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GLOBAL WELL-POSEDNESS FOR THE 2D QUASI-GEOSTROPHIC EQUATION IN A CRITICAL BESOV SPACE

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Abstract. We show that the 2D quasi-geostrophic equation has global and unique strong solution when the (large) data belongs in the critical scale invariant space $\dot{B}^{2-2\alpha}_{2,\infty} \cap L^{2/(2\alpha-1)}$.

1. Introduction

In this paper we are concerned with the mathematical properties of the Cauchy problem for the quasi-geostrophic equation in two spatial dimensions

$$\begin{gathered} \theta_t + \kappa(-\Delta)^\alpha \theta + (J(\theta)\cdot\nabla)\theta = 0 \quad (t,x) \in \mathbf{R}^+ \times \mathbb{R}^2 \\ \theta(0,x) = \theta^0(x), \end{gathered} \tag{1.1}$$

where $\theta : \mathbb{R}^2 \to \mathbb{R}$ is a scalar, real-valued function, and $J(\theta) = (-R_2\theta, R_1\theta)$, $\alpha \in [0,1]$ and R_1, R_2 are the Riesz transforms defined via the Fourier transform by $\widehat{R_j f}(\xi) = \xi_j|\xi|^{-1}\hat{f}(\xi)$, see also Section 2.3 for additional details.

The physical meaning and derivation of (1.1) has been discussed extensively in the literature. We refer the interested reader to the classical book of Pedlosky, [14].

Depending on the value of the parameter α, one distinguishes between the subcritical case $\alpha > 1/2$, the critical case $\alpha = 1/2$, and the supercritical case $\alpha < 1/2$. It is known that the critical case $\alpha = 1/2$ is especially relevant from a physical point of view, as it is a direct analogue of the 3 D Navier-Stokes equations. On the other hand, considering the family of equations (1.1) with $\alpha \in [0,1]$ allows us to understand better the influence of the diffusion on the evolution.

An important scale invariance associated with problem (1.1) is that $\theta^\lambda(t,x) = \lambda^{2\alpha-1}\theta(\lambda^{2\alpha}t, \lambda x)$ is a solution if θ is. It follows that the space $\dot{H}^{2-2\alpha}(\mathbb{R}^2)$ is critical for the problem at hand. A heuristic argument can be made to show that a well-posedness theory for initial data in H^s, $s < 2 - 2\alpha$ should not hold. Thus, we concentrate our attention to the case $s \geq 2 - 2\alpha$.

The theory for existence of solutions and their uniqueness vary greatly, according to the criticality of the index α. For the critical and supercritical case, the question

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has been studied in [2, 6, 7, 8, 17, 18] among others. The results are that when the data is large and belongs to H^s, $s > 2 - 2\alpha$, then one has at least a local solution, which may blow up after finite time. For small data in the critical space (or some Besov variant), Chae-Lee, [2] and then J. Wu, [17, 18] have been able to show existence of global solutions.

We would like to mention that the majority of these results have been subsequently refined to include Besov spaces of initial data with the same level of regularity and scaling as the corresponding Sobolev spaces. Also, various uniqueness and blow-up criteria have been developed, see for example Section 2 below. However, the fundamental question for existence of global, smooth solutions in the supercritical case remains open. We note that very recently, in the critical case $\alpha = 1/2$, Kiselev, Nazarov and Volberg, [12] have shown the existence of global and smooth solutions for any smooth (large) initial data. The smoothness assumption in [12] is essentially at the level of $H^2(\mathbb{R}^2)$, while the critical case, the critical Sobolev space is $H^1(\mathbb{R}^2)$.

In the subcritical case, $\alpha > 1/2$, which is of main concern for us, the quasi-geostrophic equation is better understood. Local and global well-posedness results, as well as L^p decay estimates for the solution has been shown.

