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STOCHASTIC EVOLUTION EQUATIONS WITH RANDOM GENERATORS

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We prove the existence of a unique mild solution for a stochastic evolution equation on a Hilbert space driven by a cylindrical Wiener process. The generator of the corresponding evolution system is supposed to be random and adapted to the filtration generated by the Wiener process. The proof is based on a maximal inequality for the Skorohod integral deduced from the Itô's formula for this anticipating stochastic integral.

1. Introduction. In this paper we study nonlinear stochastic evolution equations of the form

$$X_t = \xi + \int_0^t (A(s)X_s + F(s, X_s))\,ds + \int_0^t B(s, X_s)\,dW_s, \qquad t \in [0, T], \tag{1.1}$$

where W is a cylindrical Wiener process on a Hilbert space U. The solution process $X = \{X_t,\ t \in [0, T]\}$ is a continuous and adapted process taking values in a Hilbert space H. The functions $F(s, \omega, x)$ and $B(s, \omega, x)$ are predictable processes satisfying suitable Lipschitz–type conditions and taking values in H and $L_2(U, H)$, respectively.

We will assume that $A(s, \omega)$ is a random family of unbounded operators on H. A notion of weak solution for (1.1) can be introduced as usual (see Definition 5.2).

In the case where (1.1) is a coercive evolution system on a normal triple (K, H, K'), we can interpret (1.1) as an evolution equation to be solved in K' (see [5] and [12]). In this case, the proof of existence of a unique weak solution for (1.1) follows closely the ideas of Pardoux [11].

When $A(s)$ is a deterministic family of operators, in order to solve Equation (1.1) one looks for a mild (or evolution) solution, which satisfies the evolution equation

$$X_t = S(t, 0)\xi + \int_0^t S(t, s)F(s, X_s)\,ds + \int_0^t S(t, s)B(s, X_s)\,dW_s, \tag{1.2}$$

where $\{S(t, s),\ 0 \le s \le t \le T\}$ is an evolution system determined by $A(t)S(t, s) = (d/dt)S(t, s)$. We refer to [1] for a basic account of this theory.

In the case of a random family of operators $\{A(t)\}$, the corresponding evolution system $S(t, s)$ is also random and $\mathscr{F}_t$-measurable (where $\{\mathscr{F}_t, t \in [0, T]\}$

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is the natural family of σ-fields determined by W). As a consequence, the process $S(t,s)B(s,X_s)$ is not $\mathscr{F}_s$-measurable, and the stochastic integral appearing in (1.2) is anticipative. That is, although both the solution process $\{X_t\}$ and the random family of operators $\{A_t\}$ are adapted, the associated stochastic evolution equation involves an *anticipating integral*.

It is well known that a mild solution of (1.2), where the anticipating integral is interpreted as a Skorohod integral, is not a weak solution of (1.1) (see [7]) because a complementary term appears. We show in Section 5 (see Proposition 5.3) that a mild solution of (1.2) where the stochastic integral is a "forward integral" is also a weak solution to (1.1). Roughly speaking, the forward integral is defined as the limit (in probability) of Riemann sums defined taking the values of the process on the left points of each interval. In the case of real-valued processes, this type of integral was studied, among other authors, by Russo and Vallois in [13]. The main difficulty in handling this stochastic integral is to obtain suitable estimates for the L^p-norm of the integral. One way to do this, in the anticipating case, consists in expresssing the forward integral as the sum of the Skorohod integral plus a complementary term.

In Section 4 we obtain an expression relating the forward and the Skorohod integrals (Proposition 4.2) and we deduce an estimate for the L^p-norm of the supremum of an indefinite forward integral (Theorem 4.4). This theorem is one of the main results of this paper and constitutes the fundamental tool for solving the stochastic evolution equation (1.2).

The Skorohod integral is an extension of the Itô integral to the case of anticipating integrands, and it was introduced by Skorohod in [14]. It turns out that this generalization of the Itô integral coincides with the adjoint of the derivative operator on the Wiener space. As a consequence, one can apply the techniques of the Malliavin calculus (see [8]) in order to construct a stochastic calculus for the Skorohod integral. This has been done by Nualart and Pardoux in [10], among others. The Skorohod integral of Hilbert-valued processes with respect to a cylindrical Wiener process has been studied by Grorud and Pardoux in [3]. In Section 2 we present the basic facts on the Malliavin calculus with respect to a cylindrical Wiener process. We need to introduce random variables with values in the space of linear operators $L(H,G)$, where H and G are real and separable Hilbert spaces, and the corresponding Sobolev spaces $\mathbb{D}^{1,2}(L(H,G))$ are more general than the spaces of Hilbert–Schmidt operators $\mathbb{D}^{1,2}(L_2(H,G))$ considered in [3].

The basic estimate for the L^p-norm of a Skorohod integral (that is used in Section 4 in order to control the L^p-norm of the forward integral) is obtained in Section 3. We need to estimate a Skorohod integral of the form $\int_0^t S(t,s)\Phi_s\,dW_s$, where $\{S(t,s),\ t\geq s\}$ is an $\mathscr{F}_t$-measurable random evolution system on a Hilbert space H and $\Phi=\{\Phi_s,\ s\in[0,T]\}$ is an $L_2(U,H)$-valued adapted process. We prove that

$$E\biggl(\sup_{0\leq t\leq T}\biggl|\int_0^t S(t,s)\Phi_s\,dW_s\biggr|_H^p\biggr)\leq C\int_0^T E\|\Phi_s\|_{\mathrm{HS}}^p\,ds, \tag{1.3}$$

assuming that $S(t, s)$ is twice-differentiable in the sense of the Malliavin calculus. The constant C depends on p, T and on the random evolution system $S(t, s)$. This estimate follows from the Itô formula for the Skorohod integral, using some ideas introduced by Hu and Nualart [4]. The semigroup property of the system $S(t, s)$ allows showing this estimate using only two derivatives of $S(t, s)$.

Inequality (1.3) plus the decomposition of the forward stochastic integral obtained in Section 4 allows us to deduce an estimate similar to (1.3) for the forward integral (see Theorem 4.4). Using this, we prove in Section 5 a result on the existence and uniqueness of a mild solution to (1.2) (Theorem 5.4).

Finally, Section 6 contains an example that satisfies the assumptions of our results. Namely, a random evolution system generated by a family of random second order differential operators.

2. Preliminaries.

In this section we present some basic elements of the stochastic calculus of variations with respect to a cylindrical Wiener process. For a more detailed account on this subject we refer to [3].

Let U be a real and separable Hilbert space. Suppose that W is a cylindrical Wiener process over U defined on a complete probability space $(\Omega, \mathscr{F}, P)$. That is, $W = \{W_t(h), h \in U, t \in [0, T]\}$ is a zero-mean Gaussian family such that

$$E(W_t(h_1)W_s(h_2)) = (s \wedge t)\langle h_1, h_2\rangle_U,$$

for all h_1, $h_2 \in U$ and $s, t \in [0, T]$. We will also assume that the σ-field $\mathscr{F}$ is generated by W.

If $u \in L^2([0, T]; U)$ we set $W(u) = \sum_{j=1}^{\infty} \int_0^T \langle u(s), e_j\rangle_U \, dW_s(e_j)$, where $\{e_j, \; j \geq 1\}$ is a complete orthonormal system on U. We will also use the notation $W(u) = \int_0^T \langle u_t, dW_t\rangle_U$.

If U_1 and U_2 are two real and separable Hilbert spaces we will denote by $U_1 \otimes U_2$ its tensor product which is isometric to the space $L_2(U_2, U_1)$ of Hilbert–Schmidt operators from U_2 to U_1.

Let K be a real and separable Hilbert space. For any $p \geq 1$ we can introduce the Sobolev space $\mathbb{D}^{1,p}(K)$ of K-valued random variables in the following way. If F is a smooth K-valued random variable of the form

$$F = \sum_{j=1}^{m} f_j(W(u_1), \ldots, W(u_m))b_j, \tag{2.1}$$

where $u_i \in L^2([0, T]; U)$, $b_j \in K$ and $f_j \in C_b^{\infty}(\mathbb{R}^m)$ (f is an infinitely differentiable function such that f is bounded together with all its partial derivatives), then the derivative of F is defined as

$$DF = \sum_{j=1}^{m} \sum_{i=1}^{m} \frac{\partial f_j}{\partial x_i}(W(u_1), \ldots, W(u_m))b_j \otimes u_i.$$

So DF is a smooth random variable with values in $L^2([0, T]; L_2(U, K))$. Then $\mathbb{D}^{1,p}(K)$ is the completion of the class of smooth K-valued random variables, denoted by $\mathscr{S}_K$, with respect to the norm

$$\|F\|_{1,p}^p = E|F|_K^p + E\left(\int_0^T \|D_t F\|_{\mathrm{HS}}^2 \, dt\right)^{p/2}.$$

For each $p \geq 1$ the operator D is closable from $\mathscr{S}_K \subset L^p(\Omega; K)$ into the space $L^p(\Omega; L^2([0,T]; L_2(U,K)))$ and for $F \in \mathbb{D}^{1,p}(K)$ we have that $DF \in L^p(\Omega; L^2([0,T]; L_2(U,K)))$.

More generally, for any natural $n \geq 1$, the Sobolev space $\mathbb{D}^{n,p}(K)$ is defined as the completion of $\mathscr{S}_K$ by the norm

$$\|F\|_{n,p}^p = E|F|_K^p + \sum_{j=1}^n E\left(\int_{[0,T]^j} \|D_{t_1} \cdots D_{t_j} F\|_{L_2(U^{\otimes j},K)}^2 dt_1 \cdots dt_j\right)^{p/2}.$$

In particular, given two real and separable Hilbert spaces H and G we can consider $K = L_2(H,G)$, and in this case, for any F in the space $\mathbb{D}^{1,p}(L_2(H,G))$ we have that $DF \in L^p(\Omega; L^2([0,T]; L_2(H, L_2(U,G))))$ because $L_2(U, L_2(H,G)) \cong L_2(H, L_2(U,G))$.

We want to introduce Sobolev spaces of random variables with values in the space $L(H,G)$ of linear bounded operators from H in G. Taking into account that $L(H,G)$ is a nonseparable Banach space, we cannot use the preceding construction.

For $p \geq 1$, $L^p(\Omega; L(H,G))$ denotes the space of all functions $F: \Omega \to L(H,G)$ such that:

(a) For every $h \in H$, $F(h)$ is a G-valued integrable random variable and there exists an element $EF \in L(H,G)$ such that $E(F(h)) = (EF)(h)$ for all $h \in H$. That is, F is Bochner integrable (see [1], page 24).

(b) $\int_\Omega \|F\|_{L(H,G)}^p \, dP < \infty$.

For more details on this definition see [1]. The following definition provides a natural way to define derivatives of $L(H,G)$-valued random variables. In order to simplify the exposition, we will restrict ourselves to the case $p = 2$. This will be sufficient for the subsequent application of these notions.

DEFINITION 2.1. Let $F \in L^2(\Omega; L(H,G))$. We say that F belongs to the Sobolev space $\mathbb{D}^{1,2}(L(H,G))$ if the following conditions hold:

(a) For every $h \in H$, $F(h)$ belongs to $\mathbb{D}^{1,2}(G)$.

(b) There exists an element $DF \in L^2([0,T] \times \Omega; L(H, L_2(U,G)))$ such that for every $h \in H$ we have

$$D_t(F(h)) = (D_t F)(h) \tag{2.2}$$

for almost all $(t,\omega) \in [0,T] \times \Omega$.

REMARKS. $\mathbb{D}^{1,2}(L_2(H,G)) \subset \mathbb{D}^{1,2}(L(H,G))$, and for any F in $\mathbb{D}^{1,2}(L_2(H,G))$ we have DF belongs to the space $L^2([0,T] \times \Omega; L_2(H, L_2(U,G)))$.

In general we have that the inclusion $\mathbb{D}^{1,2}(L_2(H,G)) \subset \mathbb{D}^{1,2}(L(H,G))$ is strict. For instance, if $G = H$ and F is the identity operator I_H on H, then $I_H \notin \mathbb{D}^{1,2}(L_2(H,H))$ because I_H is not a Hilbert–Schmidt operator, but $I_H(h) = h \in \mathbb{D}^{1,2}(H)$ for any $h \in H$ and $DI_H = 0$.

We will make use of the following technical lemmas concerning the derivative operator. We will denote by H, G, J real and separable Hilbert spaces.

LEMMA 2.2. *If $\varphi \in L_2(J,H)$ and $F \in \mathbb{D}^{1,2}(L(H,G))$ then we have $F\varphi \in \mathbb{D}^{1,2}(L_2(J,G))$ and $D(F\varphi) = (DF)\varphi$.*

PROOF. Let $\{j_k, \ k \geq 1\}$ be a complete orthonormal system on J. Clearly $F\varphi$ is a random element with values in $L_2(J,G)$ and $\|F\varphi\|_{\mathrm{HS}} \leq \|F\|_{L(H,G)}\|\varphi\|_{\mathrm{HS}}$ which implies that $F\varphi \in L^2(\Omega; L_2(J,G))$. On the other hand, for each $k \geq 1$ we have $(F\varphi)(j_k) \in \mathbb{D}^{1,2}(G)$ and $D[(F\varphi)(j_k)] = (DF)(\varphi(j_k))$. Hence

$$\begin{aligned} E\int_0^T \sum_{k=1}^{\infty} \|D_s((F\varphi)(j_k))\|^2_{L_2(U,G)}\,ds &= E\int_0^T \sum_{k=1}^{\infty} \|(D_sF)(\varphi(j_k))\|^2_{L_2(U,G)}\,ds \\ &\leq E\int_0^T \|D_sF\|^2_{L(H,L_2(U,G))}\,ds\|\varphi\|^2_{L_2(J,H)} < \infty, \end{aligned}$$

which implies the result. □

LEMMA 2.3. *Consider a smooth $L_2(J,H)$-valued random element φ and let $F \in \mathbb{D}^{1,2}(L(H,G))$. Then $F\varphi \in \mathbb{D}^{1,2}(L_2(J,G))$, and*

$$D(F\varphi) = (DF)\varphi + F(D\varphi). \tag{2.3}$$

PROOF. Without loss of generality, we can assume that $\varphi = Rb$ where $b \in L_2(J,H)$ and R is a real-valued smooth random variable of the form $R = f(W(u_1), \dots, W(u_m))$ with $u_i \in L^2([0,T];U)$ and $f \in C_b^\infty(\mathbb{R}^m)$. Clearly, the composition $F\varphi$ belongs to $L^2(\Omega; L_2(J,G))$.

