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SON OF GEORGE AND $V = L$

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Abstract. This paper has three parts. In this first part, we formulate and prove from $V=L$ a new combinatorial principle, $\diamondsuit^{++}$. In the second part, we discuss the topological problem which led to the formulation of $\diamondsuit^{++}$. Finally, we use $\diamondsuit^{++}$ to construct a space solving the topological problem.

§1. **$\diamondsuit^{++}$: Formulation and proof from $V = L$.** The combinatorial principles $\diamondsuit^*$ and $\diamondsuit^+$ can be thought of as giving an ω_1-sequence of countable approximations to the set $\mathscr{P}(\omega_1)$. To construct the space of this paper, this is not enough; we need filters which approximate the club filter and which define an accurate notion of stationary. Specifically what we need is $\diamondsuit^{++}$, which asserts

There are A and C such that:

1(a) *A is a function with domain ω_1; for all $\alpha \in \omega_1$, $A(\alpha) \in [\mathscr{P}(\alpha)]^{\leq\omega}$.*

(b) *C is a function from $\mathscr{P}(\omega_1)$ to the family of club subsets of ω_1.*

(c) *For all $X \in \mathscr{P}(\omega_1)$, if $\gamma \in C(X)$, then $X \cap \gamma \in A(\gamma)$ and $C(X) \cap \gamma \in A(\gamma)$.*

Part 1 is simply a statement of $\diamondsuit^+$. Given A and C as above, define for $\delta \in \omega_1$, $\mathscr{C}_\delta = \{c \in A(\delta)\colon c \text{ is club in } \delta\}$, and for $X \in \mathscr{P}(\omega_1)$, define $S(X) = \{\delta \in \omega_1\colon \text{for all } c \in \mathscr{C}_\delta,\ c \cap X \neq \varnothing\}$.

$\diamondsuit^{++}$, continued. *Additionally, there is D such that:*

2(a) *D is a stationary subset of ω_1.*

(b) *For all $\delta \in D$, $\mathscr{C}_\delta$ is a filter.*

(c) *If $\mathscr{X}$ is a countable family of stationary subsets of ω_1, then $\bigcap\{S(X)\colon X \in \mathscr{X}\} \cap D$ is stationary.*

In outline, the proof of $\diamondsuit^{++}$ from $V = L$ is the same as that of $\diamondsuit^+$. However, to get part 2 of $\diamondsuit^{++}$, we need a few definitions and a lemma.

For $\alpha \in \omega_1$, set

$$S_\alpha = \{\nu \in \omega_1\colon \alpha = \omega_1^{L_\nu} \text{ and } L_\nu \models \mathrm{ZF}^-\}.$$

Set $D = \{\alpha\colon S_\alpha \neq \varnothing \text{ and } S_\alpha \text{ has no last element}\}$.

LEMMA 1. *D is not empty. Moreover, for any $a \in L_{\omega_2}$, there is $\alpha \in D$ with a cofinal subset, N, of S_α such that if $\nu \in N$, then there is an elementary embedding of L_ν to a transitive model of $\mathrm{ZF}^- + V = L$ containing a in the range.*

PROOF. We will prove the second assertion. Let $a \in L_{\omega_2}$. Define μ_n, $n \in \omega$, so that

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for each $n \in \omega$, μ_n is the least ordinal such that $\{\omega_1\} \cup \{a\} \cup \{\mu_j : j < n\} \subset L_{\mu_n} \models$ ZF$^-$. Construct in the usual way, for each n, a continuous increasing sequence, $(M^n_\gamma)_{\gamma<\omega_1}$, of countable elementary submodels of L_{μ_n} containing a. Let $C^n = \{M^n_\gamma \cap \omega_1 : \gamma < \omega_1\}$; for each n, C^n is club. Let α be the least element of $\bigcap_n C^n$.

For each $n \in \omega$, there is $\nu_n \in S_\alpha$ such that L_{ν_n} is the transitive collapse of an elementary submodel of L_{μ_n}. Let $N = \{\nu_n : n \in \omega\}$; we must show that N is cofinal in S_α.

First, $\rho = \sup_n \nu_n \notin S_\alpha$ because N is an ω-sequence cofinal in ρ definable from $a \in L_\rho$; hence $L_\rho \not\models$ Replacement. Second, if $\rho' > \rho$, then $\rho' \notin S_\alpha$ because the sequence $\{C_n \cap \alpha : n \in \omega\}$ is definable in $L_{\rho'}$; hence if $L_{\rho'} \models$ ZF$^-$ then $L_{\rho'} \models$ "α is countable." □

K. Kunen gave me the idea of the above proof.

THEOREM 1. *$V = L$ implies* $\diamondsuit^{++}$.

