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ERGODIC BOUNDARY/POINT CONTROL OF STOCHASTIC SEMILINEAR SYSTEMS*

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Abstract. A controlled Markov process in a Hilbert space and an ergodic cost functional are given for a control problem that is solved where the process is a solution of a parameter-dependent semilinear stochastic differential equation and the control can occur only on the boundary or at discrete points in the domain. The linear term of the semilinear differential equation is the infinitesimal generator of an analytic semigroup. The noise for the stochastic differential equation can be distributed, boundary and point. Some ergodic properties of the controlled Markov process are shown to be uniform in the control and the parameter. The existence of an optimal control is verified to solve the ergodic control problem. The optimal cost is shown to depend continuously on the system parameter.

Key words. ergodic control, stochastic semilinear equations, Markov processes in Hilbert spaces, invariant measures, boundary control

AMS subject classifications. 93E20, 93C20, 60H15

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1. Introduction.

An ergodic control problem for a stochastic process in a Hilbert space H is formulated and solved where the process is a solution of a parameter-dependent semilinear stochastic differential equation in H. The problem in the general setting is motivated by ergodic control problems for processes governed by stochastic partial differential equations (SPDEs) with control and noise occurring in the boundary conditions or at discrete points in the domain.

For example, consider the stochastic parabolic equation

$$\frac{\partial v}{\partial t}(t,\xi) = Lv(t,\xi) + F(\alpha, v(t,\xi)) + n(t,\xi) \tag{1.1}$$

for $(t,\xi) \in \mathbb{R}_+ \times (0,1)$ with initial and boundary conditions

$$v(0,\xi) = v_0(\xi), \tag{1.2}$$

$$\frac{\partial v}{\partial \xi}(t,0) = h_1(\alpha, v(t,\cdot), u(v(t,\cdot))) + \eta_1(t), \tag{1.3}$$

$$\frac{\partial v}{\partial \xi}(t,1) = h_2(\alpha, v(t,\cdot), u(v(t,\cdot))) + \eta_2(t), \tag{1.4}$$

where n denotes a space-dependent Gaussian noise that is white in time, η_1 and η_2 are one-dimensional standard Wiener processes, and these three processes are mutually independent. Furthermore,

$$Lv = a(\xi)\frac{\partial^2}{\partial \xi^2}v + b(\xi)\frac{\partial}{\partial \xi}v + c(\xi)v$$

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is a second-order uniformly elliptic operator, where $a, b, c \in C^\infty([0,1])$, $a > 0$, $c < 0$, $F : \mathcal{A} \times \mathbb{R} \to \mathbb{R}$, $h_i : \mathcal{A} \times H \times \mathcal{K} \to \mathbb{R}$, $i = 1, 2$, where $H = L^2(0,1)$, $\mathcal{A} \subset \mathbb{R}^{d_1}$, and $\mathcal{K} \subset \mathbb{R}^k$ are compact. The control problem is to minimize the ergodic cost functional

$$J(x, u, \alpha) = \limsup_{T\to\infty} \mathbb{E}\frac{1}{T}\int_0^T c(v(t), u(v(t)))dt$$

over the set of Markov controls $\mathcal{U} = \{u : H \to \mathcal{K} \mid u \text{ is Borel measurable}\}$, where $c : H \times \mathcal{K} \to \mathbb{R}$. The $\alpha \in \mathcal{A}$ in (1.1)–(1.4) represents a parameter.

The equations (1.1), (1.3), and (1.4) are only formal because the noise terms n, η_1, and η_2 are not well-defined stochastic processes (random fields). A standard approach for the rigorous treatment of the problem is to rewrite (1.1) as a controlled stochastic differential equation in the Hilbert space H, and to define the noise terms using Wiener processes with infinite-dimensional state spaces and the solution to the equation as a mild solution, using the semigroup theory (cf. [10, 27]).

In the present paper, this general framework is used. The controlled Markov process is defined by a Hilbert space-valued stochastic differential equation ((2.1) below). The linear term of the equation is the infinitesimal generator of an analytic semigroup. The general setting allows us to cover, as special cases, stochastic boundary/point control problems like the above example (see Examples 7.1 and 7.2). The noise for the stochastic differential equation can be distributed, boundary and point. The parameter-dependence occurs in the distributed and the boundary or the point drift terms. The control occurs only in the boundary or point drift term. The fact that the control is not distributed would seem to allow for more physically meaningful models. The noise is allowed to occur in both distributed and discrete forms to ensure more flexibility of the models. Since the H-valued Markov process depends on the control and the parameter, it is shown that some ergodic properties of the process are uniform in these quantities. For the solution of an ergodic control problem the existence of an optimal control is verified. It is shown that the optimal cost depends continuously on the system parameter.

Continuity of the optimal cost on the parameter is an important step in solving the adaptive control problem when the parameter is unknown. This verification is important to show the optimality of an adaptive control defined by means of a family of strongly consistent estimators of the unknown parameter α. In the case when the control and noise are distributed, the existence of an optimal control has been proven in [13], while the continuity of the optimal cost is new for this case.

The continuity of the optimal cost follows readily from the continuous dependence of the invariant measures for the controlled Markov process on the parameter α, uniform in the controls, in the norm of total variation of measures. This result can be of some independent interest and it may be interesting to note that even in some very simple cases the situation for Hilbert space–valued processes is significantly different from the finite-dimensional case. For example, consider the linear stochastic heat equation (without control)

$$\frac{\partial w}{\partial t}(t,\xi) = \alpha\frac{\partial^2 w}{\partial \xi^2}(t,\xi) + n(t,\xi), \quad (t,\xi) \in \mathbb{R}_+ \times (0,1),$$

with initial and boundary conditions $w(0,\xi) = w_0(\xi)$, $w(t,0) = w(t,1) = 0$, where $\alpha \in [1/2, 1]$ and n is a space-time white noise. It is well known (see, e.g., [28]) that for each value of α, the probability laws (in the state space $H = L_2(0,1)$)

of the solutions converge in the norm of total variation to the Gaussian invariant measure $\mu(\alpha) = N(0, Q(\alpha))$, where $Q(\alpha) = \alpha^{-1}Q,\ Q = \int_0^\infty S(2t)dt$, and $S(\cdot)$ is the semigroup generated by the operator of the second derivative on $(0,1)$, with zero Dirichlet boundary conditions.

However, by the dichotomy result for Gaussian measures it is easy to see that the invariant measures $\mu(\alpha)$ are singular for different values of $\alpha \in [1/2, 1]$, so there is no continuous dependence on α in the norm of total variation (see Remark 4.11 for some comparison between the finite- and infinite-dimensional state spaces).

A brief outline of the paper is given now. In section 2 the control problem is formulated and the basic assumptions are made and explained. The controlled process is the unique, weak, mild solution of the stochastic differential equation and induces a Markov process in H. Some estimates are made of this process, and an approximation of the transition probability function for the Markov process solution of the stochastic differential equation by transition functions of the solutions of the stochastic differential equation with bounded drifts is given, where the approximation is uniform in the control and the parameter. In section 3 the existence and uniqueness of the mild (backward) Kolmogorov equation for the controlled Markov process are verified. An estimate of the derivative of the mild solution of the Kolmogorov equation is given. In section 4 the results of section 3 are used to verify a uniform version of the strong Feller property and the strong (i.e., variation norm) continuity of the transition measures with respect to the parameter that is uniform in the control. The invariant measures of the controlled Markov process are shown to be continuous with respect to the parameter in the variation norm topology that is uniform in the control. In section 5 some tightness properties are verified. Initially it is shown that a "tightness" on balls implies tightness. A Lyapunov-type condition is shown to imply the tightness for the family of invariant measures depending on the parameter and the control. Section 6 contains the main results of the paper: the existence of an optimal control for a fixed parameter and the continuous dependence of the optimal cost on the parameter are verified using the results proven in sections 2, 3, and 4. In section 7 two examples are given that satisfy the assumptions that are made for the control problem: in Example 7.1 the control problem (1.1)–(1.4) is treated, and Example 7.2 contains a similar control problem, where the control and noise occur at given discrete points in the domain rather than on the boundary.

A brief description and a comparison of some previous results on these topics are given now. Similar results for the existence and the uniqueness of the weak, mild solutions to stochastic differential equations with only distributed noise and control are given in [10, 17, 18]. Some results for the existence and the uniqueness of mild solutions for semilinear stochastic equations with boundary or point noise are given in [11, 22, 27]. In [27] an existence result for the invariant measures is given. The methods to obtain the mild solution of the Kolmogorov equation are similar to the methods used in [6, 8, 9] for a fixed stochastic equation without parameter dependency. The approach to verifying the existence of an optimal control uses a standard procedure (see, e.g., [25, 32] for a finite-dimensional process and [13] for an infinite-dimensional process). There seems to be a fairly limited amount of work on infinite-time horizon control problems in infinite-dimensional spaces. Some work is devoted to discounted cost functionals. For this latter problem the existence of an optimal stationary control is shown in [4], and the stationary Hamilton–Jacobi–Bellman equation is investigated in [7, 20]. It seems that the ergodic control problem is only considered in [13], where a distributed control is used.

2. Preliminaries. Consider a controlled, infinite-dimensional process $(X(t),\, t \geq 0)$ that satisfies the stochastic differential equation

(2.1)
$$\mathrm{d}X(t) + AX(t)\mathrm{d}t = (f(\alpha, X(t)) + Bh(\alpha, X(t), u(X(t))))\mathrm{d}t + B\mathrm{d}V(t) + Q^{1/2}\mathrm{d}W(t),$$
$$X(0) = x,$$

where $X(0), X(t) \in H$, H is a separable, infinite-dimensional Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $| \cdot |$, $\alpha \in \mathcal{A} \subset \mathbb{R}^d$ is a parameter and $\mathcal{A}$ is compact, U is a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_U$ and norm $| \cdot |_U$, $\mathcal{K}$ is a compact product of intervals in $\mathbb{R}^k$, $-A : \mathrm{Dom}(-A) \to H$ is the infinitesimal generator of an analytic semigroup $(S(t),\, t \geq 0)$ such that $A^{-1} \in \mathcal{L}(H)$, which is often denoted $A > 0$,

$$f : \mathcal{A} \times H \to H,$$
$$h : \mathcal{A} \times H \times \mathcal{K} \to U$$

are Borel measurable functions, $B \in \mathcal{L}(U, D_A^{\varepsilon-1})$, the family of bounded linear operators from U to $D_A^{\varepsilon-1}$, where $\varepsilon \in (0,1]$ is given and D_A^δ for $\delta \geq 0$ is the domain of the fractional power A^δ with the topology induced by the graph norm $|x|_{D_A^\delta} = |A^\delta x|$, while for $\delta < 0$ it is a completion of H in the norm $| \cdot |_{D_A^\delta}$. It is assumed that $Q \in \mathcal{L}(H)$ is positive and self-adjoint and $(V(t),\, t \geq 0)$ and $(W(t),\, t \geq 0)$ are independent, standard cylindrical Wiener processes in the spaces U and H, respectively, that are defined on a filtered, complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. The family of controls, $\mathcal{U}$, is

$$\mathcal{U} = \{u : H \to \mathcal{K} \mid u \text{ is Borel measurable}\}.$$

The control problem is to minimize, over $u \in \mathcal{U}$, the ergodic cost functional

$$J(x, u, \alpha) = \limsup_{T\to\infty} \mathbb{E}\frac{1}{T}\int_0^T c(X(s), u(X(s)))\mathrm{d}s, \tag{2.2}$$

where $c : H \times \mathcal{K} \to \mathbb{R}_+$ is bounded and Borel measurable.

The following assumptions, (A1)–(A7), are used selectively in this paper.

(A1) There exist a $\gamma \in (0, 1/2]$ and a $\Delta \in (0, 1/2]$ such that $B \in \mathcal{L}_2(U, D_A^{\gamma-1/2})$ and $Q^{1/2} \in \mathcal{L}_2(H, D_A^{\Delta-1/2})$, where $\mathcal{L}_2(\cdot, \cdot)$ is the family of Hilbert–Schmidt operators.

(A2) For each $\alpha \in \mathcal{A}$ the function $h(\alpha, \cdot, \cdot) : H \times \mathcal{K} \to U$ is continuous and $f(\alpha, \cdot) : H \to H$ is Lipschitz continuous on the bounded subsets of H, and there are constants k, k_f, k_h, and $\tilde{k}(\alpha)$ such that $|f(\alpha, x)| \leq k + k_f|x|$, $|h(\alpha, x, u)|_U \leq k + k_h|x|$, and $|h(\alpha, x, u)|_U \leq \tilde{k}(\alpha)$ for all $x \in H$, $u \in \mathcal{K}$, and $\alpha \in \mathcal{A}$.

By (A1) and the analyticity of $-A$, the composition $S(r)B$ is well defined for $r > 0$, and furthermore, $S(r)B \in \mathcal{L}_2(U, H)$, $S(r)Q^{1/2} \in \mathcal{L}_2(H)$, and

$$\int_0^t |S(r)B|^2_{\mathcal{L}_2(U,H)} dr + \int_0^t |S(r)Q^{1/2}|^2_{\mathcal{L}_2(H)} dr < \infty$$

for $t > 0$. Therefore, the family of operators $(Q_t, t \geq 0)$

$$Q_t = \int_0^t S(r)BB^*S^*(r)\mathrm{d}r + \int_0^t S(r)QS^*(r)\mathrm{d}r \tag{2.3}$$

is well defined and $Q_t \in \mathcal{L}_2(H)$ for each $t \geq 0$.

(A3) The following are satisfied:

$$\mathcal{R}(\tilde{S}(t)) \subset \mathcal{R}(Q_t^{1/2}), \qquad |Q_t^{-1/2} S(t) A^{1-\varepsilon}|_{\mathcal{L}(H)} \leq \frac{c}{t^\beta}$$

for $t \in (0,T]$ for some $T > 0$, $c > 0$, and $\beta < 1$, where $(\tilde{S}(t),\, t \geq 0)$ is the restriction of $(S(t),\, t \geq 0)$ to the space $D_A^{1-\varepsilon}$ and $\mathcal{R}(\cdot)$ is the range.