Let us first prove that the right-hand side of (2.3) belongs to $L^2([0,T] \times \Omega; L_2(J, L_2(U,G)))$. We have that $D\varphi$ is a bounded random element with values in $L^2([0,T]; L_2(J, L_2(U,H)))$ given by $D\varphi = b \otimes DR$ (i.e., for each $j \in J$, $(D\varphi)(j) = b(j) \otimes DR$). As a consequence,

$$F(D\varphi) = (Fb) \otimes DR,$$

where $Fb \in L^2(\Omega; L_2(J,G))$, and

$$\begin{aligned} E\int_0^T \|F(D_s\varphi)\|^2_{L_2(U, L_2(J,G))}\,ds &= E\int_0^T \|D_sR\|^2_U \ \|Fb\|^2_{L_2(J,G)}\,ds \\ &\leq C(\varphi)E\|Fb\|^2_{L_2(J,G)} < \infty, \end{aligned}$$

where $C(\varphi)$ is a constant. On the other hand, $(DF)(\varphi) = R(DF)b$, and

$$E\int_0^T \|(D_sF)(\varphi)\|^2_{L_2(J,\,L_2(U,\,G))}\,ds \le \|R\|^2_\infty \|b\|^2_{L_2(J,\,H)} E\int_0^T \|D_sF\|^2_{L(H,\,L_2(U,\,G))}\,ds < \infty.$$

For any $j \in J$ we have that $(F\varphi)(j) = R(Fb)(j)$ belongs to $\mathbb{D}^{1,2}(G)$ and by Lemma 2.2 we can write

$$D[(F\varphi)(j)] = (Fb)(j) \otimes DR + R(DF)(b(j)). \tag{2.4}$$

Hence, it suffices to show that the right-hand side of (2.3) applied to j coincides with the right-hand side of (2.4), and this is true because

$$[(DF)\varphi](j) = R(DF \circ b)(j) = R(DF)(b(j)),$$

and $F(D\varphi)(j) = (Fb)(j) \otimes DR$. $\square$

LEMMA 2.4. *Let $A \in \mathbb{D}^{1,2}(L(H,G))$ and $F \in \mathbb{D}^{1,2}(H)$. Suppose that $\|A\|_{L(H,G)} \le M$ and $|F|_H \le M$ for some constant $M > 0$. Then $AF \in \mathbb{D}^{1,2}(G)$ and*

$$D(AF) = (DA)F + A(DF). \tag{2.5}$$

PROOF. We can find a sequence $\{F_n\}$ of H-valued smooth random variables such that $|F_n|_H \le M + 1$, F_n converges to F in $L^2(\Omega; H)$ and DF_n converges to DF in $L^2([0,T] \times \Omega; L_2(U,H))$.

Clearly $AF \in L^2(\Omega; G)$, and AF_n converges to AF in $L^2(\Omega; G)$. By Lemma 2.3 (with $J = \mathbb{R}$) we deduce that $AF_n \in \mathbb{D}^{1,2}(G)$, and

$$D(AF_n) = (DA)F_n + A(DF_n). \tag{2.6}$$

Finally, from our hypotheses we get that the right-hand side of (2.6) converges to that of (2.5) in $L^2([0,T] \times \Omega; L_2(U,G))$ as n tends to infinity, which completes the proof. $\square$

LEMMA 2.5. *Let $A \in \mathbb{D}^{1,2}(L(H,G))$ and $B \in \mathbb{D}^{1,2}(L(J,H))$. Suppose that $\|A\|_{L(H,G)} \le M$ and $\|B\|_{L(J,H)} \le M$ for some constant $M > 0$. Then $AB \in \mathbb{D}^{1,2}(L(J,G))$, and*

$$D(AB) = (DA)B + A(DB).$$

PROOF. Clearly $AB \in L^2(\Omega; L(J,G))$. Fix $j \in J$. We know that $Bj \in \mathbb{D}^{1,2}(H)$ and $|Bj|_H \le M|j|_J$. By Lemma 2.4 we have $AB(j) \in \mathbb{D}^{1,2}(G)$ and

$$D[AB(j)] = (DA)(Bj) + A(DB(j)). \tag{2.7}$$

Finally notice that $(DA)B + A(DB)$ is an element of the space $L^2([0,T] \times \Omega; L(J, L_2(U,G)))$ and $[A(DB) + (DA)B](j)$ coincides with the right-hand side of (2.7). $\square$

For any subinterval $I \subset [0, T]$ we denote by $\mathscr{F}_I$ the σ-field generated by the family of random variables $\{W(u), \operatorname{supp} u \subset I\}$.

LEMMA 2.6. *Let $A \in \mathbb{D}^{1,2}(L(H, G))$, and suppose that A is $\mathscr{F}_I$-measurable for some subinterval $I \subset [0, T]$. Then $D_t A = 0$ for almost all $(t, \omega) \in I^c \times \Omega$.*

PROOF. Let $h \in H$. Then, by hypothesis, $A(h)$ is an $\mathscr{F}_I$-measurable random element belonging to $\mathbb{D}^{1,2}(G)$. This implies that, for every $\varphi \in L^2([0, T])$ such that $\operatorname{supp} \varphi \subset I^c$, $0 = \int_0^T \varphi(s) D_s(A(h))\, ds$. Thus, the fact that H is separable and $D(A(h)) = (DA)(h)$ give the result. □

In the sequel $\{e_i,\ i \geq 1\}$ will denote a complete orthonormal system on U. We will write $D^e F(h) := (DF)(h)(e)$ for any $F \in \mathbb{D}^{1,2}(L(H, G))$, and for each $h \in H$, $e \in U$. Notice that $D^e F$ belongs to $L^2([0, T] \times \Omega; L(H, G))$.

LEMMA 2.7. *Let $A \in \mathbb{D}^{1,2}(L(H, G))$ such that*

$$E \sum_{i=1}^{\infty} \int_0^T \|D_s^{e_i} A\|_{L(H,G)}^2\, ds < \infty. \tag{2.8}$$

Then, the adjoint of A, A^*, belongs to $\mathbb{D}^{1,2}(L(G, H))$ and $D^e A^* = [D^e A]^*$ for each $e \in U$.

PROOF. Clearly A^* belongs to $L^2(\Omega; L(G, H))$. Let $F \in \mathbb{D}^{1,2}(G)$, $g \in G$ and $h \in H$. Then, it is not difficult to see that $\langle F, g\rangle_G h \in \mathbb{D}^{1,2}(H)$ and $D(\langle F, g\rangle_G h) = h \otimes [DF]^*(g)$. Hence $\langle A^*(g), h\rangle_H h = \langle g, A(h)\rangle_G h \in \mathbb{D}^{1,2}(H)$ and

$$D(\langle A^*(g), h\rangle_H h) = h \otimes [D(A(h))]^*(g) = \langle [DA]^*(g), h\rangle_H h. \tag{2.9}$$

This implies that $A^*(g)$ belongs to $\mathbb{D}^{1,2}(H)$, and $D(A^*(g)) = (DA)^*(g)$. Finally, we have to show that $(DA)^*$ belongs to $L^2([0, T] \times \Omega; L(G, L_2(U, H)))$. This follows from condition (2.8):

$$E \int_0^T \|(D_s A)^*\|_{L(G, L_2(U,H))}^2 \leq E \sum_{i=1}^{\infty} \int_0^T \|D_s^{e_i} A\|_{L(H,G)}^2\, ds < \infty.$$

Thus the proof is complete. □

As in [3] we will denote by δ_H the adjoint of the derivative operator D acting on $\mathbb{D}^{1,2}(H)$. That is, the domain of δ_H is the space of processes u in $L^2([0, T] \times \Omega; L_2(U, H))$ such that

$$\left| E \int_0^T \langle D_t F, u_t\rangle_{\mathrm{HS}}\, dt \right| \leq c_u \|F\|_{L^2(\Omega; H)},$$

for any smooth H-valued random variable F. Then $\delta_H(u)$ is the element of $L^2(\Omega; H)$ determined by the duality relationship

$$E \int_0^T \langle D_t F, u_t\rangle_{\mathrm{HS}}\, dt = E\langle F, \delta_H(u)\rangle_H,$$

for any $F \in \mathbb{D}^{1,2}(H)$. The operator δ_H is also called the H-Skorohod integral. It is an extension of the Itô stochastic integral of H-valued adapted processes in the sense that $L_a^2([0,T]\times\Omega;\ L_2(U,H)) \subset \operatorname{Dom}\delta_H$, where $L_a^2([0,T]\times\Omega; L_2(U,H))$ denotes the space of adapted processes in $L^2([0,T]\times\Omega; L_2(U,H))$.

We will make use of the following property of the Skorohod integral.

PROPOSITION 2.8. *Let $A \in \mathbb{D}^{1,2}(L(H,G))$ and let B be an $L_2(U,H)$-valued process which belongs to the domain of δ_H. Suppose the following conditions hold:*

(i) $AB \in L^2([0,T]\times\Omega; L_2(U,G))$;
(ii) $A\delta_H(B) \in L^2(\Omega;G)$;
(iii) $E(\sum_{i=1}^{\infty}\int_0^T \|D_s^{e_i}A\|_{L(H,G)}^2\,ds)^2 < \infty$ *and* $B \in L^4([0,T]\times\Omega; L_2(U,H))$.

Then $AB \in \operatorname{Dom}\delta_G$ *and*

$$\delta_G(AB) = A\delta_H(B) - \sum_{i=1}^{\infty}\int_0^T (D_s^{e_i}A)\,B_s(e_i)\,ds. \tag{2.10}$$

PROOF. Note first that by condition (iii) the right-hand side of (2.10) belongs to $L^2(\Omega;G)$. Let F be a smooth G-valued random variable. We can write

$$\begin{aligned} E\int_0^T \langle AB_s, D_sF\rangle_{L_2(U,G)}\,ds &= E\sum_{i=1}^{\infty}\int_0^T \langle AB_s(e_i), D_s^{e_i}F\rangle_G\,ds \\ &= E\sum_{i=1}^{\infty}\int_0^T \langle B_s(e_i), A^*D_s^{e_i}F\rangle_H\,ds \\ &= E\sum_{i=1}^{\infty}\int_0^T \langle B_s(e_i), D_s^{e_i}(A^*F)\rangle_H\,ds \\ &\quad - E\sum_{i=1}^{\infty}\int_0^T \langle B_s(e_i), (D_s^{e_i}A^*)F\rangle_H\,ds, \end{aligned}$$

where $A^* \in \mathbb{D}^{1,2}(L(G,H))$ is the adjoint of A, and we have used Lemma 2.3 in order to compute $D(A^*F)$. Notice that, by Lemma 2.7, $D_s^{e_i}A^* = (D_s^{e_i}A)^*$. Hence, we obtain

$$\begin{aligned} E\int_0^T \langle AB_s, D_sF\rangle_{L_2(U,G)}\,ds &= E\langle \delta_H(B), A^*F\rangle_H \\ &\quad - E\sum_{i=1}^{\infty}\int_0^T \langle (D_s^{e_i}A)(B_se_i), F\rangle_G\,ds \\ &= E(\langle R, F\rangle_G), \end{aligned}$$

where R denotes the right-hand side of (2.10). □

REMARK. Condition (iii) of Proposition 2.8 can be replaced by the following:
(iii)′ $\sum_{i=1}^{\infty} \|D_s^{e_i} A\|^2_{L(H,G)} \le M < \infty$ for all $s \in [0,T]$ and $B \in L^2([0,T] \times \Omega; L_2(U,H))$ for some constant $M > 0$.

The Sobolev spaces $\mathbb{D}^{k,2}(L(H,G))$ for any integer $k \geq 1$ are defined as in Definition 2.1, replacing U by $U^{\otimes k}$ and D by D^k in (2.2). If $F \in \mathbb{D}^{k,2}(L(H,G))$, and $p \geq 2$, we define

$$\|F\|^p_{k,p} := E\|F\|^p_{L(H,G)} + \sum_{j=1}^{k} E\left(\int_{[0,T]^j} \|D^j_{s_1 \cdots s_j} F\|^p_{L(H, L_2(U^{\otimes j}, G))} ds_1 \cdots ds_j\right)^{p/2}.$$

Let us recall Itô's formula for anticipating Hilbert-valued processes (see [3], Proposition 4.10). We will use the notation

$$\mathbb{L}^{k,p}(J) = L^p([0,T]; \mathbb{D}^{k,p}(J))$$

for any $p \geq 1$, k a positive integer and J a real and separable Hilbert space. For any $B \in \operatorname{Dom} \delta_H$ we will write $\delta_H(B) = \int_0^T B_s \, dW_s$.

PROPOSITION 2.9. *Let $\Phi \in C^2(H)$ and let $X = \{X_t, \ t \in [0,T]\}$ be the stochastic process defined by*

$$X_t = X_0 + \int_0^t A_s \, ds + \int_0^t B_s \, dW_s,$$

where we have the following:

(i) $X_0 \in \mathbb{D}^{1,2}(H)$;
(ii) $A \in \mathbb{L}^{1,2}(H)$;
(iii) $B \in \mathbb{L}^{2,4}(L_2(U,H))$.

Then

$$\Phi(X_t) = \Phi(X_0) + \int_0^t \langle \Phi'(X_s), A_s \rangle_H \, ds + \int_0^t \Phi'(X_s) B_s \, dW_s + \frac{1}{2} \int_0^t \langle \Phi''(X_s)(\nabla X)_s, B_s \rangle_{L_2(U,H)} \, ds,$$

with

$$(\nabla X)_t = 2 D_t X_0 + 2 \int_0^t D_t A_s \, ds + 2 \int_0^t D_t B_s \, dW_s + B_t.$$

REMARK. The hypotheses of Proposition 2.9 are slightly more general than those in Proposition 4.10 of [3]. The validity of the Itô's formula under these more general assumptions follows from the finite-dimensional Itô's formula established in [9] under these kind of assumptions.

We will make use of the following Fubini-type theorem for the Skorohod integral whose proof is a straightforward consequence of the duality relationship.

LEMMA 2.10. *Let $u(t,x)$ be an $L_2(U,H)$-valued random field parameterized by $(t,x) \in [0,T] \times G$, where G is a bounded d-dimensional rectangle. Sup-*

pose that $u \in L^2([0,T]\times\Omega\times G)$, and for almost all $x \in G$ the stochastic process $u(\cdot, x)$ belongs to the domain of δ_H. Suppose also that $E\int_G |\delta_H(u(\cdot,x))|_H^2\,dx < \infty$. Then $\{\int_G u(t,x)\,dx, t \in [0,T]\}$ belongs to the domain of δ_H and

$$\int_0^T \left(\int_G u(t,x)\,dx\right) dW_t = \int_G \left(\int_0^T u(t,x)\,dW_t\right) dx.$$

3. An estimate for the Skorohod integral. Let H, U be real and separable Hilbert spaces. Let W be a cylindrical Wiener process over U on the time interval $[0,T]$. We will make use of the notation $\Delta = \{(t,s) \in [0,T]^2\colon t \geq s\}$.

DEFINITION 3.1. A random evolution system is a random family of operators $\{S(t,s);\ 0 \leq s \leq t \leq T\}$ on H verifying the following properties:

(i) $S\colon \Delta \times \Omega \to L(H,H)$ is strongly measurable;
(ii) $S(t,s)$ is strongly $\mathscr{F}_t$-measurable for each $t \geq s$;
(iii) For each $\omega \in \Omega$, $\{S(t,s), (t,s) \in \Delta\}$ is an evolution system in the following sense:
(a) $S(s,s) = I$ and $S(t,r) = S(t,s)S(s,r)$ for any $0 \leq r \leq s \leq t \leq T$.
(b) For all $h \in H$, $(t,s) \mapsto S(t,s)h$ is continuous from Δ into H.

Let us introduce the following hypotheses on a given random evolution system.

