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MALLIAVIN CALCULUS FOR BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS AND APPLICATION TO NUMERICAL SOLUTIONS

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In this paper we study backward stochastic differential equations with general terminal value and general random generator. In particular, we do not require the terminal value be given by a forward diffusion equation. The randomness of the generator does not need to be from a forward equation, either. Motivated from applications to numerical simulations, first we obtain the L^p-Hölder continuity of the solution. Then we construct several numerical approximation schemes for backward stochastic differential equations and obtain the rate of convergence of the schemes based on the obtained L^p-Hölder continuity results. The main tool is the Malliavin calculus.

1. Introduction. The backward stochastic differential equation (BSDE, for short) we shall consider in this paper takes the following form:

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r)\, dr - \int_t^T Z_r\, dW_r, \qquad 0 \le t \le T, \tag{1.1}$$

where $W = \{W_t\}_{0\le t\le T}$ is a standard Brownian motion, ξ is the given terminal value and f is the given (random) generator. To solve this equation is to find a pair of adapted processes $Y = \{Y_t\}_{0\le t\le T}$ and $Z = \{Z_t\}_{0\le t\le T}$ satisfying the above equation (1.1).

Linear backward stochastic differential equations were first studied by Bismut [3] in an attempt to solve some optimal stochastic control problem through the method of maximum principle. The general nonlinear backward stochastic differential equations were first studied by Pardoux and Peng [15].

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Since then there have been extensive studies of this equation. We refer to the review paper by El Karoui, Peng and Quenez [7], to the books of El Karoui and Mazliak [6] and of Ma and Yong [12] and the references therein for more comprehensive presentation of the theory.

A current important topic in the applications of BSDEs is the numerical approximation schemes. In most work on numerical simulations, a certain forward stochastic differential equation of the following form:

$$X_t = X_0 + \int_0^t b(r, X_r, Y_r)\, dr + \int_0^t \sigma(r, X_r)\, dW_r \tag{1.2}$$

is needed. Usually it is assumed that the generator f in (1.1) depends on X_r at the time r: $f(r, Y_r, Z_r) = f(r, X_r, Y_r, Z_r)$, where $f(r, x, y, z)$ is a deterministic function of (r, x, y, z), and f is global Lipschitz in (x, y, z). If in addition the terminal value ξ is of the form $\xi = h(X_T)$, where h is a deterministic function, a so-called four-step numerical scheme has been developed by Ma, Protter and Yong in [11]. A basic ingredient in this paper is that the solution $\{Y_t\}_{0\le t\le T}$ to the BSDE is of the form $Y_t = u(t, X_t)$, where $u(t, x)$ is determined by a quasi-linear partial differential equation of parabolic type. Recently, Bouchard and Touzi [4] propose a Monte-Carlo approach which may be more suitable for high-dimensional problems. Again in this forward–backward setting, if the generator f has a quadratic growth in Z, a numerical approximation is developed by Imkeller and Dos Reis [9] in which a truncation procedure is applied.

In the case where the terminal value ξ is a functional of the path of the forward diffusion X, namely, $\xi = g(X.)$, different approaches to construct numerical methods have been proposed. We refer to Bally [1] for a scheme with a random time partition. In the work by Zhang [16], the L^2-regularity of Z is obtained, which allows one to use deterministic time partitions as well as to obtain the rate estimate (see Bender and Denk [2], Gobet, Lemor and Warin [8] and Zhang [16] for different algorithms). We should also mention the works by Briand, Delyon and Mémin [5] and Ma et al. [10], where the Brownian motion is replaced by a scaled random walk.

The purpose of the present paper is to construct numerical schemes for the general BSDE (1.1), without assuming any particular form for the terminal value ξ and generator f. This means that ξ can be an arbitrary random variable, and $f(r, y, z)$ can be an arbitrary $\mathcal{F}_r$-measurable random variable (see Assumption 2.2 in Section 2 for precise conditions on ξ and f). The natural tool that we shall use is the Malliavin calculus. We emphasize that the main difficulty in constructing a numerical scheme for BSDEs is usually the approximation of the process Z. It is necessary to obtain some regularity properties for the trajectories of this process Z. The Malliavin calculus turns out to be a suitable tool to handle these problems because the random variable Z_t can be expressed in terms of the trace of the Malliavin derivative of Y_t, namely, $Z_t = D_t Y_t$. This relationship was proved in the paper by El Karoui, Peng and Quenez [7] and was used by these authors to obtain estimates for the moments of Z_t. We shall further exploit this identity to obtain the L^p-Hölder continuity of the process Z, which is the critical ingredient for the rate estimate of our numerical schemes.

Our first numerical scheme was inspired by the paper of Zhang [16], where the author considers a class of BSDEs whose terminal value ξ takes the form $g(X.)$, where X is a forward diffusion of the form (1.2), and g satisfies a Lipschitz condition with respect to the L^∞ or L^1 norms (similar assumptions for f). The discretization scheme is based on the regularity of the process Z in the mean square sense; that is, for any partition $\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$, one obtains

$$\sum_{i=0}^{n-1} \mathbb{E} \int_{t_i}^{t_{i+1}} [|Z_t - Z_{t_i}|^2 + |Z_t - Z_{t_{i+1}}|^2] \, dt \le K|\pi|, \tag{1.3}$$

where $|\pi| = \max_{0 \le i \le n-1}(t_{i+1} - t_i)$, and K is a constant independent of the partition π.

We consider the case of a general terminal value ξ which is twice differentiable in the sense of Malliavin calculus, and the first and second derivatives satisfy some integrability conditions; we also made similar assumptions for the generator f (see Assumption 2.2 in Section 2 for details). In this sense our framework extends that of [13] and is also natural. In this framework, we are able to obtain an estimate of the form

$$\mathbb{E}|Z_t - Z_s|^p \le K|t - s|^{p/2}, \tag{1.4}$$

where K is a constant independent of s and t. Clearly, (1.4) with $p = 2$ implies (1.3). Moreover, (1.4) implies the existence of a γ-Hölder continuous version of the process Z for any $\gamma < \frac{1}{2} - \frac{1}{p}$. Notice that, up to now the path regularity of Z has been studied only when the terminal value and the generator are functional of a forward diffusion.

After establishing the regularity of Z, we consider different types of numerical schemes. First we analyze a scheme similar to the one proposed in [16] [see (3.2)]. In this case we obtain a rate of convergence of the following type:

$$\mathbb{E} \sup_{0 \le t \le T} |Y_t - Y_t^\pi|^2 + \int_0^T \mathbb{E}|Z_t - Z_t^\pi|^2 \, dt \le K(|\pi| + \mathbb{E}|\xi - \xi^\pi|^2).$$

Notice that this result is stronger than that in [16] which can be stated as (when $\xi^\pi = \xi$)

$$\sup_{0 \le t \le T} \mathbb{E}|Y_t - Y_t^\pi|^2 + \int_0^T \mathbb{E}|Z_t - Z_t^\pi|^2 \, dt \le K|\pi|.$$

We also propose and study an "implicit" numerical scheme [see (4.1) in Section 4 for the details]. For this scheme we obtain a much better result on the rate of convergence,

$$\mathbb{E}\sup_{0\le t\le T}|Y_t-Y_t^\pi|^p+\mathbb{E}\biggl(\int_0^T|Z_t-Z_t^\pi|^2\,dt\biggr)^{p/2}\le K(|\pi|^{p/2}+\mathbb{E}|\xi-\xi^\pi|^p),$$

where $p>1$ depends on the assumptions imposed on the terminal value and the coefficients.

In both schemes, the integral of the process Z is used in each iteration, and for this reason they are not completely discrete schemes. In order to implement the scheme on computers, one must replace an integral of the form $\int_{t_i}^{t_{i+1}} Z_s^\pi\,ds$ by discrete sums, and then the convergence of the obtained scheme is hardly guaranteed. To avoid this discretization we propose a truly discrete numerical scheme using our representation of Z_t as the trace of the Malliavin derivative of Y_t (see Section 5 for details). For this new scheme, we obtain a rate of convergence result of the form

$$\mathbb{E}\max_{0\le i\le n}\{|Y_{t_i}-Y_{t_i}^\pi|^p+|Z_{t_i}-Z_{t_i}^\pi|^p\}\le K|\pi|^{p/2-\varepsilon}$$

for any $\varepsilon>0$. In fact, we have a slightly better rate of convergence (see Theorem 5.2),

$$\mathbb{E}\max_{0\le i\le n}\{|Y_{t_i}-Y_{t_i}^\pi|^p+|Z_{t_i}-Z_{t_i}^\pi|^p\}\le K|\pi|^{p/2-p/(2\log(1/|\pi|))}\biggl(\log\frac{1}{|\pi|}\biggr)^{p/2}.$$

However, this type of result on the rate of convergence applies only to some classes of BSDEs, and thus this scheme remains to be further investigated.

In the computer realization of our schemes or any other schemes, an extremely important procedure is to compute the conditional expectation of form $\mathbb{E}(Y|\mathcal{F}_{t_i})$. In this paper we shall not discuss this issue but only mention the papers [2, 4] and [8].

The paper is organized as follows. In Section 2 we obtain a representation of the martingale integrand Z in terms of the trace of the Malliavin derivative of Y, and then we get the L^p-Hölder continuity of Z by using this representation. The conditions that we assume on the terminal value ξ and the generator f are also specified in this section. Some examples of application are presented to explain the validity of the conditions. Section 3 is devoted to the analysis of the approximation scheme similar to the one introduced in [16]. Under some differentiability and integrability conditions in the sense of Malliavin calculus on ξ and the nonlinear coefficient f, we establish a better rate of convergence for this scheme. In Section 4, we introduce an "implicit" scheme and obtain the rate of convergence in the L^p norm. A completely discrete scheme is proposed and analyzed in Section 5.

Throughout the paper for simplicity we consider only scalar BSDEs. The results obtained in this paper can be easily extended to multi-dimensional BSDEs.

2. The Malliavin calculus for BSDEs.

2.1. *Notations and preliminaries.* Let $W = \{W_t\}_{0 \le t \le T}$ be a one-dimensional standard Brownian motion defined on some complete filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{0 \le t \le T})$. We assume that $\{\mathcal{F}_t\}_{0 \le t \le T}$ is the filtration generated by the Brownian motion and the P-null sets, and $\mathcal{F} = \mathcal{F}_T$. We denote by $\mathcal{P}$ the progressive σ-field on the product space $[0, T] \times \Omega$.

For any $p \ge 1$ we consider the following classes of processes:

- $M^{2,p}$, for any $p \ge 2$, denotes the class of square integrable random variables F with a stochastic integral representation of the form
$$F = \mathbb{E}F + \int_0^T u_t \, dW_t,$$
where u is a progressively measurable process satisfying $\sup_{0 \le t \le T} \mathbb{E}|u_t|^p < \infty$.
- $H^p_{\mathcal{F}}([0, T])$ denotes the Banach space of all progressively measurable processes $\varphi : ([0, T] \times \Omega, \mathcal{P}) \to (\mathbb{R}, \mathcal{B})$ with norm
$$\|\varphi\|_{H^p} = \left(\mathbb{E}\left(\int_0^T |\varphi_t|^2 \, dt \right)^{p/2} \right)^{1/p} < \infty.$$
- $S^p_{\mathcal{F}}([0, T])$ denotes the Banach space of all the RCLL (right continuous with left limits) adapted processes $\varphi : ([0, T] \times \Omega, \mathcal{P}) \to (\mathbb{R}, \mathcal{B})$ with norm
$$\|\varphi\|_{S^p} = \left(\mathbb{E} \sup_{0 \le t \le T} |\varphi_t|^p \right)^{1/p} < \infty.$$

Next, we present some preliminaries on Malliavin calculus, and we refer the reader to the book by Nualart [14] for more details.

Let $\mathbf{H} = L^2([0, T])$ be the separable Hilbert space of all square integrable real-valued functions on the interval $[0, T]$ with scalar product denoted by $\langle \cdot, \cdot \rangle_{\mathbf{H}}$. The norm of an element $h \in \mathbf{H}$ will be denoted by $\|h\|_{\mathbf{H}}$. For any $h \in \mathbf{H}$ we put $W(h) = \int_0^T h(t) \, dW_t$.

We denote by $C_p^\infty(\mathbb{R}^n)$ the set of all infinitely continuously differentiable functions $g : \mathbb{R}^n \to \mathbb{R}$ such that g and all of its partial derivatives have polynomial growth. We make use of the notation $\partial_i g = \frac{\partial g}{\partial x_i}$ whenever $g \in C^1(\mathbb{R}^n)$.

Let $\mathcal{S}$ denote the class of smooth random variables such that a random variable $F \in \mathcal{S}$ has the form

$$F = g(W(h_1), \dots, W(h_n)), \tag{2.1}$$

where g belongs to $C_p^\infty(\mathbb{R}^n)$, $h_1, \dots, h_n$ are in $\mathbf{H}$ and $n \ge 1$.

The Malliavin derivative of a smooth random variable F of the form (2.1) is the $\mathbf{H}$-valued random variable given by

$$DF = \sum_{i=1}^n \partial_i g(W(h_1), \dots, W(h_n)) h_i.$$

For any $p \geq 1$ we will denote the domain of D in $L^p(\Omega)$ by $\mathbb{D}^{1,p}$, meaning that $\mathbb{D}^{1,p}$ is the closure of the class of smooth random variables $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p} = (\mathbb{E}|F|^p + \mathbb{E}\|DF\|_{\mathbf{H}}^p)^{1/p}.$$

We can define the iteration of the operator D in such a way that for a smooth random variable F, the iterated derivative $D^k F$ is a random variable with values in $\mathbf{H}^{\otimes k}$. Then for every $p \geq 1$ and any natural number $k \geq 1$ we introduce the seminorm on $\mathcal{S}$ defined by

$$\|F\|_{k,p} = \left(\mathbb{E}|F|^p + \sum_{j=1}^{k} \mathbb{E}\|D^j F\|_{\mathbf{H}^{\otimes j}}^p \right)^{1/p}.$$

We will denote by $\mathbb{D}^{k,p}$ the completion of the family of smooth random variables $\mathcal{S}$ with respect to the norm $\|\cdot\|_{k,p}$.

Let μ be the Lebesgue measure on $[0,T]$. For any $k \geq 1$ and $F \in \mathbb{D}^{k,p}$, the derivative

$$D^k F = \{D^k_{t_1,\dots,t_k} F, t_i \in [0,T], i = 1,\dots,k\}$$

is a measurable function on the product space $[0,T]^k \times \Omega$, which is defined a.e. with respect to the measure $\mu^k \times P$.

We use $\mathbb{L}_a^{1,p}$ to denote the set of real-valued progressively measurable processes $u = \{u_t\}_{0 \leq t \leq T}$ such that:

(i) For almost all $t \in [0,T]$, $u_t \in \mathbb{D}^{1,p}$.

(ii) $\mathbb{E}((\int_0^T |u_t|^2\,dt)^{p/2} + (\int_0^T \int_0^T |D_\theta u_t|^2\,d\theta\,dt)^{p/2}) < \infty$.

Notice that we can choose a progressively measurable version of the $\mathbf{H}$-valued process $\{Du_t\}_{0 \leq t \leq T}$.

2.2. *Estimates on the solutions of BSDEs.* The generator f in the BSDE (1.1) is a measurable function $f : ([0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}, \mathcal{P} \otimes \mathcal{B} \otimes \mathcal{B}) \to (\mathbb{R}, \mathcal{B})$, and the terminal value ξ is an $\mathcal{F}_T$-measurable random variable.

DEFINITION 2.1. A solution to the BSDE (1.1) is a pair of progressively measurable processes (Y,Z) such that $\int_0^T |Z_t|^2\,dt < \infty$, $\int_0^T |f(t,Y_t,Z_t)|\,dt < \infty$, a.s. and

$$Y_t = \xi + \int_t^T f(r,Y_r,Z_r)\,dr - \int_t^T Z_r\,dW_r, \qquad 0 \leq t \leq T.$$

The next lemma provides a useful estimate on the solution to the BSDE (1.1).

LEMMA 2.2. *Fix $q \geq 2$. Suppose that $\xi \in L^q(\Omega)$, $f(t,0,0) \in H^q_{\mathcal{F}}([0,T])$ and f is uniformly Lipschitz in (y,z); namely, there exists a positive number L such that $\mu \times P$ a.e.*

$$|f(t,y_1,z_1) - f(t,y_2,z_2)| \leq L(|y_1 - y_2| + |z_1 - z_2|)$$

for all $y_1, y_2 \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{R}$. Then there exists a unique solution pair $(Y,Z) \in S^q_{\mathcal{F}}([0,T]) \times H^q_{\mathcal{F}}([0,T])$ to (1.1). Moreover, we have the following estimate for the solution:

$$\begin{aligned} &\mathbb{E} \sup_{0\leq t\leq T} |Y_t|^q + \mathbb{E}\left(\int_0^T |Z_t|^2\,dt\right)^{q/2} \\ &\qquad \leq K\left(\mathbb{E}|\xi|^q + \mathbb{E}\left(\int_0^T |f(t,0,0)|^2\,dt\right)^{q/2}\right), \end{aligned} \tag{2.2}$$

where K is a constant depending only on L, q and T.

PROOF. The proof of the existence and uniqueness of the solution (Y,Z) can be found in [7], Theorem 5.1, with the local martingale $M \equiv 0$, since the filtration here is the filtration generated by the Brownian motion W. Estimate (2.2) can be easily obtained from Proposition 5.1 in [7] with $(f^1, \xi^1) = (f, \xi)$ and $(f^2, \xi^2) = (0,0)$. □

As we will see later, for a given BSDE the process Z will be expressed in terms of the Malliavin derivative of the solution Y, which will satisfy a linear BSDE with random coefficients. To study the properties of Z we need to analyze a class of linear BSDEs.

Let $\{\alpha_t\}_{0\leq t\leq T}$ and $\{\beta_t\}_{0\leq t\leq T}$ be two progressively measurable processes. We will make use of the following integrability conditions:

ASSUMPTION 2.1.

