s,x)D^{\frac{1}{2}-H}_{-}\psi(s,x)dsdx, \tag{2.4}$$

where

$$c_2(H) = \left[\Gamma\left(H+\frac{1}{2}\right)\right]^2\left(\int_0^{\infty}\left((1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}\right)^2 ds + \frac{1}{2H}\right)^{-1}.$$

Observe that if $\varphi = I^{\beta}_{-}\psi$ for some $\psi \in L^2(\mathbb{R}_+ \times \mathbb{R})$, then by Theorem 6.1 in [84] we have

$$D^{\beta}_{-}\varphi = D^{\beta}_{-}(I^{\beta}_{-}\psi) = \psi$$

and, hence,

$$\int_{\mathbb{R}_+\times\mathbb{R}}\left[D^{\beta}_{-}\varphi(x,s)\right]^2 dsdx = \int_{\mathbb{R}_+\times\mathbb{R}}\psi^2(s,x)dsdx < \infty.$$

Based on the previous observation and relation (2.4), we introduce a new set of functions. Indeed, let $\dot{H}^{\beta}$ be the class of functions $\varphi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ such that there exists $\psi \in L^2(\mathbb{R}_+ \times \mathbb{R})$ satisfying $\varphi(s,x) = I^{\beta}_{-}\psi(s,x)$. The relation between $\dot{H}^{\beta}$ and our noise W which corresponds to the specific case $\beta = \frac{1}{2} - H$ is given in the following proposition.

Proposition 6.1. *The class of functions $\dot{H}^\beta$ is a linear space with the inner product*

$$\langle \varphi, \psi \rangle_{\dot{H}^\beta} := c_{2,\beta} \int_{\mathbb{R}_+ \times \mathbb{R}} D_-^\beta \varphi(s,x) D_-^\beta \psi(s,x) ds dx. \tag{2.5}$$

The space $\dot{H}^\beta$ is complete and $\mathscr{D}([0,\infty) \times \mathbb{R})$ is dense in $\dot{H}^\beta$. Moreover if $\dot{H}_0^\beta$ denotes the class of functions $\varphi \in L^2(\mathbb{R}_+ \times \mathbb{R})$ such that $\int_{\mathbb{R}_+ \times \mathbb{R}} |\mathscr{F}\varphi(s,\xi)|^2 |\xi|^{2\beta} d\xi ds < \infty$, then $\dot{H}_0^\beta$ is not complete and the inclusion $\dot{H}_0^\beta \subset \dot{H}^\beta$ is strict. Also for any $\varphi, \psi \in \dot{H}_0^\beta$,

$$\langle \varphi, \psi \rangle_{\dot{H}^\beta} = c_{1,\beta} \int_{\mathbb{R}_+ \times \mathbb{R}} \mathscr{F}\varphi(s,\xi) \overline{\mathscr{F}\psi(s,\xi)} |\xi|^{2\beta} d\xi ds, \tag{2.6}$$

For the proof of this proposition, we refer to [78]. Note that in [78], the functions considered there are from $\mathbb{R}$ to $\mathbb{R}$, but by scrutinizing the proofs we see that it is easy to be extended to our case, i.e. the functions from $\mathbb{R}_+ \times \mathbb{R}$ to $\mathbb{R}$. We omit the details. For later reference, if the function f under consideration only depends on the space variable, then the norm $\dot{H}^\beta$ is similarly defined, i.e.,

$$\|f\|_{\dot{H}^\beta}^2 = c_{2,\beta} \int_{\mathbb{R}} |D_-^\beta f(x)|^2 dx.$$

From the propositions above we see that the Gaussian family W can be extended as an isonormal Gaussian process $W = \{W(h); h \in \dot{H}^{\frac{1}{2}-H}\}$ indexed by the Hilbert space $\dot{H}^{\frac{1}{2}-H}$. Actually in most cases in the sequel, we will take $\beta = \frac{1}{2} - H$, and we will abbreviate $\dot{H}^{\frac{1}{2}-H}$ as $\dot{H}$.

Let us now turn to the stochastic integration with respect to W. Since we are handling a Brownian motion in time, one can start by integrating elementary processes.

Definition 6.2. *Let $\mathscr{F}_t$ be the σ-algebra generated by W up to time t. An elementary process g is given by*

$$g(s,x) = \sum_{i=1}^{n} \sum_{j=1}^{m} X_{i,j} \mathbf{1}_{(a_i,b_i]}(s) \mathbf{1}_{(h_j,l_j]}(x),$$

where n and m are finite positive integers, $-\infty < a_1 < b_1 < \cdots < a_n < b_n < \infty$, $h_j < l_j$ and $X_{i,j}$ are $\mathscr{F}_{a_i}$-measurable random variables for $i = 1, \ldots, n$. The integral of such a process with respect to W is defined as

$$\begin{aligned} \int_{\mathbb{R}_+} \int_{\mathbb{R}} g(s,x) W(ds,dx) &= \sum_{i=1}^{n} \sum_{j=1}^{m} X_{i,j} W\left(\mathbf{1}_{(a_i,b_i]} \otimes \mathbf{1}_{(h_j,l_j]}\right) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} X_{i,j} \left[W(b_i,l_j) - W(a_i,l_j) - W(b_i,h_j) + W(a_i,h_j)\right]. \end{aligned} \tag{2.7}$$

We can now extend the notion of integral with respect to W to a broad class of adapted processes.

Proposition 6.3. *Let Λ_H be the space of predictable processes g defined on $\mathbb{R}_+ \times \mathbb{R}$ such that almost surely $g \in \dot{H}$ and $\mathbf{E}[\|g\|_{\dot{H}}^2] < \infty$. Then, we have:*

(i) *The space of elementary processes defined in Definition 6.2 is dense in Λ_H.*

(ii) *For $g \in \Lambda_H$, the stochastic integral $\int_{\mathbb{R}_+} \int_{\mathbb{R}} g(s,x) W(ds,dx)$ is defined as a $L^2(\Omega)$-limit of Riemann sums along elementary processes approximating g, and we have:*

$$E\left[\left(\int_{\mathbb{R}_+} \int_{\mathbb{R}} g(s,x) W(ds,dx)\right)^2\right] = E\left[\|g\|_{\dot{H}}^2\right]. \tag{2.8}$$

Proof. Let us prove item (i). To this aim, consider $g \in \Lambda_{\mathscr{H}}$ and set $\varphi(t,x) = D_{-}^{1/2-H} g(t,x)$. According to the definition of Λ_H, we have: $\mathbf{E}[\int_{\mathbb{R}_+} \int_{\mathbb{R}} |\varphi(s,x)|^2 dx ds] < \infty$. Then we will show that $g(t,x)$ can be approximated by elementary processes in $L^2(\Omega; \dot{H})$ in three steps.

Step 1. Let $\mathscr{H}_1$ be the class of functions $f \in L^2(\mathbb{R})$, such that there exists $h \in L^2(\mathbb{R})$ satisfying $f = I_-^{\frac{1}{2}-H} h$. We show that the process g can be approximated in $L^2(\Omega; \dot{H})$ by functions with the form

$$\psi_m(s,x;\omega) = \sum_{i=1}^{m} \mathbf{1}_{(a_i,b_i]}(s)\phi_i(x;\omega)\,, \tag{2.9}$$

where for each i, $\phi_i(x;\omega)$ is an $\mathscr{F}_{a_i}$-measurable $L^2(\Omega;\mathscr{H}_1)$-valued random field. To see this, we just set

$$\psi_m(s,x;\omega) = \sum_{k=1}^{m2^m} \mathbf{1}_{((k-1)2^{-m},k2^{-m}]}(s) 2^m \int_{(k-1)2^{-m}}^{k2^{-m}} g(r,x;\omega)dr\,,$$

and we easily get that $D_-^{\frac{1}{2}-H}\psi_m(s,x;\omega) \to D_-^{\frac{1}{2}-H} g(s,x;\omega)$ in $L^2(\Omega\times\mathbb{R}_+\times\mathbb{R})$ as m tends to infinity. In this way we get the desired approximation.

Step 2. We show that each $\psi_m(s,x;\omega)$ of the form (2.9) can be approximated by a linear combination of elements of the form $X\mathbf{1}_{(a,b]}(s)h(x)$, in $L^2(\Omega;\dot{H})$. Indeed, for each $\phi_i(x;\omega)$, we notice that since

$$E\int_{\mathbb{R}} |D_-^{\frac{1}{2}-H}\phi_i(x;\omega)|^2 dx < \infty\,,$$

$D_-^{\frac{1}{2}-H}\phi_i(x;\omega)$ can be approximated by functions with the form

$$\sum_{j=1}^{N} X_j h_j(x)$$

in $L^2(\Omega;L^2(\mathbb{R}))$, where each X_j is an $\mathscr{F}_{a_i}$-measurable random variable and each h_j is an element in $L^2(\mathbb{R})$. Thus, it is easily seen that $\phi_i(x;\omega)$ can be approximated a sequence of functions of the form

$$\sum_{j=1}^{N} X_j I_{-}^{\frac{1}{2}-H} h_j(x)\,.$$

So we conclude that $\psi_m(s,x;\omega)$ can be approximated by

$$\sum_{i=1}^{m} \mathbf{1}_{(a_i,b_i]}(s) \sum_{j=1}^{N} X_{i,j} I_{-}^{\frac{1}{2}-H} h_{i,j}(x)$$

in $L^2(\Omega;\mathscr{H})$, where $X_{i,j}$ are $\mathscr{F}_{a_i}$-measurable random variables and $h_{i,j} \in L^2(\mathbb{R})$.

Step 3. Owing to Theorem 3.3 in [78] we know that

$$\mathrm{Span}\left\{ D_{-}^{\frac{1}{2}-H} \mathbf{1}_{(h,l]}; h < l \right\}$$

is dense in $\Lambda_0 := \{D_{-}^{\frac{1}{2}-H} f : f \in \mathscr{H}_1\}$, in $L^2(\mathbb{R})$ norm. This observation and the results in Step 2 immediately shows that $\psi_m(s,x;\omega)$ can be approximated by elementary processes in $L^2(\Omega;\dot{H})$. This completes the proof.

□

With this stochastic integral defined, we are ready to state the definition of the solution to equation (1.1).

Definition 6.4. *Let $u = \{u(t,x), 0 \le t \le T, x \in \mathbb{R}\}$ be a real-valued predictable stochastic process such that for all $t \in [0,T]$ and $x \in \mathbb{R}$ the process $\{p_{t-s}(x-y)\sigma(u(t,y))\mathbf{1}_{[0,t]}(s), 0 \le s \le t, y \in \mathbb{R}\}$ is an element of Λ_H, where $p_t(x)$ is the heat kernel on the real line. We say that u is a mild solution of (1.1) if for all $t \in [0,T]$ and $x \in \mathbb{R}$ we have:*

$$u(t,x) = p_t u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u(s,y))W(ds,dy) \quad a.s. \tag{2.10}$$

6.2.2 Elements of Malliavin calculus

We recall that the Gaussian family W can be extended to $\dot{H}$ and this produces an isonormal Gaussian process, where $\dot{H}$ is the Hilbert space introduced in Proposition 6.1. We refer to [74] for a detailed account of the Malliavin calculus with respect to a Gaussian process. On our Wiener space, the smooth and cylindrical random variables F are of the form:

$$F = f(W(\phi_1), \dots, W(\phi_n)),$$

with $\phi_i \in \dot{H}$, $f \in C_p^\infty(\mathbb{R}^n)$ (namely f and all its partial derivatives have polynomial growth). For this kind of random variable, the derivative operator D in the sense of Malliavin calculus is the $\dot{H}$-valued random variable defined by:

$$DF = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(W(\phi_1), \dots, W(\phi_n))\phi_j .$$

The operator D is closable from $L^2(\Omega)$ into $L^2(\Omega; \dot{H})$ and we define the Sobolev space $\mathbb{D}^{1,2}$ as the closure of the space of smooth and cylindrical random variables under the norm

$$\|DF\|_{1,2} = \sqrt{\mathbf{E}[F^2] + \mathbf{E}[\|DF\|_{\dot{H}}^2]} .$$

We denote by δ the adjoint of the derivative operator (or divergence) given by the duality formula:

$$\mathbf{E}\left[\delta(u)F\right] = \mathbf{E}\left[\langle DF, u\rangle_{\dot{H}}\right], \tag{2.11}$$

for any $F \in \mathbb{D}^{1,2}$ and any element $u \in L^2(\Omega; \dot{H})$ in the domain of δ.

For any integer $n \geq 0$ we denote by $\mathbf{H}_n$ the nth Wiener chaos of W. We recall that $\mathbf{H}_0$ is simply $\mathbb{R}$ and for $n \geq 1$, $\mathbf{H}_n$ is the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_n(W(\phi)); \phi \in \dot{H}, \|\phi\|_{\dot{H}} = 1\}$, where H_n is the nth Hermite polynomial. For any $n \geq 1$, we denote by $\dot{H}^{\otimes n}$ (resp. $\dot{H}^{\odot n}$) the nth tensor product (resp. the nth symmetric tensor product) of $\dot{H}$. Then, the mapping $I_n(\phi^{\otimes n}) = H_n(W(\phi))$ can be extended to a linear isometry between $\dot{H}^{\odot n}$ (equipped with the modified norm $\sqrt{n!}\|\cdot\|_{\dot{H}^{\otimes n}}$) and $\mathbf{H}_n$.

Consider now a random variable $F \in L^2(\Omega)$ which is measurable with respect to the σ-field $\mathscr{F}$ generated by W. This random variable can be expressed as:

$$F = \mathbf{E}[F] + \sum_{n=1}^{\infty} I_n(f_n), \tag{2.12}$$

where the series converges in $L^2(\Omega)$, and the elements $f_n \in \dot{H}^{\odot n}$, $n \geq 1$, are determined by F. This identity is called the Wiener-chaos expansion of F.

The Skorohod integral (or divergence) of a random field u can be computed by using the Wiener chaos expansion. More precisely, suppose that $u = \{u(t,x); (t,x) \in \mathbb{R}_+ \times \mathbb{R}\}$ is a random field such that for each (t,x), $u(t,x)$ is an $\mathscr{F}_t$-measurable and square integrable random variable. Then, for each (t,x) we have a Wiener chaos expansion of the form

$$u(t,x) = \mathbf{E}[u(t,x)] + \sum_{n=1}^{\infty} I_n(f_n(\cdot,t,x)). \tag{2.13}$$

Suppose also that $\mathbf{E}[\|u\|_{\dot{H}}^2]$ is finite. Then, we can interpret u as a square integrable random function with values in $\dot{H}$ and the kernels f_n in the expansion (2.13) are functions in $\dot{H}^{\otimes(n+1)}$ which are symmetric in the first n variables. In this situation, u belongs to the domain of the divergence (that is, u is Skorohod integrable with respect to W) if and only if the following series converges in $L^2(\Omega)$:

$$\delta(u) = \int_0^{\infty}\int_{\mathbb{R}^d} u(t,x)\,\delta W(t,x) = W(\mathbf{E}[u]) + \sum_{n=1}^{\infty} I_{n+1}(\widetilde{f}_n(\cdot,t,x)), \tag{2.14}$$

where $\widetilde{f}_n$ denotes the symmetrization of f_n in all its $n+1$ variables. We note that whenever $u \in \Lambda_H$ the integral $\delta(u)$ coincides with the Itô integral.

Along the paper we denote by C a generic constant that may vary from line to line.

6.3 Some a priori estimates of the solution

6.3.1 Moment bound of the solution

If $\beta \in (0,1)$ and f belongs to $\dot{H}^{\beta}$, then there exists a constant C_β such that

$$\|f\|_{\dot{H}^\beta}^2 = C_\beta \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x+y) - f(x)|^2 |y|^{-1-2\beta} dxdy. \tag{3.1}$$

We refer to Proposition 1.37 in [2] for the proof of this identity. The following inequality is essential in our approach.

Proposition 6.5. *Let $p \geq 2$, f is a predictable random field. Then*

$$\left\| \int_0^t \int_{\mathbb{R}} f(s,y) W(ds,dy) \right\|_{L^p(\Omega)} \leq \sqrt{4p} \left(\int_0^t \|f(s,\cdot)\|_{\dot{H}L^p(\Omega)}^2 ds \right)^{\frac{1}{2}} \tag{3.2}$$

Proof. Applying Burkholder inequality, we have

$$\left\| \int_0^t \int_{\mathbb{R}} f(s,y) W(ds,dy) \right\|_{L^p(\Omega)} \leq \sqrt{4p} \left\| \int_0^t \|f(s,\cdot)\|_{\dot{H}}^2 ds \right\|_{L^{\frac{p}{2}}(\Omega)}^{\frac{1}{2}}.$$

The claim then follows from Minkowski inequality. □

For $p \geq 2$, $\theta > 0$ and $\varepsilon > 0$, we consider the space $\mathfrak{X}_{\theta,\varepsilon}^p$ consists of all space-time random fields $(u(t,x); t \geq 0, x \in \mathbb{R})$ with finite norm

$$\|u\|_{\mathfrak{X}_{\theta,\varepsilon}^p}$$

$$
\begin{aligned}
:= &\quad \sup_{t\ge 0, x\in\mathbb{R}} e^{-\theta t}\|u(t,x)\|_{L^p(\Omega)} \\
&\quad +\varepsilon \sup_{t\ge 0, x\in\mathbb{R}} e^{-\theta t}\left(\int_{\mathbb{R}} \|u(t,x+y)-u(t,x)\|^2_{L^p(\Omega)}|y|^{2H-2}dy\right)^{\frac{1}{2}} . \qquad (3.3)
\end{aligned}
$$

Remark 6.6. *(a) The change of the value of* ε *does not change the elements in the space* $\mathfrak{X}^p_{\theta,\varepsilon}$*, and in the case* $\varepsilon = 1$*, we simply write* $\mathfrak{X}^p_{\theta}$*.*

(b) The second term in the norm in (3.3) *is not invariant by scaling while the first term is. Indeed, denote* $f_\lambda(x) = f(\lambda x)$*, then*

$$
\begin{aligned}
&\sup_{x\in\mathbb{R}}\left(\int_{\mathbb{R}} \|f_\lambda(x+h)-f_\lambda(x)\|^2_{L^p(\Omega)}|h|^{2H-2}dh\right)^{1/2} \\
= &\ \lambda^{\frac{1}{2}-H}\sup_{x\in\mathbb{R}}\left(\int_{\mathbb{R}} \|f(x+h)-f(x)\|^2_{L^p(\Omega)}|h|^{2H-2}dh\right)^{1/2} .
\end{aligned}
$$

This is the very reason why various orders of $(t-s)$ *appear in the proof of Proposition 6.7 below. We bypass this technical difficulty by the introduction of an additional scaling factor* ε *in* (3.3)*.*

(c) Another way to see the role of ε *is via dimensional analysis. Suppose that the amplitude of* f *has unit L, the spatial variable* x *has unit S, while the randomness* ω *is dimensionless. Then the first term in* (3.3) *has unit L while the second term has unit* L/S^β*. Hence, in order for the two terms to have the same dimension, we multiply the second term with a constant* ε *having unit of* S^β*.*

Proposition 6.7. *Let* $\beta \in (0,1)$*,* $p \ge 2$*,* f *be an predictable random field. We denote*

$$
A(t,x) = \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) f(s,y) W(ds,dy) .
$$

Then the following inequality holds.

$$\|A\|_{\mathfrak{X}^p_{\theta,\varepsilon}} \le C_0\sqrt{p}\|f\|_{\mathfrak{X}^p_{\theta,\varepsilon}} \left(\kappa^{\frac{H}{2}-\frac{1}{2}}\theta^{-\frac{H}{2}} + \varepsilon^{-1}\kappa^{-\frac{1}{4}}\theta^{-\frac{1}{4}} + \varepsilon\kappa^{H-\frac{3}{4}}\theta^{\frac{1}{4}-H}\right), \tag{3.4}$$

where C_0 is a universal constant.

