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LINEAR-QUADRATIC CONTROL FOR STOCHASTIC EQUATIONS IN A HILBERT SPACE WITH FRACTIONAL BROWNIAN MOTIONS*

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Abstract. A linear-quadratic control problem with a finite time horizon for some infinite-dimensional controlled stochastic differential equations driven by a fractional Gaussian noise is formulated and solved. The feedback form of the optimal control and the optimal cost are given explicitly. The optimal control is the sum of the well-known linear feedback control for the associated deterministic linear-quadratic control problem and a suitable prediction of the adjoint optimal system response to the future noise. The covariance of the noise as well as the control operator in the system equation can in general be unbounded, so the results can also be applied where the noise or the control are on the boundary of the domain or at discrete points in the domain. Some examples of controlled stochastic partial differential equations are given.

Key words. linear-quadratic control, fractional Brownian motion, stochastic partial differential equations

AMS subject classifications. 60H15, 60G18, 60G15, 93H20

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1. Introduction.

The linear-quadratic Gaussian (LQG) control problem for the control of a finite-dimensional linear stochastic system with a Brownian motion (white Gaussian noise) and a quadratic cost functional of the state and the control (e.g., [13]) is the most well-known and basic solvable stochastic control problem for stochastic systems with continuous sample paths. Similarly, the LQG control of an infinite-dimensional linear stochastic system with a Brownian motion and a quadratic cost functional of the state and the control is the most well-known and basic solvable stochastic control problem for infinite-dimensional stochastic systems with continuous sample paths.

The noise or perturbations of a system are typically modeled by a Brownian motion because such a process is Gauss–Markov and has independent increments. However, empirical data from many physical phenomena suggest that Brownian motion is often inappropriate to use in the mathematical models of these phenomena. A family of processes that has empirical evidence of wide physical applicability is the collection of fractional Brownian motions. Fractional Brownian motions are a family of Gaussian processes that were defined by Kolmogorov [20] in his study of turbulence. While this family of processes includes Brownian motion, it also includes other processes that describe behavior that is bursty or has a long-range dependence. These other processes are neither Markov nor semimartingales. The first empirical evidence of the usefulness of these latter processes was provided by Hurst [15] in his statistical analysis of rainfall along the Nile River. Subsequently empirical justifications for modeling with fractional Brownian motions have been noted for a wide variety of physical

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phenomena, such as economic data, flicker noise in electronic devices, turbulence, Internet traffic, biology, and medicine.

The study of the solutions of stochastic equations in an infinite-dimensional space with a (cylindrical) fractional Brownian motion (for example, stochastic partial differential equations) has been relatively limited. For the Hurst parameter $H \in (1/2, 1)$, linear and semilinear equations with an additive fractional Gaussian noise (the formal derivative of a fractional Brownian motion) are considered in [3, 10, 11, 14, 25]. Random dynamical systems described by such stochastic equations and their fixed points are studied in [23]. A pathwise (or nonprobabilistic) approach is used in [22, 24, 27] to study a parabolic equation with a fractional Gaussian noise where the stochastic term is a nonlinear function of the solution. Strong solutions of bilinear evolution equations with a fractional Brownian motion are obtained in [5, 9], and the same type of equation is studied in [28], where a fractional Feynman–Kac formula is obtained. A stochastic wave equation with a fractional Gaussian noise is considered in [1], and a stochastic heat equation with a multiparameter fractional Gaussian noise is studied in [16, 17].

The control problems for stochastic equations driven by fractional noise have been studied only recently (cf. [19], where a one-dimensional problem is investigated, and [6, 7, 8], where a multidimensional problem is investigated) and no results seem to be available for infinite-dimensional systems (e.g., stochastic partial differential equations) that are considered in this paper.

It should be noted that the control problem solved in this paper is not a straightforward generalization of the corresponding finite-dimensional problem. Although the controlled equation has the same formal form as in finite dimensions, the operators here are in general unbounded and only densely defined, which means that the stochastic differentials here are only formal and the solutions must be defined in a generalized (mild) sense. Furthermore, the corresponding Riccati equation and an auxiliary nonautonomous deterministic equation (4.5) are only formal and mild solutions are required for these equations. This major difficulty occurs because the results are required to be applicable to stochastic partial differential equations where the operator in the drift term is unbounded, and if boundary control and noise are considered, then the operators in the diffusion and the control terms are also unbounded operators. These unbounded operators and the noise which may include a white noise in the space variable means that there are basic questions even with the existence and the regularity of the solutions. The conditions imposed in this paper are balanced so that the formulation can include families of linear stochastic partial differential equations but can still be mathematically tractable. An Ito formula, which is usually the basic technical tool in controlled stochastic diffusion problems, is not available here. Even for the approximating systems, because of technical difficulties with the infinite-dimensional fractional Brownian motions an Ito formula is not available. Instead, polygonal approximations of the noise and Yosida approximations of the operators are used to obtain the desired basic equality (4.51). Furthermore, the auxiliary equation (4.5) does not seem to satisfy any standard theory for two-parameter evolution operators so its required properties are proved here. It should also be mentioned that the corresponding theory of backward stochastic differential equations in infinite dimensions has not been developed, so the approach used in the one-dimensional problem in [19] cannot be used.

This paper consists of five sections. Section 2 contains some preliminaries, notation, and conditions that are assumed throughout the paper. In section 3 the existence and uniqueness of the optimal control is proved by a classical method of minimization of a quadratic cost functional (e.g., [30]). The main result is contained in section 4, where (under slightly more restrictive assumptions than in sections 2 and 3) the form of the feedback control is described and the optimal cost is given. In contrast to the Markov case, the feedback control contains a suitable prediction of the adjoint optimal system response to the future noise. Three examples are given in section 5: a controlled stochastic heat equation, a controlled stochastic wave equation, and a deterministic equation with boundary control and/or noise.

2. Preliminaries. Let U, V, and $\mathscr{H}$ denote real, separable Hilbert spaces and consider the state equation

$$\begin{aligned} dX(t) &= (AX(t) + Bu(t))dt + CdB^H(t), \\ X(0) &= x \in \mathscr{H}, \end{aligned} \tag{2.1}$$

in the space $\mathscr{H}$, where $t \geqslant 0$, $A : D_A \subset \mathscr{H} \to \mathscr{H}$ is a linear (in general) unbounded operator that is the infinitesimal generator of a strongly continuous semigroup $(S(t), t \geqslant 0)$, and $(B_t^H, t \geqslant 0)$ is a cylindrical fractional Brownian motion on the space V and is defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}(t))_{t \geqslant 0}, \mathbb{P})$, that is,

$$B^H(t) = \sum_{i=1}^{\infty} e_i \sqrt{\lambda_i} \beta_i(t), \; t > 0, \tag{2.2}$$

where $\{e_i, i \in \mathbb{N}\}$ is a complete orthonormal basis in V, $(\beta_i(t), i \in \mathbb{N}, t \geqslant 0)$ is a family of (stochastically) independent, real-valued, standard fractional Brownian motions with the Hurst parameter $H \in (\frac{1}{2}, 1)$ fixed and $\lambda_i \geqslant 0$, $(\lambda_i, i \in \mathbb{N})$ is a bounded sequence in $\mathbb{R}_+$. The linear operators B and C and the family of admissible controls $\mathscr{U}$ are specified subsequently. It can be assumed that the filtration $(\mathscr{F}(t), t \geqslant 0)$ satisfies the so-called usual conditions (e.g., [18]). For the control problem here it is natural to assume that $(\mathscr{F}(t), t \geqslant 0)$ is the $\mathbb{P}$-completion of $(\sigma(B(s), s \leqslant t), t \geqslant 0)$. The incremental covariance $\widetilde{Q}$ of $(B^H(t), t \geqslant 0)$ is defined by

$$\widetilde{Q} e_n = \lambda_n e_n, \; n \in \mathbb{N}. \tag{2.3}$$

It is not required that $\widetilde{Q}$ is a trace class operator on V, so the series in (2.2) may diverge in the space V; cf. [5] and [11] for the basic theory of fractional Brownian motions and stochastic integrals driven by fractional Brownian motions that is relevant in the present case.

In this paper the following notation is used. If Y, Z are Hilbert spaces, then let $\langle \cdot, \cdot \rangle_Y$ and $|\cdot|_Y$ denote the inner product and norm on the space Y, let $\mathscr{L}(Y, Z)$ and $\mathscr{L}_2(Y, Z)$ denote the spaces of bounded linear and Hilbert–Schmidt operators from Y to Z, respectively, and let $\mathscr{L}(Y) = \mathscr{L}(Y, Y)$ and $\mathscr{L}_2(Y) = \mathscr{L}_2(Y, Y)$.

In a latter part of this paper it is assumed that the semigroup $(S(t), t \geqslant 0)$ is analytic. In that case there exists $\hat{\beta} \geqslant 0$ such that $\hat{A} := A - \hat{\beta} I$ is a strictly negative operator. Let D_A^α, $\alpha \geqslant 0$ be the domain of the fractional power $(-\hat{A})^\alpha$ equipped with the norm $|x|_{D_A^\alpha} := |(-\hat{A})^\alpha x|_{\mathscr{H}}$ (and similarly $D_{A^*}^\alpha$, $|\cdot|_{D_{A^*}^\alpha}$ for the adjoint A^*).

Some assumptions are given now:

(A1) *One of the following two conditions is satisfied for* B *and* C *in* (2.1):

(a) $B \in \mathscr{L}(U, \mathscr{H})$, $C \in \mathscr{L}(V, \mathscr{H})$, *where* $U = (U, \langle \cdot, \cdot \rangle_U, |\cdot|_U)$ *(the state space of controls) is a Hilbert space.*

(b) $(S(t), t \geqslant 0)$ *is an analytic semigroup and there are constants* $\alpha \in (0,1)$ *and* $\beta \in (0,1]$ *such that* $B \in \mathscr{L}(U, D_A^{\alpha-1})$ *and* $C \in \mathscr{L}(V, D_A^{\beta-1})$.

Note that in the case (A1)(b), B and C may be unbounded as operators $U \to \mathscr{H}$ and $V \to \mathscr{H}$, respectively, but the formulation requires the semigroup $(S(t), t \geqslant 0)$ to be analytic.

For the family of admissible controls, $\mathscr{U}$, the following assumption is introduced:

(A2) $u \in \mathscr{U} := L^p_{\mathscr{F}} = L^p_{\mathscr{F}}((0,T) \times \Omega; U)$, *where* $p > \frac{1}{\alpha}$, $p \geqslant 2$, *is fixed and* $L^p_{\mathscr{F}}$ *denotes the closed linear subspace of all* $(\mathscr{F}(t))$*-progressively measurable processes in* $L^p((0,T) \times \Omega; U)$. *If* $B \in \mathscr{L}(U, \mathscr{H})$, *then* p *can be chosen to satisfy only* $p \geqslant 2$.

Clearly, if $B \in \mathscr{L}(U, \mathscr{H})$ or if $\alpha > 1/2$, then it is appropriate to take $p = 2$ because it is not reasonable to reduce the space of controls. However, if $\alpha \leqslant 1/2$, then the operator B is too singular and it is necessary to restrict the space of controls so that the solution to the controlled equation is well defined.

The following condition is used for the stochastic convolution integral (that occurs in the variation of constants formula for the controlled system (2.1)) to be well defined.

(A3) *There exist some* $T_0 > 0$ *and* $\eta > 0$ *such that*

$$\int_0^{T_0} \int_0^{T_0} r^{-\eta} s^{-\eta} |S(r) C \widetilde{Q}^{1/2}|_{\mathscr{L}_2(V,\mathscr{H})} |S(s) C \widetilde{Q}^{1/2}|_{\mathscr{L}_2(V,\mathscr{H})} \phi_H(r-s) dr ds < \infty,$$

where $\phi_H(r) := H(2H-1)|r|^{2H-2}$.

Remark 2.1. (a) By (A3), the stochastic convolution integral with respect to the cylindrical fractional Brownian motion $(B^H_t, t \geqslant 0)$,

$$Z(t) := \int_0^t S(t-r) C dB^H(r), \; t \in [0,T], \tag{2.4}$$

is a well-defined Gaussian process with sample paths in $C([0,T], \mathscr{H})$ (cf. [5]).