(A4) There is a continuous, increasing function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ with $\omega(0) = 0$ such that

$$|f(\alpha, x) - f(\beta, x)| + |h(\alpha, x, u) - h(\beta, x, u)|_U \leq \omega(|\alpha - \beta|)(1 + |x|)$$

for all $\alpha, \beta \in \mathcal{A}$, $x \in H$, and $u \in \mathcal{K}$.

(A5) For each $u \in \mathcal{U}$ and $\alpha \in \mathcal{A}$ there is an invariant measure $\mu(\alpha, u)$ for the process $(X(t),\, t \geq 0)$ that satisfies (2.1), and the family of measures $(\mu(\alpha, u),\ \alpha \in \mathcal{A},\, u \in \mathcal{U})$ is tight.

(A6) The function $c : H \times \mathcal{K} \to \mathbb{R}_+$ given in (2.2) is bounded and Borel measurable and $c(x, \cdot) : \mathcal{K} \to \mathbb{R}_+$ is continuous for each $x \in H$.

(A7) The set $h(\alpha, x, \mathcal{K}) \times c(x, \mathcal{K}) \subset U \times \mathbb{R}_+$ is convex for each $\alpha \in \mathcal{A}$ and $x \in H$.

Some comments on the above assumptions (A1)–(A7) are given now. Assumption (A1) is a standard condition guaranteeing that the solution of the linear version of the equation (2.1) (i.e., with $f = 0$ and $h = 0$) is an H-valued stochastic process (otherwise it is only a cylindrical process; see, e.g., [12]). Note that (A1) implies that the above-defined operators Q_t are trace class operators on H. They are covariance operators of the (Gaussian) probability distribution of the solution to the linear equation. (Some discussion on the verification of (A1) is contained, for example, in [12, 27]; (A1) is also verified in Examples 7.1, 7.2 of the present paper.)

The assumption (A2) is used to verify that there exists a unique, weak, mild solution to the equation (2.1) (below in this section).

The assumption (A3) is used in section 3 to prove some suitable smoothing properties of the mild backward Kolmogorov equation corresponding to the stochastic equation (2.1), which is needed to show the ergodicity of the solutions to (2.1) and some continuity properties of the transition probability kernels. The assumption is also rather standard in the context of the perturbation methods; for instance, for $\epsilon = 1$ the results of section 3 have been proven in [9, 10]. A class of examples in which (A3) can be easily verified is given also in section 3 (Proposition 3.4).

The assumption (A4) is a continuous dependence of the coefficients of the equation (2.1) on the parameter α. It is used for the verification of the results that are related to the continuous dependence of the optimal cost on the parameter.

The assumption (A5) is a kind of stability assumption that is usually needed in ergodic control problems. In section 5 (A5) is verified in terms of some more explicit conditions on the coefficients of equation (2.1) (Lyapunov-type conditions).

The assumptions (A7) and (A8) are typical conditions that are used in the ergodic control theory ((A7) is sometimes called the Roxin-type condition) and they are used to establish the existence of an optimal control for the given control problem.

Consider the following two stochastic differential equations:

$$\begin{aligned} \mathrm{d}Z(t) + AZ(t)\mathrm{d}t &= B\mathrm{d}V(t) + Q^{1/2}\mathrm{d}W(t), \\ Z(0) &= x, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \mathrm{d}X(t) + AX(t)\mathrm{d}t &= f(\alpha, X(t))\mathrm{d}t + B\mathrm{d}V(t) + Q^{1/2}\mathrm{d}W(t), \\ X(0) &= x. \end{aligned} \tag{2.5}$$

Under the assumptions (A1) and (A2) it is easy to verify that each of the equations (2.4) and (2.5) has one and only one mild solution on the probability space $(\Omega, \mathcal{F}, P)$, that is, the solutions to the integral equations

$$Z(t) = S(t)x + \int_0^t S(t-r)B\mathrm{d}V(r) + \int_0^t S(t-r)Q^{1/2}\mathrm{d}W(r), \qquad t \geq 0, \tag{2.6}$$

and

$$\begin{aligned} X(t) = S(t)x &+ \int_0^t S(t-r)f(\alpha, X(t))\mathrm{d}t + \int_0^t S(t-r)B\mathrm{d}V(r) \\ &+ \int_0^t S(t-r)Q^{1/2}\mathrm{d}W(r), \quad t \geq 0. \end{aligned} \tag{2.7}$$

These solutions are D_A^δ-valued processes that belong to $C([0,T], L^p(\Omega, H)) \cap C((0,T], L^p(\Omega, D_A^\delta))$ for any $p \geq 1$, $T > 0$, and $\delta \in [0, \min(\varepsilon, \Delta, \gamma))$ (cf. [27]). Furthermore, the processes $(X(t),\, t \geq 0)$ and $(Z(t),\, t \geq 0)$ have D_A^δ-continuous versions (cf. [11, 30]), and in H they induce two Markov processes in the usual way.

Let $P_\alpha : \mathbb{R}_+ \times H \times \mathcal{B}(H) \to [0,1]$ be the transition probability function for $(X(t),\, t \geq 0)$ in (2.7), that is,

$$P_\alpha(t, x, \Gamma) = \mathbb{P}_x(X(t) \in \Gamma), \tag{2.8}$$

and let $(T(t),\, t \geq 0)$ be the Markov transition semigroup for $(Z(t),\, t \geq 0)$ in (2.6), that is,

$$T_t\varphi(x) = \mathbb{E}_x\varphi(Z(t)), \tag{2.9}$$

where $x \in H$ stands for the initial value of $X(\cdot)$, $t \geq 0$, and $\varphi \in \mathcal{M}(H)$, the bounded, Borel measurable functions on H. It is clear that

$$T_t 1_\Gamma(x) = N(S(t)x, Q_t)(\Gamma),$$

where $t \geq 0$, $\Gamma \in \mathcal{B}(H)$, $x \in H$, and Q_t is given by (2.3) so it is self-adjoint, nonnegative, and nuclear, and $N(S_t x, Q_t)$ is the Gaussian measure on H with mean $S_t x$ and covariance Q_t.

Let $\xi_T^{\alpha,u}$ be the random variable as follows:

$$\xi_T^{\alpha,u} = \int_0^T \langle h(\alpha, X(t), u(X(t))), \mathrm{d}V(t)\rangle_U - \frac{1}{2}\int_0^T |h(\alpha, X(t), u(X(t)))|_U^2 \mathrm{d}t \tag{2.10}$$

for $\alpha \in \mathcal{A}$, $u \in \mathcal{U}$, and $T > 0$, where $(X(t),\, t \in [0,T])$ is the solution of (2.7). A weak solution of (2.1) is constructed following the standard procedure of an absolutely continuous change of probability measure (cf. [10, 15, 17, 23]). For control problems, the method was initiated in [1, 14]. Note that $\mathbb{E}\exp(\xi_T^{\alpha,u}) = 1$ by (A2). There is a probability measure $\mathbb{P}_x^{\alpha,u}$ on $\mathcal{F}$ such that the restriction of $\mathbb{P}_x^{\alpha,u}$ to $\mathcal{F}_T$ is given by

$$\mathbb{P}_x^{\alpha,u}(\mathrm{d}\omega) = \exp(\xi_T^{\alpha,u})\mathbb{P}(d\omega), \tag{2.11}$$

the process $(V^*(t),\, t \geq 0)$ given by

$$V^*(t) = V(t) - \int_0^t h(\alpha, X(s), u(X(s)))\mathrm{d}s$$

is a cylindrical Wiener process on U, and using $\mathbb{P}_x^{\alpha,u}$ and the solution of (2.7), it follows that

(2.12)
$$X(t) = S(t)x + \int_0^t S(t-r)f(\alpha, X(r))\mathrm{d}t + \int_0^t S(t-r)Bh(\alpha, X(r), u(X(r)))\mathrm{d}r + \int_0^t S(t-r)B\mathrm{d}V^*(r) + \int_0^t S(t-r)Q^{1/2}\mathrm{d}W(r).$$

So there is a weak solution to (2.1) which is weakly unique and induces a Markov process on H whose Markov transition semigroup is denoted as

$$P_t^{\alpha,u}\varphi(x) = \mathbb{E}_x^{\alpha,u}\varphi(X(t)) \tag{2.13}$$

for $t \geq 0$ and $\varphi \in \mathcal{M}(H)$, where $\mathbb{E}_x^{\alpha,u}$ is the expectation using the probability measure $\mathbb{P}_x^{\alpha,u}$ and

$$P^{\alpha,u}(t, x, \Gamma) = P_t^{\alpha,u}1_\Gamma(x) \tag{2.14}$$

for $t \geq 0$, $\Gamma \in \mathcal{B}(H)$, and $x \in H$ is the corresponding transition probability function.

In the remainder of the section, three technical lemmas are given that are useful in what follows. Initially, Proposition 2.2 of [27] is given as the following lemma.

LEMMA 2.1. *If* (A1) *and* (A2) *are satisfied and* $\delta \in [0, \min(\gamma, \Delta, \varepsilon))$, $p > \max((\Delta-\delta)^{-1}, (\gamma-\delta)^{-1}, (\varepsilon-\delta)^{-1})$, *and* $x \in H$, *then for each* $T > 0$ *there is a constant* $C = \hat{C}(T, p, \delta)$ *such that*

$$\mathbb{E}|A^\delta X(T)|^p \leq C(1 + |x|^p), \tag{2.15}$$

where $(X(t),\, t \geq 0)$ *is the solution of* (2.7). *If* $\delta = 0$ *then* C *does not depend on* T *from finite intervals.*

The following two lemmas reduce some of the subsequent proofs to the case where f and h are uniformly bounded.

LEMMA 2.2. *If* (A1) *and* (A2) *are satisfied, then for each* $T > 0$ *and* $R > 0$

$$\lim_{N\to\infty} \inf \mathbb{P}_x^{\alpha,u}\left(\sup_{t\in[0,T]} |X(t)| \leq N\right) = 1, \tag{2.16}$$

where the infimum is taken over $\alpha \in \mathcal{A}$, $u \in \mathcal{U}$, *and* $x \in H$ *with* $|x| \leq R$.

Proof. Recall the equation (2.12) for $(X(t),\, t \geq 0)$. Let $\Omega_{x,N}^{\alpha,u} \subset \Omega$ be given by

$$\Omega_{x,N}^{\alpha,u} = \left\{ \sup_{t\in[0,T]} \left(\left| \int_0^t S(t-r)B\mathrm{d}V^*(r) \right| + \left| \int_0^t S(t-r)Q^{1/2}\mathrm{d}W(r) \right| \right) \leq N \right\}. \tag{2.17}$$

By a maximal inequality (Lemma 2.2 of [30])

$$\mathbb{E}_x^{\alpha,u} \sup_{t\in[0,T]} \left(\left| \int_0^t S(t-r)B\mathrm{d}V^*(r) \right|^2 + \left| \int_0^t S(t-r)Q^{1/2}\mathrm{d}W(r) \right|^2 \right)$$
$$\leq \int_0^T |S(r)B|^2_{\mathcal{L}_2(U,H)}\mathrm{d}r + \int_0^T |S(r)Q^{1/2}|^2_{\mathcal{L}_2(H)}\mathrm{d}r \leq M \tag{2.18}$$

for some M that does not depend on α, x, and u by (A1) and the analyticity of

$(S(t),\, t \geq 0)$. Thus

(2.19)
$$\mathbb{P}_x^{\alpha,u}\left(\sup_{t\in[0,T]}\left(\left|\int_0^t S(t-r)B\mathrm{d}V^*(r)\right| + \left|\int_0^t S(t-r)Q^{1/2}\mathrm{d}W(r)\right|\right) \geq N\right) \leq \frac{2M}{N^2}.$$

By (A1), (A2), and (2.17) on the set $\Omega_{x,N}^{\alpha,u}$ it follows that

$$|X(t)| \leq c_1|x| + c_2 + c_3\int_0^t \frac{|X(s)|}{(t-s)^{1-\varepsilon}}\mathrm{d}s + N$$

for $t \in [0,T]$, where the constants c_1, c_2, and c_3 depend only on T. The generalized Gronwall lemma (Theorem 7.1 of [21]) implies that

$$|X(t)| \leq c_4|x| + c_5 + N \tag{2.20}$$

for $t \in [0,T]$ on $\Omega_{x,N}^{\alpha,u}$, where c_4 and c_5 only depend on T. Since $\mathbb{P}_x^{\alpha,u}(\Omega_{x,N}^{\alpha,u}) \geq 1 - 2M/N^2$ the equality (2.16) follows. □

By (A2) it follows that there is a sequence $(f_m, h_m,\, m \in \mathbb{N})$ such that for each $m \in \mathbb{N}$

$$(f_m(\alpha,x), h_m(\alpha,x,u)) = (f(\alpha,x), h(\alpha,x,u)) \tag{2.21}$$

for $\alpha \in \mathcal{A}$, $u \in \mathcal{K}$, $x \in H$ with $|x| \leq m$ and

$$|f_m| + |h_m|_U \leq M_m, \tag{2.22}$$

where M_m is a constant depending only on m, $f_m(\alpha,\cdot)$ is Lipschitz continuous, and $h_m(\alpha,\cdot,\cdot)$ is continuous for each $m \in \mathbb{N}$ and

$$|f_m(\alpha,x) - f_m(\beta,x)| + |h_m(\alpha,x,u) - h_m(\beta,x,u)|_U \leq \tilde{\omega}_m(|\alpha-\beta|) \tag{2.23}$$

for $\alpha, \beta \in \mathcal{A}$, $x \in H$, and $u \in \mathcal{K}$, where $\tilde{\omega}_m$ has the same properties as ω in (A4) for each $m \in \mathbb{N}$. It is clear that if f and h are replaced by f_m and h_m, respectively, in (2.1), then the same procedure gives a unique weak solution inducing a Markov process on H.