PROOF. For $\alpha < \omega_1$ we define $\beta(\alpha)$ by cases: if $\alpha \in D$, then $\beta(\alpha) = \sup S_\alpha$; if $\alpha \notin D$, then $\beta(\alpha)$ is the least β such that $\alpha \in L_\beta \prec L_{\omega_1}$. Let $A(\alpha) = \mathscr{P}(\alpha) \cap L_{\beta(\alpha)}$. Now we can repeat (with minor notation changes) pp. 56–57 of Devlin [D_1] to prove that $\{A(\alpha) : \alpha < \omega_1\}$ satisfies $\diamondsuit^+$.

To see that D is stationary, let C be an arbitrary club subset of ω_1, let $C \in a$, and apply Lemma 1. Then $C \cap \alpha$ is club in α, hence $\alpha \in C \cap D$. Next, let δ be any element of D. If $c, c' \in \mathscr{C}_\delta$, then there is $\nu \in S_\delta$ with $c, c' \in L_\nu$, so $c \cap c'$ is club in δ.

Toward 2(c), let $\mathscr{X}$ be a countable family of stationary subsets of ω_1. Let C be an arbitrary club subset of ω_1, let $a = \mathscr{X} \cup \{C\}$, and apply Lemma 1. As in the previous paragraph, $\alpha \in C$. If $c \in \mathscr{C}_\alpha$, then there is $\nu \in N$ with $c \in L_\nu$. Let $\pi : L_\nu \to L_\mu$ be as in Lemma 1. Then $\pi(c)$ is club in ω_1, hence $\pi(c) \cap X \neq \varnothing$ for all $X \in \mathscr{X}$. By elementarity, $c \cap \pi^{-1}(X) \neq \varnothing$. Since c is arbitrary, $\alpha \in \bigcap\{S(X) : X \in \mathscr{X}\}$. Finally $\bigcap\{S(X) : X \in \mathscr{X}\} \cap D$ is stationary because C is arbitrary. □

Let us call a subset X of ω_1 costationary if $\omega_1 - X$ is stationary. In §3 we will need to know that D is costationary. The D constructed in Lemma 1 is costationary, so we can just add "D is costationary" to the statement of $\diamondsuit^{++}$. Under a different formulation of $\diamondsuit^+$ one can show that D must be costationary (see [D_2, Lemma 1]), but I have been unable to prove that D is costationary from the assumption that A, C, D satisfy 1, 2 stated above. (It would be nice to get a proof not using 2(c).)

However, we present two ways to show that a given D can be replaced by a costationary set D'. First we note that D can be replaced by any costationary subset D' and 1, 2(a) and 2(b) will remain valid. From 2(c) it follows that the family of sets of the form $S(X) \cap C \cap D$, where $X \subset \omega_1$ and C is club, generate a countably closed filter $\mathscr{F}$ on D. Hence by Ulam's argument, D can be partitioned into ω_1 subsets, each meeting each element of $\mathscr{F}$.

A second method is to show that for at least one $X \subset \omega_1$, $S(X)$ is costationary. Then we can replace D with $D \cap S(X)$. Let us define X inductively so that $X = \{\delta \in D : (\exists \in \mathscr{C}_\delta)\ c \cap x = \varnothing\}$ (note that whether $\delta \in X$ depends only on $X \cap \delta$). Towards showing that X is stationary, assume that C is a club set, $C \cap X = \varnothing$. If $\delta \in C$ then $\delta \notin X$; if $\delta \in C(C) \cap D$ then $\delta \in X$; hence $C \cap C(C) \cap D = \varnothing$, contradiction. A similar argument gives $S(X) \cap X = \varnothing$; hence $S(X)$ is costationary.

If for all stationary sets X, $S(X)$ contained a club set, then the notion "stationary" would be $\boldsymbol{\Sigma}^1_1$. The argument of the last paragraph is from a letter from K. Kunen. He showed, moreover, that assuming $V = L$, for every regular uncountable cardinal κ, the notion "stationary in κ" is maximal $\boldsymbol{\Pi}^1_1$, hence not $\boldsymbol{\Sigma}^1_1$.