(H1) For any $\lambda > 0$,

$$C_\lambda := \mathbb{E}\exp\left(\lambda \int_0^T (|\alpha_t| + \beta_t^2)\,dt\right) < \infty.$$

(H2) For any $p \geq 1$,

$$K_p := \sup_{0\leq t\leq T} \mathbb{E}(|\alpha_t|^p + |\beta_t|^p) < \infty.$$

Under condition (H1), we denote by $\{\rho_t\}_{0\leq t\leq T}$ the solution of the linear stochastic differential equation

$$\begin{cases} d\rho_t = \alpha_t \rho_t\,dt + \beta_t \rho_t\,dW_t, & 0 \leq t \leq T, \\ \rho_0 = 1. \end{cases} \tag{2.3}$$

The following theorem is a critical tool for the proof of the main theorem in this section, and it has also its own interest.

THEOREM 2.3. *Let $q > p \geq 2$ and let $\xi \in L^q(\Omega)$ and $f \in H^q_{\mathcal{F}}([0,T])$. Assume that $\{\alpha_t\}_{0\leq t\leq T}$ and $\{\beta_t\}_{0\leq t\leq T}$ are two progressively measurable processes satisfying conditions* (H1) *and* (H2) *in Assumption 2.1. Suppose that the random variables $\xi\rho_T$ and $\int_0^T \rho_t f_t\,dt$ belong to $M^{2,q}$, where $\{\rho_t\}_{0\leq t\leq T}$ is the solution to (2.3). Then the following linear BSDE,*

$$Y_t = \xi + \int_t^T [\alpha_r Y_r + \beta_r Z_r + f_r]\,dr - \int_t^T Z_r\,dW_r, \qquad 0 \leq t \leq T, \tag{2.4}$$

has a unique solution pair (Y,Z), and there is a constant $K > 0$ such that

$$\mathbb{E}|Y_t - Y_s|^p \leq K|t-s|^{p/2} \tag{2.5}$$

for all $s,t \in [0,T]$.

We need the following lemma to prove the above result.

LEMMA 2.4. *Let $\{\alpha_t\}_{0\leq t\leq T}$ and $\{\beta_t\}_{0\leq t\leq T}$ be two progressively measurable processes satisfying condition* (H1) *in Assumption 2.1, and $\{\rho_t\}_{0\leq t\leq T}$ be the solution of (2.3). Then, for any $r \in \mathbb{R}$ we have*

$$\mathbb{E} \sup_{0\leq t\leq T} \rho_t^r < \infty. \tag{2.6}$$

PROOF. Let $t \in [0,T]$. The solution to (2.3) can be written as

$$\rho_t = \exp\left\{\int_0^t \left(\alpha_s - \frac{\beta_s^2}{2}\right) ds + \int_0^t \beta_s\,dW_s\right\}.$$

For any real number r, we have

$$\begin{aligned}
\mathbb{E} \sup_{0\leq t\leq T} \rho_t^r &= \mathbb{E} \sup_{0\leq t\leq T} \exp\left\{\int_0^t r\left(\alpha_s - \frac{\beta_s^2}{2}\right) ds + r\int_0^t \beta_s\,dW_s\right\} \\
&\leq \mathbb{E}\biggl(\exp\left\{|r|\int_0^T |\alpha_s|\,ds + \frac{1}{2}(|r| + r^2)\int_0^T \beta_s^2\,ds\right\} \\
&\qquad \times \sup_{0\leq t\leq T} \exp\left\{r\int_0^t \beta_s\,dW_s - \frac{r^2}{2}\int_0^t \beta_s^2\,ds\right\}\biggr).
\end{aligned}$$

Then, fixing any $p > 1$ and using Hölder's inequality, we obtain

$$\mathbb{E} \sup_{0\leq t\leq T} \rho_t^r \leq C\left(\mathbb{E} \sup_{0\leq t\leq T} \exp\left\{rp\int_0^t \beta_s\,dW_s - \frac{pr^2}{2}\int_0^t \beta_s^2\,ds\right\}\right)^{1/p}, \tag{2.7}$$

where

$$C = \left(\mathbb{E}\exp\left\{q|r|\int_0^T |\alpha_s|\,ds + \frac{q}{2}(|r| + r^2)\int_0^T \beta_s^2\,ds\right\}\right)^{1/q}$$

and $\frac{1}{p} + \frac{1}{q} = 1$.

Set $M_t = \exp\{r\int_0^t \beta_s\,dW_s - \frac{r^2}{2}\int_0^t \beta_s^2\,ds\}$. Then $\{M_t\}_{0\le t\le T}$ is a martingale due to (H1). We can rewrite (2.7) into

$$\mathbb{E}\sup_{0\le t\le T}\rho_t^r \le C\Big(\mathbb{E}\sup_{0\le t\le T}M_t^p\Big)^{1/p}. \tag{2.8}$$

By Doob's maximal inequality, we have

$$\mathbb{E}\sup_{0\le t\le T}M_t^p \le c_p\mathbb{E}M_T^p \tag{2.9}$$

for some constant $c_p>0$ depending only on p. Finally, choosing any $\gamma>1$, $\lambda>1$ such that $\frac{1}{\gamma}+\frac{1}{\lambda}=1$ and applying again the Hölder inequality yield

$$\begin{aligned}
\mathbb{E}M_T^p &= \mathbb{E}\bigg(\exp\bigg\{rp\int_0^T\beta_s\,dW_s - \frac{\gamma}{2}p^2r^2\int_0^T\beta_s^2\,ds\bigg\}\\
&\qquad\qquad \times\exp\bigg\{\frac{\gamma p-1}{2}pr^2\int_0^T\beta_s^2\,ds\bigg\}\bigg)\\
&\le \bigg(\mathbb{E}\exp\bigg\{rp\gamma\int_0^T\beta_s\,dW_s - \frac{1}{2}\gamma^2p^2r^2\int_0^T\beta_s^2\,ds\bigg\}\bigg)^{1/\gamma}\\
&\quad\times\bigg(\mathbb{E}\exp\bigg\{\frac{\lambda(\gamma p-1)}{2}pr^2\int_0^T\beta_s^2\,ds\bigg\}\bigg)^{1/\lambda}\\
&= \bigg(\mathbb{E}\exp\bigg\{\frac{\lambda(\gamma p-1)}{2}pr^2\int_0^T\beta_s^2\,ds\bigg\}\bigg)^{1/\lambda}<\infty.
\end{aligned}$$

Combining this inequality with (2.8) and (2.9) we complete the proof. $\square$

Proof of Theorem 2.3. The existence and uniqueness is well known. We are going to prove (2.5). Let $t\in[0,T]$. Denote $\gamma_t=\rho_t^{-1}$, where $\{\rho_t\}_{0\le t\le T}$ is the solution to (2.3). Then $\{\gamma_t\}_{0\le t\le T}$ satisfies the following linear stochastic differential equation:

$$\begin{cases} d\gamma_t = (-\alpha_t+\beta_t^2)\gamma_t\,dt - \beta_t\gamma_t\,dW_t, & 0\le t\le T,\\ \gamma_0=1.\end{cases}$$

For any $0\le s\le t\le T$ and any positive number $r\ge 1$, we have, using (H2), the Hölder inequality, the Burkholder–Davis–Gundy inequality and Lemma 2.4 applied to the process $\{\gamma_t\}_{0\le t\le T}$,

$$\begin{aligned}
\mathbb{E}|\gamma_t-\gamma_s|^r &= \mathbb{E}\bigg|\int_s^t(-\alpha_u+\beta_u^2)\gamma_u\,du - \int_s^t\beta_u\gamma_u\,dW_u\bigg|^r\\
&\le 2^{r-1}\bigg[\mathbb{E}\bigg|\int_s^t(-\alpha_u+\beta_u^2)\gamma_u\,du\bigg|^r + C_r\mathbb{E}\bigg|\int_s^t\beta_u^2\gamma_u^2\,du\bigg|^{r/2}\bigg] \qquad (2.10)\\
&\le C(t-s)^{r/2},
\end{aligned}$$

where C_r is a constant depending only on r, and C is a constant depending on T, r and the constants appearing in conditions (H1) and (H2).

From (2.3), (2.4) and by Itô's formula, we obtain

$$d(Y_t\rho_t) = -\rho_t f_t\,dt + (\beta_t\rho_t Y_t + \rho_t Z_t)\,dW_t.$$

As a consequence,

$$Y_t = \rho_t^{-1}\mathbb{E}\left(\xi\rho_T + \int_t^T \rho_r f_r\,dr\Big|\mathcal{F}_t\right) = \mathbb{E}\left(\xi\rho_{t,T} + \int_t^T \rho_{t,r} f_r\,dr\Big|\mathcal{F}_t\right), \tag{2.11}$$

where we write $\rho_{t,r} = \rho_t^{-1}\rho_r = \gamma_t\rho_r$ for any $0 \le t \le r \le T$.

Now, fix $0 \le s \le t \le T$. We have

$$\begin{aligned}
\mathbb{E}|Y_t - Y_s|^p &= \mathbb{E}\left|\mathbb{E}\left(\xi\rho_{t,T} + \int_t^T \rho_{t,r} f_r\,dr\Big|\mathcal{F}_t\right) - \mathbb{E}\left(\xi\rho_{s,T} + \int_s^T \rho_{s,r} f_r\,dr\Big|\mathcal{F}_s\right)\right|^p \\
&\le 2^{p-1}\Bigg[\mathbb{E}|\mathbb{E}(\xi\rho_{t,T}|\mathcal{F}_t) - \mathbb{E}(\xi\rho_{s,T}|\mathcal{F}_s)|^p \\
&\qquad + \mathbb{E}\left|\mathbb{E}\left(\int_t^T \rho_{t,r} f_r\,dr\Big|\mathcal{F}_t\right) - \mathbb{E}\left(\int_s^T \rho_{s,r} f_r\,dr\Big|\mathcal{F}_s\right)\right|^p\Bigg] \\
&= 2^{p-1}(I_1 + I_2).
\end{aligned}$$

First we estimate I_1. We have

$$\begin{aligned}
I_1 &= \mathbb{E}|\mathbb{E}(\xi\rho_{t,T}|\mathcal{F}_t) - \mathbb{E}(\xi\rho_{s,T}|\mathcal{F}_s)|^p \\
&= \mathbb{E}|\mathbb{E}(\xi\rho_{t,T}|\mathcal{F}_t) - \mathbb{E}(\xi\rho_{s,T}|\mathcal{F}_t) + \mathbb{E}(\xi\rho_{s,T}|\mathcal{F}_t) - \mathbb{E}(\xi\rho_{s,T}|\mathcal{F}_s)|^p \\
&\le 2^{p-1}[\mathbb{E}|\mathbb{E}(\xi\rho_{t,T}|\mathcal{F}_t) - \mathbb{E}(\xi\rho_{s,T}|\mathcal{F}_t)|^p + \mathbb{E}|\mathbb{E}(\xi\rho_{s,T}|\mathcal{F}_t) - \mathbb{E}(\xi\rho_{s,T}|\mathcal{F}_s)|^p] \\
&\le 2^{p-1}[\mathbb{E}|\xi(\rho_{t,T} - \rho_{s,T})|^p + \mathbb{E}|\mathbb{E}(\xi\rho_{s,T}|\mathcal{F}_t) - \mathbb{E}(\xi\rho_{s,T}|\mathcal{F}_s)|^p] \\
&= 2^{p-1}(I_3 + I_4).
\end{aligned}$$

Using the Hölder inequality, Lemma 2.4 and the estimate (2.10) with $r = \frac{2pq}{q-p}$, the term I_3 can be estimated as follows:

$$\begin{aligned}
I_3 &\le (\mathbb{E}|\xi|^q)^{p/q}(\mathbb{E}|\rho_{t,T} - \rho_{s,T}|^{pq/(q-p)})^{(q-p)/q} \\
&\le (\mathbb{E}|\xi|^q)^{p/q}(\mathbb{E}|\gamma_t - \gamma_s|^{2pq/(q-p)})^{(q-p)/(2q)}(\mathbb{E}\rho_T^{2pq/(q-p)})^{(q-p)/(2q)} \\
&\le C|t-s|^{p/2},
\end{aligned}$$

where C is a constant depending only on p, q, T, $\mathbb{E}|\xi|^q$ and the constants appearing in conditions (H1) and (H2).

In order to estimate the term I_4 we will make use of the condition $\xi\rho_T \in M^{2,q}$. This condition implies that

$$\xi\rho_T = \mathbb{E}(\xi\rho_T) + \int_0^T u_r\,dW_r,$$

where u is a progressively measurable process satisfying $\sup_{0\le t\le T}\mathbb{E}|u_t|^q<\infty$. Therefore, by the Burkholder–Davis–Gundy inequality, we have

$$\begin{aligned}
&\mathbb{E}|\mathbb{E}(\xi\rho_T|\mathcal{F}_t)-\mathbb{E}(\xi\rho_T|\mathcal{F}_s)|^q\\
&\qquad=\mathbb{E}\left|\int_s^t u_r\,dW_r\right|^q\le C_q\mathbb{E}\left|\int_s^t u_r^2\,dr\right|^{q/2}\\
&\qquad\le C_q(t-s)^{(q-2)/2}\mathbb{E}\left(\int_s^t |u_r|^q\,dr\right)\\
&\qquad\le C_q(t-s)^{q/2}\sup_{0\le t\le T}\mathbb{E}|u_t|^q.
\end{aligned}$$

As a consequence, from the definition of I_4 we have

$$\begin{aligned}
I_4&=\mathbb{E}|\gamma_s[\mathbb{E}(\xi\rho_T|\mathcal{F}_t)-\mathbb{E}(\xi\rho_T|\mathcal{F}_s)]|^p\\
&\le(\mathbb{E}\gamma_s^{pq/(q-p)})^{(q-p)/q}(\mathbb{E}|\mathbb{E}(\xi\rho_T|\mathcal{F}_t)-\mathbb{E}(\xi\rho_T|\mathcal{F}_s)|^q)^{p/q}\\
&\le C|t-s|^{p/2},
\end{aligned}$$

where C is a constant depending on $p,q,T,\sup_{0\le t\le T}\mathbb{E}|u_t|^q<\infty$ and the constants appearing in conditions (H1) and (H2).

The term I_2 can be decomposed as follows:

$$\begin{aligned}
I_2&=\mathbb{E}\left|\mathbb{E}\left(\int_t^T\rho_{t,r}f_r\,dr\middle|\mathcal{F}_t\right)-\mathbb{E}\left(\int_s^T\rho_{s,r}f_r\,dr\middle|\mathcal{F}_s\right)\right|^p\\
&\le 3^{p-1}\left[\mathbb{E}\left|\mathbb{E}\left(\int_t^T\rho_{t,r}f_r\,dr\middle|\mathcal{F}_t\right)-\mathbb{E}\left(\int_t^T\rho_{s,r}f_r\,dr\middle|\mathcal{F}_t\right)\right|^p\right.\\
&\qquad+\mathbb{E}\left|\mathbb{E}\left(\int_t^T\rho_{s,r}f_r\,dr\middle|\mathcal{F}_t\right)-\mathbb{E}\left(\int_s^T\rho_{s,r}f_r\,dr\middle|\mathcal{F}_t\right)\right|^p\\
&\qquad\left.+\mathbb{E}\left|\mathbb{E}\left(\int_s^T\rho_{s,r}f_r\,dr\middle|\mathcal{F}_t\right)-\mathbb{E}\left(\int_s^T\rho_{s,r}f_r\,dr\middle|\mathcal{F}_s\right)\right|^p\right]\\
&=3^{p-1}(I_5+I_6+I_7).
\end{aligned}$$

Let us first estimate the term I_5. Suppose that $p<p'<q$. Then, using (2.10) and the Hölder inequality, we can write

$$\begin{aligned}
I_5&=\mathbb{E}\left|\mathbb{E}\left(\int_t^T\rho_{t,r}f_r\,dr\middle|\mathcal{F}_t\right)-\mathbb{E}\left(\int_t^T\rho_{s,r}f_r\,dr\middle|\mathcal{F}_t\right)\right|^p\\
&\le\mathbb{E}\left|\int_t^T(\rho_{t,r}-\rho_{s,r})f_r\,dr\right|^p=\mathbb{E}\left(|\gamma_t-\gamma_s|^p\left|\int_t^T\rho_rf_r\,dr\right|^p\right)\\
&\le\{\mathbb{E}|\gamma_t-\gamma_s|^{pp'/(p'-p)}\}^{(p'-p)/p'}\left\{\mathbb{E}\left|\int_t^T\rho_rf_r\,dr\right|^{p'}\right\}^{p/p'}
\end{aligned}$$

$$
\begin{aligned}
&\le C|t-s|^{p/2}\biggl\{\mathbb{E}\biggl(\int_t^T \rho_r^2\,dr\biggr)^{p'q/(2(q-p'))}\biggr\}^{p(q-p')/(p'q)}\\
&\quad\times\biggl\{\mathbb{E}\biggl(\int_t^T f_r^2\,dr\biggr)^{q/2}\biggr\}^{p/q}\\
&\le \widehat{C}|t-s|^{p/2}\|f\|_{H^q}^p,
\end{aligned}
$$

where $\widehat{C}$ is a constant depending on p, p', q, T and the constants appearing in conditions (H1) and (H2).

Now we estimate I_6. Suppose that $p < p' < q$. We have, as in the estimate of the term I_5,

$$
\begin{aligned}
I_6 &= \mathbb{E}\biggl|\mathbb{E}\biggl(\int_t^T \rho_{s,r} f_r\,dr\Big|\mathcal{F}_t\biggr) - \mathbb{E}\biggl(\int_s^T \rho_{s,r} f_r\,dr\Big|\mathcal{F}_t\biggr)\biggr|^p\\
&\le \mathbb{E}\biggl|\int_s^t \rho_{s,r} f_r\,dr\biggr|^p = \mathbb{E}\biggl(\rho_s^{-p}\biggl|\int_s^t \rho_r f_r\,dr\biggr|^p\biggr)\\
&\le \{\mathbb{E}\rho_s^{-pp'/(p'-p)}\}^{(p'-p)/p'}\biggl\{\mathbb{E}\biggl|\int_s^t \rho_r f_r\,dr\biggr|^{p'}\biggr\}^{p/p'}\\
&= C\biggl\{\mathbb{E}\biggl|\int_s^t \rho_r f_r\,dr\biggr|^{p'}\biggr\}^{p/p'}\\
&\le C|t-s|^{p/2}\Bigl\{\mathbb{E}\sup_{0\le t\le T}\rho_t^{p'q/(q-p')}\Bigr\}^{p(q-p')/(p'q)}\|f\|_{H^q}^p\\
&= \widehat{C}|t-s|^{p/2},
\end{aligned}
$$

where $\widehat{C}$ is a constant depending on p, p', q, T and the constants appearing in conditions (H1) and (H2).

