Error-prevention scheme with two pairs of qubits

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A scheme is presented for protecting one-qubit quantum information against decoherence due to a general environment and local exchange interactions. The scheme operates essentially by distributing information over two pairs of qubits and through error-prevention procedures. In the scheme, quantum information is encoded through a decoherence-free subspace for collective phase errors and exchange errors affecting the qubits in pairs; leakage out of the encoding space due to amplitude damping is reduced by quantum Zeno effect. In addition, how to construct decoherence-free states for *n*-qubit information against phase and exchange errors is discussed.

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Quantum computing has attracted much interest in cavity QED, trapped ion systems, NMR systems, and solid state systems using nuclear spins, quantum dots, superconducting quantum interference devices, Josephson junctions, and single-Cooper-pair devices. It is realized that one of the main obstacles in realizing a quantum computer is the decoherence resulting from the coupling of the system with environment. Among methods designed to protect information, there are theoretical proposals for preventing quantum information against errors by using the quantum Zeno effect [1–4]. Compared with conventional error-correction schemes, the decoherence-reducing strategies based on the Zeno effect are significantly simpler since they only require making tests on a system and no error-correction steps are needed. The most important point is that they can reduce the number of qubits involved in the encoding of a quantum state.

Recently, using the Zeno effect, Hwang *et al.* [4] considered how to protect information in an error model where phase errors are dominant but other errors are still non-negligible. Their schemes are based on encoding one-qubit information $\alpha|0\rangle+\beta|1\rangle$ through a code $|0_L\rangle=|01\rangle$ and $|1_L\rangle=|10\rangle$. Without doubt, their schemes work perfectly if there is no qubit-qubit exchange interaction [5,6]. However, it is obvious that exchange interaction (exchanging the qubits) turns the encoded state $\alpha|01\rangle+\beta|10\rangle$ into $\alpha|10\rangle+\beta|01\rangle$, which leads to potentially fatal consequences [i.e., another term for bit-flipping error appears in the resulting state (8) of Ref. [4], not only the phase errors as mentioned there]. Therefore, their schemes cannot work in the presence of exchange interaction.

In this paper, an alternative scheme is proposed for protecting one-qubit information against decoherence due to a general environment and local exchange interaction, based on the method of pairing qubits [7–9] and the Zeno effect. In this scheme, the original message is encoded through two pairs of qubits (a four-qubit encoding). The present code forms a decoherence-free subspace (DFS) [7,10–12] for collective phase and exchange errors, if the following approximations apply: (a) the exchange interaction between the two pairs can be made negligible (this is possible by setting the two pairs apart, since the exchange effects generally decrease rapidly as the qubit-qubit distance increases [6]), and (b) the two qubits in each pair are close to each other so that each pair undergoes collective decoherence.

Consider two separate pairs I and II each containing two qubits. The four identical qubits are labeled by 1, 1′, 2 and 2′. Qubits 1 and 1′ form the pair I while qubits 2 and 2′ constitute the other pair II. The two qubits in either pair are assumed to be close to each other so that they will undergo collective decoherence. Under the assumption that the exchange interaction between the two pairs is small enough to be negligible, the Hamiltonian for the qubit system and the environment is therefore of the form

$$H=H_S+H_B+H_{SB}+H_{ex}, \tag{1}$$

where H_S and H_B denote the qubit system and the environment-free Hamiltonians, respectively; H_{SB} is the interaction Hamiltonian, and the operator H_{ex} corresponds to local exchange interactions between the two qubits in either pair. If the two pairs are physically identical, i.e., the separation of the qubits in each pair is the same, the operator H_{ex} will act simultaneously and identically on both pairs of qubits. In this case, H_{ex} acts as a collective exchange operator which has the following form:

$$H_{ex}=J(E_{11'}+E_{22'}) \tag{2}$$

(J is a constant; E_{ij} is an independent exchange operator for two identical qubits i and j, which has the property $E_{ij}|\epsilon_i\epsilon_j\rangle=|\epsilon_j\epsilon_i\rangle$, $\epsilon_i\in\{0,1\}$ [6]). The expressions for H_S and H_{SB} are as follows:

$$H_S=\epsilon_0(\sigma_I^z+\sigma_{II}^z),$$

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$$H_{SB}=\lambda_1^z\sigma_I^z\otimes V_z+\lambda_1^+\sigma_I^+\otimes V_+ +\lambda_1^-\sigma_I^-\otimes V_- +\lambda_2^z\sigma_{II}^z\otimes V_z' +\lambda_2^+\sigma_{II}^+\otimes V_+' +\lambda_2^-\sigma_{II}^-\otimes V_-' . \tag{3}$$