Proof. Applying inequality (3.2), we have

$$\|A(t,x)\|_{L^p(\Omega)} \le \sqrt{4p}\left(\int_0^t \|p_{t-s}(x-\cdot)f(s,\cdot)\|^2_{\dot{H}L^p(\Omega)}\,ds\right)^{\frac{1}{2}}. \tag{3.5}$$

Using (3.1), we have

$$\begin{aligned}
&\|p_{t-s}(x-\cdot)f(s,\cdot)\|^2_{\dot{H}}\\
&\quad\le C\int_{\mathbb{R}}\int_{\mathbb{R}} |p_{t-s}(x-y-z)f(s,y+z)-p_{t-s}(x-y)f(s,y)|^2|z|^{2H-2}dydz\\
&\quad\le C\int_{\mathbb{R}}\int_{\mathbb{R}} |p_{t-s}(x-y-z)-p_{t-s}(x-y)|^2|f(s,y+z)|^2|z|^{2H-2}dydz\\
&\qquad+\int_{\mathbb{R}}\int_{\mathbb{R}} |p_{t-s}(x-y)|^2|f(s,y+z)-f(s,y)|^2|z|^{2H-2}dydz.
\end{aligned}$$

We then apply Minkowski inequality to obtain

$$\begin{aligned}
\|p_{t-s}(x-\cdot)f(s,\cdot)\|_{\dot{H}L^p(\Omega)} &= \left\|\|p_{t-s}(x-\cdot)f(s,\cdot)\|^2_{\dot{H}}\right\|^{\frac{1}{2}}_{L^{\frac{p}{2}}(\Omega)}\\
&\le C(J_1)^{\frac{1}{2}} + C(J_2)^{\frac{1}{2}},
\end{aligned} \tag{3.6}$$

where

$$J_1 = \int_{\mathbb{R}}\int_{\mathbb{R}} |p_{t-s}(x-y-z)-p_{t-s}(x-y)|^2\|f(s,y+z)\|^2_{L^p(\Omega)}|z|^{2H-2}dydz$$

and

$$J_2=\int_{\mathbb{R}}\int_{\mathbb{R}} p_{t-s}^2(x-y)\|f(s,y+z)-f(s,y)\|_{L^p(\Omega)}^2|z|^{2H-2}dydz.$$

To estimate J_1, we use Fourier transform to get

$$\begin{aligned} J_1 &\leq C\sup_{x\in\mathbb{R}}\|f(s,x)\|_{L^p(\Omega)}^2\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\kappa(t-s)|\xi|^2}|e^{-i\xi z}-1|^2|z|^{2H-2}dzd\xi \\ &= C\sup_{x\in\mathbb{R}}\|f(s,x)\|_{L^p(\Omega)}^2\int_{\mathbb{R}} e^{-\kappa(t-s)|\xi|^2}|\xi|^{1-2H}d\xi \\ &= C\sup_{x\in\mathbb{R}}\|f(s,x)\|_{L^p(\Omega)}^2[\kappa(t-s)]^{H-1}, \end{aligned}$$

J_2 can be bounded by

$$J_2\leq C\sup_{x\in\mathbb{R}}\int_{\mathbb{R}}|f(s,x+z)-f(s,x)|^2|z|^{2H-2}dz[\kappa(t-s)]^{-\frac{1}{2}}, \tag{3.7}$$

Hence, from the above estimates we have

$$\begin{aligned} & e^{-\theta t}\sup_{x\in\mathbb{R}}\|A(t,x)\|_{L^p(\Omega)} \\ \leq\ & C\sqrt{p}\sup_{t\geq 0,x\in\mathbb{R}} e^{-\theta t}\|f(t,x)\|_{L^p(\Omega)}\left(\int_0^t e^{-2\theta(t-s)}[\kappa(t-s)]^{H-1}ds\right)^{\frac{1}{2}} \\ & +C\sqrt{p}\varepsilon\sup_{t\geq 0,x\in\mathbb{R}} e^{-\theta t}\left(\int_{\mathbb{R}}\|f(t,x+z)-f(t,x)\|_{L^p(\Omega)}^2|z|^{2H-2}dz\right)^{\frac{1}{2}} \\ & \times\frac{1}{\varepsilon}\left(\int_0^t e^{-2\theta(t-s)}[\kappa(t-s)]^{-\frac{1}{2}}ds\right)^{\frac{1}{2}} \\ \leq\ & C\sqrt{p}\|f\|_{\mathfrak{X}^p_{\theta,\varepsilon}}(\kappa^{\frac{H}{2}-\frac{1}{2}}\theta^{-\frac{H}{2}}+\frac{1}{\varepsilon}\kappa^{-\frac{1}{4}}\theta^{-\frac{1}{4}}). \end{aligned}$$

Next, we estimate the second term in the norm $\|A\|_{\mathfrak{X}^{p,\theta}_{\varepsilon}}$. For every $h\in\mathbb{R}$, we apply inequality (3.2) to get

$$\|A(t,x+h)-A(t,x)\|_{L^p(\Omega)}$$

$$\leq \quad \sqrt{4p}\left(\int_0^t \|[p_{t-s}(x+h-\cdot)-p_{t-s}(x-\cdot)]f(s,\cdot)\|^2_{\dot{H}L^p(\Omega)}\,ds\right)^{\frac{1}{2}} \tag{3.8}$$

The computations are carried out as before. From (3.1), we can write

$$\begin{aligned}
&\|[p_{t-s}(x+h-y)-p_{t-s}(x-y)]f(s,y)\|_{\dot{H}}\\
&\leq C\int_{\mathbb{R}}\int_{\mathbb{R}}\Big|[p_{t-s}(x+h-y-z)-p_{t-s}(x-y-z)]f(s,y+z)\\
&\qquad\qquad -[p_{t-s}(x+h-y)-p_{t-s}(x-y)]f(s,y)\Big|^2|z|^{2H-2}dydz\\
&\leq \int_{\mathbb{R}}\int_{\mathbb{R}}|p_{t-s}(x+h-y-z)-p_{t-s}(x-y-z)\\
&-p_{t-s}(x+h-y)+p_{t-s}(x-y)|^2|f(s,y+z)|^2|z|^{2H-2}dydz\\
&\quad+\int_{\mathbb{R}}\int_{\mathbb{R}}|p_{t-s}(x+h-y)-p_{t-s}(x-y)|^2|f(s,y+z)-f(s,y)|^2|z|^{2H-2}dydz
\end{aligned}$$

Using Minkowski inequality, we see that

$$\|[p_{t-s}(x+h-y)-p_{t-s}(x-y)]f(s,y)\|_{\dot{H}L^p(\Omega)} \leq (J_1')^{1/2}+(J_2')^{1/2} \tag{3.9}$$

where

$$\begin{aligned}
J_1' = \int_{\mathbb{R}}\int_{\mathbb{R}}&|p_{t-s}(x+h-y-z)-p_{t-s}(x-y-z)-p_{t-s}(x+h-y)+p_{t-s}(x-y)|^2\\
&\|f(s,y+z)\|^2_{L^p(\Omega)}|z|^{2H-2}dydz,
\end{aligned}$$

and

$$J_2' = \int_{\mathbb{R}}\int_{\mathbb{R}}|p_{t-s}(x+h-y)-p_{t-s}(x-y)|^2\|f(s,y+z)-f(s,y)\|^2_{L^p(\Omega)}|z|^{2H-2}dydz.$$

J_1' and J_2' can be estimated similarly to J_1 and J_2,

$$J_1' \le \sup_{x\in\mathbb{R}} \|f(s,x)\|_{L^p(\Omega)}^2 [\kappa(t-s)]^{2H-\frac{3}{2}},$$

and

$$J_2' \le \sup_{x\in\mathbb{R}} \int_{\mathbb{R}} \|f(s,x+z)-f(s,x)\|_{L^p(\Omega)}^2 |z|^{2H-2} dz [\kappa(t-s)]^{H-1}.$$

Combining these estimates for J_1', J_2' and (3.9), (3.8), similarly as the estimate for $e^{-\theta t}\|A(t,x)\|_{L^p(\Omega)}$ we obtain

$$\begin{aligned} & \sup_{x\in\mathbb{R}} e^{-\theta t} \left(\int_{\mathbb{R}} \|A(t,x+h)-A(t,x)\|_{L^p(\Omega)}^2 |h|^{2H-2} dh \right)^{1/2} \\ \le \; & C\sqrt{p}\|f\|_{\mathfrak{X}^p_{\theta,\varepsilon}} \kappa^{H-\frac{3}{4}} \theta^{\frac{1}{4}-H} + C\sqrt{p}\frac{1}{\varepsilon}\|f\|_{\mathfrak{X}^p_{\theta,\varepsilon}} \kappa^{\frac{H}{2}-\frac{1}{2}} \theta^{-\frac{H}{2}}. \end{aligned}$$

Combining altogether yields (3.4). □

6.3.2 Hölder continuity estimates

Proposition 6.8. *Let $p \ge 2$ and f be a predictable random field in $\mathfrak{X}^p_{\theta_0}$, here θ_0 is any positive number. We denote*

$$A(t,x) = \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) f(s,y) W(ds,dy).$$

Then for every $x,h \in \mathbb{R}$, $t_1,t_2 \in [0,T]$,

$$\|A([t_1,t_2],x+h)-A([t_1,t_2],x)\|_{L^p(\Omega)} \le C\|f\|_{\mathfrak{X}^p_{\theta_0}} e^{\theta_0 T} |t_2-t_1|^{\frac{2H-\beta}{4}} |h|^{\frac{\beta}{2}}, \quad \forall \beta \in [0,2H]. \tag{3.10}$$

In the above, the implied constant depends on T and here we are using the notation

$$A([t_1,t_2],x) = A(t_2,x) - A(t_1,x)\,.$$

In particular, if we let $t_1 = 0$, we get the Hölder estimate of the space variable. For the Hölder estimate of the time variable, we have

$$\|A(t_2,x) - A(t_1,x)\|^{L^p(\Omega)} \le C\|f\|_{\mathfrak{X}^p_{\theta_0}} e^{\theta_0 T}|t_2 - t_1|^{\frac{H}{2}}\,. \tag{3.11}$$

Proof. To prove (3.10), without loss of generality, we assume $t_1 < t_2$ and denote $\Delta t = t_2 - t_1$. We denote

$$\begin{aligned} V_1(f) &= \sup_{t\le T}\sup_{x\in\mathbb{R}} \|f(t,x)\|_{L^p(\Omega)}\,, \\ V_2(f) &= \sup_{t\le T}\sup_{x\in\mathbb{R}} \left(\int_{\mathbb{R}} \|f(t,x+y) - f(t,x)\|^2_{L^p(\Omega)}|y|^{2H-2}dy\right)^{1/2} \end{aligned}$$

and $V(f) = V_1(f) + V_2(f)$. We first decompose $A([t_1,t_2],x+h) - A([t_1,t_2],x) = A_1 + A_2$ where

$$A_1 = \int_0^{t_1}\int_{\mathbb{R}} [p_{[t_1-s,t_2-s]}(x+h-y) - p_{[t_1-s,t_2-s]}(x-y)]f(s,y)W(ds,dy)\,,$$

and

$$A_2 = \int_{t_1}^{t_2}\int_{\mathbb{R}} [p_{t_2-s}(x+h-y) - p_{t_2-s}(x-y)]f(s,y)W(ds,dy)\,.$$

The computations are carried out analogously to the proof of Proposition 6.7. We have

$$\|A_1\|^2_{L^p(\Omega)} \le C\int_0^{t_1}(A_{11} + A_{12})ds$$

where

$$A_{11} = \int_{\mathbb{R}}\int_{\mathbb{R}} |p_{[t_1-s,t_2-s]}(x+h-y-z) - p_{[t_1-s,t_2-s]}(x-y-z)$$
$$- p_{[t_1-s,t_2-s]}(x+h-y) + p_{[t_1-s,t_2-s]}(x-y)|^2 \|f(s,y+z)\|^2_{L^p(\Omega)} |z|^{2H-2} dydz,$$

and

$$A_{12} = \int_{\mathbb{R}}\int_{\mathbb{R}} |p_{[t_1-s,t_2-s]}(x+h-y) - p_{[t_1-s,t_2-s]}(x-y)|^2$$
$$\|f(s,y+z)-f(s,y)\|^2_{L^p(\Omega)} |z|^{2H-2} dydz.$$

Using Fourier transform, these terms can be estimated as follows,

$$\begin{aligned} A_{11} \le\, & CV_1^2(f) \\ & \int_{\mathbb{R}}\int_{\mathbb{R}} |p_{[t_1-s,t_2-s]}(h+y-z) - p_{[t_1-s,t_2-s]}(y-z) \\ & - p_{[t_1-s,t_2-s]}(h+y) + p_{[t_1-s,t_2-s]}(y)|^2 |z|^{2H-2} dydz \\ & \le CV_1^2(f) \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\kappa(t_1-s)|\xi|^2} |e^{-\frac{t_2-t_1}{2}\kappa|\xi|^2} - 1|^2 |e^{-i\xi z} - 1|^2 |e^{i\xi h} - 1|^2 |z|^{2H-2} d\xi dz \\ & \le CV_1^2(f) \int_{\mathbb{R}} e^{-\kappa(t_1-s)|\xi|^2} |e^{-\frac{t_2-t_1}{2}\kappa|\xi|^2} - 1|^2 |e^{i\xi h} - 1|^2 |\xi|^{1-2H} d\xi, \end{aligned}$$

since

$$\int_0^{t_1} |e^{-\frac{t_2-s}{2}\kappa|\xi|^2} - e^{-\frac{t_1-s}{2}\kappa|\xi|^2}|^2 ds \le \frac{1}{\kappa|\xi|^2} |e^{-\frac{\Delta t \kappa}{2}|\xi|^2} - 1|^2,$$

we obtain

$$\int_0^{t_1} A_{11} ds \le C\kappa^{-1} V_1^2(f) \int_{\mathbb{R}} |e^{-\frac{\Delta t \kappa}{2}|\xi|^2} - 1|^2 |e^{i\xi h} - 1|^2 |\xi|^{-1-2H} d\xi$$

$$\leq C\kappa^{-1}V_1^2(f)\int_{\mathbb{R}}|1-e^{-\frac{\Delta t\kappa}{2}|\xi|^2}|^2|\sin^2(\xi h/2)|\xi|^{-1-2H}d\xi\,.$$

By a change of variable the integral

$$I:=\int_{\mathbb{R}}|1-e^{-\frac{\Delta t\kappa}{2}|\xi|^2}|^2|\sin^2(\xi h/2)|\xi|^{-1-2H}d\xi$$

equals

$$|h|^{2H}\int_{\mathbb{R}}|1-e^{-\frac{\kappa\Delta t}{2|h|^2}|\xi|^2}|^2\sin^2(\xi/2)|\xi|^{-1-2H}d\xi\,,$$

we then bound $1-e^{-\frac{\kappa\Delta t}{2h^2}}$ by 1 to obtain $I\leq C|h|^{2H}$.

On the other hand, another change of variable leads to

$$I=(\kappa\Delta t)^H\int_{\mathbb{R}}|1-e^{-\xi^2/2}|^2\sin^2\big(\frac{h}{2(\kappa\Delta t)^{1/2}}\xi\big)|\xi|^{-1-2H}d\xi\,.$$

We bound the trigonometric function $\sin^2$ by 1 to obtain $I\leq(\kappa\Delta t)^H$. Interpolating these two estimates for I, we see that

$$\int_{\mathbb{R}}|1-e^{-\frac{\Delta t\kappa}{2}|\xi|^2}|^2|\sin^2(\xi h/2)|\xi|^{-1-2H}d\xi\leq C(\kappa\Delta t)^{\frac{2H-\beta}{2}}|h|^{\beta}\,,\quad\forall\beta\in[0,2H]\,.$$

Hence, we have shown

$$\int_0^{t_1}A_{11}ds\leq C\kappa^{-1}(\kappa\Delta t)^{\frac{2H-\beta}{2}}|h|^{\beta}V_1^2(f)\,,\quad\forall\beta\in[0,2H]\,.$$

Similarly,

$$\begin{aligned}\int_0^{t_1}A_{12}ds&\leq CV_2^2(f)\int_0^{t_1}\int_{\mathbb{R}}|p_{[t_1-s,t_2-s]}(h+y)-p_{[t_1-s,t_2-s]}(y)|^2dyds\\&\leq CV_2^2(f)\int_{\mathbb{R}}\int_0^{t_1}|e^{-\frac{t_2-s}{2}\kappa|\xi|^2}-e^{-\frac{t_1-s}{2}\kappa|\xi|^2}|^2ds|e^{i\xi h}-1|^2d\xi\,,\end{aligned}$$

$$\leq C\kappa^{-1}V_2^2(f)\int_{\mathbb{R}}|1-e^{-\frac{\Delta t\kappa}{2}|\xi|^2}|^2\sin^2(h\xi/2)|\xi|^{-2}d\xi\,.$$

The integral on the right hand side can be estimated as before and we get

$$\int_0^{t_1}A_{12}ds\leq CV_2^2(f)(\Delta t)^{\frac{1-\beta'}{2}}|h|^{\beta'}\,,\quad\forall\beta'\in[0,1]\,.$$

Since $1>2H-1$, we may choose $\beta'=\beta$ to obtain

$$\int_0^{t_1}A_{12}ds\leq C\kappa^{-1}(\kappa\Delta t)^{\frac{2H-\beta}{2}}|h|^{\beta}V_2^2(f)\,,\quad\forall\beta\in[0,2H]\,.$$

Hence, altogether yields

$$\|A_1\|^2_{L^p(\Omega)}\leq CV^2(f)(\Delta t)^{\frac{2H-\beta}{2}}h^{\beta}\,,\quad\forall\beta\in[0,2H]\,.$$

$\|A_2\|^2_{L^p(\Omega)}$ can be estimated analogously,

$$\|A_2\|^2_{L^p(\Omega)}\leq CV^2(f)\int_0^{\Delta t}\int_{\mathbb{R}}e^{-s\kappa|\xi|^2}\sin^2(h\xi/2)(|\xi|^{1-2H}+1)d\xi ds\,.$$

Taking integration in time first, we see that

$$\|A_2\|^2_{L^p(\Omega)}\leq C\kappa^{-1}V^2(f)\int_{\mathbb{R}}(1-e^{-\Delta t\kappa|\xi|^2})\sin^2(h\xi/2)(|\xi|^{-1-2H}+|\xi|^{-2})d\xi\,.$$

These two integrals can be estimated as before, thus we have

$$\|A_2\|^2_{L^p(\Omega)}\leq CV^2(f)(\Delta t)^{\frac{2H-\beta}{2}}|h|^{\beta}\,,\quad\forall\beta\in[0,2H]\,.$$

Let us remark that the constants in all previous estimates depends on T, p and κ^{-1}. In addition, as functions of (p,κ^{-1}), these constants grow at most polynomial. Hence, the estimates for $\|A_1\|^2_{L^p(\Omega)}$ and $\|A_2\|^2_{L^p(\Omega)}$ imply the result.

Next we show (3.11). Again, we assume that $t_1 < t_2$. We begin by writing

$$
\begin{aligned}
& \|A(t_2,x)-A(t_1,x)\|_{L^p(\Omega)} \\
= & \left\| \int_0^{t_2}\int_{\mathbb{R}} p_{t_2-s}(x-y)f(s,y)W(ds,dy) - \int_0^{t_1}\int_{\mathbb{R}} p_{t_1-s}(x-y)f(s,y)W(ds,dy) \right\|_{L^p(\Omega)} \\
= & \left\| \int_0^{t_1}\int_{\mathbb{R}} p_{[t_1-s,t_2-s]}(x-y)f(s,y)W(ds,dy) \right\|_{L^p(\Omega)} \\
& + \left\| \int_{t_1}^{t_2}\int_{\mathbb{R}} p_{t_2-s}(x-y)f(s,y)W(ds,dy) \right\|_{L^p(\Omega)} \\
:= & B_1 + B_2 \,.
\end{aligned}
$$

For B_1, using Burkholder's inequality and Minkowski's inequality we have

$$
\begin{aligned}
B_1 \le\ & C\Bigg\| \int_0^{t_1}\int_{\mathbb{R}}\int_{\mathbb{R}} \left| p_{[t_1-s,t_2-s]}(x-y)f(s,y) - p_{[t_1-s,t_2-s]}(x-y-z)f(s,y+z)\right|^2 \\
& \times |z|^{2H-2} dzdyds \Bigg\|^{\frac{1}{2}}_{L^{\frac{p}{2}}(\Omega)} \\
\le\ & C\left\| \int_0^{t_1}\int_{\mathbb{R}}\int_{\mathbb{R}} p^2_{[t_1-s,t_2-s]}(x-y)|f(s,y)-f(s,y+z)|^2 |z|^{2H-2} dzdyds \right\|^{\frac{1}{2}}_{L^{\frac{p}{2}}(\Omega)} \\
& + C\Bigg\| \int_0^{t_1}\int_{\mathbb{R}}\int_{\mathbb{R}} |p_{[t_1-s,t_2-s]}(x-y) - p_{[t_1-s,t_2-s]}(x-y-z)|^2 |f(s,y+z)|^2 \\
& \times |z|^{2H-2} dzdyds \Bigg\|^{\frac{1}{2}}_{L^{\frac{p}{2}}(\Omega)} \\
\le\ & C\left(\int_0^{t_1}\int_{\mathbb{R}}\int_{\mathbb{R}} p^2_{[t_1-s,t_2-s]}(x-y)\|f(s,y)-f(s,y+z)\|^2_{L^p(\Omega)} |z|^{2H-2} dzdyds \right)^{\frac{1}{2}} \\
& + C\Bigg(\int_0^{t_1}\int_{\mathbb{R}}\int_{\mathbb{R}} |p_{[t_1-s,t_2-s]}(x-y) - p_{[t_1-s,t_2-s]}(x-y-z)|^2 \\
& \times \|f(s,y+z)\|^2_{L^p(\Omega)} |z|^{2H-2} dzdyds \Bigg)^{\frac{1}{2}}
\end{aligned}
$$

$$\leq \quad C\left(B_{11}V_2(f)+B_{12}V_1(f)\right),$$

here

$$\begin{aligned} B_{11} &= \left(\int_0^{t_1}\int_{\mathbb{R}}|p_{[t_1-s,t_2-s]}(x-y)|^2 dyds\right)^{\frac{1}{2}} \\ &= C\left(\int_0^{t_1}\int_{\mathbb{R}}\left|e^{-\frac{t_2-s}{2}\kappa|\xi|^2}-e^{-\frac{t_1-s}{2}\kappa|\xi|^2}\right|^2 d\xi ds\right)^{\frac{1}{2}} \\ &= C(t_2-t_1)^{\frac{1}{4}}, \end{aligned}$$

and

$$\begin{aligned} B_{12} &= \left(\int_0^{t_1}\int_{\mathbb{R}}\int_{\mathbb{R}}|p_{[t_1-s,t_2-s]}(x-y)-p_{[t_1-s,t_2-s]}(x-y-z)|^2|z|^{2H-2}dzdyds\right)^{\frac{1}{2}} \\ &= C\left(\int_0^{t_1}\int_{\mathbb{R}}\int_{\mathbb{R}}\left|e^{-\frac{t_2-s}{2}\kappa|\xi|^2}-e^{-\frac{t_1-s}{2}\kappa|\xi|^2}\right|^2|e^{-i\xi z}-1|^2|z|^{2H-2}dzd\xi ds\right)^{\frac{1}{2}} \\ &= C\left(\int_0^{t_1}\int_{\mathbb{R}}\left|e^{-\frac{t_2-s}{2}\kappa|\xi|^2}-e^{-\frac{t_1-s}{2}\kappa|\xi|^2}\right|^2|\xi|^{1-2H}d\xi ds\right)^{\frac{1}{2}} \\ &= C(t_2-t_1)^{\frac{H}{2}}, \end{aligned}$$

where we have used Fourier transform and some change of variables. Combining these estimates we obtain

$$B_1 \leq C(t_2-t_1)^{\frac{H}{2}}\|f\|_{\mathfrak{X}^p_{\theta_0}}e^{\theta_0 T}$$

since $H<\frac{1}{2}$. We estimate B_2 similarly to get

$$B_2 \leq C(t_2-t_1)^{\frac{H}{2}}\|f\|_{\mathfrak{X}^p_{\theta_0}}e^{\theta_0 T}.$$

Combining the estimates for B_1 and B_2 yields the result. □

6.4 Existence and uniqueness of the solution

In this section we will first given a result regarding the uniqueness of the solution. Then we will describe the structure of some new spaces which will be used to show the existence of the solution.

6.4.1 Uniqueness of the solution

In this subsection we give some results about the uniqueness of the solution assuming that the solution is in some given space. To this end, we first introduce a norm $\|\cdot\|_{\mathscr{Z}_T^p}$ on the random field $u(t,x)$.

$$\|u\|_{\mathscr{Z}_T^p} = \sup_{t\le T}\|u(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})} + \sup_{t\le T}\left(\int_{\mathbb{R}} \frac{\|u(t,\cdot)-u(t,\cdot+h)\|^2_{L^p(\Omega\times\mathbb{R})}}{|h|^{2-2H}}dh\right)^{\frac{1}{2}}. \quad (4.1)$$

Then the space $\mathscr{Z}_T^p$ will consist all the random fields such that the above quantity is finite. The uniqueness result is the following.