The fact that $\int_0^T \rho_r f_r\,dr$ belongs to $M^{2,q}$ implies that

$$
\int_0^T \rho_r f_r\,dr = \mathbb{E}\int_0^T \rho_r f_r\,dr + \int_0^T v_r\,dW_r,
$$

where $\{v_t\}_{0\le t\le T}$ is a progressively measurable process satisfying

$$
\sup_{0\le t\le T}\mathbb{E}|v_t|^q < \infty.
$$

Then, by the Burkholder–Davis–Gundy inequality we have

$$
\begin{aligned}
&\mathbb{E}\biggl|\mathbb{E}\biggl(\int_s^T \rho_r f_r\,dr\Big|\mathcal{F}_t\biggr) - \mathbb{E}\biggl(\int_s^T \rho_r f_r\,dr\Big|\mathcal{F}_s\biggr)\biggr|^q\\
&\qquad = \mathbb{E}\biggl|\mathbb{E}\biggl(\int_0^T \rho_r f_r\,dr\Big|\mathcal{F}_t\biggr) - \mathbb{E}\biggl(\int_0^T \rho_r f_r\,dr\Big|\mathcal{F}_s\biggr)\biggr|^q\\
&\qquad = \mathbb{E}\biggl|\int_s^t v_r\,dW_r\biggr|^q \le C_q(t-s)^{q/2}\sup_{0\le t\le T}\mathbb{E}|v_t|^q.
\end{aligned}
$$

Finally, we estimate I_7 as follows:

$$
\begin{aligned}
I_7 &= \mathbb{E}\left|\mathbb{E}\left(\int_s^T \rho_{s,r} f_r\,dr\middle|\mathcal{F}_t\right) - \mathbb{E}\left(\int_s^T \rho_{s,r} f_r\,dr\middle|\mathcal{F}_s\right)\right|^p \\
&= \mathbb{E}\left|\rho_s^{-1}\left(\mathbb{E}\left(\int_s^T \rho_r f_r\,dr\middle|\mathcal{F}_t\right) - \mathbb{E}\left(\int_s^T \rho_r f_r\,dr\middle|\mathcal{F}_s\right)\right)\right|^p \\
&\le \{\mathbb{E}\rho_s^{-pq/(q-p)}\}^{(q-p)/p} \\
&\qquad \times \left\{\mathbb{E}\left|\mathbb{E}\left(\int_s^T \rho_r f_r\,dr\middle|\mathcal{F}_t\right) - \mathbb{E}\left(\int_s^T \rho_r f_r\,dr\middle|\mathcal{F}_s\right)\right|^q\right\}^{p/q} \\
&\le C\left\{\mathbb{E}\left|\mathbb{E}\left(\int_s^T \rho_r f_r\,dr\middle|\mathcal{F}_t\right) - \mathbb{E}\left(\int_s^T \rho_r f_r\,dr\middle|\mathcal{F}_s\right)\right|^q\right\}^{p/q} \\
&\le \widehat{C}|t-s|^{p/2},
\end{aligned}
\tag{2.12}
$$

where $\widehat{C}$ is a constant depending on p, q, T, $\sup_{0\le t\le T}\mathbb{E}|v_t|^q$ and the constants appearing in conditions (H1) and (H2).

As a consequence, we obtain for all $s,t\in[0,T]$

$$\mathbb{E}|Y_t - Y_s|^p \le K|t-s|^{p/2},$$

where K is a constant independent of s and t. □

2.3. *The Malliavin calculus for BSDEs.* We return to the study of (1.1). The main assumptions we make on the terminal value ξ and generator f are the following:

ASSUMPTION 2.2. Fix $2 \le p < \frac{q}{2}$.

(i) $\xi \in \mathbb{D}^{2,q}$, and there exists $L > 0$, such that for all $\theta, \theta' \in [0,T]$,

$$\mathbb{E}|D_\theta\xi - D_{\theta'}\xi|^p \le L|\theta-\theta'|^{p/2}, \tag{2.13}$$

$$\sup_{0\le\theta\le T}\mathbb{E}|D_\theta\xi|^q < \infty \tag{2.14}$$

and

$$\sup_{0\le\theta\le T}\sup_{0\le u\le T}\mathbb{E}|D_u D_\theta\xi|^q < \infty. \tag{2.15}$$

(ii) The generator $f(t,y,z)$ has continuous and uniformly bounded first- and second-order partial derivatives with respect to y and z, and $f(\cdot,0,0)\in H^q_{\mathcal{F}}([0,T])$.

(iii) Assume that ξ and f satisfy the above conditions (i) and (ii). Let (Y,Z) be the unique solution to (1.1) with terminal value ξ and generator f. For each $(y,z)\in\mathbb{R}\times\mathbb{R}$, $f(\cdot,y,z)$, $\partial_y f(\cdot,y,z)$ and $\partial_z f(\cdot,y,z)$ belong to $\mathbb{L}^{1,q}_a$, and the Malliavin derivatives $Df(\cdot,y,z)$, $D\partial_y f(\cdot,y,z)$ and $D\partial_z f(\cdot,y,z)$ satisfy

$$\sup_{0\le\theta\le T}\mathbb{E}\left(\int_\theta^T |D_\theta f(t,Y_t,Z_t)|^2\,dt\right)^{q/2}<\infty, \tag{2.16}$$

$$\sup_{0\le\theta\le T}\mathbb{E}\left(\int_\theta^T |D_\theta \partial_y f(t,Y_t,Z_t)|^2\,dt\right)^{q/2}<\infty, \tag{2.17}$$

$$\sup_{0\le\theta\le T}\mathbb{E}\left(\int_\theta^T |D_\theta \partial_z f(t,Y_t,Z_t)|^2\,dt\right)^{q/2}<\infty, \tag{2.18}$$

and there exists $L>0$ such that for any $t\in(0,T]$, and for any $0\le\theta,\theta'\le t\le T$

$$\mathbb{E}\left(\int_t^T |D_\theta f(r,Y_r,Z_r)-D_{\theta'} f(r,Y_r,Z_r)|^2\,dr\right)^{p/2}\le L|\theta-\theta'|^{p/2}. \tag{2.19}$$

For each $\theta\in[0,T]$, and each pair of (y,z), $D_\theta f(\cdot,y,z)\in\mathbb{L}_a^{1,q}$ and it has continuous partial derivatives with respect to y,z, which are denoted by $\partial_y D_\theta f(t,y,z)$ and $\partial_z D_\theta f(t,y,z)$, and the Malliavin derivative $D_u D_\theta f(t,y,z)$ satisfies

$$\sup_{0\le\theta\le T}\sup_{0\le u\le T}\mathbb{E}\left(\int_{\theta\vee u}^T |D_u D_\theta f(t,Y_t,Z_t)|^2\,dt\right)^{q/2}<\infty. \tag{2.20}$$

The following property is easy to check and we omit the proof.

REMARK 2.5. Conditions (2.17) and (2.18) imply

$$\sup_{0\le\theta\le T}\mathbb{E}\left(\int_\theta^T |\partial_y D_\theta f(t,Y_t,Z_t)|^2\,dt\right)^{q/2}<\infty$$

and

$$\sup_{0\le\theta\le T}\mathbb{E}\left(\int_\theta^T |\partial_z D_\theta f(t,Y_t,Z_t)|^2\,dt\right)^{q/2}<\infty,$$

respectively.

The following is the main result of this section.

THEOREM 2.6. *Let Assumption 2.2 be satisfied.*

(a) *There exists a unique solution pair $\{(Y_t,Z_t)\}_{0\le t\le T}$ to the BSDE (1.1), and Y,Z are in $\mathbb{L}_a^{1,q}$. A version of the Malliavin derivatives $\{(D_\theta Y_t,$*

$D_\theta Z_t)\}_{0\le\theta,t\le T}$ *of the solution pair satisfies the following linear BSDE:*

$$\begin{aligned} D_\theta Y_t = D_\theta\xi + \int_t^T [&\partial_y f(r,Y_r,Z_r) D_\theta Y_r \\ &+ \partial_z f(r,Y_r,Z_r) D_\theta Z_r + D_\theta f(r,Y_r,Z_r)]\,dr \\ &- \int_t^T D_\theta Z_r\,dW_r, \qquad 0\le\theta\le t\le T; \end{aligned} \tag{2.21}$$

$$D_\theta Y_t = 0, \qquad D_\theta Z_t = 0, \qquad 0\le t<\theta\le T. \tag{2.22}$$

Moreover, $\{D_t Y_t\}_{0\le t\le T}$ *defined by (2.21) gives a version of* $\{Z_t\}_{0\le t\le T}$*, namely,* $\mu\times P$ *a.e.*

$$Z_t = D_t Y_t. \tag{2.23}$$

(b) *There exists a constant* $K>0$*, such that, for all* $s,t\in[0,T]$,

$$\mathbb{E}|Z_t - Z_s|^p \le K|t-s|^{p/2}. \tag{2.24}$$

PROOF. Part (a): The proof of the existence and uniqueness of the solution (Y,Z), and $Y,Z\in\mathbb{L}_a^{1,2}$ is similar to that of Proposition 5.3 in [7], and also the fact that $(D_\theta Y_t, D_\theta Z_t)$ is given by (2.21) and (2.22). In Proposition 5.3 in [7] the exponent q is equal to 4, and one assumes that $\int_0^T \|D_\theta f(\cdot,Y,Z)\|_{H^2}^2\,d\theta<\infty$, which is a consequence of (2.16) and the fact that $Y,Z\in\mathbb{L}_a^{1,2}$.

Furthermore, from conditions (2.14) and (2.16) and the estimate in Lemma 2.2, we obtain

$$\sup_{0\le\theta\le T}\left\{\mathbb{E}\sup_{\theta\le t\le T}|D_\theta Y_t|^q + \mathbb{E}\left(\int_\theta^T |D_\theta Z_t|^2\,dt\right)^{q/2}\right\}<\infty. \tag{2.25}$$

Hence, by Proposition 1.5.5 in [14], Y and Z belong to $\mathbb{L}_a^{1,q}$.

Part (b): Let $0\le s\le t\le T$. In this proof, $C>0$ will be a constant independent of s and t, and may vary from line to line.

By representation (2.23) we have

$$Z_t - Z_s = D_t Y_t - D_s Y_s = (D_t Y_t - D_s Y_t) + (D_s Y_t - D_s Y_s). \tag{2.26}$$

From Lemma 2.2 and equation (2.21) for $\theta = s$ and $\theta' = t$, respectively, we obtain, using conditions (2.13) and (2.19),

$$\begin{aligned} &\mathbb{E}|D_t Y_t - D_s Y_t|^p + \mathbb{E}\left(\int_t^T |D_t Z_r - D_s Z_r|^2\,dr\right)^{p/2} \\ &\quad\le C\Biggl[\mathbb{E}|D_t\xi - D_s\xi|^p \\ &\qquad\quad + \mathbb{E}\left(\int_t^T |D_t f(r,Y_r,Z_r) - D_s f(r,Y_r,Z_r)|^2\,dr\right)^{p/2}\Biggr] \\ &\quad\le C|t-s|^{p/2}. \end{aligned} \tag{2.27}$$

Denote $\alpha_u = \partial_y f(u, Y_u, Z_u)$ and $\beta_u = \partial_z f(u, Y_u, Z_u)$ for all $u \in [0,T]$. Then, by Assumption 2.2(ii), the processes α and β satisfy conditions (H1) and (H2) in Assumption 2.1, and from (2.21) we have for $r \in [s,T]$

$$D_s Y_r = D_s\xi + \int_r^T [\alpha_u D_s Y_u + \beta_u D_s Z_u + D_s f(u, Y_u, Z_u)]\,du - \int_r^T D_s Z_u\,dW_u.$$

Next, we are going to use Theorem 2.3 to estimate $\mathbb{E}|D_s Y_t - D_s Y_s|^p$. Fix p' with $p < p' < \frac{q}{2}$ (notice that $p' < \frac{q}{2}$ is equivalent to $\frac{p'}{q-p'} < 1$). From conditions (2.14) and (2.16), it is obvious that $D_s\xi \in L^q(\Omega) \subset L^{p'}(\Omega)$ and $D_s f(\cdot, Y, Z) \in H^q([0,T]) \subset H^{p'}([0,T])$ for any $s \in [0,T]$. We are going to show that, for any $s \in [0,T]$, $\rho_T D_s\xi$ and $\int_s^T \rho_u D_s f(u, Y_u, Z_u)\,du$ are elements in $M^{2,p'}$, where

$$\rho_r = \exp\left\{\int_0^r \beta_u\,dW_u + \int_0^r \left(\alpha_u - \frac{1}{2}\beta_u^2\right) du\right\}.$$

For any $0 \le \theta \le r \le T$, let us compute

$$\begin{aligned} D_\theta \rho_r = \rho_r \biggl\{ &\int_\theta^r [\partial_{yz} f(u, Y_u, Z_u) D_\theta Y_u \\ &\qquad + \partial_{zz} f(u, Y_u, Z_u) D_\theta Z_u + D_\theta \partial_z f(u, Y_u, Z_u)]\,dW_u \\ &+ \partial_z f(\theta, Y_\theta, Z_\theta) \\ &+ \int_\theta^r (\partial_{yy} f(u, Y_u, Z_u) - \partial_{yz} f(u, Y_u, Z_u)\beta_u) D_\theta Y_u\,du \\ &+ \int_\theta^r (\partial_{yz} f(u, Y_u, Z_u) - \partial_{zz} f(u, Y_u, Z_u)\beta_u) D_\theta Z_u\,du \\ &\quad + \int_\theta^r (D_\theta \partial_y f(u, Y_u, Z_u) - \beta_u D_\theta \partial_z f(u, Y_u, Z_u))\,du \biggr\}. \end{aligned}$$

By the boundedness of the first- and second-order partial derivatives of f with respect to y and z, (2.17), (2.18), (2.25), Lemma 2.4, the Hölder inequality and the Burkholder–Davis–Gundy inequality, it is easy to show that for any $p'' < q$,

$$\sup_{0 \le \theta \le T} \mathbb{E} \sup_{\theta \le r \le T} |D_\theta \rho_r|^{p''} < \infty. \tag{2.28}$$

By the Clark–Ocone–Haussman formula, we have

$$\begin{aligned} \rho_T D_s\xi &= \mathbb{E}(\rho_T D_s\xi) + \int_0^T \mathbb{E}(D_\theta(\rho_T D_s\xi)|\mathcal{F}_\theta)\,dW_\theta \\ &= \mathbb{E}(\rho_T D_s\xi) + \int_0^T \mathbb{E}(D_\theta\rho_T D_s\xi + \rho_T D_\theta D_s\xi|\mathcal{F}_\theta)\,dW_\theta \\ &= \mathbb{E}(\rho_T D_s\xi) + \int_0^T u_\theta^s\,dW_\theta \end{aligned}$$

and

$$\begin{aligned}
&\int_s^T \rho_r D_s f(r, Y_r, Z_r)\, dr \\
&\quad = \mathbb{E}\int_s^T \rho_r D_s f(r, Y_r, Z_r)\, dr \\
&\qquad + \int_0^T \mathbb{E}\bigg(D_\theta \int_s^T \rho_r D_s f(r, Y_r, Z_r)\, dr \bigg| \mathcal{F}_\theta \bigg)\, dW_\theta \\
&\quad = \mathbb{E}\int_s^T \rho_r D_s f(r, Y_r, Z_r)\, dr \\
&\qquad + \int_0^T \mathbb{E}\bigg(\int_s^T [D_\theta \rho_r D_s f(r, Y_r, Z_r) \\
&\qquad\qquad + \rho_r \partial_y D_s f(r, Y_r, Z_r) D_\theta Y_r \\
&\qquad\qquad + \rho_r \partial_z D_s f(r, Y_r, Z_r) D_\theta Z_r \\
&\qquad\qquad\quad + \rho_r D_\theta D_s f(r, Y_r, Z_r)]\, dr \bigg| \mathcal{F}_\theta \bigg)\, dW_\theta \\
&\quad = \mathbb{E}\int_s^T \rho_r D_s f(r, Y_r, Z_r)\, dr + \int_0^T v_\theta^s\, dW_\theta.
\end{aligned}$$

We claim that $\sup_{0\le\theta\le T}\mathbb{E}|u_\theta^s|^{p'} < \infty$ and $\sup_{0\le\theta\le T}\mathbb{E}|v_\theta^s|^{p'} < \infty$. In fact,

$$\begin{aligned}
\mathbb{E}|u_\theta^s|^{p'} &= \mathbb{E}|\mathbb{E}(D_\theta \rho_T D_s \xi + \rho_T D_\theta D_s \xi | \mathcal{F}_\theta)|^{p'} \\
&\le 2^{p'-1}(\mathbb{E}|D_\theta \rho_T D_s \xi|^{p'} + \mathbb{E}|\rho_T D_\theta D_s \xi|^{p'}) \\
&\le 2^{p'-1}((\mathbb{E}|D_\theta \rho_T|^{p'q/(q-p')})^{(q-p')/q} (\mathbb{E}|D_s \xi|^q)^{p'/q} \\
&\qquad + (\mathbb{E}\rho_T^{p'q/(q-p')})^{(q-p')/q} (\mathbb{E}|D_\theta D_s \xi|^q)^{p'/q}).
\end{aligned}$$