To summarize the latest results, Constantin and Wu, [4] have shown global well-posedness for the inhomogeneous version[1] of (1.1) whenever the data is in $H^s : s > 2 - 2\alpha$. For small data, there are plethora of results, which we will not review here, since we are primarily interested in the large data regime. On the other hand, time-decay estimates for $\|\theta(t)\|_{L^p}$ have been shown in [4] and [7], see Section 2 below for further details. Finally, we mention a local well-posedness result for large data in $H^{2-2\alpha} \cap L^2$, due to Ning Ju, [10]. Note that the space $H^{2-2\alpha}$ is not scale invariant (due to the L^2 part of it) and thus, such solutions cannot be rescaled to global ones.

In this work, we show that the quasi-geostrophic equation is globally well-posed in the critical space $\dot{B}^{2-2\alpha}_{2,\infty} \cap L^{2/(2\alpha-1)}$, that is whenever the data θ^0 belongs to the space, there is a global and unique[2] solution in the same space.

Theorem 1.1. *Let $\alpha \in (1/2, 1)$. Then for any initial data $\theta^0 \in \dot{B}^{2-2\alpha}_{2,\infty}(\mathbb{R}^2) \cap L^{2/(2\alpha-1)}(\mathbb{R}^2)$, the quasi-geostrophic equation (1.1) has a global solution*

$$\theta \in L^\infty([0,\infty); \dot{B}^{2-2\alpha}_{2,\infty}(\mathbb{R}^2) \cap L^{2/(2\alpha-1)}(\mathbb{R}^2))$$

Moreover, the solution satisfies the a priori estimate

$$\|\theta(t)\|_{\dot{B}^{2-2\alpha}_{2,\infty} \cap L^{2/(2\alpha-1)}} \leq C_{\kappa,\alpha}(\|\theta^0\|_{\dot{B}^{2-2\alpha}_{2,\infty} \cap L^{2/(2\alpha-1)}} + \|\theta^0\|^{M(\alpha)}_{L^{2/(2\alpha-1)}}), \tag{1.2}$$

for all $t > 0$ and $M(\alpha) = \max(2, 1/(2\alpha-1))$. In particular, the norms remain bounded for $0 < t < \infty$.

In addition, if $\theta_0 \in L^2(\mathbb{R}^2)$, then $\theta \in L^2((0,\infty), H^\alpha(\mathbb{R}^2))$, in fact

$$\|\theta\|_{L^2, H^\alpha(\mathbb{R}^2))} \leq \|\theta^0\|_{L^2(\mathbb{R}^2)}. \tag{1.3}$$

For a fixed $T > 0$, the solution is unique class of weak solutions on $[0,T]$ satisfying $\theta \in L^\infty([0,T], L^2(\mathbb{R}^2)) \cap L^\infty([0,T], L^{2/(2\alpha-1)}) \cap L^2((0,T), H^\alpha(\mathbb{R}^2))$.

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[1] That is, the authors also consider the equation with right-hand side not necessarily equal to zero.

[2] For the uniqueness one has to assume in addition $\theta^0 \in L^2(\mathbb{R}^2)$

Several remarks are in order.

(1) Note that global solutions exist and are unique in the space $\dot{B}^{2-2\alpha}_{2,\infty}(\mathbb{R}^2) \cap L^{2/(2\alpha-1)}(\mathbb{R}^2)$, when the data is in the same scale invariant space. Note that such space properly contains $\dot{H}^{2-2\alpha}(\mathbb{R}^2)$. In other words, taking data in $\dot{H}^{2-2\alpha}(\mathbb{R}^2)$ guarantees the existence of global solution, but by (1.2) we only know that the slightly smaller norm $\|\theta(t)\|_{\dot{B}^{2-2\alpha}_{2,\infty}\cap L^{2/(2\alpha-1)}}$ stays bounded.

(2) It is an interesting question, whether Theorem 1.1 and more precisely (1.2) hold in in the case of the Sobolev space $\dot{H}^{2-2\alpha}$ or even for some Besov space in the form $B^{2-2\alpha}_{2,r}$ for some $r < \infty$. We note that the main difficulty is proving estimate (1.2) for smooth solutions. Once (1.2) is established, one easily deduce the global existence and uniqueness by standard arguments.