(b) It easily follows (e.g., [5]) that the condition (A3) is satisfied if the mapping $t \mapsto t^{-\eta} S_t C \widetilde{Q}^{1/2}$ belongs to $L^{\frac{1}{H}}(0,T; \mathscr{L}_2(V, \mathscr{H}))$. If $\widetilde{Q}^{1/2} \in \mathscr{L}_2(V)$ (which corresponds to the case when $(B^H(t), t \geqslant 0)$ is a V-valued stochastic process) and (A2) holds, then the assumption (A3) is satisfied if

$$\beta > 1 - H. \tag{2.5}$$

It follows that for $T > 0$, $\eta > 0$

$$\begin{aligned}\int_0^T |t^{-\eta} S(t) C \widetilde{Q}^{1/2}|^{\frac{1}{H}}_{\mathscr{L}_2(V,\mathscr{H})} dt &\leqslant \int_0^T |t^{-\eta} S(t) C|^{\frac{1}{H}}_{\mathscr{L}(V,\mathscr{H})} |\widetilde{Q}^{1/2}|^{\frac{1}{H}}_{\mathscr{L}_2(V)} dt \\ &\leqslant \text{const} \int_0^T \frac{dt}{t^{(1-\beta+\eta)/H}}\end{aligned}$$

and the final integral on the far right of the inequalities is finite for $\eta > 0$ sufficiently small, such that $\frac{1-\beta+\eta}{H} < 1$.

The solution to (2.1) is defined in the mild sense, that is, for $t \in [0,T]$ by the formula

$$X(t) = S(t)x + \int_0^t S(t-r) B u(r) dr + Z(t), \tag{2.6}$$

where $(Z(t), t \geqslant 0)$ is given in (2.4). Note that $(X_t, t \in [0,T])$ is a well-defined $\mathscr{H}$-valued and $\mathscr{H}$-continuous process, which follows immediately from Remark 2.1(a).

Consider (2.1) with feedback controls of the form $u(t) = K(t)X(t) + h(t)$, where $(X(t), t \geqslant 0)$ satisfies the equation for $t \in [0, T]$

$$dX(t) = (AX(t) + B(K(t)X(t) + h(t)))dt + CdB^H(t), \quad X_0 = x \in \mathscr{H}, \tag{2.7}$$

$h \in \mathscr{U}$ and $K \in C_s([0,T], \mathscr{L}(\mathscr{H}, U))$ (where $C_s([0,T], \mathscr{L}(Y, X))$ is the space of strongly continuous $\mathscr{L}(Y, X)$-valued operators, where Y and Z are Banach spaces). The (mild) solution to (2.7) is defined in the usual way by the mild formula for $t \in [0, T]$,

$$X(t) = S(t)x + \int_0^t S(t-r)B(K(r)X(r) + h(r))dr + Z(t) \tag{2.8}$$

(which coincides with (2.6) for $u(t) = K(t)X(t) + h(t)$). Since the mapping $(t, x) \mapsto f(t, x) := K(t)x$ is continuous as a map $[0, T] \times \mathscr{H} \to U$ and $f(t, \cdot)$ is Lipschitz for each $t \in [0, T]$, it is easy to see that (2.8) has a pathwise unique $\mathscr{H}$-continuous solution.

The cost functional is defined as

$$J(x, u) := \frac{1}{2}\mathbb{E}\int_0^T (\langle QX(s), X(s)\rangle_{\mathscr{H}} + \langle Ru(s), u(s)\rangle_U)ds + \frac{1}{2}\mathbb{E}\langle GX(T), X(T)\rangle_{\mathscr{H}} \tag{2.9}$$

for $x \in \mathscr{H}$ and $u \in U$, where Q, R, and G are linear operators satisfying

(A4) $Q, G \in \mathscr{L}(\mathscr{H})$, $Q \geqslant 0$, $G \geqslant 0$, $R \in \mathscr{L}(U)$, $R \geqslant 0$, Q, G, *and* R *are self-adjoint.*

The problem is to minimize the cost functional $J(x, u)$, that is,

$$\widetilde{J}(x) := \inf_{u \in \mathscr{U}} J(x, u), \tag{2.10}$$

and (for a given $x \in \mathscr{H}$) to find an optimal control $\hat{u} \in \mathscr{U}$ that achieves the infimum in (2.10), that is, $J(x, \hat{u}) = \widetilde{J}(x)$.

3. Existence and uniqueness of the optimal control. In this section a proof is given of the existence of an optimal control in the family of controls $\mathscr{U} = L^p_{\mathscr{F}}$ under the general assumptions (A1)–(A4), which are assumed to hold throughout this section. Furthermore, a proof is given of pathwise uniqueness of the optimal system (where uniqueness is in the space $L^p_{\mathscr{F}}$).

The mild solution (2.6) can be rewritten as

$$X(\cdot) = (\widetilde{S}x)(\cdot) + Lu(\cdot) + Z(\cdot), \tag{3.1}$$

$$X(T) = S(T)x + L(T)u + Z(T), \tag{3.2}$$

where $x \in \mathscr{H}$, $u \in \mathscr{U}$, $Z = (Z(t), t \in [0, T])$ is the stochastic convolution integral defined by (2.4), and the linear operators L and L_T are given by

$$Lu(t) := \int_0^t S(t-r)Bu(r)dr, \ \ t \in [0, T], \tag{3.3}$$

$$L_T u = L(T)u := \int_0^T S(T-r)Bu(r)dr, \tag{3.4}$$

and

$$(\widetilde{S}x)(t) := S(t)x. \tag{3.5}$$

Let $L^q_{\mathscr{F}}(\mathfrak{X}) := L^q_{\mathscr{F}}((0,T)\times\Omega;\mathfrak{X})$ for any $q \geqslant 1$ and a separable Hilbert space $\mathfrak{X}$, and let $L^q_{\mathscr{F}}(\Omega;\mathfrak{X})$ denote the linear subspace of $L^q(\Omega;\mathfrak{X})$ consisting of all $\mathscr{F}_T$-measurable elements.

LEMMA 3.1. *If* (A1)–(A3) *are satisfied, then the following four statements are satisfied:*

(i) $L \in \mathscr{L}(L^p_{\mathscr{F}}(U), L^2_{\mathscr{F}}(\mathscr{H}))$,
(ii) $L(T) \in \mathscr{L}(L^p_{\mathscr{F}}(U), L^2_{\mathscr{F}}(\Omega;\mathscr{H}))$,
(iii) $\widetilde{S} \in \mathscr{L}(\mathscr{H}, L^2(0,T,\mathscr{H}))$,
(iv) $Z \in L^2_{\mathscr{F}}(\mathscr{H})$, $Z_T \in L^2_{\mathscr{F}}(\Omega;\mathscr{H})$.

Proof. The statement (iii) is obvious and the statement (iv) follows from [5] (in fact, Z has an $\mathscr{H}$-continuous version; cf. Remark 2.1). By the analyticity of the semigroup $(S(t), t \geqslant 0)$, (A1), and (A2), the inequalities

$$\begin{aligned} |Lu(t)|_{\mathscr{H}} &\leqslant \int_0^t |S(t-r)Bu(r)|_{\mathscr{H}} dr \leqslant \int_0^t |S(t-r)B|_{\mathscr{L}(U,\mathscr{H})} \cdot |u(r)|_U dr \\ &\leqslant c\int_0^t \frac{|u(r)|_U}{(t-r)^{1-\alpha}} dr \leqslant c\left(\int_0^t \frac{dr}{r^{(1-\alpha)q}}\right)^{1/q}\left(\int_0^t |u(r)|_U^p dr\right)^{1/p} \\ &\leqslant c_T |u|_{L^p(0,t,U)},\ t\in[0,T], \end{aligned} \tag{3.6}$$

are satisfied for some constants c, c_T depending only on T, where $q = \frac{p}{p-1}$ and q satisfies $(1-\alpha)q < 1$. Now, (i) and (ii) follow directly. If (A1)(a) is satisfied, then (i) and (ii) follow directly. ∎

It will be useful to characterize explicitly the adjoint operators $L^* \in \mathscr{L}(L^2_{\mathscr{F}}(\mathscr{H}), L^q_{\mathscr{F}}(U))$ and $L^*_T \in \mathscr{L}(L^2_{\mathscr{F}}(\Omega;\mathscr{H}), L^q_{\mathscr{F}}(U))$, where $q = \frac{p}{p-1}$.

LEMMA 3.2. *If* (A1)–(A3) *are satisfied, then the following two equalities are satisfied:*

$$L^*\xi(t) = \mathbb{E}\left[\int_t^T B^*S^*(r-t)\xi(r)dr|\mathscr{F}(t)\right],\ \xi \in L^2_{\mathscr{F}}(\mathscr{H}), \tag{3.7}$$

for almost all $t \in [0,T]$ *and*

$$L^*_T\eta(t) = B^*S^*(T-t)\mathbb{E}[\eta|\mathscr{F}(t)],\ t\in[0,T],\ \eta \in L^2(\Omega;\mathscr{H}). \tag{3.8}$$

Proof. For an arbitrary $u \in L^p_{\mathscr{F}}(U)$ and $\xi \in L^\infty_{\mathscr{F}}(\mathscr{H})$

$$\mathbb{E}\int_0^T \langle Lu(\cdot)(t), \xi(t)\rangle_{\mathscr{H}} dt = \mathbb{E}\int_0^T \left\langle \int_0^t S(t-r)Bu(r)dr, \xi(t)\right\rangle_{\mathscr{H}} dt. \tag{3.9}$$

By (3.6) the sample paths $r \mapsto S(t-r)Bu(r)$ are in $L^2(0,T,\mathscr{H})$; thus

$$\begin{aligned} &\mathbb{E}\int_0^T \langle Lu(\cdot)(t), \xi(t)\rangle_{\mathscr{H}} dt = \mathbb{E}\int_0^T \int_0^t \langle S(t-r)Bu(r), \xi(t)\rangle_{\mathscr{H}} drdt \\ &= \mathbb{E}\int_0^T \int_r^T \langle S(t-r)Bu(r), \xi(t)\rangle_{\mathscr{H}} dtdr = \mathbb{E}\int_0^T \int_r^T \langle u(r), B^*S^*(t-r)\xi(t)\rangle_U dtdr \\ &= \mathbb{E}\int_0^T \langle u(r), \int_r^T B^*S^*(t-r)\xi(t)dt\rangle_U dr, \end{aligned}$$

where it is used that $|B^*S^*(t-r)|_{\mathscr{L}(\mathscr{H},U)} \leqslant \text{const.}\ (t-r)^{1-\alpha}$ for all $t, r \in (0,T)$, $t > r$. Clearly

$$\text{(3.10)}\quad \mathbb{E}\int_0^T \left\langle u(r), \int_r^T B^*S^*(t-r)\xi(t)dt\right\rangle_U dr \\ = \mathbb{E}\int_0^T \left\langle u(r), \mathbb{E}\left[\int_r^T B^*S^*(t-r)\xi(t)dt|\mathscr{F}_r\right]\right\rangle_U dr,$$

so it follows that

$$\text{(3.11)}\qquad L^*\xi(r) = \mathbb{E}\left[\int_r^T B^*S^*(t-r)\xi(t)dt|\mathscr{F}(r)\right],\ r\in[0,T],$$

for $\xi \in L^\infty_{\mathscr{F}}(\mathscr{H})$. If $\xi \in L^2_{\mathscr{F}}(\mathscr{H})$, then the proof of (3.7) can be completed by a suitable passage to the limit. Choosing a sequence $(\xi_n, n\in\mathbb{N})$ with elements in $L^\infty_{\mathscr{F}}(\mathscr{H})$, such that $\xi_n \to \xi$ in $L^2_{\mathscr{F}}(\mathscr{H})$, $|\xi_n|_{\mathscr{H}} \leqslant |\xi|_{\mathscr{H}}$, it follows that

$$\left|\int_r^T B^*S^*(t-r)(\xi_n(t)-\xi(t))dt\right|_U \leqslant c\int_T^t \frac{\varphi_n(t)}{(t-r)^{1-\alpha}}dt = I^\alpha_{T-}(\varphi_n)(r),$$

where $\varphi_n(t) = |\xi_n(t)-\xi(t)|_{\mathscr{H}}$ and I^α_{T-} is a Riemann–Liouville fractional integral of order α. Thus, by some well-known properties of I^α_{T-} (cf., e.g., [26])

$$\int_0^T \left|\int_r^T B^*S^*(t-r)(\xi_n(t)-\xi(t))dt\right|_U^2 dr \leqslant \text{const.}\ \int_0^T \varphi_n^2(r)dr \to 0,\ n\to\infty.$$