LEMMA 2.3. *Let $P_m^{\alpha,u} : \mathbb{R}_+ \times H \times \mathcal{B}(H)$ be the transition probability function for the Markov process that is the solution of* (2.1) *with f and h replaced by f_m and h_m, respectively, which are described above. If* (A1) *and* (A2) *are satisfied then*

$$\lim_{m\to\infty} \|P_m^{\alpha,u}(t,x,\cdot) - P^{\alpha,u}(t,x,\cdot)\| = 0 \tag{2.24}$$

uniformly in $\alpha \in \mathcal{A}$, $u \in \mathcal{U}$, and x from bounded sets in H where $\|\cdot\|$ is the variation norm.

The proof of this lemma follows easily from Lemma 2.2 and the local uniqueness theorem for stochastic integrals. Let $(X_m(t),\, t \geq 0)$ be the solution of (2.5) with f replaced by f_m. It easily follows that

$$\Omega_N = \left\{\sup_{\substack{t\in[0,T]\\ \alpha\in\mathcal{A},\ |x|\leq R}} |X_m(t)| \leq N\right\} = \left\{\sup_{\substack{t\in[0,T]\\ \alpha\in\mathcal{A},\ |x|\leq R}} |X(t)| \leq N\right\}$$

for $m \geq N > R > 0$ because the trajectories of $(X_m(t),\, t \geq 0)$ and $(X(t),\, t \geq 0)$ coincide for $t \in [0, T]$ on Ω_N. Lemma 2.2 implies that $\mathbb{P}_x^{\alpha,u}(\Omega_N) \to 1$ as $N \to \infty$ uniformly in $\alpha \in \mathcal{A}$, $u \in \mathcal{U}$, $x \in H$ with $|x| < R$. Defining $\mathbb{P}_{x,m}^{\alpha,u}$ in the same way as $\mathbb{P}_x^{\alpha,u}$ by replacing h by h_m, it follows that the probabilities $\mathbb{P}_{x,m}^{\alpha,u}$ and $\mathbb{P}_x^{\alpha,u}$ restricted to Ω_N coincide for $m \geq N$. Given $\varepsilon > 0$, choose $N \geq 0$ such that $\mathbb{P}_{x,m}^{\alpha,u}(\Omega_N) \geq 1 - \varepsilon$ for $\alpha \in \mathcal{A}$, $u \in \mathcal{U}$, $|x| \leq R$, and $m \geq N$. It follows that

$$|P_m^{\alpha,u}(T, x, \Gamma) - P^{\alpha,u}(T, x, \Gamma)| < \varepsilon$$

for each $\Gamma \in \mathcal{B}(H)$, $|x| \leq R$, $\alpha \in \mathcal{A}$, $u \in \mathcal{U}$, and $m \geq N$.

3. The mild Kolmogorov equation. In this section a version of the mild Kolmogorov equation is considered. The existence and the uniqueness of the solution of this equation is verified, as is an estimate on the derivatives which is important to establish a uniform version of the strong Feller property. Many of the results of this section are verified similarly to the verifications that are used for a single Hilbert space (cf. [6, 8, 9]), so some details are omitted. Recall the definitions of $(T_t,\, t \geq 0)$ in (2.9) and $(P_t^{\alpha,u},\, t \geq 0)$ in (2.13). Let D_x be the Fréchet derivative on H and let $\mathcal{U}_c = \{u \in \mathcal{U} : u \in C(H, \mathcal{K})\}$. Let

$$\mathcal{H} = \Bigg\{ \psi \mid \psi : (0,T] \to C_b^1(H),\ D_x\psi : (0,T] \to C_b(H, D_{A^*}^{1-\varepsilon}),$$
$$|\psi|_{\mathcal{H}} := \sup_{\substack{t \in (0,T) \\ x \in H}} (t^\beta |\psi(t,x)| + t^\beta |D_x\psi(t,x)|_{D_{A^*}^{1-\varepsilon}}) < \infty \Bigg\},$$

where $\beta \in (0, 1)$ is given in (A3), which is assumed to be satisfied throughout this section.

PROPOSITION 3.1. *Let $\varphi \in C_b(H)$ and $n(t, x) = T_t\varphi(x)$ for $t \geq 0$ and $x \in H$. Then $n \in \mathcal{H}$ and*

$$|D_x n(t,x)|_{D_{A^*}^{1-\varepsilon}} \leq \frac{c}{t^\beta} \sup |\varphi| \tag{3.1}$$

for $t \in (0, T]$ and $x \in H$, where the constant c does not depend on φ.

Proof. By the absolute continuity of measures it follows that

$$\begin{aligned} n(t,x) &= \int \varphi(y) N(S(t)x, Q_t)(\mathrm{d}y) \\ &= \int \varphi(y) \exp\left[\langle \Gamma_t x, Q_t^{-(1/2)} y\rangle - \frac{1}{2}|\Gamma_t x|^2 \right] N(0, Q_t)(\mathrm{d}y), \end{aligned} \tag{3.2}$$

where $\Gamma_t = Q_t^{-(1/2)} S_t \in \mathcal{L}(H)$ by (A3). Applying D_x to (3.2) it follows (cf. [10]) that

$$D_x n(t,x)h = \int \langle \Gamma_t h, Q_t^{-(1/2)} y\rangle \varphi(S(t)x + y) N(0, Q_t)(\mathrm{d}y)$$

for $h \in H$, so that (A3) yields

$$\begin{aligned} \sup_{|h|\leq 1} |D_x n(t,x)(A^{1-\varepsilon}h)| &\leq c_1 \sup_{|h|\leq 1} \int |\langle \Gamma_t A^{1-\varepsilon} h, Q_t^{-(1/2)} y\rangle| N(0,Q_t)(\mathrm{d}y) \sup|\varphi| \\ &\leq c_2 \sup|\varphi|\, |\Gamma_t A^{1-\varepsilon}|_{\mathcal{L}(H)} \leq \frac{c_3}{t^\beta} \sup|\varphi| \end{aligned} \tag{3.3}$$

for $t \in (0,T]$, where $c_i, i = 1,2,3$, are constants independent of φ and t. The inequality (3.1) follows because $(D_A^{\varepsilon-1})' = D_{A^*}^{1-\varepsilon}$. □

Consider the mild Kolmogorov equation of the form

(3.4)
$$v(t,x) = T_t\varphi(x) + \int_0^t T_{t-s}(\langle D_x v(s,\cdot), f(\alpha,\cdot)\rangle + \langle D_x v(s,\cdot), Bh(\alpha,\cdot,u(\cdot))\rangle)(x)\mathrm{d}s$$

for $t \geq 0$, where $\varphi \in C_b(H)$, $\langle\cdot,\cdot\rangle$ is used for the duality between the corresponding domains of the fractional powers of A and A^* as well as the inner product on H, and for notational convenience, the dependence of v on α and u is suppressed. The solution $v(t,x)$ of (3.4) is shown to be $P_t^{\alpha,u}\varphi(x)$.

Proposition 3.2. *If* (A1)–(A3) *are satisfied,* $u \in \mathcal{U}_c$, $\varphi \in C_b(H)$, *and* $|f|$ *and* $|h|_U$ *are bounded independent of* $\alpha \in \mathcal{A}$ *and* $u \in \mathcal{U}_c$, *then the equation* (3.4) *has one and only one solution* $v(t,x) = P_t^{\alpha,u}\varphi(x)$ *in* $\mathcal{H}$ *that satisfies*

$$|D_x v(t,x)|_{D_{A^*}^{1-\varepsilon}} \leq \frac{\tilde{c}}{t^\beta}\sup|\varphi| \tag{3.5}$$

for $t \in (0,T]$, *where the constant* $\tilde{c}$ *does not depend on* $\varphi, u \in \mathcal{U}_c$ *or* $\alpha \in \mathcal{A}$.

Proof. To verify the existence and uniqueness of the solution of (3.4), the Banach fixed point theorem is used for the Banach space $(\mathcal{H}, |\cdot|_{\mathcal{H}})$. Define the mapping $\Phi : \mathcal{H} \to \mathcal{H}$ as follows:

$$\Phi v(t,x) = T_t\varphi(x) + \int_0^t T_{t-s}\psi(D_x v(s,\cdot))(x)\mathrm{d}s \tag{3.6}$$

for $t \in (0,T]$, where

$$\psi(D_x v(s,\cdot)) = \langle f(\alpha,\cdot), D_x v(s,\cdot)\rangle + \langle D_x v(s,\cdot), Bh(\alpha,\cdot,u(\cdot))\rangle \tag{3.7}$$

and the dependence of ψ on $\alpha \in \mathcal{A}$ and $u \in \mathcal{U}_c$ is suppressed. For $v_1, v_2 \in \mathcal{H}$ it follows that

(3.8)
$$\begin{aligned}|\Phi(v_1) - \Phi(v_2)|_{\mathcal{H}} = \sup_{\substack{t\in(0,T)\\ x\in H}} & [t^\beta|\int_0^t T_{t-s}(\psi(D_x v_1(s,\cdot)) - \psi(D_x v_2(s,\cdot)))(x)\mathrm{d}s| \\ & + t^\beta \int_0^t |(A^*)^{1-\varepsilon} D_x T_{t-s}(\psi(D_x v_1(s,\cdot)) - \psi(D_x v_2(s,\cdot)))(x)|\mathrm{d}s].\end{aligned}$$

Note that

$$\begin{aligned}&|\psi(D_x v_1(s,\cdot)) - \psi(D_x v_2(s,\cdot))| \leq c_1|D_x v_1(s,\cdot) - D_x v_2(s,\cdot)| \\ &\quad + |(A^*)^{1-\varepsilon}(D_x v_1(s,\cdot) - D_x v_2(s,\cdot))| \leq c_2|D_x v_1(s,\cdot) - D_x v_2(s,\cdot)|_{D_{A^*}^{1-\varepsilon}}\end{aligned}$$

for suitable constants c_1 and c_2. Applying this inequality to (3.8) yields

$$\begin{aligned}|\Phi(v_1) - \Phi(v_2)|_{\mathcal{H}} \leq\ & c_2\int_0^t t^\beta \sup_{s,x}|D_x v_1(s,x) - D_x v_2(s,x)|_{D_{A^*}^{1-\varepsilon}}\mathrm{d}s \\ & + c_2 c\int_0^t \frac{t^\beta}{(t-s)^\beta}\sup_{s,x}|D_x v_1(s,x) - D_x v_2(s,x)|_{D_{A^*}^{1-\varepsilon}}\mathrm{d}s \\ \leq\ & c_3|v_1 - v_2|_{\mathcal{H}}\left(\int_0^t \frac{t^\beta}{s^\beta}\mathrm{d}s + t^\beta\int_0^t \frac{\mathrm{d}s}{(t-s)^\beta s^\beta}\right).\end{aligned}$$

Thus Φ is a contraction for $t > 0$ sufficiently small. The fact that $\Phi(\mathcal{H}) \subset \mathcal{H}$ is verified similarly. Therefore, for $T > 0$ sufficiently small, there is a unique solution of (3.4). For arbitrary $T > 0$ the interval $[0, T]$ is subdivided into a finite number of small intervals.

To verify (3.5) it follows by (3.1) that

(3.9)
$$\begin{aligned}\sup_x |D_x v(t,x)|_{D_{A^*}^{1-\varepsilon}} &\leq \sup_x |D_x T_t \varphi(x)|_{D_{A^*}^{1-\varepsilon}} + \sup_x \int_0^t |D_x T_{t-s} \psi(D_x v(s,\cdot))(x)|_{D_{A^*}^{1-\varepsilon}} \mathrm{d}s \\ &\leq c t^{-\beta} \sup |\varphi| + c_4 \int_0^t \sup_x |D_x v(s,x)|_{D_{A^*}^{1-\varepsilon}} \frac{\mathrm{d}s}{(t-s)^\beta}\end{aligned}$$

for $t \in (0, T)$ and c_4 is a constant. Applying the generalized Gronwall lemma (Theorem 7.1 of [21]) to the function $\lambda(t) = \sup_x |D_x v(t,x)|_{D_{A^*}^{1-\varepsilon}}$, it follows that

$$\sup_{x \in H} |D_x v(t,x)|_{D_{A^*}^{1-\varepsilon}} \leq \frac{c_5}{t^\beta} \sup |\varphi| \tag{3.10}$$

for $t \in (0, T]$, where the constant c_5 does not depend on $t \in (0, T]$, $\varphi \in C_b(H)$, $\alpha \in \mathcal{A}$, and $u \in \mathcal{U}_c$, though it may depend on the bounds for $|f|$ and $|h|_U$. While it remains to show that $v(t,x)$ is $P_t^{\alpha,u}\varphi(x)$, this verification is identical to the proof of (Theorem 4 of [6]) and is omitted. Only note that (A1) implies that $B \in \mathcal{L}_2(U, D_A^{-1})$ and $Q^{1/2} \in \mathcal{L}_2(H, D_A^{-1})$, which is used here. ∎

Proposition 3.2 is essential in the following result, which gives a strong Feller property that is uniform for $u \in \mathcal{U}_c$. It is improved in the next section.

LEMMA 3.3. *Let $t > 0$ and $y \in H$ be fixed. If* (A1)–(A2) *are satisfied then there is a function $\tilde{\omega} : \mathbb{R}_+ \to \mathbb{R}_+$ that is increasing and continuous with $\tilde{\omega}(0) = 0$ such that*

$$\|P^{\alpha,u}(t,x,\cdot) - P^{\alpha,u}(t,y,\cdot)\| \leq \tilde{\omega}(|x-y|) \tag{3.11}$$

for all $\alpha \in \mathcal{A}$, $u \in \mathcal{U}_c$, and $x \in H$, where $\|\cdot\|$ is the variation norm.