(3) The results in Theorem 1.1 apply may be readily extended to $\mathbf{T}^2$. We omit the details, as they amount to a minor modification of the proof presented below.

2. Preliminaries

2.1. The 2D quasigeostrophic equation - existence and maximum principles. We start this section by recalling the Resnick's theorem, [15] for existence of weak solutions. That is whenever $\theta^0 \in L^2(\mathbb{R}^2)$ and for any $T > 0$, there exists a function $\theta \in L^\infty([0,T], L^2(\mathbb{R}^2)) \cap L^2[[0,T], H^\alpha(\mathbb{R}^2))$, so that for any test function φ,

$$\int_{\mathbb{R}^2} \theta(T)\varphi(T) - \int_0^T \int_{\mathbb{R}^2} \theta(J(\theta)\nabla\varphi) + \kappa \int_0^T \int_{\mathbb{R}^2} ((-\Delta)^{\alpha/2}\theta)((-\Delta)^{\alpha/2}\varphi) = \int \theta^0 \varphi(0,x).$$

In his dissertation, [15], Resnick also established *the maximum principle for L^p norms*, that is for smooth solutions of (1.1) and $1 \le p < \infty$, one has

$$\|\theta(t)\|_{L^p(\mathbb{R}^2)} \le \|\theta^0\|_{L^p(\mathbb{R}^2)}. \tag{2.1}$$

This was later generalized by Constanin-Wu, [4], [5] for the case $p = 2$ and by Córdoba-Córdoba, [6] in the case $p = 2^n$ and N. Ju, [8] for all $p \ge 2$ to actually imply a power rate of decay for $\|\theta(t)\|_{L^p(\mathbb{R}^2)}$ and an exponential rate of decay, when one considers the equation (1.1) on the torus $\mathbf{T}^2$. In the sequel, we use primarily (2.1), but is nevertheless interesting question to determine the optimal rates of decay for these norms. Note that Constantin and Wu have shown in [4], that the optimal rate for $\|\theta(t)\|_{L^2(\mathbb{R}^2)}$ is $< t >^{-1/2\alpha}$. Ning Ju has proved in [8], that[3] $\|\theta(t)\|_{L^p(\mathbb{R}^2)} \le C(\|\theta^0\|_{L^p})(1+t)^{-(p-2)/2p\alpha}$.

2.2. The uniqueness theorem of Constantin-Wu. Recall the uniqueness theorem by Constantin-Wu (Theorem 2.2, in [4]).

Theorem 2.1. *(Constantin-Wu) Assume that $\alpha \in (1/2, 1]$ and p, q satisfy $p \ge 1, q > 1$ and $1/p + \alpha/q = \alpha - 1/2$. Then for every $T > 0$, there is at most one weak*

[3] For example, $(p-2)/2p\alpha \to 0$ as $p \to 2$, whereas the optimal rate is $(2\alpha)^{-1}$, as shown by Constantin and Wu. On the other hand, we must note that the rate of L^p decay obtained by Ning Ju holds under the assumption that $\theta_0 \in L^2(\mathbb{R}^2)$, while Constantin-Wu assume that $\theta_0 \in L^1(\mathbb{R}^2)$.

solution of (1.1) *in* $[0,T]$*, satisfying*

$$\theta \in L^\infty([0,T], L^2(\mathbb{R}^2)) \cap L^2[[0,T], H^\alpha(\mathbb{R}^2)) \cap L^q([0,T], L^p(\mathbb{R}^2)).$$

In particular, one can take $q=\infty, 1/p=\alpha-1/2$ to obtain uniqueness for weak solutions satisfying $\theta \in L^\infty([0,T], L^p(\mathbb{R}^2))$.

2.3. **Some Fourier Analysis.** Define the Fourier transform by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx$$

and its inverse by

$$f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix\cdot\xi} d\xi.$$

For a positive, smooth and even function $\chi : \mathbb{R}^2 \to \mathbb{R}$, supported in $\{\xi : |\xi| \le 2\}$ and so that $\chi(\xi) = 1$ for all $|\xi| \le 1$. Define $\varphi(\xi) = \chi(\xi) - \chi(2\xi)$, which is supported in the annulus $1/2 \le |\xi| \le 2$. Clearly $\sum_{k\in\mathcal{Z}} \varphi(2^{-k}\xi) = 1$ for all $\xi \ne 0$.