Hence (possibly, for a subsequence) it follows that

$$\text{(3.12)}\qquad \int_r^T B^*S^*(t-r)\xi_n(t)dt \to \int_r^T B^*S^*(t-r)\xi(t)dt$$

for almost all $r \in [0,T]$ $\mathbb{P}$-a.s. and (3.7) is obtained by (3.11) and the Lebesgue dominated convergence theorem. Similarly, for $u \in L^p_{\mathscr{F}}(U)$, $\eta \in L^\infty_{\mathscr{F}}(\Omega;\mathscr{H})$,

(3.13)
$$\mathbb{E}\langle L_T u, \eta\rangle_{\mathscr{H}} = \mathbb{E}\left\langle \int_0^T S(T-r)Bu(r)dr, \eta\right\rangle_{\mathscr{H}} = \mathbb{E}\int_0^T \langle u(r), B^*S^*(T-r)\eta\rangle_U dr \\ = \mathbb{E}\int_0^T \langle u(r), B^*S^*(T-r)\mathbb{E}[\eta|\mathscr{F}(r)]\rangle_U dr,$$

noting that $|B^*S^*(T-r)|_{\mathscr{L}(\mathscr{H},U)} \leqslant \text{const.}\ (T-r)^{1-\alpha}$, $r\in(0,T)$. Hence (3.8) follows for $\eta \in L^\infty_{\mathscr{F}}(\Omega,\mathscr{H})$. For arbitrary $\eta \in L^2_{\mathscr{F}}(\Omega,\mathscr{H})$, choose a sequence $(\eta_n \in \mathbb{N})$ of elements in $L^\infty_{\mathscr{F}}(\Omega,\mathscr{H})$ such that $\eta_n \to \eta$ in $L^2_{\mathscr{F}}(\Omega,\mathscr{H})$. Then $\tilde\eta_n := \mathbb{E}[\eta_n|\mathscr{F}(T)] \to \tilde\eta := \mathbb{E}[\eta|\mathscr{F}(T)]$ in $L^2(\Omega,\mathscr{H})$. Hence

$$\text{(3.14)}\quad \mathbb{E}\int_0^T |B^*S^*(T-r)(\tilde\eta_n - \tilde\eta)|_U^q dr \leqslant \mathbb{E}\int_0^T \frac{c}{(T-r)^{(1-\alpha)q}}|\tilde\eta_n - \eta|_U^q dr \to 0$$

as $n\to\infty$ because $q \leqslant 2$ and $(1-\alpha)q < 1$ by the choice of $p = \frac{q}{q-1}$ in (A2), and (3.8) follows. □

Now rewrite the cost functional $J(x,u)$ in terms of the operators L, L_T, and $\widetilde{S}$ to obtain

$$J(x,u) = \frac{1}{2}\{|R^{1/2}u|^2_{L^2_{\mathscr{F}}(U)} + |Q^{1/2}(\widetilde{S}x + Lu + Z)|^2_{L^2_{\mathscr{F}}(\mathscr{H})} + |G^{1/2}(S(T)x + L_T u + Z(T))|^2_{L^2(\Omega,\mathscr{H})}\} \tag{3.15}$$

for $x \in \mathscr{H}$, $u \in U$. Therefore, denoting by $\langle \cdot,\cdot\rangle_{q,p}$ the duality between $L^q_{\mathscr{F}}(U)$ and $L^p_{\mathscr{F}}(U)$, it follows by completion of squares that

$$\begin{aligned} J(x,u) &= \frac{1}{2}\{\langle Ru,u\rangle_{q,p} + \langle L^*QL_u,u\rangle_{q,p} + \langle L_T^*GL_Tu,u\rangle_{q,p}\} \\ &+ \langle L^*Q\widetilde{S}x,u\rangle_{q,p} + \langle L^*QZ,u\rangle_{q,p} + \langle L_T^*GS_Tx,u\rangle_{q,p} + \langle L_T^*GZ_T,u\rangle_{q,p} \\ &+ \frac{1}{2}\{\langle QZ,Z\rangle_{L^2_{\mathscr{F}}(\mathscr{H})} + 2\langle Q\widetilde{S}x,Z\rangle_{L^2_{\mathscr{F}}(\mathscr{H})} + \langle S_T^*GS_Tx,x\rangle_{\mathscr{H}} \\ &\quad + \langle GZ_T,Z_T\rangle_{L^2_{\mathscr{F}}(\Omega,\mathscr{H})} + 2\langle GS_Tx,Z_T\rangle_{L^2_{\mathscr{F}}(\Omega,\mathscr{H})}\} \\ &= \frac{1}{2}\langle Nu,u\rangle_{q,p} + \langle \Phi x,u\rangle_{q,p} + \Psi(x), \end{aligned} \tag{3.16}$$

where $N \in \mathscr{L}(L^p_{\mathscr{F}}(U), L^q_{\mathscr{F}}(U))$, $\Phi x \in L^q_{\mathscr{F}}(U)$, $\Psi(x) \in \mathbb{R}$,

$$Nu := Ru + L^*QLu + L_T^*GL_Tu, \tag{3.17}$$

$$\Phi x := L^*Q\widetilde{S}x + L^*QZ + L_T^*GS(T) + L_T^*GZ(T), \tag{3.18}$$

and

$$\Psi(x) := \frac{1}{2}\{\langle QZ,Z\rangle_{L^2_{\mathscr{F}}(\mathscr{H})} + 2\langle Q\widetilde{S}x,Z\rangle_{L^2_{\mathscr{F}}(\mathscr{H})} + \langle S_T^*GS_Tx,x\rangle_{\mathscr{H}} + \langle GZ_T,Z_T\rangle_{L^2_{\mathscr{F}}(\Omega,\mathscr{H})} + 2\langle GS_Tx,Z_T\rangle_{L^2_{\mathscr{F}}(\Omega,\mathscr{H})}\}. \tag{3.19}$$

Now we fomulate the main result of this section. If the spaces $\mathscr{H}, V$, and U are finite dimensional and $p = q = 2$ the result is well known (cf. [30]).

THEOREM 3.3. *If* (A1)–(A4), *then the following three statements are valid:*

(i) $N \in \mathscr{L}(L^p_{\mathscr{F}}(U), L^q_{\mathscr{F}}(U))$, *where* $q = \frac{p}{p-1}$, $N = N^*$ *is self-adjoint (i.e.,* $\langle Nu,v\rangle_{q,p} = \langle u,N^*v\rangle_{p,q} = \langle u,Nv\rangle_{p,q}$ *for* $u,v \in L^p_{\mathscr{F}}(U) \cap L^q_{\mathscr{F}}(U)$, *and* $N \geqslant 0$, *i.e.,* $\langle Nu,u\rangle_{q,p} \geqslant 0$ *for* $u \in L^p_{\mathscr{F}}(U)$.

(ii) *For any initial value* $x \in \mathscr{H}$, *there exists a (unique) optimal control for the problem* (2.1)–(2.9) *if and only if there exists a (unique) optimal control* $\bar{u} \in U$ *such that*

$$N\bar{u} + \Phi x = 0. \tag{3.20}$$

(iii) *If* $\bar{u}$ *is an optimal control for the problem* (2.1)–(2.9), *the optimal cost is*

$$\widetilde{J}(x) = J(x,\bar{u}) = -\frac{1}{2}\langle \Phi x,\bar{u}\rangle_{q,p} + \Psi(x), \tag{3.21}$$

where Φ *and* Ψ *are given by* (3.18) *and* (3.19), *respectively.*

Proof. From the properties of the operators L and L_T established in Lemma 3.2 it easily follows that $N \in \mathscr{L}(L^p_{\mathscr{F}}(U), L^q_{\mathscr{F}}(U))$ (note that $p \geqslant q$), and the self-adjointness and nonnegativity of N follow immediately from its form (3.17). To prove (ii), initially assume that (for a fixed $x \in \mathscr{H}$) $\bar{u} \in \mathscr{U} = L^p_{\mathscr{F}}(U)$ is an optimal control. Then the directional derivative $D_h J(x, \bar{u})$ at the point $\bar{u}$ in any direction $h \in L^p_{\mathscr{F}}(U)$ must be zero, and using the equality (3.16) it follows that

$$D_h J(x, \bar{u}) = \frac{1}{2}(\langle N\bar{u}, h\rangle_{q,p} + \langle Nh, \bar{u}\rangle_{q,p}) + \langle \Phi x, h\rangle_{q,p} = 0, \tag{3.22}$$

which, by the self-adjointness of N, yields

$$D_h J(x, \bar{u}) = \langle N\bar{u}, h\rangle_{q,p} + \langle \Phi x, h\rangle_{q,p} = 0, \;\; h \in L^p_{\mathscr{F}}(U), \tag{3.23}$$

and (3.20) follows. Conversely, if (3.20) is satisfied for arbitrary $u \in \mathscr{U}$ it follows that

(3.24)

$$\begin{aligned} &J(x, u) - J(x, \bar{u}) = J(x, \bar{u} + u - \bar{u}) - J(x, \bar{u}) \\ &= \frac{1}{2}\langle N(\bar{u} + u - \bar{u}), u + u - \bar{u}\rangle_{q,p} + \langle \Phi x, \bar{u} + u - u\rangle_{q,p} - \frac{1}{2}\langle N\bar{u}, \bar{u}\rangle_{q,p} - \langle \Phi x, \bar{u}\rangle_{q,p} \\ &= \frac{1}{2}\langle N\bar{u} + \Phi x, u - \bar{u}\rangle_{q,p} + \frac{1}{2}\langle N(u - \bar{u}), u - \bar{u}\rangle_{q,p} \geqslant 0 \end{aligned}$$

by (3.20) and nonnegativity of N; hence $\bar{u} \in U$ is optimal. The third statement (iii) follows immediately from the second statement (ii). $\square$

There is also a feedback form for the optimal control. Assume that (3.20) is satisfied. Then the following equality is satisfied:

$$\begin{aligned} O = (N\bar{u} + \Phi x) &= (R + L^*QL + L_T^*GL_T)\bar{u} + L^*Q\widetilde{S}x + L^*QZ \\ &+ L_T^*G(S(T)x + Z(T)) = R\bar{u} + L^*Q(\widetilde{S}x + L\bar{u} + Z) \\ &+ L_T^*G(S(T)x + L_T\bar{u} + Z(T)) = R\bar{u} + L^*QX^{\bar{u}} + L_T^*GX^{\bar{u}}(T), \end{aligned} \tag{3.25}$$

where $X^{\bar{u}}$ satisfies (2.1) with $u = \bar{u}$. It can be easily checked (similarly as in (3.6)) that $X^{\bar{u}} \in L^p(\Omega, C([0,T], \mathscr{H}))$. Using the form of the operators L^* and L_T^* from Lemma 3.2 the following corollary is obtained.

COROLLARY 3.4. *Let* (A1)–(A4) *be satisfied, let $x \in \mathscr{H}$ be given, and let there exist an optimal control $\bar{u}$ for the problem* (2.1)–(2.9) *(or equivalently let there exist a solution $\bar{u}$ to* (3.20)*). Then the following equality is satisfied:*

$$R\bar{u}(t) = -\int_t^T B^*S^*(r-t)Q\mathbb{E}[X^{\bar{u}}(r)|\mathscr{F}(t)]dr - B^*S^*(T-t)G\mathbb{E}[X^{\bar{u}}(T)|\mathscr{F}(t)] \tag{3.26}$$

for each $t \in [0, T]$.