Theorem 6.9. *Assume*

1. *For some sufficiently big p, the initial condition u_0 is in $L^p(\mathbb{R})$ and*

$$\int_{\mathbb{R}} \|u_0(\cdot)-u_0(\cdot+h)\|^2_{L^p(\mathbb{R})}|h|^{2H-2} < \infty. \quad (4.2)$$

2. *σ is differentiable and the derivative of σ is Lipschitz and $\sigma(0)=0$.*

3. *u and v are two solutions of (1.1) and $u,v\in\mathscr{Z}_T^p$.*

Then for every $t\in[0,T]$ and $x\in\mathbb{R}$, $u(t,x)=v(t,x), a.s.$

Proof. Assume that u solves (1.1) and $u \in \mathscr{Z}_T^p$. From the mild formulation of the solution

$$u(t,x) = p_t u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u(s,y))W(ds,dy)\,,$$

we denote the stochastic integral in the above equation by $\Phi(t,x)$ and using Fubini's theorem we can write

$$\Phi(t,x) = \frac{\sin(\pi\alpha)}{\pi} \int_0^t \int_{\mathbb{R}} (t-r)^{\alpha-1} p_{t-r}(x-z) Y(r,z)dzdr\,,$$

where

$$Y(r,z) = \int_0^r \int_{\mathbb{R}} (r-s)^{-\alpha} p_{r-s}(z-y)\sigma(u(s,y))W(ds,dy)\,,$$

the value of α will be chosen later. We are going to prove that

$$\sup_{0\le t\le T, x\in\mathbb{R}} \int_{\mathbb{R}} |\Phi(t,x)-\Phi(t,x+h)|^2 |h|^{2H-2} dh < \infty \quad a.s. \tag{4.3}$$

First we bound the difference $\Phi(t,x)-\Phi(t,x+h)$ as

$$\begin{aligned} & |\Phi(t,x)-\Phi(t,x+h)| \\ = \; & \frac{\sin(\alpha\pi)}{\pi} \left| \int_0^t \int_{\mathbb{R}} (t-r)^{\alpha-1} \left(p_{t-r}(x-z) - p_{t-r}(x+h-z)\right) Y(r,z) dzdr \right| \\ \le \; & \frac{\sin(\alpha\pi)}{\pi} \int_0^t (t-r)^{\alpha-1} \left\| p_{t-r}(\cdot) - p_{t-r}(\cdot+h) \right\|_{L^q(\mathbb{R})} \left\| Y(r,\cdot) \right\|_{L^p(\mathbb{R})} dr\,, \end{aligned}$$

here $\frac{1}{q}+\frac{1}{p}=1$. So using Minkowski's inequality, for some p which will be chosen later,

$$\begin{aligned} & \int_{\mathbb{R}} |\Phi(t,x)-\Phi(t,x+h)|^2 |h|^{2H-2} dh \\ = \; & \left(\frac{\sin(\alpha\pi)}{\pi}\right)^2 \end{aligned}$$

$$
\begin{aligned}
&\times \int_{\mathbb{R}} \left(\int_0^t (t-r)^{\alpha-1} \|p_{t-r}(x-\cdot) - p_{t-r}(x+h-\cdot)\|_{L^q(\mathbb{R})} \|Y(r,\cdot)\|_{L^p(\mathbb{R})} dr \right)^2 \\
&\times |h|^{2H-2} dh \\
\leq \; & \left(\frac{\sin(\alpha\pi)}{\pi} \right)^2 \\
&\quad \times \left(\int_0^t (t-r)^{\alpha-1} \left(\int_{\mathbb{R}} \|p_{t-r}(x-\cdot) - p_{t-r}(x+h-\cdot)\|^2_{L^q(\mathbb{R})} \right.\right. \\
&\left.\left. \times \|Y(r,\cdot)\|^2_{L^p(\mathbb{R})} |h|^{2H-2} dh \right)^{\frac{1}{2}} dr \right)^2 .
\end{aligned}
$$

For the integral

$$
\int_{\mathbb{R}} \|p_{t-r}(x-z) - p_{t-r}(x+h-z)\|^2_{L^q(\mathbb{R},dz)} |h|^{2H-2} dh,
$$

with the change of variable $z \to \sqrt{t-r}z$ and $h \to \sqrt{t-r}h$, it yields that

$$
\begin{aligned}
& \int_{\mathbb{R}} \|p_{t-r}(x-z) - p_{t-r}(x+h-z)\|^2_{L^q(\mathbb{R},dz)} |h|^{2H-2} dh \\
= \; & (t-r)^{-\frac{3}{2}+\frac{1}{q}+H} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left| \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} - \frac{1}{\sqrt{2\pi}} e^{-\frac{(z+h)^2}{2}} \right|^q dz \right)^{\frac{2}{q}} |h|^{2H-2} dh \\
= \; & C(t-r)^{-\frac{1}{2}-\frac{1}{p}+H}.
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
& \int_{\mathbb{R}} |\Phi(t,x) - \Phi(t,x+h)|^2 |h|^{2H-2} dh \\
\leq \; & C \left(\frac{\sin(\alpha\pi)}{\pi} \right)^2 \left(\int_0^t (t-r)^{\alpha-1+\frac{1}{2}(H-\frac{1}{p}-\frac{1}{2})} \|Y(r,\cdot)\|_{L^p(\mathbb{R})} dr \right)^2 \\
\leq \; & C \left(\int_0^t (t-r)^{q[\alpha-1+\frac{1}{2}(H-\frac{1}{2}-\frac{1}{p})]} dr \right)^{\frac{2}{q}} \left(\int_0^t \|Y(r,\cdot)\|^p_{L^p(\mathbb{R})} dr \right)^{\frac{2}{p}},
\end{aligned}
$$

the first integral is finite uniformly in $0 < t \le T$ if and only if

$$\alpha > \frac{3}{2p} + \frac{1}{4} - \frac{H}{2}. \tag{4.4}$$

With this choice of α, we get

$$\int_{\mathbb{R}} |\Phi(t,x) - \Phi(t,x+h)|^2 |h|^{2H-2} dh \le C \left(\int_0^t \|Y(r,\cdot)\|_{L^p(\mathbb{R})}^p dr \right)^{\frac{2}{p}},$$

so

$$\sup_{0\le t\le T, x\in\mathbb{R}} \int_{\mathbb{R}} |\Phi(t,x) - \Phi(t,x+h)|^2 |h|^{2H-2} dh \le C \left(\int_0^T \|Y(r,\cdot)\|_{L^p(\mathbb{R})}^p dr \right)^{\frac{2}{p}}.$$

Next we will show that

$$\mathbf{E} \int_0^T \|Y(r,\cdot)\|_{L^p(\mathbb{R})}^p dr < \infty, \tag{4.5}$$

this means that

$$\mathbf{E} \left(\sup_{0\le t\le T, x\in\mathbb{R}} \int_{\mathbb{R}} |\Phi(t,x) - \Phi(t,x+h)|^2 |h|^{2H-2} dh \right)^{\frac{p}{2}} < \infty,$$

which proves the claim (4.3).

To show (4.5), we note that using the assumption on σ, Minkowski's inequality, we have

$$\begin{aligned}
& \mathbf{E} \int_{\mathbb{R}} |Y(r,z)|^p dz \\
= \; & \mathbf{E} \int_{\mathbb{R}} \left| \int_0^r \int_{\mathbb{R}} (r-s)^{-\alpha} p_{r-s}(z-y) \sigma(u(s,y)) W(ds,dy) \right|^p dz \\
\le \; & C\mathbf{E} \int_{\mathbb{R}} \left(\int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha} |p_{r-s}(z-y)\sigma(u(s,y)) - p_{r-s}(z-y-h)\sigma(u(s,y+h))|^2 \right.
\end{aligned}$$

$$
\begin{aligned}
&\quad \times |h|^{2H-2} dhdyds\Big)^{\frac{p}{2}} dz \\
&\leq \; C\mathbf{E}\int_{\mathbb{R}}\left(\int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha}\,|p_{r-s}(z-y)-p_{r-s}(z-y-h)|^2\right. \\
&\quad \left.\times |\sigma(u(s,y+h))|^2\,|h|^{2H-2} dhdyds\right)^{\frac{p}{2}} dz \\
&\quad + C\mathbf{E}\int_{\mathbb{R}}\left(\int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha} p^2_{r-s}(z-y)\big|\sigma(u(s,y+h))-\sigma(u(s,y))\right. \\
&\quad \left.\times \big|^2 |h|^{2H-2} dhdyds\right)^{\frac{p}{2}} dz \\
&:= \; I + II\,.
\end{aligned}
$$

Using the assumptions on σ, Minkowski's inequality and the change of variable $y \to y+z$, we have

$$
\begin{aligned}
I \;&\leq\; C\mathbf{E}\int_{\mathbb{R}}\left(\int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha}\,|p_{r-s}(z-y)-p_{r-s}(z-y-h)|^2\right. \\
&\quad \left.|u(s,y+h)|^2\,|h|^{2H-2} dhdyds\right)^{\frac{p}{2}} dz \\
&\leq\; C\int_{\mathbb{R}}\left(\int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha}\,|p_{r-s}(y)-p_{r-s}(y+h)|^2\,\|u(s,y+z+h)\|^2_{L^p(\Omega)}\right. \\
&\quad \left.|h|^{2H-2} dhdyds\right)^{\frac{p}{2}} dz \\
&\leq\; C\left(\int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha}\,|p_{r-s}(y)-p_{r-s}(y+h)|^2\,\|u(s,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})}\right. \\
&\quad \left.|h|^{2H-2} dhdyds\right)^{\frac{p}{2}} \\
&=\; C\left(\int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha+H-1}\,\|u(s,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})}\, ds\,.\right)^{\frac{p}{2}}
\end{aligned}
$$

Similarly we get the estimate for II,

$$II \le C\left(\int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha-\frac{1}{2}} \|u(s,\cdot+h)-u(s,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})} |h|^{2H-2} dhds\right)^{\frac{p}{2}}.$$

Combining the estimates for I and II we obtain

$$\begin{aligned} &\mathbf{E}\int_{\mathbb{R}} |Y(r,z)|^p dz \\ \le\ & C\left(\int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha+H-1} \|u(s,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})} ds.\right)^{\frac{p}{2}} \\ &+C\left(\int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha-\frac{1}{2}} \|u(s,\cdot+h)-u(s,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})} |h|^{2H-2} dhds\right)^{\frac{p}{2}}, \end{aligned}$$

for the integrability consideration, here we need to assume that $\alpha < \frac{H}{2}$, combining the restriction (4.4) we see that $\frac{3}{2p}+\frac{1}{4}-\frac{H}{2} < \alpha < \frac{H}{2}$, this is possible since $H > \frac{1}{4}$ and p is big enough. By assumption $u \in \mathscr{Z}_T^p$, the above expression is finite, uniformly in $r \in [0,T]$. So (4.5) holds and thus (4.3) is proved.

Next we will prove the uniqueness of the solution. Assume that $u(t,x)$ and $v(t,x)$ are two solutions of equation (1.1). Define the stopping times

$$\begin{aligned} T_k &= \inf\Big\{0 \le t \le T : \sup_{0\le s\le t, x\in\mathbb{R}} \int_{\mathbb{R}} |u(s,x)-u(s,x+h)|^2 |h|^{2H-2} dh \ge k \\ &\qquad \text{or} \quad \sup_{0\le s\le t, x\in\mathbb{R}} \int_{\mathbb{R}} |v(s,x)-v(s,x+h)|^2 |h|^{2H-2} dh \ge k\Big\}. \end{aligned}$$

Then we note that using the property of Itô integral we have

$$\begin{aligned} I_1 &:= \mathbf{E}\left|\mathbf{1}_{\{t<T_k\}} u(t,x) - \mathbf{1}_{\{t<T_k\}} v(t,x)\right|^2 \\ &\le \mathbf{E}\left|\int_0^{t\wedge T_k} \int_{\mathbb{R}} p_{t-s}(x-y)\left[\sigma(u(s,y))-\sigma(v(s,y))\right] W(ds,dy)\right|^2 \end{aligned}$$

$$
\begin{aligned}
&= \mathbf{E}\left|\int_0^t\int_{\mathbb{R}} p_{t-s}(x-y)\mathbf{1}_{\{s<T_k\}}\left[\sigma(u(s,y))-\sigma(v(s,y))\right]W(ds,dy)\right|^2 \\
&= \mathbf{E}\left|\int_0^t\int_{\mathbb{R}} p_{t-s}(x-y)\left[\sigma(\mathbf{1}_{\{s<T_k\}}u(s,y))-\sigma(\mathbf{1}_{\{s<T_k\}}v(s,y))\right]W(ds,dy)\right|^2 \\
&\leq C\mathbf{E}\int_0^t\int_{\mathbb{R}^2}|p_{t-s}(x-y)-p_{t-s}(x-y-h)|^2 \\
&\quad\times\left|\sigma\left(\mathbf{1}_{\{s<T_k\}}u(s,y+h)\right)-\sigma\left(\mathbf{1}_{\{s<T_k\}}v(s,y+h)\right)\right|^2|h|^{2H-2}dhdyds \\
&\quad+C\mathbf{E}\int_0^t\int_{\mathbb{R}^2}p_{t-s}^2(x-y)\big[\sigma\left(\mathbf{1}_{\{s<T_k\}}u(s,y)\right)-\sigma\left(\mathbf{1}_{\{s<T_k\}}v(s,y)\right) \\
&\quad-\sigma\left(\mathbf{1}_{\{s<T_k\}}u(s,y+h)\right)+\sigma\left(\mathbf{1}_{\{s<T_k\}}u(s,y+h)\right)\big]^2|h|^{2H-2}dhdyds,
\end{aligned}
$$

then using the assumption of σ, we have the following estimate

$$
|\sigma(a)-\sigma(b)-\sigma(c)+\sigma(d)|\leq C|a-b-c+d|+C|a-b|(|a-c|+|b-d|),
$$

applying this estimate to the above estimation, we obtain

$$
\begin{aligned}
&I_1\leq\mathbf{E}\left|\mathbf{1}_{\{t<T_k\}}u(t,x)-\mathbf{1}_{\{t<T_k\}}v(t,x)\right|^2 \\
&\leq C\mathbf{E}\int_0^t\int_{\mathbb{R}^2}|p_{t-s}(x-y)-p_{t-s}(x-y-h)|^2 \\
&\quad\times\left|\mathbf{1}_{\{s<T_k\}}u(s,y+h)-\mathbf{1}_{\{s<T_k\}}v(s,y+h)\right|^2|h|^{2H-2}dhdyds \\
&\quad+C\mathbf{E}\int_0^t\int_{\mathbb{R}^2}p_{t-s}^2(x-y)\big[\mathbf{1}_{\{s<T_k\}}u(s,y)-\mathbf{1}_{\{s<T_k\}}v(s,y) \\
&\quad-\mathbf{1}_{\{s<T_k\}}u(s,y+h)+\mathbf{1}_{\{s<T_k\}}v(s,y+h)\big]^2|h|^{2H-2}dhdyds \\
&\quad+C\mathbf{E}\int_0^t\int_{\mathbb{R}^2}p_{t-s}^2(x-y)\left|\mathbf{1}_{\{s<T_k\}}u(s,y)-\mathbf{1}_{\{s<T_k\}}v(s,y)\right|^2 \\
&\quad\times\mathbf{1}_{\{s<T_k\}}\left(|u(s,y)-u(s,y+h)|^2+|v(s,y)-v(s,y+h)|^2\right)|h|^{2H-2}dhdyds \\
&\leq C\int_0^t(t-s)^{H-1}\sup_{y\in\mathbb{R}}\mathbf{E}\left|\mathbf{1}_{\{s<T_k\}}u(s,y)-\mathbf{1}_{\{s<T_k\}}v(s,y)\right|^2ds \\
&\quad+C\int_0^t(t-s)^{-\frac{1}{2}} \\
&\quad\sup_{y\in\mathbb{R}}\mathbf{E}\int_{\mathbb{R}}\mathbf{1}_{\{s<T_k\}}|u(s,y)-v(s,y)-u(s,y+h)+v(s,y+h)|^2|h|^{2H-2}dhds
\end{aligned}
$$

$$+CK\int_0^t (t-s)^{-\frac{1}{2}} \sup_{y\in\mathbb{R}} \mathbf{E}\left|\mathbf{1}_{\{s<T_k\}}u(s,y)-\mathbf{1}_{\{s<T_k\}}v(s,y)\right|^2 ds.$$

Similar with the term I_1 we estimate the term

$$\begin{aligned}
I_2 \;:=\; & \mathbf{E}\int_{\mathbb{R}}\left|\mathbf{1}_{\{t<T_k\}}u(t,x)-\mathbf{1}_{\{t<T_k\}}v(t,x)-\mathbf{1}_{\{t<T_k\}}u(t,x+h)+\mathbf{1}_{\{t<T_k\}}v(t,x+h)\right|^2 \\
& |h|^{2H-2}dh \\
\leq\; & \mathbf{E}\int_{\mathbb{R}}\left|\int_0^t\int_{\mathbb{R}}[p_{t-s}(x-y)-p_{t-s}(x+h-y)]\right. \\
& \quad \left.\times\left[\sigma(\mathbf{1}_{\{s<T_k\}}u(s,y))-\sigma(\mathbf{1}_{\{s<T_k\}}v(s,y))\right]W(ds,dy)\right|^2|h|^{2H-2}dh \\
\leq\; & C\int_{\mathbb{R}}\int_0^t\int_{\mathbb{R}^2}|p_{t-s}(x-y)-p_{t-s}(x+h-y)|^2 \\
& \quad \times\mathbf{E}\big|\sigma\left(\mathbf{1}_{\{s<T_k\}}u(s,y)\right)-\sigma\left(\mathbf{1}_{\{s<T_k\}}v(s,y)\right)-\sigma\left(\mathbf{1}_{\{s<T_k\}}u(s,y+l)\right) \\
& \qquad +\sigma\left(\mathbf{1}_{\{s<T_k\}}v(s,y+l)\right)\big|^2|l|^{2H-2}|h|^{2H-2}dldydsdh \\
& +C\int_{\mathbb{R}}\int_0^t\int_{\mathbb{R}^2}|p_{t-s}(x-y)-p_{t-s}(x+h-y)-p_{t-s}(x-y-l)+p_{t-s}(x+h-y-l)|^2 \\
& \quad \times\mathbf{E}\left|\sigma\left(\mathbf{1}_{\{s<T_k\}}u(s,y+l)\right)-\sigma\left(\mathbf{1}_{\{s<T_k\}}v(s,y+l)\right)\right|^2|l|^{2H-2}|h|^{2H-2}dldydsdh,
\end{aligned}$$

then we proceed as in the estimate for I_1 to get

$$\begin{aligned}
I_2 \;\leq\; & C\int_0^t (t-s)^{H-1}\sup_{y\in\mathbb{R}}\mathbf{E}\int_{\mathbb{R}}\left(\mathbf{1}_{\{s<T_k\}}\left|u(s,y)-v(s,y)-u(s,y+l)+v(s,y+l)\right|^2\right) \\
& |l|^{2H-2}dhds \\
& +CK\int_0^t (t-s)^{H-1}\sup_{y\in\mathbb{R}}\mathbf{E}\left|\mathbf{1}_{\{s<T_k\}}u(s,y)-\mathbf{1}_{\{s<T_k\}}v(s,y)\right|^2 ds \\
& +C\int_0^t (t-s)^{2H-\frac{3}{2}}\sup_{y\in\mathbb{R}}\mathbf{E}\left|\mathbf{1}_{\{s<T_k\}}u(s,y)-\mathbf{1}_{\{s<T_k\}}v(s,y)\right|^2 ds.
\end{aligned}$$

Then using Gronwall's lemma we conclude that

$$\mathbf{E}\left|\mathbf{1}_{\{t<T_k\}}u(t,x)-\mathbf{1}_{\{t<T_k\}}v(t,x)\right|^2=0, \tag{4.6}$$

then we let $k \to \infty$ to conclude that

$$\mathbf{E}\,|u(t,x) - v(t,x)|^2 = 0\,. \tag{4.7}$$

This proves the uniqueness. □

6.4.2 Space-time function spaces

We introduce here the function spaces which form the underlying framework of our treatments for the uniqueness of the solution. Since these spaces do not belong to standard classes of function spaces, we describe them in detail.

We denote by $C_{\mathrm{uc}}([0,T]\times\mathbb{R})$ the space of all continuous functions on $\mathbb{R}$ equipped with the topology of convergence uniformly over compact intervals.

Let $(B, \|\cdot\|)$ be a Banach space equipped with the norm $\|\cdot\|$. Let $\beta \in (0,1)$ be a fixed number. For every $\delta \in (0,\infty]$ and every function $f : \mathbb{R} \to B$, we introduce the function $V^{(\beta)}_{f,\delta} : \mathbb{R} \to [0,\infty]$

$$V^{(\beta)}_{f,\delta}(x) = \left(\int_{|h|\le\delta} \|f(x+h) - f(x)\|^2 |h|^{-1-2\beta} dh \right)^{\frac{1}{2}}.$$

When the value of β is clear from the context, we will write $V_{f,\delta}$ instead of $V^{(\beta)}_{f,\delta}$. As we will see later along the development of the paper, $V_{f,\delta}(x)$ plays a role analogous to the modulus of continuity of f at x around a distance δ. When $\delta = \infty$, we write $V_f = V_{f,\infty}$. It follows from Minkowski inequality that V satisfies

$$|V_{f,\delta}(x) - V_{g,\delta}(x)| \le V_{f-g,\delta}(x) \tag{4.8}$$

for all $\delta \in (0,\infty]$, functions f, g and x in $\mathbb{R}$. Thus, V is a seminorm.

Suppose for instance that a function f has modulus of continuity around x as $|h|^{\beta}\omega(h)$. Then $V_{f,1}^{2}(x)$ is majorized by $\int_0^1 \omega^2(h)h^{-1}dh$. Thus, in order for $V_{f,1}(x)$ to be finite, it is sufficient that $\omega^2(h)h^{-1}$ is integrable near 0. On the other hand, if $V_{f,1}$ is bounded over a domain, then f is necessary Hölder continuous.

Proposition 6.10. *Let I be a non-empty open interval of $\mathbb{R}$ and $\delta \in (0,\infty]$. Let f be a function on $\mathbb{R}$ such that $\sup_{x\in\bar{I}} V_{f,\delta}(x)$ is finite. Then*

$$\sup_{x\in I;|y|\le\frac{\delta}{3}\wedge \mathrm{dist}(x,\partial I)} \frac{\|f(x+y)-f(x)\|}{|y|^{\beta}} \le c(\beta)\sup_{x\in\bar{I}} V_{f,\delta}^{(\beta)}(x) \tag{4.9}$$

for some finite constant $c(\beta)$ depends only on β.

Proof. For every $x\in I$ and positive R, $R\le\delta$, we denote $f_{x,R}=\frac{1}{2R}\int_{-R}^{R} f(y+x)dy$. We first estimate $\|f(x)-f_{x,R}\|$ as follows

$$\begin{aligned}
\|f(x)-f_{x,R}\| &\le \frac{1}{2R}\int_{-R}^{R}\|f(x)-f(x+y)\|dy\\
&\le \frac{1}{2R}\left(\int_{-R}^{R}\|f(x)-f(x+y)\|^2|y|^{-1-2\beta}dy\right)^{1/2}\left(\int_{-R}^{R}|y|^{1+2\beta}dy\right)^{1/2}\\
&\le \frac{1}{2\sqrt{(1+\beta)}}R^{\beta}\sup_{x\in\bar{I}} V_{f,\delta}^{(\beta)}(x)\,.
\end{aligned} \tag{4.10}$$

Let us now fix $x\in I$ and $y\in\mathbb{R}$ such that $|y|\le\delta/3\wedge \mathrm{dist}(x,\partial I)$. We also choose $R=|y|$. It follows from triangle inequality that

$$\|f(x+y)-f(x)\| \le \|f(x+y)-f_{x+y,R}\|+\|f_{x+y,R}-f_{x,R}\|+\|f(x)-f_{x,R}\|\,. \tag{4.11}$$

For the second term, we apply Minkowski inequality and Cauchy-Schwartz inequality to get

$$
\begin{aligned}
&\|f_{x+y,R}-f_{x,R}\| \\
&\le \frac{1}{4R^2}\int_{-R}^{R}\int_{-R}^{R}\|f(x+y+z)-f(x+w)\|dzdw \\
&\le \frac{1}{4R^2}\int_{-R}^{R}\left(\int_{-R}^{R}\|f(x+y+z)-f(x+w)\|^2|y+z-w|^{-2\beta-1}dz\right)^{\frac{1}{2}} \\
&\left(\int_{-R}^{R}|y+z-w|^{2\beta+1}dz\right)^{\frac{1}{2}}dw\,.
\end{aligned}
$$

Because of the restrictions on the variables, $|y+z-w|\le 3R\le\delta$ and $x+w\in\bar{I}$. Hence

$$
\|f_{x+y,R}-f_{x,R}\|\le C_\beta\sup_{t\in\bar{I}}V_{f,\delta}(t)R^\beta\,.
$$

The first and third terms in the right hand side of (4.11) are estimated in (4.10). Combining these estimates with (4.11) yields (4.9). □

We introduce here a new space which will be used later.