Proof. If $|f|$ and $|h|_U$ are bounded then by (3.5) it follows that

$$\|P^{\alpha,u}(t,x,\cdot) - P^{\alpha,u}(t,y,\cdot)\| = \sup_{\varphi \in C_b,\ |\varphi| \leq 1} |P_t^{\alpha,u}\varphi(x) - P_t^{\alpha,u}\varphi(y)| \leq \frac{\tilde{c}}{t^\beta}|x-y|_{D_A^{\varepsilon-1}}$$

for $x \in H$, which easily implies (3.11) ($\tilde{c}$ may depend on the bounds for $|f|$ and $|h|_U$). If $|f|$ and $|h|_U$ are not bounded, then use Lemma 2.3 to approximate $P^{\alpha,u}(t,x,\cdot)$ and $P^{\alpha,u}(t,y,\cdot)$ by $P_k^{\alpha,u}(t,x,\cdot)$ and $P_k^{\alpha,u}(t,y,\cdot)$, respectively, uniformly with respect to $\alpha \in \mathcal{A}$, $u \in \mathcal{U}_c$, and x from bounded sets in H. ∎

This section is concluded with a simple result which can be useful in some cases to verify (A3).

PROPOSITION 3.4. *If $\varepsilon > 1/2$ and $Q^{1/2} = A^{-\eta}\Gamma$, where $\eta \in [0, \varepsilon - 1/2)$ and $\Gamma, \Gamma^{-1} \in \mathcal{L}(H)$, then* (A3) *is satisfied.*

Proof. Let

$${}^1Q_t = \int_0^t S(r)BB^*S^*(r)\mathrm{d}r$$

and

$${}^2Q_t = \int_0^t S(r)QS^*(r)\mathrm{d}r.$$

It is clear that $Q_t = {}^1Q_t + {}^2Q_t$ and ${}^1Q$ and ${}^2Q$ are nonnegative and self-adjoint. It suffices to verify (A3) with Q_t replaced by ${}^2Q_t$. By the minimum energy principle (cf. Remark B9 of [10]) it follows that

$$|{}^2Q_t^{-(1/2)}S(t)y| \leq \left(\int_0^t |u(s)|^2 \mathrm{d}s\right)^{1/2}, \tag{3.12}$$

where $u \in L^2([0,t],H)$ is arbitrary such that the solution $(z(s),\ s\in[0,t])$ of

$$\dot{z} + Az = Q^{1/2}u, \qquad z(0) = y \tag{3.13}$$

satisfies $z(t) = 0$. The existence of such a function is a necessary condition for ${}^2Q_t^{-(1/2)}S(t) \in \mathcal{L}(H)$. For $x \in D_A^{1-\varepsilon}$ define $\tilde{u}(r) = -(1/t)Q^{-1/2}S(r)A^{1-\varepsilon}x$. Clearly $\tilde{u} \in L^2([0,t],H)$ and u gives $z(t)=0$ if $y = A^{1-\varepsilon}x$. Thus

$$|{}^2Q_t^{-(1/2)}S(t)A^{1-\varepsilon}x| \leq \left(\int_0^t |\tilde{u}(r)|^2\mathrm{d}r\right)^{1/2} \leq |x|\frac{\tilde{c}}{t^{\eta+(3/2)-\varepsilon}}$$

for a constant $\tilde{c}$, so (A3) is satisfied with $\beta = \eta + (3/2) - \varepsilon < 1$, ∎

4. The continuous dependence of some measures on a parameter. In this section, the verifications are made for the continuous dependence of $P^{\alpha,u}(t,x,\cdot)$ on the parameter α and the uniform strong Feller property, which yield (under the tightness condition (A5)) the uniform continuity of the invariant measures with respect to the parameter $\alpha \in \mathcal{A}$. This last result is used in section 6 to prove continuity of the optimal cost for the control problem (2.1), (2.2).

LEMMA 4.1. *If* (A1) *and* (A2) *are satisfied then for each* $t>0$, $\alpha\in\mathcal{A}$, *and* $x \in H$

$$\lim_{u_n\to u} \|P^{\alpha,u_n}(t,x,\cdot) - P^{\alpha,u}(t,x,\cdot)\| = 0, \tag{4.1}$$

where $u_n \in \mathcal{U}$ *for all* $n\in\mathbb{N}$ *and* $u_n \to u$ *pointwise.*

Proof. By Lemma 2.3 it can be assumed that $|f|$ and $|h|_U$ are bounded uniformly in $\alpha\in\mathcal{A}$ and $u\in\mathcal{U}$. It easily follows that

$$\begin{aligned}\|P^{\alpha,u_n}(t,x,\cdot) - P^{\alpha,u}(t,x,\cdot)\| &= \sup_{\varphi\in C_b,\|\varphi\|\leq 1} |P_t^{\alpha,u_n}\varphi(x) - P_t^{\alpha,u}\varphi(x)| \\ &\leq |\mathbb{E}\varphi(X(t))\exp(\xi_t^{\alpha,u_n}) - \mathbb{E}\varphi(X(t))\exp(\xi_t^{\alpha,u})| \leq \mathbb{E}|\exp(\xi_t^{\alpha,u_n}) - \exp(\xi_t^{\alpha,u})|,\end{aligned} \tag{4.2}$$

where $(X(t),\ t\geq 0)$ satisfies (2.5). Since $\mathbb{E}\exp(2\xi_t^{\alpha,u_n}) \leq \exp(t\sup|h|)$ the sequence $(\exp(\xi_t^{\alpha,u_n}),\ n\in\mathbb{N})$ is uniformly integrable, so for every $\varepsilon>0$ there is an $R>0$ such that

$$\mathbb{E}|\exp(\xi_t^{\alpha,u_n}) - \exp(\xi_t^{\alpha,u})| \leq \mathrm{e}^R\mathbb{E}|\xi_t^{\alpha,u_n} - \xi_t^{\alpha,u}| + \varepsilon.$$

From (A2), the boundedness of $|h|_U$, and the dominated convergence theorem, (4.2) is verified. ∎

The following result is a uniform version of the strong Feller property.

LEMMA 4.2. *If* (A1)–(A3) *are satisfied, then for each* $t>0$, $y\in H$, *there is a continuous, increasing function* $\tilde{\omega}:\mathbb{R}_+\to\mathbb{R}_+$ *with* $\tilde{\omega}(0)=0$ *such that*

$$\|P^{\alpha,u}(t,x,\cdot) - P^{\alpha,u}(t,y,\cdot)\| \leq \tilde{\omega}(|x-y|)$$

for all $\alpha\in\mathcal{A}$, $u\in\mathcal{U}$, *and* $x\in H$.

Proof. Take the function $\tilde{\omega}$ from Lemma 3.3 (for the fixed $t > 0$ and $y \in H$) and let $\mathcal{U}' \in \mathcal{U}$ be the set of controls satisfying

$$\sup_{u \in \mathcal{U}'} \|P^{\alpha,u}(t,x,\cdot) - P^{\alpha,u}(t,y,\cdot)\| \leq \tilde{\omega}(|x-y|)$$

for all $x \in H$ and $\alpha \in \mathcal{A}$. By Lemma 3.3, $\mathcal{U}_c \subset \mathcal{U}'$ and by Lemma 4.1, $\mathcal{U}'$ is closed with respect to pointwise convergence. Since the families of Baire and Borel functions $H \to \mathcal{K}$ coincide (cf. [24, Theorem 2.31.IX]) it follows that $\mathcal{U}' = \mathcal{U}$.

PROPOSITION 4.3. *Denote by $\mathcal{P}(H)$ the space of probability measures on the Borel subsets of H endowed with the metric of total variation of measures. If* (A1)–(A4) *are satisfied, then for each $T > 0$ the function*

$$\eta : \mathcal{A} \to \mathcal{P}(H)$$

given by

$$\eta(\alpha) := P^{\alpha,u}(T,x,\cdot) \tag{4.3}$$

is continuous uniformly in $u \in \mathcal{U}$ and $x \in K$ for each compact set $K \subset H$.

Proof. By Lemma 2.3 it can be assumed that $|f|$ and $|h|_U$ are bounded and

$$|f(\alpha,x) - f(\beta,x)| + |h(\alpha,x,u) - h(\beta,x,u)|_U \leq \omega(|\alpha-\beta|) \tag{4.4}$$

for $x \in H$ and $\alpha, \beta \in \mathcal{A}$. Initially, the uniform continuity of (4.3) is verified for $u \in \mathcal{U}_c$. For $v_{\alpha,u}(t,x) = P_t^{\alpha,u}\varphi(x)$ for $x \in H$ and $\varphi \in C_b(H)$ it follows by Proposition 3.2 that

$$v_{\alpha,u}(t,x) = T_t\varphi(x) + \int_0^t T_{t-s}(\psi_{\alpha,u}(D_x v_{\alpha,u}(s,\cdot)))(x)\mathrm{d}s \tag{4.5}$$

for $t \in [0,T]$, where

$$\psi_{\alpha,u}(D_x v_{\alpha,u}(s,\cdot)) = \langle D_x v_{\alpha,u}(s,\cdot), f(\cdot)\rangle + \langle D_x v_{\alpha,u}(s,\cdot), Bh(\alpha,\cdot,u(\cdot))\rangle \tag{4.6}$$

and

$$|D_x v_{\alpha,u}(t,\cdot)|_{D_{A^*}^{1-\varepsilon}} \leq ct^{-\beta}\sup|\varphi| \tag{4.7}$$

for $t \in (0,T]$, where $c > 0$ does not depend on $t \in (0,T]$, $\alpha \in \mathcal{A}$, $u \in \mathcal{U}_c$, and $\varphi \in C_b(H)$. By Proposition 3.1 it follows that

$$\begin{aligned} &\sup_x |v_{\alpha,u}(t,x) - v_{\alpha_0,u}(t,x)| + \sup_x |D_x v_{\alpha,u}(t,x) - D_x v_{\alpha_0,u}(t,x)|_{D_{A^*}^{1-\varepsilon}} \\ &\quad \leq \sup_x \int_0^t |T_{t-s}(\psi_{\alpha,u}(D_x v_{\alpha,u}(s,\cdot)) - \psi_{\alpha_0,u}(D_x v_{\alpha_0,u}(s,\cdot)))(x)|\mathrm{d}s \\ &\quad + \sup_x \int_0^t |D_x T_{t-s}(\psi_{\alpha,u}(D_x v_{\alpha,u}(s,\cdot)) - \psi_{\alpha_0,u}(D_x v_{\alpha_0,u}(s,\cdot)))(x)|_{D_{A^*}^{1-\varepsilon}}\mathrm{d}s. \end{aligned} \tag{4.8}$$

By (4.4) and (4.7) it follows that

$$\begin{aligned} &\sup_x |\psi_{\alpha,u}(D_x v_{\alpha,u}(s,x)) - \psi_{\alpha_0,u}(D_x v_{\alpha_0,u}(s,x))| \\ &\leq c_1 \sup_x |D_x v_{\alpha,u}(s,x) - D_x v_{\alpha_0,u}(s,x)|_{D_{A^*}^{1-\varepsilon}} + c_2 s^{-\beta}\omega(|\alpha-\alpha_0|) \end{aligned} \tag{4.9}$$

for some constants c_1, c_2 depending only on the bounds for $|f|$, $|h|_U$, and $|B|_{\mathcal{L}(U,D_A^{\epsilon-1})}$. Let $\lambda_{\alpha,u}(\cdot)$ be the left-hand side of (4.8). By (4.8) and (4.9) it follows that

$$\lambda_{\alpha,u}(t) \leq \int_0^t \frac{k_1}{(t-s)^\beta}\lambda_{\alpha,u}(s)\ ds + \omega(|\alpha-\alpha_0|)\int_0^t \frac{k_2}{(t-s)^\beta s^\beta}\mathrm{d}s \tag{4.10}$$

for $t \in (0,T]$ for some constants k_1 and k_2. By the generalized Gronwall lemma (Theorem 7.1 of [21]) it follows that

$$\lambda_{\alpha,u}(t) \leq k_3\omega(|\alpha-\alpha_0|)$$

for $t\in(0,T]$, so

$$\|P^{\alpha,u}(T,x,\cdot)-P^{\alpha_0,u}(T,x,\cdot)\| = \sup_{|\varphi|\leq 1}|P_T^{\alpha,u}\varphi(x)-P_T^{\alpha_0,u}\varphi(x)| \leq k_4\omega(|\alpha-\alpha_0|)$$

for some constants k_3 and k_4 that are independent of $x\in H$ and $u\in\mathcal{U}_c$. The last estimate is extended to $u\in\mathcal{U}$ using Lemma 4.1 by the same argument as in the proof of Lemma 4.2.

The following result is a version of the Itô formula that is applicable to functions of the solution of (2.1).

PROPOSITION 4.4. *If* (A1) *and* (A2) *are satisfied,* $g\in C^2(H)$, $D_xg(x)\in D_{A^*}^{1-\varepsilon}$ *for* $x\in H$, $D_xg : H\to D_{A^*}^{1-\varepsilon}$ *is continuous,* $D_{xx}g : H\to \mathcal{L}(D_A^{-\delta},D_{A^*}^{\delta})\cap\mathcal{L}(H,D_{A^*}^{1-\varepsilon})$ *is continuous for* $\delta=\max((1/2)-\Delta,(1/2)-\gamma)$, *where* D_{xx} *is the second Fréchet derivative,* $\langle A\,\cdot, D_xg(\cdot)\rangle : D_A^1\to\mathbb{R}$ *can be extended to a continuous function* $\Phi : H\to\mathbb{R}$, *and*

$$|\Phi(x)|+|g(x)|+|D_xg(x)|_{D_{A^*}^{1-\varepsilon}}+|D_{xx}g(x)|_{\mathcal{L}(D_A^{-\delta},D_{A^*}^{\delta})} \leq \hat{k}(1+|x|^p) \tag{4.11}$$

for $x\in H$ *and some* $\hat{k}$ *and* $p>0$, *then the following equality is satisfied:*

$$\begin{aligned}\mathbb{E}_x^{\alpha,u}g(X(t))-g(x) = {} & \mathbb{E}_x^{\alpha,u}\int_0^t(-\Phi(X(s))+\langle f(\alpha,X(s)),D_xg(X(s))\rangle\\ & + \langle h(\alpha,X(s),u(X(s))),B^*D_xg(X(s))\rangle_U\\ & + \frac{1}{2}\mathrm{tr}[(A^*)^{1/2-\gamma}D_{xx}g(X(s))BB^*(A^*)^{\gamma-1/2}]\\ & + \frac{1}{2}\mathrm{tr}[(A^*)^{1/2-\Delta}D_{xx}g(X(s))Q(A^*)^{\Delta-1/2}])\mathrm{d}s\end{aligned} \tag{4.12}$$

for $t\geq 0$, $\alpha\in\mathcal{A}$, $u\in\mathcal{U}$, *and* $x\in H$.

