

# Essays on Monotone Comparative Statics for Constrained Optimization Problems with Applications

By

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## Abstract

My dissertation consists of three chapters. Chapters 1 and 2 both address extensions to existing monotone comparative statics results for constrained optimization problems using lattice programming techniques. Chapter 3 applies monotone comparative statics results to the question how environmental regulation affects investment in innovation in imperfectly competitive markets.

Generally we can distinguish between two types of comparative statics problems. The first type of problem considers the change of the optimal solution to a maximization problem as the objective function changes, the other type considers the change due to a change in the constraint set. Lattice-based comparative statics theorems have been developed for both types of problems in the literature. The strengths of these lattice-theoretic comparative statics results are that they don't depend on the usual smoothness, interiority and concavity assumptions as required by the classical approach based on the Implicit Function Theorem, as well as convexity of the constraint set. Moreover, these comparative statics results also apply in the case of non-unique solutions. Quah (2007) expanded existing results by Milgrom and Shannon (1994) by making them applicable to some non-lattice constraint sets. In the first chapter, I extend existing comparative statics theorems to parametrized objective functions and non-lattice constraint sets. This generalization makes it possible to analyze a variety of economic optimization problems that fall into this class of problems which cannot be addressed using existing lattice-based techniques. I provide examples from consumer theory, producer theory and environmental economics that show the result's broad scope of applications.

The second chapter studies monotone comparative statics with respect to price changes in the consumer's utility maximization problem. Most attempts to derive this property rely on aspects of the demand curve, and it has been hard to derive this property using assumptions on the primitive utility function. Using new results on the comparative statics of demand in Quah (2007), I provide simple and easy conditions on utility functions that yield the gross substitutes property. Quah (2007) provides conditions on utility functions that yield normal demand. I add an assumption on elasticity of marginal rate of substitution, which combined with Quah's assumptions yields gross substitutes. I apply this assumption to the family of constant elasticity of substitution preferences.

My approach is grounded in the standard comparative statics decomposition of a change in demand due to a change in price into a substitution effect and an income effect. Quah's assumptions are helpful to sign the income effect. Combined with the elasticity assumption, we can sign the overall effect. As a by-product, I also present conditions which yield the gross complements property.

Chapter 3 is an application of monotone comparative statics results to the question how environmental regulation affects incentives for R&D investment. For decades, there has been debate among economists whether environmental regulation hurts firms by restricting their choices or provides them with a comparative advantage through investment in innovation in more efficient technology. This chapter studies the effect of environmental regulation on firms' investment in R&D in imperfectly competitive output markets using a monotone comparative statics approach. I provide necessary and sufficient conditions on the profit function that guarantee nondecreasing R&D investment as regulation is tightened and find that a form of weak complementarity between environmental R&D investment and the policy variable plays a crucial role. Moreover, I provide properties of such profit functions through assumptions on demand, Cournot output and cost functions.

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# Chapter 1

## Extending the Scope of Monotone Comparative Statics Results

### 1.1 Introduction

Comparative statics of constrained optimization problems is a question at the heart of economic analysis. Oftentimes we are not only interested in the optimizers of the problem themselves, but also how they will be affected by changes in exogenous parameters. Generally, we distinguish between two types of comparative statics problems. The first type of problem considers the change of the optimal solution to a maximization problem as the objective function changes, the other type the change due to changes in the constraint set.

Comparative statics theorems based on lattice programming methods have been developed for both types of problems. The strengths of these results are that they don't depend on the usual smoothness, interiority and concavity assumptions as required by the classical approach based on the Implicit Function Theorem, as well as convexity of the constraint set. Moreover, these comparative statics results also apply in the case of non-unique solutions.

Based on Topkis (1978), Vives (1990) and Milgrom and Roberts (1990) use a lattice-based approach to establish monotone increasing best responses in games with strategic complements

under the cardinal assumptions of supermodularity and increasing differences on the objective function when strategy spaces are lattices. Milgrom and Roberts (1990) also focus on monotone comparative statics results under these cardinal assumptions in supermodular games. Milgrom and Shannon (1994) then extend these results to the ordinal case. They provide necessary and sufficient conditions on the primitives for nondecreasing solutions in the parameters of the problem, generalizing the previously used cardinal counterparts. In many cases however, constraint sets in economic optimization problems, such as the budget set in the consumer problem, are not lattices and therefore Milgrom and Shannon's result cannot be applied. Quah (2007) addresses these types of problems and develops comparative statics with respect to changes in non-lattice constraint sets. His result provides necessary and sufficient conditions for nondecreasing solutions using a weaker set order, that in particular, can be used to establish normality of demand under assumptions on the utility function.

The contribution of this chapter is that it extends Quah's result to include both non-lattice constraint sets and parametrized objective functions. The result shows its power through many applications.

Quah (2007) generalizes the monotonicity theorem of Milgrom and Shannon (1994) to some types of non-lattice constraint sets, i.e. the consumer's budget set. However, his result does not address comparative statics with respect to parameters in the objective function. In economics, we naturally encounter a variety of questions that involve parameter changes in the objective function, such as changes in consumer preferences or technology shocks that affect production costs, when the constraint sets are not lattices. My result extends Quah's work to these types of optimization problems by including parametrized objective functions.

Importantly, this generalization allows us to analyze a variety of economic optimization problems that fall into this class of problems which cannot be addressed using existing lattice-based techniques. A natural example from consumer theory is the consumer's utility maximization problem with Stone-Geary preferences. The question, how income and essential consumption basket changes affect consumer demand, for example, cannot be addressed by existing results.

My comparative statics theorem provides necessary and sufficient conditions for nondecreasing demand in this case. Moreover, I can use this comparative statics result to generalize known lattice-based versions of the LeChatelier principle, that expresses the idea that long run factor demand is more responsive to price changes than in the short run. In producer theory, my result can be applied to multiple-plant production problems and price discrimination with capacity constraints to answer the question when production quantities are nondecreasing in demand or cost parameters and the constraint set. Many other applications can be found in the area of environmental economics such as production regulation through emissions standards and cost-efficient emissions regulation. Here the result provides necessary and sufficient conditions for nondecreasing factor demand or emissions reduction as demand, cost parameters and emissions standards change.

The remainder of this chapter is organized as follows: Section 2 introduces the theoretical framework, section 3 gives the generalized monotone comparative statics results and section 4 provides a variety of applications.

