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Asymptotic stability of small gap solitons in nonlinear Dirac equations

Dmitry E. Pelinovsky[1] and Atanas Stefanov[2]
[1]*Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario L8S 4K1, Canada*
[2]*Department of Mathematics, University of Kansas, 1460 Jayhawk Blvd., Lawrence, Kansas 66045–7523, USA*

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We prove dispersive decay estimates for the one-dimensional Dirac operator and use them to prove asymptotic stability of small gap solitons in the nonlinear Dirac equations with quintic and higher-order nonlinear terms. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4731477]

I. INTRODUCTION

Asymptotic stability of solitary waves in the nonlinear Schrödinger equation has been considered in the space of three dimensions with a number of analytical techniques.[30,37,14,12,20] Only recently, the asymptotic stability of solitary waves was extended to the space of two dimensions[27,19] and one dimension.[6,11,26]

Relatively little is known about the asymptotic stability of solitary waves in the nonlinear Dirac equations, which can be considered as a relativistic version of the nonlinear Schrödinger equation. Asymptotic stability of small bound states in the nonlinear Dirac equations in three dimensions was constructed by Boussaid,[3,4] Boussaid and Cuccagna.[5] Global existence and scattering to zero for small initial data were obtained by Machihara *et al.*,[23,24] also in the space of three dimensions. Recently, Komech and Komech[21] proved the existence of global attractors in a linear Dirac equation coupled nonlinearly to a harmonic oscillator.

Local well-posedness of general Dirac equations was considered by Goodman *et al.*[16] and reviewed by Pelinovsky.[29] For a particular version of the nonlinear Dirac equations (called the massive Thirring model), local well-posedness was considered in the works of Bournaveas,[2] Selberg and Tesfahun,[34] Machihara *et al.*,[25] and Candy.[7] Spectral properties of linearized Dirac operators were studied by Saito and Umeda[33] and recently by Berkolainko and Comech,[1] and Comech.[9,10]

In this work, we shall consider the asymptotic stability of solitary waves in the nonlinear Dirac equations in one dimension. Since the energy functional of the Dirac equations is sign-indefinite at the linear wave spectrum, it is generally believed that the solitary waves (referred to as gap solitons) must be energetically (and nonlinearly) unstable. Indeed, gap solitons are more disposed to spectral instabilities in the sense that unstable eigenvalues may exist in a large subset of the existence domain.[8] However, the limit of small gap solitons corresponds to the nonrelativistic limit, when the nonlinear Dirac equations can be reduced to the nonlinear Schrödinger equation.[24] In this limit, when the cubic nonlinear terms are considered, the gap solitons in one dimension are typically stable both spectrally and orbitally.[35,40] It is hence an interesting question to study the nonlinear asymptotic stability of the spectrally stable small gap solitons.

The spectral information is difficult in the case of the nonlinear Dirac equation even in the limit of small gap solitons. Isolated nonzero eigenvalues and resonances at the end points of the continuous spectrum occur commonly in the problem.[1,8] To simplify the spectral information, we add a bounded exponentially decaying potential to the one-dimensional nonlinear Dirac equations and consider a local bifurcation of the small gap solitons from an isolated eigenvalue of the self-adjoint Dirac operator. In this way, our approach is similar to the one used by Mizumachi[26] for the nonlinear Schrödinger equation and by us[18] for the discrete nonlinear Schrödinger equation (see also Ref. 13 for similar results).

We shall avoid the dispersive decay estimates in weighted L^2 spaces, which are difficult for the nonlinear Dirac equations (in contrast with the nonlinear Schrödinger equations).[22] We shall instead derive the Strichartz estimates directly from the Mizumachi estimates. The balance between Strichartz and Mizumachi estimates allows us to control both the nonlinear terms and the modulation equations for small gap solitons and thus to prove their asymptotic stability for the nonlinear Dirac equations with quintic and higher-order nonlinear terms.

The article is organized as follows. Section II introduces the nonlinear Dirac equations. Section III contains information about the small gap solitons. Section IV reports on linearization and spectral stability for small gap solitons. Section V derives the modulation equations for parameters of gap solitons as well as the time evolution equation for the dispersive remainder term. Section VI describes the main result. Section VII describes the spectral theory for the one-dimensional Dirac operator. Section VIII deals with the linear dispersive estimates for the semi-group associated with the Dirac operator. Section IX gives the proof of the main theorem.

We finish this section with a list of useful notations. The inner product for complex-valued functions in $L^2(\mathbb{R})$ is denoted by

$$\forall f, g \in L^2(\mathbb{R}): \quad \langle f, g\rangle_{L^2} := \int_{\mathbb{R}} \bar{f}(x) g(x) dx.$$

For any $f \in L^2(\mathbb{R})$, we define the Fourier transform and its inverse by

$$\hat{f}(k) \equiv \mathcal{F}(f) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixk} dx, \quad \check{f}(x) \equiv \mathcal{F}^{-1}(\hat{f}) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ixk} dk.$$

Sobolev spaces are denoted by $W^{s,p}(\mathbb{R})$ for $s \geq 0$ and $1 < p < \infty$ so that $H^s(\mathbb{R}) \equiv W^{s,2}(\mathbb{R})$ and $L^p(\mathbb{R}) \equiv W^{0,p}(\mathbb{R})$. Beside Sobolev spaces, we will use Strichartz spaces $L^p_t L^q_x$ and $L^q_x L^p_t$ defined for $1 \leq p, q \leq \infty$ by the norms

$$\|f\|_{L^p_t L^q_x} := \left(\int_0^T \|f(\cdot, t)\|^p_{L^q_x} dt \right)^{1/p}, \quad \|f\|_{L^q_x L^p_t} := \left(\int_{\mathbb{R}} \|f(x, \cdot)\|^q_{L^p_t} dx \right)^{1/q},$$

where $T > 0$ is an arbitrary time including $T = \infty$.

Notation $\langle x\rangle = (1 + x^2)^{1/2}$ is used for the weights in L^q_x norms. The constant $C > 0$ is a generic constant, which may change from one line to another line. A ball of radius $\delta > 0$ in function space X centered at $0 \in X$ is denoted by $B_\delta(X)$.

Pauli matrices are defined by

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The 2-by-2 identity matrix is denoted by Id.

Scalar functions are denoted by plain letters and vector functions with two components are denoted by bold letters. For clarity of notations, we do not write the second arguments for $W^{s,p}(\mathbb{R})$, $H^s(\mathbb{R})$, and $L^2(\mathbb{R})$ when it is used for scalar or vector functions.

II. THE NONLINEAR DIRAC EQUATIONS

Consider the nonlinear Dirac equations

$$\begin{cases} i(u_t + u_x) + v = \partial_{\bar{u}} W(u, v), \\ i(v_t - v_x) + u = \partial_{\bar{v}} W(u, v), \end{cases} \tag{1}$$

where $(x, t) \in \mathbb{R}^2$, $(u, v) \in \mathbb{C}^2$, and $W(u, v) : \mathbb{C}^2 \to \mathbb{R}$ is a nonlinear function which satisfies the following three conditions:

- symmetry $W(u, v) = W(v, u)$;
- gauge invariance $W(e^{i\theta}u, e^{i\theta}v) = W(u, v)$ for any $\theta \in \mathbb{R}$;
- polynomial in (u, v) and $(\bar{u}, \bar{v})$.

A general expansion of the nonlinear function $W(u, v)$ satisfying the three properties above starts with quadratic and quartic terms

$$W = \beta(x)(|u|^2 + |v|^2) + \gamma(x)(\bar{u}v + u\bar{v}) + W_N(u, v), \tag{2}$$

with

$$W_N = \alpha_1(|u|^4 + |v|^4) + \alpha_2|u|^2|v|^2 + \alpha_3(\bar{u}v + u\bar{v})^2 + \alpha_4(|u|^2 + |v|^2)(\bar{u}v + u\bar{v}), \tag{3}$$

where $\beta(x), \gamma(x) : \mathbb{R} \to \mathbb{R}$ are bounded and decaying potentials and $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{R}^4$ are numerical coefficients.

The standard example of the nonlinear term occurs in the context of Bragg gratings, where $\beta(x)$ and $\gamma(x)$ model optical defects in the periodic grating, whereas

$$W_N = \alpha(|u|^4 + 4|u|^2|v|^2 + |v|^4), \quad \alpha \in \mathbb{R} \tag{4}$$

models the nonlinear coupling terms.[16,17]

Another example is relevant to the massive Gross–Neveu model for spinors in relativity theory,[1]

$$W_N = \alpha(\bar{u}v + u\bar{v})^2, \quad \alpha \in \mathbb{R}. \tag{5}$$

In other applications, W_N may start with terms of the sixth and higher orders. The following nonlinear potential is derived in the context of the Feshbach resonance for Bose–Einstein condensates,[31]

$$W_N = \alpha(|u|^2 + |v|^2)|u|^2|v|^2, \quad \alpha \in \mathbb{R}. \tag{6}$$

Let us introduce the 2-by-2 Dirac operator in one dimension

$$\mathcal{H} = \begin{bmatrix} -i\partial_x + \beta(x) & \gamma(x) - 1 \\ \gamma(x) - 1 & i\partial_x + \beta(x) \end{bmatrix} \equiv D + V(x), \tag{7}$$

where

$$D = \begin{bmatrix} -i\partial_x & -1 \\ -1 & i\partial_x \end{bmatrix}, \quad V(x) = \begin{bmatrix} \beta(x) & \gamma(x) \\ \gamma(x) & \beta(x) \end{bmatrix}. \tag{8}$$

The nonlinear Dirac equations can be rewritten in the abstract evolutionary form

$$i\frac{d\mathbf{u}}{dt} = \mathcal{H}\mathbf{u} + \mathbf{N}(\mathbf{u}), \quad \mathbf{N}(\mathbf{u}) = \nabla_{\bar{\mathbf{u}}} W_N(u, v), \quad \mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix}, \quad \nabla_{\bar{\mathbf{u}}} = \begin{bmatrix} \partial_{\bar{u}} \\ \partial_{\bar{v}} \end{bmatrix}. \tag{9}$$

where $\mathbf{N}(\mathbf{u}) = \mathcal{O}(\|\mathbf{u}\|^3)$ as $\|\mathbf{u}\| \to 0$ in any norm that forms Banach algebra (e.g., in $H^s(\mathbb{R})$ for $s > \frac{1}{2}$). For the potentials (4) and (5), we have explicitly

$$\mathbf{N}(\mathbf{u}) = 2\alpha \begin{bmatrix} (|u|^2 + 2|v|^2)u \\ (2|u|^2 + |v|^2)v \end{bmatrix}, \quad \mathbf{N}(\mathbf{u}) = 2\alpha \begin{bmatrix} |v|^2 u + v^2\bar{u} \\ |u|^2 v + u^2\bar{v} \end{bmatrix}.$$

Unfortunately, these cubic nonlinear functions are not sufficiently small when (u, v) decays to zero in the one-dimensional nonlinear Dirac equation. As a result, we consider a more general class of the homogeneous polynomials of W_N (of even degree). Our arguments that follow from the linear estimates for the Dirac operator $\mathcal{H}$ will be valid for the quintic nonlinear functions which are generated from the polynomial W_N of degree six, e.g., from the function (6).

Local existence of solutions in Sobolev space can be proved with standard methods.[16]

Proposition 1: Let $\mathbf{u}_0 \in H^s(\mathbb{R})$ *for a fixed* $s > \frac{1}{2}$ *and assume that W satisfies the three conditions above. There exists a* $T > 0$ *such that the nonlinear Dirac equations (9) admits a unique solution*

$$\mathbf{u}(t) \in C([0, T], H^s(\mathbb{R})) \cap C^1([0, T], H^{s-1}(\mathbb{R})),$$

where $\mathbf{u}(t)$ *depends continuously on the initial data* $\mathbf{u}(0) = \mathbf{u}_0$.

If the nonlinear functions $W_N(u, v)$ depends on $|u|^2$ and $|v|^2$ only, e.g., for (4) and (6), global well-posedness in $H^s(\mathbb{R})$ with $s \in \mathbb{N}$ can be proved.[16] Little is known about the global solutions even for small initial data for the general nonlinear Dirac equations, e.g., for (5) (Ref. 29).

III. STATIONARY SMALL GAP SOLITONS

Under the assumptions that $\beta(x), \gamma(x) \in L^\infty(\mathbb{R})$, Dirac operator $\mathcal{H}$ is a densely defined, self-adjoint operator in $L^2(\mathbb{R})$ with the domain $H^1(\mathbb{R})$. We shall further assume that

$$\beta(x), \gamma(x) \to 0 \quad \text{as} \quad |x| \to \infty$$

at an exponential rate. The potentials represent a relatively compact perturbation to the unbounded differential operator. By Weyl's Theorem, the spectrum $\sigma(\mathcal{H}) \subset \mathbb{R}$ contains the continuous spectrum at

$$\sigma_c(\mathcal{H}) \equiv (-\infty, -1] \cup [1, \infty).$$

To simplify the construction of stationary small gap solitons, we assume that $\mathcal{H}$ admits only one simple isolated eigenvalue in the gap $(-1, 1)$ of the continuous spectrum $\sigma_c(\mathcal{H})$ and no resonances at ± 1. Examples of Dirac operators $\mathcal{H}$ that satisfy this assumption are constructed by Goodman *et al.*,[17] who explore that, under the constraint

$$\text{Im}(i V_S'(x)) = 0, \quad \text{where} \quad V_S(x) = (\gamma(x) - 1) e^{2i \int_0^x \beta(x') dx'},$$

the squared Dirac operator $\mathcal{H}^2$ can be diagonalized into two uncoupled Schrödinger operators.

Assumption 1: Assume that

- $\beta, \gamma \in L^\infty(\mathbb{R})$ *and there is* $C > 0$ *and* $\kappa > 0$ *such that*

$$|\beta(x)| + |\gamma(x)| \leq C e^{-\kappa |x|}, \quad x \in \mathbb{R}.$$

- $\sigma(\mathcal{H}) \backslash \sigma_c(\mathcal{H}) = \{\omega_0\}$, *where* $\omega_0 \in (-1, 1)$ *is a simple eigenvalue of* $\mathcal{H}$ *with the* L^2*-normalized eigenfunction* $\mathbf{u}_0 \in H^1(\mathbb{R})$.
- *No resonances occur at the end points* ± 1 *of* $\sigma_c(\mathcal{H})$ *in the sense that no solutions of* $\mathcal{H}\mathbf{u} = \pm \mathbf{u}$ *exist in* $L^\infty(\mathbb{R})$.

Remark 1: It follows from the symmetry of the Dirac operator $\mathcal{H}$ *that if* $\boldsymbol{u}_0$ *is the eigenvector of* $\mathcal{H}$*, then so is* $\sigma_1 \bar{\mathbf{u}}_0$ *for the same eigenvalue. If* ω_0 *is a simple eigenvalue, then* $\sigma_1 \bar{\mathbf{u}}_0 = \mathbf{u}_0$*, or* $u(x) = \bar{v}(x)$ *in the component form.*

Remark 2: It follows from the symmetry of the Dirac operator $\mathcal{H}$ *in the case* $\beta(x) \equiv 0$ *that if* $\boldsymbol{u}_0$ *is the eigenvector of* $\mathcal{H}$ *for eigenvalue* ω_0*, then* $\sigma_3 \bar{\mathbf{u}}_0$ *and* $\sigma_2 \boldsymbol{u}_0$ *are the eigenvectors of* $\mathcal{H}$ *for eigenvalue* $-\omega_0$*. Therefore, if* ω_0 *is the only eigenvalue of* $\mathcal{H}$*, then* $\omega_0 = 0$ *and this eigenvalue is simple if* $\mathbf{u}_0 = \sigma_3 \bar{\mathbf{u}}_0 = -\sigma_2 \mathbf{u}_0$*, or* $u(x) = -i v(x) = \bar{u}(x)$ *in the component form.*

Stationary gap solitons are given by

$$u(x, t) = U(x) e^{-i\omega t}, \quad v(x, t) = V(x) e^{-i\omega t}, \tag{10}$$

where $\omega \in \mathbb{R}$ is a parameter and $\mathbf{U} = [U, V]^T \in \mathbb{C}^2$ satisfies the system of differential equations

$$(\mathcal{H} - \omega I)\mathbf{U} + \mathbf{N}(\mathbf{U}) = \mathbf{0}. \tag{11}$$

If $\mathbf{U} \in H^1(\mathbb{R})$, then $\mathbf{U} \in C(\mathbb{R})$ and $\mathbf{U}(x) \to 0$ as $|x| \to \infty$ thanks to Sobolev's embedding of $H^1(\mathbb{R})$ to $C_b^0(\mathbb{R})$. By Lemma 3.1 in Ref. 8, the stationary solution $\mathbf{U} \in H^1(\mathbb{R})$ satisfies the symmetry

$$U(x) = \bar{V}(x), \quad x \in \mathbb{R}. \tag{12}$$

For the example of nonlinear function (4) with $\alpha = \frac{1}{3}$, when linear potentials are absent with $\beta(x), \gamma(x) \equiv 0$, the stationary gap solitons are given in the explicit form for any $\omega \in (-1, 1)$,

$$U(x) = \frac{\sqrt{1-\omega^2}}{\sqrt{1-\omega}\cosh(\sqrt{1-\omega^2}x) + i\sqrt{1+\omega}\sinh(\sqrt{1-\omega^2}x)} = \bar{V}(x). \tag{13}$$

In particular, $\|\mathbf{U}\|_{L^\infty} \to 0$ as $\omega \to -1$, which indicates the limit of small gap solitons.