The k^{th} Littlewood-Paley projection is $\widehat{P_k f}(\xi) = \varphi(2^{-k}\xi)\hat{f}(\xi)$. Similarly $P_{<k} = \sum_{l\le k} P_l$ given by the multiplier $\chi(2^{-k}\xi)$. Note that the kernels of $P_k, P_{<k}$ are uniformly integrable and thus $P_k, P_{<k} : L^p \to L^p$ for $1 \le p \le \infty$ and $\|P_k\|_{L^p\to L^p} \le C\|\hat{\chi}\|_{L^1}$. In particular, the bounds are independent of k.

The kernels of P_k are smooth and real-valued[4] and P_k commutes with differential operators with constant coefficients. We will frequently use the notation $\psi_k(x)$ instead of $P_k\psi$, when this will not create confusion.

It is convenient to define the (homogeneous and inhomogeneous) Sobolev norms in terms of the Littlewood-Paley operators. Namely for any $s \ge 0$, define for every Schwartz function ψ th norms

$$\|\psi\|_{\dot{H}^s} := \Big(\sum_{k=-\infty}^{\infty} 2^{2ks}\|\psi_k\|_{L^2}^2\Big)^{1/2}$$

$$\|\psi\|_{H^s} := \Big(\|\psi\|_{L^2}^2 + \sum_{k=0}^{\infty} 2^{2ks}\|\psi_k\|_{L^2}^2\Big)^{1/2}$$

and the corresponding spaces are then obtained as the closure of the set of all Schwartz functions in these norms. Clearly $H^s = L^2 \cap \dot{H}^s$.

Introduce the operator Λ acting via $\widehat{\Lambda\psi}(\xi) := |\xi|\hat{\psi}(\xi)$. Clearly, by the uniform boundedness of P_k in the scale of L^p spaces, $\|\Lambda^s \psi_k\|_{L^p} \sim 2^{ks}\|\psi_k\|_{L^p}$.

Next, we introduce some basic facts from the theory of the paraproducts, which will be useful for us, when estimating the contribution of the nonlinearity.
Write for any two Schwartz functions f, g and any integer k,

$$P_k(fg) = P_k(\sum_{l_1,l_2} f_{l_1} g_{l_2}) = P_k(\sum_{l_1,l_2:|l_1-l_2|\le 3} f_{l_1} g_{l_2}) + P_k(\sum_{l_1,l_2:|l_1-l_2|>3} f_{l_1} g_{l_2})$$

But

$$P_k(\sum_{l_1,l_2:|l_1-l_2|\le 3} f_{l_1} g_{l_2}) = P_k(\sum_{l_1,l_2:|l_1-l_2|\le 3, \min(l_1,l_2)>k-3} f_{l_1} g_{l_2})$$

[4]Thus for a real valued function ψ, $P_k\psi$ is a real-valued function as well.

since by the properties of the convolution $2^{l_1+1}+2^{l_2+1}$ must be at least 2^{k-1} and

$$P_k(\sum_{l_1,l_2:|l_1-l_2|>3} f_{l_1}g_{l_2}) = P_k(\sum_{l_1,l_2:|l_1-l_2|>3,|\max(l_1,l_2)-k|\le 3} f_{l_1}g_{l_2})$$

since otherwise $supp\widehat{f_{l_1}g_{l_2}} \subset \{\xi : |\xi| \sim 2^{\max(l_1,l_2)}\}$, which would be away from the set $\{\xi : |\xi| \sim 2^k\}$ and thus $P_k(f_{l_1}g_{l_2}) = 0$.