Proof. Fix $\alpha\in\mathcal{A}$ and $u\in\mathcal{U}$. Choose a sequence of functions $(h_n,\ n\in\mathbb{N})$ such that $h_n : H\to U$ is globally Lipschitz continuous and $h_n\to h$ pointwise as $n\to\infty$ and h is bounded by the constant $\tilde{k}(\alpha)$ from (A2). It follows as in Proposition 3.4 of [12] and Proposition 1.5 of [27] that

$$\begin{aligned}\mathbb{E}g(X(t))\exp(\xi_{n,t})-g(x) = {} & \mathbb{E}\int_0^t(-\Phi(X(s))\\ & + \langle f(\alpha,X(s)),D_xg(X(s))\rangle + \langle h_n(X(s)),B^*D_xg(X(s))\rangle_U\\ & + \frac{1}{2}\mathrm{tr}[(A^*)^{1/2-\gamma}D_{xx}g(X(s))BB^*(A^*)^{\gamma-1/2}]\\ & + \frac{1}{2}\mathrm{tr}[(A^*)^{1/2-\Delta}D_{xx}g(X(s))Q(A^*)^{\Delta-1/2}])\exp(\xi_{n,s})\mathrm{d}s\end{aligned} \tag{4.13}$$

for $t \geq 0$, where

$$\xi_{n,s} = \int_0^t \langle h_n(\alpha, X(r), u(X(r))), \mathrm{d}V(r)\rangle_U - \frac{1}{2}\int_0^t |h_n(\alpha, X(r), u(X(r)))|_U^2 \mathrm{d}r.$$

The remainder of the proof investigates the particular terms above as $n \to \infty$. As in the proof of Lemma 4.1, it can be shown that

$$\lim_{n\to\infty} \int_0^t \mathbb{E}|\exp(\xi_{n,s}) - \exp(\xi_s^{\alpha,u})|\mathrm{d}s = 0,$$

so there is a subsequence such that $\exp(\xi_{n_k,t}) \to \exp(\xi_t^{\alpha,u})$ on $[0,T]\times\Omega$, $\lambda\times\mathbb{P}$ almost everywhere, where λ is the Lebesgue measure on $\mathbb{R}$. It remains to verify the uniform integrability of the terms on the right-hand side of (4.13). It can be assumed that p in (4.11) is sufficiently large. Thus, for example,

$$\begin{aligned}|\langle f(\alpha, X(s)), D_x g(X(s))\rangle \exp(\xi_{n,s})|^2 &\leq c_1(k + k_f|X(s)|)^2\hat{k}^2(1 + |X(s)|^p)^2 \exp(2\xi_{n,s})\\ &\leq c_2 + c_3|X(s)|^{2p+2}\exp(2\xi_{n,s}).\end{aligned}$$

By Lemma 2.1 it follows that

$$\sup_{\substack{n\in\mathbb{N}\\ s\in[0,t]}} \mathbb{E}|X(s)|^{2p+2}\exp(2\xi_{n,s}) < \infty.$$

The uniform integrability of the other terms in (4.13) is verified in a similar way. □

REMARK. If the operator A^{-1} is compact and $D_{xx}g(x)BB^*$ can be extended to a nuclear operator on H for all $x \in H$, then

$$\mathrm{tr}[(A^*)^{1/2-\gamma}D_{xx}g(x)BB^*(A^*)^{\gamma-1/2}] = \mathrm{tr}D_{xx}g(x)BB^*$$

(Theorem iii.8.2 of [19]) and the analogous equality is satisfied for the last term on the right-hand side of (4.12). The equality (4.12) then has the usual form, which is called the Itô formula.

Choose and fix $\alpha_1 \in \mathcal{A}$ and let $\eta = P_{\alpha_1}(1, 0, \cdot)$ (recall (2.8)). Note that by [27] and (A3) all of the transition functions $P_\alpha(t, x, \cdot)$, $\alpha \in \mathcal{A}$, $t > 0$, and $x \in H$ are equivalent. The following lemma is Lemma 3 of [13].

LEMMA 4.5. *Let $\varphi : H \to U$ and $G : H \to U$ be bounded, Borel measurable functions and let $(G_n, n \in \mathbb{N})$ be a sequence of bounded, Borel measurable functions that converge to G in $\sigma(L^\infty(H,\eta,H), L^1(H,\eta,H))$ (i.e., in the weak* topology of $L^\infty(H,\eta,H)$). If* (A1)–(A3) *are satisfied, then*

$$\lim_{n\to\infty} \mathbb{E}\left(\int_0^t \langle\varphi(X(s)), G_n(X(s)) - G(X(s))\rangle_U \mathrm{d}s\right)^2 = 0, \tag{4.14}$$

where $(X(t), t \geq 0)$ satisfies (2.5) *and $\alpha \in \mathcal{A}$, $x \in H$ are arbitrary.*

The following result is a technical lemma which will play an important role in the proofs of the existence of an optimal control and the uniformly continuous dependence of the invariant measures on the parameter α.

PROPOSITION 4.6. *Let $(\alpha_n, n \in \mathbb{N})$ be a sequence in $\mathcal{A}$ that converges to $\alpha_0 \in \mathcal{A}$ and let $(h(\alpha_0, \cdot, u_n(\cdot)), n \in \mathbb{N})$ be a sequence that converges to $h(\alpha_0, \cdot, u(\cdot))$ in $\sigma(L^\infty(H,\eta,H), L^1(H,\eta,H))$. If* (A1)–(A3) *are satisfied then*

$$\lim_{n\to\infty} P_t^{\alpha_n,u_n}\varphi(x) = P_t^{\alpha_0,u}\varphi(x) \tag{4.15}$$

for each $\varphi \in \mathcal{M}(H)$, $x \in H$, and $t > 0$.

Proof. It easily follows that

$$
\begin{aligned}
|P_t^{\alpha_n,u_n}\varphi(x) - P_t^{\alpha_0,u}\varphi(x)| &\le |P_t^{\alpha_n,u_n}\varphi(x) - P_t^{\alpha_0,u_n}\varphi(x)| + |P_t^{\alpha_0,u_n}\varphi(x) - P_t^{\alpha_0,u}\varphi(x)| \\
&\le \sup|\varphi| \| P^{\alpha_n,u_n}(t,x,\cdot) - P^{\alpha_o,u_n}(t,x,\cdot)\| \\
&\quad + |P_t^{\alpha_0,u_n}\varphi(x) - P_t^{\alpha_0,u}\varphi(x)|.
\end{aligned} \tag{4.16}
$$

By Proposition 4.3 the first term on the right-hand side of (4.16) tends to zero as $n \to \infty$, so it suffices to show that for any subsequence $(u_{n_k},\, k \in \mathbb{N})$

$$
\lim_{k\to\infty} \mathbb{E}\varphi(X(t)) \exp(\xi_t^{\alpha_0,u_{n_k}}) = \mathbb{E}\varphi(X(t))\exp(\xi_t^{\alpha_0,u}), \tag{4.17}
$$

where $(X(t),\, t \ge 0)$ is a solution of (2.1) with $\alpha = \alpha_0$. The sequence $(\exp(\xi_t^{\alpha_0,u_k}),\, k \in \mathbb{N})$ is bounded in $L^1(\Omega,\mathbb{P})$, so there is a subsequence denoted as the full sequence and a $Z \in L^1(\Omega,\mathbb{P})$ such that

$$
\lim_{n\to\infty} \exp(\xi_t^{\alpha_0,u_n}) = Z \tag{4.18}
$$

in $\sigma(L^1(\Omega,\mathbb{P}), L^\infty(\Omega,\mathbb{P}))$. Since φ is bounded, the equality (4.17) follows if $Z = \exp(\xi_t^{\alpha_0,u})$. Let $g = \bar{g}(\langle x,e_1\rangle,\dots,\langle x,e_n\rangle)$, where $(e_i,\, i \in \mathbb{N})$ is a complete orthonormal basis in H such that $e_i \in D^1_{A^*}$, $\bar{g} \in C_0^\infty(\mathbb{R}^n)$ is arbitrary, and $n \in \mathbb{N}$. By Proposition 4.4 it follows that

$$
\begin{aligned}
g(X(r)) - \int_0^r &(\langle -A^* D_x g(X(s)), X(s)\rangle + \langle D_x g(X(s)), f(\alpha_0, X(s))\rangle \\
&+ \langle B^* D_x g(X(s)), h(\alpha_0, X(s), u_n(X(s)))\rangle_U \\
&+ \frac{1}{2}\mathrm{tr}[(A^*)^{1/2-\gamma} D_{xx} g(X(s)) BB^*(A^*)^{\gamma-1/2}] \\
&+ \frac{1}{2}\mathrm{tr}[(A^*)^{1/2-\Delta} D_{xx} g(X(s)) Q (A^*)^{\Delta-1/2}])\mathrm{d}s
\end{aligned} \tag{4.19}
$$

for $r \in [0,t]$ is a martingale with respect to $\mathbb{P}_x^{\alpha_0,u_n}$. Apply Lemma 4.1 with $\varphi(y) = B^* D_x g(y)$, $G_n(y) = h(\alpha_0, y, u_n(y))$, and $G(y) = h(\alpha_0, y, u(y))$ to obtain

$$
\lim_{n\to\infty} \mathbb{E}\left(\int_0^t \langle B^* D_x g(X(s)), h(\alpha_0, X(s), u_n(X(s)) - h(\alpha_0, X(s), u(X(s)))\rangle_U \right)^2 \mathrm{d}s = 0
$$

so there is a subsequence such that for each $\Gamma \in \mathcal{F}$

$$
\begin{aligned}
\lim_{k\to\infty} &\int_\Gamma \int_0^r \langle B^* D_x g(X(s)), h(\alpha_0, X(s), u_{n_k}(X(s)))\rangle_U \exp(\xi_t^{\alpha_0,u_{n_k}})\mathrm{d}\mathbb{P} \\
&= \int_\Gamma \int_0^r \langle B^* D_x g(X(s)), h(\alpha_0, X(s), u(X(s)))\rangle_U Z \mathrm{d}\mathbb{P}
\end{aligned} \tag{4.20}
$$

for all $r \in [0,t]$ by (4.34) of [2] and (33) of [32]. It follows that (4.19) is a continuous martingale with respect to $Z\mathbb{P}(\mathrm{d}\omega)$, and by the weak uniqueness of the solutions of (2.1) it follows that $Z = \exp(\xi_t^{\alpha_0,u})$ (cf. [18]). ∎

REMARK 4.7. In the remainder of this section some continuity properties of the invariant measures corresponding to the solution of (2.1) are verified. One of the basic assumptions here is the tightness condition (A5). Using a Lyapunov condition, (A5) is verified in section 5. Furthermore, if (A1)–(A3) are satisfied, then for each $\alpha \in \mathcal{A}$ and $u \in \mathcal{U}$ the transition probabilities $(P^{\alpha,u}(t,x,\cdot),\ t>0,\ x \in H)$ are equivalent, which follows from the equivalence of the transition probabilities $(P_\alpha(t,x,\cdot),\ t>0,\ x\in H)$. This latter fact is an immediate consequence of the strong Feller property (a special case of Lemma 4.2) and irreducibility (Proposition 2.7 of [28]). From the equivalence of $(P^{\alpha,u}(t,x,\cdot),\ t>0,\ x \in H)$, it follows that for each $\alpha \in \mathcal{A}$ and $u \in \mathcal{U}$ the invariant measure $\mu(\alpha,u)$ is ergodic and unique (Proposition 2.5 of [31]).

The following lemma follows basically from Roxin [29] (cf. also the Appendix in [2]).

LEMMA 4.8. *Let $\alpha \in \mathcal{A}$ be fixed. If* (A2), (A6), *and* (A7) *are satisfied then*

$$\{(h(\alpha,\cdot,u(\cdot)), c(\cdot,u(\cdot))) : u \in \mathcal{U}\} \subset L^\infty(H,\eta,U\times\mathbb{R})$$

is compact in the $\sigma(L^\infty(H,\eta,U\times\mathbb{R}), L^1(H,\eta,U\times\mathbb{R}))$ topology.

PROPOSITION 4.9. *If* (A1)–(A7) *are satisfied then*

$$\lim_{\alpha\to\alpha_0}\sup_{u\in\mathcal{U}}\rho_*(\mu(\alpha,u),\mu(\alpha_0,u)) = 0 \tag{4.21}$$

and

$$\lim_{n\to\infty}\rho_*(\mu(\alpha_0,\hat{u}_n),\mu(\alpha_0,u_0)) = 0, \tag{4.22}$$

where μ is the invariant measure and ρ_* is a metric for the weak convergence of measures, $\alpha_0 \in \mathcal{A}$, $u_0 \in \mathcal{U}$ and $(\hat{u}_n,\ n\in\mathbb{N})$ is a sequence in $\mathcal{U}$ such that*

$$\lim_{n\to\infty} h(\alpha_0,\cdot,\hat{u}_n(\cdot)) = h(\alpha_0,\cdot,u_0(\cdot))$$

in the $\sigma(L^\infty(H,\eta,U), L^1(H,\eta,U))$ topology.