As we explained in Sec. I, the spectral information is difficult in the case of Dirac equations with $\beta(x), \gamma(x) \equiv 0$. If $\beta(x)$ and $\gamma(x)$ are nonzero and Assumption 1 is used, the stationary gap solitons are not known in the explicit form but the local bifurcation technique allows us to find a family of small gap solitons in a one-sided neighborhood of $\omega = \omega_0$. To make it more precise, let us assume that the nonlinear function is a homogeneous polynomial in its variables.

Assumption 2: Assume that

$$\mathbf{N}(a\mathbf{U}) = a^{2p+1}\mathbf{N}(\mathbf{U}), \quad a \in \mathbb{R},$$

for a fixed integer $p \geq 1$.

Proposition 2: Let Assumptions 1 and 2 be satisfied and

$$\langle \mathbf{u}_0, \mathbf{N}(\mathbf{u}_0)\rangle_{L^2} > 0. \tag{14}$$

For sufficiently small $\epsilon > 0$, there is a family of solutions $\mathbf{U} \in H^1(\mathbb{R})$ of system (11) for any $\omega \in (\omega_0, \omega_0 + \epsilon)$ such that the map $(\omega_0, \omega_0 + \epsilon) \ni \omega \mapsto \mathbf{U} \in H^1(\mathbb{R})$ is defined implicitly by small parameter $a \in \mathbb{R}$ and by the asymptotic expansion,

$$\|\mathbf{U} - a\mathbf{u}_0\|_{H^1} = \mathcal{O}(a^{2p+1}), \quad |\omega - \omega_0 - a^{2p}\langle \mathbf{u}_0, \mathbf{N}(\mathbf{u}_0)\rangle_{L^2}| = \mathcal{O}(a^{4p}), \quad \text{as} \quad a \to 0. \tag{15}$$

Proof: Thanks to Assumption 1, we use the decomposition

$$\mathbf{U} = a\mathbf{u}_0 + \mathbf{V}, \quad a \in \mathbb{R}, \quad \langle \mathbf{u}_0, \mathbf{V}\rangle_{L^2} = 0.$$

Let $P_0 : L^2(\mathbb{R}) \to \text{Ran}(\mathcal{H} - \omega_0 I) \subset L^2(\mathbb{R})$ be the orthogonal projection operator so that $\mathbf{V} = P_0\mathbf{V} \in \text{Ran}(\mathcal{H} - \omega_0 I)$. The stationary Eq. (11) becomes the following system of two equations:

$$\begin{cases} P_0(\mathcal{H} - \omega I)P_0\mathbf{V} + P_0\mathbf{N}(a\mathbf{u}_0 + \mathbf{V}) = \mathbf{0}, \\ (\omega_0 - \omega)a + \langle \mathbf{u}_0, \mathbf{N}(a\mathbf{u}_0 + \mathbf{V})\rangle_{L^2} = 0. \end{cases}$$

Operator $P_0(\mathcal{H} - \omega_0 I)P_0 : H^1(\mathbb{R}) \to L^2(\mathbb{R})$ is invertible for ω near ω_0, whereas $\mathbf{N}(\mathbf{U})$ is a C^{2p+1} function near $\mathbf{0} \in H^1(\mathbb{R})$. By the implicit function theorem, there is a unique C^{2p+1} map $\mathbb{R} \ni a \mapsto \mathbf{V} \in H^1(\mathbb{R}) \cap \text{Ran}(L_0 - \omega_0 I)$ such that $\mathbf{V}$ satisfies the first equation of the system and there are $a_0 > 0$ and $C > 0$ such that for all $a \in (-a_0, a_0)$,

$$\|\mathbf{V} - a^{2p+1}P_0(\mathcal{H} - \omega_0 I)^{-1}P_0\mathbf{N}(\mathbf{u}_0)\|_{H^1} \leq Ca^{4p+1}. \tag{16}$$

Let us substitute the map $\mathbb{R} \ni a \mapsto \mathbf{V} \in H^1(\mathbb{R}) \cap \text{Ran}(L_0 - \omega_0 I)$ to the scalar equation

$$F(a, \omega) = (\omega_0 - \omega) + a^{-1}\langle \mathbf{u}_0, \mathbf{N}(a\mathbf{u}_0 + \mathbf{V})\rangle_{L^2} = 0.$$

Thanks to the bound (16) and Assumption 2, there are $a_0 > 0$ and $C > 0$ such that for all $a \in (-a_0, a_0)$, there is only one solution of $F(a, \omega) = 0$ for $\omega = \omega(a)$ satisfying the bound

$$|\omega_0 + a^{2p}\langle \mathbf{u}_0, \mathbf{N}(\mathbf{u}_0)\rangle_{L^2} - \omega| \leq Ca^{4p}. \tag{17}$$

Under condition (14), we have $\omega > \omega_0$ and the bounds (15) follow from (16) and (17). □

Remark 3: Proposition 2 is valid if

$$\langle \mathbf{u}_0, \mathbf{N}(\mathbf{u}_0)\rangle_{L^2} < 0, \tag{18}$$

but the family of solutions $\mathbf{U} \in H^1(\mathbb{R})$ *of system* (11) *exist for* $\omega \in (\omega_0 - \epsilon, \omega_0)$ *under the condition* (18).

IV. LINEARIZATION AND SPECTRAL STABILITY

Linearization is performed after writing

$$\begin{cases} u(x,t) = e^{-i\omega t}\left[U(x) + \tilde{u}(x,t)\right], \\ v(x,t) = e^{-i\omega t}\left[V(x) + \tilde{v}(x,t)\right], \end{cases}$$

and neglecting quadratic terms with respect to the perturbations $\tilde{u}$ and $\tilde{v}$. Separating variables like

$$\tilde{u}(x,t) = U_1(x)e^{\lambda t}, \quad \bar{\tilde{u}}(x,t) = U_2(x)e^{\lambda t}, \quad \tilde{v}(x,t) = V_1(x)e^{\lambda t}, \quad \bar{\tilde{v}}(x,t) = V_2(x)e^{\lambda t},$$

and substituting this decomposition to the nonlinear Dirac equations (9), we obtain the linear eigenvalue problem

$$\begin{cases} i\lambda \mathbf{U}_1 = (\mathcal{H} - \omega I)\mathbf{U}_1 + V_{11}\mathbf{U}_1 + V_{12}\mathbf{U}_2, \\ -i\lambda \mathbf{U}_2 = (\bar{\mathcal{H}}_0 - \omega I)\mathbf{U}_2 + \bar{V}_{12}\mathbf{U}_1 + \bar{V}_{11}\mathbf{U}_2, \end{cases} \tag{19}$$

where $\mathbf{U}_{1,2} = [U_{1,2}, V_{1,2}]^T \in \mathbb{C}^2$,

$$V_{11} = \begin{bmatrix} \partial^2_{\bar{U}U} W_N & \partial^2_{\bar{U}V} W_N \\ \partial^2_{\bar{V}U} W_N & \partial^2_{\bar{V}V} W_N \end{bmatrix} = \bar{V}_{11}^T, \quad V_{12} = \begin{bmatrix} \partial^2_{\bar{U}\bar{U}} W_N & \partial^2_{\bar{U}\bar{V}} W_N \\ \partial^2_{\bar{V}\bar{U}} W_N & \partial^2_{\bar{V}\bar{V}} W_N \end{bmatrix} = V_{12}^T.$$

We should distinguish the self-adjoint operator $H_\omega : H^1(\mathbb{R}) \to L^2(\mathbb{R})$ given by

$$H_\omega = \begin{bmatrix} \mathcal{H} - \omega I & 0 \\ 0 & \bar{H}_0 - \omega I \end{bmatrix} + \begin{bmatrix} V_{11} & V_{12} \\ \bar{V}_{12} & \bar{V}_{11} \end{bmatrix}$$

and the non-self-adjoint linearization operator $L_\omega = -i\sigma H_\omega : H^1(\mathbb{R}) \to L^2(\mathbb{R})$, where

$$\sigma = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}. \tag{20}$$

Both operators act on $[\mathbf{U}_1, \mathbf{U}_2]^T$.

Symmetry (12) imply that

$$\partial^2_{\bar{U}U} W_N = \partial^2_{\bar{V}V} W_N, \quad \partial^2_{\bar{U}\bar{U}} W_N = \partial^2_{VV} W_N, \quad \partial^2_{UV} W_N = \partial^2_{\bar{U}\bar{V}} W_N.$$

By Theorem 4.1 in Ref. 8, the self-adjoint operator H_ω and the linearized operator L_ω can be block-diagonalized. Let S be an orthogonal matrix given by

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}.$$

Direct computations show that

$$S^{-1} H_\omega S = \begin{bmatrix} H_+ & 0 \\ 0 & H_- \end{bmatrix}, \tag{21}$$

$$S^{-1}\sigma H_\omega S = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} 0 & H_- \\ H_+ & 0 \end{bmatrix}, \tag{22}$$

where $H_\pm$ are two-by-two Dirac operators given by

$$H_\pm = \begin{bmatrix} -i\partial_x + \beta(x) - \omega & \pm(\gamma(x)-1) \\ \pm(\gamma(x)-1) & i\partial_x + \beta(x) - \omega \end{bmatrix} + V_\pm(x), \tag{23}$$

and $V_\pm(x)$ are 2-by-2 matrices with exponentially decaying coefficients given by

$$V_\pm = \begin{bmatrix} \partial^2_{\bar{U}U} W_N \pm \partial^2_{\bar{U}\bar{V}} W_N & \partial^2_{\bar{U}\bar{U}} W_N \pm \partial^2_{\bar{U}V} W_N \\ \partial^2_{\bar{U}U} W_N \pm \partial^2_{\bar{U}\bar{V}} W_N & \partial^2_{\bar{U}\bar{U}} W_N \pm \partial^2_{\bar{U}V} W_N \end{bmatrix}. \tag{24}$$

Thanks to the symmetry of the nonlinear Dirac equations (1) with respect to the gauge transformation, the linearized operator L_ω has a nontrivial kernel because

$$\mathbf{F} \equiv i \begin{bmatrix} \mathbf{U} \\ -\bar{\mathbf{U}} \end{bmatrix} \in \mathrm{Ker}(L_\omega) \equiv \mathrm{Ker}(H_\omega), \tag{25}$$

or explicitly

$$(\mathcal{H} - \omega I)\mathbf{U} + V_{11}\mathbf{U} - V_{12}\bar{\mathbf{U}} = \mathbf{0}. \tag{26}$$

The eigenvector $\mathbf{F}$ generates a two-dimensional generalized kernel

$$\mathrm{span}\{\mathbf{F}, \mathbf{G}\} \subset N_g(L_\omega), \quad \mathbf{G} = -\partial_\omega \begin{bmatrix} \mathbf{U} \\ \bar{\mathbf{U}} \end{bmatrix}, \tag{27}$$

such that $L_\omega \mathbf{G} = \mathbf{F}$, or explicitly

$$(\mathcal{H} - \omega I)\partial_\omega \mathbf{U} + V_{11}\partial_\omega \mathbf{U} + V_{12}\partial_\omega \bar{\mathbf{U}} = \mathbf{U}. \tag{28}$$

The Jordan block is two-dimensional, that is, no $\mathbf{H} \in H^1(\mathbb{R})$ solving $L_\omega \mathbf{H} = \mathbf{G}$ exists, if

$$\frac{d}{d\omega}\|\mathbf{U}\|^2_{L^2} \neq 0. \tag{29}$$

Constraint (29) is satisfied for small a in Proposition 2 under condition (14), because

$$\frac{d\mathbf{U}}{d\omega} = \frac{(\omega-\omega_0)^{\frac{1}{2p}-1}}{2p\langle \mathbf{u}_0, \mathbf{N}(\mathbf{u}_0)\rangle_{L^2}^{\frac{1}{2p}}}\mathbf{u}_0 + \mathcal{O}((\omega-\omega_0)^{\frac{1}{2p}}) \quad \text{as} \quad \omega \to \omega_0$$

and

$$\frac{d}{d\omega}\|\mathbf{U}\|^2_{L^2} = \frac{(\omega-\omega_0)^{\frac{1}{p}-1}}{p\langle \mathbf{u}_0, \mathbf{N}(\mathbf{u}_0)\rangle_{L^2}^{\frac{1}{p}}} + \mathcal{O}((\omega-\omega_0)^{\frac{1}{p}}) \quad \text{as} \quad \omega \to \omega_0.$$

In the same limit, the spectra of the linearized operator L_ω and the self-adjoint operator H_ω are characterized in the following proposition.

Proposition 3: Let assumptions of Proposition 2 be satisfied. For sufficiently small $\epsilon > 0$ and for any $\omega \in (\omega_0, \omega_0 + \epsilon)$, we have

$$\sigma(iL_\omega) = (-\infty, -1-\omega] \cup (-\infty, -1+\omega] \cup \{0\} \cup [1-\omega, \infty) \cup [1+\omega, \infty)$$

and

$$\sigma(H_+) = (-\infty, -1-\omega] \cup \{\omega_1\} \cup [1-\omega, \infty), \quad \sigma(H_-) = (-\infty, -1-\omega] \cup \{0\} \cup [1-\omega, \infty),$$

where $\omega_1 = \mathcal{O}(|\omega - \omega_0|)$. The zero eigenvalue is double for L_ω and simple for H_-, whereas the eigenvalue ω_1 is simple. No resonances exist at the end points of the continuous spectrum of L_ω and $H_\pm$.

Proof: The proof holds by perturbation theory. The self-adjoint operator H_+ is represented by

$$H_+ = \mathcal{H} - \omega I + V_+,$$

where $\|V_+\|_{L^\infty} = \mathcal{O}(a^{2p})$ as $a \to 0$ (parameter a is used in Proposition 2). We recall the resolvent identity,

$$R_+(\lambda) = (I + R_{\mathcal{H}}(\lambda + \omega)V_+)^{-1} R_{\mathcal{H}}(\lambda + \omega),$$

where $R_+(\lambda) = (H_+ - \lambda I)^{-1}$ and $R_{\mathcal{H}}(\lambda) = (\mathcal{H} - \lambda I)^{-1}$. By Assumption 1, no resonances exist in $\mathcal{H}$, hence $R_{\mathcal{H}}(\lambda)$ near $\lambda = \pm 1$ is a bounded operator from $L^2_\alpha(\mathbb{R})$ to $L^2_{-\alpha}(\mathbb{R})$ for any $\alpha > \frac{1}{2}$. Together with the smallness of V_+, this implies that $R_+(\lambda)$ near $\lambda = \pm 1 - \omega$ is also a bounded operator from $L^2_\alpha(\mathbb{R})$ to $L^2_{-\alpha}(\mathbb{R})$ for any $\alpha > \frac{1}{2}$. Therefore, no new eigenvalues of H_+ exist near the non-resonant points $\pm 1 - \omega$ for small $a > 0$. By the perturbation theory and the smallness of $\omega - \omega_0 = \mathcal{O}(a^{2p})$, the only eigenvalue at 0 for $a = 0$ becomes the eigenvalue $\omega_1 = \mathcal{O}(a^{2p})$ as $a \to 0$.