## 1.2 Preliminaries

Monotone comparative statics results rely on order theoretical concepts from lattice theory. A lattice  $X$  is a partially ordered set in which any two elements  $x$  and  $y$  have a supremum ( $x \vee y$ ) and an infimum ( $x \wedge y$ ). To obtain monotone comparative statics results with respect to changes in the constraint set we need to be able to order these sets. Milgrom and Shannon (1994) uses the strong set order by Veinott (1989), while Quah (2007) introduces a weaker concept to extend the comparative statics results from Milgrom and Shannon (1994) to problems with non-lattice constraint sets. This case is easily encountered even in simple optimization problems like the consumer's utility maximization problem.

**Definition 1. Set Orders**

1. Strong Set Order

Let  $X$  be a lattice and consider two subsets  $S$  and  $S'$ . Then  $S'$  dominates  $S$  in the strong set order ( $S' \geq_S S$ ) if and only if for all  $x \in S$  and for all  $y \in S'$ ,  $x \wedge y \in S$  and  $x \vee y \in S'$ .

2.  $(\Delta, \nabla)$ -induced Strong Set Order

Let  $X$  be a partially ordered set and consider two non-empty subsets of  $X$ ,  $S$  and  $S'$ . The operations  $\Delta$  and  $\nabla$  induce the following set order on  $S$  and  $S'$ .  $S'$  dominates  $S$  by the  $(\Delta, \nabla)$ -induced strong set order ( $S' \geq_{\Delta, \nabla} S$ ) if and only if for all  $x \in S$  and for all  $y \in S'$ ,  $x \Delta y \in S$  and  $x \nabla y \in S'$ .

3.  $C_i$  ( $C$ )-flexible Set Order

Let  $X \subseteq R^l$  be a convex set and define the operations  $\nabla_i^\lambda$  and  $\Delta_i^\lambda$  on  $X$  as follows for  $\lambda$  in  $[0, 1]$ :

$$x \nabla_i^\lambda y = \begin{cases} y & \text{if } x_i \leq y_i \\ \lambda x + (1 - \lambda)(x \vee y) & \text{if } x_i > y_i \end{cases}$$

$$x \Delta_i^\lambda y = \begin{cases} x & \text{if } x_i \leq y_i \\ \lambda y + (1 - \lambda)(x \wedge y) & \text{if } x_i > y_i \end{cases}$$

(a) Let  $S'$  and  $S$  be subsets of the convex sublattice  $X$ . Then  $S'$  dominates  $S$  in the  $C_i$ -flexible set order ( $S' \geq_i S$ ) if for any  $x$  in  $S$  and  $y$  in  $S'$ , there exists  $(\nabla_i^\lambda, \Delta_i^\lambda)$  in  $C_i$  such that  $x \nabla_i^\lambda y$  is in  $S'$  and  $x \Delta_i^\lambda y$  is in  $S$ .

(b)  $S'$  dominates  $S$  in the  $C$ -flexible set order ( $S' \geq S$ ) if  $S' \geq_i S$  for all  $i$ .

To add some intuition graphically to the operations  $\nabla_i^\lambda$  and  $\Delta_i^\lambda$ , the points  $x$ ,  $y$ ,  $x\Delta_i^\lambda y$  and  $x\nabla_i^\lambda y$  form a backward-bending parallelogram instead of the rectangle generated by  $x$ ,  $y$  and their join and meet (see Figure 1.1). Notice that  $x\Delta_i^\lambda y$  and  $x\nabla_i^\lambda y$  are linear combinations of  $y$  and  $x \wedge y$  or  $x \wedge y$  and  $x$  respectively. As  $\lambda$  increases from 0 to 1,  $x\Delta_i^\lambda y$  moves from  $x \wedge y$  towards  $y$ . Likewise, increasing  $\lambda$  moves  $x\nabla_i^\lambda y$  closer to point  $x$ .

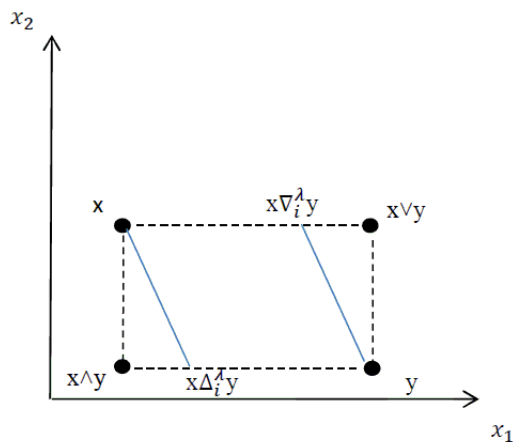


Figure 1.1:  $C_i$ -flexible set order

Quah (2007) describes the requirements of the  $C_i$ -flexible set order as that for any pair of unordered points  $x \in S$  and  $y \in S'$ , we can find  $x\Delta_i^\lambda y$  and  $x\nabla_i^\lambda y$  such that  $x\Delta_i^\lambda y$  is in  $S$  and  $x\nabla_i^\lambda y$  is in  $S'$ , with the four points forming a backward-bending parallelogram.

Besides an order on the constraint sets, we also need assumptions on the objective function for monotone comparative statics. The monotonicity theorem in Milgrom and Shannon (1994) requires quasisupermodularity, which can be interpreted as weak complementarity between the choice variables. Quah (2007) introduces a similar property needed when extending Milgrom and Shannon's result to a non-lattice context.

**Definition 2.**  $(\Delta, \nabla)$ -(Quasi)Supermodularity

Let  $X, T$  be partially ordered sets and let  $\Delta$  and  $\nabla$  be two operations on  $X$ . Consider a function  $f : X \rightarrow \mathbb{R}$ . Then define the following properties for  $f$ :

1.  $f$  is  $(\Delta, \nabla)$ -supermodular if for all  $x, y \in X$ ,  $f(x \nabla y) - f(y) \geq f(x) - f(x \Delta y)$ .
2.  $f$  is  $(\Delta, \nabla)$ -quasisupermodular if for all  $x, y \in X$ ,  $f(x) \geq (>)f(x \Delta y) \Rightarrow f(x \nabla y) \geq (>)f(y)$ .

Analogously to the previous case, Quah defines properties of a function  $f : X \rightarrow \mathbb{R}$ , where  $X \subseteq \mathbb{R}^l$  is a convex set, with respect to the operations  $\nabla_i^\lambda, \Delta_i^\lambda$ .