Proof. Let $(\alpha_n,\ n\in\mathbb{N})$ be a sequence in $\mathcal{A}$ that converges to α_0 and let $(u_n,\ n\in\mathbb{N})$ be a sequence in $\mathcal{U}$. By Lemma 4.8 there exist subsequences (again denoted $(\alpha_n,\ n\in\mathbb{N})$ and $(u_n,\ n\in\mathbb{N})$) so that $\alpha_n\to\alpha_0$ and $(h(\alpha_0,\cdot,u_n(\cdot)),\ n\in\mathbb{N})$ converges to $h(\alpha_0,\cdot,u(\cdot))$ for some $u\in\mathcal{U}$ in the $\sigma(L^\infty(H,\eta,U), L^1(H,\eta,U))$ topology. To verify (4.21) it is necessary to show that from any such sequences there are subsequences $(\alpha_{n_k},\ k\in\mathbb{N})$ and $(u_{n_k},\ k\in\mathbb{N})$ such that

$$\lim_{k\to\infty}\rho_*(\mu(\alpha_{n_k},u_{n_k}),\mu(\alpha_0,u_{n_k})) = 0. \tag{4.23}$$

By the tightness condition (A5) there are measures ν_1 and ν_2 such that $\mu(\alpha_{n_k},u_{n_k}) \to \nu_1$ and $\mu(\alpha_0,u_{n_k}) \to \nu_2$ as $k\to\infty$ in the weak* topology. It is shown that ν_1 is an invariant measure for $P_t^{\alpha_0,u}$; that is, for each $\varphi\in C_b(H)$,

$$\int \varphi \mathrm{d}\nu_1 = \int P_t^{\alpha_0,u}\varphi \mathrm{d}\nu_1 \tag{4.24}$$

for $t\ge 0$. Again, for notational simplicity, let the subsequences be denoted as $(\alpha_n,\ n\in\mathbb{N})$ and $(u_n,\ n\in\mathbb{N})$. It easily follows that

$$\left|\int \varphi \mathrm{d}\nu_1 - \int P_t^{\alpha_0,u}\varphi \mathrm{d}\nu_1\right| \leq \left|\int \varphi \mathrm{d}\nu_1 - \int \varphi \mathrm{d}\mu(\alpha_n,u_n)\right|$$
$$+\left|\int \varphi \mathrm{d}\mu(\alpha_n,u_n) - \int P_t^{\alpha_n,u_n}\varphi \mathrm{d}\mu(\alpha_n,u_n)\right|$$
$$+\left|\int P_t^{\alpha_n,u_n}\varphi \mathrm{d}\mu(\alpha_n,u_n) - \int P_t^{\alpha_0,u}\varphi \mathrm{d}\mu(\alpha_n,u_n)\right| \tag{4.25}$$
$$+\left|\int P_t^{\alpha_0,u}\varphi \mathrm{d}\mu(\alpha_n,u_n) - \int P_t^{\alpha_0,u}\varphi \mathrm{d}\nu_1\right|$$
$$:= I_n^1 + I_n^2 + I_n^3 + I_n^4.$$

It follows that $I_n^1 + I_n^4 \to 0$ as $n \to \infty$ because $\mu(\alpha_n, u_n) \to \nu_1$ in the weak* topology and $I_n^2 \equiv 0$ because $\mu(\alpha_n, u_n)$ is $P_t^{\alpha_n,u_n}$ invariant. Furthermore,

$$I_n^3 \leq \int_K |P_t^{\alpha_n,u_n}\varphi - P_t^{\alpha_0,u}\varphi| \mathrm{d}\mu(\alpha_n,u_n) + 2\max|\varphi|\mu(\alpha_n,u_n)(H \setminus K) \tag{4.26}$$

for any compact $K \subset H$. By Proposition 4.6 and Lemma 4.2, $P_t^{\alpha_n,u_n}\varphi \to P_t^{\alpha_0,u}\varphi$ uniformly on compact subsets of H, so this fact and (A5) imply that

$$\lim_{n\to\infty} I_n^3 = 0.$$

Therefore, (4.24) is satisfied. Since $\mu(\alpha_0, u)$ is the unique invariant measure for $P_t^{\alpha_0,u}$, $\nu_1 = \mu(\alpha_0, u)$. In the same way it follows that $\nu_2 = \mu(\alpha_0, u)$, which verifies (4.23) and thereby (4.21). To verify (4.22) note that given any sequence $(\mu(\alpha_0, \hat{u}_n),\, n \in \mathbb{N})$ there is a subsequence $(\mu(\alpha_0, \hat{u}_{n_k}),\, k \in \mathbb{N})$ converging to a measure ν_3 in the weak* topology. By analogy to (4.25) it can be shown that ν_3 is $P_t^{\alpha_0,u_0}$ invariant so $\nu_3 = \mu(\alpha_0, u_0)$. □

In Proposition 4.9, (4.21) gives the uniformly continuous dependence of invariant measures on the parameter α. Using Propositions 4.9 and 4.3 a strong version of (4.21) is obtained now.

PROPOSITION 4.10. *If* (A1)–(A7) *are satisfied then*

$$\lim_{\alpha\to\alpha_0} \sup_{u\in\mathcal{U}} \|\mu(\alpha,u) - \mu(\alpha_0,u)\| = 0, \tag{4.27}$$

where $\|\cdot\|$ *is the variation norm.*

Proof. It easily follows that

$$\sup_{u\in\mathcal{U}} \|\mu(\alpha,u) - \mu(\alpha_0,u)\| = \sup_{u\in\mathcal{U}} \sup_{\substack{|\varphi|\leq 1\\ \varphi\in C_b}} \left|\int_H \varphi \mathrm{d}\mu(\alpha,u) - \int_H \varphi \mathrm{d}\mu(\alpha_0,u)\right|$$
$$= \sup_{u\in\mathcal{U}} \sup_{|\varphi|\leq 1} \left|\int_H P_1^{\alpha,u}\varphi \mathrm{d}\mu(\alpha,u) - \int_H P_1^{\alpha_0,u}\varphi \mathrm{d}\mu(\alpha_0,u)\right| \tag{4.28}$$
$$\leq 2\sup_{\alpha,u} \mu(\alpha,u)(H\setminus K) + \int_K \sup_u \|P^{\alpha,u}(1,x,\cdot) - P^{\alpha_0,u}(1,x,\cdot)\| \mu(\alpha,u)(\mathrm{d}x)$$
$$+ \sup_{u,\varphi}\left|\int_K P_1^{\alpha,u}\varphi \mathrm{d}\mu(\alpha,u) - \int_K P_1^{\alpha_0,u}\varphi \mathrm{d}\mu(\alpha_0,u)\right|$$

for any compact set $K \subset H$. By (A5) the first term on the right-hand side of (4.28) can be made arbitrarily small by choosing a suitable compact set K, and by Proposition 4.3 the second term converges to zero almost surely as $\alpha \to \alpha_0$. Furthermore, by Lemma 4.2 the family of functions $(P_1^{\alpha_0,u}\varphi, |\varphi| \le 1, u \in \mathcal{U})$ is uniformly continuous on K, so for sequences $(u_n, n \in \mathbb{N})$ and $(\varphi_n, n \in \mathbb{N})$, where $u_n \in \mathcal{U}$ and $\varphi_n \in C_b$ for $n \in \mathbb{N}$, there are subsequences $(u_{n_k}, k \in \mathbb{N})$ and $(\varphi_{n_k}, k \in \mathbb{N})$ and a $\psi \in C_b(K)$ such that $P_1^{\alpha_0,u_{n_k}}\varphi_{n_k}(x) \to \psi(x)$, as $k \to \infty$, uniformly in $x \in K$. Now the third term on the right-hand side of (4.28) is shown to converge to zero.

$$\begin{aligned}
&\left|\int_K P_1^{\alpha_0,u_{n_k}}\varphi_{n_k}\mathrm{d}\mu(\alpha,u_{n_k}) - \int_K P_1^{\alpha_0,u_{n_k}}\varphi_{n_k}\mathrm{d}\mu(\alpha_0,u_{n_k})\right| \\
&\quad\le \int_K |P_1^{\alpha_0,u_{n_k}}\varphi_{n_k} - \psi|\mathrm{d}\mu(\alpha,u_{n_k}) + \left|\int_K \psi\mathrm{d}\mu(\alpha,u_{n_k}) - \int_K \psi\mathrm{d}\mu(\alpha_0,u_{n_k})\right| \\
&\qquad + \int_K |\psi - P_1^{\alpha_0,u_{n_k}}\varphi_{n_k}|\mathrm{d}\mu(\alpha_0,u_{n_k}) \\
&\quad := I_n^1 + I_n^2 + I_n^3.
\end{aligned} \tag{4.29}$$

By the uniform convergence $P_1^{\alpha_0,u_{n_k}}\varphi_{n_k} \to \psi$ on K it follows that $I_n^1 + I_n^3 \to 0$ as $n \to \infty$, and by (4.21) it follows that $I_n^2 \to 0$ as $n \to \infty$. This proves that the last term on the right-hand side of (4.28) tends to zero as $\alpha \to \alpha_0$. ☐

REMARK 4.11. The strong continuous dependence of the invariant measures on a parameter in Proposition 4.10 can be of independent interest even for equations without control. If the parameter occurs linearly in the generator of even a very simple example of an Ornstein–Uhlenbeck process then the invariant measures may not depend continuously on α in the variation norm. For example, consider the stochastic differential equation

$$\mathrm{d}X(t) + \alpha AX(t)\mathrm{d}t = \mathrm{d}W(t), \qquad X(0) = x, \tag{4.30}$$

where A and $(W(t), t \ge 0)$ are the same as in (2.1) and $\alpha \in [1/2, 2]$. If

$$\int_0^\infty |S(t)|^2_{\mathcal{L}_2(H)}\mathrm{d}t < \infty \tag{4.31}$$

then (4.30) has a unique mild solution that is a continuous H-valued process. If (4.31) is satisfied and $\alpha \in [1/2, 2]$, then there is a unique invariant measure $\mu(\alpha)$ for the solution of (4.30), where $\mu(\alpha) = N(0, \alpha^{-1}\tilde{Q})$ and $\tilde{Q} = \int_0^\infty S(t)S^*(t)\mathrm{d}t$. It is easy to verify that the family of measures $(\mu(\alpha), \alpha \in [1/2, 2])$ is tight and $\mu(\alpha) \xrightarrow{w^*} \mu(1)$ as $\alpha \to 1$. However, the variation norm convergence $\mu(\alpha) \to \mu(1)$ occurs if and only if $\dim H < \infty$ because the operator $(\alpha^{-1}\tilde{Q})\tilde{Q}^{-1} - I = (\alpha^{-1} - 1)I$ is not Hilbert–Schmidt for $\alpha \ne 1$ and $\dim H = \infty$, and so $\mu(\alpha)$ and $\mu(1)$ are singular by the well-known dichotomy for Gaussian measures. This occurs even in the strong Feller case when the solution of (4.30) converges in law to the invariant measure in the variation norm for each fixed α. For a specific example of this, consider the linear SPDE

$$\frac{\partial w}{\partial t}(t,\xi) = \alpha\frac{\partial^2 w}{\partial \xi^2}(t,\xi) + n(t,\xi), \tag{4.32}$$

where $\alpha \in [1/2, 2]$, $(t,\xi) \in \mathbb{R}_+ \times (0,1)$, $w(0,\xi) = w_0(\xi)$, $w(t,0) = v(t,1) = 0$, and $(n(t,\xi), t \ge 0, \xi \in [0,1])$ is a space-time white noise which can be expressed as an equation of the form (4.30) for $H = L^2(0,1)$ (cf. Example 7.1).

5. Existence and tightness of invariant measures. In this section some more explicit sufficient conditions for the validity of (A5) are given by means of some Lyapunov-type inequalities. Throughout this section it is assumed that

$$A^{-1} \text{ is compact.} \tag{T1}$$

Since the semigroup $S(\cdot)$ generated by $-A$ is assumed to be analytic and exponentially stable, there exist some $M > 0$ and $\omega > 0$ such that

$$|S(t)|_{\mathcal{L}(D_A^{-\delta},H)} \leq M\mathrm{e}^{-\omega t}t^{-\delta} \tag{T2}$$

for all $t > 0$ and $\delta \leq 0$. (The constants M and ω will play some role in the Lyapunov-type conditions given below.)

While in the other sections of this paper the negativity of $-A$ is assumed merely for convenience (because $A + \beta I$ can be used instead of A, and βI can be added to f), in this section it is essential.

Define $\mu_T^{\alpha,u}$ as follows:

$$\mu_T^{\alpha,u}(\cdot) = \frac{1}{T}\int_0^T P^{\alpha,u}(t,0,\cdot)\mathrm{d}t \tag{5.1}$$

for $\alpha \in \mathcal{A}$, $u \in \mathcal{U}$, and $T > 0$. Since the solution of (2.1) is Feller, to verify (A5) it suffices to show that the family of measures $(\mu_T^{\alpha,u},\ \alpha \in \mathcal{A},\ u \in \mathcal{U},\ T \geq 1)$ is tight. In the following proposition it is shown that the tightness of $(\mu_T^{\alpha,u},\ \alpha \in \mathcal{A},\ u \in \mathcal{U},\ T \geq 1)$ follows from a similar property, where compact sets are replaced by balls (5.2). Note that (5.2) does not guarantee the existence of an invariant measure in general (cf. [33]).