Let $\mathfrak{X}_T^\beta(B)$ be the space of all continuous functions $f:[0,T]\times\mathbb{R}\to B$ such that

1. $(t,x)\mapsto V_f^{(\beta)}(t,x)$ is finite and bounded on $[0,T]\times\mathbb{R}$;

2. $\|f\|_{\mathfrak{X}_T^\beta(B)}:=\sup_{t\le T;x\in\mathbb{R}}\|f(t,x)\|+\sup_{t\ge 0;x\in\mathbb{R}}V_f^{(\beta)}(t,x)$ is finite.

We equip $\mathfrak{X}_T^\beta(B)$ with the norm $\|\cdot\|_{\mathfrak{X}_T^\beta(B)}$ defined as above. Then $\mathfrak{X}_\theta^\beta(B)$ is a normed vector space. In fact, these spaces are complete.

Proposition 6.11. *$\mathfrak{X}_T^\beta(B)$ is a Banach space.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in $\mathfrak{X}_T^\beta(B)$. Since the space $C_b([0,T]\times\mathbb{R};B)$ of bounded continuous functions from $[0,T]\times\mathbb{R}$ to B is complete, there exists a bounded continuous function $f : [0,T] \times \mathbb{R} \to B$ such that

$$\lim_{n\to\infty} \sup_{t\in[0,T];x\in\mathbb{R}} \|f_n(t,x) - f(t,x)\| = 0\,.$$

Fix $\varepsilon > 0$, there exists $n_0 > 0$ such that

$$\sup_{x\in\mathbb{R}} V_{f_n - f_m}(t,x) < \varepsilon$$

for all $m,n > n_0$. It follows from Fatou's lemma that

$$V_{f_n - f}(t,x) \le \liminf_{m\to\infty} V_{f_n - f_m}(t,x) \le \varepsilon$$

for every $t \in [0,T]$, $x \in \mathbb{R}$ and $n > n_0$. This implies that $\lim_{n\to\infty} \sup_{t\le T;x\in\mathbb{R}} |V_{f_n - f}(t,x)| = 0$ which means f_n converges to f in $\mathfrak{X}_T^\beta(B)$. $\square$

When $B = L^p(\Omega)$ with $p \in [1,\infty)$, we use the notations $\mathfrak{X}_T^{\beta,p} = \mathfrak{X}_T^\beta(L^p(\Omega))$. A function f in $\mathfrak{X}_T^{\beta,p}$ can be considered as a stochastic process indexed by (t,x) in $[0,T] \times \mathbb{R}$ such that

$$\sup_{t,x} \|f(t,x)\|_{L^p(\Omega)} + \sup_{t,x} \left(\int_{\mathbb{R}} \|f(t,x+y) - f(t,x)\|_{L^p(\Omega)}^2 |y|^{-2\beta-1} dy \right)^{\frac{1}{2}} < \infty\,.$$

In the case when σ is affine (i.e. $\sigma(u) = au + b$ for some constants a,b), these spaces are sufficient to show existence and uniqueness for equation (1.1). On the other hand, the case of general Lipschitz function σ leads to the considerations of additional spaces, to which we now turn.

For every $h \in \mathbb{R}$, let τ_h be the translation map in the spatial variable, that is $\tau_h f(t,x) = f(t,x-h)$. Let X_T^β be the space of all real valued continuous functions f on : $[0,T] \times \mathbb{R}$ such that

1. $(t,x) \mapsto V^{(\beta)}_{f,1}(t,x)$ is finite and continuous on $[0,T] \times \mathbb{R}$;

2. For every positive R, $\lim_{h \downarrow 0} \sup_{t \le T; x \in [-R,R]} V_{\tau_h f - f,1}(t,x) = 0$.

We equip X_T^β with the following topology: a sequence $\{f_n\}$ in X_T^β converges to f in X_T^β if for all $R > 0$, the sequences $\{f_n\}$ and $\{V_{f_n - f,1}\}$ converge uniformly on $[0,T] \times [-R,R]$ to f and 0 respectively. We define a metric on X_T^β as follows

$$d_\beta(f,g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|f-g\|_{n,\beta}}{1 + \|f-g\|_{n,\beta}}, \tag{4.12}$$

where $\|\cdot\|_{n,\beta}$ is the seminorm

$$\|f\|_{n,\beta} := \sup_{t \in [0,T]; x \in [-n,n]} |f(t,x)| + \sup_{t \in [0,T]; x \in [-n,n]} V^{(\beta)}_{f,1}(t,x).$$

Since functions in X_T^β are locally bounded, the topology of X_T^β is not altered if in the previous definition, $V_{f,1}$ is replaced by $V_{f,\delta}$ for every *finite* positive δ. We emphasize that replacing δ by ∞ would create a strictly smaller space.

Proposition 6.12. *X_T^β is a complete metric space.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in X_T^β. Since the space $C_{\mathrm{uc}}([0,T] \times \mathbb{R})$ is complete, there exists continuous function $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ such that for all compact intervals I,

$$\lim_{n \to \infty} \sup_{t \in [0,T]; x \in I} |f_n(t,x) - f(t,x)| = 0.$$

Let us fix a compact interval $I = [-N,N]$. Fix $\varepsilon > 0$, there exists $n_0 > 0$ such that

$$\sup_{x \in I, t \in [0,T]} V_{f_n - f_m,1}(t,x) < \varepsilon$$

for all $m,n > n_0$. It follows from Fatou's lemma that

$$V_{f_n - f,1}(t,x) \le \liminf_{m\to\infty} V_{f_n - f_m,1}(t,x) \le \varepsilon$$

for every $t \in [0,T]$, $x \in I$ and $n > n_0$. This implies that $V_{f_n-f,1}$ converges to 0 uniformly on $[0,T]\times I$. In addition, from (4.8), it follows that $V_{f_n,1}$ converges to V_f uniformly on $[0,T]\times I$, thus the continuity of $V_{f_n,1}$ implies continuity of $V_{f,1}$.

It remains to check that f satisfies the condition (2). For every $\varepsilon > 0$ and $|h| \le 1$, choose n sufficiently large so that $\sup_{t\in[0,T];x\in[N-1,N+1]} V_{f_n-f,1}(t,x) < \varepsilon$. Applying Minkowski inequality, for every $(t,x) \in [0,T]\times[-N,N]$, we have

$$V_{\tau_h f - f,1}(t,x) \le V_{\tau_h f - \tau_h f_n,1}(t,x) + V_{\tau_h f_n - f_n,1}(t,x) + V_{f_n - f,1}(t,x) \le 2\varepsilon + V_{\tau_h f_n - f_n,1}(t,x).$$

Since f_n belongs to X_T^β, $\lim_{h\to 0}\sup_{t\in[0,T];x\in[-N,N]} V_{\tau_h f_n - f_n,1}(t,x) = 0$ which implies f belongs to X_T^β.

The next results give some characterizations of the space X_T^β. □

Lemma 6.13. *Let $f : [0,T]\times\mathbb{R} \to B$ be a continuous function such that $t \mapsto V_f(t,x)$ is continuous for every fixed x. Suppose in addition that for every $R > 0$,*

$$\lim_{\delta\downarrow 0} \sup_{t\in[0,T];x\in[-R,R]} \int_{-\delta}^{\delta} \|f(t,x+y) - f(t,x)\|^2 |y|^{-2\beta-1} dy = 0\,.$$

Then $V_{f,1}$ is continuous and f belongs to X_T^β.

Proof. Fix $R > 0$ and $\varepsilon > 0$, choose δ such that

$$\sup_{t\in[0,T];x\in[-R-1,R+1]} \int_{-\delta}^{\delta} \|f(t,x+y) - f(t,x)\|^2 |y|^{-2\beta-1} dy < \varepsilon\,.$$

Then for every $t \in [0,T]; x \in [-R,R]$ and $|h| \leq 1$

$$|V_{\tau_h f - f,1}(t,x)|^2 \leq \varepsilon + \sup_{t\in[0,T];x\in[-R,R]} \|\tau_h f(t,x) - f(t,x)\| \int_{|y|>\delta} |y|^{-2\beta-1} dy\,.$$

Since f is continuous, $\lim_{h\to 0}\sup_{t\in[0,T];x\in[-R,R]} \|\tau_h f(x) - f(x)\| = 0$. Together with the previous estimate, this yields $\lim_{h\to 0}\sup_{t\in[0,T];x\in[-R,R]} V_{\tau_h f - f,1}(t,x) = 0$ which on one hand, together with (4.8) implies the continuity of V_f. On the other hand, it obviously implies $f \in X_T^\beta$. □

Proposition 6.14. *Let $\phi \in C^\infty(\mathbb{R})$ be supported in $[-1,1]$, $\int \phi(x)dx = 1$, $0 \leq \phi \leq 1$ and $\phi_n(x) = n\phi(nx)$. Then*

(i) If f is a function from $[0,\infty)\times\mathbb{R}$ to $\mathbb{R}$ and $f \in X_T^\beta$, then $f * \phi_n \to f$ in X_T^β as $n \to \infty$. where $*$ denotes the convolution with respect to space variable only.

(ii) $C_0^{0,1}([0,T]\times\mathbb{R})$ i.e., the functions which are continuous in time and continuous differentiable in space and they go to 0 as the space variable goes to infinity, is dense in X_T^β.

(iii) Suppose that f is a continuous function on $[0,T]\times\mathbb{R}$ such that $t \mapsto V_{f,1}(t,x)$ is finite and continuous in time for every fixed $x \in \mathbb{R}$. Then f belongs to X_T^β if and only if for every $R > 0$

$$\lim_{\delta\downarrow 0}\sup_{t\in[0,T];x\in[-R,R]} \int_{-\delta}^{\delta} |f(t,x+y) - f(t,x)|^2 |y|^{-2\beta-1} dy = 0\,. \tag{4.13}$$

Proof. We denote $f_n = f * \phi_n$. To show ((i)), we observe that

$$\begin{aligned} &f_n(t,x+y) - f_n(t,x) - f(t,x+y) + f(t,x) \\ &\quad = \int_{\mathbb{R}} [\tau_h f(t,x+y) - \tau_h f(t,x) - f(t,x+y) + f(t,x)]\phi_n(h) dh \end{aligned}$$

and hence, for every $x \in [-R,R]$, applying Jensen's inequality, we get

$$\begin{aligned}\int_{-1}^{1} &|f_n(t,x+y) - f_n(t,x) - f(t,x+y) + f(t,x)|^2 |y|^{-2\beta-1} dy \\ &\le \int_{\mathbb{R}} \int_{-1}^{1} |\tau_h f(t,x+y) - \tau_h f(t,x) - f(t,x+y) + f(t,x)|^2 |y|^{-2\beta-1} dy \phi_n(h) dh \\ &\le \int_{\mathbb{R}} \sup_{r\in[0,T];z\in[-R-1,R+1]} V^2_{\tau_h f - f,1}(r,z) \phi_n(h) dh\,.\end{aligned}$$

By assumption f belongs to X_T^β, this integral converges to 0 when $n \to \infty$, which proves ((i)).

To show ((ii)), we first show that X_T^β contains $C_0^{0,1}([0,T]\times\mathbb{R})$. Indeed, if g is a function in $C_0^{0,1}([0,T]\times\mathbb{R})$, by dominated convergence theorem, it is easy to show that $V_g^\beta(t,x)$ is finite and continuous in time for every fixed x. Moreover, for every $R > 0$, we have

$$\sup_{t\in[0,T];x\in[-R,R]} \int_{-\delta}^{\delta} |g(t,x+y) - g(t,x)|^2 |y|^{-2\beta-1} dy \le \|\partial_x g\|_\infty \int_{|y|\le\delta} |y|^{1-2\beta} dy\,.$$

Lemma 6.13 implies g belongs to X_T^β. Together with ((i)), this yields ((ii)).

The sufficiency of ((iii)) is in fact Lemma 6.13. We focus on the necessity of (4.13). Assume that f belongs to X_T^β, fix $R > 0$, $\varepsilon > 0$ and choose g in $C_0^{0,1}$ so that $\sup_{t\in[0,T];x\in[-R,R]} V_{f-g,1}(t,x) < \varepsilon$. Then for every $\delta > 0$ we have

$$\begin{aligned}\sup_{t\le T;\,|x|\le R} &\int_{-\delta}^{\delta} |f(t,x+y) - f(t,x)|^2 |y|^{-2\beta-1} dy \\ &\le 2\varepsilon^2 + \sup_{t\le T;\,|x|\le R} \int_{-\delta}^{\delta} |g(t,x+y) - g(t,x)|^2 |y|^{-2\beta-1} dy\,.\end{aligned}$$

Since g is $C_0^{0,1}$, the last term converges to 0 when $\delta \downarrow 0$ and since ε can be chosen arbitrarily small, this implies that f satisfies the condition (4.13). □

Corollary 6.15. *X_T^β is a Polish (complete and separable) space.*

Proof. Completeness comes from Proposition 6.12. For separability, we invoke Proposition 6.14((ii)) and the fact that the functions in $C_0^{0,1}([0,T]\times\mathbb{R})$ can be approximated by polynomials with rational coefficients. □

Proposition 6.16. *The inclusion $X_T^\beta \subset X_T^\alpha$ holds continuously for $\beta > \alpha$.*

Proof. Suppose f belongs to X_T^β. Fix $n \geq 1$, by Proposition 6.10, we see that

$$\sup_{t\leq T;|x|\leq n} |f(t,x+y)-f(t,x)| \leq C \sup_{t\leq T;|x|\leq n+1} V_{f,3}^{(\beta)}(x)|y|^\beta$$

for every $|y|\leq 1$. Hence for every $t\leq T$, $|x|\leq n$

$$\int_{|y|\leq 1} |f(t,x+y)-f(t,x)|^2|y|^{-2\alpha-1}dy \leq C \sup_{t\leq T;|x|\leq n+1} V_{f,3}^{(\beta)}(x)\,.$$

is finite. The continuity of $(t,x)\mapsto \int_{|y|\leq 1}|f(x+y)-f(t,x)|^2|y|^{-2\alpha-1}dy$ follows at once from dominated convergence theorem. □

Next we derive a compactness criteria for X_T^β. We first recall some well-known facts. An *ε-cover* of a metric space is a cover of the space consisting of sets of diameter at most ε. A metric space is called *totally bounded* if it admits a finite ε-cover for every $\varepsilon > 0$. It is well known that a metric space is compact if and only if it is complete and totally bounded. The following lemma is the key ingredient for many compactness results

Lemma 6.17. *Let X be a metric space. Assume that, for every $\varepsilon > 0$, there exists some $\delta > 0$, a metric space W, and a mapping $\Phi : X \to W$ so that $\Phi(X)$ is totally bounded, and whenever $x,y \in X$ are such that $d(\Phi(x),\Phi(y)) < \delta$, then $d(x,y) < \varepsilon$. Then X is totally bounded.*

The proof of this lemma is elementary, we refer readers to Lemma 1 in [48] for details. The following result characterize compact sets in X_T^β.

Proposition 6.18. *A set $\mathfrak{F}$ in X_T^β is relatively compact if*

1. $\sup_{f\in\mathfrak{F}}|f(0,0)|$ *is finite;*

2. *For every fixed $x\in\mathbb{R}$, $\{f(\cdot,x): f\in\mathfrak{F}\}$ is equicontinuous in time;*

3. *For every $R>0$, $\lim_{\delta\downarrow 0}\sup_{f\in\mathfrak{F}}\sup_{t\in[0,T];x\in[-R,R]}\int_{-\delta}^{\delta}|f(t,x+y)-f(t,x)|^2\frac{dy}{|y|^{1+2\beta}}=0$.*

Proof. Suppose that $\mathfrak{F}$ satisfies the 3 conditions. We first observe that the condition 3 together with (4.9) implies the following equicontinuity property: for every $R>0$, for every $\varepsilon>0$, there exists $\eta>0$ such that $\sup_{t\in[0,T]}|f(t,x)-f(t,y)|<\varepsilon$ whenever $f\in\mathfrak{F}$ and $x,y\in[-R,R]$ such that $|x-y|<\eta$. Together with condition 2, this implies equicontinuity for $\mathfrak{F}$ in $(t,x)\in[0,T]\times[-R,R]$. Indeed, take N to be a sufficiently large integer, we can set $x_i=-R+\frac{j}{N}R$, $j=0,1,\dots,2N$, according to (2), $\{f(\cdot,x_i): f\in\mathfrak{F}\}$ is equicontinuous in time, uniformly for $j=0,1,\dots,2N$. By writing

$$|f(t,x)-f(s,x)|\le|f(t,x)-f(t,x_i)|+|f(t,x_i)-f(s,x_i)|+|f(s,x_i)-f(s,x)|,$$

and x_i is chosen such that $|x-x_i|<\eta$, this shows the uniformity in x.

Fix $R>0$ and $\varepsilon>0$, from condition 3, we can choose a positive number $\delta_1=\delta_1(\varepsilon)$, $\delta_1<1$, such that

$$2\sup_{f\in\mathfrak{F}}\sup_{t\in[0,T];x\in[-R,R]}\int_{-\delta_1}^{\delta_1}|f(t,x+y)-f(t,x)|^2\frac{dy}{|y|^{1+2\beta}}\le\varepsilon^2\,.$$

We now choose $\delta_2 \le \varepsilon$ such that

$$2\int_{|y|>\delta_1} (3\delta_2)^2 \frac{dy}{|y|^{1+2\beta}} \le \varepsilon^2 .$$

By the equicontinuity, we can choose a positive number $\eta = \eta(\varepsilon)$, $\eta < 1$, such that $\|f(t,x) - f(s,y)\| \le \delta_2$ whenever $f \in \mathfrak{F}$ and $(t,x),(s,y) \in [0,T] \times [-R-2,R+2]$ so that $|t-s| + |x-y| < \eta$. Since $[0,T] \times [-R-2,R+2]$ is compact, we can find a finite set of points $\{(t_a,x_i) : 1 \le a,i \le n\}$ in $[0,T] \times [-R-2,R+2]$ such that every $(t,x) \in [0,T] \times [-R-1,R+1]$, there is some (t_a,x_j) so that $|t-t_a| + |x-x_j| < \eta$ and $[x_j - 1, x_j + 1] \subset [-R-2,R+2]$.

Define $\Phi : \mathfrak{F} \to \mathbb{R}^{n^2}$ by

$$\Phi(f) = \left(f(t_a,x_i) : 1 \le a,i \le n\right).$$

Condition 1 and equicontinuity imply the image $\Phi(\mathfrak{F})$ is bounded and thus totally bounded in $\mathbb{R}^{n^2}$.

Furthermore, if $f,g \in \mathfrak{F}$ with $\|\Phi(f) - \Phi(g)\|_\infty < \delta_2$, then since for any $(t,x) \in [0,T] \times [-R-1,R+1]$ there are some a, j so that $|t-t_a| + |x-x_j| < \eta$,

$$|f(t,x)-g(t,x)| \le |f(t,x)-f(t_a,x_j)| + |f(t_a,x_j)-g(t_a,x_j)| + |g(t_a,x_j)-g(t,x)| \le 3\delta_2 ,$$

and so $\sup_{t\in[0,T];x\in[-R-1,R+1]} |f(t,x) - g(t,x)| \le 3\delta_2$. In addition, for every $(t,x) \in [0,T] \times [-R,R]$,

$$V^{(\beta)}_{f-g,1}(t,x) \le 4 \sup_{h=f or g} \int_{|y|\le\delta_1} |h(t,x+y) - h(t,x)|^2 \frac{dy}{|y|^{1+2\beta}} + 4 \sup_{r\in[0,T];z\in[-R-1,R+1]} |f(r,z) - g(r,z)|^2 \int_{|y|\ge\delta_1} \frac{dy}{|y|^{1+2\beta}}$$

$$\leq 2\varepsilon^2 .$$

We have shown

$$\sup_{t\in[0,T];x\in[-R,R]} \int_{\mathbb{R}} |f(t,x+y)-f(t,x)-g(t,x+y)+g(t,x)|^2 \frac{dy}{|y|^{1+2\beta}} \leq 12\varepsilon^2 ,$$

provided that $\|\Phi(f)-\Phi(g)\|_\infty < \delta_2$. Therefore, by Lemma 6.17, the set $\mathfrak{F}$ is totally bounded in X_T^β. $\square$

A useful consequence is the following

Corollary 6.19. *Suppose $\alpha > \beta$. Let $\mathfrak{F}$ be a subset of X_T^α such that $\mathfrak{F}$ is equicontinuous in time for every fixed x and $\sup_{f\in\mathfrak{F}}\sup_{t\leq T;|x|\leq R} V_{f,1}^{(\alpha)}(t,x)$ is finite for every positive R. Then $\mathfrak{F}$ is relatively compact in X_T^β.*

Proof. It suffices to check that $\mathfrak{F}$ satisfies condition 3 in Proposition 6.18. Applying (4.9), for δ small enough, the assumption on $\mathfrak{F}$ implies

$$\sup_{f\in\mathscr{F}} \sup_{t\leq T;|x|\leq R} |f(t,x+y)-f(t,x)| \leq C|y|^\alpha$$

for all $|y|\leq\delta$. Hence,

$$\sup_{f\in\mathscr{F}} \sup_{t\leq T;|x|\leq R} \int_{|y|\leq\delta} |f(t,x+y)-f(t,x)|^2 |y|^{-2\beta-1} dy \leq C \int_{|y|\leq\delta} |y|^{2(\alpha-\beta)-1} dy$$

which clearly implies condition 3 in Proposition 6.18 since $\alpha > \beta$. $\square$

Let $\bar{\mathfrak{X}}_T^\beta(B)$ be a subset of $\mathfrak{X}_T^\beta(B)$ consisting of functions f in $\mathfrak{X}_T^\beta(B)$ such that $V_f^{(\beta)}$ is continuous and bounded on $[0,T]\times\mathbb{R}$. We equip $\bar{\mathfrak{X}}_T^\beta(B)$ with the topology of uniform convergence over compact sets. More precisely, a sequence $\{f_n\}$ is convergent in $\bar{\mathfrak{X}}_T^\beta(B)$ if and only if f_n and $V_{f_n - f}$ converge uniformly over compact intervals of $[0,T] \times \mathbb{R}$ to f and 0 respectively. Similar to Proposition 6.11, it is easy to check that $\bar{\mathfrak{X}}_T^\beta(B)$ is complete with this topology.