The self-adjoint operator H_- is given by

$$H_- = \sigma_3 \mathcal{H} \sigma_3 - \omega I + V_-,$$

where $\|V_-\|_{L^\infty} = \mathcal{O}(a^{2p})$ as $a \to 0$. The same perturbation theory applies to self-adjoint operator H_-, except of the fact that the only eigenvalue at 0 for $a = 0$ is preserved at 0 for $a > 0$ thanks to the gauge invariance, which results in the exact relation

$$H_- \begin{bmatrix} U \\ -V \end{bmatrix} = \mathbf{0}.$$

Similarly, the double zero eigenvalue of L_ω is preserved at 0 by the gauge invariance as the generalized kernel (27), whereas the continuous spectrum does not lead to resonances at the end points or to new eigenvalues for small $a > 0$. □

V. PROJECTIONS AND MODULATION EQUATIONS

By Proposition 3, we have

$$N_g(L_\omega) = \text{span}\{\mathbf{F}, \mathbf{G}\}. \tag{30}$$

Recalling matrix σ from (20), we obtain the adjoint operator

$$L_\omega^* = (-i\sigma H_\omega)^* = i H_\omega^* \sigma^* = i H_\omega \sigma,$$

which has the generalized kernel

$$N_g(L_\omega^*) = \text{span}\{\sigma\mathbf{F}, \sigma\mathbf{G}\}. \tag{31}$$

Any vector $[\mathbf{U}_1, \mathbf{U}_2]^T$ in the invariant subspace of the linearized operator L_ω in $L^2(\mathbb{R})$, which is an orthogonal complement of the generalized null space $N_g(L_\omega)$, has to satisfy the symplectic orthogonality conditions

$$\begin{cases} \langle \mathbf{U}, \mathbf{U}_1 \rangle_{L^2} + \langle \bar{\mathbf{U}}, \mathbf{U}_2 \rangle_{L^2} = 0, \\ \langle \partial_\omega \mathbf{U}, \mathbf{U}_1 \rangle_{L^2} - \langle \partial_\omega \bar{\mathbf{U}}, \mathbf{U}_2 \rangle_{L^2} = 0. \end{cases} \tag{32}$$

If $\mathbf{U}_2 = \bar{\mathbf{U}}_1$, the symplectic orthogonality conditions (32) can be rewritten in the explicit form

$$\text{Re}\langle \mathbf{U}, \mathbf{U}_1 \rangle_{L^2} = 0, \quad \text{Im}\langle \partial_\omega \mathbf{U}, \mathbf{U}_1 \rangle_{L^2} = 0. \tag{33}$$

Using symplectic orthogonality conditions, we now set up modulation equations for nonlinear dynamics of small gap solitons. By Proposition 1, we have at least local solutions of the nonlinear

Dirac equations (9). Now we look for local solutions in the form

$$\begin{cases} u(x,t) = e^{-i\theta(t)}\left[U(x;\omega(t)) + U_1(x,t)\right], \\ v(x,t) = e^{-i\theta(t)}\left[V(x;\omega(t)) + V_1(x,t)\right], \end{cases} \tag{34}$$

where we write explicitly the dependence of the stationary solution $\mathbf{U} = [U, V]^T$ on ω. The time evolution problem for $\mathbf{U}_1 = [U_1, V_1]^T$ is given by

$$i\frac{d\mathbf{U}_1}{dt} = (\mathcal{H} - \omega I)\mathbf{U}_1 - i\dot{\omega}\partial_\omega \mathbf{U} - (\dot{\theta} - \omega)(\mathbf{U} + \mathbf{U}_1) + \mathbf{N}(\mathbf{U} + \mathbf{U}_1) - \mathbf{N}(\mathbf{U}). \tag{35}$$

Using the symplectic orthogonality condition (33) on $\mathbf{U}_1$, we obtain the modulation equations on $\omega(t)$ and $\theta(t)$:

$$\begin{cases} \dot{\omega}\mathrm{Re}\langle \partial_\omega \mathbf{U}, \mathbf{U} - \mathbf{U}_1\rangle_{L^2} + (\dot{\theta} - \omega)\mathrm{Im}\langle \mathbf{U}, \mathbf{U}_1\rangle_{L^2} = \Omega_1, \\ \dot{\omega}\mathrm{Im}\langle \partial_\omega^2 \mathbf{U}, \mathbf{U}_1\rangle_{L^2} + (\dot{\theta} - \omega)\mathrm{Re}\langle \partial_\omega \mathbf{U}, \mathbf{U} + \mathbf{U}_1\rangle_{L^2} = \Omega_2, \end{cases} \tag{36}$$

where

$$\Omega_1 = \mathrm{Im}\left[\langle \mathbf{U}, \mathbf{N}(\mathbf{U} + \mathbf{U}_1) - \mathbf{N}(\mathbf{U})\rangle_{L^2} + \langle \bar{V}_{12}\bar{\mathbf{U}} - V_{11}\mathbf{U}, \mathbf{U}_1\rangle_{L^2}\right],$$
$$\Omega_2 = \mathrm{Re}\left[\langle \partial_\omega \mathbf{U}, \mathbf{N}(\mathbf{U} + \mathbf{U}_1) - \mathbf{N}(\mathbf{U})\rangle_{L^2} - \langle V_{12}\partial_\omega \bar{\mathbf{U}} + V_{11}\partial_\omega \mathbf{U}, \mathbf{U}_1\rangle_{L^2}\right]$$

and Eqs. (26) and (28) have been used. The following result shows that the right-hand side of system (36) is quadratic with respect to the perturbation vector $\mathbf{U}_1$.

Proposition 4: Let assumptions of Proposition 2 be satisfied. Fix small $\epsilon > 0$ *and* $\delta > 0$. *For any* $\omega \in (\omega_0, \omega_0 + \epsilon)$ *and any* $\mathbf{U}_1 \in B_\delta(L^\infty)$, *there is* $C_{\epsilon,\delta} > 0$ *such that*

$$|\Omega_1| + |\Omega_2| \leq C_{\epsilon,\delta}\left|\langle \mathbf{U}_1^2, \mathbf{U}\rangle_{L^2}\right|. \tag{37}$$

Proof: In order to show that the linear terms in $\mathbf{U}_1$ vanish in the expression for Ω_1 and Ω_2, we use the expansion

$$\mathbf{N}(\mathbf{U} + \mathbf{U}_1) = \mathbf{N}(\mathbf{U}) + V_{11}\mathbf{U}_1 + V_{12}\bar{\mathbf{U}}_1 + \mathcal{O}(\|\mathbf{U}_1\|^2),$$

constraints on matrices $V_{11} = \bar{V}_{11}^T$ and $V_{12} = V_{12}^T$, and elementary properties of inner product. For instance, Ω_1 has the following linear terms in $\mathbf{U}_1$:

$$\mathrm{Im}\left[\langle \mathbf{U}, V_{11}\mathbf{U}_1 + V_{12}\bar{\mathbf{U}}_1\rangle_{L^2} + \langle \bar{V}_{12}\bar{\mathbf{U}} - V_{11}\mathbf{U}, \mathbf{U}_1\rangle_{L^2}\right]$$
$$= \mathrm{Im}\left[\langle \bar{V}_{11}^T\mathbf{U}, \mathbf{U}_1\rangle_{L^2} - \langle V_{11}\mathbf{U}, \mathbf{U}_1\rangle_{L^2} + \langle \bar{V}_{12}^T\mathbf{U}, \bar{\mathbf{U}}_1\rangle_{L^2} + \langle \bar{V}_{12}\bar{\mathbf{U}}, \mathbf{U}_1\rangle_{L^2}\right] = 0.$$

Similar computations holds for linear terms of Ω_2. Together with smoothness in Assumption 2, this computation shows that both terms Ω_1, Ω_2 are quadratic in $\mathbf{U}_1$ in the sense of (37). □

VI. MAIN RESULT

Setting $\mathbf{U}_1 = \mathbf{Y}e^{i\theta}$, we rewrite the time-evolution Eq. (35) in the equivalent form

$$i\frac{d\mathbf{Y}}{dt} = \mathcal{H}\mathbf{Y} + e^{-i\theta}\mathbf{F}, \quad \mathbf{F} = -i\dot{\omega}\partial_\omega \mathbf{U} - (\dot{\theta} - \omega)\mathbf{U} + \mathbf{N}(\mathbf{U} + \mathbf{Y}e^{i\theta}) - \mathbf{N}(\mathbf{U}). \tag{38}$$

The terms $\dot{\omega}$ and $\dot{\theta} - \omega$ are uniquely determined from the system of modulation equations (36).

We are now ready to formulate the main theorem of this article.

Theorem 1: *Assume Assumption* 1, *Assumption* 2 *with* $p \geq 2$, *and condition* (14). *Fix* $\epsilon > 0$ *and* $\delta > 0$ *sufficiently small such that* $\theta(0) = 0$, $\omega(0) \in (\omega_0, \omega_0 + \epsilon)$, *and* $\mathbf{Y}(0) \in B_\delta(H^1)$. *There exist* $\epsilon_0 > 0$, $\theta_\infty \in \mathbb{R}$, $\omega_\infty \in (\omega_0, \omega_0 + \epsilon_0)$, $(\omega, \theta) \in C^1(\mathbb{R}_+, \mathbb{R}^2)$, *and*

$$\mathbf{Y}(t) \in C(\mathbb{R}_+, H^1) \cap L^4(\mathbb{R}_+, L^\infty)$$

such that $(\omega, \theta)(t)$ solve the modulation Eqs. (36), $\mathbf{Y}(t)$ solves the evolution Eq. (38), and

$$\lim_{t\to\infty}\left(\theta(t) - \int_0^t \omega(s)ds\right) = \theta_\infty, \quad \lim_{t\to\infty}\omega(t) = \omega_\infty, \quad \lim_{t\to\infty}\|\mathbf{Y}(t)\|_{L^\infty} = 0.$$

Remark 4: The space $H^1(\mathbb{R})$ used in Theorem 1 is appropriate both for local well-posedness (Proposition 1) in $H^s(\mathbb{R})$ for $s > \frac{1}{2}$ and for the Strichartz estimates (Lemma 4) in $H^s(\mathbb{R})$ for $s > \frac{3}{4}$.

We shall prove this theorem in the remainder of the article. To do so, we shall develop first the spectral theory for the Dirac operator $\mathcal{H}$ and obtain the dispersive decay estimates for the semi-group $e^{-it\mathcal{H}}$ acting on the continuous spectrum of $\mathcal{H}$.

VII. SPECTRAL THEORY FOR OPERATOR $\mathcal{H}$

Let us consider the spectral problem $\mathcal{H}\mathbf{u} = \lambda\mathbf{u}$ or explicitly,

$$\begin{cases} -iu'(x) + \beta(x)u(x) + (\gamma(x) - 1)v(x) = \lambda u(x), \\ iv'(x) + \beta(x)v(x) + (\gamma(x) - 1)u(x) = \lambda v(x), \end{cases} \quad x \in \mathbb{R}. \tag{39}$$

Recall that

$$\sigma_c(\mathcal{H}) \equiv (-\infty, -1] \cup [1, \infty).$$

Here we develop the scattering theory of wave operators for the Dirac operator $\mathcal{H}$. A similar theory for the Schrödinger operators on an infinite line goes back to the works of Weder[38,39] and Goldberg and Schlag.[15]

Let us first define the Jost functions for $\lambda \in (-\infty, -1]$ at one branch of $\sigma_c(\mathcal{H})$. To do so, let us parameterize $(-\infty, -1]$ by $\lambda = -\sqrt{1+k^2}$ for $k \in \mathbb{R}$ and consider solutions of system (39) according to the boundary conditions

$$\mathbf{u}^\pm(x;k) \to \begin{bmatrix} 1 \\ \alpha_\pm(k) \end{bmatrix} e^{\pm ikx} \quad \text{as} \quad x \to \pm\infty, \tag{40}$$

where $\alpha_\pm(k) := \sqrt{1+k^2} \pm k$. The boundary conditions (40) arise naturally in system (39) for $\beta(x)$, $\gamma(x) \equiv 0$. The following proposition gives the construction of Jost functions for nonzero β and γ.

Proposition 5: For any $k \in \mathbb{R}$, there exist unique Jost functions $\mathbf{u}^\pm(x; k)$ such that

$$\lim_{x\to\pm\infty}\left[\mathbf{u}^\pm(\cdot;k) - [1, \alpha_+]^T e^{\pm ikx}\right] = 0.$$

Moreover,

- *If $k \neq 0$, then $\mathbf{u}^\pm(\cdot;k) \in L^\infty(\mathbb{R})$.*
- *If $k = 0$, then $\mathbf{u}^\pm(x; 0)$ may grow at most linearly in x as $x \to \mp\infty$.*
- *As $k \to \pm\infty$, both $\mathbf{u}^+(x; k)_1$ and $\mathbf{u}^-(x; k)_1$ are bounded, $\mathbf{u}^\pm(x; k)_2$ grows linearly in k, and $\mathbf{u}^\mp(x; k)_2$ decays inverse linearly in k.*

Proof: Setting $\mathbf{u}^\pm(x; k) = \mathbf{m}^\pm(x; k)e^{\pm ikx}$ and using Green's function for the unperturbed problem with $\beta(x)$, $\gamma(x) \equiv 0$, we obtain an integral equation for the Jost functions $\mathbf{m}^\pm(x; k)$

$$\mathbf{m}^\pm(x;k) = \begin{bmatrix} 1 \\ \alpha_\pm \end{bmatrix} + \int_x^{\pm\infty} G^\pm(x-y;k)V(y)\mathbf{m}^\pm(y;k)dy, \tag{41}$$

where

$$G^\pm(x;k) = \frac{1}{2ik}\begin{bmatrix} \alpha_\mp - \alpha_\pm e^{\mp 2ikx} & 1 - e^{\mp 2ikx} \\ 1 - e^{\mp 2ikx} & \alpha_\pm - \alpha_\mp e^{\mp 2ikx} \end{bmatrix}. \tag{42}$$

Under the assumption of fast decay of $V(x)$ to 0 as $|x| \to \infty$, the standard theory gives solutions $\mathbf{m}^{\pm}(\cdot; k) \in L^{\infty}(\mathbb{R})$ of the integral Eqs. (41) for $k \neq 0$.

If $k = 0$, the Jost functions $\mathbf{m}^{\pm}(x; 0)$ satisfy the integral equation

$$\mathbf{m}^{\pm}(x;0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \int_x^{\pm\infty} G^{\pm}(x-y;0)V(y)\mathbf{m}^{\pm}(y;0)dy, \tag{43}$$

where

$$G^{\pm}(x;0) = \pm \begin{bmatrix} x+i & x \\ x & x-i \end{bmatrix}. \tag{44}$$

Thanks to the fast decay of $V(x)$, existence of locally bounded function $\mathbf{m}^{\pm}(x; 0)$ follows again from the standard theory. The linear growth of $\mathbf{m}^{\pm}(x; 0)$ as $x \to \mp\infty$ follows from the integral Eqs. (43).

Finally, as $k \to \pm\infty$, $\alpha_{\pm}$ grows linearly in k, $\alpha_{\mp}$ decays inverse linearly in k, whereas $G^{\pm}(x; k)$ remains bounded. The asymptotic behavior of solutions $\mathbf{m}^{\pm}(x; k)$ of the integral Eqs. (41) follows the asymptotic behavior of the limiting functions (40) in k as $k \to +\infty$. □

Remark 5: Proposition 5 eliminates the possibility of embedded eigenvalues in the continuous spectrum $\sigma_c(\mathcal{H})$ because the space of solutions of the Dirac system (39) for $\lambda < -1$ is spanned by the two fundamental solutions $\mathbf{u}^{\pm}(x; k)$ with no decay to zero as $x \to \pm\infty$.

It follows from the integral Eqs. (41) that the Jost functions satisfy the scattering relation for all $x \in \mathbb{R}$ including $x \to \mp\infty$

$$\begin{cases} \mathbf{m}^{+}(x;k) = a^{+}(k)\mathbf{m}^{-}(x;-k) + b^{+}(k)\mathbf{m}^{-}(x;k)e^{-2ikx}, \\ \mathbf{m}^{-}(x;k) = a^{-}(k)\mathbf{m}^{+}(x;-k) + b^{-}(k)\mathbf{m}^{+}(x;k)e^{2ikx}, \end{cases} \tag{45}$$

where

$$a^{\pm}(k) = 1 \pm \frac{1}{2ik}\int_{\mathbb{R}} \left(\alpha_{\mp}[V(x)\mathbf{m}^{\pm}(x;k)]_1 + [V(x)\mathbf{m}^{\pm}(x;k)]_2\right) dx,$$

$$b^{\pm}(k) = \mp\frac{1}{2ik}\int_{\mathbb{R}} \left(\alpha_{\pm}[V(x)\mathbf{m}^{\pm}(x;k)]_1 + [V(x)\mathbf{m}^{\pm}(x;k)]_2\right) e^{\pm 2ikx} dx.$$

The following proposition lists some constraints on the scattering coefficients in the scattering relation (45).

