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SINGULAR FUNCTION MORTAR FINITE ELEMENT METHODS

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Dedicated to Raytcho Lazarov on the occasion of his 60th birthday.

Abstract — We consider the Poisson equation with Dirichlet boundary conditions on a polygonal domain with one reentrant corner. We introduce new nonconforming finite element discretizations based on mortar techniques and singular functions. The main idea introduced in this paper is the replacement of cut-off functions by mortar element techniques on the boundary of the domain. As advantages, the new discretizations do not require costly numerical integrations and have smaller a priori error estimates and condition numbers. Based on such an approach, we prove $O(h)$ ($O(h^2)$) optimal accuracy error bounds for the discrete solution in the $H^1(\Omega)$ ($L^2(\Omega)$) norm. Based on such techniques, we also derive new extraction formulas for the stress intensive factor. We establish $O(h^2)$ optimal accuracy for the computed stress intensive factor. Numerical examples are presented to support our theory.

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1. Introduction

Consider the following elliptic variational problem: Find $u^* \in H^1(\Omega)$, such that

$$\begin{aligned} a(u^*, v) &= f(v) \quad \forall v \in H_0^1(\Omega), \\ u^* &= u_0^* \qquad \text{on} \quad \partial\Omega, \end{aligned} \tag{1}$$

where

$$a(u^*,v)=\int_\Omega \nabla u^*\cdot\nabla v\,dx,\quad f(v)=\int_\Omega fv\,dx.$$

We assume that the function $f\in L^2(\Omega)$ and the boundary data function u_0^* has an extension in $H^2(\Omega)$ which we also denote by u_0^*. We let the domain Ω be an L-shaped domain in $\Re^2$ with coordinate vertices $V_1=\{0,0\},V_2=\{1,0\},V_3=\{1,1\},V_4=\{-1,1\},V_5=\{-1,-1\}$, and $V_6=\{0,-1\}$. It is well known that the solution u^* of (1) does not necessarily belong to $H^2(\Omega)$ due to the nonconvexity of the domain Ω at the corner V_1 [18, 24, 25, 29]. As a consequence, standard finite element discretizations do not give optimal accurate schemes [10, 33]. Theoretical and numerical works on corner singularities are well known and several different approaches were proposed, such as integral equations [19, 34], primal and dual singular functions [8, 9, 11, 12, 14, 15, 18, 20, 22, 28, 33], local mesh refinements or graded meshes [3, 30, 31], high-order polynomial approximations [2, 4, 33], and others; see also references therein.

In this paper, we adopt the approach of singular functions to improve the accuracy of the finite element method for problem (1). We note that the solution u^* of (1) does not necessarily belong to $H^2(\Omega)$ even if f and u_0^* are very smooth. To see this, consider the primal singular function defined as $\psi^+(r,\theta)=r^{\frac{2}{3}}\sin(\frac{2}{3}\theta)$, where $x=r\cos(\theta)$ and $y=r\sin(\theta)$. The function ψ^+ is smooth everywhere in Ω except near the nonconvex corner V_1. It is possible to check that $\psi^+\in H^{5/6-\epsilon}(\Omega)$ if and only if ϵ is positive. In addition, ψ^+ is harmonic, i.e., $-\triangle\psi^+\equiv 0$ on Ω. We note that ψ^+ vanishes on the intervals $[V_1,V_2]$ and $[V_6,V_1]$, and it is smooth in the rest of the boundary $\partial\Omega$. Introducing the C^2 cut-off function ρ

$$\rho(x,y)=\begin{cases}1, & 0\leqslant r\leqslant\dfrac{1}{4},\\[2mm] -192r^5+480r^4-440r^3+180r^2-\dfrac{135}{4}r+\dfrac{27}{8}, & \dfrac{1}{4}\leqslant r\leqslant\dfrac{3}{4},\\[2mm] 0, & \dfrac{3}{4}\leqslant r\end{cases}$$

we define the smoothed cut-off primal singular function $s^+=\rho\psi^+$. The function s^+ is an example where the optimal regularity fails: s^+ vanishes on $\partial\Omega$, $-\Delta s^+\in L^2(\Omega)$ while s^+ does not belong to $H^2(\Omega)$. The function ψ^+ is another example. Note also that $\psi^+-s^+\in H^2(\Omega)$. In fact, all the functions that make the optimal regularity fail are of the form $\lambda s^++H^2(\Omega)$ and it is well known [18, 24, 25, 29] that the solution u^* of (1) for a L-shaped domain with $u_0^*\equiv 0$ has the following unique representation:

$$u^*=v_{u^*}+\lambda_{u^*}s^+,$$

where $v_{u^*}\in H^2(\Omega)$ and $\lambda_{u^*}\in\Re$, and also with the following regularity estimate

$$\|v_{u^*}\|_{H^2(\Omega)}+|\lambda_{u^*}|\leqslant C\|f\|_{L^2(\Omega)},$$

where C here and below is a positive generic constant independent of the mesh size or the functions under consideration. Taking into account that $s^+-\psi^+\in H^2(\Omega)$ and that u_0^* has an extension on $H^2(\Omega)$, we also have the following unique representation:

$$u^*=w_{u^*}+\lambda_{u^*}\psi^+,$$

where $w_{u^*} \in H^2(\Omega)$ and $\lambda_{u^*} \in \Re$, and also with the following regularity estimates

$$\|w_{u^*}\|_{H^2(\Omega)} \leqslant C\left(\|f + \Delta u_0^*\|_{L^2(\Omega)} + \|u_0^*\|_{H^2(\Omega)}\right) \tag{2}$$

and

$$|\lambda_{u^*}| \leqslant C\|f + \Delta u_0^*\|_{L^2(\Omega)}. \tag{3}$$

The major difference between the methods proposed in this paper and the other methods in the literature is in how we treat the boundary conditions. Here we add the primal singular function ψ^+ as a basis function to the regular finite element space and enforce boundary conditions through mortar techniques. One of the methods proposed here is a variation of the method described in Chapter 8 of [33]. There, the smoothed cut-off primal singular function s^+ is added as a basis function to regular finite element spaces, and since the function s^+ belongs to $H_0^1(\Omega)$, the treatment of boundary conditions is standard. Here, instead, we use mortar finite element techniques at the boundary of $\partial\Omega$ to enforce, in a weak sense, the boundary condition. The mortar technique will also be used here to compute the stress intensive factor. In another method proposed here, we use the dual singular functions ψ^- with the mortar treatment. Numerical experiments confirm that the use of mortar techniques, as opposed to the use of the cut-off dual singular functions $s^- = \rho\psi^-$, improves dramatically the accuracy of the computed stress intensive factor. Here, the dual singular function $\psi^-(r,\theta) = r^{-\frac{2}{3}}\sin(\frac{2}{3}\theta)$ and the smoothed cut-off dual singular function $s^- = \rho\psi^-$. It will be recalled that mortar techniques [6, 5, 7, 35] have been widely used in recent years for obtaining optimal discretization on nonoverlapping nonmatching grids. Several applications have been reported, for example, fluid dynamics [1], sliding meshes [13], overlapping nonmatching grids [16], preconditionings [17, 32], finite volume discretizations [21, 26], plate problems [27], and others; see references therein. The proposed work is another application of mortar techniques for obtaining accurate schemes for problems with known singular behavior. We concentrate our discussion in this paper on the Poisson problem on a L-shaped domain, however, it can be adapted to more general equations, boundary conditions and domains (e.g., domains with cracks), provided that the singular function representation and extraction formulas are available.