Proposition 6.20. *Suppose that a set $\mathfrak{F}$ in $\bar{\mathfrak{X}}_T^\beta(B)$ satisfies*

1. *For every $t \in [0,T]; x \in \mathbb{R}$, $\mathfrak{F}(t,x) := \{f(t,x) : f \in \mathfrak{F}\}$ is relative compact in B;*
2. *For every fixed $x \in \mathbb{R}$, $\{f(\cdot,x) : f \in \mathfrak{F}\}$ is equicontinuous in time;*
3. *For every $R > 0$, $\lim_{\delta \downarrow 0} \sup_{f \in \mathfrak{F}} \sup_{t\in[0,T];x\in[-R,R]} \int_{-\delta}^{\delta} \|f(t,x+y) - f(t,x)\|^2 \frac{dy}{|y|^{1+2\beta}} = 0.$*

Then $\mathfrak{F}$ is relatively compact in $\bar{\mathfrak{X}}_T^\beta(B)$.

Proof. We first observe that the condition (3) together with (4.9) implies the following equicontinuity property: for every $R > 0$, for every $\varepsilon > 0$, there exists $\eta > 0$ such that $\sup_{t\in[0,T]} \|f(t,x) - f(t,y)\| < \varepsilon$ whenever $f \in \mathfrak{F}$ and $x, y \in [-R,R]$ such that $|x - y| < \eta$. Together with (2), this implies equicontinuity for $\mathfrak{F}$ in (t,x).

Fix $R > 0$ and $\varepsilon > 0$, from (3), we can choose a positive number $\delta_1 = \delta_1(\varepsilon)$, $\delta_1 < 1$, such that

$$4 \sup_{f\in\mathscr{F}} \sup_{t\in[0,T];x\in[-R,R]} \int_{-\delta_1}^{\delta_1} \|f(t,x+y) - f(t,x)\|^2 \frac{dy}{|y|^{1+2\beta}} \le \varepsilon^2 .$$

We now choose $\delta_2 \le \varepsilon$ such that

$$4 \int_{|y|>\delta_1} (3\delta_2)^2 \frac{dy}{|y|^{1+2\beta}} \le \varepsilon^2$$

By the equicontinuity, we can choose a positive number $\eta = \eta(\varepsilon)$, $\eta < 1$, such that $\|f(t,x) - f(s,y)\| \le \delta_2$ whenever $f \in \mathfrak{F}$ and $(t,x), (s,y) \in [0,T] \times [-R-2, R+2]$ so that $|t-s| + |x-y| < \eta$. Since $[0,T] \times [-R-2, R+2]$ is compact, we can find a finite set of points $\{(t_a,x_i) : 1 \le a,i \le n\}$ in $[0,T]\times[-R-2,R+2]$ such that every $(t,x)\in[0,T]\times[-R-1,R+1]$, there is some (t_a,x_j) so that $|t-t_a|+|x-x_j|<\eta$ and $[x_j-1,x_j+1]\subset[-R-2,R+2]$.

Define $\Phi : \mathfrak{F} \to B^{n^2}$ by

$$\Phi(f) = (f(t_a,x_i) : 1 \le a,i \le n)\,.$$

Condition (1) implies the image $\Phi(\mathfrak{F})$ is relative compact and thus totally bounded in B^{n^2}.

Furthermore, if $f,g\in\mathfrak{F}$ with $\|\Phi(f)-\Phi(g)\|_\infty<\delta_2$, then since for any $(t,x)\in[0,T]\times[-R-1,R+1]$ there are some a,j so that $|t-t_a|+|x-x_j|<\eta$,

$$\|f(t,x)-g(t,x)\| \le \|f(t,x)-f(t_a,x_j)\|+\|f(t_a,x_j)-g(t_a,x_j)\|+\|g(t_a,x_j)-g(t,x)\| \le 3\delta_2\,,$$

and so $\sup_{t\in[0,T];x\in[-R-1,R+1]}\|f(t,x)-g(t,x)\|\le 3\delta_2$. In addition, for every $(t,x)\in[0,T]\times[-R,R]$,

$$\begin{aligned}
V^2_{f-g}(t,x) &\le 4\sup_{h=f\,or\,g}\int_{|y|\le\delta_1}\|h(t,x+y)-h(t,x)\|^2\frac{dy}{|y|^{1+2\beta}}\\
&\quad +4\sup_{r\in[0,T];z\in[-R-1,R+1]}\|f(r,z)-g(r,z)\|^2\int_{|y|\ge\delta_1}\frac{dy}{|y|^{1+2\beta}}\\
&\le 2\varepsilon^2\,.
\end{aligned}$$

We have shown

$$\sup_{t\in[0,T];x\in[-R,R]}\int_{-1}^{1}\|f(t,x+y)-f(t,x)-g(t,x+y)+g(t,x)\|^2\frac{dy}{|y|^{1+2\beta}}\le 12\varepsilon^2\,,$$

provided that $\|\Phi(f)-\Phi(g)\|_\infty < \delta_2$. Therefore, by Lemma 6.17, the set $\mathfrak{F}$ is totally bounded in $\bar{\mathfrak{X}}_T^\beta$. $\square$

Proposition 6.21 (Left composition)**.** *Let σ be a Lipschitz function on $\mathbb{R}$ and f be a function in X_T^β. Suppose that for every fixed x, the map $t \mapsto V_{\sigma\circ f,1}(t,x)$ is continuous. Then $\sigma\circ f$ belongs to X_T^β. Furthermore, if f_n is a sequence converging to f in X_T^β, then for every positive R*

$$\lim_n \sup_{t\le T;|x|\le R} V_{\sigma\circ f_n-\sigma\circ f,1}(t,x)=0\,.$$

Proof. We first show that $\sigma\circ f$ belongs to X_T^β. We have

$$\int_{|y|\le\delta} |\sigma(f(t,x+y))-\sigma(f(t,x))|^2|y|^{-2\beta-1}dy \le \|\sigma\|_{Lip}^2 V_{f,\delta}^2(t,x)$$

which together with the criterion ((iii)) in Proposition 6.14 imply that $\sigma(f)$ belongs to X_T^β.

For the second assertion, for every positive R, fix $\varepsilon>0$, choose $\delta>0$ so that

$$\begin{aligned}
\sup_{t\le T;|x|\le R} V_{\sigma(f_n)-\sigma(f),\delta}(t,x) &\le \sup_{t\le T;|x|\le R} V_{\sigma(f_n),\delta}(t,x)+\sup_{t\le T;|x|\le R} V_{\sigma(f),\delta}(t,x)\\
&\le \sup_{t\le T;|x|\le R} V_{f_n,\delta}(t,x)+\sup_{t\le T;|x|\le R} V_{f,\delta}(t,x)\\
&\le \sup_{t\le T;|x|\le R} V_{f_n-f,\delta}(t,x)+2\sup_{t\le T;|x|\le R} V_{f,\delta}(t,x)\\
&< \varepsilon\,,
\end{aligned}$$

then we bound

$$\begin{aligned}
&\sup_{t\le T;|x|\le R} V_{\sigma(f_n)-\sigma(f),1}(t,x)\\
\le\ & C\varepsilon + C\|\sigma\|_{Lip}\sup_{t\le T;|x|\le R+1}|f_n(t,x)-f(t,x)|\left(\int_{|y|>\delta}|y|^{-2\beta-1}dy\right)^{1/2}.
\end{aligned}$$

We conclude the proof by passing to the limit $n \to \infty$. □

6.4.3 Probability measures on X_T^β

To show the existence of solution to equation (1.1) we need some tightness arguments of the probability measures defined on X_T^β. Let $\mathbf{P}_n$ be probability measures on X_T^β. We have the following result.

Theorem 6.22. *The sequence $\{\mathbf{P}_n\}$ is tight if these three conditions hold:*

1. *For each positive η, there exist a and n_0 such that*

$$\mathbf{P}_n(f \in X_T^\beta : |f(0,0)| \geq a) \leq \eta, \quad n \geq n_0, \tag{4.14}$$

2. *For every $x \in \mathbb{R}$, positive ε and η, there exist δ, $0 < \delta < 1$ and n_0 such that*

$$\mathbf{P}_n\left(f \in X_T^\beta : \sup_{s,t \leq T; |t-s| < \delta} |f(t,x) - f(s,x)| \geq \varepsilon\right) \leq \eta, \quad n \geq n_0, \tag{4.15}$$

3. *For every $R > 0$, for each positive ε and η, there exist $\delta \in (0,1)$ and n_0 such that*

$$\mathbf{P}_n\left(f \in X_T^\beta : \sup_{t \leq T; |x| \leq R} \int_{-\delta}^{\delta} |f(t,x+y) - f(t,x)|^2 |y|^{-2\beta-1} dy \geq \varepsilon\right) \leq \eta, \quad n \geq n_0, \tag{4.16}$$

Proof. Without loss of generality we assume $n_0 = 1$. For a given small positive number η, choose a so that

$$\mathbf{P}_n\left(f \in X_T^\beta : |f(0,0)| \geq a\right) \leq \eta, \quad n \geq 1,$$

fix such a, we denote

$$B = \left\{f \in X_T^\beta : |f(0,0)| < a\right\},$$

according to condition (3), for any integer k, N, we choose and fix δ_{kN} such that

$$\mathbf{P}_n \left(f \in X_T^\beta : \sup_{t \le T; |x| \le N} \int_{-\delta_{kN}}^{\delta_{kN}} |f(t, x+y) - f(t,x)|^2 |y|^{-2\beta - 1} dy \ge \frac{1}{k^2} \right) \text{ł} \le \eta 2^{-k-N}, \quad n \ge 1,$$

for such δ_{kN}, we denote

$$A_{kN} = \left\{ f \in X_T^\beta : \sup_{t \le T; |x| \le N} \int_{-\delta_{kN}}^{\delta_{kN}} |f(t, x+y) - f(t,x)|^2 |y|^{-2\beta - 1} dy \le \frac{1}{k^2} \right\}.$$

Then for each $\tilde{x} \in [-N, N] \cap \frac{\delta_{kN}}{3}\mathbb{Z}$, where $\mathbb{Z}$ is the set of integers, we note that the number of such $\tilde{x}$ has order $\frac{N}{\delta_{kN}}$, we choose $\delta'_{kN}(\tilde{x})$ according to condition (2) such that

$$\mathbf{P}_n \left(f \in X_T^\beta : \sup_{t,s, \le T, |t-s| \le \delta'_{kN}(\tilde{x})} |f(t, \tilde{x}) - f(s, \tilde{x})| \ge \frac{1}{k^2} \right) \le \eta \delta_{kN} 2^{-k-N}, \quad n \ge 1,$$

and denote

$$B_{kN}(\tilde{x}) = \left\{ f \in X_T^\beta : \sup_{t,s, \le T, |t-s| \le \delta'_{kN}(\tilde{x})} |f(t, \tilde{x}) - f(s, \tilde{x})| \le \frac{1}{k^2} \right\},$$

set $B_{kN} = \cap_{\tilde{x} \in [-N,N] \cap \frac{\delta_{kN}}{3}\mathbb{Z}} B_{kN}(\tilde{x})$, it is easy to see that

$$\mathbf{P}_n(B_{kN}^c) \le \sum_{\tilde{x} \in [-N,N] \cap \frac{\delta_{kN}}{3}\mathbb{Z}} \mathbf{P}_n(B_{kN}^c(\tilde{x})) \le C \frac{N}{\delta_{kN}} \eta \delta_{kN} 2^{-k-N} \le C \eta 2^{-k-N} N,$$

then we set $A = \cap_{k,N} (A_{kN} \cap B_{kN}) \cap B$, then according to Proposition 6.18 we see that the closure of A is compact in X_T^β, and $\mathbf{P}_n(A^c) \ge 1 - C\eta$. This shows the tightness of $\mathbf{P}_n$. □

The next lemma gives the criterion that if a process is in $\mathfrak{X}_T^{\alpha,p}$, then its paths almost surely lie in the space X_T^β for some β.

Lemma 6.23. *Let f be a stochastic process in $\mathfrak{X}_T^{\alpha,p}$ with $p\alpha > 1$. Assume that f is continuous in time almost surely. Then f has a version $\tilde{f}$ such that with probability one, $\tilde{f}$ belongs to X_T^{β} for every $\beta < \alpha - \frac{1}{p}$.*

Proof. Since f belongs to $\mathfrak{X}_T^{\alpha,p}$, inequality (4.9) implies

$$\sup_{t\le T}\sup_{x,y\in\mathbb{R}} \frac{\|f(t,x+y)-f(t,x)\|_{L^p(\Omega)}}{|y|^{\alpha}} \le C \sup_{t\le T;x\in\mathbb{R}} \int_{\mathbb{R}} \|f(t,x+y)-f(t,x)\|_{L^p(\Omega)}^2 |y|^{-2\alpha-1} dy\,.$$

Then by Kolmogorov continuity criteria, f has a version $\tilde{f}$ such that with probability one, $\tilde{f}$ satisfies

$$\sup_{t\le T;|x|\le n} |\tilde{f}(t,x+y)-\tilde{f}(t,x)| \le C|y|^{\beta'}$$

for every n and $|y|\le 1$, where β' is fixed such that $\beta<\beta'<\alpha-1/p$. This implies a.s. $V_{\tilde{f},1}^{(\beta)}(t,x)$ is finite and $V_{\tilde{f},\delta}^{(\beta)}$ satisfies the condition (4.13). The continuity of $V_{\tilde{f},1}^{(\beta)}$ follows from dominated convergence theorem. These facts imply $\tilde{f}$ belongs to X_T^{β} almost surely. □

The following proposition states that under some conditions, a sequence of processes u_n can be regarded as a tight sequence of probability measures on the space X_T^{β}.

Proposition 6.24. *Assume that $\alpha,\lambda\in(0,1)$, $p\alpha>1$, $p\lambda>1$ and $\beta<\alpha-\frac{1}{p}$. Let $\{u_n\}$ be a sequence of stochastic processes such that*

1. $\lim_{\delta\to\infty}\limsup_{n}\mathbf{P}(|u_n(0,0)|>\delta)=0\,,$
2. *For every $s,t\in[0,T]$ and $x\in\mathbb{R}$,* $\sup_n \|u_n(t,x)-u_n(s,x)\|_{L^p(\Omega)} \le C|t-s|^{\lambda}\,,$
3. $\sup_n \|u_n\|_{\mathfrak{X}_T^{\alpha,p}}$ *is finite.*

From Lemma 6.23, the law of u_n can be considered as a probability measure on X_T^{β}. In addition, as probability measures on X_T^{β}, $\{u_n\}$ is tight.

Proof. This proposition can be easily proved using the same idea as in the proof of Lemma 6.23 and Theorem 6.22, we omit the details. □

6.4.4 Existence of the solution

The main result of this subsection is the the following existence result.

Theorem 6.25. *Assume that for equation (1.1),*

1. *For some sufficiently big p, the initial condition u_0 is in $L^p(\mathbb{R})$ and*

$$\int_{\mathbb{R}} \|u_0(\cdot) - u_0(\cdot + h)\|^2_{L^p(\mathbb{R})} |h|^{2H-2} < \infty. \tag{4.17}$$

2. *σ is differentiable and the derivative of σ is Lipschitz and $\sigma(0) = 0$.*

Then there exists a solution u to (1.1) with paths in the space $X_T^{\frac{1}{2}-H}$, where $X_T^{\frac{1}{2}-H}$ is defined by the condition (2) and with the metric (4.12).

Proof. We will adopt the idea from [44]. We consider the regularization of the noise, which is smoothed in space. Indeed, for $\varepsilon > 0$ and $\varphi \in \dot{H}$ we define:

$$W^\varepsilon(\varphi) = \int_0^t \int_{\mathbb{R}} [\rho_\varepsilon * \varphi](s,x) W(ds,dy) = \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(s,x) \rho_\varepsilon(x-y) W(ds,dy) dx, \tag{4.18}$$

where $\rho_t(x) = (2\pi t)^{-\frac{1}{2}} e^{-x^2/2t}$. Notice that relation (4.18) can be also read (either in Fourier or direct coordinates) as:

$$\begin{aligned} \mathbb{E}[W^\varepsilon(\varphi) W^\varepsilon(\psi)] &= c_H \int_0^t \int_{\mathbb{R}} \mathscr{F}\varphi(s,\xi) \overline{\mathscr{F}\psi(s,\xi)} e^{-\varepsilon|\xi|^2} |\xi|^{1-2H} d\xi ds \\ &= c_H \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(s,x) f_\varepsilon(x-y) \psi(s,y) \, dx dy ds, \end{aligned}$$

where f_ε is given by: $f_\varepsilon(x) = \mathscr{F}^{-1}(e^{-\varepsilon|\xi|^2}|\xi|^{1-2H})$. In other words, our noise is still a white noise in time but its space covariance is now dictated by f_ε. Note that f_ε is a real positive definite function, but is not necessarily positive. We however have

$$E|W^\varepsilon(\varphi)|^2 \le E|W(\varphi)|^2 \tag{4.19}$$

for all φ in $\dot{H}$.

For every fixed $\varepsilon > 0$, the noise W^ε induces an approximation to equation (2.10), namely:

$$u^\varepsilon(t,x) = p_t u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u^\varepsilon(s,y))\, W^\varepsilon(ds,dy), \tag{4.20}$$

where the integral is understood in the Itô sense. Since $|\xi|^{1-2H}e^{-\varepsilon|\xi|^2}$ is in $L^1(\mathbb{R})$, $|f_\varepsilon|$ is bounded, using Picard iteration it is easy to see that (4.20) has a unique random field solution, and by estimating the pth moment of $|u^\varepsilon(t,x) - u^\varepsilon(t,x')|$, we see that each solution $u^\varepsilon(t,x)$ is Hölder continuous in space with order β for all $\beta \in (0,1)$, thus we conclude that u^ε in $\mathfrak{X}_T^{\beta,p}$ for all $\beta \in (0,1)$. However, using (4.19) and (3.4) (it is not hard to see that (3.4) also applies to the approximate solution $u^\varepsilon(t,x)$)and Gronwall's lemma, we can obtain the following uniform bound

$$\sup_{\varepsilon>0} \|u^\varepsilon\|_{\mathfrak{X}_T^{\beta,p}} < \infty$$

for all $\frac{1}{2} - H < \beta < H$.

Then Proposition 6.10 together with Kolmogorov continuity criteria implies that almost surely, u^ε belongs to X_T^β for $\beta \in [1/2 - H, H)$. In addition, we can show that u^ε is Hölder continuous in time. With these properties, we can check that the three conditions in Theorem 6.22 are satisfied, thus u_ε (considered as probabilities measures on X_T^β) are tight and hence weakly relatively compact (i.e. relatively compact in the topology of distributions)(see also Proposition 6.24) .

We will use Lemma 6.26 to prove that the sequence u_ε actually converges in probability. To apply this lemma, we consider now two sequences $u_m := u_{\varepsilon_m}$ and $u_l := u_{\varepsilon_l}$, ε_m and ε_k converge to 0 as m and k go to infinity respectively. The triplet (u_m, u_l, W) is considered as probability measure on $\mathscr{B} := X_T^{\frac{1}{2}-H} \times X_T^{\frac{1}{2}-H} \times C_{uc}([0,T]\times\mathbb{R})$. Since $\{u_\varepsilon\}$ is weakly relatively compact, there exist a subsequence $(u_{m(k)}, u_{l(k)}, W)$ which converges in distribution as $k \to \infty$. Thus, by Skorokhod embedding theorem, there is a probability space $(\Omega', \mathscr{F}', P')$ and a sequence of probability measures $z_k = (u'_{m(k)}, u'_{l(k)}, W')$ on $\mathscr{B}$ such that z_k has the same distribution as $(u_{m(k)}, u_{l(k)}, W)$ and z_k converges almost surely (in the topology of $\mathscr{B}$) to (u', v', W'). By Lemma 6.28 we see that both u' and v' are solutions to equation (2.10), with W replaced by W'. Then by Lemma 6.27 and the uniqueness result Theorem 6.9 we see that $u' = v'$ in $X_T^{\frac{1}{2}-H}$. From Lemma 6.26 we see that u_ε converges to some random field u in $X_T^{\frac{1}{2}-H}$, in probability, passing to a subsequence if necessary, we see that u_ε converges to u in $X_T^{\frac{1}{2}-H}$ a.s., then another application of Lemma 6.28 we see that u satisfies equation (2.10). Thus the existence of the solution is proved.

□

Lemma 6.26. *Let $\mathbb{E}$ be a Polish space equipped with the Borel σ-algebra. A sequence of $\mathbb{E}$-valued random elements z_n converges in probability if and only if for every pair of subsequences z_l, z_m there exists a subsequence $w_k := (z_{l(k)}, z_{m(k)})$ converging weakly to a random element w supported on the diagonal $\{(x,y) \in \mathbb{E}\times\mathbb{E} : x = y\}$.*

Lemma 6.27. *The approximate solutions u_ε satisfy the condition*

$$\sup_\varepsilon \|u_\varepsilon\|_{\mathscr{Z}_T^p} < \infty. \tag{4.21}$$

Furthermore, if $u_\varepsilon \to u$ in $X_T^{\frac{1}{2}-H}$ a.s., then u is also in $\mathscr{Z}_T^p$.