Proposition 6: For any $k \in \mathbb{R}$, we have

$$a^{+}(k) = a^{-}(k), \quad b^{+}(k) = -b^{-}(-k), \tag{46}$$

$$a^{+}(-k) = \bar{a}^{+}(k), \quad b^{+}(-k) = \frac{\sqrt{1+k^2}-k}{\sqrt{1+k^2}+k}\bar{b}^{+}(k), \tag{47}$$

and

$$|a^{+}(k)|^2 = 1 + \frac{\sqrt{1+k^2}-k}{\sqrt{1+k^2}+k}|b^{+}(k)|^2. \tag{48}$$

Proof: Inverting the scattering relation (45), we obtain the constraints on the scattering coefficients for all $k \in \mathbb{R}$,

$$\begin{cases} a^{+}(k)a^{-}(-k) + b^{+}(k)b^{-}(k) = 1, \\ a^{+}(k)b^{-}(-k) + b^{+}(k)a^{-}(k) = 0. \end{cases} \tag{49}$$

Let $W(\mathbf{u}_1, \mathbf{u}_2)$ denote the Wronskian determinant of any two solutions $\mathbf{u}_1$ and $\mathbf{u}_2$ of the Dirac system (39). It is clear that $W(\mathbf{u}_1, \mathbf{u}_2)$ is constant in $x \in \mathbb{R}$. Therefore, $W(\mathbf{u}_1, \mathbf{u}_2)$ can be computed in the limits $x \to \pm\infty$. Using boundary values (40) and scattering relations (45), we obtain

$$W(\mathbf{u}^+, \mathbf{u}^-) = u_1^+(x;k)u_2^-(x;k) - u_2^+(x;k)u_1^-(x;k) = -2ka^+(k) = -2ka^-(k). \tag{50}$$

This result together with the second equation of system (49) gives relations (46). The first equation of system (49) implies now for all $k \in \mathbb{R}$ that

$$a^+(k)a^+(-k) - b^+(k)b^+(-k) = 1. \tag{51}$$

Let (u_k, v_k) denote any solution of the Dirac system (39) for $\lambda = -\sqrt{1+k^2}$. It is checked directly that

$$|u_k|^2 - |v_k|^2 \quad \text{and} \quad \bar{u}_{-k}u_k - \bar{v}_{-k}v_k$$

are constant in x. Using boundary values (40) and scattering relations (45) again, we obtain

$$(\sqrt{1+k^2}+k)(1-|a^+(k)|^2) + (\sqrt{1+k^2}-k)|b^+(k)|^2 = 0, \tag{52}$$

$$(\sqrt{1+k^2}+k)\overline{b^-(k)} + (\sqrt{1+k^2}-k)b^+(k) = 0. \tag{53}$$

These identities together with Eq. (51) give relations (47) and (48). □

Remark 6: Identity (48) shows that $|a^+(k)| \geq 1$ for all $k \in \mathbb{R}$. This excludes embedded resonant states with $a^+(k) = 0$.

It follows from the explicit expressions for $a^\pm(k)$ and $b^\pm(k)$ that

$$a^\pm(k) \to \frac{\pm\gamma^\pm}{2ik}, \quad b^\pm(k) \to \frac{\mp\gamma^\pm}{2ik} \quad \text{as} \quad k \to 0, \tag{54}$$

where

$$\gamma^\pm = \int_{\mathbb{R}} (\beta(x) + \gamma(x))(m_1^\pm(x;0) + m_2^\pm(x;0))dx. \tag{55}$$

It is clear from (46) that $\gamma^+ = -\gamma^-$.

Remark 7: If $\gamma^+ = \gamma^- = 0$, then $\lim_{k\to 0} a^\pm(k)$ exists. At the same time, there exists a bounded solution of the integral Eqs. (43). We recall that the end points ± 1 are said to be resonances if there exist a solution $\mathbf{u} \in L^\infty(\mathbb{R})$ of the spectral problem (39) for $\lambda = \pm 1$. Therefore, Assumption 1 is satisfied under condition $\gamma^+ \neq 0$.

We shall now define the Jost functions for $\lambda \in [1, \infty)$ at the other branch of $\sigma_c(\mathcal{H})$. Similarly to the analysis for $\lambda \in (-\infty, -1]$, we can parameterize $[1, \infty)$ by $\lambda = \sqrt{1+k^2}$ for $k \in \mathbb{R}$ and consider solutions of system (39) according to the boundary conditions

$$\mathbf{v}^\pm(x;k) \to \begin{bmatrix} -\alpha_\pm \\ 1 \end{bmatrix} e^{\pm ikx} \quad \text{as} \quad x \to \pm\infty.$$

Using a similar Green's function, Proposition 5 can be extended to functions $\mathbf{v}^\pm(x;k)$. In what follows, we will not treat functions $\mathbf{v}^\pm(x;k)$ for $\lambda \in [1, \infty)$ but will only be working with functions $\mathbf{u}^\pm(x;k)$ for $\lambda \in (-\infty, -1]$. This approach does not limit any generality. Moreover, we note the particularly remarkable case.

Remark 8: If $\beta(x) \equiv 0$, the Jost functions are related by

$$\mathbf{v}^\pm(x;k) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{u}^\pm(x;k),$$

thanks to the symmetry of the Dirac system (39).

Let us denote again the resolvent operator by $R_{\mathcal{H}}(\lambda) = (\mathcal{H} - \lambda I)^{-1}$. $R_{\mathcal{H}}(\lambda)$ is defined as a bounded operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ for any $\lambda \notin \sigma(\mathcal{H})$. Using the Jost functions, we will show that the resolvent operator can be extended to the continuous spectrum as a bounded operator from $L^2_\alpha(\mathbb{R})$ to $L^2_{-\alpha}(\mathbb{R})$ for any $\alpha > \frac{1}{2}$. Let us denote the limiting operators by

$$R^{\pm}_{\mathcal{H}}(\lambda) := \lim_{\epsilon \downarrow 0} R_{\mathcal{H}}(\lambda \pm i\epsilon), \quad \lambda \in \sigma_c(\mathcal{H}),$$

depending on whether $\lambda \to \sigma_c(\mathcal{H})$ from the upper or lower halves of the complex plane of λ.

The following proposition allows us to express $R^{\pm}_{\mathcal{H}}(\lambda)$ for $\lambda \in \sigma_c(\mathcal{H})$ in terms of the Jost functions. According to the previous remarks, it is sufficient to consider $\lambda \in (-\infty, -1]$. The arguments for $\lambda \in [1, \infty)$ can be developed similarly.

Proposition 7: For any $\lambda \in (-\infty, -1)$ and any fixed $\alpha > \frac{1}{2}$, operators $R^{\pm}_{\mathcal{H}}(\lambda) : L^2_\alpha(\mathbb{R}) \to L^2_{-\alpha}(\mathbb{R})$ can be represented by the integral kernel in the form

$$[R^{\pm}_{\mathcal{H}}(\lambda)](x, y) = \frac{\pm 1}{2ika^+(\pm k)} \begin{cases} \mathbf{u}^+(x; \pm k)[\sigma_1 \mathbf{u}^-(y; \pm k)]^T, & x > y, \\ \mathbf{u}^-(x; \pm k)[\sigma_1 \mathbf{u}^+(y; \pm k)]^T, & x < y, \end{cases} \tag{56}$$

where $k \le 0$ and $\lambda = -\sqrt{1 + k^2}$.

Proof: Let us consider the solutions of the linear system for a fixed $y \in \mathbb{R}$ and $\lambda = -\sqrt{1 + k^2}$,

$$(\mathcal{H} - \lambda I)[R^{+}_{\mathcal{H}}(\lambda)](x, y) = \delta(x - y) Id, \tag{57}$$

which satisfy the asymptotic behavior,

$$[R^{+}_{\mathcal{H}}(\lambda)](x, y) \sim e^{ik|x-y|}, \quad \text{as} \quad |x - y| \to \infty. \tag{58}$$

The function $[R^{+}_{\mathcal{H}}(\lambda)](x, y)$ decays exponentially as $|x - y| \to \infty$ if k is extended off the real axis with $\mathrm{Im}(k) > 0$. Since $\mathrm{Re}(\lambda)\mathrm{Im}(\lambda) = \mathrm{Re}(k)\mathrm{Im}(k)$ and $\mathrm{Re}(\lambda) \le -1$, we understand that the behavior (58) recovers the limiting resolvent operator $R^{+}_{\mathcal{H}}(\lambda)$ defined for $\mathrm{Im}(\lambda) \ge 0$ if $\mathrm{Re}(k) \le 0$.

For the first column vector of the linear system (57), denoted by (u, v), we obtain

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{cases} c(y, k)\mathbf{u}^+(x; k), & x > y, \\ d(y, k)\mathbf{u}^-(x; k), & x < y, \end{cases} \tag{59}$$

where the behavior (58) is satisfied thanks to the boundary conditions (40). Parameters (c, d) are to be determined.

Matching conditions across the point $x = y$ sets up the linear system for c and d with the unique solution,

$$c(y, k) = \frac{i\mathbf{u}^-(y; k)_2}{W(\mathbf{u}^+, \mathbf{u}^-)}, \quad d(y, k) = \frac{i\mathbf{u}^+(y; k)_2}{W(\mathbf{u}^+, \mathbf{u}^-)},$$

where $W(\mathbf{u}^+, \mathbf{u}^-) = -2ka^+(k)$ by identity (50).

Similarly for the second column vector of the linear system (57), we obtain the same expression (59) with a different solution of the linear system for (c, d),

$$c(y, k) = \frac{i\mathbf{u}^-(y; k)_1}{W(\mathbf{u}^+, \mathbf{u}^-)}, \quad d(y, k) = \frac{i\mathbf{u}^+(y; k)_1}{W(\mathbf{u}^+, \mathbf{u}^-)}.$$

Using the Pauli matrix σ_1, we arrive to the expression (56) for $R^{+}_{\mathcal{H}}(\lambda)$. The expression for $R^{-}_{\mathcal{H}}(\lambda)$ is found by the replacement of k by $-k$. The exponential decay as $|x - y| \to \infty$ occurs now for $\mathrm{Im}(k) < 0$. The limiting resolvent operator $R^{-}_{\mathcal{H}}(\lambda)$ is defined for $\mathrm{Im}(\lambda) \le 0$ if $\mathrm{Re}(k) \le 0$. □

The following proposition describes λ-uniform bounds on the limiting resolvent operators $R^{\pm}_{\mathcal{H}}(\lambda)$ in weighted spaces. In order to exclude problems at the end points $\lambda = \pm 1$, we assume that no end-point resonances occur at $k = 0$ (Assumption 1). Thanks to Remark 7, it is equivalent to assume that $\gamma^+ \neq 0$.

Proposition 8: Let $\gamma^+ \neq 0$ *in (55). For any* $\alpha > \frac{3}{2}$*, there exists constant* $C_\alpha > 0$ *such that*

$$\sup_{|\lambda|\geq 1} \|R^{\pm}_{\mathcal{H}}(\lambda)\|_{L^2_\alpha \to L^2_{-\alpha}} \leq C_\alpha. \tag{60}$$

In addition, for any $\alpha \geq 1$*, there exists constant* $C_\alpha > 0$ *such that*

$$\sup_{|\lambda|\geq 1} \|R^{\pm}_{\mathcal{H}}(\lambda)\|_{L^1_\alpha \to L^\infty_{-\alpha}} \leq C_\alpha. \tag{61}$$

Proof: We recall that $|a^+(k)| \geq 1$ (Remark 6). Thanks to the asymptotic expansion (54), if $\gamma^+ \neq 0$, then $ka^+(k) \neq 0$ for any $k \in \mathbb{R}$. Using this result and Proposition 7, we construct

$$\hat{R}^{\pm}_{\mathcal{H},\alpha}(x, y) \equiv \frac{[R^{\pm}_{\mathcal{H}}(\lambda)](x, y)}{(1 + x^2)^{\alpha/2}(1 + y^2)^{\alpha/2}}.$$

By Proposition 5, $\mathbf{u}^{\pm}(\cdot; k) \in L^\infty(\mathbb{R})$ for every $k \neq 0$ and $\mathbf{u}^{\pm}(x; 0)$ grow at most linearly in x as $x \to \mp\infty$. Therefore, $\hat{R}^{\pm}_{\mathcal{H},\alpha}(x, y)$ is a kernel of a Hilbert–Schmidt operator for any fixed $\lambda \in (-\infty, -1]$ and $\alpha > \frac{3}{2}$.

It remains to show that $\hat{R}^{\pm}_{\mathcal{H},\alpha}(x, y)$ is uniformly bounded in the limit $\lambda \to -\infty$ ($k \to -\infty$) for any $x, y \in \mathbb{R}$. Note that

$$\mathbf{u}^+(x;k)[\sigma_1 \mathbf{u}^-(y;k)]^T = \begin{bmatrix} \mathbf{u}^+(x;k)_1 \mathbf{u}^-(y;k)_2 & \mathbf{u}^+(x;k)_1 \mathbf{u}^-(y;k)_1 \\ \mathbf{u}^+(x;k)_2 \mathbf{u}^-(y;k)_2 & \mathbf{u}^+(x;k)_2 \mathbf{u}^-(y;k)_1 \end{bmatrix}$$

and a similar formula for $\mathbf{u}^-(x; k)[\sigma_1 \mathbf{u}^+(y; k)]^T$. By Proposition 5, this matrix grows linearly in k as $k \to -\infty$ for any $x, y \in \mathbb{R}$. On the other hand, $ka_+(k)$ grows at least linearly as $|k| \to \infty$, which implies the λ-uniform bound (60).

To prove bound (61), we can see from the linear growth of $\mathbf{u}^{\pm}(x; 0)$ as $x \to \mp\infty$ that $\hat{R}^{\pm}_{\mathcal{H},\alpha}(x, y)$ is a kernel of a bounded operator from $L^1(\mathbb{R})$ to $L^\infty(\mathbb{R})$ for any $\alpha \geq 1$. The mapping is also bounded as $k \to -\infty$. □

Let $P_{a.c.}(\mathcal{H}) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the orthogonal projection operator to the continuous spectrum of $\mathcal{H}$. We recall the Cauchy formula,

$$e^{-it\mathcal{H}} P_{a.c.}(\mathcal{H}) f = \frac{1}{2\pi i} \left(\int_{-\infty}^{-1} + \int_{1}^{\infty} \right) e^{-it\lambda} \left[R^+_{\mathcal{H}}(\lambda) - R^-_{\mathcal{H}}(\lambda) \right] f \, d\lambda, \tag{62}$$

where the integral is understood in the norm of the mapping from $L^2_\alpha(\mathbb{R})$ to $L^2_{-\alpha}(\mathbb{R})$ for $\alpha > \frac{3}{2}$. The interval $(-\infty, -1]$ for λ can be parameterized by $(-\infty, 0]$ for k using the substitution

$$\lambda = -\sqrt{1 + k^2} \quad \Rightarrow \quad d\lambda = -\frac{k\,dk}{\sqrt{1 + k^2}}.$$

These representations are used for the derivation of linear dispersive decay estimates for the semi-group $e^{-it\mathcal{H}} P_{\mathrm{a.c.}}(\mathcal{H})$.

VIII. LINEAR ESTIMATES FOR THE OPERATOR $\mathcal{H}$

We shall need two preliminary results, which will be useful in our arguments for this section.

A. Preliminaries

The first result that we need is the Christ-Kiselev lemma. We actually state a version due to Smith and Sogge.[36]

Lemma 1: Let X, Y be Banach spaces and $\mathcal{K} : L^p(\mathbb{R}; X) \to L^q(\mathbb{R}, Y)$ be a linear operator such that $\mathcal{K} f(t) = \int_{-\infty}^{\infty} K(t,s) f(s) ds$. Then, the operator

$$\tilde{\mathcal{K}} f(t) = \int_0^t K(t,s) f(s) ds, \tag{63}$$

is bounded from $L^p(\mathbb{R}; X)$ to $L^q(\mathbb{R}, Y)$, provided $p < q$. Moreover, there is $C_{p,q} > 0$ such that

$$\|\tilde{\mathcal{K}}\|_{L^p(\mathbb{R};X)\to L^q(\mathbb{R},Y)} \leq C_{p,q} \|\mathcal{K}\|_{L^p(\mathbb{R};X)\to L^q(\mathbb{R},Y)}.$$

The second lemma is a technical statement, which is complementary to Lemma 1, when the condition $p < q$ is violated (most notably when $p = q$). This is stated for the Schrödinger operator $-\partial_x^2 + V(x)$ by Mizumachi (Lemma 11 in Ref. 26), but it applies equally well to an arbitrary self-adjoint operator $\mathcal{L}$.

