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A T_B SPACE WHICH IS NOT KATETOV T_B

WILLIAM G. FLEISSNER

In 1943, E. Hewitt [**1**] proved the beautiful theorem that a compact Hausdorff space is minimal Hausdorff and maximal compact. Restating this result in more detail, if (X, τ) is a compact Hausdorff space and (X, τ') and (X, τ'') are spaces $\tau' \subsetneq \tau \subsetneq \tau''$, then (X, τ') is not Hausdorff, and (X, τ'') is not compact. The converses to this theorem are appealing but false. There are noncompact minmal Hausdorff spaces [**2**] and non Hausdorff maximal compact spaces [**2**].

A compact space is maximal compact if every compact set is closed [**3**]. Let us call spaces in which all compact sets are closed T_B spaces, as this notion can be thought of as a separation axiom between T_1 and T_2. They are also called KC spaces. R. Larson [**4**] asked whether a space is maximal compact iff it is minimal T_B. A related question is whether every T_B topology is Katetov T_B, that is whether every T_B topology contains a minimal T_B topology. The author wishes to thank Douglas Cameron for bringing these questions to his attention. In this paper we construct a T_B not Katetov T_B tpace.

The point set of all spaces in this paper will be the countable ordinals. To avoid ambiguity, we will refer to the first uncountable ordinal (and cardinal) as ω_1, and to the point set of the spaces as Ω. A typical point of Ω will be x_α, where $\alpha < \omega_1$. The point set $\{x_\beta : \beta < \alpha\}$ will be called $P(\alpha)$, the predecessors of α; and the point set $\{x_\beta \colon \beta > \alpha\}$ will be called $S(\alpha)$, the successors of α. The usual topology on Ω, generated by $\{P(\alpha)\colon \alpha < \omega_1\} \cup \{S(\alpha) : \alpha < \omega_1\}$ will be called $\mathscr{u}$. The cardinality of a set S will be denoted $|S|$.

LEMMA 1. *If $\tau' \subset \tau$ and K is τ compact, K is τ' compact.*

LEMMA 2. *A compact T_B space is a minimal T_B space.*

If $S \subset \Omega$, we denote the subspace of (Ω, τ) with point set S by $(S, \tau|S)$. Equivalently, $\tau|S = \{U \cap S\colon U \in \tau\}$. We say that τ, τ' agree on countable sets if for all $S \subset \Omega$ with $|S| \leqq \omega$,

$$(S, \tau|S) = (S, \tau'|S).$$

LEMMA 3. *Suppose $\tau \subset \mathscr{u}$ and (Ω, τ) is T_B. Then τ, $\mathscr{u}$ agree on countable sets.*

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LEMMA 4. *If* τ, $\mathscr{u}$ *agree on countable sets, then for all* α, $S(\alpha) \in \tau$.

LEMMA 5. *Suppose* (Ω, τ) *and* G *satisfy*

i) $G \in \tau \subset \mathscr{u}$,

ii) τ, $\mathscr{u}$ *agree on countable sets,*

iii) $|\Omega - G| = \omega_1$,

iv) $\forall x_\alpha \in (\Omega - G)\, \exists G_\alpha \in \tau (x_\alpha \in G_\alpha \subset P(\alpha + 1) \cup G)$.

Then (Ω, τ) *is* T_B.

PROOF. Suppose K is τ compact. By ii) it is sufficient to show that K is countable.

First, by considering the τ open cover $\{G\} \cup \{G_\alpha : x_\alpha \in \Omega - G\}$, we may conclude that $|K - G| < \omega_1$. Aiming for a contradiction, we assume $|K| = \omega_1$. With this assumption we can find $\{\alpha(i) : i \in \omega\}$, $\alpha = \sup\{\alpha(i) : i \in \omega\}$ so that $\{x_{\alpha(i)} : i \in \omega\} \subset K \cap G$, $x_\alpha \notin K \cup G$. Now $P(\alpha + 1)$ is $\mathscr{u}$ closed, hence τ closed. Then $P(\alpha + 1) \cap K$ is τ compact. But $P(\alpha + 1) \cap K$ is not $\mathscr{u}$ compact, and τ and $\mathscr{u}$ agree on countable sets.

Let I be the set of isolated points of $(\Omega, \mathscr{u})$. We define $(\Omega, \mathscr{t})$, the space referred to in the title, by defining

$$\mathscr{t} = \{U \in \mathscr{u} : (x_0 \notin U \text{ and } x_1 \notin U) \text{ or } |I - U| < \omega_1\}.$$

Clearly $\mathscr{t}$ is a topology and $\mathscr{t} \subset \mathscr{u}$. And $\mathscr{t} \neq \mathscr{u}$ as $\{x_0\} \in \mathscr{u} - \mathscr{t}$. By Lemma 5, $(\Omega, \mathscr{t})$ is T_B.

Henceforth let τ be a topology with $\tau \subset \mathscr{t}$.