By (2.14), (2.15), (2.28) and Lemma 2.4, we have $\sup_{0\le s\le T}\sup_{0\le\theta\le T}\mathbb{E}|u_\theta^s|^{p'} < \infty$. On the other hand,

$$\begin{aligned}
\mathbb{E}|v_\theta^s|^{p'} &= \mathbb{E}\bigg|\mathbb{E}\bigg(\int_s^T [D_\theta \rho_r D_s f(r, Y_r, Z_r) \\
&\qquad\qquad + \rho_r \partial_y D_s f(r, Y_r, Z_r) D_\theta Y_r \\
&\qquad\qquad + \rho_r \partial_z D_s f(r, Y_r, Z_r) D_\theta Z_r \\
&\qquad\qquad\quad + \rho_r D_\theta D_s f(r, Y_r, Z_r)]\, dr \bigg| \mathcal{F}_\theta \bigg) \bigg|^{p'} \\
&\le 4^{p'-1}[J_1 + J_2 + J_3 + J_4],
\end{aligned}$$

where

$$J_1 = \mathbb{E}\left|\int_s^T D_\theta \rho_r D_s f(r, Y_r, Z_r)\, dr\right|^{p'},$$

$$J_2 = \mathbb{E}\left|\int_s^T \rho_r \partial_y D_s f(r, Y_r, Z_r) D_\theta Y_r\, dr\right|^{p'},$$

$$J_3 = \mathbb{E}\left|\int_s^T \rho_r \partial_z D_s f(r, Y_r, Z_r) D_\theta Z_r\, dr\right|^{p'}$$

and

$$J_4 = \mathbb{E}\left|\int_s^T \rho_r D_\theta D_s f(r, Y_r, Z_r)\, dr\right|^{p'}.$$

For J_1, we have

$$\begin{aligned}
J_1 &\le \mathbb{E}\left(\sup_{\theta\le r\le T} |D_\theta \rho_r|^{p'} \left|\int_s^T D_s f(r, Y_r, Z_r)\, dr\right|^{p'}\right) \\
&\le \left(\mathbb{E} \sup_{\theta\le r\le T} |D_\theta \rho_r|^{p'q/(q-p')}\right)^{(q-p')/q} \\
&\quad \times \left(\mathbb{E}\left|\int_s^T D_s f(r, Y_r, Z_r)\, dr\right|^{q}\right)^{p'/q} \\
&\le T^{p'/2} \left(\mathbb{E} \sup_{\theta\le r\le T} |D_\theta \rho_r|^{p'q/(q-p')}\right)^{(q-p')/q} \\
&\quad \times \left(\mathbb{E}\left(\int_0^T |D_s f(r, Y_r, Z_r)|^2\, dr\right)^{q/2}\right)^{p'/q}.
\end{aligned}$$

For J_2, we have

$$\begin{aligned}
J_2 &\le \mathbb{E}\left(\sup_{\theta\le r\le T} |D_\theta Y_r|^{p'} \left(\sup_{0\le r\le T} \rho_r \int_s^T |\partial_y D_s f(r, Y_r, Z_r)|\, dr\right)^{p'}\right) \\
&\le \left(\mathbb{E} \sup_{\theta\le r\le T} |D_\theta Y_r|^{q}\right)^{p'/q} \\
&\quad \times \left(\mathbb{E}\left(\sup_{0\le r\le T} \rho_r \int_s^T |\partial_y D_s f(r, Y_r, Z_r)|\, dr\right)^{p'q/(q-p')}\right)^{(q-p')/q} \\
&\le \left(\mathbb{E} \sup_{\theta\le r\le T} |D_\theta Y_r|^{q}\right)^{p'/q} \left(\mathbb{E} \sup_{0\le r\le T} \rho_r^{p'q/(q-2p')}\right)^{(q-2p')/q} \\
&\quad \times \left(\mathbb{E}\left(\int_s^T |\partial_y D_s f(r, Y_r, Z_r)|\, dr\right)^{q}\right)^{p'/q}
\end{aligned}$$

$$\le T^{p'/2}\Big(\mathbb{E}\sup_{\theta\le r\le T}|D_\theta Y_r|^q\Big)^{p'/q}\Big(\mathbb{E}\sup_{0\le r\le T}\rho_r^{p'q/(q-2p')}\Big)^{(q-2p')/q}$$
$$\times\left(\mathbb{E}\left(\int_0^T|\partial_y D_s f(r,Y_r,Z_r)|^2\,dr\right)^{q/2}\right)^{p'/q}.$$

Using a similar techniques as before, we obtain that

$$J_3\le T^{p'/2}\left(\mathbb{E}\left(\int_0^T|D_\theta Z_r|^2\,dr\right)^{q/2}\right)^{p'/q}\Big(\mathbb{E}\sup_{0\le r\le T}\rho_r^{p'q/(q-2p')}\Big)^{(q-2p')/q}$$
$$\times\left(\mathbb{E}\left(\int_0^T|\partial_z D_s f(r,Y_r,Z_r)|^2\,dr\right)^{q/2}\right)^{p'/q}$$

and

$$J_4\le T^{p'/2}\Big(\mathbb{E}\sup_{0\le r\le T}\rho_r^{p'q/(q-p')}\Big)^{(q-p')/q}$$
$$\times\left(\mathbb{E}\left(\int_0^T|D_\theta D_s f(r,Y_r,Z_r)|^2\,dr\right)^{q/2}\right)^{p'/q}.$$

By (2.16), (2.17)–(2.20), (2.28) and Lemma 2.4, we obtain that

$$\sup_{0\le s\le T}\sup_{0\le\theta\le T}\mathbb{E}|v_\theta^s|^{p'}<\infty.$$

Therefore, $\rho_T\xi$ and $\int_0^T\rho_u D_s f(u,Y_u,Z_u)\,du$ belong to $M^{2,p'}$.

Thus by Theorem 2.3 with $p<p'$, there is a constant $C(s)>0$, such that

$$\mathbb{E}|D_sY_t-D_sY_s|^p\le C(s)|t-s|^{p/2}$$

for all $t\in[s,T]$. Furthermore, taking into account the proof of the estimates I_k $(k=3,4,\dots,7)$ in the proof of Theorem 2.3, we can show that $\sup_{0\le s\le T}C(s)=:C<\infty$. Thus we have

$$\mathbb{E}|D_sY_t-D_sY_s|^p\le C|t-s|^{p/2} \tag{2.29}$$

for all $s,t\in[0,T]$. Combining (2.29) with (2.26) and (2.27), we obtain that there is a constant $K>0$ independent of s and t, such that

$$\mathbb{E}|Z_t-Z_s|^p\le K|t-s|^{p/2}$$

for all $s,t\in[0,T]$. □

COROLLARY 2.7. *Under the assumptions in Theorem 2.2, let $(Y,Z)\in S_{\mathcal{F}}^q([0,T])\times H_{\mathcal{F}}^q([0,T])$ be the unique solution pair to (1.1). If $\sup_{0\le t\le T}\mathbb{E}|Z_t|^q<\infty$, then there exists a constant C, such that, for any $s,t\in[0,T]$,*

$$\mathbb{E}|Y_t-Y_s|^q\le C|t-s|^{q/2}. \tag{2.30}$$

PROOF. Without loss of generality we assume $0 \le s \le t \le T$. $C > 0$ is a constant independent of s and t, which may vary from line to line. Since

$$Y_s = Y_t + \int_s^t f(r, Y_r, Z_r)\, dr - \int_s^t Z_r\, dW_r,$$

we have, by the Lipschitz condition on f,

$$\begin{aligned}
\mathbb{E}|Y_t - Y_s|^q &= \mathbb{E}\left|\int_s^t f(r, Y_r, Z_r)\, dr - \int_s^t Z_r\, dW_r\right|^q \\
&\le 2^{q-1}\left(\mathbb{E}\left|\int_s^t f(r, Y_r, Z_r)\, dr\right|^q + \mathbb{E}\left|\int_s^t Z_r\, dW_r\right|^q\right) \\
&\le C_q\left(|t-s|^{q/2}\mathbb{E}\left(\int_s^t |f(r, Y_r, Z_r)|^2\, dr\right)^{q/2} + \mathbb{E}\left(\int_s^t |Z_r|^2\, dr\right)^{q/2}\right) \\
&\le C\Bigg\{|t-s|^{q/2}\Bigg[\mathbb{E}\left(\int_s^t |Y_r|^2\, dr\right)^{q/2} + \mathbb{E}\left(\int_s^t |Z_r|^2\, dr\right)^{q/2} \\
&\qquad\qquad + \mathbb{E}\left(\int_s^t |f(r,0,0)|^2\, dr\right)^{q/2}\Bigg] \\
&\qquad\qquad + |t-s|^{q/2} \sup_{0 \le r \le T} \mathbb{E}|Z_r|^q\Bigg\} \\
&\le C|t-s|^{q/2}.
\end{aligned}$$

The proof is complete. □

REMARK 2.8. From Theorem 2.6 we know that $\{(D_\theta Y_t, D_\theta Z_t)\}_{0 \le \theta \le t \le T}$ satisfies equation (2.21) and $Z_t = D_t Y_t$, $\mu \times P$ a.e. Moreover, since (2.14) and (2.16) hold, we can apply the estimate (2.2) in Lemma 2.2 to the linear BSDE (2.21) and deduce $\sup_{0 \le t \le T} \mathbb{E}|Z_t|^q < \infty$. Therefore, by Lemma 2.7, the process Y satisfies the inequality (2.30). By Kolmogorov's continuity criterion this implies that Y has Hölder continuous trajectories of order γ for any $\gamma < \frac{1}{2} - \frac{1}{q}$.

2.4. *Examples.* In this section we discuss three particular examples where Assumption 2.2 is satisfied.

EXAMPLE 2.9. Consider equation (1.1). Assume that:

(a) $f(t, y, z) : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a deterministic function that has uniformly bounded first- and second-order partial derivatives with respect to y and z, and $\int_0^T f(t, 0, 0)^2\, dt < \infty$.

(b) The terminal value ξ is a multiple stochastic integral of the form

$$\xi = \int_{[0,T]^n} g(t_1,\dots,t_n)\, dW_{t_1}\cdots dW_{t_n}, \tag{2.31}$$

where $n \geq 2$ is an integer and $g(t_1,\dots,t_n)$ is a symmetric function in $L^2([0,T]^n)$, such that

$$\sup_{0\le u\le T}\int_{[0,T]^{n-1}} g(t_1,\dots,t_{n-1},u)^2\, dt_1\cdots dt_{n-1} < \infty,$$

$$\sup_{0\le u,v\le T}\int_{[0,T]^{n-2}} g(t_1,\dots,t_{n-2},u,v)^2\, dt_1\cdots dt_{n-2} < \infty,$$

and there exists a constant $L > 0$ such that for any $u, v \in [0,T]$

$$\int_{[0,T]^{n-1}} |g(t_1,\dots,t_{n-1},u) - g(t_1,\dots,t_{n-1},v)|^2\, dt_1\cdots dt_{n-1} < L|u-v|.$$

From (2.31), we know that

$$D_u\xi = n\int_{[0,T]^{n-1}} g(t_1,\dots,t_{n-1},u)\, dW_{t_1}\cdots dW_{t_{n-1}}.$$

The above assumption implies Assumption 2.2, and therefore, Z satisfies the Hölder continuity property (2.24).

EXAMPLE 2.10. Let $\Omega = C_0([0,1])$ equipped with the Borel σ-field and Wiener measure. Then, Ω is a Banach space with supremum norm $\|\cdot\|_\infty$, and $W_t = \omega(t)$ is the canonical Wiener process. Consider equation (1.1) on the interval $[0,1]$. Assume that:

(g1) $f(t,y,z) : [0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a deterministic function that has uniformly bounded first- and second-order partial derivatives with respect to y and z, and $\int_0^1 f(t,0,0)^2\, dt < \infty$.

(g2) $\xi = \varphi(W)$, where $\varphi : \Omega\to\mathbb{R}$ is twice Fréchet differentiable, and the first- and second-order Fréchet derivatives $\delta\varphi$ and $\delta^2\varphi$ satisfy

$$|\varphi(\omega)| + \|\delta\varphi(\omega)\| + \|\delta^2\varphi(\omega)\| \le C_1 \exp\{C_2\|\omega\|_\infty^r\}$$

for all $\omega\in\Omega$ and some constants $C_1 > 0$, $C_2 > 0$ and $0 < r < 2$, where $\|\cdot\|$ denotes the operator norm (total variation norm).

(g3) If λ denotes the signed measure on $[0,1]$ associated with $\delta\varphi$, there exists a constant $L > 0$ such that for all $0 \le \theta \le \theta' \le 1$,

$$\mathbb{E}|\lambda((\theta,\theta'])|^p \le L|\theta-\theta'|^{p/2}$$

for some $p \geq 2$.

It is easy to show that $D_\theta \xi = \lambda((\theta,1])$ and $D_u D_\theta \xi = \nu((\theta,1] \times (u,1])$, where ν denotes the signed measure on $[0,1] \times [0,1]$ associated with $\delta^2 \varphi$. From the above assumptions and Fernique's theorem, we can get Assumption 2.2, and therefore, the Hölder continuity property (2.24) of Z.

EXAMPLE 2.11. Consider the following forward–backward stochastic differential equation (FBSDE for short):

$$\begin{cases} X_t = X_0 + \int_0^t b(r, X_r)\, dr + \int_0^t \sigma(r, X_r)\, dW_r, \\ Y_t = \varphi\left(\int_0^T X_r^2\, dr\right) + \int_t^T f(r, X_r, Y_r, Z_r)\, dr - \int_t^T Z_r\, dW_r, \end{cases} \tag{2.32}$$

where b, σ, φ and f are deterministic functions, and $X_0 \in \mathbb{R}$.

We make the following assumptions:

(h1) b and σ has uniformly bounded first- and second-order partial derivatives with respect to x, and there is a constant $L > 0$, such that, for any $s, t \in [0,T]$, $x \in \mathbb{R}$,

$$|\sigma(t,x) - \sigma(s,x)| \le L|t-s|^{1/2}.$$

(h2) $\sup_{0 \le t \le T}\{|b(t,0)| + |\sigma(t,0)|\} < \infty$.

(h3) φ is twice differentiable, and there exist a constant $C > 0$ and a positive integer n such that

$$\left|\varphi\left(\int_0^T X_t^2\, dt\right)\right| + \left|\varphi'\left(\int_0^T X_t^2\, dt\right)\right| + \left|\varphi''\left(\int_0^T X_t^2\, dt\right)\right| \le C(1 + \|X\|_\infty)^n,$$

where $\|x\|_\infty = \sup\{|x(t)|, 0 \le t \le T\}$ for any $x \in C([0,T])$.

(h4) $f(t,x,y,z)$ has continuous and uniformly bounded first- and second-order partial derivatives with respect to x, y and z and $\int_0^T f(t,0,0,0)^2\, dt < \infty$.

Notice that in this example, $\Phi(X) = \varphi(\int_0^T X_t^2\, dt)$ is not necessarily globally Lipschitz in X, and the results of [16] cannot be applied directly.

Under the above assumptions, (h1) and (h4), equation (2.32) has a unique solution triple (X, Y, Z), and we have the following classical results: for any real number $r > 0$, there exists a constant $C > 0$ such that

$$\mathbb{E} \sup_{0 \le t \le T} |X_t|^r < \infty, \qquad \mathbb{E}|X_t - X_s|^r \le C|t-s|^{r/2}$$

for any $t, s \in [0,T]$. For any fixed $(y,z) \in \mathbb{R} \times \mathbb{R}$, we have $D_\theta f(t, X_t, y, z) = \partial_x f(t, X_t, y, z) D_\theta X_t$. Then, under all the assumptions in this example, by Theorem 2.2.1 and Lemma 2.2.2 in [14] and the results listed above, we can verify Assumption 2.2. Therefore, Z has the Hölder continuity property (2.24).

Note that in the multidimensional case we do not require the matrix $\sigma\sigma^T$ to be invertible.

3. An explicit scheme for BSDEs. In the remaining part of this paper, we let $\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ be a partition of the interval $[0, T]$ and $|\pi| = \max_{0\le i\le n-1} |t_{i+1} - t_i|$. Denote $\Delta_i = t_{i+1} - t_i, 0 \le i \le n-1$.

From (1.1), we know that, when $t \in [t_i, t_{i+1}]$,

$$Y_t = Y_{t_{i+1}} + \int_t^{t_{i+1}} f(r, Y_r, Z_r)\, dr - \int_t^{t_{i+1}} Z_r\, dW_r. \tag{3.1}$$

Comparing with the numerical schemes for forward stochastic differential equations, we could introduce a numerical scheme of the form

$$Y_{t_n}^{1,\pi} = \xi^\pi,$$

$$Y_{t_i}^{1,\pi} = Y_{t_{i+1}}^{1,\pi} + f(t_{i+1}, Y_{t_{i+1}}^{1,\pi}, Z_{t_{i+1}}^{1,\pi})\Delta_i - \int_{t_i}^{t_{i+1}} Z_r^{1,\pi}\, dW_r,$$

$$t \in [t_i, t_{i+1}), i = n-1, n-2, \ldots, 0,$$

where $\xi^\pi \in L^2(\Omega)$ is an approximation of the terminal condition ξ. This leads to a backward recursive formula for the sequence $\{Y_{t_i}^{1,\pi}, Z_{t_i}^{1,\pi}\}_{0\le i\le n}$. In fact, once $Y_{t_{i+1}}^{1,\pi}$ and $Z_{t_{i+1}}^{1,\pi}$ are defined, then we can find $Y_{t_i}^{1,\pi}$ by

$$Y_{t_i}^{1,\pi} = \mathbb{E}(Y_{t_{i+1}}^{1,\pi} + f(t_{i+1}, Y_{t_{i+1}}^{1,\pi}, Z_{t_{i+1}}^{1,\pi})\Delta_i | \mathcal{F}_{t_i}),$$

and $\{Z_r^{1,\pi}\}_{t_i\le r<t_{i+1}}$ is determined by the stochastic integral representation of the random variable

$$Y_{t_i}^{1,\pi} - Y_{t_{i+1}}^{1,\pi} - f(t_{i+1}, Y_{t_{i+1}}^{1,\pi}, Z_{t_{i+1}}^{1,\pi})\Delta_i.$$

Although $\{Z_r^{1,\pi}\}_{t_i\le r<t_{i+1}}$ can be expressed explicitly by Clark–Ocone–Haussman formula, its computation is a hard problem in practice. On the other hand, there are difficulties in studying the convergence of the above scheme.