Lemma 2: Let $\mathcal{L}$ be a self-adjoint operator and $P_{a.c}(\mathcal{L})$ be a projection to the absolute continuous spectrum of $\mathcal{L}$. Let $g(t, x) = g_1(t) g_2(x)$, where g_1, g_2 are in Schwartz's class, and define the function

$$U(t,x) = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-it\lambda} \check{g}_1(\lambda) \left(\left[R_{\mathcal{L}}^+(\lambda) + R_{\mathcal{L}}^-(\lambda) \right] g_2 \right)(x) d\lambda. \tag{64}$$

Then, we have

$$U(t,\cdot) = 2\int_0^t e^{-i(t-s)\mathcal{L}} P_{a.c.}(\mathcal{L}) g(s,\cdot) ds + \left(\int_{-\infty}^0 - \int_0^{\infty} \right) e^{-i(t-s)\mathcal{L}} P_{a.c.}(\mathcal{L}) g(s,\cdot) ds.$$

We use the resolvent analysis of the Dirac operator $\mathcal{H}$ to derive some linear estimates, which are used in the proof of the main theorem.

B. Mizumachi estimates

We refer to Mizumachi's work[26] in the context of the one-dimensional NLS equation, which was used in our work[18] in the context of the discrete NLS equation. For a different approach based on the classical Kato smoothing, see the work of Cuccagna and Tarulli.[13]

The following estimates are developed to control quadratic nonlinearities in the time-evolution Eq. (35), which have fast spatial decay. Thus, the challenge here is to achieve L_t^2 temporal decay, in the presence of the exponential spatial decay.

Lemma 3: Fix $\alpha > \frac{3}{2}$. There is $C_\alpha > 0$ such that

$$\| \langle x \rangle^{-\alpha} e^{-it\mathcal{H}} P_{a.c.}(\mathcal{H}) f \|_{L_x^\infty L_t^2} \leq C_\alpha \|f\|_{L_x^2} \tag{65}$$

and

$$\left\| \langle x \rangle^{-\alpha} \int_0^t e^{-i(t-\tau)\mathcal{H}} P_{a.c.}(\mathcal{H}) F(\tau,\cdot) d\tau \right\|_{L_x^\infty L_t^2} \leq C_\alpha \| \langle x \rangle^{\alpha} F\|_{L_x^1 L_t^2}. \tag{66}$$

Proof: The proof of Lemma 3 proceeds via analysis of the contribution of the high energy part and the low energy part.

Let $\chi(x)$ be an even C^∞ function with $\chi(x) = 1$ for $|x| < 1$ and $\chi(x) = 0$ for $|x| > 2$. Fix $M > 2$, let $\chi_M(x) = \chi(x/M)$ and decompose

$$e^{-it\mathcal{H}} P_{a.c.}(\mathcal{H}) f = \chi_M e^{-it\mathcal{H}} P_{a.c.}(\mathcal{H}) f + (1-\chi_M) e^{-it\mathcal{H}} P_{a.c.}(\mathcal{H}) f.$$

In order to show (65), we need the following two estimates:

$$\|(1-\chi_M) e^{-it\mathcal{H}} P_{a.c.}(\mathcal{H}) f\|_{L_x^\infty L_t^2} \leq C \|f\|_{L_x^2}, \tag{67}$$

$$\| < x >^{-\alpha} \chi_M e^{-it\mathcal{H}} P_{a.c.}(\mathcal{H}) f\|_{L_x^\infty L_t^2} \leq C \|f\|_{L_x^2}. \tag{68}$$

Combining bounds (67) and (68), we complete the proof of estimate (65). Bounds (67) and (68) are proven below.

The proof of estimate (66) is based upon Proposition 8 and Lemma 2. By Lemma 2, we can write (with $\mathcal{L} = \mathcal{H}$)

$$\int_0^t e^{-i(t-\tau)\mathcal{H}} P_{a.c.}(\mathcal{H})F(\tau,\cdot)d\tau = \frac{1}{2}U + \frac{1}{2}\left(\int_0^\infty - \int_{-\infty}^0\right) e^{-i(t-\tau)\mathcal{H}} P_{a.c.}(\mathcal{H})F(\tau,\cdot)d\tau,$$

where U is the function defined by (64). Let us first control the last two terms. Since they are similar, we only need to control one of the terms. By the estimate (65), we have

$$\begin{aligned}\left\| \langle x\rangle^{-\alpha} e^{-it\mathcal{H}} \int_0^\infty e^{i\tau\mathcal{H}} P_{a.c.}(\mathcal{H})F(\tau,\cdot)d\tau \right\|_{L^\infty_x L^2_t} &\le C \left\| \int_0^\infty e^{i\tau\mathcal{H}} P_{a.c.}(\mathcal{H})F(\tau,\cdot)d\tau \right\|_{L^2_x} \\ &\le C\|\langle x\rangle^\alpha F\|_{L^1_x L^2_t},\end{aligned}$$

where in the last step, we have used the dual estimate to (65). In order to control the U term, we observe that the set of all functions $\{g_1(t)g_2(x) : g_1 \in L^2_t, g_2 \in L^1_x\}$ is dense in $L^1_x L^2_t$. The estimate that we need follows from

$$\left\| \langle x\rangle^{-\alpha} \int_{-\infty}^\infty e^{-it\lambda} \check{g}_1(\lambda)\left[R^+_{\mathcal{H}}(\lambda) + R^-_{\mathcal{H}}(\lambda)\right] g_2 d\lambda \right\|_{L^\infty_x L^2_t} \le C\|g_1\|_{L^2_t} \|\langle x\rangle^\alpha g_2\|_{L^1_x}.$$

The left-hand side is controlled by Minkowski's inequality and Plancherel's theorem in the time variable,

$$\|\langle x\rangle^{-\alpha}\|\check{g}_1(\lambda)\left[R^+_{\mathcal{H}}(\lambda) + R^-_{\mathcal{H}}(\lambda)\right] g_2\|_{L^2_\lambda}\|_{L^\infty_x} \le C\|\check{g}_1(\lambda)\|_{L^2_\lambda} \sup_{\lambda\in\mathbb{R}} \|R^\pm_{\mathcal{H}}(\lambda)\|_{L^1_\alpha \mapsto L^\infty_{-\alpha}} \|\langle x\rangle^\alpha g_2\|_{L^\infty_x}.$$

Using bound (61) of Proposition 8 for any $\alpha \ge 1$, we bound the last expression by $C\|g_1\|_{L^2_t}\|\langle x\rangle g_2\|_{L^1_x}$, which completes the proof of estimate (66). □

Proof of (67). Using the Cauchy formula (62) for

$$g_{x,t}(\lambda) := (1 - \chi_M(\lambda))e^{-it\mathcal{H}} P_{a.c.}(\mathcal{H})f,$$

we can see that for each fixed value of x, this function is a multiple of the Fourier transform of the function

$$g_x(\lambda) := (1 - \chi_M(\lambda))\left(\left[R^+_{\mathcal{H}}(\lambda) - R^-_{\mathcal{H}}(\lambda)\right] f\right)(x),$$

evaluated at t. Therefore, by Plancherel's theorem, we have

$$\|(1 - \chi_M(\lambda))e^{-it\mathcal{H}} P_{a.c.}(\mathcal{H})f\|_{L^2_t} = C\|g_x\|_{L^2_\lambda}.$$

It is sufficient to control

$$\sup_{x\in\mathbb{R}} \|(1 - \chi_M(\lambda))R^\pm_{\mathcal{H}}(\lambda)f(x)\|_{L^2_\lambda} \le C\|f\|_{L^2_x},$$

which we will do next. By iterating the resolvent identities,

$$R_{\mathcal{H}} = R_0 - R_{\mathcal{H}} V R_0 = R_0 - R_0 V R_{\mathcal{H}},$$

we get the representation formula

$$R_{\mathcal{H}} = R_0 - R_{\mathcal{H}} V R_0 = R_0 - R_0 V R_0 + R_0 V R_{\mathcal{H}} V R_0, \tag{69}$$

where R_0 is the resolvent of the free Dirac operator D defined by (8). For the first term, we have

$$\sup_{x\in\mathbb{R}} \|(1 - \chi_M(\lambda))R^\pm_0(\lambda)f(x)\|_{L^2_\lambda}.$$

By symmetry, it suffices to consider only positive values of λ, whence we need to control

$$\sup_{x\in\mathbb{R}} \int_M^\infty |R^\pm_0(\lambda)f(x)|^2 d\lambda.$$

We compute the resolvent $R_0(\mu)$

$$R_0(\mu) = (D-\mu)^{-1} = (1-\partial_x^2-\mu^2)^{-1}\begin{pmatrix} -i\partial_x+\mu & -1 \\ -1 & i\partial_x+\mu \end{pmatrix}$$

for $\mu \notin \sigma(D) = [-\infty, -1] \cup [1, \infty]$. By analytic continuation, we may define the resolvent for values in the spectrum of $\sigma(D)$. Since we need such a formula for values of $\mu \in (M, \infty)$, it is convenient to introduce a change of variables $\mu = \sqrt{k^2+1}$. Note that $d\mu = k(k^2 + 1)^{-1/2}dk \sim dk$ and the interval of integration becomes $(\sqrt{M^2-1}, \infty)$. Now, since the resolvent operator $(-\partial_x^2 - k^2 \pm 0)^{-1}$ is given by a convolution with the explicit kernel $\frac{e^{\pm ik|\cdot|}}{2ik}$, it is clear that $R_0^{\pm}(\mu)f$ is a linear combination of convolution operators with kernels

$$e^{\pm ik|x|}\mathrm{sgn}(x), \quad \frac{e^{\pm ik|x|}}{k}, \quad \frac{e^{\pm ik|x|}\sqrt{k^2+1}}{k}. \tag{70}$$

We shall consider the first type of operators, the second one has a stronger decay, while the third one is basically the same as the first one. By Plancherel's theorem applied to the functions $f(y)\chi_{y<x}$ and $f(y)\chi_{y>x}$, we have

$$\begin{aligned} &\int_{\sqrt{M^2-1}}^{\infty} \left| \int_{-\infty}^{\infty} e^{\pm ik|x-y|}\mathrm{sgn}(x-y)f(y)dy \right|^2 dk \\ &\leq 2\int_{\sqrt{M^2-1}}^{\infty} \left(\left| \int_{-\infty}^{x} e^{\mp iky} f(y)dy \right|^2 + \left| \int_{x}^{\infty} e^{\pm iky} f(y)dy \right|^2 \right) dk \leq C\|f\|^2_{L^2_x}. \end{aligned}$$

Similarly, we estimate the contribution of the second term R_0VR_0 in the expansion (69). Again, we have to deal with different terms of the convolution operators, but the hardest one is again $e^{ik|x|}\mathrm{sgn}(x)$. We get

$$\begin{aligned} &\int_{\sqrt{M^2-1}}^{\infty} \left| \int_{-\infty}^{\infty} e^{\pm ik|x-y|}\mathrm{sgn}(x-y)V(y)\int e^{\pm ik|y-z|}\mathrm{sgn}(y-z)f(z)dzdy \right|^2 dk \\ &\leq C\|V\|^2_{L^1_x} \sup_{y\in\mathbb{R}} \int_{\sqrt{M^2-1}}^{\infty} \left| \int e^{\pm ik|y-z|}\mathrm{sgn}(y-z)f(z)dz \right|^2 dk \leq C\|V\|^2_{L^1_x}\|f\|^2_{L^2_x}, \end{aligned}$$

where in the first inequality, we have applied Minkowski's inequality and at the second inequality, we have applied our previous estimate.

In order to estimate the last term in the expansion (69), we use bound (60) of Proposition 8 and get

$$\begin{aligned} &\int_{\sqrt{M^2-1}}^{\infty} \left| \int e^{\pm ik|x-y|}\mathrm{sgn}(x-y)V(y)R^{\pm}_{\mathcal{H}}(\sqrt{1+k^2})V(R_0^{\pm}(\sqrt{1+k^2})f)(y)dy \right|^2 dk \\ &\leq C\| <x>^{\alpha} V\|^2_{L^2_x} \int_{\sqrt{M^2-1}}^{\infty} \left\| <y>^{-\alpha} R^{\pm}_{\mathcal{H}}(\sqrt{1+k^2})VR_0^{\pm}(\sqrt{1+k^2})f \right\|^2_{L^2_y} dk \\ &\leq C\| <x>^{\alpha} V\|^2_{L^2_x} \sup_{y\in\mathbb{R}} \int_{\sqrt{M^2-1}}^{\infty} |R_0^{\pm}(\sqrt{1+k^2})f(y)|^2 dk \\ &\leq C\| <x>^{\alpha} V\|^2_{L^2_x}\|f\|^2_{L^2_x}. \end{aligned}$$

This concludes the proof of bound (67).

Proof of (68). To prove bound (68), we shall prove that

$$\sup_{x\in\mathbb{R}} <x>^{-3/2} \|\chi_M(\lambda)(R^{\pm}_{\mathcal{H}}(\lambda)f)(x)\|_{L^2_\lambda} \leq C\|f\|_{L^2_x}, \tag{71}$$

which implies bound (68) by Plancherels' theorem and Cauchy's formula (62). To prove (71) for $\lambda \leq -1$, we use representation (56) and write explicitly

$$\|\chi_M(\lambda)(R^+_{\mathcal{H}}(\lambda)f)(x)\|^2_{L^2_\lambda} = \int_{-\infty}^{-1} \chi^2_M(\lambda) \left|(R^+_{\mathcal{H}}(\lambda)f)(x)\right|^2 d\lambda = \int_{-\sqrt{M^2-1}}^{0} \frac{|\tilde{\mathbf{f}}(x,k)|^2|k|dk}{4k^2|a^+(k)|^2\sqrt{1+k^2}},$$

where

$$\tilde{\mathbf{f}}(x,k) := \mathbf{u}^+(x;k)\int_{-\infty}^{x}[\sigma_1\mathbf{u}^-(y;k)]^T f(y)dy + \mathbf{u}^-(x;k)\int_{x}^{\infty}[\sigma_1\mathbf{u}^+(y;k)]^T f(y)dy.$$

For definiteness, let us assume that $x \geq 0$. We represent

$$\int_{-\infty}^{x}[\sigma_1\mathbf{u}^-(y;k)]^T f(y)dy = \int_0^x [\sigma_1\mathbf{u}^-(y;k)]^T f(y)dy + \int_{-\infty}^0 [\alpha_-,1]f(y)e^{-iky}dy$$
$$+\int_{-\infty}^0 \left([\sigma_1\mathbf{m}^-(y;k)]^T - [\alpha_-,1]\right) f(y)e^{-iky}dy \equiv I_1 + I_2 + I_3$$

and

$$\int_x^\infty [\sigma_1\mathbf{u}^+(y;k)]^T f(y)dy = \int_x^\infty [\alpha_+,1]f(y)e^{iky}dy + \int_x^\infty \left([\sigma_1\mathbf{m}^+(y;k)]^T - [\alpha_+,1]\right) f(y)e^{iky}dy$$
$$\equiv I_4 + I_5.$$

Using Proposition 5 and Cauchy–Schwarz inequality, we have

$$|I_1| \leq \|\mathbf{u}^-(\cdot;k)\|_{L^2_x(0,x)}\|f\|_{L^2_x} \leq C\langle x\rangle^{3/2}\|f\|_{L^2_x},$$
$$|I_3| \leq \|\mathbf{m}^-(\cdot;k) - [1,\alpha_-]^T\|_{L^2_x(\mathbb{R}_-)}\|f\|_{L^2_x} \leq C\|\langle x\rangle^3 V\|_{L^\infty_x}\|f\|_{L^2_x},$$
$$|I_5| \leq \|\mathbf{m}^+(\cdot;k) - [1,\alpha_+]^T\|_{L^2_x(\mathbb{R}_+)}\|f\|_{L^2_x} \leq C\|\langle x\rangle^3 V\|_{L^\infty_x}\|f\|_{L^2_x},$$

where the estimates for I_3 and I_5 follow from the bound

$$\|\mathbf{m}^+(\cdot;k) - [1,\alpha_+]^T\|_{L^2_x(\mathbb{R}_+)} + \|\mathbf{m}^-(\cdot;k) - [1,\alpha_-]^T\|_{L^2_x(\mathbb{R}_-)} \leq C\|\langle x\rangle^3 V\|_{L^\infty_x}, \tag{72}$$

which we prove now. We need only control the first term, the other one is controlled in a similar matter.

By the formula (42), for all $x \in \mathbb{R}$ and all $k \in \mathbb{R}$ near $k = 0$, there is $C > 0$ such that

$$|G^+(x;k)| \leq C\langle x\rangle.$$

By Proposition 5, for all $x > 0$, there is $C > 0$ such that

$$|\mathbf{m}^+(x,k)| = |\mathbf{u}^+(x,k)| \leq C.$$

Thus, by the integral equation (41), we get for all $x > 0$,

$$|\mathbf{m}^+(x;k) - [1,\alpha_+]^T| \leq C\int_x^\infty \langle x-y\rangle|V(y)dy \leq C\|\langle x\rangle^3 V\|_{L^\infty_x}\int_0^\infty \langle z\rangle\frac{1}{\langle x+z\rangle^3}dz$$
$$\leq C\|\langle x\rangle^3 V\|_{L^\infty_x} < x >^{-1}.$$

This computation completes the proof of the first inequality in (72).