All in all,

$$\begin{gathered} P_k(fg) = P_k(\sum_{l=k-3}^{\infty} P_l f P_{l-3\le\cdot\le l+3} g) + \\ P_k(\sum_{j=-3}^{3} P_{k+j} f P_{<k+j-3} g) + P_k(\sum_{j=-3}^{3} P_{k+j} g P_{<k+j-3} f). \end{gathered} \tag{2.2}$$

We will refer to the first term as "high-high interaction" term, while the second and the third terms represent the "high-low interaction" term. We have the following lemma, which is an application of the representation formula (2.2).

Lemma 2.2. *For every $0 < s \le 1$, $2 < p, q < \infty : 1/p + 1/q = 1/2$, there is the estimate*

$$|\int P_k\psi_k[J(\psi)\cdot\nabla\psi]dx| \le C2^{k(1-s)}\|\psi_k\|_{L^p}(\sum_{l\ge k-3} 2^{-s(l-k)}\|\Lambda^s\psi_l\|_{L^2})\|\psi\|_{L^q}.$$

for some absolute constant C.

Proof. Integration by parts and $\operatorname{div}(J(\theta)) = 0$ yield

$$\int P_k\psi_k[J(\psi)\cdot\nabla\psi]dx = -\int \nabla\psi_k\cdot P_k[J(\psi)\psi]dx$$

At this point, by the boundedness of the Riesz transform on L^p, we treat $J(\psi)$ as $T\psi$, where $T : L^r \to L^r$ for all $1 < r < \infty$ and ignore the vector structure. By Hölder's inequality,

$$\big|\int \nabla\psi_k P_k[T(\psi)\cdot\psi]dx\big| \lesssim 2^k\|\psi_k\|_{L^p}\|P_k[T(\psi)\psi]\|_{L^{p'}}.$$

By (2.2),

$$\begin{aligned} &\|P_k[T(\psi)\psi]\|_{L^{p'}} \\ &\le \|\sum_{l=k-3}^{\infty} P_l T\psi P_{l-3\le\cdot\le l+3}\psi\|_{L^{p'}} + \big\|\sum_{j=-3}^{3} P_{k+j}(T\psi)P_{<k+j-3}\psi\big\|_{L^{p'}} \\ &\quad + \big\|\sum_{j=-3}^{3} P_{k+j}(\psi)P_{<k+j-3}T\psi\big\|_{L^{p'}} \\ &\le \sum_{l=k-3}^{\infty} \|P_l\psi\|_{L^2}\|P_{l-3\le\cdot\le l+3}\psi\|_{L^q} + \sum_{j=-3}^{3} \|P_{k+j}\psi\|_{L^2}\|P_{<k+j-3}\psi\|_{L^q} \\ &\le C(\sum_{l\ge k-3} \|\psi_l\|_{L^2})\|\psi\|_{L^q}. \end{aligned}$$

The Lemma follows by the observation $\|\psi_l\|_{L^2} \sim 2^{-ls}\|\Lambda^s\psi_l\|_{L^2}$ and by reshufling the 2^{ks}. $\square$

3. Proof of Theorem 1.1

The main step of the proof of Theorem 1.1 is the energy estimate (1.2).

We start with the assumption that we are given a smooth solution $\theta(t,x)$, corresponding to an initial data θ^0 up to time T and we will prove (1.2) based on it. Assume (1.2) for a moment for such smooth solutions. We will show that the global existence and uniqueness follows in a standard way from an approximation argument and the Constantin-Wu uniqueness result, Theorem 2.1.

Indeed, for a given initial data θ^0, take an approximating sequence in $\dot{B}^{2-2\alpha}_{2,\infty} \cap L^{2/(2\alpha-1)}$, $\{\theta^0_l\}$ of smooth functions (say in the Schwartz class $\mathcal{S}$). By the Constantin-Wu existence result for data in $H^s : s > (2-2\alpha)$, we have global and smooth solutions $\theta_l(t)$. In addition, they will satisfy the energy estimate (1.2). Moreover, by the L^p maximum principle, $\|\theta_l(t)\|_{L^q} \leq \|\theta_l(0)\|_{L^q}$ for all $1 < q < \infty$, in particular for $q = 2, q = 2/(2\alpha - 1)$.

