Miscellany

Comparative Statics When the Objective Function Is Concave: Old Wine in Old Bottles?

The purpose of this note is to provide an alternative to the harassment that students are subjected to at the hands of Hessians, bordered or otherwise. The idea is to utilize the fact that differentiable concave function lies below the tangent plane to derive qualitative relations between two optima corresponding to two values of the parameters.[1] We shall use consumer's behavior theory as an illustration. Let x^A be the bundle which maximizes the consumer's utility when income is m and when prices are p^A, where x^A and p^A are n-vectors, subject to the budget constraint: $p^A \cdot x \leq m$ and to $x \geq 0$. Let x^B be the optimal bundle when income is m and prices are p^B. *Assuming that the utility function*, denoted by $U(x)$, *is concave and differentiable*,[2] it is necessary and sufficient for x^A to be optimal that:

$$U_{x_i^A} \leq \lambda^A p_i^A$$

with equality when

$$x_i^A > 0,$$

where

$$U_{x_i^A} = \partial U / \partial x_i \mid x = x^A, \lambda^A \geq 0, \lambda^A m = \lambda^A (p^A \cdot x^A). \tag{1}$$

Now let $x^{B'}$ be the optimal bundle under p^B when income is changed to $m^{B'}$, so that the consumer gets to the old indifference curve, that is, $U^{B'} = U^A$, where $U^{B'}$ denotes $U(x^{B'})$ and U^A denotes $U(x^A)$. The vector $x^{B'} - x^A$ is then the compensated change in consumption and is denoted by $\Delta' x$. Let Δp denote $p^B - p^A$. That $x^{B'}$ is an optimum is equivalent to:

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[1] See, for example, Fleming 1965.

[2] Admittedly, concave functions do not stay concave as they undergo monotone transformation, but that should not be interpreted as lack of invariance on their part if invariance means getting the same demand functions. Still, assuming concavity is worse than assuming quasi-concavity.

$$U_{x_i}^{B\prime} \leqq \lambda^{B\prime} p_i^B,$$

with equality when

$$x^{B\prime} > 0, \lambda^{B\prime} \geqq 0, \lambda^{B\prime} m^{B\prime} = \lambda^{B\prime} p^B x^{B\prime}. \tag{2}$$

In view of compensation and concavity of U we have:

$$0 = U^{B\prime} - U^A \leqq U_x^A (x^{B\prime} - x^A) = U_x^A \Delta' x. \tag{3}$$

From (1) and (3) we have:

$$\lambda^A (p^A \cdot \Delta' x) \geqq 0 \tag{4}$$

Assuming *the consumer not to be satiated at any point*, it follows[3] from (1) that $\lambda^A > 0$. Thus:

$$p^A \cdot \Delta' x \geqq 0. \tag{5}$$

Again, by concavity, and compensation we have:

$$0 = U^A - U^{B\prime} \leqq U_x^{B\prime} \cdot (x^A - x^{B\prime}) = U_x^{B\prime} \cdot (-\Delta' x). \tag{6}$$

By (2) and (6) we have

$$-\lambda^{B\prime} p^B \cdot \Delta' x \geqq 0. \tag{7}$$

By nonsatiation, concavity and (3), $\lambda^{B\prime} > 0$. Thus:

$$-p^B \cdot \Delta' x \geqq 0. \tag{8}$$

Adding (5) and (8) we have:

$$(p^A - p^B) \cdot \Delta' x \geqq 0. \tag{9}$$

Noting that $p^A - p^B = -\Delta p$, (9) implies:

$$\Delta p \cdot \Delta' x \leqq 0. \tag{10}$$

Relation (1) could be used to show that the own substitution term is nonpositive, if all prices except one are unchanged Δp will have zero components except one (say the ith) and then (10) implies $\Delta p_i \Delta' x_i \leqq 0$, or to show that the Slutsky matrix is negative semidefinite. To accomplish this last task we follow Samuelson's discussion.[4] We could write (10) as:

$$dp \cdot dx \leqq 0. \tag{11}$$

Assuming the demand function to be differentiable we get, by compensation and definition of the Slutsky terms (denoted by k_{ij}):

$$dx_i = dh_i = \sum_j K_{ij} dp_j. \tag{12}$$

[3] By contradiction, since if λ^A is not positive then, as $\lambda^A \geqslant 0, \lambda^A = 0$. By (1), then $U_{x_i}^A \leqq 0$, with equality if $x_i^A > 0$, for $i = 1, \ldots, n$. This, by concavity, implies that U has an absolute maximum at x^A subject to $x \geqslant 0$ which means that the consumer is satiated at x^A.

[4] See Samuelson 1947.

Multiplying both sides of (12) by dp_i and summing over i we get

$$\Sigma dp_i dx_i = \sum_{i,j} K_{ij} dp_i dp_j. \tag{13}$$

By (11) and (12) the Slutsky matrix is negative semidefinite.

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References

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Samuelson, Paul A. *Foundations of Economic Analysis.* Cambridge, Mass.: Harvard Univ. Press, 1947.