To estimate I_2 and I_4, we note that, for any finite $M > 1$, Plancherel's theorem gives

$$\int_{-\sqrt{M^2-1}}^0 \left(|I_2|^2 + |I_4|^2\right)dk \leq C\|f\|^2_{L^2_x}.$$

Since $ka^+(k)$ is bounded away from zero as $k \to 0$ and $|a^+(k)| \geq 1$, we obtain

$$\int_{-\sqrt{M^2-1}}^0 \frac{|\tilde{\mathbf{f}}(x,k)|^2|k|dk}{4k^2|a^+(k)|^2\sqrt{1+k^2}} \leq C(1+\langle x\rangle^3)\|f\|^2_{L^2_x},$$

which concludes the proof of bound (71) and hence of bound (68).

C. Strichartz estimates

We use the following standard definition.

Definition 1: We say that a pair (q, r) is Strichartz admissible for the nonlinear Dirac equations if

$$q \geq 2, \quad r \geq 2 \quad \text{and} \quad \frac{2}{q} + \frac{1}{r} \leq \frac{1}{2}.$$

In particular, (q, r) = (4, ∞) and (q, r) = (∞, 2) are end-point Strichartz pairs.

Lemma 4: Let (q, r) be a Strichartz admissible pair, $s \geq 0$, and $\epsilon > 0$. Then, there are constants $C_\epsilon > 0$ and $C > 0$ such that

$$\|e^{-it\mathcal{H}} P_{a.c.}(\mathcal{H}) f\|_{L^4_t L^\infty_x} \leq C_\varepsilon \|f\|_{H_x^{3/4+\varepsilon}}, \tag{73}$$

$$\|e^{-it\mathcal{H}} P_{a.c.}(\mathcal{H}) f\|_{L^\infty_t H^s_x} \leq C \|f\|_{H^s_x}, \tag{74}$$

$$\left\| \int_0^t e^{-i(t-\tau)\mathcal{H}} P_{a.c.}(\mathcal{H}) F(\tau, \cdot) d\tau \right\|_{L^\infty_t H^1_x \cap L^q_t L^r_x} \leq C \|F\|_{L^1_t H^1_x}. \tag{75}$$

Proof: Let us first comment on the estimates (74) and (75). It is easy to see by the self-adjointness of $\mathcal{H}$ that (74) is trivial for $s = 0$. We easily extend to all integer values of s by the observation that ∂_x behaves like $\mathcal{H}$ and commuting $\mathcal{H}$ with $e^{-it\mathcal{H}} P_{a.c.}(\mathcal{H})$. This is made precise in formula (80) below. We then conclude by interpolation to obtain (74) for all nonnegative values of s. Regarding (75), it follows by an easy application of Lemma 1 combined with the dual estimate of (73).

Thus, it remains to show (73). We will in fact deduce this Strichartz estimate for the perturbed Dirac operator $\mathcal{H}$ by using the corresponding result for the free Dirac operator D, in addition to the weighted estimates in Lemma 3. This is in essence the approach taken by Rodnianski and Schlag.[32] Let us first record the Strichartz estimates for the Dirac operator D,

$$\|e^{-itD} f\|_{L^q_t L^r_x} \leq C_\delta \|f\|_{H_x^{s(q,r)}}, \quad s(q,r) = \frac{1}{2} + \frac{1}{q} - \frac{1}{r}, \tag{76}$$

for all Strichartz admissible pairs (q, r), so that $q \geq 4 + \delta$. This of course looks exactly the same as the estimates that one gets from interpolating between (73) and (74). We refer the reader to recent work of Nakamura-Ozawa,[28] (more specifically Lemma 2.1 with $\theta = 1$, $\lambda = 3/2$, $n = 1$) for a reference for this result. Note that this result would not extend to the full range $q = 4$, $r = \infty$, unless we are willing to replace the L^∞ by the Besov space $B^0_{\infty,2}$ (which we are avoiding for the purpose of simplicity). In order to extend this to the useful endpoint $q = 4$, $r = \infty$, we must introduce slight loss of smoothness, so we have

$$\|e^{-itD} f\|_{L^4_t L^\infty_x} \leq C_\varepsilon \|f\|_{H_x^{3/4+\varepsilon}}. \tag{77}$$

Fix now $\epsilon > 0$ and take a test function $f = P_{a.c.}(\mathcal{H}) f \in H^{3/4+\varepsilon}$. Recall that since $\mathcal{H} = D + V(x)$, we may write

$$e^{-it\mathcal{H}} f = e^{-itD} f - i \int_0^t e^{-i(t-s)D} V e^{-is\mathcal{H}} f ds.$$

Furthermore, we may write the symmetric matrix $V(x)$ as the product of $V_1(x)$ and $V_2(x)$, where both $V_1(x)$ and $V_2(x)$ are C^1-smooth and have fast decay at spatial infinity. For instance, one may pick $V_1(x) = V(x)\langle x\rangle^{10}$ and $V_2(x) = \langle x\rangle^{-10} Id$. We have

$$\|e^{-it\mathcal{H}} P_{a.c.}(\mathcal{H}) f\|_{L^4_t L^\infty_x} \le \|e^{-itD} f\|_{L^4_t L^\infty_x} + \left\| \int_0^t e^{-i(t-s)D} V_1 V_2 e^{-is\mathcal{H}} P_{a.c.}(\mathcal{H}) f ds \right\|_{L^4_t L^\infty_x}$$

$$\le C_\varepsilon \|f\|_{H^{3/4+\varepsilon}_x} + \left\| \int_0^t e^{-i(t-s)D} V_1 V_2 e^{-is\mathcal{H}} P_{a.c.}(\mathcal{H}) f ds \right\|_{L^4_t L^\infty_x}.$$

At this stage, in order to estimate the second term, we will use Lemma 1. Let $K(t, s) = e^{-i(t-s)D} V_1$ be considered as acting between $L^2_t H^{3/4+\varepsilon}_x$ to $L^4_t L^\infty_x$. The Duhamel's term that we need to estimate is

$$M(t) = \int_0^t K(t, s) V_2 e^{-is\mathcal{H}} P_{a.c.}(\mathcal{H}) f ds = \tilde{K} V_2 e^{-it\mathcal{H}} P_{a.c.}(\mathcal{H}) f,$$

where $\tilde{K}$ is defined by (63). It follows from Lemma 1 (since $q = 4 > 2 = p$, this lemma can be applied) that

$$\|M\|_{L^4_t L^\infty_x} \le C \|K\|_{L^2_t H^{3/4+\varepsilon}_x \to L^4_t L^\infty_x} \|V_2 e^{-it\mathcal{H}} P_{a.c.}(\mathcal{H}) f\|_{L^2_t H^{3/4+\varepsilon}_x}.$$

We need estimate then the operator norm $\|K\|_{L^2_t H^{3/4+\varepsilon}_x \to L^4_t L^\infty_x}$. We have by (77)

$$\|KG\|_{L^4_t L^\infty_x} = \left\| e^{-itD} \int_{-\infty}^{\infty} e^{isD} V_1 G(s, \cdot) ds \right\|_{L^4_t L^\infty_x} \le C_\varepsilon \left\| \int_{-\infty}^{\infty} e^{isD} V_1 G(s, \cdot) ds \right\|_{H^{3/4+\varepsilon}_x}.$$

We will show that for $0 \le s \le 1$,

$$\left\| \int_{-\infty}^{\infty} e^{isD} V_1 G(s, \cdot) ds \right\|_{H^s_x} \le C_{s,V_1} \|G\|_{L^2_t H^s_x}, \tag{78}$$

and

$$\|V_2 e^{-it\mathcal{H}} P_{a.c.}(\mathcal{H}) f\|_{L^2_t H^s_x} \le C_{V_2} \|f\|_{H^s_x}, \tag{79}$$

which implies what is needed. Indeed, for $s = 3/4 + \varepsilon$, we deduce

$$\|M\|_{L^4_t L^\infty_x} \le C_{V_1} \|V_2 e^{-it\mathcal{H}} P_{a.c.}(\mathcal{H}) f\|_{L^2_t H^{3/4+\varepsilon}_x} \le C_{V_1,V_2} \|f\|_{H^{3/4+\varepsilon}_x}.$$

It thus suffices to establish (78) and (79). By interpolation, it suffices to check both only for $s = 0$ and $s = 1$. The statements for $s = 0$ in fact follow from the corresponding arguments for $s = 1$, so we concentrate on $s = 1$. For (78), (observe that $\partial_x e^{itD} = e^{itD} \partial_x$), we have

$$\left\| \int_{-\infty}^{\infty} e^{isD} V_1 G(s, \cdot) ds \right\|_{H^1_x} \le \left\| \int_{-\infty}^{\infty} e^{isD} V_1 G(s, \cdot) ds \right\|_{L^2_x} + \left\| \int_{-\infty}^{\infty} e^{isD} \partial_x [V_1 G(s, \cdot)] ds \right\|_{L^2_x}.$$

By the dual estimate to (76) (recall $s(\infty, 2) = 0$), the right-hand side of the last inequality is estimated by

$$C(\|V_1 G(s, \cdot)\|_{L^1_t L^2_x} + \|\partial_x [V_1 G(s, \cdot)]\|_{L^1_t L^2_x}) \le C(\|V_1\|_{L^\infty_x} + \|V_1'\|_{L^\infty_x}) \|G\|_{L^1_t H^1_x}.$$

This is the proof of (78).

Next, we need to deal with derivatives in the estimates for the perturbed evolution. From the formula $D = \mathcal{H} - V(x)$, we have the equivalence

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{H^1} \sim \left\| \mathcal{H} \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{L^2} + \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{L^2}, \tag{80}$$

which will be used repeatedly in the arguments to follow. Regarding (79) for $s = 1$, we use (80) to obtain

$$\|\partial_x [V_2 e^{-it\mathcal{H}} f]\|_{L^2_t L^2_x} \le \|V_2' e^{-it\mathcal{H}} f\|_{L^2_t L^2_x} + \|V_2 e^{-it\mathcal{H}} \mathcal{H} f\|_{L^2_t L^2_x} + \|V_2 e^{-it\mathcal{H}} f\|_{L^2_t L^2_x}.$$

Now, since $|V_2'(x)| + |V_2(x)| \leq \langle x \rangle^{-10}$, we estimate the last three quantities by

$$C\|\langle x\rangle^{-5}\|_{L^2_x}(\|\langle x\rangle^{-5}e^{-it\mathcal{H}}f\|_{L^\infty_x L^2_t} + \|\langle x\rangle^{-5}e^{-it\mathcal{H}}\mathcal{H}f\|_{L^\infty_x L^2_t} + \|\langle x\rangle^{-5}e^{-it\mathcal{H}}f\|_{L^\infty_x L^2_t}) \leq C(\|f\|_{L^2_x} + \|\mathcal{H}f\|_{L^2_x}) \leq C\|f\|_{H^1_x},$$

where bound (65) and Hölder's inequality are used. This computation establishes (79) and hence Lemma 4. □

We now formulate a slight variation of Lemma 4, which will be useful in the sequel.

Corollary 1: Let (q, r) be admissible in the sense of Definition 1, $q > 4 + \delta$ and $s(q,r) = \frac{1}{2} + \frac{1}{q} - \frac{1}{r}$. Then, there exists $C = C_\delta$, so that

$$\|e^{-it\mathcal{H}}P_{a.c.}(\mathcal{H})f\|_{L^q_t W^{-s(q,r),r}_x} \leq C\|f\|_{L^2_x}. \tag{81}$$

Remark 9: The only difference of Corollary 1 with Lemma 4 is that the left-hand side is measured in a negative order Sobolev space, but the right hand-side is only in L^2. Of course, since

$$< \nabla >^{-s(q,r)} e^{-it\mathcal{H}}P_{a.c.}(\mathcal{H}) \neq e^{-it\mathcal{H}}P_{a.c.}(\mathcal{H}) < \nabla >^{-s(q,r)}, \quad \widehat{< \nabla >^{\alpha}} h(\xi) := < \xi >^{\alpha} \hat{h}(\xi),$$

we cannot deduce (81) from its counterparts in Lemma 4.

Proof: We can essentially repeat the proof of Lemma 4, so we just indicate the main points. Starting from (76), we may rewrite it

$$\| < \nabla >^{-s(q,r)} e^{-itD}f\|_{L^q_t L^r_x} \leq C_\delta \|f\|_{L^2}. \tag{82}$$

Thus, following the approach of Lemma 4, we estimate

$$\begin{aligned}\| < \nabla >^{-s(q,r)} e^{-it\mathcal{H}}P_{a.c.}(\mathcal{H})f\|_{L^q_t L^r_x} &\leq \| < \nabla >^{-s(q,r)} e^{-itD}f\|_{L^q_t L^r_x} + \\ &+ \left\| < \nabla >^{-s(q,r)} \int_0^t e^{-i(t-s)D}V_1V_2e^{-is\mathcal{H}}P_{a.c.}(\mathcal{H})f ds\right\|_{L^q_t L^r_x} \\ &\leq C\|f\|_{L^2} + \| < \nabla >^{-s(q,r)} e^{-i(t-s)D}V_1\|_{L^2_t L^2_x \to L^q_t L^r_x}\|V_2e^{-it\mathcal{H}}P_{a.c.}(\mathcal{H})f\|_{L^2_{tx}},\end{aligned}$$

where

$$\| <\nabla>^{-s(q,r)} e^{-i(t-s)D}V_1\|_{L^2_t L^2_x \to L^q_t L^r_x} = \sup_{G:\|G\|_{L^2_{sx}}=1} \| < \nabla >^{-s(q,r)} e^{-itD}\int_{-\infty}^{\infty} e^{isD}V_1G(s,\cdot)ds\|_{L^q_t L^r_x}$$

By (82) and (78) (for $s = 0$), we conclude

$$\| < \nabla >^{-s(q,r)} e^{-i(t-s)D}V_1\|_{L^2_t L^2_x \to L^q_t L^r_x} \leq C,$$

whereas by (79) (for $s = 0$), we have $\|V_2e^{-it\mathcal{H}}P_{a.c.}(\mathcal{H})f\|_{L^2_t L^2_x} \leq C\|f\|_{L^2_x}$. □

D. Additional estimates

Mizumachi estimates and Strichartz estimates admit a number of useful corollaries.

Corollary 2: Let (q, r) be and admissible Strichartz pair such that $q \geq 4 + \delta$ and $I \subset \mathbb{R}^1$ is an arbitrary interval. For each $\delta > 0$, there is $C_\delta > 0$ (independent of I) such that

$$\left\|\int_I e^{it\mathcal{H}}P_{a.c.}(\mathcal{H})F(t,\cdot)dt\right\|_{L^2_x} \leq C_\delta\|F\|_{L^{q'}_t W^{s(q,r),r'}_x}, \quad s(q,r) = \frac{1}{2} + \frac{1}{q} - \frac{1}{r}, \tag{83}$$

where (q′, r′) are duals of (q, r).