Proposition 5.1. *If* (A1), (A2), *and* (T1) *are satisfied and*

$$\lim_{n\to\infty} \mu_T^{\alpha,u}(H \setminus B_n) = 0, \tag{5.2}$$

where the convergence is uniform in $\alpha \in \mathcal{A}$, $u \in \mathcal{U}$, and $T \geq 1$, and $B_n = \{x \in H : |x| \leq n\}$, then the family of measures $(\mu_T^{\alpha,u},\ \alpha \in \mathcal{A},\ u \in \mathcal{U},\ T \geq 1)$ is tight.

Proof. The weak solution of (2.1) satisfies the equation

$$\begin{aligned} X(t) = S(t)x &+ \int_0^t S(t-r)f(\alpha, X(r))\mathrm{d}r \\ &+ \int_0^t S(t-r)Bh(\alpha, X(r), u(X(r)))\mathrm{d}r + Z_1(t) + Z_2(t), \end{aligned} \tag{5.3}$$

where

$$Z_1(t) = \int_0^t S(t-r)B\mathrm{d}V^*(r)$$

and

$$Z_2(t) = \int_0^t S(t-r)Q^{1/2}\mathrm{d}W(r)$$

for $t \geq 0$. By (A1) and Lemma 2.2 of [30] it follows that

$$\mathbb{E}_x^{\alpha,u}|Z_1(t)|_\delta^2 + \mathbb{E}_x^{\alpha,u}|Z_2(t)|_\delta^2 \leq M_1$$

for $t \in [0,T]$, where $T > 0$ is fixed and the constant M_1 (as well as $M_2,\dots,M_5$ below) does not depend on $\alpha \in \mathcal{A}$, $u \in \mathcal{U}$, and $x \in H$, and $|\cdot|_\delta$ is the D_A^δ norm and

$\delta \in (0, \min(\varepsilon, \gamma, \Delta))$ is fixed. It follows that

$$\mathbb{E}_x^{\alpha,u}|X(t)|_\delta \leq M_2|x|t^{-\delta} + \int_0^t \frac{M_3}{(t-s)^{1-\varepsilon+\delta}}\mathbb{E}_x^{\alpha,u}|X(s)|_\delta \mathrm{d}s + M_4 \tag{5.4}$$

for $t \in (0, T]$ and $x \in H$, so the generalized Gronwall lemma (Theorem 7.1 of [21]) implies that

$$\mathbb{E}_x^{\alpha,u}|X(T)|_\delta \leq M_5(1+|x|) \tag{5.5}$$

for $\alpha \in \mathcal{A}$, $u \in \mathcal{U}$, and $x \in H$. By the Chebyshev inequality it follows that

$$\sup_{|y|\leq R} \mathbb{P}_y^{\alpha,u}(|X(t)|_\delta \geq n) \leq \frac{1}{n}M_5(1+R) \tag{5.6}$$

for $n \in \mathbb{N}$, $R > 0$, $\alpha \in \mathcal{A}$, and $u \in \mathcal{U}$.

Let $K_n \subset H$ be given by

$$K_n = \mathrm{Cl}_H A^{-\delta} B_n \tag{5.7}$$

for $n \in \mathbb{N}$, where Cl_H is the closure in H. Since $A^{-\delta}$ is a compact operator, K_n is compact in H. It follows that

(5.8)
$$\begin{aligned}\frac{1}{T}\int_1^T P^{\alpha,u}(t,0,H\setminus K_n)\mathrm{d}t &= \frac{1}{T}\int_1^T\int_H P^{\alpha,u}(1,y,H\setminus K_n)P^{\alpha,u}(t-1,0,\mathrm{d}y)\mathrm{d}t \\ &= \frac{T-1}{T}\int_H P^{\alpha,u}(1,y,H\setminus K_n)\mu_{T-1}^{\alpha,u}(\mathrm{d}y) \\ &\leq \mu_{T-1}^{\alpha,u}(H\setminus B_R) + \mu_{T-1}^{\alpha,u}(B_R)\sup_{|y|\leq R} P^{\alpha,u}(1,y,H\setminus K_n) \\ &\leq \mu_{T-1}^{\alpha,u}(H\setminus B_R) + \frac{1}{n}M_5(1+R)\end{aligned}$$

for each $R > 0$. By (5.2) the right-hand side tends to zero as $n \to \infty$ uniformly in $\alpha \in \mathcal{A}$, $u \in \mathcal{U}$, and $T \geq 1$. ◻

In Theorem 5.3 below, the condition (5.2) is verified by a Lyapunov functional that completes the verification of (A5). Let V be given by

$$V = 2\int_0^\infty S(r)S^*(r)\mathrm{d}r. \tag{5.9}$$

If (T2) is satisfied then $V \in \mathcal{L}(H)$ is well defined, $V = V^*$, and $V \geq 0$.

The following estimates are easily verified.

LEMMA 5.2. *For $\beta, \lambda \in \mathbb{R}_+$ with $\beta + \lambda < 1$, $V \in \mathcal{L}(D_A^{-\beta}, D_A^\lambda)$ and the following inequality is satisfied:*

$$|V|_{\mathcal{L}(D_A^{-\beta}, D_A^\lambda)} \leq 2M^2(2\omega)^{\beta+\lambda-1}\Gamma(1-\beta-\lambda), \tag{5.10}$$

where Γ is the gamma function, M and ω are given in (T2)*, and V is given by* (5.9)*. Furthermore, if A is self-adjoint, then $V = A^{-1}$ and*

$$|V|_{\mathcal{L}(D_A^{-\beta}, D_A^\lambda)} \leq \omega^{\beta+\lambda-1} \tag{5.11}$$

and ω is the first eigenvalue of A.

THEOREM 5.3. *If* (A1), (A2), (T1), (T2) *are satisfied and either*

$$M^2\omega^{-1}k_f + 2^{1-\varepsilon}M^2\omega^{-\varepsilon}|B|_{\mathcal{L}(U,D_A^{\varepsilon-1})}\Gamma(\varepsilon)k_h < 1 \tag{5.12}$$

for A not self-adjoint or

$$\omega^{-1}k_f + |B|_{\mathcal{L}(U,D_A^{\varepsilon-1})}\omega^{-\varepsilon}k_h < 1 \tag{5.13}$$

for A self-adjoint, where M and ω are given in (T2) *and Γ is the gamma function, then the condition* (A5) *is satisfied. In particular,* (5.12) *and* (5.13) *are satisfied if $|f|$ and $|h|_U$ are bounded uniformly with respect to $\alpha \in \mathcal{A}$ and $u \in \mathcal{U}$.*

Proof. By Proposition 5.1 it suffices to verify (5.2). Use Proposition 4.4 with $g(x) = \langle Vx, x\rangle$ so that $D_x g(x) = Vx$, $D_{xx}g(x) = V$,

$$|V|_{\mathcal{L}(H,D_{A^*}^{1-\varepsilon})} \le 2M^2(2\omega)^{-\varepsilon}\Gamma(\varepsilon), \tag{5.14}$$

and $\langle Ax, D_x g(x)\rangle = |x|^2$ for $x \in D_A^1$, and by Lemma 5.2 $V \in \mathcal{L}(D_A^{\gamma-1/2}, D_{A^*}^{1/2-\gamma}) \cap \mathcal{L}(D_A^{\Delta-1/2}, D_{A^*}^{1/2-\Delta})$. Thus the assumptions of Proposition 4.4 are satisfied, and by (A2) and (5.14),

$$\begin{aligned} &\mathbb{E}_x^{\alpha,u}\langle VX(t), X(t)\rangle - \langle Vx, x\rangle \\ &\le \mathbb{E}_x^{\alpha,u}\int_0^t (|X(s)|^2(-1 + M^2\omega^{-1}k_f + 2^{1-\varepsilon}M^2\omega^{-\varepsilon}|B|_{\mathcal{L}(U,D_A^{\varepsilon-1})}\Gamma(\varepsilon)k_h) \\ &\quad + c_1|X(s)| + c_2)\mathrm{d}s \end{aligned} \tag{5.15}$$

for $t \ge 0$, where the constants c_1 and c_2 (as well as the constants c_3 and c_4 below) do not depend on $\alpha \in \mathcal{A}$ and $u \in \mathcal{U}$. Choosing r such that $M^2\omega^{-1}k_f + 2^{1-\varepsilon}M^2\omega^{-2}|B|_{\mathcal{L}(U,D_A^{\varepsilon-1})}\Gamma(\varepsilon)\cdot k_h < r < 1$, it follows that

$$\mathbb{E}_x^{\alpha,u}\langle VX(t), X(t)\rangle - \langle Vx, x\rangle \le \mathbb{E}_x^{\alpha,u}\int_0^t ((r-1)|X(t)|^2 + c_3)\mathrm{d}s$$

for $t \ge 0$, and since $V \ge 0$ it follows that

$$\sup_{t\ge 1}\frac{1}{t}\int_0^t \mathbb{E}_x^{\alpha,u}|X(s)|^2\mathrm{d}s \le \sup_{t\ge 1}\frac{\langle Vx, x\rangle}{t(1-r)} + \frac{c_3}{1-r} \le c_4. \tag{5.16}$$

By (5.16) and the Chebyshev inequality it follows that (5.2) is satisfied. If A is self-adjoint then (5.11) can be used instead of (5.10). ∎

6. The existence of an optimal control. Recall that the control problem is described by the system (2.1) and the cost functional

$$J(\alpha, u) = \limsup_{T\to\infty}\mathbb{E}_x^{\alpha,u}\frac{1}{T}\int_0^T c(X(s), u(X(s)))\mathrm{d}s, \tag{6.1}$$

and the optimal cost is $J^*(\alpha) = \inf_{u\in\mathcal{U}} J(\alpha, u)$. If (A1)–(A3), (A5), and (A6) are satisfied then the following equality is satisfied:

$$J(\alpha, u) = \int_H c(y, u(y))\mu(\alpha, u)(\mathrm{d}y) \tag{6.2}$$

(cf. Remark 4.7), so the cost $J(\alpha, u)$ does not depend on the initial condition $X(0) = x \in H$. In this section the existence of an optimal control for the control problem (2.1) and (6.1) with a fixed parameter $\alpha \in \mathcal{A}$ and the continuity of the optimal cost $J^* : \mathcal{A} \to \mathbb{R}$ are verified. In Lemma 6.1 and Theorem 6.2 the parameter is fixed, so it is suppressed for notational convenience.

Recall that $P(t, x, \Gamma)$ is given in (2.8) and $\eta = P(1, 0, \cdot)$.

LEMMA 6.1. *Let* $(A_n, n \in \mathbb{N})$ *be a sequence in* $\mathcal{B}(H)$ *such that* $\eta(A_n) \to 0$ *as* $n \to \infty$. *If* (A1)–(A3) *and* (A5) *are satisfied then*

$$\lim_{n\to\infty} \sup_{u\in\mathcal{U}} \mu(u)(A_n) = 0. \tag{6.3}$$

Proof. Since $P(1, \cdot, \cdot) : H \to \mathcal{P}(H)$ is continuous in the variation norm and $P(1, x, \cdot)$ and η are equivalent for each $x \in K$, where $K \subset H$ is compact, it follows that

$$\lim_{n\to\infty} \sup_{x\in K} P(1, x, A_n) = 0. \tag{6.4}$$

Since, for a fixed $\alpha \in \mathcal{A}$, $|h|_U$ is bounded it follows that

$$\begin{aligned} \sup_{u\in\mathcal{U},\ x\in K} P^u(1, x, A_n) &= \sup_{u\in\mathcal{U}} \sup_{x\in K} \mathbb{E}_x 1_{A_n}(X(1)) \exp(\xi_1^u) \\ &\le \sup_{x\in K} (P(1, x, A_n))^{1/2} \exp(\sup |h|^2). \end{aligned} \tag{6.5}$$

The right-hand side of this inequality tends to zero as $n \to \infty$ by (6.4).

Finally it follows that

$$\begin{aligned} \sup_{u\in\mathcal{U}} \mu(u)(A_n) &= \sup_{u\in\mathcal{U}} \int_H P^u(1, x, A_n)\mu(u)(\mathrm{d}x) \\ &\le \sup_{u\in\mathcal{U}} \mu(u)(H \setminus K) + \int_K \sup_{u\in\mathcal{U}} P^u(1, x, A_n)\mu(u)(\mathrm{d}x). \end{aligned}$$

By (6.5) and the tightness of the family of measures $(\mu(u), u \in \mathcal{U})$ the right-hand side of this inequality tends to zero as $n \to \infty$.

THEOREM 6.2. *If* (A1)–(A3) *and* (A5)–(A7) *are satisfied for each* $\alpha \in \mathcal{A}$, *then there is an optimal control for the control problem given by* (2.1) *and* (6.1).