An alternative scheme is introduced in [16], where the approximating pairs (Y^π, Z^π) are defined recursively by

$$Y_{t_n}^\pi = \xi^\pi, \qquad Z_{t_n}^\pi = 0,$$

$$Y_t^\pi = Y_{t_{i+1}}^\pi + f\left(t_{i+1}, Y_{t_{i+1}}^\pi, \mathbb{E}\left(\frac{1}{\Delta_{i+1}} \int_{t_{i+1}}^{t_{i+2}} Z_r^\pi\, dr \middle| \mathcal{F}_{t_{i+1}}\right)\right)\Delta_i - \int_t^{t_{i+1}} Z_r^\pi\, dW_r, \qquad t \in [t_i, t_{i+1}), i = n-1, n-2, \ldots, 0, \tag{3.2}$$

where, by convention, $\mathbb{E}(\frac{1}{\Delta_{i+1}} \int_{t_{i+1}}^{t_{i+2}} Z_r^\pi\, dr | \mathcal{F}_{t_{i+1}}) = 0$ when $i = n-1$. In [16] the following rate of convergence is proved for this approximation scheme, assuming that the terminal value ξ and the generator f are functionals of a forward diffusion associated with the BSDE,

$$\max_{0\le i\le n} \mathbb{E}|Y_{t_i} - Y_{t_i}^\pi|^2 + \mathbb{E}\int_0^T |Z_t - Z_t^\pi|^2\, dt \le K|\pi|. \tag{3.3}$$

The main result of this section is the following, which on one hand improves the above rate of convergence, and on the other hand extends terminal value ξ and generator f to more general situation.

THEOREM 3.1. *Consider the approximation scheme (3.2). Let Assumption 2.2 be satisfied, and let the partition π satisfy $\max_{0\le i\le n-1}\Delta_i/\Delta_{i+1}\le L_1$, where L_1 is a constant. Assume that a constant $L_2>0$ exists such that*

$$|f(t_2,y,z)-f(t_1,y,z)|\le L_2|t_2-t_1|^{1/2} \tag{3.4}$$

for all $t_1,t_2\in[0,T]$ and $y,z\in\mathbb{R}$. Then there are positive constants K and δ, independent of the partition π, such that, if $|\pi|<\delta$, then

$$\mathbb{E}\sup_{0\le t\le T}|Y_t-Y_t^\pi|^2+\mathbb{E}\int_0^T|Z_t-Z_t^\pi|^2\,dt\le K(|\pi|+\mathbb{E}|\xi-\xi^\pi|^2). \tag{3.5}$$

PROOF. In this proof, $C>0$ will denote a constant independent of the partition π, which may vary from line to line. Inequality (2.24) in Theorem 2.6(b) yields the following estimate (Theorem 3.1 in [16]) with $p=2$:

$$\sum_{i=0}^{n-1}\mathbb{E}\int_{t_i}^{t_{i+1}}(|Z_t-Z_{t_i}|^2+|Z_t-Z_{t_{i+1}}|^2)\,dt\le C|\pi|.$$

Using this estimate and following the same argument as the proof of Theorem 5.3 in [16], we can obtain the following result:

$$\max_{0\le i\le n}\mathbb{E}|Y_{t_i}-Y_{t_i}^\pi|^2+\mathbb{E}\int_0^T|Z_t-Z_t^\pi|^2\,dt\le C(|\pi|+\mathbb{E}|\xi-\xi^\pi|^2). \tag{3.6}$$

Denote

$$\widetilde{Z}_{t_i}^\pi=\begin{cases}0, & \text{if } i=n;\\ \mathbb{E}\left(\dfrac{1}{\Delta_i}\displaystyle\int_{t_i}^{t_{i+1}}Z_r^\pi\,dr\Big|\mathcal{F}_{t_i}\right), & \text{if } i=n-1,n-2,\dots,0.\end{cases} \tag{3.7}$$

If $t_i\le t<t_{i+1}$, $i=n-1,n-2,\dots,0$, then, by iteration, we have

$$\begin{aligned}Y_t^\pi&=Y_{t_{i+1}}^\pi+f(t_{i+1},Y_{t_{i+1}}^\pi,\widetilde{Z}_{t_{i+1}}^\pi)\Delta_i-\int_t^{t_{i+1}}Z_r^\pi\,dW_r\\&=\xi^\pi+\sum_{k=i+1}^n f(t_k,Y_{t_k}^\pi,\tilde{Z}_{t_k}^\pi)\Delta_{k-1}-\int_t^T Z_r^\pi\,dW_r.\end{aligned} \tag{3.8}$$

Therefore,

$$Y_t^\pi=\mathbb{E}\left(\xi^\pi+\sum_{k=i+1}^n f(t_k,Y_{t_k}^\pi,\widetilde{Z}_{t_k}^\pi)\Delta_{k-1}\Big|\mathcal{F}_t\right),\qquad t\in[t_i,t_{i+1}).$$

We rewrite the BSDE (1.1) as follows:

$$
\begin{aligned}
Y_t &= \xi + \int_t^T f(r, Y_r, Z_r)\, dr - \int_t^T Z_r\, dW_r \\
&= \xi + \sum_{k=i+1}^{n} f(t_k, Y_{t_k}, Z_{t_k})\Delta_{k-1} - \int_t^T Z_r\, dW_r + R_t^\pi,
\end{aligned} \tag{3.9}
$$

where

$$
\begin{aligned}
|R_t^\pi| &= \left| \int_t^T f(r, Y_r, Z_r)\, dr - \sum_{k=i+1}^{n} f(t_k, Y_{t_k}, Z_{t_k})\Delta_{k-1} \right| \\
&= \left| \sum_{k=i+1}^{n} \int_{t_{k-1}}^{t_k} [f(r, Y_r, Z_r) - f(t_k, Y_{t_k}, Z_{t_k})]\, dr - \int_{t_i}^{t} f(r, Y_r, Z_r)\, dr \right| \\
&\le \sum_{k=i+1}^{n} \int_{t_{k-1}}^{t_k} |f(r, Y_r, Z_r) - f(t_k, Y_{t_k}, Z_{t_k})|\, dr + \int_{t_i}^{t_{i+1}} |f(r, Y_r, Z_r)|\, dr.
\end{aligned}
$$

By Lemma 2.2 and the Lipschitz condition on f, we have

$$
\mathbb{E}\left(\int_0^T |f(r, Y_r, Z_r)|^2\, dr \right)^{p/2} < \infty,
$$

and hence,

$$
\begin{aligned}
&\mathbb{E} \max_{0\le i\le n-1} \left(\int_{t_i}^{t_{i+1}} |f(r, Y_r, Z_r)|\, dr \right)^p \\
&\qquad \le |\pi|^{p/2} \mathbb{E}\left(\int_0^T |f(r, Y_r, Z_r)|^2\, dr \right)^{p/2}.
\end{aligned} \tag{3.10}
$$

Define a function $\{t(r)\}_{0\le r\le T}$ by

$$
t(r) = \begin{cases} T, & \text{if } r = T, \\ t_{i+1}, & \text{if } t_i \le r < t_{i+1},\ i = n-1, \dots, 0. \end{cases}
$$

By the Hölder inequality, the boundedness of the first-order partial derivatives of f, (3.4), (2.24), Remark 2.8 and (3.10), it is easy to see that

$$
\begin{aligned}
\mathbb{E} \sup_{0\le t\le T} |R_t^\pi|^p &\le 2^{p-1} \left[\mathbb{E}\left(\int_0^T |f(r, Y_r, Z_r) - f(t(r), Y_{t(r)}, Z_{t(r)})|\, dr \right)^p \right. \\
&\qquad \left. + \mathbb{E} \max_{0\le i\le n-1} \left(\int_{t_i}^{t_{i+1}} |f(r, Y_r, Z_r)|\, dr \right)^p \right] \\
&\le (2T)^{p-1} \mathbb{E} \int_0^T |f(r, Y_r, Z_r) - f(t(r), Y_{t(r)}, Z_{t(r)})|^p\, dr
\end{aligned} \tag{3.11}
$$

$$
\begin{aligned}
&\quad + 2^{p-1}|\pi|^{p/2}\mathbb{E}\biggl(\int_0^T |f(r,Y_r,Z_r)|^2\,dr\biggr)^{p/2}\\
&\le C|\pi|^{p/2},
\end{aligned}
$$

where, by convention, $R_T = 0$. In particular, we obtain

$$
\mathbb{E}\sup_{0\le t\le T}|R_t^\pi|^2 \le C|\pi|. \tag{3.12}
$$

To simplify the notation we denote

$$
\delta Y_t^\pi = Y_t - Y_t^\pi, \qquad \delta Z_t^\pi = Z_t - Z_t^\pi \qquad \text{for all } t\in[0,T]
$$

and

$$
\widehat{Z}_{t_i}^\pi = Z_{t_i} - \widetilde{Z}_{t_i}^\pi \qquad \text{for } i = n, n-1,\dots,0.
$$

Then, when $t_i \le t < t_{i+1}$, by (3.8) and (3.9) we can write

$$
\begin{aligned}
\delta Y_t^\pi = {}& \sum_{k=i+1}^n [f(t_k, Y_{t_k}, Z_{t_k}) - f(t_k, Y_{t_k}^\pi, \widetilde{Z}_{t_k}^\pi)]\Delta_{k-1}\\
&- \int_t^T \delta Z_r^\pi\,dW_r + R_t^\pi + \delta\xi^\pi,
\end{aligned}
$$

where $\delta\xi^\pi = \xi - \xi^\pi$. Therefore, we obtain

$$
\delta Y_t^\pi = \mathbb{E}\Biggl(\sum_{k=i+1}^n [f(t_k, Y_{t_k}, Z_{t_k}) - f(t_k, Y_{t_k}^\pi, \widetilde{Z}_{t_k}^\pi)]\Delta_{k-1} + R_t^\pi + \delta\xi^\pi \Big| \mathcal{F}_t\Biggr). \tag{3.13}
$$

Denote $\widetilde{f}_{t_k}^\pi = f(t_k, Y_{t_k}, Z_{t_k}) - f(t_k, Y_{t_k}^\pi, \widetilde{Z}_{t_k}^\pi)$. From equality (3.13) for $t_j \le t < t_{j+1}$, where $i \le j \le n-1$, and taking into account that $\delta Y_T^\pi = \delta Y_{t_n}^\pi = \delta\xi^\pi$, we obtain

$$
\sup_{t_i\le t\le T}|\delta Y_t^\pi| \le \sup_{t_i\le t\le T}\mathbb{E}\Biggl(\sum_{k=i+1}^n |\widetilde{f}_{t_k}^\pi|\Delta_{k-1} + \sup_{0\le r\le T}|R_r^\pi| + |\delta\xi^\pi|\Big|\mathcal{F}_t\Biggr).
$$

The above conditional expectation is a martingale if it is considered as a process indexed by $t\in[t_i,T]$. Thus, using Doob's maximal inequality, we obtain

$$
\begin{aligned}
\mathbb{E}\sup_{t_i\le t\le T}|\delta Y_t^\pi|^2 &\le \mathbb{E}\sup_{t_i\le t\le T}\Biggl[\mathbb{E}\Biggl(\sum_{k=i+1}^n |\widetilde{f}_{t_k}^\pi|\Delta_{k-1} + \sup_{0\le r\le T}|R_r^\pi| + |\delta\xi^\pi|\Big|\mathcal{F}_t\Biggr)\Biggr]^2\\
&\le C\mathbb{E}\Biggl(\sum_{k=i+1}^n |\widetilde{f}_{t_k}^\pi|\Delta_{k-1} + \sup_{0\le r\le T}|R_r^\pi| + |\delta\xi^\pi|\Biggr)^2\\
&\le C\Biggl\{\mathbb{E}\Biggl(\sum_{k=i+1}^n |\widetilde{f}_{t_k}^\pi|\Delta_{k-1}\Biggr)^2 + \mathbb{E}\sup_{0\le r\le T}|R_r^\pi|^2 + \mathbb{E}|\delta\xi^\pi|^2\Biggr\}.
\end{aligned}
$$

From (3.12), we deduce

$$\mathbb{E}\sup_{t_i\le t\le T}|\delta Y_t^\pi|^2 \le C\left\{\mathbb{E}\left(\sum_{k=i+1}^{n}|\widetilde{f}_{t_k}^\pi|\Delta_{k-1}\right)^2 + \mathbb{E}|\delta\xi^\pi|^2 + |\pi|\right\}.$$

Using the Lipschitz condition on f, we obtain

$$\begin{aligned}\mathbb{E}\sup_{t_i\le t\le T}|\delta Y_t^\pi|^2 \le C\Bigg\{&(T-t_i)^2\mathbb{E}\sup_{i+1\le k\le n}|\delta Y_{t_k}^\pi|^2\\ &+\mathbb{E}\left(\sum_{k=i+1}^{n-1}|\widehat{Z}_{t_k}^\pi|\Delta_{k-1}\right)^2 + \mathbb{E}|\widehat{Z}_{t_n}|^2\Delta_{n-1}^2\Bigg\}\\ &+C(\mathbb{E}|\delta\xi^\pi|^2+|\pi|).\end{aligned}\tag{3.14}$$

Notice that

$$\begin{aligned}\mathbb{E}\left(\sum_{k=i+1}^{n-1}|\widehat{Z}_{t_k}^\pi|\Delta_{k-1}\right)^2 &= \mathbb{E}\left(\sum_{k=i+1}^{n-1}\left|Z_{t_k}-\frac{1}{\Delta_k}\int_{t_k}^{t_{k+1}}\mathbb{E}(Z_u^\pi|\mathcal{F}_{t_k})\,du\right|\Delta_{k-1}\right)^2\\ &\le \mathbb{E}\left(\sum_{k=i+1}^{n-1}\frac{\Delta_{k-1}}{\Delta_k}\int_{t_k}^{t_{k+1}}\mathbb{E}(|Z_{t_k}-Z_u^\pi||\mathcal{F}_{t_k})\,du\right)^2\\ &\le L_1^2\mathbb{E}\left(\sum_{k=i+1}^{n-1}\int_{t_k}^{t_{k+1}}\mathbb{E}(|Z_{t_k}-Z_u^\pi||\mathcal{F}_{t_k})\,du\right)^2\\ &\le 2L_1^2\Bigg\{\mathbb{E}\left(\sum_{k=i+1}^{n-1}\int_{t_k}^{t_{k+1}}\mathbb{E}(|Z_{t_k}-Z_u||\mathcal{F}_{t_k})\,du\right)^2\\ &\qquad+\mathbb{E}\left(\sum_{k=i+1}^{n-1}\int_{t_k}^{t_{k+1}}\mathbb{E}(|Z_u-Z_u^\pi||\mathcal{F}_{t_k})\,du\right)^2\Bigg\}\\ &= 2L_1^2(I_1+I_2).\end{aligned}\tag{3.15}$$

Now the Minkowski and the Hölder inequalities yield

$$\begin{aligned}I_1 &\le \mathbb{E}\left(\sum_{k=i+1}^{n-1}\left\{\int_{t_k}^{t_{k+1}}(\mathbb{E}(|Z_{t_k}-Z_u||\mathcal{F}_{t_k}))^2\,du\right\}^{1/2}\Delta_k^{1/2}\right)^2\\ &\le (T-t_i)\sum_{k=i+1}^{n-1}\int_{t_k}^{t_{k+1}}\mathbb{E}(\mathbb{E}(|Z_{t_k}-Z_u||\mathcal{F}_{t_k}))^2\,du\\ &\le (T-t_i)\sum_{k=i+1}^{n-1}\int_{t_k}^{t_{k+1}}\mathbb{E}|Z_{t_k}-Z_u|^2\,du\end{aligned}\tag{3.16}$$

$$\le C(T-t_i)\sum_{k=i+1}^{n-1}\int_{t_k}^{t_{k+1}}|t_k-u|\,du \le C|\pi|.$$

In a similar way and by (3.6), we obtain

$$(3.17)\quad \begin{aligned} I_2 &\le (T-t_i)\sum_{k=i+1}^{n-1}\int_{t_k}^{t_{k+1}}\mathbb{E}|Z_u-Z_u^\pi|^2\,du \\ &= (T-t_i)\int_{t_{i+1}}^{T}\mathbb{E}|\delta Z_u^\pi|^2\,du \le C|\pi|. \end{aligned}$$

On the other hand,

$$(3.18)\quad \mathbb{E}(\widehat{Z}_{t_n}^\pi\Delta_{n-1})^2 = \mathbb{E}|Z_{t_n}|^2|\Delta_{n-1}|^2 \le C|\pi|^2.$$

From (3.14)–(3.18), we have

$$(3.19)\quad \begin{aligned} \mathbb{E}\sup_{t_i\le t\le T}|\delta Y_t^\pi|^2 &\le C_1(T-t_i)^2\mathbb{E}\sup_{i+1\le k\le n}|\delta Y_{t_k}^\pi|^2 \\ &\quad + C_2(\mathbb{E}|\delta\xi^\pi|^2+|\pi|), \end{aligned}$$

where C_1 and C_2 are two positive constants independent of the partition π.

We can find a constant $\delta>0$ independent of the partition π, such that $C_1(3\delta)^2<\frac{1}{2}$ and $T>2\delta$. Denote $l=[\frac{T}{2\delta}]$ ($[x]$ means the greatest integer no larger than x). Then $l\ge 1$ is an integer independent of the partition π. If $|\pi|<\delta$, then for the partition π we can choose $n-1>i_1>i_2>\cdots>i_l\ge 0$, such that, $T-2\delta\in(t_{i_1-1},t_{i_1}]$, $T-4\delta\in(t_{i_2-1},t_{i_2}],\dots,T-2\delta l\in[0,t_{i_l}]$ (with $t_{-1}=0$).