Proof. Recall the covariance of the regularized noise is given by

$$\mathbf{E}[\dot{W}^\varepsilon(s,y)\dot{W}^\varepsilon(t,x)] = \delta(s-t)f_\varepsilon(x-y)\,,$$

where f_ε is the inverse Fourier transform of $|\xi|^{1-2H}e^{-\varepsilon|\xi|^2}$, which is bounded for each fixed ε. We will use Picard iteration to show that for each ε, $u_\varepsilon \in \mathscr{Z}_T^p$. Then we will use the Gronwall's lemma to show that the norms in $\mathscr{Z}_T^p$ of u_ε is bounded uniformly in ε. To this end, we first define

$$u_\varepsilon^0(t,x) = u_0(x)\,,$$

and

$$u_\varepsilon^{n+1}(t,x) = p_t u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u_\varepsilon^n(s,y))W_\varepsilon(ds,dy)\,.$$

Then by Burkholder's inequality we have

$$\begin{aligned}
& \mathbf{E}|u_\varepsilon^{(n+1)}(t,x) - u_\varepsilon^{(n)}(t,x)|^p \\
= \; & \mathbf{E}\left|\int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)[\sigma(u_\varepsilon^{(n)}(s,y)) - \sigma(u_\varepsilon^{(n-1)}(s,y))]W_\varepsilon(ds,dy)\right|^p \\
\leq \; & C\mathbf{E}\Big|\int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)p_{t-s}(x-z)[\sigma(u_\varepsilon^{(n)}(s,y)) - \sigma(u_\varepsilon^{(n-1)}(s,y))] \\
& \quad \times[\sigma(u_\varepsilon^{(n)}(s,z)) - \sigma(u_\varepsilon^{(n-1)}(s,z))]f_\varepsilon(y-z)dydzds\Big|^{\frac{p}{2}} \\
\leq \; & C_\varepsilon \mathbf{E}\left|\int_0^t \left(\int_{\mathbb{R}} p_{t-s}(x-y)|\sigma(u_\varepsilon^{(n)}(s,y)) - \sigma(u_\varepsilon^{(n-1)}(s,y))|dy\right)^2 ds\right|^{\frac{q}{2}} \\
\leq \; & C_\varepsilon \mathbf{E}\left|\int_0^t \left(\int_{\mathbb{R}} p_{t-s}(y)|u_\varepsilon^{(n)}(s,x+y) - u_\varepsilon^{(n-1)}(s,x+y)|dy\right)^2 ds\right|^{\frac{q}{2}}\,,
\end{aligned}$$

taking the integral with respect to the space variable and using Minkowski inequality we obtain

$$
\begin{aligned}
&\mathbf{E}\left\|u_\varepsilon^{(n+1)}(t,\cdot)-u_\varepsilon^{(n)}(t,\cdot)\right\|_{L^p(\mathbb{R})}^p \\
\leq\;& C_\varepsilon \mathbf{E}\left\|\int_0^t\left(\int_{\mathbb{R}} p_{t-s}(y)|u_\varepsilon^{(n)}(s,y+\cdot)-u_\varepsilon^{(n-1)}(s,y+\cdot)|dy\right)^2 ds\right\|_{L^{\frac{p}{2}}(\mathbb{R})}^{\frac{p}{2}} \\
\leq\;& C_\varepsilon \mathbf{E}\left(\int_0^t\left(\int_{\mathbb{R}} p_{t-s}(y)\left\|u_\varepsilon^{(n)}(s,\cdot)-u_\varepsilon^{(n-1)}(s,\cdot)\right\|_{L^p(\mathbb{R})}dy\right)^2 ds\right)^{\frac{p}{2}} \\
\leq\;& C_\varepsilon\left(\int_0^t\left\|u_\varepsilon^{(n)}(s,\cdot)-u_\varepsilon^{(n-1)}(s,\cdot)\right\|_{L^p(\Omega\times\mathbb{R})}^2 ds\right)^{\frac{p}{2}},
\end{aligned}
$$

hence we conclude that

$$
\left\|u_\varepsilon^{(n)}(t,\cdot)-u_\varepsilon^{(n-1)}(t,\cdot)\right\|_{L^p(\Omega\times\mathbb{R})}^2 \leq C_\varepsilon\int_0^t\left\|u_\varepsilon^{(n)}(s,\cdot)-u_\varepsilon^{(n-1)}(s,\cdot)\right\|_{L^p(\Omega\times\mathbb{R})}^2 ds,
$$

Using Gronwall's lemma we see that for each fixed ε,

$$
\sup_{t\leq T}\|u_\varepsilon(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})}<\infty.
$$

Next we estimate the second part in the norm of $\mathscr{Z}_T^p$. Instead of Picard iteration, we directly estimate u_ε to obtain

$$
\begin{aligned}
&\int_{\mathbb{R}}\mathbf{E}|u_\varepsilon(t,x)-u_\varepsilon(t,x+h)|^p dx \\
\leq\;& C\int_{\mathbb{R}}|p_t u_0(x)-p_t u_0(x+h)|^p dx \\
&+C_\varepsilon\int_{\mathbb{R}}\mathbf{E}\left|\int_0^t\left(\int_{\mathbb{R}}|p_{t-s}(y)-p_{t-s}(y+h)||u_\varepsilon(s,y+x)|dy\right)^2 ds\right|^{\frac{p}{2}} dx \\
\leq\;& C\int_{\mathbb{R}}|p_t u_0(x)-p_t u_0(x+h)|^p dx
\end{aligned}
$$

$$+\left(\int_0^t\left(\int_{\mathbb{R}}|p_{t-s}(y)-p_{t-s}(y+h)|dy\right)^2\|u_\varepsilon(s,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})}ds\right)^{\frac{p}{2}},$$

thus we obtain

$$\begin{aligned}
&\int_{\mathbb{R}}\frac{\|u_\varepsilon(t,\cdot)-u_\varepsilon(t,\cdot+h)\|^2_{L^p(\Omega\times\mathbb{R})}}{|h|^{2-2H}}dh\\
\le\ & C\int_{\mathbb{R}}\frac{\|p_tu_0(\cdot)-p_tu_0(\cdot+h)\|^2_{L^p(\mathbb{R})}}{|h|^{2-2H}}dh\\
&+C_\varepsilon\int_0^t\int_{\mathbb{R}}\frac{(\int_{\mathbb{R}}|p_{t-s}(y)-p_{t-s}(y+h)|dy)^2}{|h|^{2-2H}}dhds\sup_{s\le T}\|u_\varepsilon(s,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})}\\
<\ & \infty.
\end{aligned}$$

So we conclude that for each fixed ε, $u_\varepsilon\in\mathscr{Z}_T^p$. To prove the norms of u_ε in $\mathscr{Z}_T^p$ are uniformly bounded in ε, we note that since u_ε satisfies the equation

$$u_\varepsilon(t,x)=p_tu_0(x)+\int_0^t\int_{\mathbb{R}}[(p_{t-s}(x-\cdot)\sigma(u_\varepsilon(s,\cdot)))*p_\varepsilon](y)W(ds,dy),$$

then

$$\begin{aligned}
&\mathbf{E}|u_\varepsilon(t,x)|^p\\
\le\ & C|p_tu_0(x)|^p\\
&+C\mathbf{E}\left(\int_0^t\int_{\mathbb{R}}|\mathscr{F}\left(p_{t-s}(x-\cdot)\sigma(u_\varepsilon(s,\cdot))\right)(\xi)|^2e^{-\varepsilon|\xi|^2}|\xi|^{1-2H}d\xi ds\right)^{\frac{p}{2}}\\
\le\ & C|p_tu_0(x)|^p\\
&+C\mathbf{E}\left(\int_0^t\int_{\mathbb{R}^2}\frac{|p_{t-s}(x-y)\sigma(u_\varepsilon(s,y))-p_{t-s}(x-y-z)\sigma(u_\varepsilon(s,y+z))|^2}{|z|^{2-2H}}dzdyds\right)^{\frac{p}{2}}\\
\le\ & C\|u_0\|^p_{L^p(\mathbb{R})}+C\left(\int_0^t(t-s)^{-\frac{1}{2}}\int_{\mathbb{R}}\|u_\varepsilon(s,\cdot)-u_\varepsilon(s,\cdot+h)\|^2_{L^p(\Omega\times\mathbb{R})}|h|^{2H-2}dhds\right)^{\frac{p}{2}}\\
&+C\left(\int_0^t(t-s)^{H-1}\|u_\varepsilon(s,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})}ds\right)^{\frac{p}{2}},
\end{aligned}$$

similarly we get

$$\begin{aligned}
&\int_{\mathbb{R}} \|u_\varepsilon(t,\cdot)-u_\varepsilon(t,\cdot+h)\|^2_{L^p(\Omega\times\mathbb{R})}|h|^{2H-2}dh \\
\le\ & C\int_{\mathbb{R}} \|u_0(\cdot)-u_0(\cdot+h)\|^2_{L^p(\mathbb{R})}|h|^{2H-2}dh \\
&+C\int_0^t\int_{\mathbb{R}}(t-s)^{H-1}\|u_\varepsilon(s,\cdot)-u_\varepsilon(s,\cdot+l)\|^2_{L^p(\Omega\times\mathbb{R})}|l|^{2H-2}dlds \\
&+C\int_0^t (t-s)^{2H-\frac{3}{2}}\|u_\varepsilon(s,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})}ds\,.
\end{aligned}$$

Thus combining the above estimates we obtain

$$\begin{aligned}
&\|u_\varepsilon(t,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})}+\int_{\mathbb{R}} \|u_\varepsilon(t,\cdot)-u_\varepsilon(t,\cdot+h)\|^2_{L^p(\Omega\times\mathbb{R})}|h|^{2H-2}dh \\
\le\ & C\|u_0\|^2_{L^p(\mathbb{R})}+C\int_{\mathbb{R}} \|u_0(\cdot)-u_0(\cdot+h)\|^2_{L^p(\mathbb{R})}|h|^{2H-2}dh \\
&+C\int_0^t (t-s)^{2H-\frac{3}{2}}\|u_\varepsilon(s,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})}ds \\
&+C\int_0^t\int_{\mathbb{R}}(t-s)^{H-1}\|u_\varepsilon(s,\cdot+h)-u_\varepsilon(s,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})}|h|^{2H-2}dhds\,,
\end{aligned}$$

since we have shown that for each fixed ε, $\|u_\varepsilon\|_{\mathscr{Z}_T^p}<\infty$, we can apply the Gronwall's lemma to the above inequality to show that

$$\sup_{\varepsilon>0}\|u_\varepsilon\|_{\mathscr{Z}_T^p}<\infty\,.$$

Next we prove that u is in $\mathscr{Z}_T^p$. Since $u_\varepsilon\to u$ in $X_T^{\frac{1}{2}-H}$ a.s., so for each t and x $u_\varepsilon(t,x)\to u(t,x)$ a.s. Thus by Fatou's lemma,

$$\begin{aligned}
\|u(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})} &= \left(\mathbf{E}\int_{\mathbb{R}}\lim_{\varepsilon\to 0}|u_\varepsilon(t,x)|^p dx\right)^{\frac{1}{p}} \\
&\le \varliminf_{\varepsilon\to 0}\left(\mathbf{E}\int_{\mathbb{R}}|u_\varepsilon(t,x)|^p dx\right)^{\frac{1}{p}}\le C\,,
\end{aligned}$$

then we conclude that $\sup_{t\le T}\|u(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})}$ is finite. On the other hand, for each x and h we have $|u_\varepsilon(t,x+h)-u_\varepsilon(t,x)|^2 \to |u(t,x+h)-u(t,x)|^2$, so by Fatou's lemma

$$\begin{aligned}
&\int_{|h|\le 1}\frac{\|u(t,\cdot+h)-u(t,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})}}{|h|^{2-2H}}dh\\
\le\ &\int_{|h|\le 1}\frac{\underline{\lim}_{\varepsilon\to 0}\|u(t,\cdot+h)-u(t,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})}}{|h|^{2-2H}}dh\\
\le\ &\underline{\lim}_{\varepsilon\to 0}\int_{|h|\le 1}\frac{\|u(t,\cdot+h)-u(t,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})}}{|h|^{2-2H}}dh,
\end{aligned}$$

for the integral when $|h|>1$, we simply bound $\|u(t,\cdot+h)-u(t,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})}$ by

$$2\|u(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})},$$

so we can conclude that

$$\sup_{t\le T}\int_{\mathbb{R}}\frac{\|u(t,\cdot+h)-u(t,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})}}{|h|^{2-2H}}dh<\infty.$$

Together with the previous estimate, we conclude that $u\in\mathscr{Z}_T^p$. □

Lemma 6.28. *Let $u_n(t,x)$ be a solution to the equation*

$$u_n(t,x)=p_tu_0(x)+\int_0^t\int_{\mathbb{R}}p_{t-s}(x-y)\sigma(u_n(s,y))W^n(ds,dy),$$

where W_n is defined in (4.18) with ε defined by ε_n such that as $n\to\infty$, $\varepsilon_n\to 0$.

We assume the following conditions

1. *with probability one, u_n converges to u in $X_T^{\frac{1}{2}-H}$,*
2. $\sup_n\|u_n\|_{\mathfrak{X}_T^{\beta,p}}<\infty$.

Then u belongs to $\mathfrak{X}_T^{\frac{1}{2}-H,2}$ *and for any fixed* $t \le T$ *and* $x \in \mathbb{R}$,

$$\int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u_n(s,y))W^n(ds,dy)$$

converges to

$$\int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u(s,y))W(ds,dy)$$

in $L^2(\Omega)$ *as* $n \to \infty$.

Proof. First we note that

$$\begin{aligned}
& \mathbf{E}\left|\int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u(s,y))W^n(ds,dy) - \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u(s,y))W(ds,dy)\right|^2 \\
= \; & \mathbf{E}\left|\int_0^t \int_{\mathbb{R}} \left[(p_{t-s}(x-\cdot)\sigma(u(s,\cdot))) * \rho_{\varepsilon_n}\right](y)W(ds,dy)\right. \\
& \left. - \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u(s,y))W(ds,dy)\right|^2 \\
= \; & C\mathbf{E}\int_0^t \int_{\mathbb{R}} \left|e^{-\frac{\varepsilon_n|\xi|^2}{2}} - 1\right|^2 \left|\mathscr{F}\left(p_{t-s}(x-\cdot)\sigma\left(u(s,\cdot)\right)\right)(\xi)\right|^2 |\xi|^{1-2H} d\xi ds,
\end{aligned}$$

which obviously converges to 0 as ε_n goes to 0 because of the finiteness of

$$\mathbf{E}\int_0^t \int_{\mathbb{R}} \left|\mathscr{F}\left(p_{t-s}(x-\cdot)\sigma\left(u(s,\cdot)\right)\right)(\xi)\right|^2 |\xi|^{1-2H} d\xi ds$$

which can be seen from (6.7) and Fatou's lemma since $u_n(t,x)$ converges to $u(t,x)$ a.s. for each t and x.

It remains to show that

$$\begin{aligned}
\lim_{n\to\infty} \mathbf{E}\bigg| & \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u_n(s,y))W^n(dy,ds) \\
& - \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u(s,y))W^n(dy,ds)\bigg|^2 = 0
\end{aligned}$$

Because of our assumption on W_n, this moment is mojorzied by

$$\mathbf{E}\left|\int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) f_n(s,y) W(ds,dy)\right|^2$$

where $f_n = \sigma \circ u_n - \sigma \circ u$. Using Proposition 6.21 we see that f_n converges to 0 in $X_T^{\frac{1}{2}-H}$. Then an application of Lemma 6.29 completes the proof. □

Lemma 6.29. *Suppose f_n is a sequence of stochastic processes in $\mathfrak{X}_T^{\beta,p} \cap \bar{\mathfrak{X}}_T^{1/2-H,2}$ with $\beta > 1/2 - H$ and $p > 2$. We assume that*

1. *With probability one, f_n converges to 0 over compact sets of $[0,T] \times \mathbb{R}$;*
2. *For every $x \in \mathbb{R}$, $\sup_n \mathbf{E}|f_n(t,x) - f_n(s,x)|^2 \le C|t-s|^\lambda$ for some positive λ,*
3. *$\sup\limits_n \|f_n\|_{\mathfrak{X}_T^{\beta,p}} \le M$ where M is a finite number.*

Then for every $t \le T$ and $x \in \mathbb{R}$

$$\int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) f_n(s,y) W(ds,dy)$$

converges to 0 in $L^2(\Omega)$.

Proof. We first show that $\{f_n\}$ is relatively compact in $\mathfrak{X}_T^{\frac{1}{2}-H,2}$. For this purpose, we verify the three conditions (1)-(3) of Proposition 6.20. Condition (2) in Proposition 6.20 is evident from (2). Condition (3) in Proposition 6.20 is verified by combining the following facts: f_n is bounded in $\mathfrak{X}_T^{\beta,p}$, $p > 2$, $\beta > 1/2 - H$ and inequality (4.9). To verify condition (1) in Proposition 6.20, we fix t,x and note that (1) implies $f_n(t,x)$ converges almost surely to 0. On the other hand, $\mathbf{E}|f_n(t,x)|^p$ is uniformly bounded. These two facts imply $\{f_n(t,x)\}$ converges to 0 in $L^2(\Omega)$, thus condition(1) in Proposition 6.20 is verified.

Furthermore, condition (1) ensures that 0 is the only possible limit point of $\{f_n\}$ in $\bar{\mathfrak{X}}_T^{\frac{1}{2}-H,2}$. We conclude that f_n converges to 0 in $\bar{\mathfrak{X}}_T^{\frac{1}{2}-H,2}$.

By Itô isometry, we see that

$$\mathbf{E}\left|\int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) f_n(s,y) W(ds,dy)\right|^2 = \mathbf{E}\int_0^t \|p_{t-s}(x-\cdot) f_n(s,\cdot)\|_{\dot{H}}^2 ds\,.$$

Using (3.1), we have

$$\begin{aligned}
&\|p_{t-s}(x-\cdot) f_n(s,\cdot)\|_{\dot{H}}^2 \\
&\quad\le C\int_{\mathbb{R}}\int_{\mathbb{R}} |p_{t-s}(x-y-z) f_n(s,y+z) - p_{t-s}(x-y) f_n(s,y)|^2 |z|^{2H-2} dydz \\
&\quad\le C\int_{\mathbb{R}}\int_{\mathbb{R}} |p_{t-s}(x-y-z) - p_{t-s}(x-y)|^2 |f_n(s,y+z)|^2 |z|^{2H-2} dydz \\
&\qquad + C\int_{\mathbb{R}}\int_{\mathbb{R}} |p_{t-s}(x-y)|^2 |f_n(s,y+z) - f_n(s,y)|^2 |z|^{2H-2} dydz\,.
\end{aligned}$$

Thus

$$\mathbf{E}\|p_{t-s}(x-\cdot) f_n(s,\cdot)\|_{\dot{H}}^2 \le C(J_1+J_2)\,,$$

where

$$J_1 = \int_{\mathbb{R}}\int_{\mathbb{R}} |p_{t-s}(x-y-z) - p_{t-s}(x-y)|^2 \mathbf{E} f_n^2(s,y+z) |z|^{2H-2} dydz$$

and

$$J_2 = \int_{\mathbb{R}}\int_{\mathbb{R}} |p_{t-s}(x-y)|^2 \mathbf{E}|f_n(s,y+z) - f_n(s,y)|^2 |z|^{2H-2} dydz\,.$$

Now for every fixed $\varepsilon > 0$ and $R > 0$, choose n so that

$$\sup_{s\le T;|y|\le R} \mathbf{E} f_n^2(s,y) + \sup_{s\le T;|y|\le R} \int_{\mathbb{R}} \mathbf{E}|f_n(s,y+z) - f_n(s,y)|^2 |y|^{2H-2} dy < \varepsilon$$

By making a shift in y,

$$\begin{aligned}J_1 &= \int_{\mathbb{R}}\int_{\mathbb{R}} |p_{t-s}(x-y)-p_{t-s}(x-y+z)|^2 \mathbf{E} f_n^2(s,y)|z|^{2H-2} dydz \\ &\leq \int_{|y-x|\leq R} V_p^2(t-s,x-y)dy \sup_{|y|\leq R} \mathbf{E} f_n^2(s,y) \\ &+ \int_{|y-x|>R} V_p^2(t-s,x-y)dy \sup_n \sup_{r\leq T; w\in\mathbb{R}} \mathbf{E} f_n^2(r,w) \\ &\leq C\varepsilon + CM\int_{|y|>R} V_p^2(t-s,y)dy\,.\end{aligned}$$

Similarly,

$$J_2 \leq C\varepsilon + CM\int_{|y|>R} |p_{t-s}(y)|^2 dy\,.$$

We now choose R sufficiently large so that

$$\int_0^t \int_{|y|>R} [|p_{t-s}(y)|^2 + V_p^2(t-s,y)]dy < \varepsilon$$

then $\mathbf{E}\|p_{t-s}(x-\cdot)f_n(s,\cdot)\|_{\dot{H}}^2 \leq C\varepsilon$ for n sufficiently large. This implies the result. □

Recall the space $\mathfrak{X}_{\theta,\varepsilon}^p$ defined in (3.3), using the same idea as in the proof of Lemma 6.27 we can show that $u \in \mathfrak{X}_{\theta,\varepsilon}^p$. Thus in Proposition 6.7 if we take f to be the solution to equation (1.1) u, and combine it with the mild formulation of the solution, we will get the bound

$$\|u\|_{\mathfrak{X}_{\theta,\varepsilon}^p} \leq C\|u_0\|_{L^\infty} + C\sqrt{p}\|u\|_{\mathfrak{X}_{\theta,\varepsilon}^p}\left(\kappa^{\frac{H}{2}-\frac{1}{2}}\theta^{-\frac{H}{2}} + \varepsilon^{-1}\kappa^{-\frac{1}{4}}\theta^{-\frac{1}{4}} + \varepsilon\kappa^{H-\frac{3}{4}}\theta^{\frac{1}{4}-H}\right),$$

now we choose $\varepsilon = \kappa^{\frac{1}{4}-\frac{H}{2}}\theta^{-\frac{1}{4}+\frac{H}{2}}$ to obtain

$$\|u\|_{\mathfrak{X}_{\theta,\varepsilon}^p} \leq C + C\sqrt{p}\|u\|_{\mathfrak{X}_{\theta,\varepsilon}^p}\kappa^{\frac{H}{2}-\frac{1}{2}}\theta^{-\frac{H}{2}},$$

then choose θ such that $C\sqrt{p}\kappa^{\frac{H}{2}-\frac{1}{2}}\theta^{-\frac{H}{2}}=\frac{1}{2}$, that is

$$\theta = Cp^{\frac{1}{H}}\kappa^{1-\frac{1}{H}}, \quad \text{and} \varepsilon = C\kappa^{\frac{1}{4H}-\frac{1}{2}}p^{\frac{1}{2}-\frac{1}{4H}},$$

the above inequality will give the bound

$$\|u\|_{\mathfrak{X}^p_{\theta,\varepsilon}} \le C.$$

Summarizing the above estimates, we have the following theorem about the moment bound.

Theorem 6.30. *Assume the conditions in Theorem 6.25, then for the solution we have the following moment bound*

$$\sup_{x\in\mathbb{R}} \|u(t,x)\|_{L^p(\Omega)} \le Ce^{Ctp^{\frac{1}{H}}\kappa^{1-\frac{1}{H}}},$$

and

$$\sup_{x\in\mathbb{R}} \left(\int_{\mathbb{R}} \|u(t,x+y)-u(t,x)\|^2_{L^p(\Omega)}|y|^{2H-2}dy\right)^{\frac{1}{2}} \le C\kappa^{\frac{1}{2}-\frac{1}{4H}}p^{\frac{1}{4H}-\frac{1}{2}}e^{Ctp^{\frac{1}{H}}\kappa^{1-\frac{1}{H}}}.$$

If, in addition, we assume that the initial condition u_0 is Hölder continuous with order γ, then by Proposition 6.7 we have

$$\|u(t,x)-u(s,y)\|_{L^p(\Omega)} \le C(|t-s|^{\frac{H}{2}\wedge\frac{\gamma}{2}} + |x-y|^{H\wedge\gamma}) \tag{4.22}$$

for all $s,t\in[0,T]$ and $x,y\in\mathbb{R}$.

6.5 The Anderson model, existence and uniqueness

In this section we will study the special case of equation (1.1) when the function σ is the identity. This is a continuous version of the so-called parabolic Anderson model. In this case equation (1.1) is reduced to

$$\frac{\partial u}{\partial t} = \frac{\kappa}{2}\frac{\partial^2 u}{\partial x^2} + u\dot{W} \tag{5.1}$$

with deterministic initial condition $u(0,x) = u_0(x)$. With some restrictions on the initial condition $u_0(x)$, the existence and uniqueness of the solution to this linear equation stems directly from Theorem 6.9 and 6.25. However, we shall prove this result again by means of two different methods: one is via Fourier transform and the other is via chaos expansion. We include these developments here for two reasons: first they lead to proofs which are shorter and more elegant than in the case of a general coefficient σ; secondly, the assumptions on initial conditions are different.