(H1) For each $(t,s) \in \Delta$, $S(t,s) \in \mathbb{D}^{2,2}(L(H,H))$, and $\int_0^t \|S(t,s)\|_{2,p}^p\,ds < \infty$ for all $p \geq 2$.
(H2) There is a version of $D_r S(t,s)$ such that for all $\omega \in \Omega$ and $h \in H$, the limit

$$D_s^- S(t,s)(h) = \lim_{\varepsilon \downarrow 0} D_s S(t, s-\varepsilon)(h)$$

exists in $L_2(U,H)$ and $D_s^- S(t,s)$ belongs to $\mathbb{D}^{1,2}(L(H, L_2(U,H)))$.
(H3) There is a constant $M > 0$ such that the following estimates hold for all $t \geq s \geq r$:
(H3a) $\|S(t,s)\|_{L(H,H)} \leq M$;
(H3b) $\|D_s S(t,r)\|_{L(H, L_2(U,H))} \leq M$;
(H3c) $\sum_{i=1}^\infty \|D_r^{e_i} D_s^- S(t,s)\|_{L(H, L_2(U,H))}^2 \leq M^2$.

REMARK. Fix $t > s-\varepsilon > r$, $\varepsilon > 0$. From property (a) of a random evolution system we have

$$S(t,r) = S(t, s-\varepsilon)S(s-\varepsilon, r).$$

Suppose that the random evolution system $S(t,s)$ satisfies the hypotheses (H1), (H2) and (H3). Applying Lemmas 2.5 and 2.6 yields

$$D_s S(t,r) = D_s S(t, s-\varepsilon) S(s-\varepsilon, r).$$

Now letting $\varepsilon \downarrow 0$ and using property (b) in the definition of a random evolution system, (H2) and (H3), we obtain

$$D_s S(t,r) = D_s^- S(t,s) S(s,r).$$

Indeed, for any $h \in H$ we have

$$\begin{aligned}
&\|D_s S(t, s-\varepsilon)S(s-\varepsilon, r)(h) - D_s^- S(t,s)S(s,r)(h)\|_{\mathrm{HS}} \\
&\quad \le \|D_s S(t, s-\varepsilon)(S(s-\varepsilon, r)(h) - S(s,r)(h))\|_{\mathrm{HS}} \\
&\qquad + \|[D_s S(t, s-\varepsilon) - D_s^- S(t,s)]S(s,r)(h)\|_{\mathrm{HS}} \\
&\quad \le \|D_s S(t, s-\varepsilon)\|_{L(H, L_2(U,H))} |(S(s-\varepsilon, r) - S(s,r))(h)|_H \\
&\qquad + \|[D_s S(t, s-\varepsilon) - D_s^- S(t,s)]S(s,r)(h)\|_{\mathrm{HS}},
\end{aligned}$$

and this converges to zero as ε tends to zero due to hypotheses (H2) and (H3).

Let us now prove the following theorem.

THEOREM 3.2. *Fix $p \ge 2$ and $\alpha \in [0, \frac{1}{2})$. Let $\Phi = \{\Phi_t,\ t \in [0,T]\}$ be an $L_2(U,H)$-valued adapted process such that $E\int_0^T \|\Phi_s\|_{\mathrm{HS}}^p\, ds < \infty$. Let $S(t,s)$ be a random evolution system satisfying the hypotheses* (H1), (H2) *and* (H3). *Then the $L_2(U,H)$-valued process $\{(t-s)^{-\alpha} S(t,s)\Phi_s I_{[0,t]}(s),\ s \in [0,T]\}$ belongs to the domain of δ_H for almost all $t \in [0,T]$, and we have*

$$E\left|\int_0^t (t-s)^{-\alpha} S(t,s)\Phi_s\, dW_s\right|_H^p \le C\int_0^t (t-s)^{-2\alpha} E\|\Phi_s\|_{\mathrm{HS}}^p\, ds, \tag{3.1}$$

for some constant $C > 0$ which depends on T, p, α and on the evolution system $S(t,s)$.

PROOF. Let us denote by $\mathscr{E}$ the class of $L_2(U,H)$-valued elementary adapted processes of the form

$$\Phi_s = \sum_{k=1}^n \sum_{i=1}^n f_{ik}(W(u_1^i), \dots, W(u_n^i)) b_k I_{(t_i, t_{i+1}]}(s), \tag{3.2}$$

where $f_{ik} \in C_b^\infty(\mathbb{R}^n)$, $b_k \in L_2(U,H)$, $0 < t_1 < \cdots < t_{n+1} < T$ and $\operatorname{supp} u_j^i \subset [0, t_i]$. Let Φ be an $L_2(U,H)$-valued adapted process such that $E\int_0^T \|\Phi_s\|_{\mathrm{HS}}^p\, ds < \infty$. We can find a sequence Φ^n of elementary adapted processes in the class $\mathscr{E}$ satisfying

$$\lim_n E\int_0^T \|\Phi_s^n - \Phi_s\|_{\mathrm{HS}}^p\, ds = 0.$$

This implies that

$$\lim_n E\int_0^T \left(\int_0^t (t-s)^{-2\alpha} \|\Phi_s^n - \Phi_s\|_{\mathrm{HS}}^p\, ds\right) dt = 0.$$

By choosing a subsequence we have that for all $t \in [0,T]$ out of a set of zero Lebesgue measure,

$$\lim_i E\int_0^t (t-s)^{-2\alpha} \|\Phi_s^{n_i} - \Phi_s\|_{\mathrm{HS}}^p\, ds = 0.$$

Hence, we can assume that Φ is of the form (3.2).

We are going to apply Itô's formula to the fuction $F(x) = |x|_H^p$ on H. Recall that

$$F'(x) = p|x|_H^{p-2} x$$

and

$$F''(x) = p(p-2)|x|_H^{p-4} x \otimes x + p|x|_H^{p-2} I_H.$$

Fix $t_0 > t_1$ in $[0, T]$, and define

$$B_s = (t_0 - s)^{-\alpha} S(t_1, s)\Phi_s I_{[0,t_1]}(s).$$

From hypothesis (H1) it follows that $B \in \mathbb{L}^{2,q}(L_2(U, H))$, for each $q \geq 2$. As a consequence, we can apply Itô's formula (Proposition 2.9) to the process $X_t = \int_0^t B_s \, dW_s$, and to the function $F(x) = |x|_H^p$. In this way we obtain, for each $t \in [0, t_1]$,

$$\begin{aligned} |X_t|_H^p = & \int_0^t p|X_s|_H^{p-2} \langle X_s, B_s \, dW_s \rangle_H \\ & + \tfrac{1}{2} \int_0^t \langle F''(X_s) \left(B_s + 2\int_0^s D_s B_r \, dW_r \right), B_s \rangle_{\mathrm{HS}} \, ds. \end{aligned} \tag{3.3}$$

We claim that the Skorohod integral appearing in (3.3), that can be written as $p \int_0^t |X_s|_H^{p-2} B_s^*(X_s) \, dW_s$, has zero expectation. This might not be true because this Skorohod integral is defined by localization. Nevertheless, our assumptions imply that the process $|X_s|_H^{p-2} B_s^*(X_s)$ belongs to $\mathbb{L}^{1,2}(U) \subset \operatorname{Dom} \delta$. In fact, we have, by [3], Proposition 4.1,

$$\begin{aligned} & E \int_0^T |X_s|_H^{2(p-2)} |B_s^*(X_s)|_U^2 \, ds \\ & \quad \leq C_1 E \int_0^T |X_s|_H^{2(p-1)} \, ds \\ & \quad \leq C_2 \left(1 + E \left(\int_0^T \int_0^T \|D_\theta B_s\|_{L_2(U \otimes U, H)}^2 \, d\theta \, ds \right)^{p-1} \right) < \infty, \end{aligned}$$

due to hypotheses (H1) and (H2). Notice that hypothesis (H1) implies that $\int_0^T E|X_s|_H^p \, ds < \infty$ for any $p \geq 2$. On the other hand we have,

$$\begin{aligned} & E \int_0^T \int_0^T \|D_\theta[|X_s|_H^{p-2} B_s^*(X_s)]\|_{U \otimes U}^2 \, d\theta \, ds \\ & \quad \leq C \Bigg[\left(E \int_0^T |X_s|_H^{4(p-2)} \, ds \right)^{1/2} \left(E \int_0^T \left(\int_0^T \|D_\theta X_s\|_{L_2(U,H)}^2 \, d\theta \right)^2 ds \right)^{1/2} \\ & \qquad + \left(E \int_0^T |X_s|_H^{4(p-1)} \, ds \right)^{1/2} \left(E \int_0^T \left(\int_0^T \|D_\theta B_s\|_{L_2(U \otimes U, H)}^2 \, d\theta \right)^2 ds \right)^{1/2} \Bigg] \\ & \quad < \infty, \end{aligned}$$

where we use the fact that $X \in \mathbb{L}^{1,4}(H)$ (see [15], Theorem 2.1). Thus, we have proved that $|X_s|_H^{p-2} B_s^*(X_s)$ belongs to $\mathbb{L}^{1,2}(U)$.

Notice that $\|F''(x)\|_{L(H,H)} \le p(p-1)|x|_H^{p-2}$. Hence, taking expectations in (3.3) yields

$$E|X_t|_H^p \le \frac{p(p-1)}{2} E\int_0^t |X_s|_H^{p-2}\left(\|B_s\|_{\mathrm{HS}}^2 + 2\|B_s\|_{\mathrm{HS}}\left\|\int_0^s D_s B_r\, dW_r\right\|_{\mathrm{HS}}\right) ds.$$

Using the inequality $2\|a\|\,\|b\| \le \|a\|^2 + \|b\|^2$ we obtain

$$\begin{aligned} E|X_t|_H^p \le{}& p(p-1)E\int_0^t |X_s|_H^{p-2}\|B_s\|_{\mathrm{HS}}^2\, ds \\ &+ \frac{p(p-1)}{2} E\int_0^t |X_s|_H^{p-2}\left\|\int_0^s D_s B_r\, dW_r\right\|_{\mathrm{HS}}^2 ds. \end{aligned}$$

Now we substitute B_s by its definition and we use the adaptability of Φ_s and Lemmas 2.3 and 2.6 to get

$$\begin{aligned} E|X_t|_H^p \le{}& p(p-1)E\int_0^t |X_s|_H^{p-2}(t_0-s)^{-2\alpha}\|S(t_1,s)\Phi_s\|_{\mathrm{HS}}^2\, ds \\ &+ \frac{p(p-1)}{2} E\int_0^t |X_s|_H^{p-2} \\ &\times \left\|\int_0^s (t_0-r)^{-\alpha}(D_s S(t_1,r))\Phi_r\, dW_r\right\|_{\mathrm{HS}}^2 ds. \end{aligned} \tag{3.4}$$

Applying Hölder's inequality to the expectation in the right-hand side of (3.4) yields

$$\begin{aligned} E|X_t|_H^p \le{}& p(p-1)\int_0^t (E|X_s|_H^p)^{1-2/p}(t_0-s)^{-2\alpha}(E\|S(t_1,s)\Phi_s\|_{\mathrm{HS}}^p)^{2/p}\, ds \\ &+ \frac{p(p-1)}{2}\int_0^t (E|X_s|_H^p)^{1-2/p} \\ &\times \left(E\left\|\int_0^s (t_0-r)^{-\alpha}(D_s\, S(t_1,r))\Phi_r\, dW_r\right\|_{\mathrm{HS}}^p\right)^{2/p} ds \\ ={}& \int_0^t (E|X_s|_H^p)^{1-2/p} A_s\, ds. \end{aligned}$$

Then the lemma proved in [16] implies that

$$E|X_t|_H^p \le \left(\frac{2}{p}\int_0^t A_s\, ds\right)^{p/2},$$

that is,

$$\begin{aligned} E|X_t|_H^p \le \Bigg\{& 2(p-1)\int_0^t (t_0-s)^{-2\alpha}(E\|S(t_1,s)\Phi_s\|_{\mathrm{HS}}^p)^{2/p}\, ds \\ &+ (p-1)\int_0^t \left(E\left\|\int_0^s (t_0-r)^{-\alpha}(D_s S(t_1,r))\Phi_r\, dW_r\right\|_{\mathrm{HS}}^p\right)^{2/p} ds\Bigg\}^{p/2} \end{aligned} \tag{3.5}$$

$$\leq 2^{p/2-1}(p-1)^{p/2}\bigg[2^{p/2}\bigg(\int_0^t (t_0-s)^{-2\alpha}(E\|S(t_1,s)\Phi_s\|_{\mathrm{HS}}^p)^{2/p}\,ds\bigg)^{p/2}$$
$$+\, t^{p/2-1}\int_0^t E\bigg(\bigg\|\int_0^s (t_0-r)^{-\alpha}(D_s S(t_1,r))\Phi_r\,dW_r\bigg\|_{\mathrm{HS}}^p\bigg)\,ds\bigg]$$
$$\leq M^p 2^{p-1}(p-1)^{p/2}\int_0^t (t_0-s)^{-2\alpha}E(\|\Phi_s\|_{\mathrm{HS}}^p)\,ds$$
$$+\, 2^{p/2-1}(p-1)^{(p/2)}t^{(p/2)-1}$$
$$\times \int_0^t E\bigg\|\int_0^s (t_0-r)^{-\alpha}(D_s S(t_1,r))\Phi_r\,dW_r\bigg\|_{\mathrm{HS}}^p\,ds.$$

Using the remark at the beginning of this section, Proposition 2.8 and hypothesis (H3), we can write

$$\bigg\|\int_0^s (t_0-r)^{-\alpha}(D_s S(t_1,r))\Phi_r\,dW_r\bigg\|_{\mathrm{HS}}$$
$$= \bigg\|\int_0^s (t_0-r)^{-\alpha}(D_s^- S(t_1,s))S(s,r)\Phi_r\,dW_r\bigg\|_{\mathrm{HS}}$$
$$= \bigg\|D_s^- S(t_1,s)\int_0^s (t_0-r)^{-\alpha}S(s,r)\Phi_r\,dW_r$$
$$- \sum_{i=1}^\infty \int_0^s (t_0-r)^{-\alpha}(D_r^{e_i}D_s^- S(t_1,s))S(s,r)\Phi_r(e_i)\,dr\bigg\|_{\mathrm{HS}}$$
$$\leq M\bigg|\int_0^s (t_0-r)^{-\alpha}S(s,r)\Phi_r\,dW_r\bigg|_H \tag{3.6}$$
$$+ \sum_{i=1}^\infty \int_0^s (t_0-r)^{-\alpha}\|D_r^{e_i}D_s^- S(t_1,s)\|_{L(H,\,L_2(U,H))}$$
$$\times \|S(s,r)\Phi_r(e_i)\|_H\,dr$$
$$\leq M\bigg|\int_0^s (t_0-r)^{-\alpha}S(s,r)\Phi_r\,dW_r\bigg|_H$$
$$+ M^2\int_0^s (t_0-r)^{-\alpha}\|\Phi_r\|_{\mathrm{HS}}\,dr.$$

Substituting (3.6) into (3.5) yields

$$E|X_t|_H^p \leq C_{M,T,p}\bigg\{\int_0^t (t_0-s)^{-2\alpha}E\|\Phi_s\|_{\mathrm{HS}}^p\,ds$$
$$+ \int_0^t E\bigg|\int_0^s (t_0-r)^{-\alpha}S(s,r)\Phi_r\,dW_r\bigg|_H^p\,ds \tag{3.7}$$
$$+ \int_0^t E\bigg(\int_0^s (t_0-r)^{-\alpha}\|\Phi_r\|_{\mathrm{HS}}\,dr\bigg)^p\,ds\bigg\}.$$

Applying Hölder's inequality [for the integral with respect to $(t_0-r)^{-\alpha}\,dr$] and Fubini's theorem to the last summand in (3.7), and taking $t_1 = t$ we get

$$\begin{aligned} E\left|\int_0^t (t_0-s)^{-\alpha} S(t,s)\Phi_s\,dW_s\right|_H^p &\le C_{M,\,p,\,T,\,\alpha}\left\{\int_0^t (t_0-s)^{-2\alpha} E\|\Phi_s\|_{\mathrm{HS}}^p\,ds + \int_0^t (t_0-s)^{-\alpha} E\|\Phi_s\|_{\mathrm{HS}}^p\,ds \right.\\ &\qquad \left. + \int_0^t E\left|\int_0^s (t_0-r)^{-\alpha} S(s,r)\Phi_r\,dW_r\right|_H^p ds\right\}. \end{aligned}$$

By Gronwall's lemma we deduce

$$E\left|\int_0^t (t_0-s)^{-\alpha} S(t,s)\Phi_s\,dW_s\right|^p \le C\int_0^t (t_0-s)^{-2\alpha} E\|\Phi_s\|_{\mathrm{HS}}^p\,ds, \tag{3.8}$$

where C is a constant depending on T, M, p and α.