Proof. Let $(u_n, n \in \mathbb{N})$ be a sequence in $\mathcal{U}$ such that there is a subsequence in $(u_n, n \in \mathbb{N})$ denoted as $(u_n, n \in \mathbb{N})$ for notational convenience, such that

$$\lim_{n\to\infty} (h(\cdot, u_n(\cdot)), c(\cdot, u_n(\cdot))) = (h(\cdot, u(\cdot)), c(\cdot, u(\cdot))) \tag{6.6}$$

in the $\sigma(L^\infty(H, \eta, U \times \mathbb{R}), L^1(H, \eta, U \times \mathbb{R}))$ topology. To verify that u is an optimal control it is necessary to prove that for any subsequence $(u_{n_k}, k \in \mathbb{N})$,

$$\lim_{k\to\infty} J(u_{n_k}) = J(u). \tag{6.7}$$

As in Lemma 4.5 it follows that

$$\lim_{n\to\infty} \mathbb{E}\left(\int_0^t (c(X(s), u_n(X(s))) - c(X(s), u(X(s))))\mathrm{d}s\right)^2 = 0,$$

where $(X(t),\, t \geq 0)$ satisfies (2.5) with $X(0) = x \in H$ arbitrary (cf. Theorem 2 of [13]). As in the passage to the limit in the proof of Proposition 4.4 it follows that

$$\lim_{n\to\infty} \mathbb{E}_x^{u_n} \int_0^t c(X(s), u_n(X(s)))\mathrm{d}s = \mathbb{E}_x^u \int_0^t c(X(s), u(X(s)))\mathrm{d}s \tag{6.8}$$

for a subsequence again denoted by $(u_n,\, n \in \mathbb{N})$. By Egorov's theorem the convergence in (6.8) is uniform in x except possibly on a set of arbitrarily small η-measure. This fact and Lemma 6.1 imply that

(6.9)

$$\lim_{n\to\infty} \int_H \left|\mathbb{E}_x^{u_n} \int_0^t c(X(s), u_n(X(s)))\mathrm{d}s - \mathbb{E}_x^u \int_0^t c(X(s), u(X(s)))\mathrm{d}s\right| \mu(u_n)(\mathrm{d}x) = 0.$$

For each fixed $t > 0$ it follows that

(6.10)

$$\begin{aligned}
&|J(u_n) - J(u)| \\
&= \left|\int_H c(y, u_n(y))\mu(u_n)(\mathrm{d}y) - \int_H c(y, u(y))\mu(u)(\mathrm{d}y)\right| \\
&\leq \left|\frac{1}{t}\int_0^t \left[\int_H \mathbb{E}_y^{u_n} c(X(s), u_n(X(s)))\mu(u_n)(\mathrm{d}y)\right.\right. \\
&\qquad \left.\left. - \int_H \mathbb{E}_y^u c(X(s), u(X(s)))\mu(u)(\mathrm{d}y)\right] \mathrm{d}s\right| \\
&\leq \frac{1}{t}\int_H \left|\mathbb{E}_y^{u_n}\int_0^t c(X(s), u_n(X(s)))\mathrm{d}s - \mathbb{E}_y^u \int_0^t c(X(s), u(X(s)))\mathrm{d}s\right| \mu(u_n)(\mathrm{d}y) \\
&\quad + \frac{1}{t}\int_0^t \left|\int_H \mathbb{E}_y^u c(X(s), u(X(s)))\mu(u_n)(\mathrm{d}y) - \int_H \mathbb{E}_y^u c(X(s), u(X(s)))\mu(u)(\mathrm{d}y)\right| \mathrm{d}s \\
&:= I_n^1 + I_n^2.
\end{aligned}$$

By (6.9) it suffices to show that $I_n^2 \to 0$ as $n \to \infty$. Since $c(\,\cdot\,, u(\,\cdot\,))$ is bounded and Borel measurable, the strong Feller property (Lemma 4.2) implies that

$$\mathbb{E}.c(X(s), u(X(s))) : H \to \mathbb{R}$$

is continuous for each $s > 0$ where $\mathbb{E}_x c(X(s), u(X(s))) = P_s^u c(\,\cdot\,, u(\,\cdot\,))(x)$. So by (4.22) and the dominated convergence theorem, $I_n^2 \to 0$ as $n \to \infty$. ☐

THEOREM 6.3. *If* (A1)–(A7) *are satisfied then the optimal cost* $J^* : \mathcal{A} \to \mathbb{R}$ *is continuous.*

Proof. It follows that

$$\sup_{u\in\mathcal{U}} |J(\alpha, u) - J(\alpha_0, u)| \leq \sup |c| \sup_{u\in\mathcal{U}} \|\mu(u, \alpha) - \mu(u, \alpha_0)\|. \tag{6.11}$$

By Proposition 4.10 it follows that the right-hand side of this inequality tends to zero as $n \to \infty$. Given $\varepsilon > 0$ there is a $\delta > 0$ such that if $|\alpha - \alpha_0| < \delta$ then

$$\sup_{u\in\mathcal{U}} |J(\alpha, u) - J(\alpha_0, u)| < \varepsilon.$$

Let $u_\alpha \in \mathcal{U}$ be an optimal control for the control problem (2.1) and (6.1), that is, $J^*(\alpha) = J(\alpha, u_\alpha)$ for $\alpha \in \mathcal{A}$. Since $J(\alpha, u_{\alpha_0}) \geq J(\alpha, u_\alpha)$ it follows that $J(\alpha_0, u_{\alpha_0}) \geq J(\alpha, u_\alpha) - \varepsilon$. Since $J(\alpha_0, u_{\alpha_0}) \leq J(\alpha_0, u_\alpha)$ it follows that $J(\alpha_0, u_{\alpha_0}) \leq J(\alpha, u_\alpha) + \varepsilon$ for $\alpha \in \mathcal{A}$ and $|\alpha - \alpha_0| < \delta$. ☐

7. Some Examples.

EXAMPLE 7.1. Consider the scalar stochastic parabolic partial differential equation

$$\frac{\partial v}{\partial t}(t,\xi) = Lv(t,\xi) + F(\alpha, v(t,\xi)) + n(t,\xi) \tag{7.1}$$

for $(t,\xi) \in \mathbb{R}_+ \times (0,1)$ with the initial and boundary conditions

$$v(0,\xi) = v_0(\xi), \tag{7.2}$$

$$\frac{\partial v}{\partial \xi}(t,0) = h_1(\alpha, v(t,\cdot), u(v(t,\cdot))) + \dot{\beta}_1(t), \tag{7.3}$$

$$\frac{\partial v}{\partial \xi}(t,1) = h_2(\alpha, v(t,\cdot), u(v(t,\cdot))) + \dot{\beta}_2(t), \tag{7.4}$$

where n denotes a space-dependent Gaussian noise that is white in time, β_1 and β_2 are one-dimensional standard Wiener processes, and these three processes are mutually independent. Furthermore,

$$Lv = a(\xi)\frac{\partial^2}{\partial \xi^2}v + b(\xi)\frac{\partial}{\partial \xi}v + c(\xi),$$

where $a,b,c \in C^\infty([0,1])$, $a > 0$, $c < 0$, $F : \mathcal{A} \times \mathbb{R} \to \mathbb{R}$, $h_i : \mathcal{A} \times H \times \mathcal{K} \to \mathbb{R}$, $i = 1,2$, where $H = L^2(0,1)$, $\mathcal{A} \subset \mathbb{R}^{d_1}$ is compact, $\mathcal{K} \subset \mathbb{R}^k$ is a compact product of intervals, $F(\alpha,\cdot) : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, $h_i(\alpha,\cdot,\cdot) : H \times \mathcal{K} \to \mathbb{R}$, $i = 1,2$, is continuous and bounded for each $\alpha \in \mathcal{A}$ with at most linear growth that is uniform with respect to $\alpha \in \mathcal{A}$, and

$$|F(\alpha,\xi) - F(\beta,\xi)| + \sum_{i=1}^{2} |h_i(\alpha,x,u) - h_i(\beta,x,u)| \leq \omega(|\alpha-\beta|)(1 + \max(|x|,|\xi|)) \tag{7.5}$$

for $\alpha,\beta \in \mathcal{A}$, $\xi \in \mathbb{R}$, $x \in H$, and $u \in \mathcal{K}$, where ω satisfies the properties in (A2). The system of equations (7.1)–(7.4) can be rewritten in the form of (2.1) in a natural way, where $H = L^2(0,1)$, $U = \mathbb{R}^2$, $A = -L$ with

$$\text{Dom}(A) = \left\{\varphi : \varphi \in H^2(0,1), \frac{\partial}{\partial \xi}\varphi(0) = \frac{\partial}{\partial \xi}\varphi(1) = 0\right\},$$

$f(\alpha,x)(\xi) = F(\alpha, X(\xi))$, $x \in H$, $\xi \in (0,1)$, and $h = [h_1, h_2]$. The operator B is defined as $B = \hat{A}N$, where $N \in \mathcal{L}(\mathbb{R}^2, D_A^\varepsilon)$, $\varepsilon < 3/4$ is the Neumann map corresponding to the elliptic Neumann problem

$$Lz(\xi) = 0, \qquad \xi \in (0,1), \tag{7.6}$$

$$\frac{\partial z}{\partial \xi}(0) = g_1, \qquad \frac{\partial z}{\partial \xi}(1) = g_2 \tag{7.7}$$

for $g_1, g_2 \in \mathbb{R}$, and $\hat{A} \in \mathcal{L}(D_A^\varepsilon, D_A^{\varepsilon-1})$ is the isomorphic extension of the operator A to D_A^ε. (See [26] for the theory of Dirichlet and Neumann maps, [16] for the identification of D_A^ε with the corresponding Sobolev spaces, and [22] or [27] for the mathematical justification of the form (2.1) for the equations (7.1)–(7.4).) Thus it follows that $B \in \mathcal{L}(U, D_A^{\varepsilon-1})$ for $\varepsilon < 3/4$ in the present case. Now it is verified that (A1) and (A3) are satisfied, where $Q^{1/2} = A^{-\eta}\Gamma$ with $\eta \geq 0$ and $\Gamma, \Gamma^{-1} \in \mathcal{L}(H)$. Since $A^{-\delta}$ is Hilbert–Schmidt for $\delta > 1/4$ (cf. Example 6.1 of [12]) it follows that

$Q^{1/2} \in \mathcal{L}_2(H, D_A^{\Delta-1/2})$ for $\Delta < 1/4 + \eta$. Since the space U is finite-dimensional, $B \in \mathcal{L}_2(U, D_A^{\gamma-1/2})$ for $\gamma < \varepsilon - 1/2$ and γ is positive if $\varepsilon > 1/2$. To verify (A3) use Proposition 3.4, which shows that (A3) is satisfied if $\eta \in [0, \varepsilon - 1/2)$ if $\varepsilon \geq 1/2$. Thus the assumptions (A1) and (A3) are satisfied for $\eta \in [0, 1/4)$ so that ε, γ, and Δ can be chosen to satisfy $\varepsilon \in (\eta + 1/2, 3/4)$, $\gamma < \varepsilon - 1/2$, and $\Delta < 1/4 + \eta$. The assumptions (A2) and (A4) are satisfied by the conditions imposed on F, h_1, and h_2. The tightness condition (A5) can be verified using Theorem 5.3 (note that A^{-1} is compact in the present case). For example, if $|F|$, $|h_1|$, and $|h_2|$ are (uniformly) bounded then (A5) is satisfied. Thus the results of the paper (in particular, Proposition 4.10, Theorems 6.2 and 6.3) can be applied for any cost functional $c : H \times \mathcal{K} \to \mathbb{R}$ that satisfies (A6) and satisfies with h_1 and h_2 the convexity condition (A7). A simple example of a boundary input (7.3), (7.4) that satisfies all the above conditions is

$$\frac{\partial v}{\partial \xi}(t, 0) = u_1(v(t, \cdot)) + \dot{\beta}_1(t), \tag{7.8}$$

$$\frac{\partial v}{\partial \xi}(t, 1) = u_2(v(t, \cdot)) + \dot{\beta}_2(t), \tag{7.9}$$

where $(u_1, u_2) : H \to [-M, M]^2 = \mathcal{K}$.

EXAMPLE 7.2. Consider the stochastic parabolic partial differential equation with pointwise noise and control

$$\frac{\partial v}{\partial t}(t, \xi) = Lv(t, \xi) + F(\alpha, v(t, \xi)) + \sum_{i=1}^{N} [h_i(\alpha, v(t, \cdot), u(v(t, \cdot))) + \dot{\beta}_i(t)]\delta_{\xi_i} + n(t, \xi) \tag{7.10}$$

for $(t, \xi) \in \mathbb{R}_+ \times (0, 1)$ with initial and boundary conditions

$$v(0, \xi) = v_0(\xi), \tag{7.11}$$

$$v(t, 0) = 0, \tag{7.12}$$

$$v(t, 1) = 0 \tag{7.13}$$

for $(t, \xi) \in \mathbb{R}_+ \times (0, 1)$, where L, F, n, β_i, and h_i are the same as in Example 7.1, and δ_{ξ_i}, $i = 1, 2, \ldots, N$, are the Dirac distributions at the points $\xi_i \in (0, 1)$, $i = 1, 2, \ldots, N$. The equation (7.10) is given a precise interpretation by using (2.1) with H and f as in Example 7.1, $V(t) = (\beta_1(t), \ldots, \beta_N(t))$, $U = \mathbb{R}^N$, $h = (h_1, \ldots, h_N)$, and $A = -L$ with $\mathrm{Dom}(A) = H^2(0, 1) \cap H_0^1(0, 1)$. It is possible to use the Neumann boundary conditions in (7.12), (7.13) as well, so that $\mathrm{Dom}(A)$ would be the same as in Example 7.1. Since the domain is one-dimensional it follows by the Sobolev imbedding theorem that $\delta_{\xi_i} \in D_A^{\varepsilon-1}$ for $\varepsilon < 3/4$ (cf. Theorem 1.1 of [5]). It trivially follows that $B \in \mathcal{L}(\mathbb{R}^N, D_A^{\varepsilon-1})$ for $B\lambda = \sum_{i=1}^{N} \lambda_i \delta_{\xi_i}$, $\lambda = (\lambda_1, \ldots, \lambda_N)$. The verification of assumptions (A1)–(A7) in the present example is almost identical to the verifications in Example 7.1 because H and f are the same and U, h, and $V(t)$ are analogous (but the dimension is N instead of 2), $A^{-\delta}$ is Hilbert–Schmidt for $\delta > 1/4$, A^{-1} is compact, and it is again required that $\varepsilon < 3/4$. If the covariance Q of the distributed Wiener process can be expressed as $Q^{1/2} = A^{-\eta}\Gamma$ for $\Gamma, \Gamma^{-1} \in \mathcal{L}(H)$, $\eta \in [0, 1/4)$, then the assumptions (A1) and (A3) are satisfied. Given an $M > 0$ and the set of controls $\mathcal{U} = \{u : H \to [-M, M]^N |\ u$ is Borel mesurable$\}$ it is now possible to apply Proposition 4.10 and Theorems 6.2 and 6.3 with any cost functional $c : H \times [-M, M]^N \to \mathbb{R}$ that satisfies (A6) and, together with h, the convexity condition (A7).

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