**Definition 3.**  $C_i$  ( $C$ )-(Quasi)Supermodularity

1. A function is  $(\nabla_i^\lambda, \Delta_i^\lambda)$ -supermodular for some  $\lambda \in [0, 1]$  if  $f(x \nabla_i^\lambda y) - f(y) \geq f(x) - f(x \Delta_i^\lambda y)$  for all  $x, y$  in  $X$ .
2. A function  $f : X \rightarrow R$  is  $C_i$ -supermodular if it is  $(\nabla_i^\lambda, \Delta_i^\lambda)$ -supermodular for all  $(\nabla_i^\lambda, \Delta_i^\lambda)$  in  $C_i = \{(\nabla_i^\lambda, \Delta_i^\lambda) : \lambda \in [0, 1]\}$ .  
If  $f$  is  $C_i$ -supermodular for all  $i$ , then it is  $C$ -supermodular.
3. A function  $f : X \rightarrow R$  is  $C_i$ -quasisupermodular if it is  $(\nabla_i^\lambda, \Delta_i^\lambda)$ -quasisupermodular for all  $(\nabla_i^\lambda, \Delta_i^\lambda)$  in  $C_i = \{(\nabla_i^\lambda, \Delta_i^\lambda) : \lambda \in [0, 1]\}$ , that is  $f(x) \geq (>)f(x \Delta_i^\lambda y) \Rightarrow f(x \nabla_i^\lambda y) \geq (>)f(y)$ .  
If  $f$  is  $C_i$ -quasisupermodular for all  $i$ , then it is  $C$ -quasisupermodular.

Intuitively, Definition 3 (1) requires the function's value to increase more along the right side than the left side of the parallelogram formed by these four points. Definition 3 (3) says that for all possible parallelograms that can be formed based on  $x$  and  $y$ , if the function is nondecreasing along the left side of the parallelogram, then it will also be nondecreasing along the right side.

Quah (2007) also shows that  $C_i$ -quasisupermodularity is implied by two properties that can easily be verified. The first one is supermodularity, the other a form of concavity.

**Definition 4.** *i*-Concavity

A function  $f$  is *i*-concave if it is concave in direction  $v \in \mathbb{R}^l$  for any  $v > 0$  with  $v_i = 0$ .

While Quah (2007) addresses comparative statics with regard to changes in the constraint set using this framework, Milgrom and Shannon (1994) also include comparative statics with respect to parameter changes in the objective function. Their result requires an additional assumption on the objective function, which is that for a parametrized objective function,  $f : X \times T \rightarrow \mathbb{R}$ , where  $X$  is a lattice and  $T$  is a partially ordered set,  $f$  needs to satisfy the Single Crossing Property in  $(x, t)$ . To work in the more general, non-lattice context, I introduce new versions of the Single Crossing Property to use throughout the remainder of the chapter.

**Definition 5.** Single Crossing Property

## 1. Standard Single Crossing Property

$f$  satisfies the Single Crossing Property (SCP) if for every  $x \leq y$  and for every  $t \leq t'$ ,  $f(y, t) \geq (>)f(x, t) \Rightarrow f(y, t') \geq (>)f(x, t')$ .

2.  $(\Delta, \nabla)$ -Single Crossing Property

$f$  satisfies the  $(\Delta, \nabla)$ -Single Crossing Property ( $(\Delta, \nabla)$ -SCP) if for every  $x, y$  with  $x \leq_{(\Delta, \nabla)} y$  and for every  $t \leq t'$ ,  $f(y, t) \geq (>)f(x, t) \Rightarrow f(y, t') \geq (>)f(x, t')$ .

3. *i*-Single Crossing Property

$f$  satisfies the *i*-Single Crossing Property (*i*-SCP) if for every  $x, y$  with  $x_i \leq y_i$  and for every  $t \leq t'$ ,  $f(y, t) \geq (>)f(x, t) \Rightarrow f(y, t') \geq (>)f(x, t')$ .

The standard Single Crossing Property is implied by increasing differences, which for twice continuously differentiable functions  $f$  is equivalent to  $\frac{\partial^2 f}{\partial x_i \partial t} \geq 0$  for all  $i$ . The following proposition shows how the *i*-SCP can be verified for the twice continuously differentiable case.

**Proposition 1.** Let  $f : \mathbb{R}^l \rightarrow \mathbb{R}$  be twice continuously differentiable. Then  $f$  satisfies the  $i$ -SCP if  $\frac{\partial^2 f}{\partial x_i \partial t} \geq 0$  and  $\frac{\partial^2 f}{\partial x_j \partial t} = 0$  for all  $j \neq i$ .

*Proof.* Let  $f : \mathbb{R}^l \rightarrow \mathbb{R}$  be twice continuously differentiable. If  $f$  is “nice”, then on  $\mathbb{R}^1$  for an interval  $[a, b]$ , the fundamental theorem of calculus gives us  $F(b) - F(a) = \int_a^b f(\lambda) d\lambda$  with  $f(\lambda) = F'(\lambda)$ . Suppose  $x$  and  $y \in \mathbb{R}^l$  with  $x_i \leq y_i$ . Let  $z(\lambda) = \lambda y + (1 - \lambda)x$ ,  $\lambda \in [0, 1]$ . Then  $f(x) : \mathbb{R}^l \rightarrow \mathbb{R}$  and  $f(z(\lambda)) : \mathbb{R}^1 \rightarrow \mathbb{R}$ .

We can then write  $f(y, t) - f(x, t) = f(z(1), t) - f(z(0), t) = \int_0^1 \frac{df}{d\lambda} d\lambda$ .

As  $\frac{df}{d\lambda} = \frac{\partial f}{\partial x_1} \cdot \frac{\partial x_1}{\partial \lambda} + \frac{\partial f}{\partial x_2} \cdot \frac{\partial x_2}{\partial \lambda} + \dots + \frac{\partial f}{\partial x_l} \cdot \frac{\partial x_l}{\partial \lambda}$ ,

$$f(y, t) - f(x, t) = \int_0^1 \left( \frac{\partial f}{\partial x_1} \cdot \frac{\partial x_1}{\partial \lambda} + \frac{\partial f}{\partial x_2} \cdot \frac{\partial x_2}{\partial \lambda} + \dots + \frac{\partial f}{\partial x_l} \cdot \frac{\partial x_l}{\partial \lambda} \right) d\lambda$$

and suppose  $f(y, t) - f(x, t) \geq 0$  for every  $x, y$  with  $x_i \leq y_i$ . Differentiating this difference with respect to  $t$  yields

$$\frac{d(f(y, t) - f(x, t))}{dt} = \int_0^1 \left( \frac{\partial^2 f}{\partial x_i \partial t} \cdot \frac{\partial x_i}{\partial \lambda} + \sum_{i \neq j} \frac{\partial^2 f}{\partial x_j \partial t} \cdot \frac{\partial x_j}{\partial \lambda} \right) d\lambda.$$