Fix $t \in [0, T)$, and take $t_0 = t+1/n$. From (3.8) for $t_0 = t+1/n$ and letting n tend to infinity we deduce that $\{(t-s)^{-\alpha} S(t,s)\Phi_s I_{[0,\,t]}(s),\ s \in [0,T]\}$ belongs to $\mathrm{Dom}\,\delta_H$ and (3.1) holds. The proof of the theorem is complete. □

Let us introduce the following hypothesis on a random evolution system $S(t,s)$ verifying (H1) and (H2).

(H3)′ Conditions (H3a) and (H3c) hold, and moreover, we have

(H3b)′ $\sum_{i=1}^{\infty} \|D_r^{e_i} S(t,s)\|_{L(H,\,H)}^2 \le M^2$, for all $t \ge s, r$ and for some constant $M > 0$.

Notice that (H3b)′ is stronger than (H3b), and it implies that

$$\sum_{i=1}^{\infty} \|D_s^- S(t,s)(e_i)\|_{L(H,\,H)}^2 \le M^2$$

for all $t \ge s$.

The following theorem provides an estimate of the L^p norm of the maximum of a Skorohod integral, and it constitutes the main result of this section.

THEOREM 3.3. *Fix $p > 2$. Let $\Phi = \{\Phi_t,\ t \in [0,T]\}$ be an $L_2(U,H)$-valued adapted process such that $E\int_0^T \|\Phi_s\|_{\mathrm{HS}}^p\,ds < \infty$. Let $S(t,s)$ be a random evolution system satisfying hypotheses* (H1), (H2) *and* (H3)′. *Then the $L_2(U,H)$-valued process $\{S(t,s)\Phi_s I_{[0,\,t]}(s),\ s \in [0,T]\}$ belongs to* $\mathrm{Dom}\,\delta_H$ *and we have*

$$E\left(\sup_{0\le t\le T}\left|\int_0^t S(t,s)\Phi_s\,dW_s\right|_H^p\right) \le CE\int_0^T \|\Phi_s\|_{\mathrm{HS}}^p\,ds,$$

for some constant $C > 0$ which depends on T, p and on the evolution system $S(t,s)$.

PROOF. We will make use of the factorization method in order to handle the supremum in t. Fix $\alpha \in (1/p, 1/2)$. We can write

$$S(t,s)\Phi_s = C_\alpha \int_s^t S(t,r)(t-r)^{\alpha-1} S(r,s)(r-s)^{-\alpha}\Phi_s\, dr, \tag{3.9}$$

where $C_\alpha = \sin \pi\alpha/\pi$. By Theorem 3.2 we know that for all $r \in [0,T]$ a.e., the process $S(r,s)(r-s)^{-\alpha}\Phi_s I_{[0,r]}(s)$ belongs to Dom δ_H. Then applying Proposition 2.8 and using hypothesis (H3)′ we obtain for almost all $r \in [0,t]$,

$$\begin{aligned} &\int_0^r S(t,r)(t-r)^{\alpha-1} S(r,s)(r-s)^{-\alpha}\Phi_s\, dW_s \\ &\quad = S(t,r)(t-r)^{\alpha-1} Y_r \\ &\qquad - \sum_{i=1}^{\infty} \int_0^r (t-r)^{\alpha-1} (D_s^{e_i} S(t,r)) S(r,s)(r-s)^{-\alpha}\Phi_s(e_i)\, ds, \end{aligned} \tag{3.10}$$

where

$$Y_r = \int_0^r S(r,s)(r-s)^{-\alpha}\Phi_s\, dW_s.$$

By Fubini's theorem for anticipating stochastic integrals (see Lemma 2.10) and using (3.9) we obtain

$$\begin{aligned} &\int_0^t S(t,s)\Phi_s\, dW_s \\ &\quad = C_\alpha \int_0^t \left(\int_s^t S(t,r)(t-r)^{\alpha-1} S(r,s)(r-s)^{-\alpha}\Phi_s\, dr \right) dW_s \\ &\quad = C_\alpha \int_0^t \left(\int_0^r S(t,r)(t-r)^{\alpha-1} S(r,s)(r-s)^{-\alpha}\Phi_s\, dW_s \right) dr. \end{aligned} \tag{3.11}$$

Substituting (3.10) into (3.11) yields

$$\begin{aligned} \int_0^t S(t,s)\Phi_s\, dW_s &= C_\alpha \int_0^t (t-r)^{\alpha-1} S(t,r) Y_r\, dr - C_\alpha \int_0^t (t-r)^{\alpha-1} \\ &\quad \times \left(\int_0^r \sum_{i=1}^{\infty} (D_s^{e_i} S(t,r)) S(r,s)(r-s)^{-\alpha}\Phi_s(e_i)\, ds \right) dr. \end{aligned} \tag{3.12}$$

Applying Hölder's inequality to the right-hand side of (3.12) and using hypothesis (H3b)′ yields

$$\begin{aligned} &\sup_{0\le t\le T} \left| \int_0^t S(t,s)\Phi_s\, dW_s \right|_H \\ &\quad \le \frac{M}{\pi} \sup_{0\le t\le T} \int_0^t (t-r)^{\alpha-1} |Y_r|_H\, dr + M^2 \int_0^T \|\Phi_s\|_{\mathrm{HS}}\, ds \\ &\quad \le \frac{M}{\pi} \left(\frac{p-1}{\alpha p - 1} \right)^{1-1/p} T^{\alpha-1/p} \left(\int_0^T |Y_r|_H^p\, dr \right)^{1/p} + M^2 \int_0^T \|\Phi_s\|_{\mathrm{HS}}\, ds, \end{aligned}$$

and hence,

$$E\left(\sup_{0\le t\le T}\left|\int_0^t S(t,s)\Phi_s\,dW_s\right|_H^p\right) \le C_{T,p,\alpha}\left(E\int_0^T |Y_r|_H^p\,dr + E\int_0^T \|\Phi_s\|_{\mathrm{HS}}^p\,ds\right). \tag{3.13}$$

From Theorem 3.2 we deduce

$$E(|Y_t|_H^p) \le C\int_0^t (t-s)^{-2\alpha} E\|\Phi_s\|_{\mathrm{HS}}^p\,ds. \tag{3.14}$$

Finally, substituting (3.14) into (3.13) and using Fubini's theorem we deduce the desired estimation. □

4. The forward integral.

Let U and H be two real and separable Hilbert spaces and let W be a cylindrical Wiener process over U on the time interval $[0,T]$. We will denote by $\{e_i,\ i\ge 1\}$ and $\{h_i,\ i\ge 1\}$ complete orthonormal systems on U and H, respectively.

DEFINITION 4.1. Let $Y\colon [0,T]\times\Omega\to L_2(U,H)$ be a measurable process such that $Y(u)\in L^1([0,T];H)$ a.s. for each $u\in U$. We say that Y belongs to $\mathrm{Dom}\,\delta^-$ if

$$Y^n := n\int_0^T \sum_{i=1}^n Y_s(e_i)(W_{(s+1/n)\wedge T}(e_i) - W_s(e_i))\,ds$$

converges in probability as n tends to infinity. The limit of the sequence Y^n is denoted by $\int_0^T Y_s\,dW_s^-$ and is called the forward integral of Y with respect to W.

The forward integral has been studied by Russo and Vallois in [13] in the case of real-valued processes. From Definition 4.1 it follows that for any process Y belonging to $\mathrm{Dom}\,\delta^-$ and for any $A\in\mathscr{F}$ such that $Y_t(\omega)=0$, $dt\times dP$-a.e. on $[0,T]\times A$ we have

$$\int_0^T Y_s\,dW_s^- = 0 \quad \text{a.s. on } A.$$

The next proposition establishes the relationship between the forward and the Shorohod integrals of a process of the form $\{S(t,s)\Phi_s I_{[0,t]}(s),\ s\in[0,T]\}$ where $S(t,s)$ is a random evolution system and Φ_s is an adapted process.

PROPOSITION 4.2. *Let $\Phi=\{\Phi_t,\ t\in[0,T]\}$ be an $L_2(U,H)$-valued adapted process such that $E\int_0^T\|\Phi_s\|_{\mathrm{HS}}^2\,ds<\infty$. Let $S(t,s)$ be a random evolution system satisfying hypotheses* (H1), (H2) *and* (H3)′. *Then for each $t\in[0,T]$,*

$\{S(t,s)\Phi_s I_{[0,t]}(s),\ s\in[0,T]\}$ *belongs to* Dom δ^- *and*

$$\int_0^t S(t,r)\Phi_r\,dW_r^- = \delta_H(S(t,\cdot)\Phi_\cdot 1_{[0,t]}(\cdot)) + \int_0^t \sum_{i=1}^\infty (D_r^- S(t,r))(e_i)\Phi_r(e_i)\,dr. \tag{4.1}$$

In order to prove (4.1) we first state the following.

LEMMA 4.3. *Let* Φ *and* $S(t,s)$ *be as in Proposition 4.2. Then for each* $t\in[0,T]$, *and each positive integer* $n\ge 1$,

$$\left(\sum_{i=1}^n 1_{[0,(t+1/n)\wedge T]}(\cdot)\int_{(\cdot-1/n)^+}^{\cdot\wedge t} S(t,s)(\Phi_s(e_i)\otimes e_i)\,ds\right)\in \text{Dom}\,\delta_H$$

and

$$\begin{aligned}\sum_{i=1}^n\int_0^{(t+1/n)\wedge T}&\left(\int_{(r-1/n)^+}^{r\wedge t} S(t,s)(\Phi_s(e_i)\otimes e_i)\,ds\right)dW_r\\ &=\sum_{i=1}^n\int_0^t S(t,s)\delta_H(1_{(s,s+1/n]}(\cdot)\Phi_s(e_i)\otimes e_i)\,ds\\ &\quad-\sum_{i=1}^n\int_0^{(t+1/n)\wedge T}\int_{(r-1/n)^+}^{r\wedge t}(D_r^{e_i}S(t,s))\Phi_s(e_i)\,ds\,dr.\end{aligned}\tag{4.2}$$

PROOF. By (H3)$'$ and Proposition 2.8 we have

$$\begin{aligned}\sum_{i=1}^n\int_0^t &S(t,s)\delta_H(1_{(s,s+1/n]}(\cdot)\Phi_s(e_i)\otimes e_i)\,ds\\ &=\sum_{i=1}^n\int_0^t\left(\int_s^{s+1/n} S(t,s)(\Phi_s(e_i)\otimes e_i)\,dW_r\right)ds\\ &\quad+\sum_{i=1}^n\int_0^t\left(\sum_{j=1}^\infty\int_s^{s+1/n}(D_r^{e_j}S(t,s))\Phi_s(e_i)\langle e_i,e_j\rangle_U\,dr\right)ds\\ &=\sum_{i=1}^n\int_0^t\left(\int_s^{s+1/n} S(t,s)(\Phi_s(e_i)\otimes e_i)\,dW_r\right)ds\\ &\quad+\sum_{i=1}^n\int_0^t\int_s^{s+1/n}(D_r^{e_i}S(t,s))\Phi_s(e_i)\,dr\,ds.\end{aligned}$$

Notice that $S(t,s)\in\mathbb{D}^{1,2}(L(H,H))$ and $I_{(s,s+1/n]}(\cdot)\Phi_s(e_i)\otimes e_i$ satisfy the assumptions (i), (ii) and (iii)$'$ of Proposition 2.8. Indeed, we have for all $s\le r\le t$,

$$\sum_{j=1}^\infty\|D_r^{e_j}S(t,s)\|^2_{L(H,H)}\le M^2,$$

due to (H3b)′. Finally, Fubini's theorem for the Skorohod integral (see Lemma 2.10) allows us to conclude the proof of the lemma. □

PROOF OF PROPOSITION 4.2. Fix $t \in (0, T]$. We only need to prove that $\Lambda_n := n\sum_{i=1}^n \int_0^t S(t,s)\Phi_s(e_i)[W_{s+1/n}(e_i) - W_s(e_i)]\,ds$ converges in probability as n tends to infinity to the right-hand side of (4.1). Actually we will show the convergence in $L^2(\Omega)$. Using (4.2) we have

$$\begin{aligned}\Lambda_n &= n\sum_{i=1}^n \int_0^{(t+1/n)\wedge T}\left(\int_{(r-1/n)^+}^{r\wedge t} S(t,s)(\Phi_s(e_i)\otimes e_i)\,ds\right)dW_r\\ &\quad + n\sum_{i=1}^n \int_0^{(t+1/n)\wedge T}\int_{(r-1/n)^+}^{r\wedge t}(D_r^{e_i}S(t,s))\Phi_s(e_i)\,ds\,dr.\end{aligned}$$

Applying Theorem 3.2 with $\alpha = 0$ and $p = 2$ yields

$$\begin{aligned}&E\left| n\sum_{i=1}^n \int_0^{(t+1/n)\wedge T}\left(\int_{(r-1/n)^+}^{r\wedge t} S(t,s)(\Phi_s(e_i)\otimes e_i)\,ds\right)dW_r - \int_0^t S(t,r)\Phi_r\,dW_r\right|_H^2\\ &\quad\le 2E\left|\int_t^{(t+1/n)\wedge T} n\left(\int_{(r-1/n)^+}^{t} S(t,s)\left(\sum_{i=1}^n \Phi_s(e_i)\otimes e_i\right)ds\right)dW_r\right|_H^2\\ &\qquad + 2CE\int_0^t \left\| n\int_{(r-1/n)^+}^{r}\left(\sum_{i=1}^n S(r,s)(\Phi_s(e_i)\otimes e_i)\right)ds - \Phi_r\right\|_{\mathrm{HS}}^2 dr\\ &\quad\le 2E\int_t^{t+1/n}\left\| n\int_{(r-1/n)^+}^{t} S(t,s)\left(\sum_{i=1}^n \Phi_s(e_i)\otimes e_i\right)ds\right\|_{\mathrm{HS}}^2 dr\\ &\qquad + 4CE\int_0^t \left\| n\int_{(r-1/n)^+}^{r} S(r,s)\Phi_s\,ds - \Phi_r\right\|_{\mathrm{HS}}^2 dr\\ &\qquad + 4CE\int_0^t \left\| n\int_{(r-1/n)^+}^{r}\left(\sum_{i=n+1}^\infty S(r,s)(\Phi_s(e_i)\otimes e_i)\right)ds\right\|_{\mathrm{HS}}^2 dr.\end{aligned}$$

This expression can be estimated by

$$\begin{aligned}&2M^2\int_t^{(t+1/n)\wedge T} E\|\Phi_s\|_{\mathrm{HS}}^2\,ds + 4C\int_0^T n\int_{(r-1/n)^+}^{r} E\|S(r,s)\Phi_s - \Phi_r\|_{\mathrm{HS}}^2\,dr\,ds\\ &\quad + 4CM^2E\int_0^T \sum_{i=n+1}^\infty |\Phi_s(e_i)|_H^2\,ds = a_1 + a_2 + a_3.\end{aligned}$$

Then terms a_1 and a_3 clearly converge to zero as n tends to infinity, uniformly with respect to $t \in [0, T]$. The convergence to zero of a_2 as n tends to infinity follows from the estimate

$$a_2 \le 8C(M^2+1)\int_0^T E\|\Phi_s\|_{\mathrm{HS}}^2\,ds,$$

which allows us to approximate Φ by a process in $C([0,T]; L^2(\Omega; L_2(U,H)))$.