The motivations for the decision to use mortar techniques instead of cut-off functions are described below. We note that ψ^+ and s^- can be approximated by the regular space of linear piecewise continuous functions with an order of $O(h^{2/3})$ [2]. Thus, when the grid size h gets smaller, ψ^+ and s^- will become more linearly dependent with respect to the regular finite element space. We note, however, that the function s^+ has a much larger energy than the function ψ^+ since a large energy is created when cut-off the harmonic function ψ^+ to define s^+ and this large energy is not related to the corner singularity. In case of using the Cauchy-Schwarz inequality, this means that the angle between the function ψ^+ and the regular space of finite elements will be larger than the angle between s^+ and the same finite element spaces. As a consequence, the constants associated with the a priori discrete error estimates will be larger for s^+ as well as the conditioning of the matrix associated with its discrete problem. Any error in numerical quadratures will have a strong effect on the numerical results and this effect will get worse when the mesh gets refined. Hence, very expensive numerical quadratures are needed. The chief advantage of using mortar techniques is not only that the discrete matrix will be less ill-conditioned but also that several of the numerical quadratures are avoided since ψ^+ or ψ^- are known harmonic functions, and, therefore, numerical integrations can be avoided through integration by parts.

We will prove later in this paper that the approximation of the discrete solution u to u^* is $O(h)$ in $H^1(\Omega)$ and $O(h^2)$ in the $L^2(\Omega)$ norm. We note, however, that discrete λ_u and w_u obtained from the associated discrete problem will not end up on second-order approximations to λ_{u^*} and w_{u^*}. To obtain second-order schemes for λ_{u^*} and w_{u^*} as well, we introduce two different post-processing approaches to modify the discrete solution u. Such post-processing approaches are based on the dual singular function ψ^-. One approach is based on mortar techniques and the other one is based on the smoothed cut-off dual singular function s^-. We will show the second-order approximation for both approaches, however, the one used with the mortar technique is much more accurate.

The paper is organized as follows. In section 2, we introduce notations and mortar techniques on $\partial\Omega$. The new algorithms are described in section 3. Section 4 is devoted to the mathematical analysis of the new algorithms. In Section 5, we discuss implementation issues. We conclude the paper in Section 6 by providing some numerical experiments.

2. Notations

In this section, we introduce some notations and tools.

2.1. Triangulation

Let $\mathcal{T}^h(\Omega)$ be a regular finite element triangulation (composed of triangles) of $\overline{\Omega}$. We assume the triangulation $\mathcal{T}^h(\Omega)$ to be regular in shape and quasi-uniform with the grid size of $O(h)$. Let $\mathcal{V}(\Omega)$, also denoted by $\mathcal{V}$, be the discrete space of piecewise continuous linear functions on $\mathcal{T}^h(\Omega)$; note that we do not assume that the functions of $\mathcal{V}$ vanish on $\partial\Omega$.

2.2. Mortar functions at the boundary

The boundary of our domain is given by $\partial\Omega = \bigcup_{m=1}^{6} \overline{D}_m$, where the open straight segments D_m are given by the intervals $D_1 = (V_1, V_2)$, $D_2 = (V_2, V_3)$, $D_3 = (V_3, V_4)$, $D_4 = (V_4, V_5)$, $D_5 = (V_5, V_6)$, and $D_6 = (V_6, V_1)$. For each interval D_m, the triangulation $\mathcal{T}_h(D_m)$ is inherited from the triangulation $\mathcal{T}_h(\Omega)$. Let us denote the space $W(D_m)$ as the trace of $\mathcal{V}$ to $\overline{D}_m$; i.e.,

$$W(D_m) = \{v \in \mathcal{C}(\overline{D}_m) : v = w_{|\overline{D}_m}, \quad w \in \mathcal{V}\}.$$

Here we denote $\mathcal{C}(\overline{D}_m)$ as the space of the continuous function on $\overline{D}_m$. The Lagrange multiplier spaces, denoted by $M(D_m)$, considered in this paper and also in the numerical experiments, are going to be the dual biorthogonal functions introduced in [35]. The number of degrees of freedom of the Lagrange multiplier spaces $M(D_m)$ is the number of interior nodes of $\mathcal{T}_h(D_m)$. For each edge D_m, the mortar projection operator $\Pi_m : \mathcal{C}(\overline{D}_m) \longrightarrow W(D_m)$ is defined by

$$v - \Pi_m v \in \mathcal{C}_0(D_m), \qquad \int_{D_m} (v - \Pi_m v)\mu_m ds = 0 \quad \forall \mu_m \in M(D_m). \tag{4}$$

Here, $\mathcal{C}_0(D_m)$ is the space of continuous functions which vanish at the two endpoints of $\overline{D}_m$. It can be shown [7, 35] that

$$\|\Pi_m v\|_{H_{00}^{1/2}(D_m)} \leqslant C\|v\|_{H_{00}^{1/2}(D_m)} \qquad \forall v \in H_{00}^{1/2}(D_m), \tag{5}$$

$$\|v - \Pi_m v\|_{H_{00}^{1/2}(D_m)} \leqslant Ch\|v\|_{H^{3/2}(D_m)} \quad \forall v \in H^{3/2}(D_m), \tag{6}$$

and

$$\inf_{\mu_m \in M(D_m)} \|v - \mu_m\|_{(H^{1/2}(D_m))'} \leqslant Ch\|v\|_{H^{1/2}(D_m)} \quad \forall v \in H^{1/2}(D_m). \tag{7}$$

Note that above we have assumed continuity at the vertices $V_k,\ k = 1, \ldots, 6$. We note that the theory presented here also holds for the old mortars [7], where we assume continuity at the vertices. The theory can also be easily extended to the new mortar generation [6] where the values at the vertices $V_k,\ k = 2, \ldots, 5$ are considered to be interior values. We note, however, that the theory cannot be extended to the case where the function value at V_1 is treated as an interior values, thus, for the theory to work, continuity at V_1 is required.

3. Singular function mortar finite element method

Once we have defined the mortar condition at the boundary, we are ready to define the new finite element space. We define the discrete enhanced finite element space $\mathcal{V}_0^+$ as follows:

$$\mathcal{V}_0^+ = \left\{ v = w + \lambda\psi^+ : w \in \mathcal{V}, \quad \lambda \in \Re, \quad \Pi_m v = 0, \quad m = 1, \cdots, 6 \right\}.$$

The functions of the space $\mathcal{V}_0^+$ vanish at the vertices $V_k,\ k = 1, \cdots, 6$ and satisfy the zero Dirichlet boundary condition in the weak discrete sense on the intervals $\overline{D}_m,\ m = 2, \ldots, 5$, and satisfy the zero Dirichlet boundary condition in the strong sense on D_1 and D_6. It is easy to see that the degrees of freedom of the space $\mathcal{V}_0^+$ are the scalar λ and nodal values of w at the interior nodes of $\mathcal{T}_h(\Omega)$; the values of w on $\overline{D}_m$ are obtained via $w = -\lambda\Pi_m\psi^+$.