Under the assumptions that  $\frac{\partial^2 f}{\partial x_i \partial t} \geq 0$  and  $\frac{\partial^2 f}{\partial x_j \partial t} = 0$  for all  $j \neq i$ ,  $\frac{d(f(y, t) - f(x, t))}{dt} \geq 0$ , which implies the  $i$ -SCP. □

### 1.3 Comparative Statics

Let  $f : X \times T \rightarrow \mathbb{R}$ , where  $X$  is a set,  $T$  is a partially ordered set and  $S \subset X$ . Consider the following general constrained optimization problem:

$$\max f(x, t) \text{ subject to } x \in S$$

For convenience, denote  $M(t, S) := \operatorname{argmax}_{x \in S} f(x, t)$ .

The main theorem of this chapter generalizes the main monotone comparative statics result in Quah (2007) for optimization problems with constraint sets that can be ordered by the  $C_i$ -flexible set order to problems with parametrized objective functions. I provide necessary and sufficient



conditions for nondecreasing optimal solutions for parameter changes in the objective function and the above mentioned type of constraint sets; thus the main theorem of the chapter provides conditions on the objective function that characterize nondecreasing optimal solutions. The proofs follow Milgrom and Shannon (1994) with some simple modifications.

**Theorem 1.** *Let  $f : X \times T \rightarrow \mathbb{R}$ , where  $X \subset \mathbb{R}^l$ ,  $T$  is a partially ordered set and  $S \subset X$ .  $f(x, t)$  is  $C_i$ -quasisupermodular and has the  $i$ -Single Crossing Property in  $(x, t)$  if and only if whenever  $S' \geq_i S$  and  $t' \geq t$ ,  $\operatorname{argmax}_{x \in S'} f(x, t') \geq_i \operatorname{argmax}_{x \in S} f(x, t)$ .*

*Proof.* ( $\Rightarrow$ ) Suppose  $S' \geq_i S$  for  $t' \geq t$  and let  $x \in M(t, S)$ ,  $y \in M(t', S')$ . As  $x \in M(t, S)$  and  $S \leq_i S'$ ,  $f(x, t) \geq f(x \Delta_i^\lambda y, t)$ . By  $C_i$ -quasisupermodularity of  $f$ , this implies  $f(x \nabla_i^\lambda y, t) \geq f(y, t)$  with  $x_i > y_i$ . As  $f$  also has the  $i$ -Single Crossing Property in  $(x, t)$ ,  $f(x \nabla_i^\lambda y, t) \geq f(y, t) \Rightarrow f(x \nabla_i^\lambda y, t') \geq f(y, t')$  for  $t' \geq t$ . Since  $y \in M(t', S')$  it follows that  $x \nabla_i^\lambda y \in M(t', S')$ .

Now suppose  $x \Delta_i^\lambda y \notin M(t, S)$  and hence  $f(x, t) > f(x \Delta_i^\lambda y, t)$ .  $C_i$ -quasisupermodularity of  $f$  implies  $f(x \nabla_i^\lambda y, t) > f(y, t)$  and by the  $i$ -Single Crossing Property it follows that  $f(x \nabla_i^\lambda y, t') > f(y, t')$  for any  $t' \geq t$ . This contradicts the assumption that  $y \in M(t', S')$ . Therefore,  $x \Delta_i^\lambda y \in M(t, S)$  and  $M(t, S) \leq_i M(t', S')$ .

( $\Leftarrow$ ) Fix  $t$ . Let  $x$  and  $y$  be two elements in  $X$  and suppose that  $f$  is not  $C_i$ -quasisupermodular for some  $\lambda^* \in [0, 1]$ . The only case we need to look at is when  $x_i > y_i$  and  $x$  and  $y$  are unordered. Also,  $x \Delta_i^{\lambda^*} y \neq x$  and  $x \nabla_i^{\lambda^*} y \neq y$ . Let  $S = \{x, x \Delta_i^{\lambda^*} y\}$  and  $S' = \{y, x \nabla_i^{\lambda^*} y\}$ . Then  $S' \geq_i S$ .

$C_i$ -quasisupermodularity of  $f$  can be violated in two ways. First, suppose  $f(x, t) \geq f(x \Delta_i^{\lambda^*} y, t)$ , but  $f(x \nabla_i^{\lambda^*} y, t) < f(y, t)$ . In this case  $x$  is a maximizer of  $f$  in  $S$  and  $y$  maximizes  $f$  uniquely in  $S'$ , which violates the  $i$ -increasing property (since  $x_i > y_i$ ). Alternatively, suppose  $f(x, t) > f(x \Delta_i^{\lambda^*} y, t)$ , but  $f(x \nabla_i^{\lambda^*} y, t) = f(y, t)$ . Now  $y$  maximizes  $f$  in  $S'$  while  $x$  is the unique maximizer in  $S$ . This again contradicts the  $i$ -increasing property. So  $f$  is  $C_i$ -quasisupermodular.

Now let  $S \equiv \{x, \bar{x}\}$  with  $x_i \leq \bar{x}_i$ . Then  $f(\bar{x}, t) - f(x, t) \geq 0$  implies  $\bar{x} \in M(t, S)$ . Since we know that  $M(t, S) \leq_i M(\bar{t}, S)$  for  $\bar{t} \geq t$  it follows that  $f(\bar{x}, \bar{t}) - f(x, \bar{t}) \geq 0$  for all  $\bar{t} \geq t$ .

Moreover,  $f(\bar{x}, t) - f(x, t) > 0$  implies that  $\bar{x} \in M(t, S)$  and is unique. Since  $M(t, S) \leq_i M(\bar{t}, S)$  for  $\bar{t} \geq t$  it follows that  $f(\bar{x}, \bar{t}) - f(x, \bar{t}) > 0$  for all  $\bar{t} \geq t$ . Thus  $f$  has the  $i$ -Single Crossing Property.  $\square$

Notice that the solution to the optimization may not be unique, as I have made no assumptions to guarantee uniqueness. In the case of a solution set,  $\underset{x \in S'}{\operatorname{argmax}} f(x, t')$  dominates  $\underset{x \in S}{\operatorname{argmax}} f(x, t)$  in the  $C_i$ -flexible set order. By Proposition 3 in Quah (2007), this implies that  $\underset{x \in S'}{\operatorname{argmax}} f(x, t')$  is  $i$ -higher<sup>1</sup> than  $\underset{x \in S}{\operatorname{argmax}} f(x, t)$ .