In a similar way we can write

$$\begin{aligned}
&\Bigg| n\sum_{i=1}^{n}\int_0^{(t+1/n)\wedge T}\int_{(r-1/n)^+}^{r\wedge t}(D_r^{e_i}S(t,s))\Phi_s(e_i)\,ds\,dr \\
&\qquad\qquad - \sum_{i=1}^{\infty}\int_0^t (D_r^- S(t,r))(e_i)\Phi_r(e_i)\,dr\Bigg|_H \\
&\quad \le \Bigg| n\sum_{i=1}^{n}\int_t^{(t+1/n)\wedge T}\int_{(r-1/n)^+}^{t}(D_r^{e_i}S(t,s))\Phi_s(e_i)\,ds\,dr\Bigg|_H \\
&\qquad + \Bigg|\sum_{i=n+1}^{\infty}\int_0^t (D_r^- S(t,r))(e_i)\Phi_r(e_i)\,dr\Bigg|_H \\
&\qquad + \Bigg|\sum_{i=1}^{n}\int_0^t n\int_{r-1/n}^{r}\big\{(D_r^{e_i}S(t,s))\Phi_s(e_i) \\
&\qquad\qquad\qquad - (D_r^- S(t,r))(e_i)\Phi_r(e_i)\big\}\,ds\,dr\Bigg|_H \\
&\quad = \Lambda_n^1 + \Lambda_n^2 + \Lambda_n^3.
\end{aligned}$$

Clearly Λ_n^1 and Λ_n^2 tend to zero in $L^2(\Omega)$ as n tends to infinity. The term Λ_n^3 can be estimated as follows:

$$\begin{aligned}
\Lambda_n^3 &\le \Bigg|\sum_{i=1}^{n}\int_0^t n\int_{r-1/n}^{r}(D_r^{e_i}S(t,s))(\Phi_s(e_i)-\Phi_r(e_i))\,ds\,dr\Bigg|_H \\
&\quad + \Bigg|\sum_{i=1}^{n}\int_0^t n\int_{r-1/n}^{r}\big[D_r^{e_i}S(t,s)-(D_r^- S(t,r))(e_i)\big]\Phi_r(e_i)\,ds\,dr\Bigg|_H \\
&\le \left(\sum_{i=1}^{\infty}\int_0^t n\int_{r-1/n}^{r}\|D_r^{e_i}S(t,s)\|_{L(H,H)}^2\,ds\,dr\right)^{1/2} \\
&\quad \times\left(\sum_{i=1}^{\infty}\int_0^t n\int_{r-1/n}^{r}|\Phi_s(e_i)-\Phi_r(e_i)|_H^2\,ds\,dr\right)^{1/2} \\
&\quad + \sum_{i=1}^{\infty}\int_0^t n\int_{r-1/n}^{r}|(D_r^- S(t,r))(e_i)(S(r,s)-I)\Phi_r(e_i)|_H\,ds\,dr \\
&\le M\sqrt{t}\left(\int_0^t n\int_{r-1/n}^{r}\|\Phi_s-\Phi_r\|_{\mathrm{HS}}^2\,ds\,dr\right)^{1/2} \\
&\quad + M\sqrt{t}\left(\sum_{i=1}^{\infty}\int_0^t n\int_{r-1/n}^{r}|(S(r,s)-I)\Phi_r(e_i)|_H^2\,ds\,dr\right)^{1/2},
\end{aligned}$$

and this converges to zero uniformly with respect to t in $L^2(\Omega)$ as n tends to infinity. □

Combining Theorem 3.3 and the expression given in Proposition 4.2 for the forward integral, we deduce the following maximal inequality for the forward integral.

THEOREM 4.4. *Fix $p > 2$. Let $\Phi = \{\Phi_t,\ t \in [0, T]\}$ be an $L_2(U, H)$-valued adapted process such that $E\int_0^T \|\Phi_s\|_{\mathrm{HS}}^p\, ds < \infty$. Let $S(t, s)$ be a random evolution system satisfying hypotheses* (H1), (H2) *and* (H3)′. *Then the $L_2(U, H)$-valued process $\{S(t, s)\Phi_s I_{[0,t]}(s),\ s \in [0, T]\}$ belongs to* Dom δ^- *and we have*

$$E\left(\sup_{0\le t\le T}\left|\int_0^t S(t, r)\Phi_r\, dW_r^-\right|_H^p\right) \le C_{S,\,p,\,T}E\int_0^T \|\Phi_s\|_{\mathrm{HS}}^p\, ds,$$

for some constant $C_{S,\,p,\,T} > 0$ depending on T, p and the random evolution system $S(t, s)$.

PROOF. By Proposition 4.2 we know that the $L_2(U, H)$-valued stochastic process $\{S(t, s)\Phi_s I_{[0,\,t]}(s), s \in [0, T]\}$ belongs to Dom δ^- and we have

$$\begin{aligned}\int_0^t S(t, r)\Phi_r\, dW_r^- &= \int_0^t S(t, r)\Phi_r\, dW_r \\ &\quad + \int_0^t \sum_{i=1}^{\infty}(D_r^- S(t, r))(e_i)\Phi_r(e_i)\, dr.\end{aligned} \tag{4.3}$$

Then the result follows from Theorem 3.3 and hypothesis (H3b)′. □

As a consequence of Theorem 4.4, we have the following continuity result.

COROLLARY 4.5. *Let Φ and $S(t, s)$ be as in Theorem 4.4. Then the H-valued process $\{\int_0^t S(t, s)\Phi_s\, dW_s^-,\ t \in [0, T]\}$ has a continuous modification.*

PROOF. Fix $\alpha \in (1/p, 1/2)$ and set

$$Y_r = \int_0^r S(r, s)(r - s)^{-\alpha}\Phi_s\, dW_s.$$

We know that the process Y_r is well defined for almost all r in $[0, T]$. From Proposition 4.2 applied to the process $\{(r - s)^{-\alpha}\Phi_s,\ s \in [0, r]\}$ we deduce that

$$\overline{Y}_r = Y_r + \int_0^r \sum_{i=1}^{\infty}(D_s^- S(r, s))(e_i)(r - s)^{-\alpha}\Phi_s(e_i)\, ds, \tag{4.4}$$

where

$$\overline{Y}_r = \int_0^r S(r, s)(r - s)^{-\alpha}\Phi_s\, dW_s^-.$$

On the other hand, substituting the relation

$$(D_s^{e_i} S(t, r))S(r, s) = (D_s^- S(t, s))(e_i) - S(t, r)(D_s^- S(r, s))(e_i)$$

into (3.12) yields

$$
\begin{aligned}
\int_0^t S(t,s)\Phi_s\,dW_s &= C_\alpha \int_0^t (t-r)^{\alpha-1} S(t,r) Y_r\,dr \\
&\quad - \int_0^t \sum_{i=1}^{\infty} (D_s^- S(t,s))(e_i)\Phi_s(e_i)\,ds \\
&\quad + C_\alpha \int_0^t (t-r)^{\alpha-1} S(t,r) \\
&\qquad \times \left(\int_0^r \sum_{i=1}^{\infty} (D_s^- S(r,s))(e_i)(r-s)^{-\alpha}\Phi_s(e_i)\,ds \right) dr \\
&= C_\alpha \int_0^t (t-r)^{\alpha-1} S(t,r)\overline{Y}_r\,dr \\
&\quad - \int_0^t \sum_{i=1}^{\infty} (D_s^- S(t,s))(e_i)\Phi_s(e_i)\,ds.
\end{aligned} \tag{4.5}
$$

Hence, from Proposition 4.2 we deduce

$$
\int_0^t S(t,s)\Phi_s\,dW_s^- = C_\alpha \int_0^t (t-r)^{\alpha-1} S(t,r)\overline{Y}_r\,dr. \tag{4.6}
$$

By (4.6), we only need to show that the right-hand side of this equation is continuous in t. Fix $0 \le t_0 < t \le T$. Then our hypotheses on the evolution system $S(t,s)$ and the dominated convergence theorem imply that

$$
\begin{aligned}
&\left| \int_0^t (t-r)^{\alpha-1} S(t,r)\overline{Y}_r\,dr - \int_0^{t_0} (t_0-r)^{\alpha-1} S(t_0,r)\overline{Y}_r\,dr \right|_H \\
&\quad \le \left| \int_{t_0}^t (t-r)^{\alpha-1} S(t,r)\overline{Y}_r\,dr \right|_H \\
&\qquad + \left| (S(t,t_0)-I) \int_0^{t_0} (t_0-r)^{\alpha-1} S(t_0,r)\overline{Y}_r\,dr \right|_H \\
&\qquad + \left| S(t,t_0) \int_0^{t_0} [(t-r)^{\alpha-1} - (t_0-r)^{\alpha-1}] S(t_0,r)\overline{Y}_r\,dr \right|_H
\end{aligned}
$$

converges to zero as $t \downarrow t_0$. In a similar way we show that the above expression converges to zero as $t \uparrow t_0$. □

5. Stochastic evolution equations with a random evolution system. In this section we will study nonlinear stochastic equations of the form

$$
X_t = \xi + \int_0^t (A(s)X_s + F(s,X_s))\,ds + \int_0^t B(s,X_s)\,dW_s, \qquad t \in [0,T], \tag{5.1}
$$

where ξ is an H-valued $\mathscr{F}_0$-measurable random variable and W is a cylindrical Wiener process over the Hilbert space U on the time interval $[0,T]$. We will assume the following conditions on the coefficients A, F and B.

(A.1) The mapping F: $[0, T] \times \Omega \times H \to H$ is $\mathscr{P}_T \times \mathscr{B}(H)$-measurable, where $\mathscr{P}_T$ denotes the predictable σ-field of $[0, T] \times \Omega$,

$$|F(t, x) - F(t, y)|_H \le C|x - y|_H,$$
$$|F(t, x)|_H^2 \le C^2(1 + |x|_H^2),$$

for some constant $C > 0$ and for all $x, y \in H$.

(A.2) The mapping B: $[0, T] \times \Omega \times H \to L_2(U, H)$ is $\mathscr{P}_T \times \mathscr{B}(H)$-measurable,

$$\|B(t, x) - B(t, y)\|_{\mathrm{HS}} \le C|x - y|_H,$$
$$\|B(t, x)\|_{\mathrm{HS}}^2 \le C^2(1 + |x|_H^2),$$

for some constant $C > 0$ and for all $x, y \in H$.

(A.3) $\{A(s, \omega),\ s \in [0, T], \omega \in \Omega\}$ is a random family of unbounded operators on H such that Dom $A^*(s) \supset H_0$ where H_0 is a dense subset of H. We assume that $A^*(\cdot)y \in L^2([0, T] \times \Omega; H)$ for all $y \in H_0$, and there exists a random evolution system $S(t, s)$ satisfying hypotheses (H1), (H2) and (H3)′ such that

$$S^*(t, s)A^*(t)y = \frac{d}{dt}S^*(t, s)y \quad \text{for all } y \in H_0.$$

Definition 5.1. We say that an adapted and continuous H-valued process $X = \{X_t,\ t \in [0, T]\}$ such that $E(\sup_{0\le t\le T} |X_t|_H^p) < \infty$ for some $p > 2$ is a mild solution to (5.1) if

$$X_t = S(t, 0)\xi + \int_0^t S(t, s)F(s, X_s)\, ds + \int_0^t S(t, s)B(s, X_s)\, dW_s^- \tag{5.2}$$

for each $t \in [0, T]$, where dW_s^- denotes the forward integral (see Section 4).

Definition 5.2. An adapted and continuous H-valued process $X = \{X_t, t \in [0, T]\}$ such that $E(\sup_{0\le t\le T} |X_t|_H^p) < \infty$ for some $p > 2$ is a weak solution to (5.1) if for each $y \in H_0$ and $t \in [0, T]$ we have

$$\langle X_t, y\rangle_H = \langle \xi, y\rangle_H + \int_0^t \langle A^*(s)y, X_s\rangle_H\, ds + \int_0^t \langle y, F(s, X_s)\rangle_H\, ds + \int_0^t \langle B^*(s, X_s)y, dW_s\rangle_U.$$

Proposition 5.3. *Under assumptions* (A.1), (A.2) *and* (A.3), *any mild solution to (5.1) is a weak solution.*

Proof. For each $n \ge 1$ we define

$$X_t^n = S(t, 0)\xi + \int_0^t S(t, s)F(s, X_s)\, ds + n \sum_{i=1}^n \int_0^t S(t, s)B(s, X_s)(e_i)(W_{s+1/n}(e_i) - W_s(e_i))\, ds.$$

Notice that

$$X_t^n = S(t,s)X_s^n + \int_s^t S(t,r)F(r,X_r)\,dr$$
$$+ n\sum_{i=1}^n \int_s^t S(t,r)B(r,X_r)(e_i)(W_{r+1/n}(e_i) - W_r(e_i))\,dr.$$

We know, by Assumption (A.3), that for all $y \in H_0$, $x \in H$ we have

$$\int_\sigma^t \langle S^*(r,\sigma)A^*(r)y, x\rangle_H\,dr = \langle S^*(t,\sigma)y, x\rangle_H - \langle y, x\rangle_H.$$

Hence, for all $y \in H_0$, we obtain

$$\begin{aligned}
\Gamma_n &:= n\sum_{i=1}^n \int_s^t \int_\sigma^t \Big\langle S^*(r,\sigma)A^*(r)y, \\
&\qquad\qquad B(\sigma,X_\sigma)(e_i)[W_{\sigma+\frac{1}{n}}(e_i) - W_\sigma(e_i)]\Big\rangle_H\,dr\,d\sigma \\
&= n\sum_{i=1}^n \int_s^t \langle S^*(t,\sigma)y - y, \\
&\qquad\qquad B(\sigma,X_\sigma)(e_i)[W_{\sigma+1/n}(e_i) - W_\sigma(e_i)]\rangle_H\,d\sigma \\
&= \langle X_t^n, y\rangle_H - \langle S(t,s)X_s^n, y\rangle_H - \left\langle \int_s^t S(t,r)F(r,X_r)\,dr, y\right\rangle_H \\
&\quad - n\sum_{i=1}^n \int_s^t \langle y, B(r,X_r)(e_i)[W_{r+1/n}(e_i) - W_r(e_i)]\rangle_H\,dr.
\end{aligned} \tag{5.3}$$