We next introduce the new finite element method using the finite–element space $\mathcal{V}_0^+$. Afterwards, we introduce two different methods to compute second-order accurate approximations for the stress intensive factor (SIF) λ_{u^*} and for the smooth part w_{u^*}.

3.1. Finite element formulation

Let us take $u_0 \in \mathcal{V}$ to be equal to $\Pi_m u_0^*$ on $\partial\Omega$, and equal to zero at the interior nodes of $\mathcal{T}_h(\Omega)$. We define the singular–function mortar finite element method as follows:

Find $u = w_u + \lambda_u\psi^+$ such that $u - u_0 \in \mathcal{V}_0^+$ and

$$a(u, v) = f(v) \quad \forall v \in \mathcal{V}_0^+. \tag{8}$$

We will prove later in this paper that problem (8) has a unique solution and the approximation of the discrete solution u to u^* is $O(h)$ in $H^1(\Omega)$ and $O(h^2)$ in the $L^2(\Omega)$ norm. To obtain the second-order approximation on $L_2(\Omega)$, we assume that $u_0^* \equiv \Pi_m u_0^*$ on D_1 and D_6, i.e., that the function u_0^* restricted to D_1 and D_6 belongs to $W(D_1)$ and $W(D_6)$, respectively. This technical assumption can be avoided if $H^2(\Omega)$ extension of u_0^* is explicitly available so that $-\Delta u_0^*$ can be computed. Hence, we can reduce the original problem (1) to the homogeneous Dirichlet boundary condition case. Even under the technical assumption or the homogeneous Dirichlet boundary condition case, λ_u and w_u obtained from (8) do not end up on second-order approximations to λ_{u^*} and w_{u^*}. Theoretically, we can only show that $|\lambda_{u^*} - \lambda_u| \leqslant Ch^{1/3}$; see (27). Hence, to obtain second-order schemes for λ_{u^*} and w_{u^*} as well, we introduce a post-processing approach to modify u to $\tilde{u}$ to obtain $\tilde{u} = w_{\tilde{u}} + \lambda_{\tilde{u}}\psi^+$, where now $w_{\tilde{u}}$, $\lambda_{\tilde{u}}$, and $\tilde{u}$ are second-order approximations for both λ_{u^*}, w_{u^*}, and u^*, respectively.

3.2. Post-processing with a cut-off function

Let $f = -\triangle u^*$ and define $f^- = -\triangle s^-$, where $s^- = \rho\psi^-$. We note that $-\Delta\psi^- \equiv 0$ and ψ^- vanishes on the intervals $[V_1, V_2]$ and $[V_6, V_1]$, and it is possible to check that $\psi^- \in H^{1/3-\epsilon}$ if and only if ϵ is positive. Applying the integration by parts to $\int_\Omega(-\Delta u^* s^- + \Delta s^- u^*)$ and using the asymptotic behavior of s^+ and s^- near the origin (see [29]), we obtain

$$\lambda_{u^*} = \frac{1}{\pi}\left(\int_\Omega (fs^- - f^-u^*) + \int_{\partial\Omega} s^-\partial_n u^- - u_0^*\partial_n s^-\right),$$

and taking into account that s^- vanishes on $\partial\Omega$, we have

$$\lambda_{u^*} = \frac{1}{\pi}\left(\int_\Omega (fs^- - f^-u^*) - \int_{\partial\Omega} u_0^*\partial_n s^-\right). \tag{9}$$

The reconstructed discrete stress intensity factor is defined as follows. We first solve (8) to obtain $u = w_u + \lambda_u\psi^+$, and then substitute this u as u^* into (9) to define the discrete stress intensity factor as

$$\lambda_{\tilde{u}} = \frac{1}{\pi}\left(\int_\Omega (fs^- - f^-u) - \int_{\partial\Omega} u_0^*\partial_n s^-\right). \tag{10}$$

The reconstruction of $w_{\tilde{u}}$ is obtained through

$$w_{\tilde{u}} = w_u + (\lambda_u - \lambda_{\tilde{u}})I_h\psi^+, \tag{11}$$

and we let

$$\tilde{u} = w_{\tilde{u}} + \lambda_{\tilde{u}}\psi^+. \tag{12}$$

The operator I_h introduced above is the standard pointwise interpolator to $\mathcal{V}$. We note that ψ^+ vanishes on the segments $\overline{D}_1$ and $\overline{D}_6$, therefore, $\tilde{u}$ satisfies the mortar condition on these segments. On the segments $\overline{D}_2$, $\overline{D}_3$, $\overline{D}_4$, and $\overline{D}_5$, $\tilde{u}$ does not satisfy the mortar condition, however, the function ψ^+ is very smooth and, hence, pointwise interpolation will not deteriorate the optimality of the approximation. Of course, if necessary, I_h can be modified only on $\partial\Omega$ to satisfy all the mortar conditions without losing the optimality of the approximation.

We next introduce another post-processing procedure to modify u to $\hat{u} = w_{\hat{u}} + \lambda_{\hat{u}}\psi^+$ to obtain the optimal-order approximation for both λ_{u^*}, w_{u^*}, and u^*. This approach gives better numerical results, but it requires that the function u_0^* vanishes on the whole $\partial\Omega$. We note that this requirement is automatically satisfied if $H^2(\Omega)$ extension of u_0^* is available since we can reduce the original problem (1) to the homogeneous Dirichlet boundary condition case.

3.3. Post-processing without a cut-off function

A in (3.2), we can use ψ^- instead of s^- and use $-\triangle\psi^- \equiv 0$ to obtain

$$\lambda_{u^*} = \frac{1}{\pi}\left(\int_\Omega f\psi^- - \int_{\partial\Omega}(u_0^*\partial_n\psi^- - \psi^-\partial_n u^*)\right). \tag{13}$$

Note that we do not know the value of $\partial_n u^*$ and, therefore, formula (13) is cannot be used to obtain the discrete stress intensity factor. A discrete approximation for $\partial_n u^*$ can be obtained via the saddle point formulation [5, 35] of (8). However, we cannot show theoretically that such an approach can end up on a second-order scheme for λ_{u^*}. Hence, we next introduce a new method that does not require the knowledge or the approximation of the $\partial_n u^*$.