Proof: The result is simply the dual statement of (81). Indeed, take a test function $F : L_t^{q'} W^{s(q,r),r'} = (L_t^q W^{s(q,r),r})^*$, so that $supp_t F(\cdot, x) \subset I$. Then (81) implies

$$\langle e^{-it\mathcal{H}} P_{a.c.}(\mathcal{H}) f, F\rangle \le C\|f\|_{L_x^2}\|F\|_{L_t^{q'} W^{s(q,r),r'}},$$

which is

$$\langle f, \int_I e^{ith} P_{a.c.}(\mathcal{H}) F(t,\cdot) dt\rangle \le C\|f\|_{L_x^2}\|F\|_{L_t^{q'} W^{s(q,r),r'}}.$$

Taking sup in the last expression with respect to $f : \|f\|_{L^2} = 1$ implies (83). □

Corollary 3: Fix $\alpha > 2$. There is $C_\alpha > 0$ such that

$$\left\| \int_0^t e^{-i(t-\tau)\mathcal{H}} P_{a.c.}(\mathcal{H}) F(\tau,\cdot) d\tau \right\|_{L_t^\infty H_x^1 \cap L_t^4 L_x^\infty} \le C_\alpha (\|\langle x\rangle^\alpha F\|_{L_x^1 L_t^2} + \|\langle x\rangle^\alpha \partial_x F\|_{L_x^1 L_t^2}). \quad (84)$$

Proof: Due to the density of $\{g_1(t) g_2(x) : g_1 \in L_t^2, g_2 \in L_x^1\}$ in $L_x^1 L_t^2$, it will suffice to show

$$\left\| \int_0^t e^{-i(t-\tau)\mathcal{H}} P_{a.c.}(\mathcal{H}) g_1(\tau) g_2 d\tau \right\|_{L_t^\infty H_x^1 \cap L_t^4 L_x^\infty} \le C\|g_1\|_{L_t^2}(\|\langle x\rangle^\alpha g_2\|_{L_x^1} + \|\langle x\rangle^\alpha \partial_x g_2\|_{L_x^1}). \quad (85)$$

By Lemma 1, we need to show that

$$\left\| \int_{-\infty}^\infty e^{-i(t-\tau)\mathcal{H}} P_{a.c.}(\mathcal{H}) g_1(\tau) g_2 d\tau \right\|_{L_t^\infty H_x^1 \cap L_t^4 L_x^\infty} \le C\|g_1\|_{L_t^2}(\|\langle x\rangle^\alpha g_2\|_{L_x^1} + \|\langle x\rangle^\alpha \partial_x g_2\|_{L_x^1}).$$

By (73) and (74), we have

$$\left\| e^{-it\mathcal{H}} \int_{-\infty}^\infty e^{i\tau\mathcal{H}} P_{a.c.}(\mathcal{H}) g_1(\tau) g_2 d\tau \right\|_{L_t^\infty H_x^1 \cap L_t^4 L_x^\infty} \le \left\| \int_{-\infty}^\infty e^{i\tau\mathcal{H}} P_{a.c.}(\mathcal{H}) g_1(\tau) g_2 d\tau \right\|_{H_x^1}.$$

Again, one may convert one derivative to $\mathcal{H} - V$ by the equivalence (80), whence we further estimate by the dual of (65),

$$\begin{aligned}
&\left\| \int_{-\infty}^\infty e^{i\tau\mathcal{H}} P_{a.c.}(\mathcal{H}) g_1(\tau) g_2 d\tau \right\|_{H_x^1} \\
&\le C \left(\left\| \int_{-\infty}^\infty e^{i\tau\mathcal{H}} P_{a.c.}(\mathcal{H}) g_1(\tau) \mathcal{H} g_2(\cdot) d\tau \right\|_{L_x^2} + \left\| V \int_{-\infty}^\infty e^{i\tau\mathcal{H}} P_{a.c.}(\mathcal{H}) g_1(\tau) g_2(\cdot) d\tau \right\|_{L_x^2} \right) \\
&\le C\|g_1\|_{L_t^2}(\|\langle x\rangle^\alpha \mathcal{H} g_2\|_{L_x^1} + \|\langle x\rangle^\alpha g_2\|_{L_x^1}) \\
&\le C\|g_1\|_{L_t^2}(\|\langle x\rangle^\alpha g_2\|_{L_x^1} + \|\langle x\rangle^\alpha \partial_x g_2\|_{L_x^1}),
\end{aligned}$$

which is the desired estimate. □

Corollary 4: Fix $\alpha > 2$. There is $C_\alpha > 0$ such that

$$\left\| \langle x\rangle^{-\alpha} \int_0^t e^{-i(t-\tau)\mathcal{H}} P_{a.c.}(\mathcal{H}) F(\tau,\cdot) d\tau \right\|_{L_x^\infty L_t^2} \le C\|F\|_{L_t^1 L_x^2}. \quad (86)$$

More generally, let (q, r) be an admissible Strichartz pair. Then,

$$\left\| \langle x\rangle^{-\alpha} \int_0^t e^{-i(t-\tau)\mathcal{H}} P_{a.c.}(\mathcal{H}) F(\tau,\cdot) d\tau \right\|_{L_x^\infty L_t^2} \le C_\alpha \|F\|_{L_t^{q'} W_x^{1,r'}}, \quad (87)$$

where (q', r') is a dual pair.

Proof: The proof of (86) is by averaging the estimate (65). More precisely, using the triangle inequality and estimate (65) yields

$$\left\| \langle x\rangle^{-\alpha} \int_0^t e^{-i(t-\tau)\mathcal{H}} P_{a.c.}(\mathcal{H}) F(\tau,\cdot) d\tau \right\|_{L^\infty_x L^2_t} \leq C \int_{-\infty}^{\infty} \|\langle x\rangle^{-\alpha} e^{-i(t-\tau)\mathcal{H}} P_{a.c.}(\mathcal{H}) F(\tau,\cdot)\|_{L^\infty_x L^2_t} d\tau$$
$$\leq C \int_{-\infty}^{\infty} \|F(\tau,\cdot)\|_{L^2_x} d\tau = C\|F\|_{L^1_t L^2_x}.$$

For the proof of (87), we use Lemma 1. It will suffice to bound the operator

$$TF(t) = \langle x_0\rangle^{-\alpha} \int_{-\infty}^{\infty} e^{-i(t-\tau)\mathcal{H}} F(\tau,\cdot) d\tau \Big|_{x=x_0} : L^{q'}_t W^{1,r'}_x \to L^2_t$$

for any fixed $x_0 \in \mathbb{R}$. We have, by (65)

$$\|TF\|_{L^2_t} \leq \left\| \langle x\rangle^{-\alpha} e^{-it\mathcal{H}} \int_{-\infty}^{\infty} e^{i\tau\mathcal{H}} F(\tau,\cdot) d\tau \right\|_{L^\infty_x L^2_t} \leq C \left\| \int_{-\infty}^{\infty} e^{i\tau\mathcal{H}} F(\tau,\cdot) d\tau \right\|_{L^2_x}.$$

By Corollary 2, we bound the last expression by

$$C\|F\|_{L^{q'}_t W^{\frac{3}{2q}+\delta,r'}_x} \leq C\|F\|_{L^{q'}_t W^{1,r'}_x},$$

as stated in (87). In the last step, we have used that if $4 \leq q \leq \infty$ and $\delta \ll 1$, then $\frac{3}{2q} + \delta < 1$. □

Corollary 5: Fix $\alpha > 2$. There is $C_\alpha > 0$ such that

$$\|\langle x\rangle^{-\alpha} \partial_x e^{-it\mathcal{H}} P_{a.c.}(\mathcal{H}) f\|_{L^\infty_x L^2_t} \leq C_\alpha \|f\|_{H^1_x}, \tag{88}$$

$$\left\| \langle x\rangle^{-\alpha} \partial_x \int_0^t e^{-i(t-\tau)\mathcal{H}} P_{a.c.}(\mathcal{H}) F(\tau,\cdot) d\tau \right\|_{L^\infty_x L^2_t} \leq C\|F\|_{L^1_t H^1_x}, \tag{89}$$

and

$$\left\| \langle x\rangle^{-\alpha} \partial_x \int_0^t e^{-i(t-\tau)\mathcal{H}} P_{a.c.}(\mathcal{H}) F(\tau,\cdot) d\tau \right\|_{L^\infty_x L^2_t} \leq C(\|\langle x\rangle^{\alpha} F\|_{L^\infty_x L^2_t} + \|\langle x\rangle^{\alpha} \partial_x F\|_{L^\infty_x L^2_t}). \tag{90}$$

Proof: The proof of the estimate (88) is based again on the equivalence (80). Since $\mathcal{H}$ commutes with all functions of $\mathcal{H}$ (by the functional calculus), we have from (80) and (65)

$$\|\langle x\rangle^{-\alpha} \partial_x e^{-it\mathcal{H}} P_{a.c.}(\mathcal{H}) f\|_{L^\infty_x L^2_t} \leq \|\langle x\rangle^{-\alpha} e^{-it\mathcal{H}} P_{a.c.}(\mathcal{H}) \mathcal{H} f\|_{L^\infty_x L^2_t} + \|\langle x\rangle^{-\alpha} V e^{-it\mathcal{H}} P_{a.c.} f\|_{L^\infty_x L^2_t}$$
$$\leq C(\|\mathcal{H} f\|_{L^2_x} + \|V\|_{L^\infty} \|f\|_{L^2_x}) \leq C_V \|f\|_{H^1_x}.$$

The proof of the estimate (89) is by averaging. Indeed, by (88),

$$\left\| \langle x\rangle^{-\alpha} \partial_x \int_0^t e^{-i(t-\tau)\mathcal{H}} P_{a.c.}(\mathcal{H}) F(\tau,\cdot) d\tau \right\|_{L^\infty_x L^2_t}$$
$$\leq C \int_{-\infty}^{\infty} \|\langle x\rangle^{-\alpha} \partial_x e^{-i(t-\tau)\mathcal{H}} P_{a.c.}(\mathcal{H}) F(\tau,\cdot)\|_{L^\infty_x L^2_t} d\tau$$
$$\leq C \int_{-\infty}^{\infty} \|F(\tau,\cdot)\|_{H^1_x} d\tau = C\|F\|_{L^1_t H^1_x}$$

For the proof of the estimate (90), we apply again the equivalence (80) and then we use (66). We have

$$\begin{aligned}
&\left\| \langle x\rangle^{-\alpha}\partial_x \int_0^t e^{-i(t-\tau)\mathcal{H}} P_{a.c.}(\mathcal{H}) F(\tau,\cdot) d\tau \right\|_{L^\infty_x L^2_t} \\
&\leq \left\| \langle x\rangle^{-\alpha} \int_0^t e^{-i(t-\tau)\mathcal{H}} P_{a.c.}(\mathcal{H}) \mathcal{H} F(\tau,\cdot) d\tau \right\|_{L^\infty_x L^2_t} \\
&\quad + \left\| \langle x\rangle^{-\alpha} V(x) \int_0^t e^{-i(t-\tau)\mathcal{H}} P_{a.c.}(\mathcal{H}) F(\tau,\cdot) d\tau \right\|_{L^\infty_x L^2_t} \\
&\leq C(\|\langle x\rangle^{\alpha}\mathcal{H}F\|_{L^1_x L^2_t} + \|\langle x\rangle^{\alpha} F\|_{L^1_x L^2_t}) \\
&\leq C(\|\langle x\rangle^{\alpha} F\|_{L^1_x L^2_t} + \|\langle x\rangle^{\alpha}\partial_x F\|_{L^1_x L^2_t}).
\end{aligned}$$

This concludes the proof of the corollary. □

IX. PROOF OF THE MAIN THEOREM

We first formulate the solution and the nonlinearity spaces. Let

$$\|\mathbf{Y}\|_{X_1} := \|\mathbf{Y}\|_{L^4_t L^\infty_x} + \|\mathbf{Y}\|_{L^\infty_t H^1_x}, \quad \|\mathbf{Y}\|_{X_2} := \|\langle x\rangle^{-\alpha}\mathbf{Y}\|_{L^\infty_x L^2_t} + \|\langle x\rangle^{-\alpha}\partial_x\mathbf{Y}\|_{L^\infty_x L^2_t},$$

and $\|\mathbf{Y}\|_X := \|\mathbf{Y}\|_{X_1} + \|\mathbf{Y}\|_{X_2}$. The nonlinearity space is defined via the norm

$$\|\mathbf{F}\|_{\mathcal{N}} := \inf_{\mathbf{F}=\mathbf{F}_1+\mathbf{F}_2} \|\mathbf{F}_1\|_{L^1_t H^1_x} + (\|\langle x\rangle^{\alpha}\mathbf{F}_2\|_{L^1_x L^2_t} + \|\langle x\rangle^{\alpha}\partial_x\mathbf{F}_2\|_{L^1_x L^2_t}).$$

Consider the Cauchy problem for the inhomogeneous linear equation, projected along the absolutely continuous spectrum of $\mathcal{H}$

$$\begin{cases} i\frac{d\mathbf{Y}}{dt} = \mathcal{H}\mathbf{Y} + P_{a.c.}(\mathcal{H})\mathbf{F}, \\ \mathbf{Y}(0) = \mathbf{Y}_0 = P_{a.c.}(\mathcal{H})\mathbf{Y}_0. \end{cases} \tag{91}$$

When one interprets correctly the results of the dispersive decay estimates (formulated and proved in Sec. VIII) in the notations above, we get that a solution to the Cauchy problem (91) satisfies

$$\|\mathbf{Y}\|_X \leq C(\|\mathbf{Y}_0\|_{H^1} + \|\mathbf{F}\|_{\mathcal{N}}). \tag{92}$$

For the proof of the main theorem, we need to show the existence of small solutions for the system of two (scalar) ordinary differential Eqs. (36) for ω and θ coupled with the partial differential Eq. (38) for $\mathbf{Y}$.

Since the right-hand side of Eq. (38) is not projected to the continuous spectrum of $\mathcal{H}$, we decompose

$$\mathbf{Y} = a\mathbf{u}_0 + \mathbf{Z}, \quad a = \langle \mathbf{u}_0, \mathbf{Y}\rangle_{L^2}, \quad \langle \mathbf{u}_0, \mathbf{Z}\rangle_{L^2} = 0, \tag{93}$$

where $\mathbf{u}_0$ is the eigenfunction of $\mathcal{H}$ for eigenvalue ω_0. Substituting (93) into (38), we obtain the system of equations

$$\begin{cases} i\dot{a} = \omega_0 a + \langle \mathbf{u}_0, e^{-i\theta}\mathbf{F}\rangle_{L^2}, \\ i\dot{\mathbf{Z}} = \mathcal{H}\mathbf{Z} + P_{a.c.}(\mathcal{H})e^{-i\theta}\mathbf{F}. \end{cases} \tag{94}$$

We now set up our problem as an iteration scheme, where we look for a fixed point in a small ball in a normed space. More precisely, this space is composed of all quadruples (ω, θ, a, $\mathbf{Z}$), equipped with the norm

$$\|(\omega,\theta,a,\mathbf{Z})\|_Z := \|\dot{\omega}\|_{L^1_t} + \|\dot{\theta}-\omega\|_{L^1_t} + \|a\|_{L^2_t\cap L^\infty_t} + \|\mathbf{Z}\|_X.$$

Note that the elements of the corresponding set are subject to the appropriate initial conditions

$$\omega(0) \in (\omega_0, \omega_0+\epsilon), \quad \theta(0) = 0, \quad a(0) = \langle \mathbf{u}_0, \mathbf{Y}(0)\rangle_{L^2}, \quad \mathbf{Z}(0) = P_{a.c.}(\mathcal{H})\mathbf{Y}(0).$$

First, observe that the matrix in front of the variables $\dot{\omega}$ and $\dot{\theta} - \omega$ in (36) has the form

$$\begin{bmatrix} \mathrm{Re}\langle \partial_\omega \mathbf{U}, \mathbf{U} - \mathbf{U}_1 \rangle_{L^2} & \mathrm{Im}\langle \partial_\omega \mathbf{U}, \mathbf{U}_1 \rangle_{L^2} \\ \mathrm{Im}\langle \partial_\omega^2 \mathbf{U}, \mathbf{U}_1 \rangle_{L^2} & \mathrm{Re}\langle \partial_\omega \mathbf{U}, \mathbf{U} + \mathbf{U}_1 \rangle_{L^2} \end{bmatrix} = \frac{1}{2}\frac{d}{d\omega}\|\mathbf{U}\|_{L^2}^2 Id + O(\|\mathbf{U}_1\|_{L^2}). \tag{95}$$

Due to the smallness of

$$\|\mathbf{U}_1\|_{L_t^\infty L_x^2} \leq \|a\|_{L_t^\infty} + \|\mathbf{Z}\|_{L_t^\infty L_x^2}$$

and the non-degeneracy condition (29), we may conclude that the matrix (95) is invertible. (Note that $\|\mathbf{Z}\|_{L_t^\infty L_x^2}$ is a part of the norm $\|\mathbf{Z}\|_X$, which is kept small in our fixed point arguments.)