For simplicity, we denote $t_{i_0}=T$ and $t_{i_{l+1}}=0$. Each interval $[t_{i_{j+1}},t_{i_j}]$, $j=0,1,\dots,l$, has length less than 3δ, that is, $|t_{i_j}-t_{i_{j+1}}|<3\delta$. On each interval $[t_{i_{j+1}},t_{i_j}]$, $j=0,1,\dots,l$, we consider the recursive formula (3.2), and (3.19) becomes

$$(3.20)\quad \begin{aligned} \mathbb{E}\sup_{t_{i_{j+1}}\le t\le t_{i_j}}|\delta Y_t^\pi|^2 &\le C_1(t_{i_j}-t_{i_{j+1}})^2\mathbb{E}\sup_{i_{j+1}+1\le k\le i_j}|\delta Y_{t_k}^\pi|^2 \\ &\quad + C_2(\mathbb{E}|\delta Y_{t_{i_j}}^\pi|^2+|\pi|). \end{aligned}$$

Using (3.20), we can obtain inductively

$$\begin{aligned} &\mathbb{E}\sup_{t_{i_{j+1}}\le t\le t_{i_j}}|\delta Y_t^\pi|^2 \\ &\quad\le C_1(t_{i_j}-t_{i_{j+1}})^2\mathbb{E}\sup_{i_{j+1}+1\le k\le i_j}|\delta Y_{t_k}^\pi|^2 + C_2(\mathbb{E}|\delta Y_{t_{i_j}}^\pi|^2+|\pi|) \\ &\quad\le C_1(t_{i_j}-t_{i_{j+1}})^2\cdots C_1(t_{i_j}-t_{i_j-1})^2\mathbb{E}|\delta Y_{t_{i_j}}^\pi|^2 \end{aligned}$$

$$
\begin{aligned}
&\quad + C_2(\mathbb{E}|\delta Y^\pi_{t_{i_j}}|^2 + |\pi|) \\
&\qquad \times (1 + C_1(t_{i_j} - t_{i_{j+1}})^2 + C_1(t_{i_j} - t_{i_{j+1}})^2 C_1(t_{i_j} - t_{i_{j+1}+1})^2 \\
&\qquad\quad + \cdots + C_1(t_{i_j} - t_{i_{j+1}})^2 C_1(t_{i_j} - t_{i_{j+1}+1})^2 \cdots C_1(t_{i_j} - t_{i_j-1})^2) \qquad (3.21)\\
&\le (C_1(3\delta)^2)^{i_j - i_{j+1}} \mathbb{E}|\delta Y^\pi_{t_{i_j}}|^2 \\
&\quad + C_2(\mathbb{E}|\delta Y^\pi_{t_{i_j}}|^2 + |\pi|) \\
&\qquad \times (1 + C_1(3\delta)^2 + (C_1(3\delta)^2)^2 + \cdots + (C_1(3\delta)^2)^{i_j - i_{j+1}}) \\
&\le \mathbb{E}|\delta Y^\pi_{t_{i_j}}|^2 + \frac{C_2}{1 - C_1(3\delta)^2}(\mathbb{E}|\delta Y^\pi_{t_{i_j}}|^2 + |\pi|) \\
&\le \mathbb{E}|\delta Y^\pi_{t_{i_j}}|^2 + 2C_2(\mathbb{E}|\delta Y^\pi_{t_{i_j}}|^2 + |\pi|) \\
&= (2C_2 + 1)\mathbb{E}|\delta Y^\pi_{t_{i_j}}|^2 + 2C_2|\pi|.
\end{aligned}
$$

By recurrence, we obtain

$$
\begin{aligned}
&\mathbb{E} \sup_{t_{i_{j+1}} \le t \le t_{i_j}} |\delta Y^\pi_t|^2 \\
&\qquad \le (2C_2 + 1)^{j+1} \mathbb{E}|\delta \xi^\pi|^2 + C_2|\pi|(1 + (2C_2 + 1) + \cdots + (2C_2 + 1)^j) \\
&\qquad \le (2C_2 + 1)^{l+1} \mathbb{E}|\delta \xi^\pi|^2 + C_2|\pi|(1 + (2C_2 + 1) + \cdots + (2C_2 + 1)^l) \qquad (3.22)\\
&\qquad \le \frac{3(2C_2 + 1)^{l+1}}{2}(\mathbb{E}|\delta \xi^\pi|^2 + |\pi|).
\end{aligned}
$$

Therefore, taking $C = \frac{3(2C_2+1)^{l+1}}{2}$, we obtain

$$
\mathbb{E} \sup_{0 \le t \le T} |\delta Y^\pi_t|^2 \le \max_{0 \le j \le l} \mathbb{E} \sup_{t_{i_{j+1}} \le t \le t_{i_j}} |\delta Y^\pi_t|^2 \le C(|\pi| + \mathbb{E}|\xi - \xi^\pi|^2).
$$

Combining the above estimate with (3.6), we know that there exists a constant $K > 0$ independent of the partition π, such that

$$
\mathbb{E} \sup_{0 \le t \le T} |Y_t - Y^\pi_t|^2 + \mathbb{E} \int_0^T |Z_t - Z^\pi_t|^2 \, dt \le K(|\pi| + \mathbb{E}|\xi - \xi^\pi|^2). \qquad \square
$$

REMARK 3.2. The numerical scheme introduced before, as other similar schemes, involves the computation of conditional expectations with respect to the σ-field $\mathcal{F}_{t_{i+1}}$. To implement this scheme in practice we need to approximate these conditional expectations. Some work has been done to solve this problem, and we refer the reader to the references [2, 4] and [8].

4. An implicit scheme for BSDEs. In this section, we propose an implicit numerical scheme for the BSDE (1.1). Define the approximating pairs

(Y^π, Z^π) recursively by

$$Y^\pi_{t_n} = \xi^\pi,$$

$$Y^\pi_t = Y^\pi_{t_{i+1}} + f\left(t_{i+1}, Y^\pi_{t_{i+1}}, \frac{1}{\Delta_i}\int_{t_i}^{t_{i+1}} Z^\pi_r\,dr\right)\Delta_i - \int_t^{t_{i+1}} Z^\pi_r\,dW_r, \tag{4.1}$$
$$t \in [t_i, t_{i+1}), i = n-1, n-2, \ldots, 0,$$

where the partition π and Δ_i, $i = n-1, \ldots, 0$, are defined in Section 3, and ξ^π is an approximation of the terminal value ξ. In this recursive formula (4.1), on each subinterval $[t_i, t_{i+1}), i = n-1, \ldots, 0$, the nonlinear "generator" f contains the information of Z^π on the same interval. In this sense, this formula is different from formula (3.2), and (4.1) is an equation for $\{(Y^\pi_t, Z^\pi_t)\}_{t_i \le t < t_{i+1}}$. When $|\pi|$ is sufficiently small, the existence and uniqueness of the solution to the above equation can be established. In fact, equation (4.1) is of the following form:

$$Y_t = \xi + g\left(\int_a^b Z_r\,dr\right) - \int_t^b Z_r\,dW_r, \qquad t \in [a,b] \text{ and } 0 \le a < b \le T. \tag{4.2}$$

For the BSDE (4.2), we have the following theorem.

THEOREM 4.1. *Let $0 \le a < b \le T$ and $p \ge 2$. Let ξ be $\mathcal{F}_b$-measurable and $\xi \in L^p(\Omega)$. If there exists a constant $L > 0$ such that $g : (\Omega \times \mathbb{R}, \mathcal{F}_b \otimes \mathcal{B}) \to (\mathbb{R}, \mathcal{B})$ satisfies*

$$|g(z_1) - g(z_2)| \le L|z_1 - z_2|$$

for all $z_1, z_2 \in \mathbb{R}$ and $g(0) \in L^p(\Omega)$, then there is a constant $\delta(p, L) > 0$, such that, when $b - a < \delta(p, L)$, equation (4.2) has a unique solution $(Y, Z) \in S^p_{\mathcal{F}}([a,b]) \times H^p_{\mathcal{F}}([a,b])$.

PROOF. We shall use the fixed point theorem for the mapping from $H^p_{\mathcal{F}}([a,b])$ into $H^p_{\mathcal{F}}([a,b])$ which maps z to Z, where (Y, Z) is the solution of the following BSDE:

$$Y_t = \xi + g\left(\int_a^b z_r\,dr\right) - \int_t^b Z_r\,dW_r, \qquad t \in [a,b]. \tag{4.3}$$

In fact, by the martingale representation theorem, there exist a progressively measurable process $Z = \{Z_t\}_{a \le t \le b}$ such that $\mathbb{E}\int_a^b Z_t^2\,dt < \infty$ and

$$\xi + g\left(\int_a^b z_r\,dr\right) = \mathbb{E}\left(\xi + g\left(\int_a^b z_r\,dr\right)\Big|\mathcal{F}_a\right) + \int_a^b Z_t\,dW_t.$$

By the integrability properties of $\xi, g(0)$ and z, one can show that $Z \in H^p_{\mathcal{F}}([a,b])$. Define $Y_t = \mathbb{E}(\xi + g(\int_a^b z_r\,dr)|\mathcal{F}_t), t \in [a,b]$. Then (Y, Z) satisfies equation (4.3). Notice that Y is a martingale. Then by the Lipschitz condition on g, the integrability of $\xi, g(0)$ and z, and Doob's maximal inequality, we can prove that $Y \in S^p_{\mathcal{F}}([a,b])$.

Let z^1, z^2 be two elements in the Banach space $H^p_{\mathcal{F}}([a,b])$, and let (Y^1, Z^1), (Y^2, Z^2) be the associated solutions, that is,

$$Y^i_t = \xi + g\left(\int_a^b z^i_r\,dr\right) - \int_t^b Z^i_r\,dW_r, \qquad t \in [a,b], i = 1,2.$$

Denote

$$\bar{Y} = Y^1 - Y^2, \qquad \bar{Z} = Z^1 - Z^2, \qquad \bar{z} = z^1 - z^2.$$

Then

$$\bar{Y}_t = g\left(\int_a^b z^1_r\,dr\right) - g\left(\int_a^b z^2_r\,dr\right) - \int_t^b \bar{Z}_r\,dW_r \tag{4.4}$$

for all $t \in [a,b]$. So

$$\bar{Y}_t = \mathbb{E}\left(g\left(\int_a^b z^1_r\,dr\right) - g\left(\int_a^b z^2_r\,dr\right)\Big|\mathcal{F}_t\right)$$

for all $t \in [a,b]$. Thus by Doob's maximal inequality, we have

$$\begin{aligned}
\mathbb{E}\sup_{a\le t\le b}|\bar{Y}_t|^p &= \mathbb{E}\sup_{a\le t\le b}\left|\mathbb{E}\left(g\left(\int_a^b z^1_r\,dr\right) - g\left(\int_a^b z^2_r\,dr\right)\Big|\mathcal{F}_t\right)\right|^p \\
&\le C\mathbb{E}\left|g\left(\int_a^b z^1_r\,dr\right) - g\left(\int_a^b z^2_r\,dr\right)\right|^p \\
&\le C\mathbb{E}\left|\int_a^b z^1_r\,dr - \int_a^b z^2_r\,dr\right|^p \\
&\le C(b-a)^{p/2}\mathbb{E}\left(\int_a^b |\bar{z}_r|^2\,dr\right)^{p/2},
\end{aligned} \tag{4.5}$$

where $C > 0$ is a generic constant depending on L and p, which may vary from line to line. From (4.4), it is easy to see

$$\bar{Y}_t = \bar{Y}_a + \int_a^t \bar{Z}_r\,dW_r$$

for all $t \in [a,b]$. Therefore, by the Burkholder–Davis–Gundy inequality and (4.5), we have

$$\begin{aligned}
\mathbb{E}\left(\int_a^b |\bar{Z}_r|^2\,dr\right)^{p/2} &\le C\mathbb{E}\sup_{a\le t\le b}\left|\int_a^t \bar{Z}_r\,dW_r\right|^p \\
&\le C\Big[\mathbb{E}|\bar{Y}_a|^p + \mathbb{E}\sup_{a\le t\le b}|\bar{Y}_t|^p\Big] \\
&\le C(b-a)^{p/2}\mathbb{E}\left(\int_a^b |\bar{z}_r|^2\,dr\right)^{p/2},
\end{aligned} \tag{4.6}$$

that is,

$$\|\bar{Z}\|_{H^p} \leq C_1(b-a)^{1/2}\|\bar{z}\|_{H^p},$$

where C_1 is a positive constant depending only on L and p.

Take $\delta(p,L) = 1/C_1^2$. It is obvious that the mapping is a contraction when $b-a < \delta(p,L)$, and hence there exists a unique solution $(Y,Z) \in S^p_{\mathcal{F}}([a,b]) \times H^p_{\mathcal{F}}([a,b])$ to the BSDE (4.2). □

Now we begin to study the convergence of the scheme (4.1).

THEOREM 4.2. *Let Assumption 2.2 be satisfied, and let π be any partition. Assume that $\xi^\pi \in L^p(\Omega)$ and there exists a constant $L_1 > 0$ such that, for all $t_1, t_2 \in [0,T]$,*

$$|f(t_2,y,z) - f(t_1,y,z)| \leq L_1|t_2 - t_1|^{1/2}.$$

Then, there are two positive constants δ and K independent of the partition π, such that, when $|\pi| < \delta$, we have

$$\mathbb{E}\sup_{0\leq t\leq T}|Y_t - Y_t^\pi|^p + \mathbb{E}\left(\int_0^T |Z_t - Z_t^\pi|^2\,dt\right)^{p/2} \leq K(|\pi|^{p/2} + \mathbb{E}|\xi - \xi^\pi|^p).$$

PROOF. If $|\pi| < \delta(p,L)$, where $\delta(p,L)$ is the constant in Theorem 4.1, then Theorem 4.1 guarantees the existence and uniqueness of (Y^π, Z^π). Denote, for $i = n-1, n-2, \ldots, 0$,

$$\widetilde{Z}^\pi_{t_{i+1}} = \frac{1}{t_{i+1} - t_{t_i}}\int_{t_i}^{t_{i+1}} Z_r^\pi\,dr.$$

Notice that $\{\widetilde{Z}^\pi_{t_i},\}_{i=n-1,n-2,\ldots,0}$ here is different from that in Section 3. Then

$$\begin{aligned} Y^\pi_{t_i} &= Y^\pi_{t_{i+1}} + f(t_{i+1}, Y^\pi_{t_{i+1}}, \widetilde{Z}^\pi_{t_{i+1}})\Delta_i \\ &\quad - \int_{t_i}^{t_{i+1}} Z_r^\pi\,dW_r, \qquad i = n-1, n-2, \ldots, 0. \end{aligned}$$

Recursively, we obtain

$$\begin{aligned} Y^\pi_{t_i} &= \xi^\pi + \sum_{k=i+1}^{n} f(t_k, Y^\pi_{t_k}, \widetilde{Z}^\pi_{t_k})\Delta_{k-1} \\ &\quad - \int_{t_i}^{T} Z_r^\pi\,dW_r, \qquad i = n-1, n-2, \ldots, 0. \end{aligned}$$

Denote

$$\delta\xi^\pi = \xi - \xi^\pi, \qquad \delta Y_t^\pi = Y_t - Y_t^\pi, \qquad \delta Z_t^\pi = Z_t - Z_t^\pi, \qquad t \in [0,T],$$

and

$$\widehat{Z}^\pi_{t_i} = Z_{t_i} - \widetilde{Z}^\pi_{t_i}, \qquad i = n-1, \ldots, 0.$$

If $t\in[t_i,t_{i+1})$, $i=n-1,n-2,\dots,0$, then by iteration, we have

$$\begin{aligned}\delta Y_t^\pi &= \delta\xi^\pi + \sum_{k=i+1}^{n}[f(t_k,Y_{t_k},Z_{t_k})-f(t_k,Y_{t_k}^\pi,\widetilde{Z}_{t_k}^\pi)]\Delta_{k-1}\\ &\quad -\int_{t_i}^{T}\delta Z_r^\pi\,dW_r + R_t^\pi,\end{aligned}\tag{4.7}$$

where R_t^π is exactly the same as that in Section 3.