We modify ψ^- to $\hat{\psi}^-$, where $\hat{\psi}^-$ vanishes on the whole $\partial\Omega$, $\hat{\psi}^-$ and ψ^- have the same singular behavior near the origin, and $-\Delta\hat{\psi}^- \equiv 0$. This is done as follows. We first solve $\delta\psi^- \in H^1(\Omega)$ so that

$$\begin{aligned} a(\delta\psi^-, v) = 0 \qquad \forall v \in H_0^1(\Omega), \\ \delta\psi^- = \psi^- \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{14}$$

Then we define $\hat{\psi}^- = \psi^- - \delta\psi^-$. Since the function ψ^- vanishes on D_1 and D_6, and is smooth on the remaining part of $\partial\Omega$, the function ψ^- has H^2 extension to Ω. Hence, the solution of (14) is of the form of $\delta\psi^- = w_{\delta\psi^-} + \lambda_{\delta\psi^-}\psi^+$, where $w_{\delta\psi^-} \in H^2(\Omega)$. In addition, the singular behavior of $\hat{\psi}^-$ near the origin is the same as that of ψ^-, since we have $\psi^- = O\left(r^{-\frac{2}{3}}\right)$ and $\delta\psi^- = O\left(r^{\frac{2}{3}}\right)$. We obtain

$$\lambda_{u^*} = \frac{1}{\pi}\left(\int_\Omega f\hat{\psi}^- - \int_{\partial\Omega} u_0^*\partial_n\hat{\psi}^-\right).$$

If we assume that the boundary value u_0^* vanishes on $\partial\Omega$, we have

$$\lambda_{u^*} = \frac{1}{\pi}\int_\Omega f\hat{\psi}^-. \tag{15}$$

Note that we do not know $\hat{\psi}^-$ and, therefore, a numerical approximation for $\hat{\psi}^-$ has to be calculated. We take $\delta\psi_{0,h}^- \in \mathcal{V}$ to be equal to $\Pi_m\psi^-$ on D_m and to zero at the interior nodes of $\mathcal{T}_h(\Omega)$. We solve $\delta\psi_h^- - \delta\psi_{h,0}^- \in \mathcal{V}_0^+$ so that

$$a(\delta\psi_h^-, v) = 0 \quad \forall v \in \mathcal{V}_0^+.$$

We define $\hat{\psi}_h^- = \psi^- - \delta\psi_h^-$, and define the discrete stress intensity factor by

$$\lambda_{\hat{u}} = \frac{1}{\pi}\int_\Omega f\hat{\psi}_h^- = \frac{1}{\pi}\left(\int_\Omega f\psi^- - f\delta\psi_h^-\right). \tag{16}$$

Note that $\lambda_{\hat{u}}$ can be obtained without computing the discrete solution u and can be only used if u_0^* vanishes on $\partial\Omega$. The reconstruction for $\hat{u}$ can be defined as

$$\hat{u} = w_{\hat{u}} + \lambda_{\hat{u}}\psi^+,$$

where

$$w_{\hat{u}} = w_u + (\lambda_u - \lambda_{\hat{u}})I_h\psi^+.$$

We next concentrate on the analysis of the algorithms.

4. Analysis

In this section we analyze the proposed algorithms. We prove optimality accuracy of the discrete solution u on the L_2 and H_1 norms. We also show that the two proposed discrete stress intensive factor formulas, given by (10) and (16), are optimal (second) order approximations for λ_{u^*}.

4.1. Uniform ellipticity

Note that $v \in \mathcal{V}_0^+$ implies that v vanishes on D_1 and D_6. Therefore, using a standard Poincaré inequality, we have:

Lemma 4.1. *There exists a constant C that does not depend on h and v so that*

$$\|v\|_{H^1(\Omega)} \leqslant C|v|_{H^1(\Omega)} \quad \forall v \in \mathcal{V}_0^+.$$

4.2. Energy discrete error

We next establish the optimal (first) order approximation of the discrete solution u on the energy error.

Theorem 4.1. *Let $u_0^* \in H^2(\Omega)$ and $f \in L^2(\Omega)$. Then the energy error is of order h, i.e.,*

$$\|u^* - u\|_{H^1(\Omega)} \leqslant Ch\left(\|f + \Delta u_0^*\|_{L^2(\Omega)} + \|u_0^*\|_{H^2(\Omega)}\right), \tag{17}$$

where u^* and u are the solutions of (1) and (8), respectively.

Proof. Note that the proposed discretization (8) is nonconformal since the space $\mathcal{V}_0^+$ is not included in $H_0^1(\Omega)$; the functions in $\mathcal{V}_0^+$ vanish on D_m, $m = 2, \cdots, 5$ only in a weak sense. To establish the H^1-discrete error estimate, we make use of the Cea's lemma (the second Strang lemma) for the nonconforming discretization [10] to obtain

$$\begin{aligned}\|u^* - u\|_{H^1(\Omega)} &\leqslant \inf_{v \in u_0 + \mathcal{V}_0^+} \|u^* - v\|_{H^1(\Omega)} + \sup_{z \in \mathcal{V}_0^+} \frac{|a(u^*, z) - f(z)|}{\|z\|_{H^1(\Omega)}} \\ &= \inf_{v \in u_0 + \mathcal{V}_0^+} \|u^* - v\|_{H^1(\Omega)} + \sup_{z \in \mathcal{V}_0^+} \frac{|\int_{\partial\Omega} z \partial_n u^* ds|}{\|z\|_{H^1(\Omega)}}.\end{aligned} \tag{18}$$

The first term of (18) is the *best approximation error* and the second term is the *consistency error*. Both errors are estimated in the following two lemmas and are of $O(h)$. □

4.2.1. Best approximation error. We next establish that the best approximation error on the energy norm is of the optimal (first) order.

Lemma 4.2. *Let $u_0^* \in H^2(\Omega)$ and $f \in L^2(\Omega)$. Then the best approximation error is of order h, i.e.*

$$\inf_{v \in u_0 + \mathcal{V}_0^+} \|u^* - v\|_{H^1(\Omega)} \leqslant Ch\left(\|f + \Delta u_0^*\|_{L^2(\Omega)} + \|u_0^*\|_{H^2(\Omega)}\right).$$

Proof. Let $\hat{v}$ be defined as

$$\hat{v} = I_h(u^* - \lambda_{u^*}\psi^+) + \lambda_{u^*}\psi^+,$$

where I_h is the standard pointwise interpolator on $\mathcal{V}$. Note that the interpolation is well-defined, since the function $w_{u^*} = u^* - \lambda_{u^*}\psi^+$ belongs to $H^2(\Omega)$ and, therefore, w_{u^*} is a continuous function. The function $\hat{v}$ is not necessarily equal to $\Pi_m u_0^*$ on D_m. This means that $\hat{v}$ does not satisfy the boundary condition in the mortar sense. We therefore modify $\hat{v}$ to $v = \hat{v} + \sum_{m=1}^{6} \mathcal{H}_m \Pi_m(u_0^* - \hat{v})$, where the operator $\mathcal{H}_m$ denote the $\mathcal{V}$-discrete harmonic extension function with boundary values $\Pi_m(u_0^* - \hat{v})$ given on $\overline{D}_m$ and zero on $\partial\Omega \backslash D_m$. Note that by construction of v we have $v \in u_0 + \mathcal{V}_0^+$. Using the triangular inequalities we also have

$$\|u^* - v\|_{H^1(\Omega)} \leqslant \|w_{u^*} - I_h w_{u^*}\|_{H^1(\Omega)} + \|\sum_{m=1}^{6} \mathcal{H}_m \Pi_m(u_0^* - \hat{v})\|_{H^1(\Omega)}. \tag{19}$$