After seeing a preliminary version of this paper, K. Devlin [D_2] formulated and proved from $V = L$ the combinatorial principle $\diamondsuit^{\#}$, which implies $\diamondsuit^{++}$, the existence of Kurepa tree with no Aronszajn subtree, $\diamondsuit$ for stationary systems, and more. Very roughly, 2(c) of $\diamondsuit^{++}$ says that $\boldsymbol{\Pi}^1_1$ notion "stationary" reflects down to some $A(\alpha)$; $\diamondsuit^{\#}$ says that all $\boldsymbol{\Pi}^2_1$ notions reflect down.

Finally, M. E. Rudin [R] has used $\diamondsuit^{++}$ to construct a screenable, normal, not paracompact space.

§2. Approximations of Bing's Example G. First, three definitions from topology. A family, $\mathscr{Y}$, of subsets of a space, X, is *discrete* if every $x \in X$ has a neighborhood meeting at most one element of $\mathscr{Y}$. A space, X, is *collectionwise normal* if every discrete collection of closed sets can be simultaneously separated by disjoint open sets. Since a finite family of disjoint closed sets is discrete, a collectionwise normal space is normal. The *character* of a space X, $\chi(X)$, is the least infinite cardinal such that every point of x has a neighborhood base of cardinality no more than $\chi(X)$.

A generalization of the normal Moore space conjecture is to find a normal, not collectionwise normal space of small character. (For a recent survey of the normal Moore space conjecture, see [T].) Three types of normal, not collectionwise normal spaces are known. First, straightforward variations of Bing's Example G; second, spaces constructed assuming MA + $\neg$CH, or using iterated forcing directly; third, spaces constructed by tying together approximations of Bing's Example G. In chronological order, spaces of the third kind are George [F_1], the space of this paper, and the space of [F_2].

Let us now define Bing's Example G. The point set of G is ${}^{\mathscr{P}(\omega_1)}2$. For each $\alpha \in \omega_1$ define $y_\alpha \in {}^{\mathscr{P}(\omega_1)}2$ by $y_\alpha(A) = 1$ iff $\alpha \in A$. Set $Y = \{y_\alpha: \alpha \in \omega_1\}$. A basis for the space consists of all the usual product basic open sets together with all singleton subsets of $G - Y$. The character of G is the cardinality of $\mathscr{P}(\omega_1)$, or 2^{ω_1}.

G is normal. Let H and K be disjoint closed subsets of G. Let $A = H \cap Y$; set $U = \{g \in G: g(A) = 1\}$ and $V = \{g \in G: g(A) = 0\}$. Then $(U \cup H) - K$ and $(V \cup K) - H$ are disjoint open sets of G separating H and K.

G *is not collectionwise normal.* Note that $\{\{y_\alpha\}: \alpha \in \omega_1\}$ is a discrete collection of closed sets. Because every basic open set containing a point $y \in Y$ has positive measure (in the usual product measure), Y cannot be simultaneously separated by disjoint open sets.

Bing's Example G is the simplest example of a normal, not collectionwise normal space, and it illustrates several general points. First, normality is built into the construction. For a canonical pair of disjoint closed sets, there is a subbasic set separating them (in G, a canonical pair (H, K) is one such that $H \cup K = Y$ and $H \cap K = \varnothing$). Second, the not collectionwise normality follows from a relatively simple argument. These two points are true of all known normal, not collectionwise normal spaces. For types one and three, there are subbasic sets for canonical pairs of disjoint closed sets. For type two, the pairs of open sets do not appear in the construction of the space, but they appear in the construction of the model of set theory. To prove that the space is not collectionwise normal, there may be some combinatorial preliminaries, but the heart is a simple measure, cardinality, or pressing down argument.

The goal is to construct a space like G, but with smaller character, by tying together smaller approximations to G. One way to do this is to take as approximations $({}^{\mathscr{P}(\alpha)}2)_{\alpha<\omega_1}$ and tie them together with the usual topology on ω_1. In [F_1], a space, George, was defined this way, and the character was reduced to 2^ω (the cardinality of $\mathscr{P}(\alpha)$).