Remark 3.5. (i) A simple sufficient condition for the (unique) solvability of (3.20) may be given if $p = q = 2$ (which is a natural choice if $\alpha > \frac{1}{2}$ in the assumption (A1)). If the operator $R \in \mathscr{L}(U)$ is uniformly positive ($R \geqslant \gamma I$ for some $\gamma > 0$), then by (3.17) the inequality

$$\langle Nu, u\rangle_{L^2_{\mathscr{F}}U} \geqslant \gamma |u|^2_{L^2_{\mathscr{F}}(U)}, \;\; u \in L^2_{\mathscr{F}}(U)$$

is satisfied and it follows that N has a bounded inverse, $N^{-1} \in \mathscr{L}(L^2_{\mathscr{F}}(U))$. Therefore, for each $x \in \mathscr{H}$, (3.20) has a unique solution

$$\bar{u} = -N^{-1}\Phi x, \tag{3.27}$$

which is the optimal control. In this case, (3.26) yields the feedback form of the optimal control for $t \in [0,T]$,

(3.28)
$$\bar{u}(t) = -R^{-1}\left(\int_t^T B^* S^*(r-t)Q\mathbb{E}[X^{\bar{u}}(r)|\mathscr{F}(t)]dr + B^* S^*(T-t)G\mathbb{E}[X^{\bar{u}}(T)|\mathscr{F}(t)]\right).$$

(ii) If, moreover, the spaces $\mathscr{H}, U$, and V are finite-dimensional, proceeding as in (3.25) it can be easily seen that

$$Nu + \Phi x = Ru - B^* p, \ u \in L^2_{\mathscr{F}}(U), \tag{3.29}$$

where p satisfies the backward stochastic equation

$$\begin{aligned} dp(t) &= (-A^* p(t) + QX(t))dt + q(t)dB^H(t), \ t \in [0,T], \\ p(T) &= GX(T). \end{aligned} \tag{3.30}$$

This equation has a solution

$$p(t) = \int_t^T S^*(r-t)\mathbb{E}(QX(r)|\mathscr{F}(t)]dr + S^*(T-t)\mathbb{E}[GX(T)|\mathscr{F}(t)], \tag{3.31}$$

Hence for the optimal control $\bar{u}$ (satisfying $N\bar{u} + \Phi x = R\bar{u} - B^* p = 0$) the same expression as in (3.28) is obtained. In the general infinite-dimensional case (with unbounded A, B, and C) the corresponding theory of equations of the form (3.30) has not yet been developed. Nonetheless, the solution for the backward stochastic differential equation is obtained directly from (3.28) by computing the operators L^* and L_T^*.

4. Optimal feedback control. The expression (3.28) for the optimal control in section 3 is not very satisfactory, because the control is not expressed directly in terms of the optimal solution. In this section an optimal control is described in the feedback form $\bar{u}_t = K(t)X_t^{\bar{u}} + \varphi(t)$, where φ is a functional of X. The hypotheses (A1)–(A4) are restricted somewhat. In addition, the following assumption is introduced.

(A5) *The following three conditions are satisfied:*
- (a) $Tr\,\widetilde{Q} < \infty$.
- (b) $\beta \geqslant \alpha > 1 - H$, *where* α, β *are defined in* (A1)(b).
- (c) R *has a bounded inverse, that is,* $R^{-1} \in \mathscr{L}(U)$, *and* $G \in \mathscr{L}(\mathscr{H}, D_{A^*}^{1-\alpha})$.

The condition (A5)(a) implies that $(B_t^H, t \geqslant 0)$ is a "genuine" V-valued fractional Brownian motion, not merely a cylindrical fractional Brownian motion. The inequality $\beta \geqslant \alpha$ means, roughly speaking, that the diffusion operator C is not "more unbounded" than the control operator B.

The formal Riccati equation in this case is

$$\begin{aligned} \dot{V} &= A^* V + VA - VBR^{-1}B^* V + Q, \\ V(0) &= G, \end{aligned} \tag{4.1}$$

which does not include the noise linear operator C, so the well-known deterministic result for this case (e.g., [12]) can be used. Let $\Sigma^+ = \{V \in \mathscr{L}(\mathscr{H}), V = V^*; V \geqslant 0\}$.

THEOREM 4.1. *If* (A1), (A4), *and* (A5) *are satisfied, then for an arbitrary* $V_0 \in \Sigma^+ \cap \mathscr{L}(\mathscr{H}, D_{A^*}^{1-\alpha})$ *there exists a unique operator-valued function* $V \in C_s([0,T], \mathscr{L}(\mathscr{H}, D_{A^*}^{1-\alpha})) \cap C_s([0,T], \Sigma^+)$ ($C_s(\cdot)$ *denotes strongly continuous) such that*

(4.2)
$$V(t) = S^*(t)V_0 S(t) + \int_0^t S^*(t-s)(Q - (B^* V(s))^* R^{-1} B^* V(s) S(t-s)ds, \ t \in [0,T],$$

or (equivalently)

(4.3)
$$\frac{d}{dt}\langle V(t)x,y\rangle_{\mathscr{H}} = \langle V(t)x, Ay\rangle_{\mathscr{H}} + \langle Ax, V(t)y\rangle_{\mathscr{H}} + \langle Qx, y\rangle_{\mathscr{H}} - \langle R^{-1}B^*V(t)x, B^*V(t)y\rangle_U$$

for all $t \in (0,T)$, $x, y \in D_A$, *and* $V(0) = V_0$.

The mapping V satisfying (4.2) (or equivalently, (4.3)) is called the mild (or weak) solution to (4.1) with $V_0 = G$. Let

$$\widetilde{B}(t) := V(t)BR^{-1}B^*. \tag{4.4}$$

By the fact that $V \in C_s([0,T], \mathscr{L}(\mathscr{H}, D_{A^*}^{1-\alpha}))$ and $V = V^*$ on $\mathscr{H}$ there is a unique extension of V (denoted again by V) such that $V \in C_s([0,T], \mathscr{L}(D_A^{\alpha-1}, \mathscr{H}))$; hence $\widetilde{B} \in C_s([0,T], \mathscr{L}(D_{A^*}^{1-\alpha}, \mathscr{H}))$.

For $t \geqslant s$ consider the differential equation

$$\begin{aligned} \dot{y}(t) &= A^*y(t) - \widetilde{B}(t)y(t), \\ y(s) &= x \in \mathscr{H}. \end{aligned} \tag{4.5}$$

The following technical lemma states the existence and some properties of the evolution operator used to solve (4.5).

Lemma 4.2. *If* (A1), (A4), *and* (A5)(c) *are satisfied, then for each* $s \in [0,T]$, $x \in D_{A^*}^{1-\alpha}$, *there exists a unique mild solution* $y(t) = U(t,s)x$ *to* (4.5), *that is, a* $D_{A^*}^{1-\alpha}$*-valued function* y *satisfying*

$$y(t) = S^*(t-s)x + \int_s^t S^*(t-r)\widetilde{B}(r)y(r)dr, \ t \in [s,T], \tag{4.6}$$

such that for each $s \in [0,T]$, *the solution is continuous, that is,* $y \in C([s,T], D_{A^*}^{1-\alpha})$.

The collection $(U(t,s),\ t \geqslant s;\ t,s \in [0,T])$ *is a strongly continuous family of evolution operators as maps* $\overline{\Delta} \to \mathscr{L}(D_{A^*}^{1-\alpha})$, *where*

$$\Delta = \{(t,s) \in \mathbb{R}_+^2;\ 0 \leqslant s < t \leqslant T\}$$

(and $\overline{\Delta}$ *is the closure of* Δ*), that is, for each convergent sequence* $(t_n, s_n) \to (t_0, s_0)$ *in* $\overline{\Delta}$ *and each* $x \in D_{A^*}^{1-\alpha}$ *there is convergence* $U(t_n,s_n)x \to U(t_0,s_0)x$ *in* $\mathscr{L}(D_{A^*}^{1-\alpha})$.

Furthermore,

$$(U(t,s),\ t > s;\ t,s \in [0,T])$$

can be uniquely extended to a strongly continuous family of evolution operators as maps $\Delta \to \mathscr{L}(\mathscr{H}, D_{A^*}^{1-\alpha})$ *(denoted again by* $U(t,s)$*) and for a constant* $c > 0$ *(independent of* $(t,s) \in \Delta$*) there is the inequality*

$$|U(t,s)|_{\mathscr{L}(\mathscr{H}, D_{A^*}^{1-\alpha})} \leqslant \frac{c}{(t-s)^{1-\alpha}},\ (t,s) \in \Delta. \tag{4.7}$$

Proof. Given $x \in D_{A^*}^{1-\alpha}$ and $s \in [0,T)$ by a standard fixed point argument as follows, (4.6) has a unique solution in $C([s,T], D_{A^*}^{1-\alpha})$. Since

(4.8)
$$\begin{aligned} |\tilde{B}(t)|_{\mathscr{L}(D_{A^*}^{1-\alpha}, \mathscr{H})} &\leqslant |V(t)|_{\mathscr{L}(D_A^{\alpha-1}, \mathscr{H})} \cdot |B|_{\mathscr{L}(U, D_A^{\alpha-1})} \cdot |R^{-1}|_{\mathscr{L}(U)} \cdot |B^*|_{\mathscr{L}(D_{A^*}^{1-\alpha}, U)} \\ &\leqslant c, \quad t \in (s,T], \end{aligned}$$

for some constant c, defining

$$\Phi(y)(t) := S^*(t-s)x + \int_s^t S^*(t-r)\tilde{B}(r)y(r)dr, \ t \in [s,T], \tag{4.9}$$

for $y \in C([s,T], D_{A^*}^{1-\alpha}) =: \mathfrak{X}$, it follows that Φ maps $\mathfrak{X}$ into itself. Moreover,

$$\begin{aligned} |\Phi(y_1)(t) - \Phi(y_2)(t)|_{D_{A^*}^{1-\alpha}} &\leqslant \int_s^t |S^*(t-r)\tilde{B}(r)(y_1(r)-y_2(r))|_{D_{A^*}^{1-\alpha}} dr \\ &\leqslant \int_s^t \frac{k}{(t-r)^{1-\alpha}} |y_1(r)-y_2(r)|_{D_{A^*}^{1-\alpha}} dr \end{aligned} \tag{4.10}$$

for some $k > 0$; hence

$$|\Phi(y_1) - \Phi(y_2)|_{\mathfrak{X}} \leqslant C_T |y_1 - y_2|_{\mathfrak{X}}$$

for $y_1, y_2 \in \mathfrak{X}$, where $C_T \to 0+$ as $T - s \to 0+$. Thus, Φ is a contraction on $\mathfrak{X}$ for $T-s$ sufficiently small. For larger T this result can be extended by subdividing the interval $[s,T]$ into smaller subintervals and iterating the solution.

Hence it follows that $U(t,s)$ is a well-defined linear operator on $D_{A^*}^{1-\alpha}$ for each $(t,s) \in \overline{\Delta}$, and it is also clear that $(U(t,s),\ (t,s) \in \overline{\Delta})$ is a family of evolution operators. Furthermore,

$$|U(t,s)x|_{D_{A^*}^{1-\alpha}} \leqslant c_1 |x|_{D_{A^*}^{1-\alpha}} + \int_s^t |S^*(t-r)\widetilde{B}(r)|_{\mathscr{L}(D_{A^*}^{1-\alpha})} |U(r,s)x|_{D_{A^*}^{1-\alpha}} dr \tag{4.11}$$

for all $(t,s) \in \overline{\Delta}$ and some constant $c_1 > 0$. Hence

$$\begin{aligned} |U(t,s)|_{\mathscr{L}(D_{A^*}^{1-\alpha})} \leqslant c_1 + \int_s^t & |S^*(t-r)|_{\mathscr{L}(\mathscr{H}, D_{A^*}^{1-\alpha})} \\ & \cdot |\widetilde{B}(r)|_{\mathscr{L}(D_{A^*}^{1-\alpha}, \mathscr{H})} |U(r,s)|_{\mathscr{L}(D_{A^*}^{1-\alpha})} dr \end{aligned} \tag{4.12}$$

and for the mapping $h(t) := |U(t,s)|_{\mathscr{L}(D_{A^*}^{1-\alpha})}$ by the analyticity of the semigroup $(S^*(t), t \geqslant 0)$

$$h(t) \leqslant c_1 + c_2 \int_s^t \frac{h(r)}{(t-r)^{1-\alpha}} dr, \ (t,s) \in \overline{\Delta}, \tag{4.13}$$

for some independent constants $c_1, c_2 > 0$. By the generalized Gronwall lemma it follows that

$$h(t) = |U(t,s)|_{\mathscr{L}(D_{A^*}^{1-\alpha})} \leqslant c_3, \quad t \geqslant s, \tag{4.14}$$

where c_3 does not depend on $(t,s) \in \overline{\Delta}$. The strong continuity of the family $(U(t,s),$ $(t,s) \in \overline{\Delta})$ in the norms mapping $\overline{\Delta} \to \mathscr{L}(D_{A^*}^{1-\alpha})$ is now a simple consequence of the uniform estimate (4.14) and the mild formula (4.6).