For the first term of (19) we use a standard approximation result on pointwise interpolation and (2) to obtain

$$\|w_{u^*} - I_h w_{u^*}\|_{H^1(\Omega)} \leqslant Ch\|w_{u^*}\|_{H^2(\Omega)} \leqslant Ch\left(\|f + \Delta u_0^*\|_{L^2(\Omega)} + \|u_0^*\|_{H^2(\Omega)}\right).$$

For the second term of (19) we use the properties of the discrete harmonic extensions and $H_{00}^{1/2}$-norm, and the stability and approximation results (5) and (6) to obtain

$$\begin{aligned}\left\|\sum_{m=1}^{6} \mathcal{H}_m \Pi_m(u_0^* - \hat{v})\right\|_{H^1(\Omega)} &\leqslant C\sum_{m=1}^{6} \|\mathcal{H}_m \Pi_m(u_0^* - \hat{v})\|_{H^1(\Omega)} \leqslant C\sum_{m=1}^{6} \|\Pi_m(u_0^* - \hat{v})\|_{H_{00}^{1/2}(D_m)} \\ &\leqslant C\sum_{m=1}^{6} \|u_0^* - \hat{v}\|_{H_{00}^{1/2}(D_m)} \leqslant Ch\|u_0^*\|_{H^{3/2}(D_m)} \leqslant Ch\|u_0^*\|_{H^2(\Omega)}.\end{aligned}$$

□

4.2.2. ***Consistency error.*** We next establish that the consistency error is of the optimal (first) order.

Lemma 4.3. *Let $u_0^* \in H^2(\Omega)$ and $f \in L^2(\Omega)$. Then the consistency error is of order h, i.e.,*

$$\sup_{z \in \mathcal{V}_0^+} \frac{\left|\int_{\partial\Omega} \partial_n u^* z ds\right|}{\|z\|_{H^1(\Omega)}} \leqslant Ch\left(\|f + \Delta u_0^*\|_{L^2(\Omega)} + \|u_0^*\|_{H^2(\Omega)}\right). \tag{20}$$

Proof. Note that $z \in \mathcal{V}_0^+$ implies that z vanishes on $\overline{D}_1$ and $\overline{D}_6$. Therefore,

$$\int_{\partial\Omega} z\partial_n u^* ds = \sum_{m=2}^{5} \int_{D_m} z\partial_n u^* ds.$$

From the definition of $\mathcal{V}_0^+$, we have $\int_{D_m} z\mu_m ds = 0 \quad \forall \mu_m \in M(D_m)$. Thus,

$$\sum_{m=2}^{5} \int_{D_m} z\partial_n u^* ds = \sum_{m=2}^{5} \int_{D_m} z(\partial_n u^* - \mu_m) ds \quad \forall \mu_m \in M(D_m),$$

and using the duality arguments, we obtain

$$\sum_{m=2}^{5}\left|\int_{D_m} z\partial_n u^* ds\right| \leqslant C\sum_{m=2}^{5}\|z\|_{H^{1/2}(D_m)} \inf_{\mu_m\in M(D_m)}\|\partial_n u^*-\mu_m\|_{(H^{1/2})'(D_m)}.$$

Let us denote $\Omega_{1/4}=\Omega\cap\{r^2=x^2+y^2\leqslant 1/16\}$, and $\Omega_{1/4}^c=\Omega\backslash\Omega_{1/4}$. Since $\psi^+\in H^2(\Omega_{1/4}^c)$, we have $u^*\in H^2(\Omega_{1/4}^c)$, and, therefore, we can use a trace theorem to obtain $\partial_n u^*\in H^{1/2}(D_m)$, $m=2,\cdots,5$. We then use the approximation property (7), the trace result, and the regularity estimates (2) and (3) to obtain

$$\begin{aligned}\inf_{\mu_m\in M(D_m)}\|\partial_n u^*-\mu_m\|_{(H^{1/2})'(D_m)} &\leqslant Ch\|\partial_n u^*\|_{H^{1/2}(D_m)}\leqslant Ch\|u^*\|_{H^2(\Omega_{1/4}^c)}\\ &\leqslant Ch(|\lambda_{u^*}|\ \|\psi^+\|_{H^2(\Omega_{1/4}^c)}+\|w_{u^*}\|_{H^2(\Omega_{1/4}^c)})\\ &\leqslant Ch\left(\|f+\Delta u_0^*\|_{L^2(\Omega)}+\|u_0^*\|_{H^2(\Omega)}\right).\end{aligned}$$

We finally take into account that $\|z\|_{H^{1/2}(D_m)}\leqslant C\|z\|_{H^1(\Omega)}$ to obtain (20).

□

4.3. Error in the L^2-norm

We also obtain an optimal (second) order error estimate in the $L^2(\Omega)$-norm for the discrete solution u of (8). Here, we assume that $u_0^*(D_m)\in W(D_m)$ or, equivalently, $u_0^*=\Pi_m u_0^*$ for $m=1$ and $m=6$.

Theorem 4.2. *Let $u_0^*\in H^2(\Omega)$ and $f\in L^2(\Omega)$. In addition, assume that $u_0^*\in W(D_m)$ for $m=1$ and $m=6$. Then the L_2 discrete error is of order h^2, i.e.,*

$$\|u^*-u\|_{L^2(\Omega)}\leqslant Ch^2\left(\|f+\Delta u_0^*\|_{L^2(\Omega)}+\|u_0^*\|_{H^2(\Omega)}\right). \tag{21}$$

Proof. Let the functions $\phi_g^*\in H_0^1(\Omega)$ and $\phi_g\in\mathcal{V}_0^+$ be weak solutions of $a(w,\phi_g^*)=(w,g)\ \ \forall w\in H_0^1(\Omega)$ and $a(w,\phi_g)=(w,g)\ \ \forall w\in\mathcal{V}_0^+$, respectively. It is easy to see that

$$\begin{aligned}(u^*-u,g)=a(u^*-u,\phi_g^*-\phi_g)&-\big(a(u^*-u,\phi_g^*)-(u^*-u,g)\big)\\ &-\big(a(u^*,\phi_g^*-\phi_g)-(f,\phi_g^*-\phi_g)\big).\end{aligned}$$

Using the Aubin-Nitche trick and integration by parts, we obtain

$$\begin{aligned}\|u^*-u\|_{L^2(\Omega)}\leqslant \sup_{g\in L^2(\Omega)}\frac{1}{\|g\|_{L^2(\Omega)}}\Bigg\{&c\|u^*-u\|_{H^1(\Omega)}\|\phi_g^*-\phi_g\|_{H^1(\Omega)}+\\ &\left|\int_{\partial\Omega}(u^*-u)\partial_n\phi_g^*\right|+\left|\int_{\partial\Omega}(\phi_g^*-\phi_g)\partial_n u^*\right|\Bigg\}.\end{aligned} \tag{22}$$