Similarly, consider the general case, where  $\Delta$  and  $\nabla$  are two operations on  $X$ . This result gives necessary and sufficient conditions for nondecreasing optimal solutions using the  $(\Delta, \nabla)$ -Single Crossing Property for operations  $\Delta$  and  $\nabla$  on  $X$ .

**Theorem 2.**  $f(x, t)$  is  $(\Delta, \nabla)$ -quasisupermodular and has the  $(\Delta, \nabla)$ -Single Crossing Property in  $(x, t)$  if and only if whenever  $S'$  dominates  $S$  in the  $(\Delta, \nabla)$ -induced set order ( $S' \geq_{(\Delta, \nabla)} S$ ) for  $t' \geq t$ ,  $\underset{x \in S'}{\operatorname{argmax}} f(x, t') \geq_{(\Delta, \nabla)} \underset{x \in S}{\operatorname{argmax}} f(x, t)$  ( $\underset{x \in S}{\operatorname{argmax}} f(x, t)$  is nondecreasing in  $(t, S)$ ).

*Proof.* See Appendix A.  $\square$

To demonstrate the applicability of his result, Quah (2007) also shows that his new concept of  $C_i$ -supermodularity arises from the combination of supermodularity and a form of concavity, two assumptions that are reasonable under many circumstances.

**Proposition 2.** (Quah)

*The function  $f : X \rightarrow R$  is  $C_i$ -supermodular if it is supermodular and  $i$ -concave.*

This proposition allows us to easily verify  $C_i$ -supermodularity, which is important in applications. While Quah's result is able to address problems with non-lattice constraint sets, the number of applications remains limited as it does not include commonly occurring parametrized objective

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<sup>1</sup>Quah (2007) defines that a set  $S'$  is  $i$ -higher than a set  $S$ , if whenever both sets are nonempty, for any  $x \in S$  there exists  $x' \in S'$  such that  $x'_i \geq x_i$  and for any  $x' \in S'$  there exists  $x \in S$  such that  $x'_i \geq x_i$ .

functions. My monotone comparative statics theorem is able to address these types of problems and opens the door to a variety of applications.

## 1.4 Applications

Applications for my monotone comparative statics result can be found in many different areas of economics. In consumer theory, parametrized utility functions such as Stone-Geary preferences are very natural applications. I also provide a lattice-based LeChatelier Principle that generalizes existing versions and an example regarding an individual's labor supply. Multi-plant production allocation and price discrimination with capacity constraints are examples from producer theory. Finally, many applications can also be found in environmental economics, i.e. production decisions and technological change under emissions standards, the effect of changes in the ethanol quota and the cost-efficient emissions allocation.

### 1.4.1 Parametrized Consumer Utility Maximization Problem

The standard utility maximization problem as discussed in Quah (2007) can be extended by parametrization of the utility function by some parameter  $\theta$ . The parameter vector of this problem is multidimensional,  $t = (\theta, w)$ .

As shown in Quah (2007), the budget set for  $w' \geq w$  does not dominate the initial one by the strong set order, because the join of two arbitrary elements may lie outside of the larger set. However,  $B(p, w')$  does dominate the smaller budget set  $B(p, w)$  in the  $C$ -flexible set order as illustrated in Figure 1.2. We see that for any (unordered) pair of  $x \in B(p, w)$  and  $y \in B(p, w')$ , we can find some  $\lambda \in [0, 1]$  such that the points  $x$ ,  $y$ ,  $x\Delta_i^\lambda y$  and  $x\nabla_i^\lambda y$  form a backwards bending parallelogram with  $x\Delta_i^\lambda y$  in the smaller budget set and  $x\nabla_i^\lambda y$  in the larger budget set.<sup>2</sup>

For my comparative statics theorem to apply, the utility function needs to satisfy  $C_i$ -quasisupermodularity and the  $i$ -Single Crossing Property. If these conditions are satisfied, by Theorem 1, we

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<sup>2</sup>The case of ordered  $x$  and  $y$  is trivial, therefore limit attention to the unordered case.

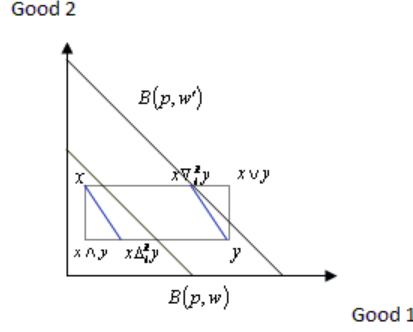


Figure 1.2: Budget set

have nondecreasing solutions to the consumer's utility maximization problem.

The class of Stone-Geary utility functions is an example of such parametrized utility functions and in the following possible interpretations of the parameter  $\theta$  will be discussed.

### Example 1. Stone-Geary Utility

Consider utility functions of the form  $u(x) = \sum_{i=1}^n \alpha_i \log(x_i - b_i)$  with  $\alpha_i > 0$ ,  $x_i - b_i > 0$  and  $\sum_{i=1}^n \alpha_i = 1$ . In this class of utility functions, the parameters  $b_1, \dots, b_n$  can be interpreted as a necessary consumption basket. An interesting question in this context is how the consumer's demand for a good changes when his income and also his necessary consumption basket increase. Thus,  $t = (b, w) \leq t' = (b', w')$ . Theorem 1 readily provides comparative statics results for this case. We can easily check that  $u$  is  $C_i$ -quasisupermodular<sup>3</sup> and satisfies the  $i$ -Single Crossing Property.<sup>4</sup> Thus, since  $B(p, w') \geq_i B(p, w)$ , my result guarantees nondecreasing demand for good  $i$  as income and necessary consumption basket go up. Notice that this comparative statics result also applies for changes in the parameter vector  $b$  only. Then my theorem yields nondecreasing demand in  $b$  as income remains unchanged. This type of comparative statics could not have been analyzed using earlier lattice-based results, as Milgrom and Shannon (1994) does not apply in cases where the constraint set is not a lattice, and Quah (2007) does not consider parameter changes in the

<sup>3</sup> $u$  is  $C_i$ -quasisupermodular if it is supermodular and  $i$ -concave. In this case, I see that  $u$  is supermodular since  $\frac{\partial^2 u}{\partial x_i \partial x_j} = 0, \forall i, j, i \neq j$ . Moreover,  $u$  is  $i$ -concave as  $\frac{\partial^2 u}{\partial x_i^2} = -\frac{\alpha_i}{(x_i - b_i)^2} \leq 0, \forall i$ .