Here, $\sigma_I^j=\sigma_1^j+\sigma_{1'}^j$, $\sigma_{II}^j=\sigma_2^j+\sigma_{2'}^j$, $(j=z,+,-)$; σ_i^j is Pauli spin operators of the qubit i; V_j and V_j' are the environment operators coupled to these degrees of freedom. This interaction Hamiltonian H_{SB} applies to the following situation: the qubits inside each pair undergo collective decoherence while the two pairs undergo independent decoherence for the case of different V_j and V_j' or imperfect collective decoherence for the case of the same V_j and V_j'.

Suppose that qubit 1 is the original information carrier, which is initially in an arbitrary unknown state $|\psi\rangle=\alpha|0\rangle+\beta|1\rangle$. The encoding is

$$|\psi\rangle_{enc}=\alpha|0\rangle_L+\beta|1\rangle_L , \tag{4}$$

where

$$|0_L\rangle=(|01\rangle+|10\rangle)_{11'}(|01\rangle-|10\rangle)_{22'} ,$$

$$|1_L\rangle=(|01\rangle-|10\rangle)_{11'}(|01\rangle+|10\rangle)_{22'} . \tag{5}$$

This encoding will protect the state (4) against collective phase errors taking place at either pair or both, since the qubits 11′ and 22′ are paired up in the decoherence-free state combinations $|01\rangle$ and $|10\rangle$. Moreover, it is obvious that the collective exchange operator (2) has the property $H_{ex}|0\rangle_L=(E_{11'}+E_{22'})|0\rangle_L=0$ and $H_{ex}|1\rangle_L=(E_{11'}+E_{22'})|1\rangle_L=0$, which shows that the independent exchange errors for each pair cancel each other due to the cooperative action between the local exchange interaction in one pair and the local exchange interaction in the other pair, i.e., the code also forms a DFS for exchange errors.

Suppose that the environment is initially in the state $|\psi_b(0)\rangle$. During a finite time T_0, perform a test N times. In a short period of time T_0/N, under the Hamiltonian (1), the encoded state (4) will evolve into

$$\begin{aligned}|\psi(T_0/N)\rangle &\approx [1-iH(T_0/N)]|\psi\rangle_{enc}\otimes|\psi_b(0)\rangle=[\alpha(|01\rangle+|10\rangle)_{11'}(|01\rangle-|10\rangle)_{22'}+\beta(|01\rangle-|10\rangle)_{11'}(|01\rangle+|10\rangle)_{22'}]\\ &\otimes[1-iH_B(T_0/N)]|\psi_b(0)\rangle-i(T_0/N)|11\rangle_{11'}(|01\rangle-|10\rangle)_{22'}\otimes\lambda_1^+\alpha V_+|\psi_b(0)\rangle-i(T_0/N)|00\rangle_{11'}(|01\rangle\\ &-|10\rangle)_{22'}\otimes\lambda_1^-\alpha V_-|\psi_b(0)\rangle-i(T_0/N)(|01\rangle-|10\rangle)_{11'}|11\rangle_{22'}\otimes\lambda_2^+\beta V_+'|\psi_b(0)\rangle-i(T_0/N)(|01\rangle\\ &-|10\rangle)_{11'}|00\rangle_{22'}\otimes\lambda_2^-\beta V_-'|\psi_b(0)\rangle.\end{aligned} \tag{6}$$

Equation (6) shows that after evolution for a short time T_0/N, if one performs a subsequent measurement to determine whether the four-qubit system has left the encoding space spanned by Eq. (5), the probability for getting a result "out of the encoding space" is of the order of $1/N^2$, and therefore the probability of obtaining such an outcome during the time T_0 is proportional to $1/N$. Taking N, the number of tests during the time T_0, large enough one can decrease the probability of such an error below any desired level. On the other hand, after the evolution of time T_0/N the state inside the encoding space remains the same as the initial encoded state, and the probability of obtaining such an outcome during the time T_0 is proportional to $1-O(1/N)$.