From the symmetry of $a(\cdot,\cdot)$ and taking into account that $g\in L^2(\Omega)$, we have $\phi_g^*=w_{\phi_g^*}+\lambda_{\phi_g^*}\psi^+$, where

$$\|w_{\phi_g^*}\|_{H^2(\Omega)}+|\lambda_{\phi_g^*}|\leqslant C\|g\|_{L^2(\Omega)},$$

and using Theorem 4.1 (with f replaced by g and $u_0^*\equiv 0$), we have

$$\|\phi_g^*-\phi_g\|_{H^1(\Omega)}\leqslant Ch\|g\|_{L^2(\Omega)}.$$

Also, using (17), we obtain

$$\|u^* - u\|_{H^1(\Omega)}\|\phi_g^* - \phi_g\|_{H^1(\Omega)} \leqslant Ch^2 \left(\|f + \Delta u_0^*\|_{L^2(\Omega)} + \|u_0^*\|_{H^2(\Omega)}\right) \|g\|_{L^2(\Omega)}.$$

We next obtain a bound for the second term of the right-hand side of (22). We now use the assumption $u_0^* = \Pi_m u_0^*$ on D_1 and D_6 to have the second term of (22) bounded by

$$\left|\int_{\partial\Omega} (u^* - u)\partial_n \phi_g^*\right| \leqslant \sum_{m=2}^{5} \left|\int_{D_m} (u^* - u)\partial_n \phi_g^*\right|.$$

In addition, by using similar arguments as in the proof of Lemma 4.3 we also have

$$\|\partial_n \phi_g^*\|_{H^{1/2}(D_m)} \leqslant \|\partial_n \phi_g^*\|_{H^2(\Omega^c_{1/4})} \leqslant C\|g\|_{L^2(\Omega)}.$$

Then by using $\|\cdot\|_{(H_{00}^{1/2})'(D_m)} \leqslant \|\cdot\|_{(H^{1/2})'(D_m)}$, (7), and taking into account that ψ_g^* is smooth on D_m, $m = 2, \cdots, 5$, we obtain

$$\begin{aligned}\sum_{m=2}^{5} \left|\int_{D_m} (u^* - u)\partial_n \phi_g^*\right| &\leqslant \sum_{m=2}^{5} \|u^* - u\|_{H_{00}^{1/2}(D_m)} \inf_{\mu_m \in M(D_m)} \|\partial_n \phi_g^* - \mu_m\|_{(H_{00}^{1/2})'(D_m)} \\ &\leqslant Ch^2 \left(\|f + \Delta u_0^*\|_{L^2(\Omega)} + \|u_0^*\|_{H^2(\Omega)}\right) \|g\|_{L^2(\Omega)}.\end{aligned}$$

Using similar ideas, the third term of (22) can be bounded by

$$\left|\int_{\partial\Omega} (\phi_g^* - \phi_g)\partial_n u^*\right| \leqslant Ch^2 \left(\|f + \Delta u_0^*\|_{L^2(\Omega)} + \|u_0^*\|_{H^2(\Omega)}\right) \|g\|_{L^2(\Omega)}.$$

□

4.4. Reconstructed stress intensive factor error

We next show that the reconstructed stress intensive factor errors $|\lambda_{u^*} - \lambda_{\tilde{u}}|$ and $|\lambda_{u^*} - \lambda_{\hat{u}}|$ are of order h^2. We also show that the L_2-error of $w_{u^*} - w_{\tilde{u}}$ and $w_{u^*} - w_{\hat{u}}$ are of order h^2.

Theorem 4.3. *Let $u_0^* \in H^2(\Omega)$ and $f \in L^2(\Omega)$. In addition, assume that $u_0^* \in W(D_m)$, for $m = 1$ and $m = 6$. Then*

$$h^{-2}\left(\|w_{u^*} - w_{\tilde{u}}\|_{L^2(\Omega)} + |\lambda_{u^*} - \lambda_{\tilde{u}}|\right) + h^{-1}|w_{u^*} - w_{\tilde{u}}|_{H^1(\Omega)} \leqslant C\left(\|f + \Delta u_0^*\|_{L^2(\Omega)} + \|u_0^*\|_{H^2(\Omega)}\right).$$

Proof. We subtract (10) from (9) to obtain

$$|\lambda_{u^*} - \lambda_{\tilde{u}}| = \left|\frac{1}{\pi}\int_{\Omega} f^-(u - u^*)\right| \leqslant \|f^-\|_{L^2(\Omega)}\|u - u^*\|_{L^2(\Omega)}.$$

From the smoothing properties of the cut-off function ρ we have that $\|f^-\|_{L^2(\Omega)} \leqslant C$. Using (21), we then obtain

$$|\lambda_{u^*} - \lambda_{\tilde{u}}| \leqslant Ch^2 \left(\|f + \Delta u_0^*\|_{L^2(\Omega)} + \|u_0^*\|_{H^2(\Omega)}\right).$$

We next find a bound estimate for $\|u^* - \tilde{u}\|_{L^2(\Omega)}$. We first use a triangular inequality to obtain

$$\|u^* - \tilde{u}\|_{L^2(\Omega)} \leqslant \|u^* - u\|_{L^2(\Omega)} + \|u - \tilde{u}\|_{L^2(\Omega)}. \tag{23}$$

By using Theorem 4.2, we obtain a second-order error estimate for the first term of the right-hand side of (23). We next show that the second term of the right-hand side of (23) is $O(h^2)$. Using (11), (12), (8), and the Cauchy-Schwarz inequality, we have

$$\|u - \tilde{u}\|_{L^2(\Omega)} \leqslant |\lambda_u - \lambda_{\tilde{u}}| \ \|\psi^+ - I_h\psi^+\|_{L^2(\Omega)}.$$

The next step is to get bounds for $\|\psi^+ - I_h\psi^+\|_{L^2(\Omega)}$ and $|\lambda_u - \lambda_{\tilde{u}}|$. We note that

$$\|\psi^+ - I_h\psi^+\|_{L^2(\Omega)} \leqslant Ch^{5/3}, \tag{24}$$

despite the fact that the function ψ^+ does not belong to $H^{5/3}(\Omega)$. To see this, we first consider elements $\tau_h \in \mathcal{T}_h(\Omega)$ that do not touch the origin. Noting that ψ^+ and $I_h\psi^+$ are harmonic functions in the interior of each τ_h, we can use the maximum principle for the harmonic functions to have

$$\|\psi^+ - I_h\psi^+\|_{L_\infty(\tau_h)} \leqslant \|\psi^+ - I_h\psi^+\|_{L_\infty(\partial\tau_h)}.$$

Since $\psi^+ = r^{2/3}\sin(\frac{2}{3}\theta)$, the second derivatives of ψ^+ can be bounded by $|\partial^2\psi^+(x,y)| \leqslant Cr^{-4/3}$. Taking into account that ψ^+ is equal to $I_h\psi^+$ at the three vertices of τ_h, we can use the Taylor theorem to obtain

$$\|\psi^+ - I_h\psi^+\|_{L_\infty(\partial\tau_h)} \leqslant Ch^2 r^{-4/3}.$$

By the maximum principle and simple integrations we obtain

$$\|\psi^+ - I_h\psi^+\|^2_{L^2(\tau_h)} \leqslant Ch^6 r^{-8/3}. \tag{25}$$

For the few elements τ_h that touch the origin, it is easy to see, for well-shaped elements, that there exist positive constants C_1 and C_2 that do not depend on h such that

$$C_1 h^{10/3} \leqslant \|\psi^+ - I_h\psi^+\|^2_{L^2(\tau_h)} \leqslant C_2 h^{10/3}. \tag{26}$$

This last result is obtained by using that the maximum of $\psi^+ - I_h\psi^h$, on an edge of the triangle τ_h that has the origin as one of its endpoints, is reached at distance ch from the origin. Where here c is positive constant and does not depend on h.