6.5.1 Existence and uniqueness via Fourier transform

In this subsection we discuss the existence and uniqueness of equation (5.1) using techniques of Fourier analysis.

The spaces of functions adapted to the linear equation (5.1) are of the following form: we denote by $\mathscr{G}$ the class of functions $f:\mathbb{R}\to\mathbb{R}$ such that there exists $g\in L^2(\mathbb{R})$ such that $f = I_-^{1/2-H}g$. On the other hand, let $\mathscr{G}_0$ be the set of functions $f\in L^2(\mathbb{R})$ such that $\int_{\mathbb{R}}|\mathscr{F}f(\xi)|^2|\xi|^{1-2H}d\xi<\infty$. These spaces are the time independent analogues to the spaces $\mathscr{H}$ and $\mathscr{H}_0$ introduced before. Recall that the inclusion $\mathscr{G}_0\subset\mathscr{G}$ is strict.

In the next theorem we show the existence and uniqueness result assuming that the initial condition belongs to $\mathscr{G}_0$ and using estimates based on the Fourier transform in the space variable. To this purpose, we need to introduce a space $\mathscr{V}(H)$ which is an analogue to $\dot{H}$ in Fourier modes. Its is defined by the following semi-norm on spatial processes defined on $\mathbb{R}$:

$$\|X\|^2_{\mathscr{V}(H)} = \int_{\mathbb{R}} \mathbf{E}\left[|\mathscr{F}X(\xi)|^2\right] \left(1+|\xi|^{1-2H}\right) d\xi .$$

Accordingly, for a fixed horizon $T > 0$, we denote by $\mathscr{V}_T(H)$ the space of $\mathscr{G}_0$-valued predictable processes u such that

$$\|u\|^2_{\mathscr{V}_T(H)} := \sup_{t\in[0,T]} \|u(t,\cdot)\|^2_{\mathscr{V}(H)} < \infty. \tag{5.2}$$

We now state a convolution lemma.

Proposition 6.31. *Consider a function $u_0 \in \mathscr{G}_0$ and $\frac{1}{4} < H < \frac{1}{2}$. For any $v \in \mathscr{V}_T(H)$ we set $V = \Gamma_\ell(v)$ in the following way:*

$$V(t,x) = p_t u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) v(s,y) W(ds,dy), \quad t \in [0,T], x \in \mathbb{R}.$$

Then Γ_ℓ is well-defined as a map from $\mathscr{V}_T(H)$ to $\mathscr{V}_T(H)$. Furthermore, there exists two positive constants c_1, c_2 such that the following estimate holds true on $[0,T]$:

$$\|V(t,\cdot)\|^2_{\mathscr{V}(H)} \le c_1 \|u_0\|^2_{\mathscr{V}(H)} + c_2 \int_0^t (t-s)^{2H-3/2} \|v(s,\cdot)\|^2_{\mathscr{V}(H)} ds . \tag{5.3}$$

Proof. Let v be a process in $\mathscr{V}_T(H)$ and set $V = \Gamma_\ell(v)$. We focus on the bound (5.3) for V.

Notice that the Fourier transform of V can be computed easily. Indeed, setting $v_0(t,x) = p_t u_0(x)$ and invoking a stochastic version of Fubini's theorem we get

$$\mathscr{F}V(t,\xi) = \mathscr{F}v_0(t,\xi) + \int_0^t \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{ix\xi}\, p_{t-s}(x-y)\, dx \right) v(s,y) W(ds,dy)\,,$$

and according to the expression of $\mathscr{F}p_t$ we obtain

$$\mathscr{F}V(t,\xi) = \mathscr{F}v_0(t,\xi) + \int_0^t \int_{\mathbb{R}} e^{-i\xi y} e^{-\frac{\kappa}{2}(t-s)\xi^2} v(s,y) W(ds,dy)\,.$$

We now evaluate the quantity $\mathbf{E}[\int_{\mathbb{R}} |\mathscr{F}V(t,\xi)|^2 |\xi|^{1-2H} d\xi]$ in the definition of $\|u_n\|_{\mathscr{V}_T(H)}$ given by (5.2). We thus write:

$$\begin{aligned} \mathbf{E}\left[\int_{\mathbb{R}} |\mathscr{F}V(t,\xi)|^2 |\xi|^{1-2H} d\xi\right] &\leq 2\int_{\mathbb{R}} |\mathscr{F}v_0(t,\xi)|^2 |\xi|^{1-2H} d\xi \\ &+ 2\int_{\mathbb{R}} \mathbf{E}\left[\Big|\int_0^t \int_{\mathbb{R}} e^{-i\xi y} e^{-\frac{\kappa}{2}(t-s)\xi^2} v(s,y) W(ds,dy)\Big|^2\right] |\xi|^{1-2H} d\xi := 2\,(I_1 + I_2)\,, \end{aligned}$$

and we handle the terms I_1 and I_2 separately.

The term I_1 can be easily bounded using that $u_0 \in \mathscr{G}_0$, that is,

$$I_1 = \int_{\mathbb{R}} |\mathscr{F}u_0(\xi)|^2 e^{-\kappa t|\xi|^2} |\xi|^{1-2H} d\xi \leq C\, \|u_0\|^2_{\mathscr{V}(H)}.$$

We thus focus on the estimation of I_2, and we set $f_\xi(s,\eta) = e^{-i\xi\eta} e^{-\frac{\kappa}{2}(t-s)\xi^2} v(s,\eta)$. Applying the isometry property (2.8) together with the Fourier transform expression for $\|h\|_{\dot{H}}$ in (2.6), we have

$$\begin{aligned} &\mathbf{E}\left[\Big|\int_0^t \int_{\mathbb{R}} e^{-i\xi y} e^{-\frac{\kappa}{2}(t-s)\xi^2} v(s,y) W(ds,dy)\Big|^2\right] \\ =\ & C_1(H) \int_0^t \int_{\mathbb{R}} \mathbf{E}\left[|\mathscr{F}_\eta f_\xi(s,\eta)|^2\right] |\eta|^{1-2H}\, ds d\eta, \end{aligned}$$

where $\mathscr{F}_\eta$ designates the Fourier transform with respect to η. It is obvious from the definition of Fourier transform that the Fourier transform of $e^{-i\xi y}V(y)$ is $\mathscr{F}V(\eta+\xi)$. Thus we have

$$\begin{aligned} I_2 &= C\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\kappa(t-s)\xi^2}\,\mathbf{E}\left[|\mathscr{F}v(s,\eta+\xi)|^2\right]|\eta|^{1-2H}|\xi|^{1-2H}\,d\eta d\xi ds \\ &= C\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\kappa(t-s)\xi^2}\,\mathbf{E}\left[|\mathscr{F}v(s,\eta)|^2\right]|\eta-\xi|^{1-2H}|\xi|^{1-2H}\,d\eta d\xi ds. \end{aligned}$$

We now bound $|\eta-\xi|^{1-2H}$ by $|\eta|^{1-2H}+|\xi|^{1-2H}$, which yields $I_2 \le I_{21}+I_{22}$ with:

$$\begin{aligned} I_{21} &= C\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\kappa(t-s)\xi^2}\,\mathbf{E}\left[|\mathscr{F}v(s,\eta)|^2\right]|\eta|^{1-2H}|\xi|^{1-2H}\,d\eta d\xi ds \\ I_{22} &= C\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\kappa(t-s)\xi^2}\,\mathbf{E}\left[|\mathscr{F}v(s,\eta)|^2\right]|\xi|^{2-4H}\,d\eta d\xi ds. \end{aligned}$$

Performing the change of variable $\xi \to (t-s)^{1/2}\xi$ and then trivially bounding the integrals of the form $\int_{\mathbb{R}}|\xi|^\beta e^{-\kappa\xi^2}d\xi$ by constants, we end up with

$$\begin{aligned} I_{21} &\le C\int_0^t (t-s)^{H-1}\int_{\mathbb{R}}\mathbf{E}\left[|\mathscr{F}v(s,\eta)|^2\right]|\eta|^{1-2H}\,d\eta\,ds \\ I_{22} &\le C\int_0^t (t-s)^{2H-3/2}\int_{\mathbb{R}}\mathbf{E}\left[|\mathscr{F}v(s,\eta)|^2\right]d\eta\,ds. \end{aligned}$$

Observe that for $H\in(1/4,1/2)$ the term $(t-s)^{2H-3/2}$ is more singular than $(t-s)^{H-1}$, but we still have $2H-3/2>-1$. Summarizing our considerations up to now, we have thus obtained

$$\begin{aligned} &\int_{\mathbb{R}}\mathbf{E}\left[|\mathscr{F}V(t,\xi)|^2\right]|\xi|^{1-2H}d\xi \\ &\quad\le C_{1,T}\,\|u_0\|^2_{\mathscr{V}(H)}+C_{2,T}\int_0^t (t-s)^{2H-3/2}\int_{\mathbb{R}}\mathbf{E}\left[|\mathscr{F}v(s,\xi)|^2\right]\,(1+|\xi|^{1-2H})\,d\xi\,ds, \end{aligned} \tag{5.4}$$

for two strictly positive constants $C_{1,T}, C_{2,T}$.

The term $\mathbf{E}[\int_{\mathbb{R}} |\mathscr{F}V(t,\xi)|^2 d\xi]$ in the definition of $\|V\|_{\mathscr{V}_T(H)}$ can be bounded with the same computations as above, and we find:

$$\int_{\mathbb{R}} \mathbf{E}\left[|\mathscr{F}V(t,\xi)|^2\right] d\xi$$
$$\leq C_{1,T} \|u_0\|^2_{\mathscr{V}(H)} + C_{2,T} \int_0^t (t-s)^{H-1} \int_{\mathbb{R}} \mathbf{E}\left[|\mathscr{F}v(s,\xi)|^2\right] (1+|\xi|^{1-2H})\, d\eta\, ds, \quad (5.5)$$

Hence, gathering our estimates (5.4) and (5.5), our bound (5.3) is easily obtained, which finishes the proof. □

As in the general case, Proposition 6.31 is the key to the existence and uniqueness result for equation (5.1).

Theorem 6.32. *Suppose that u_0 is an element of $\mathscr{G}_0$ and $\frac{1}{4} < H < \frac{1}{2}$. Fix $T > 0$. Then there is a unique process u in the space $\mathscr{V}_T(H)$ such that for all $t \in [0,T]$,*

$$u(t,\cdot) = p_t u_0 + \int_0^t \int_{\mathbb{R}} p_{t-s}(\cdot - y) u(s,y) W(ds,dy). \quad (5.6)$$

Proof. The proof follows from the standard Picard iteration scheme, where we just set $u_{n+1} = \Gamma_\ell(u_n)$. Details are left to the reader for sake of conciseness. □

6.5.2 Existence and uniqueness via chaos expansions

Next we provide another way to prove the existence and uniqueness of the solution to equation (5.1), by means of chaos expansions. This will enable us to obtain moment estimates. Before stating our main theorem in this direction, let us label an elementary lemma borrowed from [53] for further use:

Lemma 6.33. *For $m \geq 1$ let $\alpha \in (-1+\varepsilon, 1)^m$ with $\varepsilon > 0$ and set $|\alpha| = \sum_{i=1}^m \alpha_i$. For $t \in [0,T]$, the mth dimensional simplex over $[0,t]$ is denoted by $T_m(t) = \{(r_1, r_2, \dots, r_m) \in \mathbb{R}^m : 0 < r_1 < \cdots < r_m < t\}$. Then there is a constant $\varkappa$ such that*

$$J_m(t,\alpha) := \int_{T_m(t)} \prod_{i=1}^m (r_i - r_{i-1})^{\alpha_i} dr \leq \frac{\varkappa^m t^{|\alpha|+m}}{\Gamma(|\alpha|+m+1)},$$

where by convention, $r_0 = 0$.

Let us now state a new existence and uniqueness theorem for our equation of interest.

Theorem 6.34. *Suppose that $\frac{1}{4} < H < \frac{1}{2}$ and that the initial condition u_0 satisfies*

$$\int_{\mathbb{R}} (1 + |\xi|^{\frac{1}{2}-H}) |\mathscr{F} u_0(\xi)| d\xi < \infty. \tag{5.7}$$

Then there exists a unique solution to equation (5.1), that is a process $u \in \Lambda_{\dot{H}}$ such that for any $(t,x) \in [0,T] \times \mathbb{R}$, relation (2.10) holds true.

Remark 6.35. *The formulation of Theorem 6.34 yields the definition of our solution u for all $(t,x) \in [0,T] \times \mathbb{R}$. This is in contrast with Theorem 6.32 which gives a solution sitting in $\mathscr{G}_0$ for every value of t, and thus defined a.e. in x only.*

Proof of Theorem 6.34. Suppose that $u = \{u(t,x), t \geq 0, x \in \mathbb{R}^d\}$ is a solution to equation (2.10) in $\Lambda_{\dot{H}}$. Then according to (2.12), for any fixed (t,x) the random variable $u(t,x)$ admits the following Wiener chaos expansion:

$$u(t,x) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t,x)), \tag{5.8}$$

where for each (t,x), $f_n(\cdot,t,x)$ is a symmetric element in $\dot{H}^{\otimes n}$. Furthermore, we have seen that Itô and Skorohod's integral coincide for processes in $\Lambda_{\dot{H}}$. Hence, thanks to (2.14) and using an iteration procedure, one can find an explicit formula for the kernels f_n for $n \geq 1$:

$$f_n(s_1,x_1,\ldots,s_n,x_n,t,x) = \frac{1}{n!} p_{t-s_{\sigma(n)}}(x-x_{\sigma(n)}) \cdots p_{s_{\sigma(2)}-s_{\sigma(1)}}(x_{\sigma(2)}-x_{\sigma(1)}) p_{s_{\sigma(1)}} u_0(x_{\sigma(1)}), \quad (5.9)$$

where σ denotes the permutation of $\{1,2,\ldots,n\}$ such that $0 < s_{\sigma(1)} < \cdots < s_{\sigma(n)} < t$ (see, for instance, formula (4.4) in [55] of formula (3.3) in [53]). Then, to show the existence and uniqueness of the solution it suffices to prove that for all (t,x) we have

$$\sum_{n=0}^{\infty} n! \|f_n(\cdot,t,x)\|^2_{\dot{H}^{\otimes n}} < \infty. \quad (5.10)$$

The remainder of the proof is devoted to derive relation (5.10).

Starting from relation (5.9), some elementary Fourier computations show that:

$$\mathscr{F} f_n(s_1,\xi_1,\ldots,s_n,\xi_n,t,x) = \frac{c_H^n}{n!} \int_{\mathbb{R}} \prod_{i=1}^{n} e^{-\frac{\kappa}{2}(s_{\sigma(i+1)}-s_{\sigma(i)})|\xi_{\sigma(i)}+\cdots+\xi_{\sigma(1)}-\zeta|^2} \mathscr{F} u_0(\zeta) e^{-\frac{s_{\sigma(1)}|\zeta|^2}{2}} d\zeta,$$

where we have set $s_{\sigma(n+1)} = t$. Hence, owing to formula (2.6) for the norm in $\dot{H}$ (in its Fourier mode version), we have:

$$n!\|f_n(\cdot,t,x)\|^2_{\dot{H}^{\otimes n}} = \frac{c_H^{2n}}{n!} \int_{[0,t]^n} \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}} \prod_{i=1}^{n} e^{-\frac{\kappa}{2}(s_{\sigma(i+1)}-s_{\sigma(i)})|\xi_i+\cdots+\xi_1-\zeta|^2} \mathscr{F} u_0(\zeta) e^{-\frac{\kappa s_{\sigma(1)}|\zeta|^2}{2}} d\zeta \right|^2 \times \prod_{i=1}^{n} |\xi_i|^{1-2H} d\xi ds, \quad (5.11)$$

where $d\xi$ denotes $d\xi_1 \cdots d\xi_n$ and similarly for ds. Then using the change of variable $\xi_i + \cdots + \xi_1 = \eta_i$, for all $i = 1, 2, \ldots, n$ and a linearization of the above expression, we obtain:

$$
\begin{aligned}
&n! \|f_n(\cdot, t, x)\|^2_{\dot{H}^{\otimes n}} \\
&\quad = \frac{c_H^{2n}}{n!} \int_{[0,t]^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^2} \prod_{i=1}^n e^{-\frac{\kappa}{2}(s_{\sigma(i+1)} - s_{\sigma(i)})(|\eta_i - \zeta|^2 + |\eta_i - \zeta'|^2)} \mathscr{F} u_0(\zeta) \overline{\mathscr{F} u_0(\zeta')} \\
&\qquad\qquad \times e^{-\frac{\kappa s_{\sigma(1)}(|\zeta|^2 + |\zeta'|^2)}{2}} \prod_{i=1}^n |\eta_i - \eta_{i-1}|^{1-2H} d\zeta d\zeta' d\eta ds,
\end{aligned}
$$

where we have set $\eta_0 = 0$. Then we use Cauchy-Schwarz inequality and bound the term $\exp(-\kappa s_{\sigma(1)}(|\zeta|^2 + |\zeta'|^2)/2)$ by 1 to get

$$
\begin{aligned}
&n! \|f_n(\cdot, t, x)\|^2_{\dot{H}^{\otimes n}} \\
&\quad \leq \frac{c_H^{2n}}{n!} \int_{\mathbb{R}^2} \left(\int_{[0,t]^n} \int_{\mathbb{R}^n} \prod_{i=1}^n e^{-\kappa(s_{\sigma(i+1)} - s_{\sigma(i)})|\eta_i - \zeta|^2} \prod_{i=1}^n |\eta_i - \eta_{i-1}|^{1-2H} d\eta ds \right)^{\frac{1}{2}} \\
&\qquad \times \left(\int_{[0,t]^n} \int_{\mathbb{R}^n} \prod_{i=1}^n e^{-\kappa(s_{\sigma(i+1)} - s_{\sigma(i)})|\eta_i - \zeta'|^2} \prod_{i=1}^n |\eta_i - \eta_{i-1}|^{1-2H} d\eta ds \right)^{\frac{1}{2}} \\
&\qquad\qquad |\mathscr{F} u_0(\zeta)| \left| \mathscr{F} u_0(\zeta') \right| d\zeta d\zeta'.
\end{aligned}
$$

Arranging the integrals again, performing the change of variables $\eta_i := \eta_i - \zeta$ and invoking the trivial bound $|\eta_i - \eta_{i-1}|^{1-2H} \leq |\eta_{i-1}|^{1-2H} + |\eta_i|^{1-2H}$, this yields:

$$
n! \|f_n(\cdot, t, x)\|^2_{\dot{H}^{\otimes n}} \leq \frac{c_H^{2n}}{n!} \left(\int_{\mathbb{R}} L_{n,t}^{\frac{1}{2}}(\zeta) \, |\mathscr{F} u_0(\zeta)| \, d\zeta \right)^2, \tag{5.12}
$$

where

$L_{n,t}(\zeta)$

$$= \int_{[0,t]^n} \int_{\mathbb{R}^n} \prod_{i=1}^{n} e^{-\kappa(s_{\sigma(i+1)} - s_{\sigma(i)})|\eta_i|^2} (|\zeta|^{1-2H} + |\eta_1|^{1-2H}) \times \prod_{i=2}^{n} (|\eta_i|^{1-2H} + |\eta_{i-1}|^{1-2H}) d\eta ds.$$

Let us expand the product $\prod_{i=2}^{n}(|\eta_i|^{1-2H} + |\eta_{i-1}|^{1-2H})$ in the integral defining $L_{n,t}(\zeta)$. We obtain an expression of the form $\sum_{\alpha \in D_n} \prod_{i=1}^{n} |\eta_i|^{\alpha_i}$, where D_n is a subset of multi-indices of length $n-1$. The complete description of D_n is omitted for sake of conciseness, and we will just use the following facts: $\mathrm{Card}(D_n) = 2^{n-1}$ and for any $\alpha \in D_n$ we have:

$$|\alpha| \equiv \sum_{i=1}^{n} \alpha_i = (n-1)(1-2H), \quad \text{and} \quad \alpha_i \in \{0, 1-2H, 2(1-2H)\}, \quad i = 1, \ldots, n.$$

This simple expansion yields the following bound:

$$L_{n,t}(\zeta) \leq |\zeta|^{1-2H} \sum_{\alpha \in D_n} \int_{[0,t]^n} \int_{\mathbb{R}^n} \prod_{i=1}^{n} e^{-\kappa(s_{\sigma(i+1)} - s_{\sigma(i)})|\eta_i|^2} \prod_{i=1}^{n} |\eta_i|^{\alpha_i} d\eta ds + \sum_{\alpha \in D_n} \int_{[0,t]^n} \int_{\mathbb{R}^n} \prod_{i=1}^{n} e^{-\kappa(s_{\sigma(i+1)} - s_{\sigma(i)})|\eta_i|^2} |\eta_1|^{1-2H} \prod_{i=1}^{n} |\eta_i|^{\alpha_i} d\eta ds.$$

Perform the change of variable $\xi_i = (\kappa(s_{\sigma(i+1)} - s_{\sigma(i)}))^{1/2} \eta_i$ in the above integral, and notice that $\int_{\mathbb{R}} e^{-\xi^2} |\xi|^{\alpha_i} d\xi$ is bounded by a constant. Changing the integral over $[0,t]^n$ into an integral over the simplex, we get

$$\begin{aligned} L_{n,t}(\zeta) \quad &\leq \quad C|\zeta|^{1-2H} n! c_H^n \sum_{\alpha \in D_n} \int_{T_n(t)} \prod_{i=1}^{n} (\kappa(s_{i+1} - s_i))^{-\frac{1}{2}(1+\alpha_i)} ds. \\ &+ Cn! c_H^n \sum_{\alpha \in D_n} \int_{T_n(t)} (\kappa(s_2 - s_1))^{-\frac{2-2H+\alpha_1}{2}} \prod_{i=2}^{n} (\kappa(s_{i+1} - s_i))^{-\frac{1}{2}(1+\alpha_i)} ds. \end{aligned}$$

We observe that whenever $\frac{1}{4} < H < \frac{1}{2}$, we have $\frac{1}{2}(1+\alpha_i) < 1$ for all $i = 2, \ldots n$, and it is easy to see that α_1 is at most $1-2H$ so $\frac{1}{2}(2-2H+\alpha_1) < 1$. Thanks to Lemma 6.33

and recalling that $\sum_{i=1}^{n} \alpha_i = n - 2nH$ for all $\alpha \in D_n$, we thus conclude that:

$$L_{n,t}(\zeta) \leq \frac{C(t^{H-\frac{1}{2}}\kappa^{H-\frac{1}{2}} + |\zeta|^{1-2H}) n! c_H^n t^{nH} \kappa^{nH-n}}{\Gamma(nH+1)}.$$

Plugging this expression into (5.12), we end up with:

$$n! \|f_n(\cdot,t,x)\|^2_{\dot{H}^{\otimes n}} \leq \frac{C c_H^n t^{nH} \kappa^{nH-n}}{\Gamma(nH+1)} \left(\int_{\mathbb{R}} (t^{H-\frac{1}{2}}\kappa^{H-\frac{1}{2}} + |\zeta|^{\frac{1}{2}-H}) |\mathscr{F} u_0(\zeta)| \, d\zeta \right)^2. \tag{5.13}$$

The proof of (5.10) is now easily completed thanks to the asymptotic behavior of the Gamma function and our assumption of u_0, and this finishes the existence and uniqueness proof. □

6.6 The Anderson model, moment bounds

In this section we derive the upper and lower bounds for the moments of the solution to equation (5.1) which allow to conclude on the intermittency of the solution. We proceed by first getting an approximation result for u, and then deriving the upper and lower bounds for the approximation.