The required projection can be performed in two steps. The first step is to prepare a test qubit (labeled by t) in the state $|0\rangle$, make it interact with each of the two qubits in the first pair I consecutively by a joint operation $C_{1t}C_{1't}$, and then perform a measurement on the test qubit. The measurement outcome $|1\rangle$ projects the whole system onto the state

$$\begin{aligned}|\psi(T_0/N)\rangle' &= a[\alpha(|01\rangle+|10\rangle)_{11'}(|01\rangle-|10\rangle)_{22'}+\beta(|01\rangle\\ &-|10\rangle)_{11'}(|01\rangle+|10\rangle)_{22'}]+b(|01\rangle\\ &-|10\rangle)_{11'}|11\rangle_{22'}+c(|01\rangle-|10\rangle)_{11'}|00\rangle_{22'} ,\end{aligned} \tag{7}$$

while $|0\rangle$ corresponds to the projection onto the state

$$|\psi(T_0/N)\rangle''=d|11\rangle_{11'}(|01\rangle-|10\rangle)_{22'}+e|00\rangle_{11'}(|01\rangle-|10\rangle)_{22'} . \tag{8}$$

Under the condition of large N, the effects of the state (8), which is outside the encoding space, can be negligible. Thus, after this test step, the four qubits and the environment will be in the state (7).

The second step follows the same procedure as described above. One needs to have the test qubit (in the zero state) interact with each of the two qubits in the second pair II by a joint operation $C_{2t}C_{2't}$ and then make a measurement on the test qubit. From Eq. (7) one can see that the measurement outcome $|0\rangle$ projects the whole system onto the state

$$b(|01\rangle-|10\rangle)_{11'}|11\rangle_{22'}+c(|01\rangle-|10\rangle)_{11'}|00\rangle_{22'} , \tag{9}$$

which is the wrong state out of the encoding space, and again the effects of this state (9) can be neglected if one frequent enough performs, on the other hand, if the test qubit is measured in the state $|1\rangle$, the four qubits will remain in the original encoded state (4). Thus, after the time T_0, the final state for the whole system will be given by

$$|\psi(T_0)\rangle\approx|\psi\rangle_{enc}\otimes|\tilde{\psi}_b\rangle, \tag{10}$$

where $|\tilde{\psi}_b\rangle$ is the state of the environment. It is clear that no errors in the encoded state (4) occur after overall time evolution. Thus, one can protect one-qubit information against decoherence without any other error-correction.

The present scheme works via the Zeno effect; thus it can deal only with "slow" noise. The characteristic time of the noise coupling has to be larger than the time interval between the projection measurements. These limitations are also required by other error-prevention schemes based on the quantum Zeno effect [1–4].

One might envision using Vaidman *et al.*'s code [1]

$$|0_L\rangle=(|00\rangle+|11\rangle)(|00\rangle+|11\rangle),$$
$$|1_L\rangle=(|00\rangle-|11\rangle)(|00\rangle-|11\rangle) \tag{11}$$

to accomplish the goal. As long as the exchange interaction between the left two qubits and the right two qubits is small enough to be negligible, this code also forms a DFS for exchange errors. It is noted that the code (11) works for the case of each qubit undergoing independent decoherence, i.e., the left or the right two qubits in Eq. (11) do not need to be set close. In this sense, the scheme of Vaidman *et al.* is better than the present scheme since it has a less strict condition. However, as was argued by Vaidman *et al.* [1], after a short-time evolution, the test qubit has to interact with *all four physical qubits* of the system consecutively to detect phase errors, as well as interacting with every two physical qubits of the system to distinguish bit-flip errors. In contrast, since the present code forms a DFS for collective phase errors, no phase errors occur and thus no such step for detecting phase errors is required. As shown above, the present scheme only needs to detect bit-flip errors by a test qubit interacting with two qubits for each test step. Therefore, the present error-prevention procedures are much simpler.

Duan and Guo [2] have shown that one-qubit information can be protected against decoherence due to a general environment with only two qubits and the assistance of an external driving field. The present scheme, however, focuses on how to protect one-qubit information without using an external driving field and how to reduce decoherence arising from the qubit-qubit exchange interaction.

Another point may need to be made here. If there is no exchange interaction, and if a general environment affects qubits independently, $|0_L\rangle$ and $|1_L\rangle$ in Eq. (4) could be the logical zero and 1 of the five-qubit [13] or seven-qubit codes [14]; or they could be the logical zero and 1 of the four-qubit code [15].