Taking weak limits will produce a weak solution $\theta(t)$ of (1.1), corresponding to initial data θ^0, so that it satisfies the energy estimate (1.2) and $\|\theta\|_{L^\infty_t L^{2/(2\alpha-1)}} \leq \|\theta^0\|_{L^{2/(2\alpha-1)}}$. This shows the existence of a weak solution with the required smoothness of the initial data.

For the uniqueness part, we should require in addition that $\theta^0 \in L^2(\mathbb{R}^2)$. Then, we will show $\|\theta\|_{L^2_t H^\alpha_x} < \|\theta^0\|_{L^2}$, which allows us to apply the Constantin-Wu uniqueness result (Theorem 2.1). That is, θ is the unique solution in the class $L^\infty([0,T], L^2(\mathbb{R}^2)) \cap L^2[[0,T], H^\alpha(\mathbb{R}^2)) \cap L^\infty([0,T], L^{2/(2\alpha-1)}(\mathbb{R}^2))$. Thus, it remains to prove (1.2) for smooth solutions and (1.3). Since, (1.3) is relatively easy, we start with (1.2).

3.1. **Proof of the energy estimate** (1.2).

Let $s_0 = 2 - 2\alpha$. Take a Littlewood-Paley operator on both sides of (1.1)

$$\partial_t \theta_k + \kappa(-\Delta)^\alpha \theta_k + P_k(J(\theta)\nabla\theta) = 0.$$

Taking a dot product with θ_k (which is real-valued!) yields

$$\partial_t \|\theta_k\|^2_{L^2} + 2\kappa \|(-\Delta)^{\alpha/2}\theta_k\|^2_{L^2} + 2\int P_k\theta_k J(\theta)\nabla\theta = 0.$$

By the properties of the Littlewood-Paley operators, $\|(-\Delta)^{\alpha/2}\theta_k\|^2_{L^2} \sim 2^{2\alpha k}\|\theta_k\|^2_{L^2}$. For the integral term, use Lemma 2.2 with $1/p = 1/2 - s_0/2, 1/q = s_0/2$. We have

$$\begin{aligned} \Big|\int P_k\theta_k J(\theta)\nabla\theta dx\Big| &\leq C2^{k(1-s_0)}\|\theta_k(t)\|_{L^q}\Big(\sum_{l\geq k-3} 2^{-s_0(l-k)}\|\Lambda^s\theta_l\|_{L^2}\Big)\|\theta(t)\|_{L^p} \\ &\leq C2^{k(1-s_0)}\|\theta_k(t)\|_{L^q}\sup_l \|\Lambda^s\theta_l\|_{L^2}\|\theta(t)\|_{L^p}. \end{aligned}$$

By the L^p maximum principle, (2.1), we have $\|\theta(t)\|_{L^p} \leq \|\theta^0\|_{L^p}$. Substituting everything in the equation allows us to conclude

$$\partial_t \|\theta_k\|^2_{L^2} + c\kappa 2^{2k\alpha}\|\theta_k\|^2_{L^2} \leq C2^{k(1-s_0)}\|\theta^0\|_{L^p}\|\theta_k(t)\|_{L^q}\sup_l \|\Lambda^{s_0}\theta_l\|_{L^2} \tag{3.1}$$

At this point, the argument splits in two cases with a threshold value of $\alpha = 3/4$. As expected, the case $3/4 \leq \alpha < 1$ proves out to be slightly simpler, so we start with it.

The case $3/4 \leq \alpha < 1$. The significance of the restriction $\alpha \geq 3/4$ is in the fact that $s_0 = 2 - 2\alpha \in (0, 1/2]$. Therefore $1/q = s_0/2 \leq 1/2 - s_0/2 = 1/p$, implying $p \leq q$. Thus, by the Sobolev embedding[5], the boundedness of P_k on L^p and the L^p maximum principle imply

$$\|\theta_k(t)\|_{L^q} \lesssim 2^{2k(1/p-1/q)}\|\theta_k(t)\|_{L^p} \lesssim 2^{k(1-2s_0)}\|\theta_k(t)\|_{L^p} \lesssim 2^{k(1-2s_0)}\|\theta^0\|_{L^p}$$