It remains to verify the behavior of the evolution family $(U(t,s))$ in the norm of $\mathscr{L}(\mathscr{H}, D_{A^*}^{1-\alpha})$. For $x \in D_{A^*}^{1-\alpha}$ it follows that similarly as in (4.12),

$$|U(t,s)x|_{D_{A^*}^{1-\alpha}} \leqslant \frac{c_4 |x|_{\mathscr{H}}}{(t-s)^{1-\alpha}} + \int_s^t |S^*(t-r)|_{\mathscr{L}(\mathscr{H}, D_{A^*}^{1-\alpha})} \tag{4.15}$$

$$\cdot |\widetilde{B}(r)|_{\mathscr{L}(D_{A^*}^{1-\alpha}, \mathscr{H})} \cdot |U(r,s)x|_{D_{A^*}^{1-\alpha}} dr \tag{4.16}$$

for $(t,s) \in \Delta$. Since $D_{A^*}^{1-\alpha}$ is dense in $\mathscr{H}$ there is the inequality

$$g(t) \leqslant \frac{c_4}{(t-s)^{1-\alpha}} + c_5 \int_s^t \frac{g(r)}{(t-r)^{1-\alpha}} dr, \quad t > s, \tag{4.17}$$

where $g(t) := |U(t,s)|_{\mathscr{L}(\mathscr{H}, D_{A^*}^{1-\alpha})}$, $t > s$, $s \in [0,T)$, and c_5 is a constant. By the generalized Gronwall lemma it follows that

$$g(t) \leqslant \frac{c_4}{(t-s)^{1-\alpha}} + c_6 \int_s^t \frac{dr}{(t-r)^{1-\alpha}(r-s)^{1-\alpha}}, \quad (t,s) \in \Delta, \tag{4.18}$$

where the constants c_4 and c_6 are independent of $(t,s) \in \Delta$, and (4.7) easily follows. The strong continuity of $(U(t,s))$ in the norm $\mathscr{L}(\mathscr{H}, D_{A^*}^{1-\alpha})$ on Δ is easily obtained by (4.6) and (4.7). □

Subsequently let $P(t) = V(T-t)$, where V is the mild solution to (4.1) with $V_0 = G$ and $U_P(s,t) = U(T-t, T-s)$ for $(s,t) \in \overline{\Delta}$ and consider the function φ defined by stochastic integral

$$\varphi(t) := \int_t^T U_P(s,t) P(s) C dB^H(s) \tag{4.19}$$

for $t \in [0,T]$. This expression for φ plays an important role in the formulation of the main result.

LEMMA 4.3. *If* (A1), (A3)–(A5) *are satisfied, then the process* φ *given by* (4.19) *is a well-defined centered Gaussian process in* $L^p(\Omega \times (0,T), D_{A^*}^{1-\alpha})$.

Proof. By (4.7) for an independent constant $\hat{c} > 0$

$$\begin{aligned} |U_P(s,t)P(s)C\widetilde{Q}^{1/2}|_{\mathscr{L}_2(V, D_{A^*}^{1-\alpha})} &\leqslant |U_P(s,t)|_{\mathscr{L}(\mathscr{H}, D_{A^*}^{1-\alpha})} \\ &\quad \cdot |P(s)|_{\mathscr{L}(D_A^{\alpha-1}, \mathscr{H})} \cdot |C|_{\mathscr{L}(V, D_A^{\alpha-1})} \cdot |\widetilde{Q}^{1/2}|_{\mathscr{L}_2(V)} \\ &\leqslant \frac{\hat{c}}{(s-t)^{1-\alpha}}, \quad (s,t) \in \Delta, \end{aligned} \tag{4.20}$$

and since $\alpha > 1 - H$ it follows that

$$U_P(\cdot, t) P(\cdot) C \widetilde{Q}^{1/2} \in L^{\frac{1}{H}}((t,T), \mathscr{L}_2(U, D_{A^*}^{1-\alpha})) \tag{4.21}$$

for each $t \in [0,T)$, so the stochastic integral (4.10) is well defined for each t (cf. Remark 2.1(b)) as a $D_{A^*}^{1-\alpha}$-valued random variable. Next it is shown that the process $t \mapsto \int_t^T U_P(s,t)P(s)CdB^H(s)$ is mean-square left continuous in $D_{A^*}^{1-\alpha}$. Choosing $0 \leqslant h < t \leqslant T$ it follows that

$$\begin{aligned} &\mathbb{E}\left| \int_t^T U_P(s,t)P(s)CdB^H(s) - \int_h^T U_P(s,h)P(s)CdB^H(s) \right|^2_{D_{A^*}^{1-\alpha}} \\ &\leqslant 2\Bigg(\mathbb{E}\left| \int_h^t U_P(s,h)P(s)CdB^H(s) \right|^2_{D_{A^*}^{1-\alpha}} \\ &\qquad + \mathbb{E}\left| \int_t^T (U_P(s,t) - U_P(s,h))P(s)CdB^H(s) \right|^2_{D_{A^*}^{1-\alpha}} \Bigg) \\ &= 2(I_1 + I_2). \end{aligned} \tag{4.22}$$

By (4.20) there is the inequality

$$
\begin{aligned}
I_1 &\leqslant \left(\int_h^t |U_P(s,h)P(s)C\widetilde{Q}^{1/2}|^{\frac{1}{H}}_{\mathscr{L}_2(V,D^{1-\alpha}_{H^*})} \right)^{2H} \\
&\leqslant \left(\int_h^t \frac{\hat{c}^{\frac{1}{H}}}{(s-h)^{\frac{1-\alpha}{H}}} ds \right)^{2H} \to 0 \text{ as } t-h \to 0
\end{aligned} \tag{4.23}
$$

since $\frac{1-\alpha}{H} < 1$. Let (e_j) be a complete orthonormal basis in V; then the following inequality is satisfied:

$$
\begin{aligned}
I_2 &\leqslant \left(\int_t^T |(U_P(s,t) - U_P(s,h))P(s)C\widetilde{Q}^{1/2}|^{\frac{1}{H}}_{\mathscr{L}_2(V,D^{1-\alpha}_{A^*})} ds \right)^{2H} \\
&= \left(\int_t^T \left(\sum_{i=1}^{\infty} |(U_P(s,t) - U_P(s,h))P(s)C\widetilde{Q}^{1/2}e_i|^2_{D^{1-\alpha}_{A^*}} \right)^{\frac{1}{2H}} ds \right)^{2H}.
\end{aligned} \tag{4.24}
$$

By the strong continuity of $U(s,\cdot)$ established in Lemma 4.2 it follows that

$$
|(U_P(s,t) - U_P(s,h))P(s)C\widetilde{Q}^{1/2}e_i|^2_{D^{1-\alpha}_{A^*}} \to 0 \text{ as } h \to t- \tag{4.25}
$$

for each $s > t$ and $i \in \mathbb{N}$. Furthermore, by (4.20)

$$
\begin{aligned}
&|(U_P(s,t) - U_P(s,h))P(s)C\widetilde{Q}^{1/2}e_i|^2_{D^{1-\alpha}_{A^*}} \\
&\leqslant 2(|U_P(s,t)P(s)C\widetilde{Q}^{1/2}e_i|^2_{D^{1-\alpha}_{A^*}} + |U_P(s,h)P(s)C\widetilde{Q}^{1/2}e_i|^2_{D^{1-\alpha}_{A^*}}) \\
&\leqslant 2(|U_P(s,t)|^2_{\mathscr{L}(\mathscr{H},D^{1-\alpha}_{A^*})} \cdot |P(s)|^2_{\mathscr{L}(D^{\alpha-1}_A,\mathscr{H})} \cdot |C|^2_{\mathscr{L}(V,D^{\alpha-1}_A)} \cdot |\widetilde{Q}^{1/2}e_i|^2_V \\
&\quad + |U_P(s,h)|^2_{\mathscr{L}(\mathscr{H},D^{1-\alpha}_{A^*})} \cdot |P(s)|^2_{\mathscr{L}(D^{\alpha-1}_A,\mathscr{H})} \cdot |C|^2_{\mathscr{L}(V,D^{\alpha-1}_A)} \cdot |\widetilde{Q}^{1/2}e_i|^2) \\
&\leqslant k|\widetilde{Q}^{1/2}e_i|^2 \left(\frac{1}{(s-t)^{1-\alpha}} + \frac{1}{(s-h)^{1-\alpha}} \right)^2 \\
&\leqslant \frac{2k}{(s-t)^{2(1-\alpha)}} \cdot |\widehat{Q}^{1/2}e_i|^2
\end{aligned} \tag{4.26}
$$

for a constant $K > 0$, which gives the convergence of the integrand in the integral on the right-hand side (r.h.s.) of (4.24), so together with (4.25) it follows that $I_2 \to 0$ as $h \to t-$. Hence the stochastic integral (4.10) is left-continuous in the mean-square and thus has a measurable version which is a Gaussian process in $L^p(0,T,D^{1-\alpha}_{A^*})$ (cf. [2, Proposition 3.6]). □

Note that the solution of the controlled equation is well defined by the mild formula (2.6) even if the control $u \in \mathscr{V} := L^p([0,T] \times \Omega, U)$ is not adapted. Using the operator-valued mapping P and the process φ the following result for a nonadapted control is obtained. It may be of independent interest but it also is used later to prove a corresponding result for adapted controls.

THEOREM 4.4. *Let* (A1)–(A5) *be satisfied, let $x \in \mathscr{H}$ and $u \in \mathscr{U}$ be arbitrary, let φ be given by* (4.19)*, and let $(X(t),\ t \geqslant 0)$ be the solution of the controlled equation* (2.1)*. Let $\mathscr{V} = L^p([0,T] \times \Omega, U)$ be the linear space of nonadapted controls. The optimal control $\bar{v} \in \mathscr{V}$ for the control problem* (2.1)–(2.9) *with the family of controls $\mathscr{U}$ replaced by $\mathscr{V}$ is*

$$
\bar{v}(t) = -R^{-1}B^*(P(t)X(t) + \varphi(t)), \tag{4.27}
$$

where $(X(t), t \in [0,T])$ *is the solution of the controlled equation* (2.1) *with* $u = \bar{v}$. *The optimal cost* $\tilde{J}$ *is*

$$\begin{aligned}(4.28)\quad \tilde{J}(x) &= \frac{1}{2}\langle P(0)x, x\rangle_{\mathscr{H}} - \frac{1}{2}\mathbb{E}\int_0^T |R^{-1}B^*\varphi(s)|_U^2 ds \\ &\quad + \int_0^T\int_0^s Tr[C^*P(s)U_P^*(s,r)C\widetilde{Q}]\phi_H(r,s)drds \\ &= \frac{1}{2}\langle P(0)x, x\rangle_{\mathscr{H}} \\ &\quad - \frac{1}{2}\int_0^T\int_s^T\int_s^T Tr(R^{-1}B^*U_P(r,s)P(r)C\widetilde{Q}C^*P(q)U_P^*(q,s)R^{-1}B) \\ &\quad \phi_H(r-q)dqdrds + \int_0^T\int_0^s Tr[C^*P(s)U_P^*(s,r)C\widetilde{Q}]\phi_H(r-s)drds.\end{aligned}$$