In what follows, our purpose is to show how to construct DF states for *n*-qubit quantum information against collective phase and exchange errors. The general state of *n* qubits is expressed as

$$|\psi\rangle=\sum_{\{i_l\}} c_{\{i_l\}}|\{i_l\}\rangle, \tag{12}$$

where $|\{i_l\}\rangle$ represents a computational basis state $|i_1\rangle\otimes|i_2\rangle\otimes\cdots\otimes|i_n\rangle$ with $i_l=0$ or 1. The state (12) is encoded into the following state of $n+2$ pairs:

$$|\psi\rangle_{enc}=\sum_{\{i_l\}} c_{\{i_l\}}|\{i_l\}\rangle_L . \tag{13}$$

Here,

$$|\{i_l\}\rangle_L=\prod_{k=1}^{n+2}|j_{kk'}\rangle$$
$$=|j_{11'}\rangle\otimes|j_{22'}\rangle\otimes\cdots\otimes|j_{(n+2)(n+2)'}\rangle. \tag{14}$$

In Eq. (14), $|j_{kk'}\rangle$ indicates the encoded zero or one of the *k*th pair, which is given by

$$|0_{kk'}\rangle\rightarrow\frac{1}{2}(|01\rangle+|10\rangle)_{kk'},$$
$$|1_{kk'}\rangle\rightarrow\frac{1}{2}(|01\rangle-|10\rangle)_{kk'}, \tag{15}$$

where kk' represents the two qubits in the *k*th pair. Clearly, such an encoding (15) on each pair ensures that the encoded state (13) is a DF state for collective phase errors if the two qubits in each pair are close to each other.

Assume that the separation of the two qubits in each pair is the same and that the exchange interaction between any two pairs is negligible. Thus the collective exchange operator H_{ex} is

$$H_{ex}=J\sum_{k=1}^{n+2} E_{kk'} . \tag{16}$$

It is worth noting that not all the DF states for phase errors are DF states for exchange errors, since exchanging the two qubits in each pair will make $|0_{kk'}\rangle\rightarrow|0_{kk'}\rangle$ while $|1_{kk'}\rangle\rightarrow-|1_{kk'}\rangle$ (for the latter, there is a phase-flip error). However, one can still expect that the encoded state (13) is a DF state for exchange errors, through an appropriate encoding on each pair and making the encoded state (13) an eigenstate of the collective exchange operator (16).

In order to have the encoded state (13) an eigenstate of the collective exchange operator (16), one needs to make each logical state in the encoded state (13) be an eigenstate of the collective exchange operator (16) with the same eigenvalue. In general, for $n+2$ pairs of qubits, one can construct C_{n+2}^m orthogonal states. Each of them takes the form (14) and all of them are eigenstates of the collective exchange operator (16) with the same eigenvalue $J(n-2m+2)$ [where $m=1,2,\ldots,(n+1)/2$ for odd n and $m=1,2,\ldots,n/2+1$ for even n]. It is easy to see that (a) C_{n+2}^m reaches maximum when $m=(n+1)/2$ for odd n or $m=n/2+1$ for even n, and (b) such a maximum satisfies the relation $n<\log_2 C_{n+2}^m<n+1$. The point (a) means that in the case when each orthogonal state is an eigenstate of the collective exchange operator (16) with the same eigenvalue J for odd n or 0 for even n, the number of such orthogonal states is maximal; the point (b) implies that all these orthogonal states, as logical states $\{|\{i_l\}\rangle_L\}$, are sufficient to encode n logical qubits. Thus, $n+2$ pairs of qubits are sufficient to encode an arbitrary state of *n* qubits into a DF state. For large *n*, the efficiency of the encoding is approximately 1/2. On the other hand, it is easy to show that $n+1$ pairs of qubits are not sufficient to do the above.

It is interesting to note that, for some kinds of entangled state of *n* (distant) qubits, the DF states for collective phase and exchange errors can be obtained by pairing each entangled qubit with an ancilla qubit and applying local operation on each pair. For example, consider the following entangled state:

$$|\Psi\rangle^{(1)} = \alpha_0|0\rangle_{1,2,\ldots,(n-1)}|1\rangle_n + \sum_{i=1}^{n-1} \alpha_i|n-2,1\rangle^{(i)}_{1,2,\ldots,(n-1)}|0\rangle_n, \qquad (17)$$