A natural idea to further reduce the character to ω is to assume $\diamondsuit^*$, take as approximations $({}^{A(\alpha)}2)_{\alpha<\omega_1}$ and tie them together with the usual topology on ω_1. The problem with this approach is that when we define the space, we need for each $\alpha \in \omega_1$ and $a \in A(\alpha)$, a neighborhood W of α such that if $\beta \in W$ then $a \cap \beta \in A(\beta)$. Axiom $\diamondsuit^*$ simply is not strong enough. One way to patch this problem is to assume $\diamondsuit^+$ and to use the club sets from it to define a new finer topology on ω_1. As often happens in mathematics, patching a problem in one place causes a new problem in another place. To define a topological space, the basic open sets must be closed under finite intersection. Two club subsets of a countable ordinal can be disjoint; this is the new problem. Parts 2(a) and 2(b) of $\diamondsuit^{++}$ simply assert that this problem does not happen on a stationary set. (I do not know whether 2(c) is necessary to show that the space has the desired properties, but it is a natural extension of the ideas above, and is quite useful in the proof presented here.)

A final comment before constructing the space. We will define a space, X, with a discrete collection $\mathscr{Y} = \{Y_i: i \in I\}$ of closed subsets, and with points of $X - \bigcup\mathscr{Y}$ isolated. To show that X is normal it does not suffice to consider pairs of closed sets of the form $\bigcup\{Y_i: i \in J\}$ and $\bigcup\{Y_i: i \in I - J\}$. We must consider also pairs H, K where for some i (possibly all i), $H \cap Y_i \neq \varnothing \neq K \cap Y_i$. This problem also arose in [F_1], where it was solved by patching. T. C. Przymusinski [P] pointed out a more elegant (and more general) solution; instead of ${}^{\mathscr{P}(I)}2$, use ${}^{\mathscr{R}}2$, where $\mathscr{R}$ is the family of clopen subsets of $\bigcup\mathscr{Y}$.

§3. Construction of a first countable normal, not cwn space from $\diamondsuit^{++}$. Suppose A', C', D' satisfy $\diamondsuit^{++}$. We begin by slightly changing them to obtain A, C, D which satisfy $\diamondsuit^{++}$ and:

3(a) *If $X \subset \omega_1$ is club, then $C(X) \subset X$.*

(b) *If $X \subset \omega_1$ is nonstationary, then $C(X) \cap X = \varnothing$.*

(c) *If $X \subset \omega_1$ and $\delta \in C(X) \cap D$, then $C(X) \cap \delta \in C_\delta$.*

(d) $D \subset \mathrm{Lim} \cap C(\varnothing)$.

We define C'': $\mathscr{P}(\omega_1) \to \mathscr{P}(\omega_1)$ by cases. If X is club, $C''(X) = C'(X) \cap X$; if X is nonstationary, let W be a club disjoint from X and set $C''(X) = C'(X) \cap W$; if X is stationary but not club, $C''(X) = C'(X)$. Recall from the end of §1 that we may assume that D' is costationary. For $X \subset \omega_1$ and $\alpha \in \omega_1$, define

$$s_X(\alpha) = \inf(C''(X) - D' - (\alpha + 1)),$$

$$C(X) = \text{range } s_X \cup \{\gamma \in \omega_1: \text{if } \alpha < \gamma \text{ then } s_X(\alpha) < \gamma\}.$$

Finally, set $A = A'$ and $D = D' \cap \mathrm{Lim} \cap C(\varnothing)$.

The following definitions and observations will facilitate the definition of the space. For $\delta \in D$, set $\mathscr{B}'_\delta = \{C(X) \cap \delta: X \subset \omega_1 \text{ and } \delta \in C(X)\}$, and let $\mathscr{B}_\delta$ be the family of finite intersections of elements from $\mathscr{B}'_\delta \cup \{\delta - \alpha: \alpha < \delta\}$. Now 3(c) has important consequences. First, $\mathscr{B}'_\delta \subset \mathscr{C}_\delta$, so $\mathscr{B}_\delta$ is countable. Second (using 2(b), also) we see that $\varnothing \notin \mathscr{B}_\delta$. Finally, if $b \in \mathscr{B}'_\delta$ and $\eta \in b \cap D$, then $b \cap \eta \in \mathscr{B}'_\eta$. Hence if $b \in \mathscr{B}_\delta$ and $\eta \in b \cap D$, then $b \cap \eta \in \mathscr{B}_\eta$.

Now we define the "tie-together" space Y. The point set is $\omega_1 = \{\alpha: \alpha < \omega_1\}$. If $\alpha \notin D$, then $\{\{\alpha\}\}$ is a local base at α; i.e. α is isolated. If $\delta \in D$, a local base is $\{\{\delta\} \cup b; b \in \mathscr{B}_\delta\}$. It follows from the observations of the previous paragraph that these local bases "fit together" to form a basis for a first countable topological space. Y is Hausdorff because its topology refines the usual topology on ω_1. The basic open sets given are closed in the usual topology on ω_1, hence also in Y. Consequently Y is regular.