By (3.1), we infer

$$\partial_t\|\theta_k\|_{L^2}^2 + c\kappa 2^{2k\alpha}\|\theta_k\|_{L^2}^2 \leq C2^{k(2-3s_0)}\|\theta^0\|_{L^p}^2 \sup_l \|\Lambda^{s_0}\theta_l(t)\|_{L^2} \tag{3.2}$$

It is a standard step now to make use of the Gronwal's inequality, namely rewrite (3.2) as

$$\partial_t(\|\theta_k\|_{L^2}^2 e^{c\kappa 2^{2k\alpha}t}) \leq C2^{k(2-3s_0)}e^{c\kappa 2^{2k\alpha}t}\|\theta^0\|_{L^p}^2 \sup_l \|\Lambda^{s_0}\theta_l(t)\|_{L^2}$$

and estimate after integration

$$\|\theta_k(t)\|_{L^2}^2 \leq C_\kappa 2^{k(2-3s_0-2\alpha)}\|\theta^0\|_{L^p}^2 \sup_{0\leq z\leq t}\sup_l \|\Lambda^{s_0}\theta_l(z)\|_{L^2} + \|\theta_k^0\|_{L^2}^2 e^{-c\kappa 2^{2k\alpha}t}. \tag{3.3}$$

Note that in the formula above $C_k \sim 1/\kappa$ and $2 - 3s_0 - 2\alpha = -2s_0$.
Introduce the functional

$$E(t) = \sup_{0\leq z\leq t}\sup_k 2^{ks_0}\|\theta_k(z)\|_{L^2}.$$

Clearly, one may deduce from (3.3) that

$$E^2(t) \leq E^2(0) + C_\kappa E(t)\|\theta^0\|_{L^p}^2,$$

hence

$$E(t) \leq 2J(0) + C_\kappa\|\theta^0\|_{L^p}^2,$$

which is

$$\sup_k 2^{k(2-2\alpha)}\|\theta_k(t)\|_{L^2} \leq 2\sup_k 2^{k(2-2\alpha)}\|\theta_k^0\|_{L^2} + C_\kappa\|\theta^0\|_{L^p}^2. \tag{3.4}$$

This is the *a priori* estimate of the solution θ, (1.2) for the case $\alpha \in [3/4, 1)$. As we have observed in the beginning of the section, it follows that the 2 D quasi-geostrophic equation (1.1) has global solution with (potentially large) data in the scale invariant space $\dot{B}^{2-2\alpha}_{2,\infty}(\mathbb{R}^2) \cap L^{2/(2\alpha-1)}(\mathbb{R}^2)$.

The case $1/2 < \alpha < 3/4$. In this case, it is clear that $s_0 = 2 - 2\alpha \in (1/2, 1)$, whence $2 < q = (1-\alpha)^{-1} < p = (\alpha - 1/2)^{-1}$. At this point, we make use of the Gagliardo-Nirenberg's inequality (see for example [13] or the classical [1]), which states that whenever $X = (X_0, X_1)_\theta$, say by the complex interpolation method, then $\|\cdot\|_X \leq \|\cdot\|_{X_0}^{1-\theta}\|\cdot\|_{X_1}^\theta$. In particular, applying this to the Sobolev spaces $\dot{W}^{p,k}$, we obtain

$$\|\theta_k\|_{L^q} \leq C\|\Lambda^{2-2\alpha}\theta_k\|_{L^2}^\gamma\|\Lambda^{-a}\theta_k\|_{L^p}^{1-\gamma},$$

with $\gamma = \frac{3-4\alpha}{2-2\alpha} \in (0,1)$ and $a = \frac{(2-2\alpha)(3-4\alpha)}{2\alpha-1}$. Thus, by $\|\Lambda^{-a}\theta_k\|_{L^p} \sim 2^{-ak}\|\theta_k\|_{L^p}$, whence it follows that

$$\|\theta_k\|_{L^q} \leq C2^{-k(3-4\alpha)}\sup_l\|\Lambda^{s_0}\theta_l\|_{L^2}^\gamma\|\theta_k(t)\|_{L^p}^{1-\gamma}.$$