To prove Theorem 4.4 some approximations to the state and the noise processes are useful. For $\lambda > \hat{\beta}$, let $R(\lambda) = \lambda R(\lambda, A)$, where $R(\lambda, A) = (\lambda I - A)^{-1}$ is the resolvent of the operator A. For $t \geqslant s$ consider the equations

$$(4.29)\qquad \begin{aligned}\dot{y}(t) &= A^*y(t) - \tilde{B}_\lambda(t)y(t), \\ y(s) &= x \in \mathscr{H}\end{aligned}$$

for $s \in [0,T]$, where $\tilde{B}_\lambda(t) := V(t)B_\lambda R^{-1}B_\lambda^*$ and $B_\lambda = R(\lambda)B$. From the properties of the Yosida approximations it follows that for each $\lambda > \hat{\beta}$, $B_\lambda \in \mathscr{L}(U, \mathscr{H})$ and $B_\lambda \to B$ in the strong operator topology of $\mathscr{L}(U, D_A^{\alpha-1})$ as $\lambda \to +\infty$. As in Lemma 4.2 it follows that there exists an evolution operator $(U_\lambda(t,s), s \geqslant 0, t \geqslant s)$ corresponding to (4.29) in the same way as $(U(t,s), s \geqslant 0, t \geqslant s)$ corresponding to (4.5). Moreover, $U_\lambda(t,s) \to U(t,s)$, as $\lambda \to \infty$, strongly in $\mathscr{L}(\mathscr{H}, D_{A^*}^{1-\alpha})$ for each $0 < s < t \leqslant T$. Now set $U_{P,\lambda}(s,t) = U_\lambda(T-t, T-s)$ and define the family of processes, $(\varphi_\lambda(t), t \in [0,T], \lambda > \hat{\beta})$,

$$(4.30)\qquad \varphi_\lambda(t) := \int_t^T U_{P,\lambda}(s,t)P(s)CdB^H(s).$$

It is easy to verify that

$$(4.31)\qquad \varphi_\lambda \to \varphi$$

as $\lambda \to \infty$ in $L^2((0,T)\times\Omega; D_{A^*}^{1-\alpha})$. Similarly, define a controlled process X_λ satisfying the equation

$$(4.32)\qquad \begin{aligned}dX_\lambda(t) &= (AX_\lambda(t) + B_\lambda u(t))dt + R(\lambda)CdB^H(t),\ t \in (0,T], \\ X_\lambda(0) &= x_\lambda := R(\lambda)x,\end{aligned}$$

which is trivially uniquely solvable and it follows that as $\lambda \to \infty$

$$(4.33)\qquad X_\lambda \to X \text{ in } L^2((0,T)\times\Omega, \mathscr{H}) \text{ and } X_\lambda(T) \to X(T) \text{ in } L^2(\Omega, \mathscr{H}).$$

The next step consists in approximating the fractional Brownian motion $(B^H(t), t \geqslant 0)$ by a sequence of continuous, piecewise linear processes. Recall that

$$(4.34)\qquad B^H(t) = \sum_{i=1}^\infty e_i\sqrt{\lambda_i}\beta_i(t)$$

and for $(i,n) \in \mathbb{N}^2$ let B_n^i denote the real-valued process such that $B_n^i(t_j^n) = \sqrt{\lambda_i}\beta_i(t_j^n)$, $t_j^n = \frac{jT}{n}$, $j = 0, \dots, n$, and B_n^i is linear and continuous on the intervals $[t_j^n, t_{j+1}^n]$, $j = 0, \dots, n-1$. Furthermore, for almost all t and each $(i,n) \in \mathbb{N}^2$ let $b_n^i(t) := \frac{\partial}{\partial t} B_n^i(t)$ for $t \in [0,T] \backslash \cup_{j=0}^n \{t_j^{(n)}\}$ (so b_n^i is a piecewise constant process) and $b_n(t) := \sum_{i=1}^\infty b_n^i(t) e_i$.

For each fixed $\lambda > \hat{\beta}$ consider the sequence of random differential equations for $t \in [0,T]$

$$\begin{aligned} \dot{X}_{\lambda,n}(t) &= AX_{\lambda,n}(t) + B_\lambda u(t) + R(\lambda)Cb_n(t), \\ X_{\lambda,n}(0) &= x_\lambda, \end{aligned} \tag{4.35}$$

and

$$\begin{aligned} &\dot{\varphi}_{\lambda,n}(t) + (A^* - P(t)B_\lambda R^{-1}B_\lambda^*)\varphi_{\lambda,n}(t) + P(t)R(\lambda)Cb_n(t) = 0, \\ &\varphi_{\lambda,n}(T) = 0. \end{aligned} \tag{4.36}$$

Both of the equations have unique mild solutions for each $n \in \mathbb{N}$ and $\lambda > \hat{\beta}$ and for $t \in (0,T)$ that can be expressed as

$$X_{\lambda,n}(t) = S(t)x_\lambda + \int_0^t S(t-r)B_\lambda u(r)dr + \int_0^t S(t-r)R(\lambda)Cb_n(r)dr \tag{4.37}$$

and

$$\varphi_{\lambda,n}(t) = \int_t^T U_{P,\lambda}(s,t)P(s)R(\lambda)Cb_n(s)ds, \tag{4.38}$$

respectively. Now it is easy to see that as $n \to \infty$

$$X_{\lambda,n} \to X_\lambda, \ \varphi_{\lambda,n} \to \varphi_\lambda \tag{4.39}$$

in $L^2((0,T)\times\Omega;\mathscr{H})$ and $L^2((0,T)\times\Omega, D_{A^*}^{1-\alpha})$, respectively, and

$$X_{\lambda,n}(T) \to X_\lambda(T) \tag{4.40}$$

in $L^2(\Omega;\mathscr{H})$. (For the passage to the limit in the stochastic integral use integration by parts or a special case of the result of [29] on Wong–Zakai approximations for multiple fractional Brownian integrals.)

LEMMA 4.5. *Under the assumptions given in the above theorem, the following equality is satisfied:*

$$\begin{aligned} &\lim_{\lambda\to\infty}\lim_{n\to\infty} \mathbb{E}\int_0^T \langle \varphi_{\lambda,n}(t), R(\lambda)Cb_n(t)\rangle_{\mathscr{H}} dt \\ &= \int_0^T \int_0^s Tr[C^*P(s)U_P^*(s,r)C\tilde{Q}]\phi_H(r-s)drds. \end{aligned} \tag{4.41}$$

Proof. The following equalities are satisfied.

$$\begin{aligned}
&\mathbb{E}\int_0^T \langle \varphi_{\lambda,n}(t), R(\lambda)Cb_n(t)\rangle_{\mathscr{H}} \\
&= \mathbb{E}\int_0^T \left\langle \int_t^T U_{P,\lambda}(s,t)P(s)R(\lambda)Cb_n(s)ds, R(\lambda)Cb_n(t)\right\rangle_{\mathscr{H}} dt \\
&= \mathbb{E}\int_0^T \int_0^s \langle C^*R^*(\lambda_1)P(s)U^*_{P,\lambda}(s,t)R(\lambda)Cb_n(t), b_n(s)\rangle_V dtds \\
&= \mathbb{E}\int_0^T \int_0^s \sum_{i,j=1}^{\infty} \langle C^*R^*(\lambda)P(s)U^*_{P,\lambda}(s,t)R(\lambda)Ce_i, e_j\rangle_V b_n^i(t)b_n^j(s)dtds.
\end{aligned} \tag{4.42}$$

A result of C. Tudor and M. Tudor (Theorem 3.1 in [29]) is used in the verification of (4.41). Specifically, Theorem 3.1 in [29] verifies that for $N \in \mathbb{N}$

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}\int_0^T \int_0^s \sum_{i,j=1}^{N} \langle C^*R^*(\lambda)P(s)U^*_{P,\lambda}(s,t)R(\lambda)Ce_i, e_j\rangle_V b_n^i(t)b_n^j(s)dtds \\
&= \mathbb{E}\sum_{i,j=1}^{N}\int_0^T \int_0^s \langle C^*R^*(\lambda)P(s)U^*_{P,\lambda}(s,t)R(\lambda)Ce_i, e_j\rangle_V \sqrt{\lambda_i\lambda_j}d\beta^i(t)\circ d\beta^j(s),
\end{aligned} \tag{4.43}$$

where the symbol $\circ$ denotes the Stratonovich integral. The only assumptions for this theorem are that the integrand is continuous and symmetric. The symmetric property follows by symmetrizing the integrand above. The trace class property of the incremental covariance $\widetilde{Q}$ allows (4.43) to be satisfied for $N = +\infty$.

Let $h^\lambda_{i,j}(s,t) = \langle C^*R^*(\lambda)P(s)U^*_{P,\lambda}(s,t)R(\lambda)C\tilde{Q}e_i, e_j\rangle_V$ and use the relation between the Stratonovich and the Skorokhod integrals [5] to rewrite the expression on the r.h.s. of (4.43) as

$$\begin{aligned}
&\mathbb{E}\sum_{i,j=1}^{\infty}\int_0^T \int_0^s h^\lambda_{ij}(s,t)d\beta^i(t)\circ d\beta^j(s) \\
&= \sum_{i,j}\left(\mathbb{E}\int_0^T \int_0^s h^\lambda_{ij}(s,t)d\beta^i(t)d\beta^j(s)\right) \\
&\quad + \sum_{i=1}^{\infty}\mathbb{E}\int_0^T \int_0^T D_r\left(\int_0^s h^\lambda_{ii}(s,t)d\beta^i(t)\right)\phi_H(r-s)drds,
\end{aligned} \tag{4.44}$$

where $\phi_H(r-s) = H(2H-1)|r-s|^{2H-2}$ and D_r denotes the Malliavin (or path) derivative operator for the fractional Brownian motion (cf. [9]). The first term on the r.h.s. of (4.44) is zero, while the second can be rewritten as

$$\begin{aligned}
&\int_0^T \int_0^s \sum_i h^\lambda_{ii}(s,r)\phi_H(r-s)drds \\
&= \int_0^T \int_0^s Tr[C^*R^*(\lambda)P(s)U^*_{P,\lambda}(s,r)R(\lambda)C\tilde{Q}]\phi_H(r-s)drds.
\end{aligned} \tag{4.45}$$

It remains to pass to the limit as $\lambda \to +\infty$. Note that for an arbitrary family $(M_\lambda(s,t), t \geqslant 0, s \geqslant t)$ of operators from $\mathscr{L}(V)$, $\lambda > \hat{\beta}$, $(r,t) \in [0,s]\times[0,T]$, such that $M_\lambda(r,t) \to M(r,t)$ as $\lambda \to \infty$ in the strong operator norm of $\mathscr{L}(V)$ and $|M_\lambda(r,t)|_{\mathscr{L}(V)} \leqslant \varphi(r,t)$ for each pair (r,t), and for an integrable function φ on $[0,s]\times[0,T]$ it follows by the Lebesgue dominated convergence theorem that there is convergence for any sequence so

$$\int_0^T \int_0^s Tr[M_\lambda(r,s)\tilde{Q}]drds \to \int_0^T \int_0^s Tr[M(r,s)\tilde{Q}]drds$$

as $\lambda \to \infty$. This convergence is satisfied for

$$M_\lambda(r,s) = C^* R^*(\lambda) P(s) U^*_{P,\lambda}(s,r) R(\lambda) C$$

by the above mentioned properties of the operators $(U_{P,\lambda}(s,r), r \geqslant 0, s \geqslant r)$ and $(R(\lambda), \lambda \geqslant \hat{\beta})$ because

$$|M_\lambda(r,s)|_{\mathscr{L}(V)} \leqslant |C^*|_{\mathscr{L}(D^{1-\beta}_{A^*},V)} \cdot |P(s)|_{\mathscr{L}(\mathscr{H},D^{1-\alpha}_{A^*})} \cdot |U_{P,\lambda}(s,r)|_{\mathscr{L}(\mathscr{H},D^{1-\beta}_{A^*})}$$