Next, we show that the quantities $\|\dot{\omega}\|_{L_t^1}$ and $\|\dot{\theta} - \omega\|_{L_t^1}$ are under control. Indeed, due to the invertibility of the matrix in the modulation Eqs. (36), and the quadratic nature of Ω_1, Ω_2 (Proposition 4), we have

$$\begin{aligned} \|\dot{\omega}\|_{L_t^1} + \|\dot{\theta} - \omega\|_{L_t^1} &\leq C(\|\Omega_1\|_{L_t^1} + \|\Omega_2\|_{L_t^1}) \leq C\int_0^\infty \int_{\mathbb{R}} |\mathbf{Y}(x,t)|^2 |\mathbf{U}(x)| dx dt \\ &\leq C\| <x>^{2\alpha} \mathbf{U}\|_{L_x^1} \| <x>^{-\alpha} \mathbf{Y}\|_{L_x^\infty L_t^2}^2 \leq C\left(\|a\|_{L_t^2}^2 + \|\mathbf{Z}\|_X^2\right). \end{aligned}$$

It follows from this bound that

$$\|\omega - \omega(0)\|_{L_t^\infty} + \|\theta - \int_0^t \omega(s) ds\|_{L_t^\infty} \leq C\left(\|a\|_{L_t^2}^2 + \|\mathbf{Z}\|_X^2\right). \tag{96}$$

Since $\dot{\omega} \in L_t^1$ and $\|\omega - \omega(0)\|_{L_t^\infty}$ is small, there exists $\epsilon_0 > 0$ and $\omega_\infty := \lim_{t\to\infty}\omega(t)$ such that $\omega_\infty \in (\omega_0, \omega_0 + \epsilon_0)$ if $\omega(0) \in (\omega_0, \omega_0 + \epsilon)$. Similarly there exists $\theta_\infty \in \mathbb{R}$ such that

$$\lim_{t\to\infty}\left(\theta(t) - \int_0^t \omega(s) ds\right) = \theta_\infty.$$

Now, we control the quantity $\|a\|_{L_t^2 \cap L_t^\infty}$. It follows from the symplectic orthogonality conditions (32) that

$$\langle \mathbf{u}_0, \mathbf{U}_1\rangle_{L_x^2} = \mathrm{Re}\langle \mathbf{u}_0 - \frac{\mathbf{U}}{\|\mathbf{U}\|_{L_x^2}}, \mathbf{U}_1\rangle_{L_x^2} + i\,\mathrm{Im}\langle \mathbf{u}_0 - \frac{\partial_\omega \mathbf{U}}{\|\partial_\omega \mathbf{U}\|_{L_x^2}}, \mathbf{U}_1\rangle_{L_x^2}.$$

By Proposition 2, for any $\alpha \geq 0$, there is $C_\alpha > 0$ such that

$$\left\| <x>^\alpha (\mathbf{u}_0 - \frac{\mathbf{U}}{\|\mathbf{U}\|_{L_x^2}})\right\|_{L_x^2} + \left\| <x>^\alpha (\mathbf{u}_0 - \frac{\partial_\omega \mathbf{U}}{\|\partial_\omega \mathbf{U}\|_{L_x^2}})\right\|_{L_x^2} \leq C|\omega - \omega_0|.$$

Therefore, we obtain

$$\begin{aligned} \|a\|_{L_t^2} &= \|\langle \mathbf{u}_0, \mathbf{Y}\rangle_{L_x^2}\|_{L_t^2} = \|\langle \mathbf{u}_0, \mathbf{U}_1\rangle_{L_x^2}\|_{L_t^2} \\ &\leq C\|\omega - \omega_0\|_{L_t^\infty}\| <x>^{-\alpha} \mathbf{U}_1\|_{L_x^\infty L_t^2} \leq C(\epsilon + \|\omega - \omega(0)\|_{L_t^\infty})\| <x>^{-\alpha} \mathbf{Y}\|_{L_x^\infty L_t^2}, \end{aligned}$$

where $\epsilon + \|\omega - \omega(0)\|_{L_t^\infty}$ is small due to smallness of ϵ and the bound (96). Similarly, we obtain

$$\begin{aligned} \|a\|_{L_t^\infty} &= \|\langle \mathbf{u}_0, \mathbf{Y}\rangle_{L_x^2}\|_{L_t^\infty} = \|\langle \mathbf{u}_0, \mathbf{U}_1\rangle_{L_x^2}\|_{L_t^\infty} \\ &\leq C\|\omega - \omega_0\|_{L_t^\infty}\|\mathbf{U}_1\|_{L_t^\infty L_x^2} \leq C(\epsilon + \|\omega - \omega(0)\|_{L_t^\infty})\|\mathbf{Y}\|_{L_t^\infty L_x^2}. \end{aligned}$$

Finally, it remains to estimate the quantity $\|\mathbf{Z}\|_X$. Due to our construction, we have $\mathbf{Z} = P_{a.c.}(\mathcal{H})\mathbf{Y}$, so that we may apply the linear estimates (92). The nonlinearity $P_{a.c.}(\mathcal{H})e^{-i\theta}\mathbf{F}$ in the residual Eq. (94) has two parts. The first part satisfies

$$\begin{aligned} \|P_{a.c.}(\mathcal{H})e^{-i\theta}(i\dot{\omega}\partial_\omega \mathbf{U} + (\dot{\theta} - \omega)\mathbf{U})\|_{L_t^1 H_x^1} &\leq C(\|\dot{\omega}\|_{L_t^1} + \|\dot{\theta} - \omega\|_{L_t^1})(\|\mathbf{U}\|_{H_x^1} + \|\partial_\omega \mathbf{U}\|_{H_x^1}) \\ &\leq C\left(\|a\|_{L_t^2}^2 + \|\mathbf{Z}\|_X^2\right). \end{aligned}$$

The second (nonlinear) part

$$\mathbf{G} := P_{a.c.}(\mathcal{H})e^{-i\theta}\left(\mathbf{N}(\mathbf{U} + \mathbf{Y}e^{i\theta}) - \mathbf{N}(\mathbf{U})\right)$$

is estimated by the sum of the smallest and largest power of $\mathbf{Y}$ as follows:

$$|\mathbf{G}| \leq C|\mathbf{Y}\mathbf{U}^{2p}| + C|\mathbf{Y}^{2p+1}|,$$

where $\mathbf{Y}$ is controlled in the X-norm by

$$\|\mathbf{Y}\|_X \leq C(\|a\|_{L_t^2 \cap L_t^\infty} + \|\mathbf{Z}\|_X).$$

Note that

$$|\mathbf{G}(x,t)| + |\partial_x \mathbf{G}(x,t)| \leq C(|\mathbf{Y}| + |\partial_x \mathbf{Y}|)(|\mathbf{U}|^{2p} + |\partial_x \mathbf{U}|^{2p}) + C(|\mathbf{Y}| + |\partial_x \mathbf{Y}|)|\mathbf{Y}|^{2p}.$$

We need to control the quantity $\|\mathbf{G}\|_{\mathcal{N}}$ in terms of $\|\mathbf{Y}\|_X$. We have

$$\begin{aligned}\|\mathbf{G}\|_{\mathcal{N}} &\leq C\| < x >^{\alpha} (|\mathbf{Y}| + |\partial_x \mathbf{Y}|)(|\mathbf{U}|^{2p} + |\partial_x \mathbf{U}|^{2p})\|_{L_x^1 L_t^2} + C\|(|\mathbf{Y}| + |\partial_x \mathbf{Y}|)|\mathbf{Y}|^{2p}\|_{L_t^1 L_x^2} \\ &\leq C\left(\|\langle x\rangle^{-\alpha}\mathbf{Y}\|_{L_x^\infty L_t^2} + \|\langle x\rangle^{-\alpha}\partial_x \mathbf{Y}\|_{L_x^\infty L_t^2}\right)\| < x >^{2\alpha} (|\mathbf{U}|^{2p} + |\partial_x \mathbf{U}|^{2p})\|_{L_x^1 L_t^\infty} \\ &\quad + C\|\mathbf{Y}\|_{L_t^\infty H_x^1}\|\mathbf{Y}\|^{2p}_{L_t^{2p} L_x^\infty}.\end{aligned}$$

It is now easy to close the argument in the norm $\|\mathbf{Y}\|_X$. Indeed, by Sobolev embedding for any $\epsilon > 0$

$$\|\mathbf{Y}\|_{L_t^\infty L_x^\infty} \leq C\|\mathbf{Y}\|_{L_x^\infty H_x^{1/2+\epsilon}} \leq C\|\mathbf{Y}\|_X.$$

We also have $\|\mathbf{Y}\|_{L_t^4 L_x^\infty} \leq \|\mathbf{Y}\|_X$ (by the definition of $\|\cdot\|_X$) and hence, for $p \geq 2$, by the log convexity of the L^q norms, we have

$$\|\mathbf{Y}\|_{L_t^{2p} L_x^\infty} \leq \|\mathbf{Y}\|^{2/p}_{L_t^4 L_x^\infty}\|\mathbf{Y}\|^{1-2/p}_{L_t^\infty L_x^\infty} \leq C\|\mathbf{Y}\|_X.$$

All in all, combining the estimates for $\|\mathbf{G}\|_{\mathcal{N}}$ with the estimates for $\|\mathbf{Y}\|_{L_t^{2p} L_x^\infty}$, we obtain

$$\|\mathbf{G}\|_{\mathcal{N}} \leq C\| < x >^{2\alpha} (|\mathbf{U}|^{2p} + |\partial_x \mathbf{U}|^{2p})\|_{L_x^1 L_t^\infty}\|\mathbf{Y}\|_X + C\|\mathbf{Y}\|_X^{2p+1}.$$

By Proposition 2, there is $C > 0$ such that

$$\| < x >^{2\alpha} (|\mathbf{U}|^{2p} + |\partial_x \mathbf{U}|^{2p})\|_{L_x^1 L_t^\infty} \leq C\|\omega - \omega_0\|_{L_t^\infty} \leq C(\epsilon + \|\omega - \omega(0)\|_{L_t^\infty}).$$

Since the last term is small due to the smallness of ϵ and the bound (96), the fixed point argument is closed, and the proof of Theorem 1 is complete.

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[1] Berkolaiko, G. and Comech, A., "On spectral stability of solitary waves of nonlinear Dirac equations in 1D," Math. Model. Nat. Phenom. **7**, 13–31 (2012).
[2] Bournaveas, N., "Local well-posedness for a nonlinear Dirac equation in spaces of almost critical dimension," Discrete Contin. Dyn. Syst. **20**, 605–616 (2008).
[3] Boussaid, N., "Stable directions for small nonlinear Dirac standing waves," Commun. Math. Phys. **268**, 757–817 (2006).
[4] Boussaid, N., "On the asymptotic stability of small nonlinear Dirac standing waves in a resonant case," SIAM J. Math. Anal. **40**, 1621–1670 (2008).
[5] Boussaid, N. and Cuccagna, S., "On stability of standing waves of nonlinear Dirac equations," Commun. Partial Differ. Equ. **37**, 1001–1056 (2012).
[6] Buslaev, V. and Sulem, C., "On the stability of solitary waves for nonlinear Schrödinger equations," Annales Institut Henri Poincaré, Analyse Nonlineaire **202**, 419–475 (2003).
[7] Candy, T., "Global existence for an L^2-critical nonlinear Dirac equation in one dimension," Adv. Diff. Eqs. **7-8**, 643–666 (2011).
[8] Chugunova, M. and Pelinovsky, D., "Block-diagonalization of the symmetric first-order coupled-mode system," SIAM J. Appl. Dyn. Syst. **5**, 66–83 (2006).
[9] Comech, A., "On the meaning of the Vakhitov–Kolokolov stability criterion for the nonlinear Dirac equation," e-print arXiv:1107/1763.
[10] Comech, A., "Linear instability of nonlinear Dirac equation in 1D with higher order nonlinearity," e-print arXiv:1203.3859.

[11] Cuccagna, S., "On asymptotic stability in energy space of ground states of NLS in 1D," J. Differ. Equations **245**, 653–691 (2008).
[12] Cuccagna, S. and Mizumachi, T., "On asymptotic stability in energy space of ground states for nonlinear Schrödinger equations," Commun. Math. Phys. **284**, 51–77 (2008).
[13] Cuccagna, S. and Tarulli, M., "On asymptotic stability of standing waves of discrete Schrödinger equation in $\mathbb{Z}$," SIAM J. Math. Anal. **41**, 861–885 (2009).
[14] Gang, Z. and Sigal, I. M., "Asymptotic stability of nonlinear Schrödinger equations with potential," Rev. Math. Phys. **17**, 1143–1207 (2005).
[15] Goldberg, M. and Schlag, W., "Dispersive estimates for Schrödinger operators in dimensions one and three," Commun. Math. Phys. **251**, 157–178 (2004).
[16] Goodman, R. H., Weinstein, M. I., and Holmes, P. J., "Nonlinear propagation of light in one-dimensional periodic structures," J. Nonlinear Sci. **11**, 123–168 (2001).
[17] Goodman, R. H., Slusher, R. E., Weinstein, M. I., and Klaus, M., "Trapping light with grating defects," Contemp. Math. **379**, 83–92 (2005).
[18] Kevrekidis, P., Pelinovsky, D., and Stefanov, A., "Asymptotic stability of small solitons in the discrete nonlinear Schrödinger equation in one dimension," SIAM J. Math. Anal. **41**, 2010–2030 (2009).
[19] Kirr, E. and Zarnescu, A., "Asymptotic stability of ground states in 2D nonlinear Schrödinger equation including subcritical cases," J. Differ. Equations **247**, 710–735 (2009).
[20] Kirr, E. and Mizrak, Ö., "Asymptotic stability of ground states in 3D nonlinear Schrödinger equation including subcritical cases," J. Funct. Anal. **257**, 3691–3747 (2009).
[21] Komech, A. and Komech, A., "Global attraction to solitary waves for a nonlinear Dirac equation with mean field interaction," SIAM J. Math. Anal. **42**, 2944–2964 (2010).
[22] Kopylova, E. A., "Weighted energy decay for 1D Dirac equation," Dyn. Partial Differ. Equ. **8**, 113–125 (2011).
[23] Machihara, S., Nakamura, M., Nakanishi, K., and Ozawa, T., "Endpoint Strichartz estimates and global solutions for the nonlinear Dirac equation," J. Funct. Anal. **219**, 1–20 (2005).
[24] Machihara, S., Nakanishi, K., and Ozawa, T., "Small global solutions and the nonrelativistic limit for the nonlinear Dirac equation," Rev. Math. Iberoam. **19**, 179–194 (2003).
[25] Machihara, S., Nakanishi, K., and Tsugawa, K., "Well-posedness for nonlinear Dirac equations in one dimension," Kyoto J. Math. **50**, 403–451 (2010).
[26] Mizumachi, T., "Asymptotic stability of small solitary waves to 1D nonlinear Schrödinger equations with potential," J. Math. Kyoto Univ. **48**, 471–497 (2008).
[27] Mizumachi, T., "Asymptotic stability of small solitons for 2D nonlinear Schrödinger equations with potential," J. Math. Kyoto Univ. **47**, 599–620 (2007).
[28] Nakamura, M. and Ozawa, T., "The Cauchy problem for nonlinear Klein-Gordon equations in the Sobolev spaces," Publ. Res. Inst. Math. Sci. **37**, 255–293 (2001).
[29] Pelinovsky, D. E., "Survey on global existence in the nonlinear Dirac equations in one dimension," in *Harmonic Analysis and Nonlinear Partial Differential Equations*, edited by T. Ozawa and M. Sugimoto (RIMS Kokyuroku Bessatsu, 2011), Vol. B26, pp. 37–50.
[30] Pillet, C. A. and Wayne, C. E., "Invariant manifolds for a class of dispersive, Hamiltonian, partial differential equations," J. Differ. Equations **141**, 310–326 (1997).
[31] Porter, M. A., Chugunova, M., and Pelinovsky, D. E., "Feshbach resonance management of Bose–Einstein condensates in optical lattices," Phys. Rev. E **74**, 036610-8 (2006).
[32] Rodnianski, I. and Schlag, W., "Time decay for solutions of Schrödinger equations with rough and time-dependent potentials," Invent. Math. **155**, 451–513 (2004).
[33] Saito, Y. and Umeda, T., "The zero modes and zero resonances of massless Dirac operators," Hokkaido Math. J. **37**, 363–388 (2008).
[34] Selberg, S. and Tesfahun, A., "Low regularity well-posedness for some nonlinear Dirac equations in one space dimension," Diff. Integral Eq. **23**, 265–278 (2010).
[35] Shatah, J. and Strauss, W., "Instability of nonlinear bound states," Commun. Math. Phys. **100**, 173–190 (1985).
[36] Smith, H. and Sogge, C., "Global Strichartz estimates for nontrapping perturbations of the Laplacian," Commun. Partial Differ. Equ. **25**, 2171–2183 (2000).
[37] Soffer, A. and Weinstein, M. I., "Selection of the ground state for nonlinear Schrödinger equations," Rev. Math. Phys. **16**, 977–1071 (2004).
[38] Weder, R., "The $W_{k,p}$-continuity of the Schrödinger wave operators on the line," Commun. Math. Phys. **208**, 507–520 (1999).
[39] Weder, R., "L^p–$L^{p'}$ estimates for the Schrödinger equation on the line and inverse scattering for the nonlinear Schrödinger equation with a potential," J. Funct. Anal. **170**, 37–68 (1999).
[40] Weinstein, M. I., "Lyapunov stability of ground states of nonlinear dispersive evolution equations," Commun. Pure Appl. Math. **39**, 51–67 (1986).
[41] Yau, H. T. and Tsai, T. P., "Asymptotic dynamics of nonlinear Schrödinger equations: resonance dominated and radiation dominated solutions," Commun. Pure Appl. Math. **55**, 1–64 (2002).