On the other hand, applying Fubini's theorem we have

$$\begin{aligned}
\Gamma_n &= n\sum_{i=1}^n \int_s^t \int_s^r \Big\langle S^*(r,\sigma)A^*(r)y, \\
&\qquad\qquad B(\sigma,X_\sigma)(e_i)[W_{\sigma+1/n}(e_i) - W_\sigma(e_i)]\Big\rangle_H\,d\sigma\,dr \\
&= n\sum_{i=1}^n \int_s^t \left\langle A^*(r)y, \int_s^r S(r,\sigma)B(\sigma,X_\sigma)(e_i)[W_{\sigma+1/n}(e_i) - W_\sigma(e_i)]\,d\sigma\right\rangle_H dr \\
&= \int_s^t \left\langle A^*(r)y, X_r^n - S(r,s)X_s^n - \int_s^r S(r,\sigma)F(\sigma,X_\sigma)\,d\sigma\right\rangle_H dr \\
&= \int_s^t \langle A^*(r)y, X_r^n\rangle_H\,dr - \int_s^t \langle A^*(r)y, S(r,s)X_s^n\rangle_H\,dr \\
&\quad - \int_s^t \left\langle A^*(r)y, \int_s^r S(r,\sigma)F(\sigma,X_\sigma)\,d\sigma\right\rangle_H dr \\
&= \int_s^t \langle A^*(r)y, X_r^n\rangle_H\,dr - \langle S(t,s)X_s^n, y\rangle_H + \langle X_s^n, y\rangle_H \\
&\quad - \int_s^t \int_\sigma^t \Big\langle A^*(r)y, S(r,\sigma)F(\sigma,X_\sigma)\Big\rangle_H\,dr\,d\sigma
\end{aligned} \tag{5.4}$$

$$= \int_s^t \langle A^*(r)y, X_r^n \rangle_H \, dr - \langle S(t,s)X_s^n, y\rangle_H + \langle X_s^n, y\rangle_H$$
$$- \int_s^t \langle y, S(t,\sigma)F(\sigma, X_\sigma)\rangle_H \, d\sigma + \int_s^t \langle y, F(\sigma, X_\sigma)\rangle_H \, d\sigma.$$

Comparing (5.3) and (5.4) yields

$$\begin{aligned} \langle X_t^n, y\rangle_H &= \langle X_s^n, y\rangle_H + \int_s^t \langle A^*(r)y, X_r^n\rangle_H \, dr + \int_s^t \langle y, F(r, X_r)\rangle_H \, dr \\ &\quad + n \sum_{i=1}^n \int_s^t \langle y, B(r, X_r)(e_i)[W_{r+1/n}(e_i) - W_r(e_i)]\rangle_H \, dr. \end{aligned} \tag{5.5}$$

We have that, by Proposition 4.2 with $S(\cdot,\cdot) \equiv I_H$, the last summand in (5.5) converges in $L^2(\Omega)$ as n tends to infinity to $\langle \int_s^t B(r, X_r)\, dW_r, y\rangle_H = \int_s^t \langle B^*(r, X_r)y, dW_r\rangle_U$. Then it suffices to show that $\sup_{0\le t\le T} E(|X_t - X_t^n|_H^2)$ converges to zero as n tends to infinity. This is a consequence of the estimates used in the proof of Proposition 4.2. □

THEOREM 5.4. *Let $S(t,s)$ be a random evolution system satisfying hypotheses* (H1), (H2) *and* (H3)′ *and let F and B satisfy* (A.1) *and* (A.2), *respectively. Then (5.1) has a unique mild solution.*

PROOF OF UNIQUENESS. Assume that X and Y are two mild solutions to (5.1). Then, for arbitrary $t \in [0, T]$ and $p > 2$ such that

$$E\Big(\sup_{0\le r\le T} |X_r|_H^p\Big) + E\Big(\sup_{0\le r\le T} |Y_r|_H^p\Big) < \infty,$$

we have

$$\begin{aligned} |X_t - Y_t|_H^p &= \bigg| \int_0^t S(t,r)\{F(r,X_r) - F(r,Y_r)\}\, dr \\ &\qquad + \int_0^t S(t,r)\{B(r,X_r) - B(r,Y_r)\}\, dW_r^- \bigg|_H^p \\ &\le 2^{p-1} \bigg| \int_0^t S(t,r)\{F(r,X_r) - F(r,Y_r)\}\, dr \bigg|_H^p \\ &\qquad + 2^{p-1} \bigg| \int_0^t S(t,r)\{B(r,X_r) - B(r,Y_r)\}\, dW_r^- \bigg|_H^p \\ &\le 2^{p-1} M^p C^p T^{p-1} \int_0^t |X_r - Y_r|_H^p \, dr \\ &\qquad + 2^{p-1} \sup_{s\in[0,T]} \bigg| \int_0^s S(s,r) I_{[0,t]}(r)\{B(r,X_r) - B(r,Y_r)\}\, dW_r^- \bigg|_H^p. \end{aligned}$$

Hence, from Theorem 4.4, we obtain

$$E|X_t - Y_t|_H^p \le 2^{p-1} M^p C^p T^{p-1} \int_0^t E|X_r - Y_r|_H^p \, dr + 2^{p-1} C_{S,\,p,\,T} \int_0^t E\|B(r, X_r) - B(r, Y_r)\|_{\mathrm{HS}}^p \, dr.$$

Therefore, using Hypothesis (A.2), we get

$$\begin{aligned} E|X_t - Y_t|_H^p &\le 2^{p-1} M^p C^p T^{p-1} \int_0^t E|X_r - Y_r|_H^p \, dr \\ &\quad + 2^{p-1} C_{S,\,p,\,T} C^p \int_0^t E|X_r - Y_r|_H^p \, dr \\ &= 2^{p-1} C^p (M^p T^{p-1} + C_{S,\,p,\,T}) \int_0^t E|X_r - Y_r|_H^p \, dr, \end{aligned}$$

which, together with Gronwall's lemma, implies $E|X_t - Y_t|_H^p = 0$, for arbitrary $t \in [0, T]$, and the proof of uniqueness is complete. □

Proof of existence. The proof of existence is similar to that for a deterministic evolution system. We begin an iteration procedure with $X_t^{(0)} = S(t,0)\xi$ and let us define, for $n \ge 1$ and $t \in [0, T]$,

$$X_t^{(n)} = S(t,0)\xi + \int_0^t S(t,r) F(r, X_r^{(n-1)}) \, dr + \int_0^t S(t,r) B(r, X_r^{(n-1)}) \, dW_r^-. \tag{5.6}$$

Using induction on n, it is easy to prove that assumptions (A.1) and (A.2), Theorem 4.4 and Corollary 4.5 imply that $X^{(n)}$ is an adapted and continuous H-valued process such that

$$\sup_{t \in [0,\,T]} E|X_t^{(n)}|_H^p < \infty.$$

Computations similar to those in the first step of this proof and Theorem 4.4 yield

$$\sum_{n=0}^{\infty} E \sup_{t \in [0,\,T]} |X_t^{(n+1)} - X_t^{(n)}|_H^p < \infty. \tag{5.7}$$

Therefore, from the Borel-Cantelli lemma, the sequence $\{X^{(n)},\ n \in \mathbb{N}\}$ is uniformly convergent in $[0, T]$, for almost all ω. Denote the limit by X_t. Since X is the uniform limit of a sequence of adapted and continuous H-valued processes, it is also adapted and continuous. The estimate (5.7) implies that X belongs to $L^p([0, T] \times \Omega)$ and that $\{X^{(n)}, n \in \mathbb{N}\}$ also converges to X in $L^p([0, T] \times \Omega)$. Finally, from (5.6) and Theorem 4.4, it is easy to show that X is a mild solution of (5.1) and so the proof is complete. □

Remark. The existence of a mild solution still holds if we suppose that conditions (A.1), (A.2) and (A.3) are true locally. That is, we assume that for all n, (A.1) and (A.2) are satisfied for any $x, y \in H$ with $|x|_H \le n$ and $|y|_H \le n$, and with some constant C_n, and on the other hand, we also assume that the random evolution system $S(t,s)$ satisfies (H1), (H2) and (H3)′ locally. This means that there exists a sequence $\{\Omega_k,\ k\in\mathbb{N}\}\subset\mathscr{F}$ and a sequence $\{S^k,\ k\in\mathbb{N}\}$ such that $\Omega_k\uparrow\Omega$, and for each k, $S=S^k$ on Ω_k a.s., and $S^k(t,s)$ is a random evolution system satisfying conditions (H1), (H2) and (H3)′.

6. Stochastic partial differential equations with random generators. Let O be a domain in $\mathbb{R}^n$ and consider the Hilbert space $H=L^2(O)$. As in the previous sections, W will be a cylindrical Wiener process over a Hilbert space U on the time interval $[0,T]$.

In this section we will first provide sufficient conditions for a random operator Λ on $L^2(O)$ given by a random kernel $f(x,y,\omega)$ to be in $\mathbb{D}^{1,2}(L(H,H))$.

LEMMA 6.1. *Let $f\colon O\times O\to\mathbb{R}_+$ be a measurable function such that the following hold:*

(i) *$f(x,\cdot)\in L^2(O)$ for all $x\in O$;*
(ii) *$\sup_{x\in O}\int_O f(x,y)\,dy<\infty$ and $\sup_{x\in O}\int_O f(y,x)\,dy<\infty$.*

Then the mapping $\Lambda\colon L^2(O)\to L^2(O)$ given by

$$(\Lambda g)(x)=\int_O f(x,y)g(y)\,dy$$

is a bounded linear operator such that

$$\|\Lambda\|_{L(H,H)}\le\left(\sup_{x\in O}\int_O f(x,y)\,dy\right)^{1/2}\left(\sup_{y\in O}\int_O f(x,y)\,dx\right)^{1/2}.$$

PROOF. This lemma is an immediate consequence of Fubini's theorem and Schwarz's inequality:

$$\begin{aligned}\int_O|(\Lambda g)(x)|^2\,dx&=\int_O\left|\int_O f(x,y)g(y)\,dy\right|^2dx\\&\le\int_O\left(\int_O f(x,y)\,dy\right)\left(\int_O f(x,y)g^2(y)\,dy\right)dx\\&\le\left(\sup_{x\in O}\int_O f(x,y)\,dy\right)\left(\sup_{y\in O}\int_O f(x,y)\,dx\right)\|g\|^2_{L^2(O)}.\end{aligned}$$ □

LEMMA 6.2. *Let $f\colon O\times O\times\Omega\to\mathbb{R}_+$ be a random measurable function verifying the following conditions:*

(i) *$f(x,\cdot)\in L^2(O)$ for every $x\in O$ a.s.;*
(ii) *there exist two nonnegative random variables M_1, M_2 such that*

$$\sup_{z\in O}\int_O f(z,y)\,dy\le M_1\quad\text{a.s.},$$
$$\sup_{z\in O}\int_O f(y,z)\,dy\le M_2\quad\text{a.s.}$$

and $E(M_1^p)<\infty$, $E(M_2^p)<\infty$ for some $p\ge 2$.

Then the random operator $\Lambda(\omega)$ on H defined by

$$(\Lambda(\omega)g)(x) = \int_O f(x, y, \omega)g(y)\, dy$$

belongs to the space $L^p(\Omega; L(H, H))$.

PROOF. First notice that by Lemma 6.1 for each $\omega \in \Omega$ a.s., $\Lambda(\omega)$ is a bounded linear operator on $H = L^2(O)$ and $\|\Lambda\|_{L(H,H)} \le (M_1 M_2)^{1/2}$ a.s. Then the result follows from the fact that f is measurable and we have

$$E\|\Lambda\|^p_{L(H,H)} \le (E(M_1^p)E(M_2^p))^{1/2} < \infty. \qquad \square$$

We can state a Hilbert-valued version of Lemma 6.2 whose proof would be identical.

LEMMA 6.3. *Let G be a real and separable Hilbert space. Consider a measurable function F: $O \times O \times \Omega \to G$ verifying the following conditions:*

(i) *$F(x, \cdot) \in L^2(O; G)$ for every $x \in O$ a.s.;*
(ii) *there exist two nonnegative random variables M_1 and M_2 such that*

$$\sup_{z\in O} \int_O |F(y, z)|_G \, dy \le M_1 \quad \text{a.s.},$$

$$\sup_{z\in O} \int_O |F(z, y)|_G \, dy \le M_2 \quad \text{a.s.}$$

and $E(M_1^p) < \infty$, $E(M_2^p) < \infty$.

Then the random operator from H to $L^2(O; G) \cong L_2(G, H)$ defined by

$$(\Lambda(\omega)g)(x) = \int_O F(x, y, \omega)g(y)\, dy$$

belongs to the space $L^p(\Omega; L(H, L^2(O; G)))$ and

$$\|\Lambda\|_{L(H, L^2(O;G))} \le (M_1 M_2)^{1/2}.$$

LEMMA 6.4. *Let f: $O \times O \times \Omega \to \mathbb{R}_+$ be a measurable mapping verifying the hypotheses of Lemma 6.2. Assume, in addition, that $f(x, y) \in \mathbb{D}^{1,2}$ for each $x, y \in O$, and that there exists a version of the derivative $D_r f(x, y)$ which is measurable from $[0, T] \times O \times O \times \Omega$ into U and verifies the following:*

(i) *$Df(x, \cdot) \in L^2([0, T] \times O \times \Omega; U)$ for all $x \in O$;*
(ii) *$\sup_{z\in O} \int_O |D_r f(x, z)|_U \, dx \le a_1(r)$ a.s., $\sup_{z\in O} \int_O |D_r f(z, x)|_U \, dx \le a_2(r)$ a.s., where $a_1(r)$ and $a_2(r)$ are nonnegative measurable processes such that $E\int_0^T (a_1(r))^2\, dr < \infty$, $E\int_0^T (a_2(r))^2\, dr < \infty$.*

Then the random operator $\Lambda(\omega)$: $L^2(O) \to L^2(O)$ defined by

$$(\Lambda g)(x) = \int_O f(x, y)g(y)\, dy \tag{6.1}$$

belongs to $\mathbb{D}^{1,2}(L(H,H))$ *and for all* (r,ω) *almost everywhere,* $D_r\Lambda(\omega)$ *is the operator in* $L(H, L_2(U,H))$ *given by the kernel* $D_r f(x,y)$.

REMARK. Notice that for all $r \in [0,T]$ a.e., the kernel $D_r f(x,y)$ verifies the assumptions of Lemma 6.3.

PROOF. By Lemma 6.2 we know that $\Lambda \in L^2(\Omega; L(H,H))$. According to Definition 2.1, in order to show that $\Lambda \in \mathbb{D}^{1,2}(L(H,H))$ we have to show that conditions (a) and (b) of this definition are satisfied.