Denote $\widetilde{f}_{t_k}^\pi = f(t_k,Y_{t_k},Z_{t_k})-f(t_k,Y_{t_k}^\pi,\widetilde{Z}_{t_k}^\pi)$. Then for $t\in[t_i,t_{i+1}), i=n-1, n-2,\dots,0$, we have

$$\delta Y_t^\pi = \mathbb{E}\left(\delta\xi^\pi + \sum_{k=i+1}^{n}\widetilde{f}_{t_k}^\pi\Delta_{k-1} + R_t^\pi\middle|\mathcal{F}_t\right).\tag{4.8}$$

From equality (4.8) for $t_j\le t<t_{j+1}$, where $i\le j\le n-1$, and taking into account that $\delta Y_T^\pi=\delta Y_{t_n}^\pi=\delta\xi^\pi$, we obtain

$$\sup_{t_i\le t\le T}|\delta Y_t^\pi| \le \sup_{t_i\le t\le T}\mathbb{E}\left(\sum_{k=i+1}^{n}|\widetilde{f}_{t_k}^\pi|\Delta_{k-1} + \sup_{0\le r\le T}|R_r^\pi| + |\delta\xi^\pi|\middle|\mathcal{F}_t\right).$$

The above conditional expectation is a martingale if it is considered as a process indexed by t for $t\in[t_i,T]$. Using Doob's maximal inequality, (3.11), and the Lipschitz condition on f, we have

$$\begin{aligned}
&\mathbb{E}\sup_{t_i\le t\le T}|\delta Y_t^\pi|^p\\
&\quad\le \mathbb{E}\sup_{t_i\le t\le T}\left[\mathbb{E}\left(\sum_{k=i+1}^{n}|\widetilde{f}_{t_k}^\pi|\Delta_{k-1} + \sup_{0\le r\le T}|R_r^\pi| + |\delta\xi^\pi|\middle|\mathcal{F}_t\right)\right]^p\\
&\quad\le C\mathbb{E}\left(\sum_{k=i+1}^{n}|\widetilde{f}_{t_k}^\pi|\Delta_{k-1} + \sup_{0\le r\le T}|R_r^\pi| + |\delta\xi^\pi|\right)^p\\
&\quad\le C\left\{\mathbb{E}\left(\sum_{k=i+1}^{n}|\widetilde{f}_{t_k}^\pi|\Delta_{k-1}\right)^p + \mathbb{E}\sup_{0\le r\le T}|R_r^\pi|^p + \mathbb{E}|\delta\xi^\pi|^p\right\}\\
&\quad\le C\left\{\mathbb{E}\left(\sum_{k=i+1}^{n}|\delta Y_{t_k}^\pi|\Delta_{k-1}\right)^p + \mathbb{E}\left(\sum_{k=i+1}^{n}|\widehat{Z}_{t_k}^\pi|\Delta_{k-1}\right)^p + |\pi|^{p/2} + \mathbb{E}|\delta\xi^\pi|^p\right\}\\
&\quad\le C\left\{(T-t_i)^p\mathbb{E}\sup_{i+1\le k\le n}|\delta Y_{t_k}^\pi|^p\right.\\
&\qquad\quad \left.+\,\mathbb{E}\left(\sum_{k=i+1}^{n}|\widehat{Z}_{t_k}^\pi|\Delta_{k-1}\right)^p + |\pi|^{p/2} + \mathbb{E}|\delta\xi^\pi|^p\right\},
\end{aligned}$$

where, and in the following, $C > 0$ denotes a generic constant independent of the partition π and may vary from line to line. On the other hand, we have, by the Hölder continuity of Z given by (2.24),

$$
\begin{aligned}
&\mathbb{E}\left(\sum_{k=i+1}^{n} |\widehat{Z}^{\pi}_{t_k}|\Delta_{k-1}\right)^p \\
&\quad = \mathbb{E}\left(\sum_{k=i+1}^{n} \left| Z_{t_k} - \frac{1}{\Delta_{k-1}} \int_{t_{k-1}}^{t_k} Z^{\pi}_r \, dr \right| \Delta_{k-1}\right)^p \\
&\quad \le \mathbb{E}\left(\sum_{k=i+1}^{n} \int_{t_{k-1}}^{t_k} |Z_{t_k} - Z_r| \, dr + \sum_{k=i+1}^{n} \int_{t_{k-1}}^{t_k} |Z_r - Z^{\pi}_r| \, dr\right)^p \\
&\quad \le C|\pi|^{p/2} + 2^{p-1}\mathbb{E}\left(\int_{t_i}^{T} |Z_r - Z^{\pi}_r| \, dr\right)^p \\
&\quad \le C|\pi|^{p/2} + 2^{p-1}(T - t_i)^{p/2}\mathbb{E}\left(\int_{t_i}^{T} |Z_r - Z^{\pi}_r|^2 \, dr\right)^{p/2} \\
&\quad = C|\pi|^{p/2} + 2^{p-1}(T - t_i)^{p/2}\mathbb{E}\left(\int_{t_i}^{T} |\delta Z^{\pi}_r|^2 \, dr\right)^{p/2}.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
&\mathbb{E} \sup_{t_i \le t \le T} |\delta Y^{\pi}_t|^p \\
&\qquad \le C_1\biggl\{(T - t_i)^p \mathbb{E} \sup_{i+1 \le k \le n} |\delta Y_{t_k}|^p \\
&\qquad\qquad + (T - t_i)^{p/2}\mathbb{E}\left(\int_{t_i}^{T} |\delta Z^{\pi}_r|^2 \, dr\right)^{p/2} \\
&\qquad\qquad + |\pi|^{p/2} + \mathbb{E}|\delta\xi^{\pi}|^p\biggr\},
\end{aligned} \tag{4.9}
$$

where C_1 is a constant independent of the partition π. By the Burkholder–Davis–Gundy inequality, we have

$$
\mathbb{E}\left(\int_{t_i}^{T} |\delta Z^{\pi}_r|^2 \, dr\right)^{p/2} \le c_p \mathbb{E}\left|\int_{t_i}^{T} \delta Z^{\pi}_r \, dW_r\right|^p. \tag{4.10}
$$

From (4.7), we obtain

$$
\int_{t_i}^{T} \delta Z^{\pi}_r \, dW_r = \delta\xi^{\pi} + \sum_{k=i+1}^{n} \widetilde{f}^{\pi}_{t_k}\Delta_{k-1} + R^{\pi}_{t_i} - \delta Y^{\pi}_{t_i}. \tag{4.11}
$$

Thus, from (4.10) and (4.11), we obtain

$$\mathbb{E}\biggl(\int_{t_i}^T |\delta Z_r^\pi|^2\,dr\biggr)^{p/2} \le C_p\Biggl\{\mathbb{E}\Biggl|\sum_{k=i+1}^n \widetilde{f}_{t_k}^\pi \Delta_{k-1}\Biggr|^p + \mathbb{E}|\delta\xi^\pi|^p + \mathbb{E}|R_{t_i}^\pi|^p + \mathbb{E}|\delta Y_{t_i}^\pi|^p\Biggr\}.$$

Similar to (4.9), we have

$$\begin{aligned}\mathbb{E}\biggl(\int_{t_i}^T |\delta Z_r^\pi|^2\,dr\biggr)^{p/2} &\le C_2\biggl\{(T-t_i)^p\mathbb{E}\sup_{i+1\le k\le n}|\delta Y_{t_k}|^p \\ &\qquad + (T-t_i)^{p/2}\mathbb{E}\biggl(\int_{t_i}^T |\delta Z_r^\pi|^2\,dr\biggr)^{p/2} + |\pi|^{p/2} + \mathbb{E}|\delta\xi^\pi|^p\biggr\},\end{aligned}$$

where C_2 is a constant independent of the partition π.

If $C_2(T-t_i)^{p/2} < \frac{1}{2}$, then we have

$$\begin{aligned}\mathbb{E}\biggl(\int_{t_i}^T |\delta Z_r^\pi|^2\,dr\biggr)^{p/2} &\le 2C_2(T-t_i)^p\mathbb{E}\sup_{i+1\le k\le n}|\delta Y_{t_k}|^p \\ &\qquad + 2C_2(|\pi|^{p/2} + \mathbb{E}|\delta\xi^\pi|^p).\end{aligned} \tag{4.12}$$

Substituting (4.12) into (4.9), we have

$$\begin{aligned}&\mathbb{E}\sup_{t_i\le t\le T}|\delta Y_t^\pi|^p \\ &\quad\le C_1(1+2C_2(T-t_i)^{p/2})(T-t_i)^p\mathbb{E}\sup_{i+1\le k\le n}|\delta Y_{t_k}|^p \\ &\qquad + C_1(1+2C_2(T-t_i)^{p/2})(|\pi|^{p/2} + \mathbb{E}|\delta\xi^\pi|^p) \\ &\quad\le 2C_1(T-t_i)^p\mathbb{E}\sup_{i+1\le k\le n}|\delta Y_{t_k}|^p + 2C_1(|\pi|^{p/2} + \mathbb{E}|\delta\xi^\pi|^p).\end{aligned} \tag{4.13}$$

We can find a positive constant $\delta < \delta(p,L)$ independent of the partition π, such that,

$$C_2(3\delta)^{p/2} < \tfrac{1}{2}, \tag{4.14}$$

$$2C_1(3\delta)^p < \tfrac{1}{2} \tag{4.15}$$

and $T > 2\delta$. Denote $l = [\frac{T}{2\delta}]$. Then $l \ge 1$ is an integer independent of the partition π. If $|\pi| < \delta$, then for the partition π we can choose $n-1 > i_1 > i_2 > \cdots > i_l \ge 0$, such that, $T-2\delta \in (t_{i_1-1}, t_{i_1}]$, $T-4\delta \in (t_{i_2-1}, t_{i_2}], \ldots, T-2\delta l \in$

$[0,t_{i_l}]$ (with $t_{-1}=0$). For simplicity, we denote $t_{i_0}=T$ and $t_{i_{l+1}}=0$. Each interval $[t_{i_{j+1}},t_{i_j}]$, $j=0,1,\dots,l$, has length less than 3δ, that is, $|t_{i_j}-t_{i_{j+1}}|<3\delta$. On $[t_{i_{j+1}},t_{i_j}]$, we consider the recursive formula (4.1). Then (4.13)–(4.15) yield

$$
\begin{aligned}
&\mathbb{E}\sup_{t_{i_{j+1}}\le t\le t_{i_j}}|\delta Y_t^\pi|^p \\
&\quad\le 2C_1(t_{i_j}-t_{i_{j+1}})^p\mathbb{E}\sup_{i_{j+1}+1\le k\le i_j}|\delta Y_{t_k}|^p+2C_1(|\pi|^{p/2}+\mathbb{E}|\delta Y_{t_{i_j}}^\pi|^p)\\
&\quad\le 2C_1(3\delta)^p\mathbb{E}\sup_{i_{j+1}+1\le k\le i_j}|\delta Y_{t_k}|^p+2C_1(|\pi|^{p/2}+\mathbb{E}|\delta Y_{t_{i_j}}^\pi|^p)\\
&\quad\le \frac{1}{2}\sup_{i_{j+1}+1\le k\le i_j}|\delta Y_{t_k}|^p+2C_1(|\pi|^{p/2}+\mathbb{E}|\delta Y_{t_{i_j}}^\pi|^p).
\end{aligned}\tag{4.16}
$$

As in the proof of (3.21) and (3.22), we have

$$
\mathbb{E}\sup_{t_{i_{j+1}}\le t\le t_{i_j}}|\delta Y_t^\pi|^p\le(4C_1+1)\mathbb{E}|\delta Y_{t_{i_j}}^\pi|^p+4C_1|\pi|^{p/2}
$$

and

$$
\mathbb{E}\sup_{t_{i_{j+1}}\le t\le t_{i_j}}|\delta Y_t^\pi|^p\le\frac{3(4C_1+1)^{l+1}}{2}(\mathbb{E}|\delta\xi^\pi|^2+|\pi|^{p/2}).
$$

Therefore, we obtain

$$
\begin{aligned}
\mathbb{E}\sup_{0\le t\le T}|\delta Y_t^\pi|^p&\le\max_{0\le j\le l}\mathbb{E}\sup_{t_{i_{j+1}}\le t\le t_{i_j}}|\delta Y_t^\pi|^p\\
&\le\frac{3(4C_1+1)^{l+1}}{2}(\mathbb{E}|\delta\xi^\pi|^p+|\pi|^{p/2}).
\end{aligned}\tag{4.17}
$$

On $[t_{i_{j+1}},t_{i_j}]$, $j=0,1,\dots,l$, based on the recursive formula (4.1) and (4.17), inequality (4.12) becomes

$$
\begin{aligned}
&\mathbb{E}\left(\int_{t_{i_{j+1}}}^{t_{i_j}}|\delta Z_r^\pi|^2\,dr\right)^{p/2}\\
&\quad\le 2C_2(t_{i_j}-t_{i_{j+1}})^p\mathbb{E}\sup_{i_{j+1}+1\le k\le i_j}|\delta Y_{t_k}|^p+2C_2(|\pi|^{p/2}+\mathbb{E}|\delta\xi^\pi|^p)\\
&\quad\le 2C_2(3\delta)^p\mathbb{E}\sup_{i_{j+1}+1\le k\le i_j}|\delta Y_{t_k}|^p+2C_2(|\pi|^{p/2}+\mathbb{E}|\delta\xi^\pi|^p)\\
&\quad\le \frac{1}{2}\mathbb{E}\sup_{i_{j+1}+1\le k\le i_j}|\delta Y_{t_k}|^p+2C_2(|\pi|^{p/2}+\mathbb{E}|\delta\xi^\pi|^p)\\
&\quad\le\left(\frac{3(4C_1+1)^{l+1}}{4}+2C_2\right)(|\pi|^{p/2}+\mathbb{E}|\delta\xi^\pi|^p).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\mathbb{E}\biggl(\int_0^T |\delta Z_t^\pi|^2\,dt\biggr)^{p/2}\\
&\qquad = \mathbb{E}\Biggl(\sum_{j=0}^{l}\int_{t_{i_{j+1}}}^{t_{i_j}} |\delta Z_t^\pi|^2\,dt\Biggr)^{p/2}\\
&\qquad \le (l+1)^{p/2-1}\sum_{j=0}^{l}\mathbb{E}\biggl(\int_{t_{i_{j+1}}}^{t_{i_j}} |\delta Z_t^\pi|^2\,dt\biggr)^{p/2}\\
&\qquad \le (l+1)^{p/2}\biggl(\frac{3(4C_1+1)^{l+1}}{4}+2C_2\biggr)(|\pi|^{p/2}+\mathbb{E}|\delta\xi^\pi|^p).
\end{aligned}
\tag{4.18}
$$

Combining (4.17) and (4.18), we know that there exists a constant

$$
K=(l+1)^{p/2}\biggl(\frac{3(4C_1+1)^{l+1}}{2}+4C_2\biggr)
$$

independent of the partition π, such that

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}|Y_t-Y_t^\pi|^p+\mathbb{E}\biggl(\int_0^T |Z_t-Z_t^\pi|^2\,dt\biggr)^{p/2}\\
&\qquad \le K(|\pi|^{p/2}+\mathbb{E}|\xi-\xi^\pi|^p).
\end{aligned}
$$

□

REMARK 4.3. The advantages of this implicit numerical scheme are:

(i) we can obtain the rate of convergence in L^p sense;
(ii) the partition π can be arbitrary ($|\pi|$ should be small enough) without assuming $\max_{0\le i\le n-1}\Delta_i/\Delta_{i+1}\le L_1$.

5. A new discrete scheme. For all the numerical schemes considered in Sections 3 and 4, one needs to evaluate processes $\{Z_t^\pi\}_{0\le t\le T}$ with continuous index t. In this section, we use the representation of Z in terms of the Malliavin derivative of Y to derive a completely discrete scheme.

From (2.21), $\{D_\theta Y_t\}_{0\le\theta\le t\le T}$ can be represented as

$$
D_\theta Y_t=\mathbb{E}\biggl(\rho_{t,T}D_\theta\xi+\int_t^T \rho_{t,r}D_\theta f(r,Y_r,Z_r)\,dr\bigg|\mathcal{F}_t\biggr),
\tag{5.1}
$$

where

$$
\rho_{t,r}=\exp\biggl\{\int_t^r \beta_s\,dW_s+\int_t^r\biggl(\alpha_s-\frac{1}{2}\beta_s^2\biggr)\,ds\biggr\}
\tag{5.2}
$$

with $\alpha_s=\partial_y f(s,Y_s,Z_s)$ and $\beta_s=\partial_z f(s,Y_s,Z_s)$.

Using that $Z_t = D_t Y_t$, $\mu \times P$ a.e., from (1.1), (5.1) and (5.2), we propose the following numerical scheme. We define recursively

$$
\begin{aligned}
&Y^\pi_{t_n} = \xi, \qquad Z^\pi_{t_n} = D_T \xi,\\
&Y^\pi_{t_i} = \mathbb{E}(Y^\pi_{t_{i+1}} + f(t_{i+1}, Y^\pi_{t_{i+1}}, Z^\pi_{t_{i+1}})\Delta_i | \mathcal{F}_{t_i}),\\
&Z^\pi_{t_i} = \mathbb{E}\left(\rho^\pi_{t_{i+1},t_n} D_{t_i}\xi + \sum_{k=i}^{n-1} \rho^\pi_{t_{i+1},t_{k+1}} D_{t_i} f(t_{k+1}, Y^\pi_{t_{k+1}}, Z^\pi_{t_{k+1}})\Delta_k \middle| \mathcal{F}_{t_i}\right),\\
&\qquad\qquad i = n-1, n-2, \ldots, 0,
\end{aligned} \tag{5.3}
$$

where $\rho^\pi_{t_i,t_i} = 1, i = 0, 1, \ldots, n$, and for $0 \le i < j \le n$,

$$
\begin{aligned}
\rho^\pi_{t_i,t_j} = \exp\Biggl\{&\sum_{k=i}^{j-1} \int_{t_k}^{t_{k+1}} \partial_z f(r, Y^\pi_{t_k}, Z^\pi_{t_k})\, dW_r\\
&+ \sum_{k=i}^{j-1} \int_{t_k}^{t_{k+1}} \left(\partial_y f(r, Y^\pi_{t_k}, Z^\pi_{t_k}) - \frac{1}{2}[\partial_z f(r, Y^\pi_{t_k}, Z^\pi_{t_k})]^2\right) dr\Biggr\}.
\end{aligned} \tag{5.4}
$$

An alternative expression for $\rho^\pi_{t_i,t_j}$ is given by the following formula:

$$
\begin{aligned}
\rho^\pi_{t_i,t_j} = \exp\Biggl\{&\sum_{k=i}^{j-1} \partial_z f(t_k, Y^\pi_{t_k}, Z^\pi_{t_k})(W_{t_{k+1}} - W_{t_k})\\
&+ \sum_{k=i}^{j-1} \left(\partial_y f(t_k, Y^\pi_{t_k}, Z^\pi_{t_k}) - \frac{1}{2}[\partial_z f(t_k, Y^\pi_{t_k}, Z^\pi_{t_k})]^2\right)\Delta_k\Biggr\}.
\end{aligned} \tag{5.5}
$$

However, we will only consider the scheme (5.3) with $\rho^\pi_{t_i,t_j}$ given by (5.4).

We make the following assumptions:

(G1) $f(t, y, z)$ is deterministic, which implies $D_\theta f(t, y, z) = 0$.

(G2) $f(t, y, z)$ is linear with respect to y and z; namely, there are three functions $g(t)$, $h(t)$ and $f_1(t)$ such that

$$f(t, y, z) = g(t)y + h(t)z + f_1(t).$$

Assume that g, h are bounded and $f_1 \in L^2([0, T])$. Moreover, there exists a constant $L_2 > 0$, such that, for all $t_1, t_2 \in [0, T]$,

$$|g(t_2) - g(t_1)| + |h(t_2) - h(t_1)| + |f_1(t_2) - f_1(t_1)| \le L|t_2 - t_1|^{1/2}.$$

(G3) $\mathbb{E}\sup_{0\le\theta\le T} |D_\theta \xi|^r < \infty$, for all $r \ge 1$.

Notice that (G1) and (G2) imply (ii) and (iii) in Assumption 2.2.