(here it is used that $\alpha \leqslant \beta$ by (A5)(b)). Therefore, letting $\lambda \to \infty$ in (4.45), (4.41) is obtained. ∎

Proof of Theorem 4.4. Fix $n \in \mathbb{N}$ and $\lambda > \hat{\beta}$; let $J_{n,\lambda}(x_\lambda, u)$ denote the cost for the approximate system that is expressed as

$$\begin{aligned} J_{n,\lambda}(x_\lambda,u) := \frac{1}{2}\mathbb{E}\left(\int_0^T (\langle QX_{\lambda,n}(s), X_{\lambda,n}(s)\rangle_{\mathscr{H}} + \langle Ru(s), u(s)\rangle_U)ds\right) \\ + \frac{1}{2}\mathbb{E}\langle GX_{\lambda,n}(T), X_{\lambda,n}(T)\rangle_{\mathscr{H}}. \end{aligned}$$

From (4.39) and (4.40) and the convergence of this resulting family of limits as $\lambda \to \infty$ it follows that

$$\lim_{\lambda\to\infty}\lim_{n\to\infty} J_{n,\lambda}(x_\lambda,u) = J(x,u),\ x \in \mathscr{H},\ u \in \mathscr{U}. \tag{4.46}$$

Assume initially that the process $u \in \mathscr{V}$ has the sample paths ($\mathbb{P}$-a.s.) in $C^1([0,T],U)$ (so there exists a strong solution to (4.35)). By taking the differentials of the processes $(\langle P(t)X_{\lambda,n}(t), X_{\lambda,n}(t)\rangle_{\mathscr{H}}, t \in [0,T])$ and $(\varphi_{\lambda,n}(t), t \in [0,T])$ for fixed $\lambda > \hat{\beta}$ and $n \in \mathbb{N}$ and integrating these differential expressions it follows that

$$\begin{aligned} & J_{n,\lambda}(x_\lambda,u) - \frac{1}{2}\langle P(0)x_\lambda, x_\lambda\rangle_{\mathscr{H}} - \mathbb{E}\langle\varphi_{\lambda,n}(0), x_\lambda\rangle_{\mathscr{H}} \\ &= \frac{1}{2}\mathbb{E}\int_0^T \Big\{\langle Ru(s), u(s)\rangle_U + \langle P(s)BR^{-1}B^*P(s)X_{\lambda,n}(s), X_{\lambda,n}(s)\rangle_{\mathscr{H}} \\ &\quad + 2\langle R^{-1}B^*_\lambda P(s)X_{\lambda,n}(s), B^*_\lambda\varphi_{\lambda,n}(s)\rangle_U + 2\langle B^*_\lambda\varphi_{\lambda,n}(s), u(s)\rangle_U \\ &\quad + 2\langle\varphi_{\lambda,n}(s), R(\lambda)Cb_n(s)\rangle_{\mathscr{H}}\Big\}ds \\ &= \frac{1}{2}\mathbb{E}\int_0^T \big\{|R^{-\frac{1}{2}}(Ru(s) + B^*_\lambda P(s)X_{n,\lambda}(s) + B^*_\lambda\varphi_{\lambda,n}(s))|^2_U \\ &\quad + 2\langle\varphi_{\lambda,n}(s), R(\lambda)Cb_n(s)\rangle_{\mathscr{H}}\big\}ds \end{aligned} \tag{4.47}$$

or equivalently

$$J_{n,\lambda}(x_\lambda, u) - \frac{1}{2}\langle P(0)x_\lambda, x_\lambda\rangle_{\mathscr{H}} - \mathbb{E}\langle \varphi_{\lambda,n}(0), x_\lambda\rangle_{\mathscr{H}} = \frac{1}{2}\mathbb{E}\int_0^T \Big\{|R^{-\frac{1}{2}}(Ru(s) + B_\lambda^* P(s)X_{n,\lambda}(s) + B_\lambda^*\varphi_{\lambda,n}(s))|_U^2 - |R^{-1/2}B_\lambda^*\varphi_{\lambda,n}(s)|_U^2\Big\}ds + M_{\lambda,n} + N_{\lambda,n}, \tag{4.48}$$

where

$$M_{\lambda,n} := -\frac{1}{2}\mathbb{E}\int_0^T \langle P(s)(B_\lambda R^{-1}B_\lambda^* - BR^{-1}B^*)P(s)X_{n,\lambda}(s), X_{n,\lambda}(s)\rangle_{\mathscr{H}} ds$$

and

$$N_{\lambda,n} := \mathbb{E}\int_0^T \langle \varphi_{\lambda,n}(s), R(\lambda)Cb_n(s)\rangle_{\mathscr{H}} ds.$$

For $\lambda > \hat{\beta}$, all the operators in (4.48) are bounded on their respective spaces, so by (4.39) and (4.40) it follows that

$$\lim_{n\to\infty} J_{n,\lambda}(x_\lambda, u) - \frac{1}{2}\langle P(0)x_\lambda, x_\lambda\rangle_{\mathscr{H}} - \mathbb{E}\langle\varphi_\lambda(0), x_\lambda\rangle_{\mathscr{H}} = \frac{1}{2}\mathbb{E}\int_0^T \Big\{|R^{-1}(Ru(s) + B_\lambda^* P(s)X_\lambda(s) + B_\lambda^*\varphi_\lambda(s))|_U^2 - |R^{-1/2}B_\lambda^*\varphi_\lambda(s)|_U^2\Big\}ds + \lim_{n\to\infty}(M_{\lambda,n} + N_{\lambda,n}). \tag{4.49}$$

Now consider $\lambda \to \infty$. Use the convergence $B_\lambda \to B$ in the strong operator topology of $\mathscr{L}(U, D_A^{\alpha-1})$ (and the equiboundedness of $|B_\lambda|_{\mathscr{L}(U,D_A^{\alpha-1})}$) and the convergence in (4.31) and (4.33). It follows that

$$J(x,u) - \frac{1}{2}\langle P(0)x, x\rangle_{\mathscr{H}} - \mathbb{E}\langle\varphi(0), x\rangle_{\mathscr{H}} = \frac{1}{2}\mathbb{E}\int_0^T \Big\{|R^{-\frac{1}{2}}(Ru(s) + B^*P(s)X(s) + B^*\varphi(s))|_U^2 - |R^{-1}B^*\varphi(s)|_U^2\Big\}ds + \lim_{\lambda\to\infty}\lim_{n\to\infty} N_{\lambda,n}, \tag{4.50}$$

and by Lemma 4.5 ($\mathbb{E}\varphi(0) = 0$) the following equality is obtained:

$$J(x,u) - \frac{1}{2}\langle P(0)x, x\rangle_{\mathscr{H}} = \frac{1}{2}\mathbb{E}\int_0^T \Big\{|R^{\frac{1}{2}}(u(s) + R^{-1}B^*P(s)X(s) + R^{-1}B^*\varphi(s))|_U^2 - |R^{-1}B^*\varphi(s)|_U^2\Big\}ds + \int_0^T\int_0^s Tr[C^*P(s)U_P^*(s,r)C\tilde{Q}]\phi_H(r-s)drds. \tag{4.51}$$

If the sample paths of $(u(t), t \geqslant 0) \in \mathscr{V}$ are not smooth the same equality can be obtained by approximating an arbitrary $u \in \mathscr{V}$ by a sequence $(u_n, n \in \mathbb{N})$ of piecewise smooth processes in the norm of $L^p((0,T)\times\Omega,U)$ and passing to the limit directly in (4.56) (the corresponding sequence of controlled processes $(X^n(t), t\in[0,T], n\in\mathbb{N}) = (X^{u_n}(t), t\in[0,T], n\in\mathbb{N})$ then converges to the limiting process in $L^2((0,T)\times\Omega;\mathscr{H})$).

From the expression (4.51) for J it is clear that the optimal control in $\mathscr{V}$ is (4.27). Note that the control $u=\bar{v}$ and the corresponding optimal solution to the controlled equation are well defined because $R^{-1}B^*\varphi\in\mathscr{V}$ and $R^{-1}B^*P(\cdot)\in C_s([0,T],\mathscr{L}(\mathscr{H},U))$ (cf. the remark following (2.8)). ∎

Using the equality (4.51) the main result stated below can be proved.

THEOREM 4.6. *If* (A1)–(A5) *are satisfied, then there exists a unique optimal control* $(\bar{u}(t), t\in[0,T])$ *in* $\mathscr{U}$ *for the control problem* (2.1)–(2.9) *that is given by*

$$\bar{u}(t) = -R^{-1}B^*P(t)X(t) - R^{-1}B^*\psi(t) \tag{4.52}$$

and

(4.53)

$$\psi(t) := \mathbb{E}[\varphi(t)|\mathscr{F}(t)] = \int_0^t s^{-(H-1/2)}(I_{t_-}^{-(H-1/2)}(I_{T_-}^{(H-1/2)}u_{H-1/2}(U_P(\cdot,t)P(\cdot)C)))(s)dB^H(s),$$

where $t\in[0,T]$, $u_a(s)=s^aI$ *for* $s>0$ *and* $a\in\mathbb{R}$, U_P *is the fundamental solution for* $dX(t)=(A-BR^{-1}B^*P(t))X(t)$, *and* $I_{t_-}^a$ *denotes the left-sided fractional Riemann–Liouville integral for* $\alpha\in(0,1)$,

$$(I_{t_-}^a h)(s) := \frac{1}{\Gamma(a)}\int_s^t (\lambda-s)^{a-1}h(\lambda)d\lambda$$

(and the left-sided fractional derivative for $\alpha\in(-1,0)$*),* $s\in(0,t)$, $h\in L^1([0,t],\mathscr{L}(U,\mathscr{H}))$, $a>0$, *and* Γ *is the (Euler) Gamma function. The optimal cost is*

$$\widetilde{J}(x) = J(x,\bar{u}) = \frac{1}{2}\langle P(0)x,x\rangle_{\mathscr{H}} - \frac{1}{2}\mathbb{E}\int_0^T |R^{-1}B^*\psi(s)|_U^2 ds + \int_0^T\int_0^s Tr[C^*P(s)U_P^*(s,r)C\tilde{Q}]\phi_H(r-s)drds \tag{4.54}$$

$$= \frac{1}{2}\langle P(0)x,x\rangle_{\mathscr{H}} - \int_0^T\int_0^s\int_0^s Tr(\eta(s,p)\tilde{Q}\eta(s,q))\phi_H(p-q)dpdqds + \int_0^T\int_0^s Tr[C^*P(s)U_P^*(s,r)C\tilde{Q}]\phi_H(r-s)drds, \tag{4.55}$$

where $\mathbb{E}[\varphi(t)|\mathscr{F}(t)] = \int_0^t \eta(t,s)dB^H(s)$*; cf.* (4.53).