<sup>4</sup> $\frac{\partial^2 u}{\partial x_i \partial b_i} = \frac{\alpha_i}{(x_i - b_i)^2} \geq 0, \frac{\partial^2 u}{\partial x_j \partial b_i} = 0$

objective function.

An application of Stone-Geary preferences can be found in Harbaugh (1998), that analyzes the prestige motive behind charitable donations. His paper considers two possible types of benefits to the donor, intrinsic benefit and prestige benefit. His utility hence depends on the amount of the public good  $x$  that he consumes, prestige  $p$  and amount donated  $d$ . The donor's utility maximization problem can then be written as

$$\max u(x, p, d) = \log x + b \log(p + k_1) + c \log(d + k_2)$$

subject to the budget constraint  $x + qd \leq w$ , where  $q$  is the after-tax price of giving and  $k_1$  and  $k_2$  are non-negative constants that capture how much the individual values prestige and his intrinsic benefit. In the framework of the above discussion of Stone-Geary utility,  $-k_1$  and  $-k_2$  are the necessary amounts of prestige and intrinsic benefit for the donor. The amount of prestige resulting from a donation depends on how charities report donations and how society converts these reports into prestige. Harbaugh (1998) considers different reporting schemes by the charities, one of which is exact reporting. In this case, prestige is equal to the amount donated ( $p = d$ ). The optimization problem can then be simplified to two variables of choice, the amount of the public good  $x$  and the donation  $d$ . The main comparative statics question that Harbaugh's paper wants to address is the impact of the prestige motive on donations, but it also provides results for the effect of income.

Notice that the question how a person's increased preference for prestige affects the amount donated cannot be addressed using existing monotone comparative statics results. Milgrom and Shannon's monotonicity theorem applies to parameter changes in the objective functions when the constraint set is a lattice, which this budget set clearly is not. Quah's result is able to address the isolated effect of an income change, but does not apply to parameter changes in the utility function and hence cannot answer the question regarding the prestige motive for donations.

My comparative statics theorem however is able to address parameter changes in the objective function when constraint sets are not lattices. As shown above, this class of utility functions satisfies  $C_i$ -quasisupermodularity and has the  $i$ -SCP in  $-k_i$ . Since  $B(q, w') \geq_i B(q, w)$ , the optimal

donation is nondecreasing in  $w$  and  $-k_i$  by Theorem 1. Hence, if a person puts more emphasis on prestige (i.e.  $-k_1$  increases) as his wealth increases, my comparative statics result yields nondecreasing donations, which is what the empirical analysis in Harbaugh (1998) concludes as well. We see from the data in his paper, that the model with the higher parameter estimate for prestige preference predicts higher donations and more donations as income increases. Overall, the theoretical predictions of my result are supported by the empirical findings in Harbaugh (1998).

### 1.4.2 LeChatelier Principle

**Example 2.** The LeChatelier principle in economics expresses the idea that long run demand is more responsive to price changes than demand in the short run. For example, consider the impact of an input price reduction on a firm's demand for that particular input. The LeChatelier principle now says, that the demand increase in the short run, when some factors of production are fixed, is smaller than the increase in the long run, when all factors can be adjusted freely. Milgrom and Roberts (1996) introduces a lattice-theory based global LeChatelier principle, which is extended to additional classes of constraints faced by the firm in the short run in Quah (2007).

Quah (2007) points out the two different types of comparative statics problems associated with the short run and long run factor demand adjustment following a price change. In the short run, the firm's objective function changes while the constraint set, based on the optimal point before the price change, stays the same. Between the short run and the long run, the objective function remains the same, but the short run constraints no longer exist; hence the constraint set changes.

Thus consider the following two increases of the parameter vector  $t = (\theta, S)$ , consisting of the cost parameter  $\theta$  and the constraint set  $S$ . In the short run,  $t$  increases to  $t' = (\theta', S) \geq t$ , as the cost parameter changes. As the constraint set changes in the long run, the parameter vector increases again from  $t'$  to  $t'' = (\theta', S')$ .

The firm's objective function can be written as  $\pi(x, \theta) = V(F(x)) - C(x, \theta)$ , where  $x$  is the firm's input vector,  $F(x)$  its production function and  $V$  the revenue from output. Quah (2007) specifies the firm's cost function as  $C(x, \theta) = p \cdot x - \theta x_1$ , where  $p$  denotes the input price vector

and  $\theta > 0$  is the considered price reduction for input 1. In the short run the firm faces constraints, since not all inputs can be varied without costs. Thus, this constraint set  $S$  needs to include the pre-price change optimal input vector  $x^*$ . So,  $S = \{x \in \mathbb{R}_+^l \mid x^* \in S\}$ . Therefore the short run adjustment from  $x^*$  to  $x^{**}$  is a comparative statics problem where the parameter  $\theta$  in the objective function changes, but the constraint set remains unchanged at  $S$ . In the long run the change of the constraint set from  $S$  to  $S'$  leads to the change in input demand from  $x^{**}$  to  $x^{***}$ .

The proof of the following proposition is in parts similar to the proof of Proposition 8 in Quah (2007).

**Proposition 3.** *Let  $x^*$  maximize  $\pi(x, \theta)$  subject to  $x \in \mathbb{R}_+^l$ . Suppose also,  $x^{**}$  and  $x^{***}$  are solutions to the problems*

1. *maximize  $\pi(x, \theta')$  subject to  $x \in S$  and*
2. *maximize  $\pi(x, \theta')$  subject to  $x \in S' = \mathbb{R}_+^l$*   
*where  $\theta' \geq \theta$ .*

*Then  $x_i^{***} \geq x_i^{**} \geq x_i^*$  if  $\pi$  is  $C_i$ -quasisupermodular, satisfies the  $i$ -Single Crossing Property and  $X_S = \{x \in \mathbb{R}_+^l \mid x \geq s \text{ for some } s \in S\} \geq_i S$ .*

*Proof.* First consider the increase from  $t = (\theta, S)$  to  $t' = (\theta', S)$ . Since  $\pi$  is  $C_i$ -quasisupermodular, satisfies the  $i$ -Single Crossing Property and the constraint set remains unchanged,  $x_i^{**} \geq x_i^*$  by Theorem 1.