REMARK 5.1. We propose condition (G1) in order to simplify $\{Z^\pi_{t_i}\}_{i=n-1,\ldots,0}$ in formula (5.3). In fact, there are some difficulties in generalizing the condition (G)s, especially (G1), to a forward–backward stochastic differential equation (FBSDE, for short) case.

If we consider a FBSDE

$$\begin{cases} X_t = X_0 + \int_0^t b(r, X_r)\, dr + \int_0^t \sigma(r, X_r)\, dW_r, \\ Y_t = \xi + \int_t^T f(r, X_r, Y_r, Z_r)\, dr - \int_t^T Z_r\, dW_r, \end{cases}$$

where $X_0 \in \mathbb{R}$, and the functions b, σ, f are deterministic, then under some appropriate conditions [e.g., (h1)–(h4) in Example 2.11] $Z^\pi_{t_i}$ for $i = n-1, \dots, 0$ in (5.3) is of the form

$$Z^\pi_{t_i} = \mathbb{E}\Bigg(\rho^\pi_{t_{i+1},t_n} D_{t_i}\xi + \sum_{k=i}^{n-1} \rho^\pi_{t_{i+1},t_{k+1}} \partial_x f(t_{k+1}, X^\pi_{t_{k+1}}, Y^\pi_{t_{k+1}}, Z^\pi_{t_{k+1}}) D_{t_i} X^\pi_{t_{k+1}} \Delta_k \Big| \mathcal{F}_{t_i}\Bigg),$$

where (X^π, Y^π, Z^π) is a certain numerical scheme for (X, Y, Z). It is hard to guarantee the existence and the convergence of Malliavin derivative of X^π, and therefore, the convergence of Z^π is difficult to derive.

THEOREM 5.2. *Let Assumption 2.2*(i) *and assumptions* (G1)–(G3) *be satisfied. Then there are positive constants K and δ independent of the partition π, such that, when $|\pi| < \delta$ we have*

$$\mathbb{E} \max_{0 \le i \le n} \{|Y_{t_i} - Y^\pi_{t_i}|^p + |Z_{t_i} - Z^\pi_{t_i}|^p\} \le K|\pi|^{p/2 - p/(2\log(1/|\pi|))} \left(\log \frac{1}{|\pi|}\right)^{p/2}.$$

PROOF. In the proof, $C > 0$ will denote a constant independent of the partition π, which may vary from line to line. Under the assumption (G1), we can see that

$$Z^\pi_{t_i} = \mathbb{E}(\rho^\pi_{t_{i+1},t_n} D_{t_i}\xi | \mathcal{F}_{t_i}), \qquad i = n-1, n-2, \dots, 0. \tag{5.6}$$

Denote, for $i = n-1, n-2, \dots, 0$,

$$\delta Z^\pi_{t_i} = Z_{t_i} - Z^\pi_{t_i}, \qquad \delta Y^\pi_{t_i} = Y_{t_i} - Y^\pi_{t_i}.$$

Since $|e^x - e^y| \le (e^x + e^y)|x - y|$, we deduce, for all $i = n-1, n-2, \dots, 0$,

$$\begin{aligned} |\delta Z^\pi_{t_i}| &= |\mathbb{E}(\rho_{t_i,t_n} D_{t_i}\xi | \mathcal{F}_{t_i}) - \mathbb{E}(\rho^\pi_{t_{i+1},t_n} D_{t_i}\xi | \mathcal{F}_{t_i})| \\ &\le \mathbb{E}(|\rho_{t_i,t_n} - \rho^\pi_{t_{i+1},t_n}||D_{t_i}\xi||\mathcal{F}_{t_i}) \\ &\le \mathbb{E}\Bigg(|D_{t_i}\xi|(\rho_{t_i,t_n} + \rho^\pi_{t_{i+1},t_n}) \\ &\qquad \times \left|\int_{t_i}^T h(r)\, dW_r + \int_{t_i}^T g(r)\, dr - \frac{1}{2}\int_{t_i}^T h(r)^2\, dr\right. \end{aligned}$$

$$
\begin{aligned}
&\quad - \sum_{k=i+1}^{n-1} \int_{t_k}^{t_{k+1}} h(r)\,dW_r - \sum_{k=i+1}^{n-1} \int_{t_k}^{t_{k+1}} g(r)\,dr \\
&\qquad + \frac{1}{2} \sum_{k=i+1}^{n-1} \int_{t_k}^{t_{k+1}} h(r)^2\,dr \bigg| \bigg| \mathcal{F}_{t_i}\bigg) \\
&\le \mathbb{E}\bigg(|D_{t_i}\xi| (\rho_{t_i,t_n} + \rho^{\pi}_{t_{i+1},t_n}) \\
&\qquad \times \bigg[\bigg| \int_{t_i}^{t_{i+1}} h(r)\,dW_r \bigg| + \int_{t_i}^{t_{i+1}} |g(r)|\,dr \\
&\qquad\qquad + \frac{1}{2} \int_{t_i}^{t_{i+1}} h(r)^2\,dr \bigg] \bigg| \mathcal{F}_{t_i}\bigg).
\end{aligned}
$$

From (G2), we have

$$
\begin{aligned}
&|D_{t_i}\xi| \rho^{\pi}_{t_{i+1},t_n} \\
&\quad \le |D_{t_i}\xi| \exp\bigg\{ \int_{t_{i+1}}^{T} h(r)\,dW_r + \sum_{k=i+1}^{n-1} \int_{t_k}^{t_{k+1}} g(r)\,dr - \frac{1}{2} \int_{t_{i+1}}^{T} h(r)^2\,dr \bigg\} \\
&\quad \le C_1 \Big(\sup_{0\le\theta\le T} |D_\theta \xi| \Big) \bigg(\sup_{0\le t\le T} \exp\bigg\{ \int_t^T h(r)\,dW_r \bigg\} \bigg),
\end{aligned}
$$

where $C_1 > 0$ is a constant independent of the partition π.

In the same way, we obtain

$$
|D_{t_i}\xi| \rho_{t_i,t_n} < C_1 \Big(\sup_{0\le\theta\le T} |D_\theta \xi| \Big) \bigg(\sup_{0\le t\le T} \exp\bigg\{ \int_t^T h(r)\,dW_r \bigg\} \bigg).
$$

Thus for $i = n-1, n-2, \dots, 0$,

$$
\begin{aligned}
|\delta Z^{\pi}_{t_i}| &\le 2C_1 \mathbb{E}\bigg(\Big(\sup_{0\le\theta\le T} |D_\theta \xi| \Big) \bigg(\sup_{0\le t\le T} \exp\bigg\{ \int_t^T h(r)\,dW_r \bigg\} \bigg) \\
&\qquad \times \bigg[\bigg| \int_{t_i}^{t_{i+1}} h(r)\,dW_r \bigg| + \int_{t_i}^{t_{i+1}} |g(r)|\,dr + \frac{1}{2} \int_{t_i}^{t_{i+1}} h(r)^2\,dr \bigg] \bigg| \mathcal{F}_{t_i}\bigg) \\
&\le 2C_1 \mathbb{E}\bigg(\Big(\sup_{0\le\theta\le T} |D_\theta \xi| \Big) \bigg(\sup_{0\le t\le T} \exp\bigg\{ \int_t^T h(r)\,dW_r \bigg\} \bigg) \\
&\qquad \times \bigg[\sup_{0\le k\le n-1} \bigg| \int_{t_k}^{t_{k+1}} h(r)\,dW_r \bigg| + \sup_{0\le k\le n-1} \int_{t_k}^{t_{k+1}} |g(r)|\,dr \\
&\qquad\qquad + \frac{1}{2} \sup_{0\le k\le n-1} \int_{t_k}^{t_{k+1}} h(r)^2\,dr \bigg] \bigg| \mathcal{F}_{t_i}\bigg).
\end{aligned}
$$

The right-hand side of the above inequality is a martingale as a process indexed by $i = n-1, n-2, \ldots, 0$.

Let $\eta_t = \exp\{-\int_0^t h(u)\, dW_u\}$. Then, η_t satisfies the following linear stochastic differential equation:

$$\begin{cases} d\eta_t = -h(t)\eta_t\, dW_t + \frac{1}{2}h(t)^2\eta_t dt, \\ \eta_0 = 1. \end{cases}$$

By (G1), (G2), the Hölder inequality and Lemma 2.4, it is easy to show that, for any $r \geq 0$,

$$\begin{aligned} &\mathbb{E}\biggl(\sup_{0\le t\le T}\exp\biggl\{\int_t^T h(u)\, dW_u\biggr\}\biggr)^r \\ &\quad = \mathbb{E}\biggl(\exp\biggl\{\int_0^T h(u)\, dW_u\biggr\}\sup_{0\le t\le T}\exp\biggl\{-\int_0^t h(u)\, dW_u\biggr\}\biggr)^r \\ &\quad \le \biggl(\mathbb{E}\exp\biggl\{2r\int_0^T h(u)\, dW_u\biggr\}\biggr)^{1/2} \qquad (5.7) \\ &\qquad \times \biggl(\mathbb{E}\sup_{0\le t\le T}\exp\biggl\{-2r\int_0^t h(u)\, dW_u\biggr\}\biggr)^{1/2} \\ &\quad = \exp\biggl\{r^2\int_0^T h(u)^2\, dr\biggr\}\Bigl(\mathbb{E}\sup_{0\le t\le T}\eta_t^{2r}\Bigr)^{1/2} < \infty. \end{aligned}$$

For any $p' \in (p, \frac{q}{2})$, by Doob's maximal inequality and the Hölder inequality, (G3) and (5.7), we have

$$\begin{aligned} &\mathbb{E}\sup_{0\le i\le n}|\delta Z_{t_i}^\pi|^p \\ &\quad \le C\mathbb{E}\biggl(\Bigl(\sup_{0\le\theta\le T}|D_\theta\xi|\Bigr)^p\biggl(\sup_{0\le t\le T}\exp\biggl\{\int_t^T h(r)\, dW_r\biggr\}\biggr)^p \\ &\qquad\qquad \times \biggl[\sup_{0\le k\le n-1}\biggl|\int_{t_k}^{t_{k+1}} h(r)\, dW_r\biggr| \\ &\qquad\qquad\qquad + \sup_{0\le k\le n-1}\int_{t_k}^{t_{k+1}}|g(r)|\, dr + \frac{1}{2}\sup_{0\le k\le n-1}\int_{t_k}^{t_{k+1}} h(r)^2\, dr\biggr]^p\biggr) \\ &\quad \le C\biggl[\mathbb{E}\biggl(\Bigl(\sup_{0\le\theta\le T}|D_\theta\xi|\Bigr)^{pp'/(p'-p)} \\ &\qquad\qquad \times \biggl(\sup_{0\le t\le T}\exp\biggl\{\int_t^T h(r)\, dW_r\biggr\}\biggr)^{pp'/(p'-p)}\biggr)\biggr]^{(p'-p)/p'} \\ &\qquad \times \biggl[\mathbb{E}\biggl(\sup_{0\le k\le n-1}\biggl|\int_{t_k}^{t_{k+1}} h(r)\, dW_r\biggr| + \sup_{0\le k\le n-1}\int_{t_k}^{t_{k+1}}|g(r)|\, dr \end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad + \frac{1}{2} \sup_{0\le k\le n-1} \int_{t_k}^{t_{k+1}} h(r)^2\,dr\Big)^{p'}\Big]^{p/p'}\\
&\le C\Big[\mathbb{E}\Big(\sup_{0\le\theta\le T}|D_\theta\xi|\Big)^{2pp'/(p'-p)}\Big]^{p'/(2(p'-p))}\\
&\quad\times\Big[\mathbb{E}\Big(\sup_{0\le t\le T}\exp\Big\{\int_t^T h(r)\,dW_r\Big\}\Big)^{2pp'/(p'-p)}\Big]^{p'/(2(p'-p))}\\
&\quad\times\Big[\mathbb{E}\sup_{0\le k\le n-1}\Big|\int_{t_k}^{t_{k+1}} h(r)\,dW_r\Big|^{p'} + \mathbb{E}\sup_{0\le k\le n-1}\Big(\int_{t_k}^{t_{k+1}}|g(r)|\,dr\Big)^{p'}\\
&\qquad\qquad + \mathbb{E}\sup_{0\le k\le n-1}\Big(\int_{t_k}^{t_{k+1}} h(r)^2\,dr\Big)^{p'}\Big]^{p/p'}\\
&= C[I_1+I_2+I_3]^{p/p'}.
\end{aligned}
$$

For any $r>1$, by the Hölder inequality we can obtain

$$
\begin{aligned}
I_1 &= \mathbb{E}\sup_{0\le k\le n-1}\Big|\int_{t_k}^{t_{k+1}} h(r)\,dW_r\Big|^{p'} \le \Big\{\mathbb{E}\sup_{0\le k\le n-1}\Big|\int_{t_k}^{t_{k+1}} h(r)\,dW_r\Big|^{p'r}\Big\}^{1/r}\\
&\le \Big\{\mathbb{E}\sum_{k=0}^{n-1}\Big|\int_{t_k}^{t_{k+1}} h(r)\,dW_r\Big|^{p'r}\Big\}^{1/r}.
\end{aligned}
$$

For any centered Gaussian variable X, and any $\gamma\ge 1$, we know that

$$
\mathbb{E}|X|^\gamma \le \tilde{C}^\gamma \gamma^{\gamma/2}(\mathbb{E}|X|^2)^{\gamma/2},
$$

where $\tilde{C}$ is a constant independent of γ. Thus, we can see that

$$
I_1 \le \Big(\tilde{C}^{p'r}(p'r)^{p'r/2}\sum_{i=0}^{n-1}\Big(\int_{t_i}^{t_{i+1}} h(r)^2\,dr\Big)^{p'r/2}\Big)^{1/r} \le Cr^{p'/2}|\pi|^{p'/2-1/r}.
$$

Take $r=\frac{2\log(1/|\pi|)}{p'}$. Assume $|\pi|$ is small enough; then we have

$$
I_1 \le C|\pi|^{p'/2-p'/(2\log(1/|\pi|))}\Big(\log\frac{1}{|\pi|}\Big)^{p'/2}.
$$

It is easy to see that

$$
I_2 = \mathbb{E}\sup_{0\le k\le n-1}\Big(\int_{t_k}^{t_{k+1}}|g(r)|\,dr\Big)^{p'} \le C|\pi|^{p'}
$$

and

$$
I_3 = \mathbb{E}\sup_{0\le k\le n-1}\Big(\int_{t_k}^{t_{k+1}} h(r)^2\,dr\Big)^{p'} \le C|\pi|^{p'}.
$$

Consequently, we obtain

$$\mathbb{E}\sup_{0\le i\le n}|\delta Z_{t_i}^\pi|^p \le C|\pi|^{p/2-p/(2\log(1/|\pi|))}\left(\log\frac{1}{|\pi|}\right)^{p/2}. \tag{5.8}$$

Applying recursively the scheme given by (5.3), we obtain

$$Y_{t_i}^\pi = \mathbb{E}\left(\xi + \sum_{k=i+1}^{n} f(t_k, Y_{t_k}^\pi, Z_{t_k}^\pi)\Delta_{k-1}\middle|\mathcal{F}_{t_i}\right), \qquad i = n-1, n-2, \dots, 0.$$

Therefore, for $i = n-1, n-2, \dots, 0$,

$$|\delta Y_{t_i}^\pi| \le \mathbb{E}\left(\sum_{k=i+1}^{n} |f(t_k, Y_{t_k}, Z_{t_k}) - f(t_k, Y_{t_k}^\pi, Z_{t_k}^\pi)|\Delta_{k-1} + |R_{t_i}^\pi| + |\delta\xi^\pi|\middle|\mathcal{F}_{t_i}\right),$$

where R_t^π is exactly the same as in Section 3 and $\delta\xi^\pi = \xi - \xi = 0$. In fact, we keep the term $\delta\xi^\pi$ to indicate the role it plays as the terminal value.

For $j = n-1, n-2, \dots, i$, we have

$$|\delta Y_{t_j}^\pi| \le \mathbb{E}\left(\sum_{k=i+1}^{n} |f(t_k, Y_{t_k}, Z_{t_k}) - f(t_k, Y_{t_k}^\pi, Z_{t_k}^\pi)|\Delta_{k-1} + \sup_{0\le t\le T}|R_t^\pi| + |\delta\xi^\pi|\middle|\mathcal{F}_{t_j}\right).$$

By Doob's maximal inequality and (5.8), we obtain

$$\begin{aligned}
&\mathbb{E}\sup_{i\le j\le n}|\delta Y_{t_j}^\pi|^p \\
&\quad\le C\mathbb{E}\left(\sum_{k=i+1}^{n} |f(t_k, Y_{t_k}, Z_{t_k}) - f(t_k, Y_{t_k}^\pi, Z_{t_k}^\pi)|\Delta_{k-1}\right)^p \\
&\qquad + C(|\pi|^{p/2} + \mathbb{E}|\delta\xi^\pi|^p) \\
&\quad\le C\left\{\mathbb{E}\left(\sum_{k=i+1}^{n}|Y_{t_k} - Y_{t_k}^\pi|\Delta_{k-1}\right)^p + \mathbb{E}\left(\sum_{k=i+1}^{n}|Z_{t_k} - Z_{t_k}^\pi|\Delta_{k-1}\right)^p\right\} \\
&\qquad + C(|\pi|^{p/2} + \mathbb{E}|\delta\xi^\pi|^p) \\
&\quad\le C_2(T-t_i)^p\mathbb{E}\sup_{i+1\le k\le n}|Y_{t_k} - Y_{t_k}^\pi|^p \\
&\qquad + C_3\left(|\pi|^{p/2-p/(2\log(1/|\pi|))}\left(\log\frac{1}{|\pi|}\right)^{p/2} + \mathbb{E}|\delta\xi^\pi|^p\right),
\end{aligned}$$

where C_2 and C_3 are constants independent of the partition π.

We can obtain the estimate for $\mathbb{E}\max_{0\le i\le n}|Y_{t_i} - Y_{t_i}^\pi|^p$ by using similar arguments to analyze (4.13) in Theorem 4.2 to get the estimate for $\mathbb{E}\sup_{0\le t\le T}|Y_t - Y_t^\pi|$. □

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