Proof. The process $(\varphi(t), t\in[0,T])$ is not $(\mathscr{F}(t))$-adapted and therefore does not belong to $\mathscr{U}$. Let Π denote the orthogonal projection of the space $L^2((0,T)\times\Omega,U) = L^2(U)$ on its closed linear subspace $L^2_{\mathscr{F}}(U)$ and let $\Pi^\perp := I-\Pi$ be the complementary orthogonal projection. By the proof of Theorem 4.4 for any $x \in \mathscr{H}$, $u \in L^2_{\mathscr{F}}(U)$

(4.56)
$$J(x,u) - \frac{1}{2}\langle P(0)x, x\rangle_{\mathscr{H}} = \frac{1}{2}\mathbb{E}\int_0^T \Big\{|R^{-\frac{1}{2}}(Ru(s) + B^*P(s)X(s) + \Pi B^*\varphi(s))|_U^2 + |\Pi^{\perp}R^{-1}B^*\varphi(s)|_U^2 - |R^{-1}B^*\varphi(s)|_U^2\Big\}ds + \int_0^T\int_0^s Tr[C^*P(s)U_P^*(s,r)C\tilde{Q}]\phi_H(r-s)drds.$$

Clearly the r.h.s. of (4.56) is minimized if u is chosen to satisfy

$$u(t) = \bar{u}(t) = -R^{-1}B^*P(t)X(t) - \Pi R^{-1}B^*\varphi(t),$$

and since the orthogonal projection Π is conditional expectation with respect to $(\mathscr{F}(t), t \in [0,T])$, it follows that

$$u(t) = \bar{u}(t) = -R^{-1}B^*P(t)X(t) - R^{-1}B^*\psi(t). \tag{4.57}$$

The stochastic integral form of the prediction in (4.53) follows from [4]. The process $(R^{-1}B^*\psi(t), t \in [0,T])$ is Gaussian, belongs to $L^p((0,T)\times\Omega, U)$, and is progressively measurable. Therefore it belongs to $\mathscr{U} = L^p_{\mathscr{F}}(U)$. It easily follows (as in the proof of Theorem 4.4) that the optimal solution with the feedback control $(\bar{u}(t), t \in [0,T])$ defined by (4.47) is well defined and $\bar{u}(t), t \in [0,T]$ belongs to $\mathscr{U}$ and is therefore the optimal control. The formula (4.54) for the optimal cost is now obtained by substituting the optimal control into (4.56). □

Remark 4.7. While it is assumed that $H > \frac{1}{2}$, the optimal control given in Theorem 4.6 is also valid for the well-known case of $H = \frac{1}{2}$ (i.e., the case when $(B^H(t), t \geqslant 0)$ is a V-valued Wiener process). Then by the independent increments of $(B^{\frac{1}{2}}(t), t \geqslant 0)$ it follows that

$$\psi(t) = \mathbb{E}[\varphi(t)|\mathscr{F}(t)] = \mathbb{E}\left[\int_t^T U_P(s,t)P(s)CdB^{1/2}(s)\right] = 0$$

so the optimal control is $\bar{u}(t) = -R^{-1}B^*P(t)X(t)$. To demonstrate the analogous correspondence for the optimal cost, it is necessary to verify (4.44), which for $H = \frac{1}{2}$ is

$$\mathbb{E}\sum_{i,j=1}^{\infty}\int_0^T\int_0^s h_{ij}^{\lambda}(s,t)d\beta^i(t)\circ d\beta^j(t) = \sum_{i=1}^{\infty}\int_0^T h_{ii}^{\lambda}(s,s)ds = \int_0^T Tr[C^*R^*(\lambda)P(s)R(\lambda)C\tilde{Q}]ds. \tag{4.58}$$

Proceeding as in the remaining part of the proof of Lemma 4.5, this family of integrals can be shown to converge to

$$\int_0^T Tr[C^*P(s)C\tilde{Q}]ds$$

as $\lambda \to \infty$. Summarizing, the well-known expression for the optimal cost is obtained:

$$\tilde{J}(x) = J(x,\bar{u}) = \frac{1}{2}\langle P(0)x, x\rangle_{\mathscr{H}} + \int_0^T Tr[C^*P(s)C\tilde{Q}]ds.$$

5. Examples.

Example 5.1. Consider the controlled stochastic heat equation

$$\frac{\partial y}{\partial t}(t,\xi) = \Delta y(t,\xi) + u(t,\xi) + \eta(t,\xi) \tag{5.1}$$

for $(t,\xi) \in \mathbb{R}_+ \times D$ with the initial condition and Dirichlet boundary conditions

$$u(0,\xi) = x(\xi) \tag{5.2}$$

for $\xi \in D$ and

$$u|_{\mathbb{R}_+ \times \partial D} = 0, \tag{5.3}$$

where $D \subset \mathbb{R}^d$ is a bounded domain with a smooth boundary, u is the control, and η is a noise process that is the formal time derivative of a space-dependent fractional Brownian motion. To provide a precise meaning to (5.1)–(5.3), the parabolic system is rewritten as an infinite-dimensional stochastic differential equation

$$dX(t) = AX(t)dt + u(t)dt + dB^H(t) \tag{5.4}$$

for $t \geqslant 0$ in the space $\mathscr{H} = L^2(D)$, where $A = \Delta|_{\mathrm{Dom}(A)}$ generates an analytic semigroup $(S(t), t \geqslant 0)$ on $\mathscr{H}$ with $\mathrm{Dom}(A) = H^2(D) \cap H_0^1(D)$, $U = V = \mathscr{H}$, and the noise η is modeled as the formal derivative $(dB^H/dt)(t)$, where $(B^H(t), t \geqslant 0)$ is a cylindrical fractional Brownian motion in V with covariance $\tilde{Q} \in \mathscr{L}(V)$. If $\tilde{Q}^{1/2} \in \mathscr{L}_2(V)$, which corresponds to the case where the fractional Brownian motion in (5.4) has a trace class covariance, then it follows from [11] that the condition (A3) is satisfied and there is an $\mathscr{H}$-continuous solution to (5.4). The condition (A1) (in the version (a)) is obviously satisfied since $B = C = I$ and $\mathscr{U} = L^2_{\mathscr{F}}(U)$ can be chosen to verify (A2). If the cost functional has the form (2.9) with an arbitrary choice of operators Q, R, and G satisfying (A4) with the above choice of the spaces $\mathscr{H}$ and U and such that R has a bounded inverse on U, all the results of the previous sections can be applied to find the unique optimal control.

If it is only assumed that $\tilde{Q} \in \mathscr{L}(U)$ so that $(B^H(t), t \geqslant 0)$ is only a cylindrical fractional Brownian motion, then it follows by well-known estimates on the Green function for $dx/dt = Ax$ that

$$|S(t)|_{\mathscr{L}_2(\mathscr{H})} \leqslant ct^{-d/4} \tag{5.5}$$

for $t \in (0,T]$, $c > 0$, and d is the dimension of the underlying space. It follows from [11] that if

$$\frac{d}{4} < H, \tag{5.6}$$

then the condition (A3) is satisfied. All other assumptions are verified as above except (A5)(a). Hence the existence and the uniqueness of the optimal control as stated in section 2, including the form of the optimal control (3.28), are available.

Example 5.2. Consider a stochastic wave equation formally described as

$$\frac{\partial^2 y}{\partial t^2}(t,\xi) = \Delta y(t,\xi) + \tilde{B}(u(t))(\xi) + \eta(t,\xi) \tag{5.7}$$

for $(t,\xi) \in \mathbb{R}_+ \times D$, where D and η satisfy the conditions in the previous example and the control operator $\tilde{B}$ belongs to $\mathscr{L}(U, L^2(D))$, U being an arbitrary control (Hilbert) space. The initial and boundary conditions are

$$\frac{\partial y}{\partial t}(0,\xi) = x_1(\xi), \tag{5.8}$$

$$y(0,\xi) = x_2(\xi), \tag{5.9}$$

$$y(t,\xi) = 0 \tag{5.10}$$

for $\xi \in D$ and $(t,\xi) \in \mathbb{R}_+ \times \partial D$, respectively. The corresponding infinite-dimensional stochastic differential equation is

$$\begin{aligned} dX(t) &= AX(t)dt + Bu(t)dt + dB^H(t), \\ X(0) &= x = (x_1, x_2) \end{aligned} \tag{5.11}$$

with the following choice of operators and spaces: let $\Lambda = \Delta|_{\mathrm{Dom}(\Lambda)}$, $\mathrm{Dom}(\Lambda) = H_0^1(D) \cap H^1(D)$, $\mathrm{Dom}(A) = \mathrm{Dom}(\Lambda) \times \mathrm{Dom}((-\Lambda)^{1/2})$, and

$$A = \begin{pmatrix} 0 & I \\ \Lambda & 0 \end{pmatrix}. \tag{5.12}$$

It is well known that A generates a strongly continuous semigroup in the space $\mathscr{H} = \mathrm{Dom}(-\Lambda)^{1/2} \times L^2(D)$. Let $(B^H(t), t \geqslant 0)$ be a fractional Brownian motion on $\mathscr{H}$ with the covariance

$$\tilde{Q} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{Q}_2 \end{pmatrix},$$

where $\tilde{Q}_2^{1/2}$ is a Hilbert–Schmidt operator on $L^2(D)$. The control operator B is defined as

$$B = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{B} \end{pmatrix}$$

and clearly $B \in \mathscr{L}(U, \mathscr{H})$. The space of controls is chosen as $\mathscr{U} = L^2_{\mathscr{F}}(U)$. It follows from [11] that all the conditions (A1)(a), (A2)–(A5) are satisfied (if it is assumed that the operators from the cost functional satisfy (A4) and R has a bounded inverse). Thus the theorems of the previous two sections can be applied.

Example 5.3. Consider the heat equation with boundary noise and control

$$\frac{\partial y}{\partial t}(t,\xi) = \Delta y(t,\xi) \tag{5.13}$$

for $(t,\xi) \in \mathbb{R}_+ \times D$ with the initial condition

$$y(0,\xi) = x(\xi)$$

for $\xi \in D$ and the boundary conditions that are either of Dirichlet type

$$y(t,\xi)|_{\mathbb{R}_+ \times \partial D} = u(t,\xi) + \eta(t,\xi) \tag{5.14}$$

or of Neumann type

$$\frac{\partial y}{\partial \nu}(t,\xi) = u(t,\xi) + \eta(t,\xi) \tag{5.15}$$

for $(t,\xi) \in \mathbb{R}_+ \times \partial D$, where $\partial/\partial\nu$ is a normal derivative, D is a bounded domain in $\mathbb{R}^d$ with smooth boundary ∂D, u is the control, and η is a noise process on ∂D.

Let $A = \Delta|_{\mathrm{Dom}(A)}$, where $\mathrm{Dom}(A) = H_0^1(D) \cap H^2(D)$ for the Dirichlet boundary condition (5.14) of $\mathrm{Dom}(A) = \{\varphi \in H^2(D) : \partial\varphi/\partial\nu = 0 \text{ on } \partial D\}$ for the Neumann boundary conditions (5.15). In both cases, A is the generator of an analytic semigroup on $\mathscr{H} = L^2(D)$. The fractional Brownian motion $(B^H(t), t \geqslant 0)$ is defined on $V = L^2(\partial D)$ and the covariance $\tilde{Q}$ is assumed to be trace class. If the underlying space dimension is $d = 1$, then $V = \mathbb{R}^2$, so this is no restriction. The semigroup $(S(t), t \geqslant 0)$ is exponentially stable for the Dirichlet boundary conditions where $\hat{\beta} = 0$ suffices and for the Neumann boundary conditions where $\hat{\beta} < 0$ is arbitrary. The operator N, which is called the Dirichlet map for (5.14) or the Neumann map for (5.15), is defined as follows: $Ng = -h$, where h satisfies the elliptic equation

$$(\Delta - \beta I)h = 0, \tag{5.16}$$

$$h|_{\partial D} = g \tag{5.17}$$

for the Neumann problem and analogously for the Dirichlet problem. The operator N satisfies $N \in \mathscr{L}(V, D_A^\alpha)$ if

$$\alpha < \frac{1}{4} \tag{5.18}$$

for the Dirichlet boundary conditions or

$$\alpha < \frac{3}{4} \tag{5.19}$$

for the Neumann boundary conditions. It is well known how to derive the equation

$$\begin{aligned} dX(t) &= AX(t)dt + Bu(t)dt + CdB^H(t) \\ X(0) &= x, \end{aligned} \tag{5.20}$$

where $U = V$, $C = B$ is defined as the composition $\tilde{A}N$, and $\tilde{A}$ is an isometric extension of the operator A, and in this case $B \in \mathscr{L}(U, D_A^{\alpha-1})$ with the above choice of α (and $\beta = \alpha$) in the respective cases (cf., e.g., [21]). Now it follows that the condition (A1)(b) is verified, and the condition (A2) is verified, e.g., with $p = 2$ in the Neumann case and arbitrary $p > 4$ in the Dirichlet case. The condition (A3) for the stochastic integral is verified in [11]: for the Dirichlet case (A3) is satisfied if $H > 3/4$ and in the Neumann case there is no restriction on the Hurst parameter H (because throughout the paper it is assumed that $H > 1/2$). Hence if the operators in the cost functional satisfy the conditions (A4) and (A5) with the above choice of spaces, using α and β above, all statements of the previous two sections can be applied to the present case.

In a similar way the problem with boundary control and distributed noise can be solved. The dual case (boundary noise and distributed control) does not satisfy the condition (A5)(b).

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