Y *is normal.* Let H and K be disjoint closed subsets of Y. We begin by showing that at least one of H and K is nonstationary. If not, we apply 2(c) of $\diamondsuit^{++}$ to get $\delta \in S(H) \cap S(K) \cap D$. Then $\delta \in \bar{H} \cap \bar{K}$, contradiction.

Suppose that it is K which is not stationary. $C(K)$ is club, hence closed in Y. By 3(b) $C(K) \cap K = \varnothing$. If $\delta \in D \cap C(K)$, then $\{\delta\} \cup (C(K) \cap \delta) \in \mathscr{B}'_\delta$. Hence $C(K)$ is open. Further, $Y - C(K)$ is a discrete union of countable, hence metrizable, subspaces. So let U_0 and V be subsets of $Y - C(K)$ separating $H - C(K)$ and K. Then $U_0 \cup C(K)$ and V are clopen sets separating H and K.

Recall that our plan is to take "countable" approximations (i.e. ${}^{A(\alpha)}2$) of Bing's Example G, and to glue them together using the space Y. In other words, G would fit on top of our space X (although it is *not* there) in such a way that for each special point $y_i \in G$ there is Y_i, a copy of Y, approaching it. On further thought, we see that to use $\diamondsuit^{++}$, we need Y_i to be a copy of $\{\alpha \in Y; i < \alpha < \omega_1\}$ rather than Y.

A final, minor problem is that we want to consider subsets of $\bigcup_i Y_i$, but $\diamondsuit^{++}$ considers instead subsets of ω_1. To deal with this routine bookkeeping, we fix a bijection $\theta: \omega_1 \times \omega_1 \simeq \omega_1$ such that for all limit ordinals $\lambda < \omega_1$, $\theta''\lambda \times \lambda = \lambda$.

Resuming the construction of X, we define a space Z with point set $\{(i, \alpha): i < \alpha < \omega_1\}$. Set $Y_i = \{(i, \alpha) \in Z: i < \alpha < \omega_1\}$; set $Z_\alpha = \{(i, \alpha) \in Z: i < \alpha < \omega_1\}$. Topologize Z so that for all $i \in \omega_1$, Y_i is open and the map of Y_i into Y taking (i, α) to α is a homeomorphism.

For each $\alpha < \omega_1$, let $\mathscr{R}(\alpha)$ be the family of clopen subsets of $\bigcup_{\beta \le \alpha} Z_\beta$ having one of the two following forms:

1. $\{i\} \times B$, where $i < \alpha$ and $B \in \mathscr{B}_\alpha$.
2. R, where R is clopen in $\bigcup_{\beta \le \alpha} Z_\beta$ and $\theta''(R \cap Z_\alpha) \in A(\alpha)$.

Note that $\mathscr{R}(\alpha)$ is countable.

For each $\alpha < \omega_1$, set $X_\alpha = {}^{\mathscr{R}(\alpha)}2$. Set $X = \bigcup_\alpha X_\alpha$. For $x \in X_\alpha$, define $\alpha(x) = \alpha$. For $i < \delta \in D$, define the function $y_{i,\delta}$ from $\mathscr{R}(\delta)$ to $\{0, 1\}$ by

$$y_{i,\delta}(R) = 1 \quad \text{if } (i, \delta) \in R,$$
$$= 0 \quad \text{if } (i, \delta) \notin R.$$

Points not of the form $y_{i,\delta}$ are isolated. The basic open neighborhoods of $y_{i,\delta} \in X_\delta$

are indexed by $a \in [\mathscr{R}(\delta)]^{<\omega}$ and $\beta < \delta$. The basic open set $B(i, \delta, a, \beta)$ is defined to be the set

$$\{x: \beta < \alpha(x) \leq \delta \text{ and, for all } R \in a,\ x(R \cap Z_{\alpha(x)+1}) = y_{i,\delta}(R)\}.$$