Now  $t'$  increases to  $t'' = (\theta', S')$ . Given that  $\pi$  is  $C_i$ -quasisupermodular, has the  $i$ -SCP and  $\text{argmax}_{x \in S'} \pi(x, \theta')$  exists, there is  $\bar{x} \in \text{argmax}_{x \in S'} \pi(x, \theta')$  such that  $\bar{x} \geq x^*$ .

Since  $x^* \in S$ ,  $\bar{x} \in \text{argmax}_{x \in X_S} \pi(x, \theta')$ . Moreover, we know that  $x^{**} \in \text{argmax}_{x \in S} \pi(x, \theta')$ .

Since  $X_S \geq_i S$ , by Theorem 1 there is  $x^{***} \in \text{argmax}_{x \in X_S} \pi(x, \theta')$  such that  $x_i^{***} \geq x_i^{**}$ .

Lastly,  $x^{***} \in \text{argmax}_{x \in S'} \pi(x, \theta')$ , because  $\bar{x} \in X_S$  and  $\bar{x} \in \text{argmax}_{x \in S'} \pi(x, \theta')$ . □

We can easily check that the above objective function is  $C_i$ -quasisupermodular if  $V \circ F$  is  $C_i$ -quasisupermodular. In the case of a perfectly competitive output market, this is the case when all inputs are complementary to each other and the production function displays decreasing marginal product for each input.<sup>5</sup> Moreover, the objective function needs to satisfy the  $i$ -SCP. This is the case whenever marginal costs of production are nonincreasing in  $\theta$  and the cost function is additively separable.<sup>6</sup>

The advantages of this version of the LeChatelier principle are that like Milgrom and Roberts (1996), it allows for more general parameter changes than Quah (2007) (who only considers the very specific case described above) while also permitting larger classes of constraint sets than Milgrom and Roberts (1996) like Quah (2007). The result in Milgrom and Roberts (1996) is limited to constraint sets  $S$  and  $X_S$  that can be ranked in the strong set order. This is the case, for example, when certain inputs are held constant in the short run. Quah (2007) however allows for more general, economically plausible constraint sets. For example, suppose there is one good that serves different roles in the production process and is therefore considered as several inputs. If in the short run, the total quantity used of these inputs cannot be changed, the constraint set can be written as  $S = \{x \in \mathbb{R}_+^l \mid \sum x_i = \sum_{i=m}^l x_i^*\}$ . In this case,  $X_S$  and  $S$  cannot be ranked by the strong set order; however,  $X_S$  dominates  $S$  in the  $C$ -flexible set order.

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<sup>5</sup>For supermodularity, I need  $\frac{\partial^2 \pi}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_j} \left( \frac{\partial V}{\partial F} \cdot \frac{\partial F}{\partial x_i} \right) \geq 0$ . Assuming a perfectly competitive output market,  $\frac{\partial V}{\partial F} = P$ , where  $P$  is the market price of output. Then  $\frac{\partial^2 \pi}{\partial x_i \partial x_j} = P \cdot \frac{\partial^2 F}{\partial x_i \partial x_j}$ , which is nonnegative if  $\frac{\partial^2 F}{\partial x_i \partial x_j} \geq 0$ .

For  $i$ -concavity in the simple two goods case with a linear cost function, I need  $\frac{\partial^2 \pi}{\partial x_j^2} \leq 0$ , for all  $j \neq i$ . In a perfectly competitive environment,  $\frac{\partial^2 \pi}{\partial x_j^2} = P \cdot \frac{\partial^2 F}{\partial x_j^2}$ , which is nonpositive if  $\frac{\partial^2 F}{\partial x_j^2} \leq 0$ . In the general case, I need to verify that the Hessian of the production function,  $H_{-i}^F$ , is negative semi-definite and that the Hessian of the cost function,  $H_{-i}^C$ , is positive semi-definite.

<sup>6</sup>The  $i$ -SCP is satisfied if  $\frac{\partial^2 \pi}{\partial x_i \partial \theta} = -\frac{\partial^2 C(x, \theta)}{\partial x_i \partial \theta} \geq 0$  and the cost function is additively separable. This is the case whenever  $\frac{\partial MC(x, \theta)}{\partial \theta} \leq 0$ .



### 1.4.3 Labor Supply with Discrete Choices

**Example 3.** Another application of my comparative statics result is an individual's labor supply decision. In the microeconomic labor supply model, an individual maximizes utility coming from income on the one hand and leisure time on the other hand while being constrained by the maximum amount of working hours.

Let  $h$  and  $l$  denote working and leisure hours respectively and let  $w$  be the wage rate,  $\bar{I}$  non-labor income and  $\bar{T}$  the time available to the individual.<sup>7</sup> Then the utility maximization problem can be written as

$$\begin{aligned} \max \quad & u(wh + \bar{I}, l) \\ \text{subject to} \quad & h + l \leq \bar{T} \end{aligned}$$

Originally, labor supply models assumed that the individual's constraint set is convex, that is they choose from the continuum of working hours (see for example Hausman (1981)). More recent works, starting with Aaberge, Dagsvik and Strøm (1995), van Soest (1995) and Hoynes (1996), use a discrete choice model, where the individual chooses between different income-leisure combinations. As van Soest (1995) points out, hours worked is always an integer in the data and therefore a choice set where work-leisure combinations that are integers is a natural choice. So in the following, let  $S = \{(h, l) \in \mathbb{N}_0 \times \mathbb{N}_0 \mid h + l \leq \bar{T}\}$ .

Comparative statics questions of interest in this model are how changes in wage, non-labor income or time endowment affect the labor supply. To apply Theorem 1, we need to verify that  $u$  is  $C_i$ -quasisupermodular and satisfies the  $i$ -SCP in the respective variable and parameter. The former is satisfied whenever  $u$  is supermodular and  $i$ -concave, that is  $\frac{\partial^2 u}{\partial l \partial h} \geq 0$ ,  $\frac{\partial^2 u}{\partial h \partial l} \geq 0$ ,  $\frac{\partial^2 u}{\partial h^2} \leq 0$  and  $\frac{\partial^2 u}{\partial l^2} \leq 0$ .  $u$  has the  $i$ -SCP in  $(h, w)$  if  $\frac{\partial^2 u}{\partial h \partial w} \geq 0$  and  $\frac{\partial^2 u}{\partial l \partial w} = 0$  and similarly for the other variable and parameters. Moreover, even though the constraint set consists of discrete points, we can still order  $S(\bar{T})$  and  $S(\bar{T}')$  with  $\bar{T} \leq \bar{T}'$  in the  $C_i$ -flexible set order.