Summing the contributions of all elements $\tau_h \in \mathcal{T}_h(\Omega)$ and using (25) and the upper bound (26), we obtain (24). Further, we have

$$|\lambda_{u^*} - \lambda_u| \leqslant Ch^{1/3}\lambda_{u^*}. \tag{27}$$

This follows from the lower bound of (26) and Theorem (4.2). Hence, using (3), (27), and (24), we obtain

$$\|u^* - \tilde{u}\|_{L^2(\Omega)} \leqslant Ch^2 \left(\|f + \Delta u_0^*\|_{L^2(\Omega)} + \|u_0^*\|_{H^2(\Omega)}\right).$$

The second-order estimate for $\|w_{u^*} - w_{\tilde{u}}\|_{L^2(\Omega)}$ follows directly from the second-order estimates for $\|u^* - \tilde{u}\|_{L^2(\Omega)}$ and $|\lambda_{u^*} - \lambda_{\tilde{u}}|$.

The fact that the error $\|u^* - \tilde{u}\|_{H^1(\Omega)}$ is of order h is proved in the same manner as for the L_2 error.

□

For the second reconstruction approach, we use similar arguments to obtain:

Theorem 4.4. *If $f \in L^2(\Omega)$, and u_0^* vanishes on $\partial\Omega$, then*

$$h^{-2}\left(\|w_{u^*} - w_{\hat{u}}\|_{L^2(\Omega)} + |\lambda_{u^*} - \lambda_{\hat{u}}|\right) + h^{-1}|w_{u^*} - w_{\hat{u}}|_{H^1(\Omega)} \leqslant C\left(\|f + \Delta u_0^*\|_{L^2(\Omega)} + \|u_0^*\|_{H^2(\Omega)}\right).$$

Proof. Taking into account that u_0^* vanishes on $\partial\Omega$ and subtracting (16) from (15), we obtain

$$|\lambda_{u^*} - \lambda_{\hat{u}}| = \left|\frac{1}{\pi}\int_\Omega f^-(\delta\psi_h^- - \delta\psi^-)\right|. \tag{28}$$

Note that $\delta\psi^-$ vanishes on D_1 and D_6 and $-\Delta\delta\psi^-$ belongs to $L_2(\Omega)$. Hence, we can use Theorem 4.2 (with $\delta\psi^-$ instead of u^*) to obtain $\|\delta\psi^- - \delta\psi_h^-\|_{L^2(\Omega)} \leqslant Ch^2$. The second-order approximation of $|\lambda_{u^*} - \lambda_{\hat{u}}|$ follows from (28) and the Cauchy-Schwarz inequality. The remaining part of the proof, i.e., L_2 and H_1 error estimates, is based on the same ideas as the proof of Theorem 4.3.

□

5. Matrix notations and implementation issues

Using matrix terminology, we can write method (8) as follows. Solve

$$\begin{pmatrix} A & b \\ b^T & c \end{pmatrix}\begin{pmatrix} w_u \\ \lambda_u \end{pmatrix} = \begin{pmatrix} f_\varphi \\ f(\psi^+) \end{pmatrix},$$

where

$$A_{ij} = \int_\Omega \nabla\varphi_i \nabla\varphi_j, \quad 1 \leqslant i \leqslant N, \quad 1 \leqslant j \leqslant N,$$

$$b_i = \int_\Omega \nabla\varphi_i \nabla\psi^+, \quad 1 \leqslant i \leqslant N,$$

and

$$c = \int_\Omega \nabla\psi^+ \nabla\psi^+.$$

Here, A is an $N \times N$ symmetric matrix and b is an $N \times 1$ vector, where N is the total number of nodes of $\mathcal{T}_h(\overline{\Omega})$ including all boundary nodes. f_φ is an $N \times 1$ vector defined by

$$f_\varphi = \begin{pmatrix} \int_\Omega f\varphi_1 \\ \vdots \\ \int_\Omega f\varphi_N \end{pmatrix},$$

and $f(\psi^+) = \int_\Omega f\psi^+$. We denote φ_i as the standard basis functions of $\mathcal{V}$.

$\nabla\psi^+$ blows up near the L-shapede corner (the origin) and, therefore, to obtain b_i and c with a good accuracy, numerical integrations should be done carefully. To calculate b_i, we do

$$b_i = \int_\Omega \nabla\varphi_i \nabla\psi^+ = \int_{\partial\Omega} \partial_n\psi^+\varphi_i - \int_\Omega \triangle\psi^+\varphi_i = \int_{\partial\Omega} \partial_n\psi^+\varphi_i.$$

Hence, b_i is zero except if the node i is a boundary node. On D_1 and D_6, $\partial_n\psi^+$ has a singular behavior and, fortunately, we can integrate $\partial_n\psi^+\varphi_i$ exactly. On $D_m, m = 2, \cdots, 5$, the $\partial_n\psi^+$ is smooth and we use a numerical quadrature to integrate $\partial_n\psi^+\varphi_i$. To obtain c, we do:

$$c = \int_\Omega \nabla\psi^+\nabla\psi^+ = \int_{\partial\Omega} \partial_n\psi^+\psi^+ - \int_\Omega \triangle\psi^+\psi^+ = \int_{\partial\Omega} \partial_n\psi^+\psi^+.$$

Only numerical integration is required on $D_m,\ m = 2, \cdots, 5$. On D_1 and D_6 the function ψ^+ vanishes.

Let $w = w_u - u_0$. We have

$$\begin{pmatrix} A & b \\ b^T & c \end{pmatrix} \begin{pmatrix} w \\ \lambda_u \end{pmatrix} = \begin{pmatrix} f_\varphi - Au_0 \\ f(\psi^+) - b^T u_0 \end{pmatrix}.$$

We now denote by w_{in} the discrete w on the interior nodes of $\mathcal{T}_h(\Omega)$ and by w_b the discrete w on the boundary nodes. Denoting $\hat{\psi}^+ = \Pi_m\psi^+$ on D_m, we have $w_b = -\lambda_u\hat{\psi}^+$. We then get the linear system for $\begin{pmatrix} w_{in} \\ \lambda_u \end{pmatrix}$ given by

$$M_A \begin{pmatrix} w_{in} \\ \lambda_u \end{pmatrix} = b_A, \tag{29}$$

where

$$M_A = \begin{pmatrix} I & 0 & 0 \\ 0 & (-\hat{\psi}^+)^T & 1 \end{pmatrix} \begin{pmatrix} A & b \\ b^T & c \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -\hat{\psi}^+ \\ 0 & 1 \end{pmatrix}$$

and

$$b_A = \begin{pmatrix} I & 0 & 0 \\ 0 & (-\hat{\psi}^+)^T & 1 \end{pmatrix} \begin{pmatrix} f_\varphi - Au_0 \\ f(\psi^+) - b^T u_0 \end{pmatrix}.$$