[5] or more appropriately the Bernstein inequality

Substituting this in (3.1) yields

$$\partial_t\|\theta_k(t)\|_{L^2}^2 + c\kappa 2^{2k\alpha}\|\theta_k(t)\|_{L^2}^2 \le 2^{k(1-s_0-3+4\alpha)}\sup_l \|\Lambda^{s_0}\theta_l\|_{L^2}^{1+\gamma}\|\theta_k(t)\|_{L^p}\|\theta^0\|_{L^p}^{1-\gamma}. \tag{3.5}$$

Using the maximum principle $\|\theta_k(t)\|_{L^p} \lesssim \|\theta^0\|_{L^p}$, this reduces to

$$\partial_t\|\theta_k(t)\|_{L^2}^2 + c\kappa 2^{2k\alpha}\|\theta_k(t)\|_{L^2}^2 \le 2^{k(1-s_0-3+4\alpha)}\sup_l \|\Lambda^{s_0}\theta_l\|_{L^2}^{1+\gamma}\|\theta^0\|_{L^p}^{2-\gamma}.$$

By the Gronwall's inequality, we deduce

$$\|\theta_k(t)\|_{L^2}^2 \le \|\theta_k^0\|_{L^2}^2 e^{-c\kappa 2^{2k\alpha}t} + C_\kappa 2^{-2ks_0}\sup_{0\le z\le t}\sup_l \|\Lambda^{s_0}\theta_l(z)\|_{L^2}^{1+\gamma}\|\theta^0\|_{L^p}^{2-\gamma}. \tag{3.6}$$

By using the same energy functional $E(t)$ defined above, we conclude that

$$E^2(t) \le E^2(0) + C_k[E(t)]^{1+\gamma}\|\theta^0\|_{L^p}^{2-\gamma}.$$

Since $1+\gamma < 2$, by Young's inequality

$$E^2(t) \le E^2(0) + \frac{E^2(t)}{2} + C_{\kappa,\gamma}\|\theta^0\|_{L^p}^{(4-2\gamma)/(1-\gamma)}.$$

whence

$$E(t) \le 2E(0) + C_{\kappa,\gamma}\|\theta^0\|_{L^p}^{(2-\gamma)/(1-\gamma)}.$$

which is

$$\sup_k 2^{k(2-2\alpha)}\|\theta_k(t)\|_{L^2} \le \sup_k 2^{k(2-2\alpha)}\|\theta_k^0\|_{L^2} + C_{\kappa,\gamma}\|\theta^0\|_{L^p}^{(2-\gamma)/(1-\gamma)}. \tag{3.7}$$

Again, this implies (1.2) with $M(\alpha) = 1/(2\alpha-1)$ and the problem (1.1) has global solution in $\dot{B}^{2-2\alpha}_{2,\infty}(\mathbb{R}^2) \cap L^{2/(2\alpha-1)}(\mathbb{R}^2)$, when the initial data is taken in the same space.

3.2. $\theta \in L^\infty([0,\infty), L^2(\mathbb{R}^2)) \cap L^2((0,\infty), H^\alpha(\mathbb{R}^2))$**.** Both of these estimates are classical for smooth solutions, but we sketch their proofs for completeness.

In fact, $\theta \in L^\infty([0,\infty), L^2(\mathbb{R}^2))$ follows from the maximum principle (2.1). For the second estimate, we multiply the equation by θ and integrate in x. We get

$$\partial_t\|\theta(t)\|_{L^2}^2 + \|\Lambda^\alpha\theta(t)\|_{L^2}^2 = -\int \theta[J(\theta)\nabla\theta]dx = 0$$

Time integration now yields

$$\int_0^T \|\Lambda^\alpha\theta(t)\|_{L^2}^2 dt \le \|\theta^0\|_{L^2}^2 - \|\theta(T)\|_{L^2}^2 < \|\theta^0\|_{L^2}^2,$$

whence $\theta \in L^2((0,\infty), H^\alpha(\mathbb{R}^2))$.

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