6.6.1 Approximation of the solution

The approximation of the solution we consider is based on an approximation of the noise W, which defined in (4.18).

The noise W^ε induces an approximation to the mild formulation of equation (5.1), namely:

$$u^\varepsilon(t,x) = p_t u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) u^\varepsilon(s,y) W^\varepsilon(ds,dy), \tag{6.1}$$

where the integral is understood (as in Section 6.5.1) in the Itô sense. We will start by a formula for the moments of u^ε.

Proposition 6.36. *Let W^ε be the noise defined by* (4.18)*, and assume $\frac{1}{4} < H < \frac{1}{2}$ and u_0 is bounded such that $\int_{\mathbb{R}}(1+|\xi|^{\frac{1}{2}-H})|\mathscr{F}u_0(\xi)|d\xi < \infty$. Then:*

(i) *Equation* (6.1) *admits a unique solution.*

(ii) *For any integer $n \geq 2$ and $(t,x) \in [0,T] \times \mathbb{R}$, we have:*

$$\mathbf{E}\left[(u^\varepsilon(t,x))^n\right] = \mathbf{E}_B\left[\prod_{j=1}^n u_0(x+B^j_{\kappa t})\exp\left(c_H \sum_{1\leq j\neq k\leq n} V^{\varepsilon,j,k}_{t,x}\right)\right], \tag{6.2}$$

with

$$V^{\varepsilon,j,k}_{t,x} = \int_0^t f_\varepsilon(B^j_{\kappa r} - B^k_{\kappa r})dr = \int_0^t \int_{\mathbb{R}} e^{-\varepsilon|\xi|^2}|\xi|^{1-2H}e^{i\xi(B^j_{\kappa r}-B^k_{\kappa r})}d\xi dr. \tag{6.3}$$

In formula (6.3)*, $\{B^j; j=1,\dots,n\}$ is a family of n independent standard Brownian motions which are also independent of W and $\mathbf{E}_B$ designates the expected value with respect to the randomness in B only.*

(iii) *The quantity $\mathbf{E}[(u^\varepsilon(t,x))^n]$ is uniformly bounded in ε. More generally, for any $a > 0$ we have:*

$$\sup_{\varepsilon>0}\mathbf{E}_B\left[\exp\left(a\sum_{1\leq j\neq k\leq n} V^{\varepsilon,j,k}_{t,x}\right)\right] \equiv c_a < \infty$$

Proof. The proof of item (i) is almost identical to the proof of Theorem 6.34, and is omitted for sake of conciseness. Moreover, in the proof of (ii) and (iii), we may take $u_0(x) \equiv 1$ for simplicity.

In order to check item (ii), set:

$$A^\varepsilon_{t,x}(r,y) = \rho_\varepsilon(B^x_{\kappa(t-r)} - y), \quad \text{and} \quad \alpha^\varepsilon_{t,x} = \|A^\varepsilon_{t,x}\|^2_{\dot{H}}. \tag{6.4}$$

Then one can prove, similarly to Proposition 5.2 in [55], that u^ε admits a Feynman-Kac representation of the form:

$$u^\varepsilon(t,x) = \mathbf{E}_B\left[\exp\left(W(A^\varepsilon_{t,x}) - \frac{1}{2}\alpha^\varepsilon_{t,x}\right)\right]. \tag{6.5}$$

Now fix an integer $n \geq 2$. According to (6.5) we have:

$$\mathbf{E}\left[\left(u^\varepsilon_{t,x}\right)^n\right] = \mathbf{E}_W\left[\prod_{j=1}^n \mathbf{E}_B\left[\exp\left(W(A^{\varepsilon,B^j}_{t,x}) - \frac{1}{2}\alpha^{\varepsilon,B^j}_{t,x}\right)\right]\right],$$

where for any $j = 1,\dots,n$, $A^{\varepsilon,B^j}_{t,x}$ and $\alpha^{\varepsilon,B^j}_{t,x}$ are evaluations of (6.4) using the Brownian motion B^j. Therefore, since $W(A^{\varepsilon,B^j}_{t,x})$ is a Gaussian random variable conditionally on B, we obtain:

$$\begin{aligned}
\mathbf{E}\left[\left(u^\varepsilon_{t,x}\right)^n\right] &= \mathbf{E}_B\left[\exp\left(\frac{1}{2}\|\sum_{j=1}^n A^{\varepsilon,B^j}_{t,x}\|^2_{\dot{H}} - \frac{1}{2}\sum_{j=1}^n \alpha^{\varepsilon,B^j}_{t,x}\right)\right] \\
&= \mathbf{E}_B\left[\exp\left(\frac{1}{2}\|\sum_{j=1}^n A^{\varepsilon,B^j}_{t,x}\|^2_{\dot{H}} - \frac{1}{2}\sum_{j=1}^n \|A^{\varepsilon,B^j}_{t,x}\|^2_{\dot{H}}\right)\right] \\
&= \mathbf{E}_B\left[\exp\left(\sum_{1\leq i<j\leq n} \langle A^{\varepsilon,B^i}_{t,x}, A^{\varepsilon,B^j}_{t,x}\rangle_{\dot{H}}\right)\right].
\end{aligned}$$

The evaluation of $\langle A^{\varepsilon,B^i}_{t,x}, A^{\varepsilon,B^j}_{t,x}\rangle_{\dot{H}}$ easily yields our claim (6.2), the last details being left to the patient reader.

Let us now prove item (iii), namely:

$$\sup_{\varepsilon>0}\sup_{t\in[0,T],x\in\mathbb{R}} \mathbf{E}\left[(u^\varepsilon(t,x))^n\right] < \infty. \tag{6.6}$$

To this aim, observe first that we have obtained an expression (6.2) which does not depend on $x \in \mathbb{R}$, so that the $\sup_{t\in[0,T],x\in\mathbb{R}}$ in (6.6) can be reduced to a sup in t only.

Next, still resorting to formula (6.2), it is readily seen that it suffices to show that for two independent Brownian motions B and $\tilde{B}$, we have:

$$\sup_{\varepsilon>0, t\in[0,T]} \mathbf{E}_B\left[\exp\left(c F_t^{\varepsilon}\right)\right] < \infty, \quad \text{with} \quad F_t^{\varepsilon} \equiv \int_0^t \int_{\mathbb{R}} e^{-\varepsilon|\xi|^2} |\xi|^{1-2H} e^{i\xi(B_{\kappa r}-\tilde{B}_{\kappa r})} d\xi dr, \tag{6.7}$$

for any positive constant c. In order to prove (6.7), we expand the exponential and write:

$$\mathbf{E}_B\left[\exp(c F_t^{\varepsilon})\right] = \sum_{l=0}^{\infty} \frac{\mathbf{E}_B\left[(c F_t^{\varepsilon})^l\right]}{l!}. \tag{6.8}$$

Next, we have:

$$\begin{aligned} \mathbf{E}_B\left[(F_t^{\varepsilon})^l\right] &= \mathbf{E}_B\left[\int_{[0,t]^l} \int_{\mathbb{R}^l} \prod_{j=1}^{l} e^{-i\xi_j(B_{\kappa r_j}-\tilde{B}_{\kappa r_j})-\varepsilon|\xi_j|^2} |\xi_j|^{1-2H} d\xi dr\right] \\ &\le \int_{[0,t]^l} \int_{\mathbb{R}^l} \prod_{j=1}^{l} e^{-\kappa(t-r_{\sigma(l)})|\xi_l+\cdots+\xi_1|^2} |\xi_j|^{1-2H} d\xi dr, \end{aligned}$$

where σ is the permutation on $\{1,2,\ldots,l\}$ such that $t \ge r_{\sigma(l)} \ge \cdots \ge r_{\sigma(1)}$. We have thus gone back to an expression which is very similar to (5.11). We now proceed as in the proof of Theorem 6.34 to show that (6.6) holds from equation (6.8).

□

Starting from Proposition 6.36, let us take limits in order to get the moment formula for the solution u to equation (5.1).

Theorem 6.37. *Assume $\frac{1}{4} < H < \frac{1}{2}$, consider $n \ge 1$, $j,k \in \{1,\ldots,n\}$ with $j \ne k$ and for $(t,x) \in [0,T]\times\mathbb{R}$ set:*

$$V_{t,x}^{j,k} = L^2(\Omega) - \lim_{\varepsilon\to 0} V_{t,x}^{\varepsilon,j,k}, \quad \textit{with} \quad V_{t,x}^{\varepsilon,j,k} = \int_0^t \int_{\mathbb{R}} e^{-\varepsilon|\xi|^2} |\xi|^{1-2H} e^{i\xi(B_{\kappa r}^j - B_{\kappa r}^k)} d\xi dr.$$

Then $\mathbf{E}[(u^{\varepsilon}(t,x))^n]$ *converges as* $\varepsilon \to 0$ *to* $\mathbf{E}[u(t,x)^n]$*, which is given by*

$$\mathbf{E}[u(t,x)^n] = \mathbf{E}_B\left[\prod_{j=1}^{n} u_0(B^j_{\kappa t}+x)\exp\left(c_H \sum_{1\le j\ne k\le n} V^{j,k}_{t,x}\right)\right]. \tag{6.9}$$

Proof. As in Proposition 6.36, we will prove the theorem for $u_0 \equiv 1$ for simplicity. For any $p \ge 1$ and $1 \le j < k \le n$, we can easily prove that $V^{\varepsilon,j,k}_{t,x}$ converges in $L^p(\Omega)$ to $V^{j,k}_{t,x}$ defined by:

$$V^{j,k}_{t,x} = \int_0^t \int_{\mathbb{R}} |\xi|^{1-2H} e^{i\xi(B^j_{\kappa r}-B^k_{\kappa r})} d\xi\, dr. \tag{6.10}$$

Indeed, this is due to the fact that $e^{-\varepsilon|\xi|^2}|\xi|^{1-2H}e^{i\xi(B^j_{\kappa r}-B^k_{\kappa r})}$ converges to $|\xi|^{1-2H}e^{i\xi(B^j_{\kappa r}-B^k_{\kappa r})}$ in the $d\xi \otimes dr \otimes d\mathbf{P}$ sense, plus standard uniform integrability arguments. Now, taking into account relation (6.2), Proposition 6.36 and the fact that $L^2(\Omega)-\lim_{\varepsilon\to 0} V^{\varepsilon,j,k}_{t,x} = V^{j,k}_{t,x}$, we obtain:

$$\begin{aligned}\lim_{\varepsilon\to 0}\mathbf{E}\left[(u^{\varepsilon}(t,x))^n\right] &= \lim_{\varepsilon\to 0}\mathbf{E}_B\left[\exp\left(c_H\sum_{1\le j\ne k\le n} V^{\varepsilon,j,k}_{t,x}\right)\right]\\ &= \mathbf{E}_B\left[\exp\left(c_H\sum_{1\le j\ne k\le n} V^{j,k}_{t,x}\right)\right].\end{aligned} \tag{6.11}$$

To end the proof, let us now identify the right hand side of (6.11) with $\mathbf{E}[u(t,x)^n]$, where u is the solution to equation (5.1). For $\varepsilon, \varepsilon' > 0$ we write:

$$\mathbf{E}\left[u^{\varepsilon}(t,x)\,u^{\varepsilon'}(t,x)\right] = \mathbf{E}_B\left[\exp\left(\langle A^{\varepsilon,B^1}_{t,x}, A^{\varepsilon',B^2}_{t,x}\rangle_{\dot{H}}\right)\right],$$

where we recall that $A^{\varepsilon,B}_{t,x}$ is defined by relation (6.4). As before we can show that this converges as $\varepsilon, \varepsilon'$ tend to zero. So, $u^{\varepsilon}(t,x)$ converges in L^2 to some limit $v(t,x)$, and the limit is actually in L^p , for all $p \ge 1$. Moreover, $\mathbf{E}[v^k(t,x)]$ equals to the right hand side of (6.11). Finally for any smooth random variable F which is a linear combination of $W(\mathbf{1}_{[a,b]}(s)\varphi(x))$, where φ is a C^∞ function with compact support, using the fact that Itô's and Skorohod's integrals coincide on the set Λ_H, plus the duality relation (2.11), we have:

$$\mathbf{E}\left[Fu^{\varepsilon}_{t,x}\right] = \mathbf{E}[F] + \mathbf{E}\left[\langle Y^{\varepsilon}, DF\rangle_{\mathscr{H}}\right], \tag{6.12}$$

where

$$Y^{t,x}(s,z) = \left(\int_{\mathbb{R}} p_{t-s}(x-y)\, p_{\varepsilon}(y-z) u^{\varepsilon}_{s,y}\, dy\right) \mathbf{1}_{[0,t]}(s).$$

Letting ε tend to zero in equation (6.12), after some easy calculation we get:

$$\mathbf{E}[Fv_{t,x}] = \mathbf{E}[F] + \mathbf{E}\left[\langle DF, vp_{t-\cdot}(x-\cdot)\rangle_{\dot{H}}\right].$$

This equation is valid for any $F \in \mathbb{D}^{1,2}$ by approximation. So the above equation implies that the process v is the solution of equation (5.1), and by the uniqueness of the solution we have $v = u$.

□

6.6.2 Intermittency estimates

In this section we prove some upper and lower bounds on the moments of the solution which entail the intermittency phenomenon.

Theorem 6.38. *Let $\frac{1}{4} < H < \frac{1}{2}$, and consider the solution u to equation (5.1) and for simplicity we assume that the initial condition is $u_0(x) \equiv 1$. Let $n \geq 2$ be an integer, $x \in \mathbb{R}$ and $t > 0$ sufficiently large. Then the following bounds hold true, for some positive constants c_1, c_2, c_3 such that $0 < c_1 < c_2$:*

$$\exp(c_1 n^{1+\frac{1}{H}} \kappa^{1-\frac{1}{H}} t) \leq \mathbf{E}\left[|u(t,x)|^n\right] \leq c_3 \exp\left(c_2 n^{1+\frac{1}{H}} \kappa^{1-\frac{1}{H}} t\right). \tag{6.13}$$

Proof. We divide this proof into upper and lower bound estimates.

Step 1: Upper bound. Recall from equation (5.8) that for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, $u(t,x)$ can be written as: $u(t,x) = \sum_{m=0}^{\infty} I_m(f_m(\cdot,t,x))$. Moreover, as a consequence of the hypercontractivity property on a fixed chaos we have (see [74, p. 62]):

$$\|I_m(f_m(\cdot,t,x))\|_{L^n(\Omega)} \leq (n-1)^{\frac{m}{2}} \|I_m(f_m(\cdot,t,x))\|_{L^2(\Omega)},$$

and plugging our bound (5.13), we end up with:

$$\|I_m(f_m(\cdot,t,x))\|_{L^n(\Omega)} \leq n^{\frac{m}{2}} \|I_m(f_m(\cdot,t,x))\|_{L^2(\Omega)} \leq \frac{c^{\frac{n}{2}} n^{\frac{m}{2}} t^{\frac{mH}{2}} \kappa^{\frac{Hm-m}{2}}}{\left[\Gamma(mH+1)\right]^{\frac{1}{2}}}.$$

Therefore recalling the elementary bound $\sum_{n\geq 0} x^n/(n!)^a \leq 2\exp(cx^{1/a})$, which can be found e.g in [4, Lemma A.1], we get:

$$\|u(t,x)\|_{L^n(\Omega)} \leq \sum_{m=0}^{\infty} \|J_m(t,x)\|_{L^n(\Omega)} \leq \sum_{m=0}^{\infty} \frac{c^{\frac{m}{2}} n^{\frac{m}{2}} t^{\frac{mH}{2}} \kappa^{\frac{Hm-m}{2}}}{\left(\Gamma(mH+1)\right)^{\frac{1}{2}}} \leq c_1 \exp\left(c_2 t n^{\frac{1}{H}} \kappa^{\frac{H-1}{H}}\right),$$

from which the upper bound in our Theorem is easily deduced.

Step 2: Lower bound for u^{ε}. For the lower bound, we start from the moment formula (6.2) for the approximate solution, and write:

$$\begin{aligned}
&\mathbf{E}\left[(u^{\varepsilon}(t,x))^n\right] \\
&= \mathbf{E}_B\left[\exp\left(\int_0^t \int_{\mathbb{R}} e^{-\varepsilon|\xi|^2} \left|\sum_{j=1}^n e^{-iB^j_{\kappa r}\xi}\right|^2 |\xi|^{1-2H} d\xi dr - nt \int_{\mathbb{R}} e^{-\varepsilon|\xi|^2} |\xi|^{1-2H} d\xi\right)\right].
\end{aligned}$$

In order to estimate the expression above, notice first that the obvious change of variable $\lambda = \varepsilon^{1/2}\xi$ yields $\int_{\mathbb{R}} e^{-\varepsilon|\xi|^2}|\xi|^{1-2H} d\xi = c_H \varepsilon^{-(1-H)}$. Now for an additional arbitrary parameter $\eta > 0$, consider the set:

$$A_\eta = \left\{ \omega;\ \sup_{1\le j\le n} \sup_{0\le r\le t} |B^j_{\kappa r}(\omega)| \le \frac{\pi}{3\eta} \right\}.$$

Observe that classical small balls inequalities for a Brownian motion (see (1.3) in [66]) yield $\mathbf{P}(A_\eta) \ge c_1 e^{-c_2\eta^2 n\kappa t}$ for a large enough η. In addition, if we assume that A_η is realized and $|\xi| \le \eta$, some elementary trigonometric identities show that the following deterministic bound hold true: $|\sum_{j=1}^n e^{-iB^j_{\kappa r}\xi}| \ge \frac{n}{2}$. Gathering those considerations, we thus get:

$$\begin{aligned}
& \mathbf{E}\left[(u^\varepsilon(t,x))^n\right] \\
\ge\ & \exp\left(c_1 n^2 \int_0^t \int_0^\eta e^{-\varepsilon|\xi|^2} |\xi|^{1-2H} d\xi dr - c_2 nt\varepsilon^{H-1} \right) \mathbf{P}(A_\eta) \\
\ge\ & C\exp\left(c_1 n^2 t\varepsilon^{-(1-H)} \int_0^{\varepsilon^{1/2}\eta} e^{-|\xi|^2} |\xi|^{1-2H} d\xi - c_2 nt\varepsilon^{-(1-H)} - c_3 n\kappa t\eta^2 \right).
\end{aligned}$$

We now choose the parameter η such that $\kappa\eta^2 = \varepsilon^{-(1-H)}$, which means in particular that $\eta \to \infty$ as $\varepsilon \to 0$. It is then easily seen that $\int_0^{\varepsilon^{1/2}\eta} e^{-|\xi|^2}|\xi|^{1-2H} d\xi$ is of order $\varepsilon^{H(1-H)}$ in this regime, and some elementary algebraic manipulations entail:

$$\mathbf{E}\left[(u^\varepsilon(t,x))^n\right] \ge C\exp\left(c_1 n^2 t\kappa^{H-1}\varepsilon^{-(1-H)^2} - c_2 nt\varepsilon^{-(1-H)} \right) \ge C\exp\left(c_3 t\kappa^{1-\frac{1}{H}} n^{1+\frac{1}{H}} \right),$$

where the last inequality is obtained by choosing $\varepsilon^{-(1-H)} = c\,\kappa^{\frac{H-1}{H}} n^{\frac{1}{H}}$ in order to optimize the second expression. We have thus reached the desired lower bound in (6.13) for the approximation u^ε in the regime $\varepsilon = c\,\kappa^{\frac{1}{H}} n^{-\frac{1}{H(1-H)}}$.

Step 3: Lower bound for u. To complete the proof, we need to show that for all sufficiently small ε, $E[(u^\varepsilon(t,x))^n] \le E[(u(t,x))^n]$. We thus start from equation (6.2) and use the series expansion of the exponential function as in (6.8). We get:

$$\mathbf{E}\left[(u^{\varepsilon}(t,x))^{n}\right] = \sum_{m=0}^{\infty} \frac{c_H^m}{m!} \mathbf{E}_B\left[\left(\sum_{1\le j\neq k\le n} V_{t,x}^{\varepsilon,j,k}\right)^m\right], \tag{6.14}$$

where we recall that $V_{t,x}^{\varepsilon,j,k}$ is defined by (6.3). Furthermore, expanding the mth power above, we have:

$$\mathbf{E}_B\left[\left(\sum_{1\le j\neq k\le n} V_{t,x}^{\varepsilon,j,k}\right)^m\right] = \sum_{\alpha\in K_{n,m}} \int_{[0,t]^m}\int_{\mathbb{R}^m} e^{-\varepsilon\sum_{l=1}^m |\xi_l|^2} \mathbf{E}_B\left[e^{iB^{\alpha}(\xi)}\right] \prod_{l=1}^m |\xi_l|^{1-2H}\, d\xi\, dr,$$

where $K_{n,m}$ is a set of multi-indices defined by:

$$K_{n,m} = \left\{\alpha = (j_1,\ldots,j_m,k_1,\ldots,k_m) \in \{1,\ldots,n\}^{2m};\ j_l < k_l \text{ for all } l = 1,\ldots,n\right\},$$

and $B^{\alpha}(\xi)$ is a shorthand for the linear combination $\sum_{l=1}^m \xi_l (B_{\kappa r_l}^{j_l} - B_{\kappa r_l}^{k_l})$. The important point here is that $E_B e^{iB^{\alpha}(\xi)}$ is positive for any $\alpha \in K_{n,m}$. We thus get the following inequality, valid for all $m \ge 1$:

$$\begin{aligned}\mathbf{E}_B\left[\left(\sum_{1\le j\neq k\le n} V_{t,x}^{\varepsilon,j,k}\right)^m\right] &\le \sum_{\alpha\in K_{n,m}} \int_{[0,t]^m}\int_{\mathbb{R}^m} \mathbf{E}_B\left[e^{iB^{\alpha}(\xi)}\right] \prod_{l=1}^m |\xi_l|^{1-2H}\, d\xi\, dr \\ &= \mathbf{E}_B\left[\left(\sum_{1\le j\neq k\le n} V_{t,x}^{j,k}\right)^m\right],\end{aligned}$$

where $V_{t,x}^{j,k}$ is defined by (6.10). Plugging this inequality back into (6.14) and recalling expression (6.9) for $\mathbf{E}[(u(t,x))^n]$, we easily deduce that $\mathbf{E}[(u^{\varepsilon}(t,x))^n] \le \mathbf{E}[(u(t,x))^n]$, which finishes the proof.

□

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