It is easy to see that the matrix M_A is a positive definite symmetric matrix due to Lemma 4.1 and, therefore, the CG algorithm can be used to solve (29). After we get w_{in} and λ_u, we let $w_b = -\lambda_u\hat{\psi}^+$ and $u = w + u_0 + \lambda_u\psi^+$, where $w = w_{in}$ at the interior nodes and $w = w_b$ at the boundary nodes. We note that optimal preconditioners for solving (29) can easily be designed and analyzed. For instance, eliminating λ_u from M_A, we reduce the discrete problem (29) to a system of the form $Bw_{in} = g_{in}$. Here, $B = A_{in,in} - dd^T$ and $A_{in,in}$ is the sub-block matrix of A associated with the interior nodes on $\mathcal{T}_h(\Omega)$. The matrix B is a one-rank perturbation of $A_{in,in}$ and thus, using the Sherman-Morrison formula [23], we can solve two linear systems with $A_{in,in}$ instead of solving one linear system with B (see also [15].). Several optimal preconditioners are well known for solving systems of the form $A_{in,in}x = b$, since such problems arise when using the regular finite element method with a zero Dirichlet boundary condition. Note that effective multigrid methods that take the advantage of smoothed cut-off primal and dual singular functions have been developed recently for solving certain nonsymmetric formulations [11, 12, 15]. The development of such a kind of multigrid methods for the mortared symmetric formulation (8) seems promising for future research.

6. Numerical experiments

In the first set of experiments, whose results are reported in Table 1, we solved the discrete Poisson equation (8) with $f = -\triangle s^+ - \triangle s_2^+ + 6x(y^2 - y^4) + (x - x^3)(12y^2 - 2)$, where the exact solution is $u^* = s^+ + s_2^+ + (x - x^3)(y^2 - y^4)$. Here, $s^+ = \rho(r)\psi^+$ and $s_2^+ = \rho(r)\psi_2^+$, where ψ_2^+ is the next singular function associated with problem (1), i.e., $\psi_2^+ = r^{4/3}\sin(\frac{4}{3}\theta)$. The integer k is the level of refinement of the mesh, where $k = 0$ corresponds to a mesh with 2 triangles per quadrant. The L^2 norm (H^1 semi-norm) discretization error on the kth level of mesh refinement is given by $e_2^k = \|u - u^*\|_{L^2(\Omega)}$ ($e_1^k = |u - u^*|_{H^1(\Omega)}$). The reconstructed discrete stress intensity factors are given by $\tilde{\lambda}^k = \lambda_{\tilde{u}}$ and $\hat{\lambda}^k = \lambda_{\hat{u}}$. For this test, we have the exact solution $\lambda_{u^*} = 1$. We also measured the rate of convergence for four discrete errors given by

$$\tilde{\sigma}^k = \log_2 \frac{|\tilde{\lambda}^{k-1} - 1|}{|\tilde{\lambda}^k - 1|}, \quad \hat{\sigma}^k = \log_2 \frac{|\hat{\lambda}^{k-1} - 1|}{|\hat{\lambda}^k - 1|}, \quad \epsilon_2^k = \log_2 \frac{e_2^{k-1}}{e_2^k}, \quad \epsilon_1^k = \log_2 \frac{e_1^{k-1}}{e_1^k}.$$

Table 1. Results with $f = -\triangle s^+ - \triangle s_2^+ + 6x(y^2 - y^4) + (x - x^3)(12y^2 - 2)$

k	$\tilde{\lambda}^k - 1$	$\tilde{\sigma}^k$	$1 - \hat{\lambda}^k$	$\hat{\sigma}^k$	e_2^k	ϵ_2^k	e_1^k	ϵ_1^k
2	2.967e-1	–	2.698e-3	–	7.512e-2	–	9.032e-1	–
3	9.457e-2	1.6497	6.914e-4	1.9642	2.415e-2	1.6380	5.027e-1	0.8454
4	2.651e-2	1.8349	1.673e-4	2.0474	6.805e-3	1.8275	2.673e-1	0.9115
5	6.862e-3	1.9497	4.083e-5	2.0152	1.764e-3	1.9475	1.361e-1	0.9736
6	1.730e-3	1.9873	1.006e-5	2.0216	4.454e-4	1.9858	6.839e-2	0.9928
7	4.341e-4	1.9952	2.550e-6	1.9832	1.116e-4	1.9958	3.424e-2	0.9980
8	1.085e-5	1.9996	6.290e-7	2.0154	2.794e-5	1.9991	1.713e-2	0.9994

Table 2. Result with $f = 1$

k	λ_k	σ_k	$\hat{\lambda}^k$	$\hat{\sigma}^k$	e_{k2}	ϵ_{k2}	e_{k1}	ϵ_{k1}
2	0.392530808	–	0.400094292	—	–	–	–	–
3	0.399690201	–	0.401466455	—	7.0033e-3	–	9.4462e-2	–
4	0.401373618	2.0884	0.401814664	1.9784	1.9044e-3	1.8787	5.0036e-2	0.9168
5	0.401790082	2.0151	0.401901770	1.9991	4.9070e-4	1.9559	2.5597e-2	0.9670
6	0.401894921	1.9900	0.401923602	1.9963	1.2427e-4	1.9820	1.2931e-2	0.9851
7	0.401921650	1.9717	0.401929119	1.9845	3.1246e-5	1.9917	6.4995e-3	0.9925

In the second set of experiments, whose results are reported in Table 2, we solved the discrete Poisson equation (8) with $f = 1$. Here, we do not know the exact solution. We estimate the error on L^2 norm (H^1 semi-norm) by Richardson quotients given by $e_2^k = \|u_k - u_{k-1}\|_{L^2(\Omega)}$ ($e_1^k = |u_k - u_{k-1}|_{H^1(\Omega)}$). For this example λ_{u^*} is close to 0.40193193 (computed on a very fine mesh). We also measured the difference quotient rate of convergence for four discrete errors given by

$$\tilde{\sigma}^k = \log_2 \frac{|\tilde{\lambda}^{k-2} - \tilde{\lambda}^{k-1}|}{|\tilde{\lambda}^{k-1} - \tilde{\lambda}^{k}|}, \quad \hat{\sigma}^k = \log_2 \frac{|\hat{\lambda}^{k-2} - \hat{\lambda}^{k-1}|}{|\hat{\lambda}^{k-1} - \hat{\lambda}^{k}|}, \quad \epsilon_2^k = \log_2(\frac{e_2^{k-1}}{e_2^k}), \quad \epsilon_1^k = \log_2(\frac{e_1^{k-1}}{e_1^k}).$$

The results of the numerical experiments reported in Tables 1 and 2 confirm the theory showing optimality of all proposed algorithms. Tables 1 and 2 show, in particular, that the computed stress intensive factor $\lambda_{\hat{u}}$ (obtained only through mortar techniques) is more accurate than the factor $\lambda_{\tilde{u}}$ (obtained through smoothed cut-off dual singular function s^-).

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