Clearly the space X is first countable. Let us show how to separate two points $y_{i,\delta}$ and $y_{j,\eta}$. If $\delta < \eta$, then the basic open sets $B(i, \delta, \emptyset, 0)$ and $B(j, \eta, \emptyset, \delta)$ do fine. If $\delta = \eta$, then $i \neq j$. Let $R \in \mathscr{R}(\delta)$ have the form $\{i\} \times B$. Now $B(i, \delta, \{R\}, 0)$ and $B(j, \delta, \{R\}, 0)$ separate the points. Thus X is Hausdorff. We will show that X is regular by showing that the basic open sets are closed. As usual, the isolated points are no problem. Suppose that $y_{j,\eta} \notin B(i, \delta, a, \beta)$. According to whether $\eta > \delta$, $\delta \geq \eta > \beta$, or $\beta \geq \eta$, the basic open set $B(j, \eta, \emptyset, \delta)$, $B(j, \eta, a, \beta)$, $B(j, \eta, \emptyset, 0)$ contains $y_{j,\eta}$ and is disjoint from $B(i, \delta, a, \beta)$. A final easy observation is that the map, Ψ, defined by $\Psi((i, \delta)) = y_{i,\delta}$ is a homeomorphism of Z into X. Via Ψ, we assume $Z \subset X$.

X is normal. Let us first note that for every $A \subset \omega_1$, the set $A\# = \{x \in X: \alpha(x) \in C(A)\}$ is clopen. Further, $X - A\#$ is a discrete union of spaces with only countably many nonisolated points, hence disjoint closed subsets of $X - A\#$ can be separated by a clopen set.

Let H and K be disjoint closed subsets of X. As usual, we may assume that $H \cup K \subset Z$. Let R be a clopen subset of Z separating H and K. Set $A = \theta''R$. Let

$$U_1 = \{B(i, \delta, \{R \cap Z_{\delta+1}\}, 0): y_{i,\delta} \in H,\ \delta \in C(A) \cap D\} \cup H \cap A\#,$$
$$V_1 = \{B(i, \delta, \{R \cap Z_{\alpha+1}\}, 0): y_{i,\delta} \in K,\ \delta \in C(A) \cap D\} \cup K \cap A\#.$$

Let U_2 and V_2 be disjoint open subsets of $X - A\#$ separating $H \cap (X - A\#)$ and $K \cap (X - A\#)$. Then $U = U_1 \cup U_2$ and $V = V_1 \cup V_2$ separate H and K in X. □

X is not collectionwise normal. Note that $\{Y_i: i \in \omega_1\}$ is a discrete collection of closed subsets of X. Suppose that for each $i \in \omega_1$, U_i is an open set containing Y_i. For each $i < \delta \in D$, choose $B(i, \delta, a(i, \delta), \beta)$ so that

(i) $y_{i,\delta} \in B(i, \delta, a(i, \delta), \beta) \subset U_i$,

(ii) $|a(i, \delta)|$ is minimal.

Now for $i \in \omega_1$, $n \in \omega$, set $A_{i,n} = \{\delta \in D: |a(i, \delta)| = n\}$. Set $M_i = \{n: A_{i,n}$ is stationary$\}$. We

Claim. For each i, M_i is finite, Aiming for a contradiction, suppose that for some i, M_i were infinite. From $\diamondsuit^{++}$ 2(c), there is $\delta_0 \in D \cap \bigcap\{S(A_{i,n}): n \in M_i\}$. Let $m = |a(i, \delta_0)|$; choose $k \in M_i$, $k > m$. Recall that $\delta \in S(X)$ means that for all $c \in \mathscr{C}_\delta$, $c \cap X \neq \emptyset$. Since $\delta_0 \in S(A_{i,k})$, there is $\gamma \in A_{i,k}$ such that $y_{i,\gamma} \in B(i, \delta_0, a(i, \delta_0), \beta)$. Define

$$a\# = \{R \cap Z_{\gamma+1}: R \in a(i, \delta_0)\}.$$

Clearly, $|a\#| \leq |a(i, \delta_0)| = m < k = |a(i, \gamma)|$, so

$$y_{i,\gamma} \in B(i, \gamma, a\#, \beta) \subset B(i, \delta_0, a(i, \delta_0), \beta) \subset U_i$$

contradicts (ii) in the choice of $a(i, \gamma)$. □ claim.

Since, for each $i \in \omega_1$, M_i is finite, we can define max $M_i \in \omega$. By routine counting we can find $j \in \omega$ and e, a subset of ω_1 such that (a) $|e| = 2^j + 1$, (b) for all $i \in e$, $\max M_i = j$. Since the union of countably many nonstationary sets is nonstationary, there is $\delta \in D - \bigcup\{A_{i,n}: i \in e, n \notin M_i\}$. Then for $i \in e$, $|a(i, \delta)| \leq j$, so each U_i meets X_δ in a set "of measure at least 2^{-j}." We conclude that the U_i's are not disjoint. □

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