For (a) we must show that for every $g \in L^2(O)$, Λg belongs to $\mathbb{D}^{1,2}(H)$. From condition (i) of Lemma 6.4 it follows that $(\Lambda g)(x) \in \mathbb{D}^{1,2}$ for each $x \in O$, and $D[(\Lambda g)(x)] = \int_O Df(x,y)g(y)\,dy$. Furthermore, using condition (ii) we get

$$
\begin{aligned}
&E\int_0^T \int_O |D_r[(\Lambda g)(x)]|_U^2\,dx\,dr \\
&\quad\le E\int_0^T \int_O\left(\int_O |D_r f(x,y)|_U \ |g(y)|\,dy\right)^2 dx\,dr \\
&\quad\le E\left(\int_0^T \left(\sup_{x\in O}\int_O |D_r f(x,y)|_U\,dy\right)\right. \\
&\qquad\qquad \left.\times\left(\sup_{y\in O}\int_O |D_r f(y,x)|_U\,dx\right) dr\right)\|g\|_{L^2(O)}^2 \\
&\quad\le E\left(\int_0^T a_1(r)a_2(r)\,dr\right)\|g\|_{L^2(O)}^2 \\
&\quad\le \left\{E\left(\int_0^T (a_1(r))^2\,dr\right)E\left(\int_0^T (a_2(r))^2\,dr\right)\right\}^{1/2}\|g\|_{L^2(O)}^2 < \infty.
\end{aligned}
$$

This implies that $\Lambda g \in \mathbb{D}^{1,2}(H)$ (see [15], Theorem 3.1).

For (b), clearly $D_r(\Lambda g) = (\hat{D}_r\Lambda)(g)$, where $\hat{D}_r\Lambda$ is the random operator belonging to the space $L(H, L_2(U,H))$ associated with the kernel $D_r f(x,y)$. Hence, it suffices to show that $\hat{D}_r\Lambda$ belongs to the space $L^2([0,T]\times\Omega;\ L(H, L_2(U,H)))$. This follows from the fact that

$$
\int_O |D_r[(\Lambda g)(x)]|_U^2\,dx \le a_1(r)a_2(r)\|g\|_{L^2(O)}^2,
$$

which implies $\|\hat{D}_r\Lambda\|_{L(H, L_2(U,H))} \le (a_1(r)a_2(r))^{1/2}$. □

We can also show a version of Lemma 6.4 for kth differentiable operators.

LEMMA 6.5. *Let* $f\colon O\times O\times\Omega \to \mathbb{R}_+$ *be a measurable mapping verifying the hypotheses of Lemma 6.2. Assume that* $f(x,y) \in \mathbb{D}^{k,2}$ *for each* $x, y \in O$ *and for some integer* $k \ge 1$, *and there exist versions of the derivatives* $D^j_{r_1\cdots r_j} f(x,y)$,

$1 \le j \le k$, which are measurable from $[0,T]^j \times O \times O \times \Omega$ into $U^{\otimes j}$ and verify the following:

(i) $D^j f(x,\cdot) \in L^2([0,T]^j \times O \times \Omega; U^{\otimes j})$ *for all* $x \in O$, $1 \le j \le k$;

(ii)
$$\sup_{z\in O} \int_O |D^j_{r_1\cdots r_j} f(x,z)|_{U^{\otimes j}}\, dx \le a_{1,j}(r_1,\dots,r_j),$$
$$\sup_{z\in O} \int_O |D^j_{r_1\cdots r_j} f(z,x)|_{U^{\otimes j}}\, dx \le a_{2,j}(r_1,\dots,r_j),$$

where $a_{1,j}$ and $a_{2,j}$ are nonnegative measurable random fields such that $E\int_{[0,T]^j}(a_{1,j}(r))^2\,dr < \infty$ and $E\int_{[0,T]^j}(a_{2,j}(r))^2\,dr < \infty$, for each $j = 1,\dots,k$. Then the random operator $\Lambda(\omega)$ on $L^2(O)$ given by (6.1) belongs to $\mathbb{D}^{k,2}(L(H,H))$, and for all $r_1,\dots,r_j,\omega$ a.e. $D^j_{r_1\cdots r_j}\Lambda(\omega)$ is the operator belonging to $L(H, L_2(U^{\otimes j}, H))$ given by the kernel $D^j_{r_1\cdots r_j} f(x,y)$.

Consider now a random second order differential operator of the form

$$A_t = \sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x,t)\frac{\partial}{\partial x_i} + c(x,t). \tag{6.2}$$

The coefficients a_{ij}, b_i and c are measurable functions from $\overline{O} \times [0,T] \times \Omega$ in $\mathbb{R}$. Let us introduce the following hypotheses on the random operator A_t.

(A1) For each $(x,t) \in \overline{O} \times [0,T]$, $a_{ij}(x,t)$, $b_i(x,t)$ and $c(x,t)$ are $\mathscr{F}_t$-measurable (*adaptability*).

(A2) The matrix $(a_{ij})_{1\le i,j\le n}$ is symmetric and uniformly elliptic. That is, there exist constants $0 < c_1 \le c_2 < \infty$ such that
$$c_1|\xi|^2 \le \sum_{i,j=1}^n a_{ij}(x,t)\xi_i\xi_j \le c_2|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n.$$

(A3) The coefficients a_{ij}, b_i and c are continuous and uniformly bounded in $\overline{O} \times [0,T]$, and, in addition they verify the following Hölder continuity property:
$$|a_{ij}(x,t) - a_{ij}(y,s)| \le K(|x-y|^\alpha + |s-t|^{\alpha/2}),$$
$$|b_i(x,t) - b_i(y,t)| \le K|x-y|^\alpha,$$
$$|c(x,t) - c(y,t)| \le K|x-y|^\alpha,$$
for some constants $0 < K < \infty$, $\alpha > 0$ and for all $x, y \in \overline{O}$, $s,t \in [0,T]$.

Furthermore $a_{ij}(\cdot,t)$ is of class C^1 with uniformly bounded partial derivatives.

(A4) For each $(x,t) \in \overline{O} \times [0,T]$ we have that $a_{ij}(x,t)$, $(\partial a_{ij}/\partial x_i)(x,t)$, $b_i(x,t)$ and $c(x,t)$ belong to $\mathbb{D}^{2,2}$, and the derivatives
$$|D_r a_{ij}(x,t)|_U,\ \left| D_r \frac{\partial a_{ij}}{\partial x_i}(x,t)\right|_U,\ |D_r b_i(x,t)|_U,\ |D_r c(x,t)|_U$$

are bounded by a nonnegative process $\Phi(r)$ such that

$$E\left(\left|\int_0^T |\Phi(r)|^2\, dr\right|^p\right) < \infty$$

for all $p \geq 2$. We also assume that

$$|D^2_{r_1 r_2} a_{ij}(x,t)|_{U\otimes U}, \quad \left|D^2_{r_1 r_2}\frac{\partial a_{ij}}{\partial x_i}(x,t)\right|_{U\otimes U},$$
$$|D^2_{r_1 r_2} b_i(x,t)|_{U\otimes U}, \quad |D^2_{r_1 r_2} c(x,t)|_{U\otimes U}$$

are bounded by a nonnegative process $\Psi(r_1, r_2)$ such that

$$E\left(\left|\int_{[0,T]^2} (\Psi(r_1,r_2))^2\, dr_1\, dr_2\right|^p\right) < \infty \quad \text{for all } p \geq 2.$$

(A4)′ We assume that the following quantities are uniformly bounded:

$$\sum_{k=1}^{\infty} \sup_{x,t}\left\{|D_r^{e_k} a_{ij}(x,t)|^2 + \left|D_r^{e_k}\frac{\partial a_{ij}}{\partial x_i}(x,t)\right|^2 + |D_r^{e_k} b_i(x,t)|^2 + |D_r^{e_k} c(x,t)|^2\right\},$$

$$\sum_{k=1}^{\infty} \sup_{x,s,t}\left\{|D_r^{e_k} D_s a_{ij}(x,t)|_U^2 + \left|D_r^{e_k} D_s\frac{\partial a_{ij}}{\partial x_i}(x,t)\right|_U^2 + |D_r^{e_k} D_s b_i(x,t)|_U^2 + |D_r^{e_k} D_s c(x,t)|_U^2\right\}.$$

In what follows we will assume that $O = \mathbb{R}^n$. The case of a bounded domain O with Dirichlet or Neuman boundary conditions would be treated in a similar way.

Suppose that A is a random second order differential operator verifying hypotheses (A1), (A2) and (A3) with $O = \mathbb{R}^n$. We will denote by $\Gamma(x,t;y,s)$ the fundamental solution of

$$\frac{\partial \Gamma}{\partial t} = A_t \Gamma, \qquad t > s, \qquad \lim_{t\downarrow s}\Gamma(x,t;y,s) = \delta_x(y). \tag{6.3}$$

(For details, see [6].)

Conditions (A2) and (A3) imply that there exist constants $c_1, c_2 > 0$ such that

$$\Gamma(x,t;y,s) \leq c_1(t-s)^{-n/2}\exp\left(-\frac{|x-y|^2}{c_2(t-s)}\right), \tag{6.4}$$

$$\left|\frac{\partial \Gamma}{\partial x_i}(x,t;y,s)\right| \leq c_1(t-s)^{-(n+1)/2}\exp\left(-\frac{|x-y|^2}{c_2(t-s)}\right). \tag{6.5}$$

PROPOSITION 6.6. *Suppose A_t is a random second order differential operator verifying hypotheses* (A1), (A2) *and* (A3). *Let $\Gamma(x, t; y, s)$ be the fundamental solution of (6.3). For any $t > s$, $t, s \in [0, T]$ let $S(t, s)$ be the random operator on $L^2(\mathbb{R}^n)$ given by*

$$(S(t,s)g)(x) = \int_{\mathbb{R}^n} \Gamma(x, t; y, s) g(y)\, dy. \tag{6.6}$$

Put $S(t, t) = Id$. Then $\{S(t, s),\ 0 \leq s \leq t \leq T\}$ is a random evolution system on $L^2(\mathbb{R}^n)$ in the sense of Definition 3.1.

PROOF. By construction (see [2]), the random kernel $\Gamma(x, t; y, s; \omega)$ is a measurable mapping from $\mathbb{R}^n \times \mathbb{R}^n \times \Omega \to \mathbb{R}_+$ for each $t > s$. From (6.4) we deduce

$$\Gamma(x, t; \cdot, s) \in L^2(\mathbb{R}^n) \quad \text{for all } x \in \mathbb{R}^n,\ t > s$$

and

$$\int_{\mathbb{R}^n} \Gamma(x, t; y, s)\, dy \leq c_1 (2\pi)^{n/2}, \tag{6.7}$$

$$\int_{\mathbb{R}^n} \Gamma(x, t; y, s)\, dx \leq c_1 (2\pi)^{n/2}. \tag{6.8}$$

Hence, by Lemma 6.1, $S(t, s)$ is a bounded linear operator on $L^2(\mathbb{R}^n)$. Moreover the mapping $(t, s, \omega) \to S(t, s, \omega)$ is strongly measurable from $\Delta \times \Omega$ in $L(H, H)$, and $S(t, s)$ is $\mathscr{F}_t$–strongly measurable from Ω in $L(H, H)$. Condition (iiia) of Definition 3.1 clearly holds, and the continuity property (iiib) is also known (see [6]). □

PROPOSITION 6.7. *Let A_t be a random second order differential operator verifying hypotheses* (A1), (A2), (A3), (A4) *and* (A4)′. *Then the random evolution system $S(t, s)$ given by (6.6) verifies hypotheses* (H1), (H2) *and* (H3)′.

The proof will be done in several steps.

PROOF OF (H1). By Lemma 6.2 and the estimates (6.7) and (6.8), we deduce that

$$\|S(t, s)\|_{L(H, H)} \leq c_1 (2\pi)^{n/2}.$$

So $S(t, s) \in L^2(\Omega; L(H, H))$ and the norm of $S(t, s)$ is uniformly bounded. In order to show that $S(t, s)$ belongs to $\mathbb{D}^{2,2}(L(H, H))$ for $t > s$ we will make use of Lemma 6.5. We have to show that $\Gamma(x, t; y, s) \in \mathbb{D}^{2,2}$ for each $x, y \in \mathbb{R}^n$, $t > s$ and that conditions (i) and (ii) of Lemma 6.5 for $j = 1, 2$ hold.

Let us first show that $\Gamma(x, t; y, s) \in \mathbb{D}^{1,2}$. We recall that $\Gamma(x, t; y, s)$ is the fundamental solution of

$$\frac{\partial \Gamma}{\partial t} = A_t \Gamma, \qquad t > s,$$
$$\lim_{t \downarrow s} \Gamma(x, t; y, s) = \delta_x(y).$$

In the sequel we will write $\Gamma_{t,s}(x, y)$ for $\Gamma(x, t; y, s)$. Using the characterization of the space $\mathbb{D}^{1,2}$ given by Sugita in [15] we can show that $\Gamma_{t,s}(x, y)$ is RAC (ray absolutely continuous), and the derivative $D_r\Gamma_{t,s}(x, y)$ verifies

$$\frac{\partial}{\partial t} D_r\Gamma_{t,s}(x, y) = A_t D_r \Gamma_{t,s}(x, y) + (D_r A_t)\Gamma_{t,s}(x, y)$$

for $r \in [0, t]$. Hence,

$$\begin{aligned} D_r\Gamma_{t,s}(x, y) = \int_{\mathbb{R}^n}\int_s^t \Gamma_{t,\tau}(x, \xi)\Bigg\{ &\sum_{i,j=1}^n D_r a_{ij}(\xi, \tau)\frac{\partial^2\Gamma_{\tau,s}}{\partial\xi_i\partial\xi_j}(\xi, y) \\ &+ \sum_{i=1}^n D_r b_i(\xi, \tau)\frac{\partial\Gamma_{\tau,s}}{\partial\xi_i}(\xi, y) \\ &+ D_r c(\xi, \tau)\Gamma_{\tau,s}(\xi, y)\Bigg\}\, d\tau\, d\xi. \end{aligned} \tag{6.9}$$

Integrating by parts this can be written as

$$\begin{aligned} D_r\Gamma_{t,s}(x, y) = \int_{\mathbb{R}^n}\int_s^t \Gamma_{t,\tau}(x, \xi)\Bigg\{ &-\sum_{i,j=1}^n D_r\left(\frac{\partial a_{ij}}{\partial\xi_i}(\xi, \tau)\right)\frac{\partial\Gamma_{\tau,s}}{\partial\xi_j}(\xi, y) \\ &+ \sum_{i=1}^n D_r b_i(\xi, \tau)\frac{\partial\Gamma_{\tau,s}}{\partial\xi_i}(\xi, y) \\ &+ D_r c(\xi, \tau)\Gamma_{\tau,s}(\xi, y)\Bigg\}\, d\tau\, d\xi \\ &- \int_{\mathbb{R}^n}\int_s^t \sum_{i,j=1}^n \frac{\partial\Gamma_{t,\tau}}{\partial\xi_i}(x, \xi)\frac{\partial\Gamma_{\tau,s}}{\partial\xi_j}(\xi, y) D_r a_{ij}(\xi, \tau)\, d\tau\, d\xi. \end{aligned} \tag{6.10}$$

From (6.4), (6.5) and (6.10) we obtain the following estimate:

$$\begin{aligned} |D_r\Gamma_{t,s}(x, y)|_U \le\ & C(t-s)^{-n/2}\exp\left(-\frac{|x-y|^2}{c(t-s)}\right) \\ &\times \int_s^t \Bigg\{ \sup_{\xi\in\mathbb{R}^n}\Bigg[\sum_{i,j=1}^n \left|D_r\frac{\partial a_{ij}}{\partial\xi_i}(\xi, \tau)\right|_U + \sum_{i=1}^n \left|D_r b_i(\xi, \tau)\right|_U\Bigg] \\ &\qquad \times (\tau - s)^{-1/2} + \sup_{\xi\in\mathbb{R}^n}|D_r c(\xi, \tau)|_U \\ &\qquad + \sup_{\xi\in\mathbb{R}^n}\sum_{i,j=1}^n |D_r a_{ij}(\xi, \tau)|_U (t-\tau)^{-1/2}(\tau - s)^{-1/2}\Bigg\}\, d\tau \\ \le\ & C(t-s)^{-n/2}\exp\left(-\frac{|x-y|^2}{c(t-s)}\right)\Phi(r), \end{aligned} \tag{6.11}$$

for some constants $c, C > 0$. Hence, conditions (i) and (ii) of Lemma 6.5 hold for $j = 1$.