LEMMA 6. *If* (Ω, τ) *is* T_B *then for all* $\alpha < \omega_1$ *there is* $V_\alpha \in \tau$ *satisfying* $x_\alpha \in V_\alpha$ *and* $|\Omega - V_\alpha| = \omega_1$.

PROOF. Suppose not, that for some α, $x_\alpha \in V \in \tau$ implies $|\Omega - V| < \omega_1$. Let $y \in \{x_0, x_1\} - \{x_\alpha\}$. We aim for the contradiction that $\Omega - \{y\}$ is τ compact but not τ closed. From $\tau \subset \mathscr{t}$ and the definition of $\mathscr{t}$, $\Omega - \{y\}$ is not τ closed.

Let $\mathscr{U}$ be a τ open cover of $\Omega - \{y\}$. Then there is $V \in \mathscr{U}$, $x_\alpha \in V \in \tau$. By hypothesis, $|\Omega - V| < \omega_1$, so there is a β with $\Omega - V \subset P(\beta + 1)$. Now $P(\beta + 1) - \{y\}$ is $\mathscr{u}$ compact, so by Lemma 1, there is $\mathscr{U}' \subset \mathscr{U}$, $\mathscr{U}'$ a finite subcover of $P(\beta + 1) - \{y\}$. Then $\mathscr{U}' \cup \{V\}$ is a finite subcover of $\Omega - \{y\}$, establishing the contradiction that $\Omega - \{y\}$ is τ compact.

We assume (Ω, τ) is T_B; we aim towards constructing a coarser T_B topology.

For all $\alpha < \omega_1$, let V_α be as asserted in Lemma 6. We define

$$\Delta = \{x_\alpha : x_\alpha \notin \bigcup_{\beta < \alpha} V_\beta\}$$

Note that $x_0 \in \Delta$. (By definition, if you like.)

LEMMA 7. $\Omega - \Delta \in \tau$.

PROOF. If $x_\alpha \in \Omega - \Delta$, then there is $\beta < \alpha$ such that $x_\alpha \in V_\beta$. Since $\tau \subset \mathscr{U}$ and τ is T_B, by Lemmas 3 and 4, $S(\beta) \in \tau$. Thus $x_\alpha \in V_\beta \cap S(\beta) \subset \Omega - \Delta$, $V_\beta \cap S(\beta) \in \tau$.

LEMMA 8. $|\Delta| = \omega_1$.

PROOF. By definition $\alpha < \omega_1$ means that there is a map f_α from ω onto $\{\beta : \beta < \alpha\}$. Let g and h be maps from $\omega - \{0\}$ to ω such that $g(i) < i$, and for all $(m, n) \in \omega \times \omega$ there are infinitely many $i \in \omega$ such that $(g(i), h(i)) = (m, n)$.

Let $\alpha(0) < \omega_1$ be arbitrary. We will establish Lemma 8 by finding $\alpha > \alpha(0)$ with $x_\alpha \in \Delta$.

For $i > 0$, we may choose, by our assumption on V_α, $\alpha(i) > \alpha(i-1)$ such that $x_{\alpha(i)} \in \Omega - V_{f_{\alpha(g(i))}(h(i))}$. Let $\alpha = \sup\{\alpha(i) : i \in \omega\}$; we claim $x_\alpha \in \Delta$. For let $\beta < \alpha$. Then $\beta < \alpha(j)$ for some $j \in \omega$, and so $\beta = f_{\alpha(j)}(k)$ for some $k < \omega$. Now $g(i) = j$, $h(i) = k$ implies $x_{\alpha(i)} \in \Omega - V_\beta$, a closed set. By our choice of g and h, $\alpha = \sup\{\alpha(i) : g(i) = j, h(i) = k\}$, so $x_\alpha \in \Omega - V_\beta$, establishing our cliam.

Now we define I' to be the set of isolated points of $(\Delta, \tau|\Delta)$. That is, $I' = \{x_\alpha : U \in \tau, U \cap \Delta = \{x_\alpha\}\}$.

LEMMA 9. $|I'| = \omega_1$.

PROOF. Let $\alpha < \omega_1$. Let $\beta = \inf\{\gamma : x_\gamma \in \Delta, \gamma > \alpha\}$. Then $\Delta \cap V_\beta \cap S(\alpha) = \{x_\beta\}$.

Finally, we define a T_B topology coarser than τ. Set $\tau' = \{U \in \tau : (x_0 \notin U) \text{ or } |I' - U| < \omega_1\}$.

Clearly τ' is a topology and $\tau' \subset \tau$. By Lemma 3 τ, $\mathscr{U}$ agree on countable sets, and by definition τ', τ agree on countable sets. Also, $\tau' \neq \tau$ because $V_0 \in \tau - \tau'$. (Ω, τ') is T_B by Lemma 5, setting $G = (\Omega - \Delta) \cup I'$, and $G_\alpha = V_\alpha - \{x_0\}$.

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INSTITUTE FOR MEDICINE AND MATHEMATICS, OHIO UNIVERSITY, ATHENS, OHIO 45701