where the number of entangled qubits $n \geqslant 3$, and $|n-2,1\rangle^{(i)}_{1,2,\ldots,(n-1)}$ denotes the ith computational basis state of the $n-1$ entangled qubits involving $n-2$ zeros and 1 ones. In the case of $|\alpha_0| = |\alpha_i| = 1/\sqrt{n}$, the states (17) are known as the entangled W states [16]. If each entangled qubit is paired with an ancilla qubit and then the two orthogonal states $|0\rangle$ and $|1\rangle$ of the original kth entangled qubit are encoded into the logical zero $|0_{kk'}\rangle$ and one $|1_{kk'}\rangle$ in Eq. (15), respectively, one can see that the resulting encoded state for the state (17) is an eigenstate of the collective exchange operator $H_{ex} = J\Sigma_{k=1}^{n} E_{kk'}$ with an eigenvalue $(n-2)J$, i.e., the encoded state is a DF state for exchange errors; and it is also a DF state for collective phase errors if collective decoherence holds for each pair.

In addition, entangled states of the form

$$|\Psi\rangle^{(2)} = \alpha|i_1, i_2, \ldots, i_n\rangle + \beta|\overline{i_1}, \overline{i_2}, \ldots, \overline{i_n}\rangle \qquad (18)$$

(which, in the case $|\alpha| = |\beta| = 1/\sqrt{2}$, are known as entangled Greenberger-Horne-Zailinger states [17]) are widely used in information processing. Here, the i_j are ones or zeros and $\overline{i_j}$ are their complements. By pairing each entangled qubit with an ancilla qubit and performing the same encoding on each pair as above, one can see that the two components $|i_1, i_2, \ldots, i_n\rangle_L$ and $|\overline{i_1}, \overline{i_2}, \ldots, \overline{i_n}\rangle_L$ in the encoded state $|\Psi\rangle^{(2)}_{enc} = \alpha|i_1, i_2, \ldots, i_n\rangle_L + \beta|\overline{i_1}, \overline{i_2}, \ldots, \overline{i_n}\rangle_L$ are eigenstates of the collective exchange operator $H_{ex} = \Sigma_{k=1}^{n} J_{kk'} E_{kk'}$ with an eigenvalue $\Sigma_{k=1}^{n}(-1)^{i_k} J_{kk'}$ for $|i_1, i_2, \ldots, i_n\rangle_L$ while $\Sigma_{k=1}^{n}(-1)^{\overline{i_k}} J_{kk'}$ for $|\overline{i_1}, \overline{i_2}, \ldots, \overline{i_n}\rangle_L$. It is easy to show that after evolving for time t the n pairs of qubits will be in the state

$$\alpha|i_1, i_2, \ldots, i_n\rangle_L + e^{i\varphi}\beta|\overline{i_1}, \overline{i_2}, \ldots, \overline{i_n}\rangle_L, \qquad (19)$$

where $\varphi = t\Sigma_{k=1}^{n}[(-1)^{i_k} - (-1)^{\overline{i_k}}]J_{kk'}$. This accumulated phase factor in the final state might not be significant for the states (18) in some applications. Furthermore, if (a) the number of the originally entangled qubits is even, (b) the number of 1's is the same as that of 0's in each of the two basis states of Eq. (18), (c) $J_{kk'} = J$, the phase factor φ will be zero. In this case, the encoded state is perfectly protected against collective phase and exchange errors during the time evolution.

So far, a three-qubit error-correction code [18–20] and a two-qubit error prevention code [1,3], which protect one-qubit information against phase damping and exchange errors, have been proposed. Compared with these schemes, the present method has the advantage of not requiring error correction or error detection. Moreover, compared with the schemes [18–20], the present method requires less qubit resource in protecting the entangled states (17) and (18), or in protecting n-qubit information $(n \geqslant 5)$. Thus, the present method is more efficient, although one has to have the two qubits in each pair close to each other and all the pairs well separated.

Finally, according to the above description, for each pair leakage out of the encoding subspace spanned by Eq. (15), due to amplitude damping, can be suppressed by frequent tests on each pair. Thus, for a general environment, n-qubit information or above n-qubit entangled states can also be protected by encoding them into the above DF states plus the Zeno effect.

In conclusion, we have presented an error-prevention scheme for protecting one-qubit information against decoherence due to a general environment and local exchange interactions. As shown above, the present error-prevention procedures are relatively simple. We have discussed how to construct DF states for n-qubit information protecting against collective phase and exchange errors. Moreover, we have shown that certain kinds of important entangled states of n (distant) qubits can be protected, by pairing each entangled qubit with only one ancilla qubit and applying only local operations on each pair.

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