For the second derivative we have

$$\frac{\partial}{\partial t} D^2_{r_1 r_2}\Gamma_{t,s}(x,y) = A_t D^2_{r_1 r_2}\Gamma_{t,s}(x,y) + (D_{r_2}A_t)D_{r_1}\Gamma_{t,s}(x,y) + (D_{r_1}A_t)D_{r_2}\Gamma_{t,s}(x,y) + (D^2_{r_1 r_2}A_t)\Gamma_{t,s}(x,y).$$

Hence,

$$\begin{aligned} D^2_{r_1 r_2}\Gamma_{t,s}(x,y) = \int_{\mathbb{R}^n}\int_s^t \Gamma_{t,\tau}(x,\xi) \times \Bigg\{ &\sum_{i,j=1}^n D_{r_2}a_{ij}(\xi,\tau)\frac{\partial^2 D_{r_1}\Gamma_{\tau,s}}{\partial\xi_i\partial\xi_j}(\xi,y) \\ &+ \sum_{i=1}^n D_{r_2}b_i(\xi,\tau)\frac{\partial D_{r_1}\Gamma_{\tau,s}}{\partial\xi_i}(\xi,y) + D_{r_2}c(\xi,\tau)D_{r_1}\Gamma_{\tau,s}(\xi,y) \\ &+ \sum_{i,j=1}^n D_{r_1}a_{ij}(\xi,\tau)\frac{\partial^2 D_{r_2}\Gamma_{\tau,s}}{\partial\xi_i\partial\xi_j}(\xi,y) \\ &+ \sum_{i=1}^n D_{r_1}b_i(\xi,\tau)\frac{\partial D_{r_2}\Gamma_{\tau,s}}{\partial\xi_i}(\xi,y) + D_{r_1}c(\xi,\tau)D_{r_2}\Gamma_{\tau,s}(\xi,y) \\ &+ \sum_{i,j=1}^n D^2_{r_1 r_2}a_{ij}(\xi,\tau)\frac{\partial^2\Gamma_{\tau,s}}{\partial\xi_i\partial\xi_j}(\xi,y) \\ &+ \sum_{i=1}^n D^2_{r_1 r_2}b_i(\xi,\tau)\frac{\partial\Gamma_{\tau,s}}{\partial\xi_i}(\xi,y) \\ &+ D^2_{r_1 r_2}c(\xi,\tau)\Gamma_{\tau,s}(\xi,y)\Bigg\}\,d\xi\,d\tau. \end{aligned}$$

Integrating by parts and using the estimates (A4) and (6.11) we get

$$\begin{aligned} &|D^2_{r_1 r_2}\Gamma_{t,s}(x,y)|_{U\otimes U} \\ &\quad \le C(t-s)^{-n/2}\exp\left(-\frac{|x-y|^2}{c(t-s)}\right)\{\Phi(r_1)\Phi(r_2)+\Psi(r_1,r_2)\}. \end{aligned} \tag{6.12}$$

Hence conditions (i) and (ii) of Lemma 6.5 hold for $j=2$. Furthermore,

$$\begin{aligned} \|S(t,s)\|^p_{2,p} &= E\|S(t,s)\|^p_{L(H,H)} + E\left(\int_0^t \|D_r S(t,s)\|^2_{L(H,L_2(U,H))}\,dr\right)^{p/2} \\ &\quad + E\left(\int_0^t\int_0^t \|D^2_{r_1 r_2}S(t,s)\|^2_{L(H,L_2(U\otimes U,H))}\,dr_1\,dr_2\right)^{p/2} \\ &\le C\Bigg\{1 + E\left|\int_0^t (\Phi(r))^2\,dr\right|^{p/2} + E\left|\int_0^t (\Phi(r))^2\,dr\right|^p \\ &\qquad + E\left|\int_0^t\int_0^t (\Psi(r_1,r_2))^2\,dr_1\,dr_2\right|^{p/2}\Bigg\} < \infty, \end{aligned}$$

and Hypothesis (H1) holds. □

PROOF OF (H2). Fix an element $h \in H = L^2(\mathbb{R}^n)$. Then

$$D_r S(t,s)(h) = \int_{\mathbb{R}^n} D_r \Gamma_{t,s}(x,y) h(y)\,dy \quad \text{for } r \in [s,t],$$

where $D_r \Gamma_{t,s}(x,y)$ is given by formula (6.9). Let us define $D_s^- S(t,s)$ as the operator given by the kernel $D_s^- \Gamma_{t,s}(x,y)$, where

$$\begin{aligned} D_s^- \Gamma_{t,s}(x,y) = \int_{\mathbb{R}^n} \int_s^t \Gamma_{t,\tau}(x,\xi) \Bigg\{ &- \sum_{i,j=1}^n D_s\left(\frac{\partial a_{ij}}{\partial \xi_i}(\xi,\tau)\right) \frac{\partial \Gamma_{\tau,s}}{\partial \xi_j}(\xi,y) \\ &+ \sum_{i=1}^n D_s b_i(\xi,\tau) \frac{\partial \Gamma_{\tau,s}}{\partial \xi_i}(\xi,y) \\ &+ D_s c(\xi,\tau) \Gamma_{\tau,s}(\xi,y) \Bigg\}\,d\tau\,d\xi \\ &- \int_{\mathbb{R}^n} \int_s^t \sum_{i,j=1}^n \frac{\partial \Gamma_{t,\tau}}{\partial \xi_i}(x,\xi) \frac{\partial \Gamma_{\tau,s}}{\partial \xi_j}(\xi,y) D_s a_{ij}(\xi,\tau)\,d\tau\,d\xi. \end{aligned}$$

The difference $D_s S(t, s-\varepsilon) - D_s^- S(t,s)$ is the operator in $L(H, L_2(U,H))$ given by the kernel

$$\begin{aligned} &\int_{\mathbb{R}^n} \int_{s-\varepsilon}^s \Gamma_{t,\tau}(x,\xi) \Bigg\{ - \sum_{i,j=1}^n D_s\left(\frac{\partial a_{ij}}{\partial \xi_i}(\xi,\tau)\right) \frac{\partial \Gamma_{\tau,s-\varepsilon}}{\partial \xi_j}(\xi,y) \\ &\qquad + \sum_{i=1}^n D_s b_i(\xi,\tau) \frac{\partial \Gamma_{\tau,s-\varepsilon}}{\partial \xi_i}(\xi,y) + D_s c(\xi,\tau) \Gamma_{\tau,s-\varepsilon}(\xi,y) \Bigg\}\,d\tau\,d\xi \\ &\quad - \int_{\mathbb{R}^n} \int_{s-\varepsilon}^s \sum_{i,j=1}^n \frac{\partial \Gamma_{t,\tau}}{\partial \xi_i}(x,\xi) \frac{\partial \Gamma_{\tau,s-\varepsilon}}{\partial \xi_j}(\xi,y) D_s a_{ij}(\xi,\tau)\,d\tau\,d\xi \\ &\quad + \int_{\mathbb{R}^n} \int_s^t \Gamma_{t,\tau}(x,\xi) \Bigg\{ - \sum_{i,j=1}^n D_s\left(\frac{\partial a_{ij}}{\partial \xi_i}(\xi,\tau)\right) \left(\frac{\partial \Gamma_{\tau,s-\varepsilon}}{\partial \xi_j} - \frac{\partial \Gamma_{\tau,s}}{\partial \xi_j}\right)(\xi,y) \\ &\qquad + \sum_{i=1}^n D_s b_i(\xi,\tau) \left(\frac{\partial \Gamma_{\tau,s-\varepsilon}}{\partial \xi_i} - \frac{\partial \Gamma_{\tau,s}}{\partial \xi_i}\right)(\xi,y) \\ &\qquad + D_s c(\xi,\tau)(\Gamma_{\tau,s-\varepsilon} - \Gamma_{\tau,s})(\xi,y) \Bigg\}\,d\tau\,d\xi \\ &\quad - \int_{\mathbb{R}^n} \int_s^t \sum_{i,j=1}^n \frac{\partial \Gamma_{t,\tau}}{\partial \xi_i}(x,\xi) \left(\frac{\partial \Gamma_{\tau,s-\varepsilon}}{\partial \xi_j} - \frac{\partial \Gamma_{\tau,s}}{\partial \xi_j}\right)(\xi,y) D_s a_{ij}(\xi,\tau)\,d\tau\,d\xi. \end{aligned}$$

Let us denote this kernel by $\Phi_{\varepsilon,s,t}(x,y)$. We have for any $h \in H$

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} \Phi_{\varepsilon,s,t}(x,y) h(y)\,dy \right|_U \\ &\quad \le C \left\{ \int_{s-\varepsilon}^s [\Phi(s)(\tau - s + \varepsilon)^{-1/2} + \Phi(s) + \Phi(s)(t-\tau)^{-1/2}(\tau - s + \varepsilon)^{-1/2}]\,d\tau \right\} \end{aligned}$$

$$\times \int_{\mathbb{R}^n} (t-s+\varepsilon)^{-n/2} \exp\left(-\frac{|x-y|^2}{c(t-s+\varepsilon)}\right) |h(y)|\, dy$$
$$+ C\left\{\int_s^t [\Phi(s)(\tau-s)^{-1/2} + \Phi(s) + \Phi(s)(t-\tau)^{-1/2}(\tau-s)^{-1/2}]\, d\tau\right\}$$
$$\times \int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} \Gamma_{s,\, s-\varepsilon}(z, y)h(y)\, dy - h(z)\right| (t-s)^{-n/2} \exp\left(-\frac{|x-z|^2}{c(t-s)}\right) dz.$$

As a consequence,

$$\int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} \Phi_{\varepsilon,\, s,\, t}(x, y)h(y)\, dy\right|_U^2 dx$$
$$\leq C(\Phi(s))^2 \left\{ \left(\sqrt{\varepsilon} + \varepsilon + (t-s)^{-1/2}\sqrt{\varepsilon}\right)^2 \|h\|_H^2 + \int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} \Gamma_{s,\, s-\varepsilon}(z, y)h(y)\, dy - h(z)\right|^2 dz \right\},$$

and this converges to zero as ε tends to zero.

On the other hand, $D_s^- S(t, s)$ belongs to $\mathbb{D}^{1,2}(L(H, L_2(U, H)))$ (see first step of the proof of Proposition 6.7). □

PROOF OF (H3)′. We have already seen that $\|S(t,s)\|_{L(H,H)} \leq c_1(2\pi)^{n/2}$. We have

$$\|D_r^{e_j} S(t, s)\|_{L(H,H)}$$
$$= \sup_{\|h\|_H \leq 1} \int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} D_r^{e_j}\Gamma_{t,\, s}(x, y)h(y)\, dy\right|^2 dx$$
$$\leq \left(\left(\sup_x \int_{\mathbb{R}^n} |D_r^{e_j}\Gamma_{t,\, s}(x, y)|\, dy\right)\left(\sup_y \int_{\mathbb{R}^n} |D_r^{e_j}\Gamma_{t,\, s}(x, y)|\, dx\right)\right)^{1/2}.$$

Hence, the boundedness of $\sum_{j=1}^\infty \|D_r^{e_i} S(t,s)\|_{L(H,H)}^2$ follows from (6.9) and Hypothesis (A4)′.

Finally, let us show that $\sum_{k=1}^\infty \|D_r^{e_k} D_s^- S(t,s)\|_{L(H,\, L_2(U,H))}^2$ is bounded. We have, for $r \leq s \leq t$,

$$D_r^{e_k} D_s^- \Gamma_{t,\, s}(x, y) = \int_{\mathbb{R}^n} \int_s^t \Gamma_{t,\, \tau}(x, \xi)$$
$$\times \left\{ \sum_{i,\, j=1}^n D_r^{e_k} a_{ij}(\xi, \tau) \frac{\partial^2 D_s^- \Gamma_{\tau,\, s}}{\partial \xi_i \partial \xi_j}(\xi, y) + \sum_{i=1}^n D_r^{e_k} b_i(\xi, \tau) \frac{\partial D_s^- \Gamma_{\tau,\, s}}{\partial \xi_i}(\xi, y) + D_r^{e_k} c(\xi, \tau) D_s^- \Gamma_{\tau,\, s}(\xi, y) \right.$$

$$+\sum_{i,j=1}^{n} D_s a_{ij}(\xi,\tau)\frac{\partial^2 D_r^{e_k}\Gamma_{\tau,s}}{\partial\xi_i\partial\xi_j}(\xi,y)$$

$$+\sum_{i=1}^{n} D_s b_i(\xi,\tau)\frac{\partial D_r^{e_k}\Gamma_{\tau,s}}{\partial\xi_i}(\xi,y)+D_s c(\xi,\tau)D_r^{e_k}\Gamma_{\tau,s}(\xi,y)$$

$$+\sum_{i,j=1}^{n} D_r^{e_k} D_s a_{ij}(\xi,\tau)\frac{\partial^2\Gamma_{\tau,s}}{\partial\xi_i\partial\xi_j}(\xi,y)$$

$$+\sum_{i=1}^{n} D_r^{e_k} D_s b_i(\xi,\tau)\frac{\partial\Gamma_{\tau,s}}{\partial\xi_i}(\xi,y)$$

$$\left.+ D_r^{e_k} D_s c(\xi,\tau)\Gamma_{\tau,s}(\xi,y)\right\} d\tau\, d\xi.$$

Again using (A4)$'$, integration by parts and (6.9) we show the boundedness of the expressions

$$\sum_{k=1}^{\infty}\sup_x \int_{\mathbb{R}^n} |D_r^{e_k} D_s^{-}\Gamma_{t,s}(x,y)|_U\, dy$$

and

$$\sum_{k=1}^{\infty}\sup_y \int_{\mathbb{R}^n} |D_r^{e_k} D_s^{-}\Gamma_{t,s}(x,y)|_U\, dx. \qquad \square$$

REMARKS. Theorem 5.4 together with Proposition 6.7 allow us to deduce the existence of a unique mild solution for stochastic partial differential equations of the form

$$\text{(6.13)}\qquad \begin{aligned} &\frac{\partial u}{\partial t} = A_t u + f(t,x,u) + g(t,x,u)\dot{W}_t(x), \qquad t\in[0,T],\ x\in D,\\ &u(0,x)=\varphi(x), \end{aligned}$$

where $D\subset\mathbb{R}^n$ is a bounded domain with smooth boundary, f, g are continuous functions on $[0,T]\times D\times\mathbb{R}$ which are Lipschitz and have linear growth in the last variable, uniformly with respect to the first two variables, and $\dot{W}$ is a Wiener process in $L^2(D)$ whose covariance operator is bounded on $D\times D$. Here A_t is a second order operator of the form (6.2) with random and adapted coefficients satisfying assumptions (A.1), (A.2), (A.3), (A.4) and (A.4)$'$.

The above method allows handling stochastic partial differential equations of the form (6.13) without motononicity or coercivity assumptions on the coefficients.

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