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<sup>7</sup>Assume that  $h, l, w, \bar{I}, \bar{T} \geq 0$

Hoynes (1996), for example, uses a Stone-Geary utility function. For

$$u(wh + \bar{I}, l) = \alpha_1 \ln(wh + \bar{I}) + \alpha_2 \ln(l) \text{ with } \alpha_1, \alpha_2 \geq 0,$$

we can easily check that  $u$  is supermodular and  $i$ -concave.<sup>8</sup>  $u$  also satisfies the  $i$ -SCP in  $(h, -\bar{I})$ .<sup>9</sup> Thus labor supply increases as non-labor income decreases. Moreover, we can verify that  $u$  also satisfies the  $i$ -SCP in  $(h, w)$ .<sup>10</sup> Hence these preferences suggest that labor supply increases in the wage rate.

Notice that the existing monotone comparative statics results cannot be applied in this case. Milgrom and Shannon (1994) addresses changes of parameters in the objective function such as wage or non-labor income in this example, however their result requires the constraint set to be a lattice, which this constraint set is not. Quah (2007) can address changes in this type of constraint set, however his result does not account for parameter changes in the objective function.

#### 1.4.4 Multiple-Plant Production

**Example 4.** Another example that my comparative statics result can be applied to is production allocation in a multiple-plant firm. Basic two firm models can be found in Patinkin (1947) and Sattler and Scott (1982). In the latter, the firm is allocating production of a given production target  $\bar{q}$  between an old and a new plant, where the new plant has lower costs than the old one. The firm faces the following cost minimization problem:

$$\min C(q_{new}, q_{old}) = C_{new}(q_{new}) + C_{old}(q_{old})$$

$$\text{subject to } q_{new} + q_{old} \geq \bar{q}.$$

Both Patinkin (1947) and Sattler and Scott (1982) mostly focus on the “broken” marginal cost curve as the firm switches from only operating one plant to both plants as total output increases. My

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<sup>8</sup>  $u$  is supermodular as  $\frac{\partial^2 u}{\partial l \partial h} = \frac{\partial^2 u}{\partial h \partial l} = 0$  and  $i$ -concave since  $\frac{\partial^2 u}{\partial h^2} = -\alpha_1 \frac{w^2}{(wh + \bar{I})^2} \leq 0$  and  $\frac{\partial^2 u}{\partial l^2} = -\alpha_2 \frac{1}{l^2} \leq 0$

<sup>9</sup>  $\frac{\partial^2 u}{\partial h \partial (-\bar{I})} = \alpha_1 \frac{w}{(wh + \bar{I})^2} \geq 0$ ,  $\frac{\partial^2 u}{\partial l \partial (-\bar{I})} = 0$

<sup>10</sup>  $\frac{\partial^2 u}{\partial h \partial w} = \alpha_1 \frac{1}{(wh + \bar{I})^2} \geq 0$ ,  $\frac{\partial^2 u}{\partial l \partial w} = 0$

comparative statics theorem on the other hand adds more general insights under what assumptions on the cost function exogenous changes will lead to increased production in both plants (in the case that both of them are in use).

Since we are interested in comparative statics results, I include parameters  $\omega_{new}$  and  $\omega_{old}$  in the cost functions that measure the level of technology (with  $\frac{\partial MC_i}{\partial \omega_i} \leq 0$ ) and rewrite the problem as

$$\max \quad \tilde{C}(q_{new}, q_{old}, \omega_{new}, \omega_{old}) = -[C_{new}(q_{new}, \omega_{new}) + C_{old}(q_{old}, \omega_{old})]$$

$$\text{subject to } q_{new} + q_{old} \geq \bar{q}.$$

Now consider a situation where production technology in the old plant gets updated and the firm increases its total production target. Hence,  $t' = (\omega'_{new}, \omega'_{old}, \bar{q}') \geq t$ , where  $t'$  is the new and  $t$  the initial parameter vector.

Since the constraint sets are not lattices, the comparative statics result by Milgrom and Shannon (1994) does not apply here. The result in Quah (2007), while allowing for non-lattice constraint sets, does not address parameter changes in the objective function. My result on the other hand is able to handle these types of problems.

We easily see that the new constraint set  $S(\bar{q}')$  dominates the original one  $S(\bar{q})$  in the  $C_i$ -flexible set order. Assuming that the cost function at each plant is convex in quantity, the objective function is  $C_i$ -quasisupermodular and satisfies the  $i$ -SCP for all  $i$ .<sup>11</sup> As a result, my comparative statics theorem yields nondecreasing optimal production quantities for either plant in this case.

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<sup>11</sup>  $\tilde{C}$  is  $C_i$ -quasisupermodular if it is supermodular and  $i$ -concave. As each cost function is independent of the quantity produced at the other plant,  $\frac{\partial^2 \tilde{C}}{\partial q_i \partial q_j} = 0$  for  $i \neq j$ , thus  $\tilde{C}$  is supermodular. Moreover, assuming convex cost functions,  $\frac{\partial^2 \tilde{C}}{\partial x_i^2} \leq 0$  for all  $i$ . Thus  $\tilde{C}$  is  $C_i$ -quasisupermodular for all  $i$ .

Additionally, since marginal cost is nonincreasing in  $\omega_i$  at both plants,  $\frac{\partial^2 \tilde{C}}{\partial x_i \partial \omega_j} = -\frac{\partial MC_i}{\partial \omega_j} = 0$  for  $i \neq j$  and  $\frac{\partial^2 \tilde{C}}{\partial x_i \partial \omega_i} = -\frac{\partial MC_i}{\partial \omega_i} \geq 0$ ,  $\frac{\partial^2 \tilde{C}}{\partial x_j \partial \omega_i} = 0$  for all  $i$ . Therefore the  $i$ -SCP holds.

### 1.4.5 Price Discrimination with Capacity Constraints

**Example 5.** Another simple application of my comparative statics result can be found in the area of price discrimination with capacity constraints. In the airline and lodging industries, this problem is known as yield management. Belobaba (1987) summarizes yield management research in the airline industry; for examples from the lodging industry, see Hanks, Cross and Noland (1992). Reece and Sobel (2000) discusses the example of an airline that practices price discrimination while facing a capacity constraint<sup>12</sup> and addresses the question how airlines should adjust the allocation of seats between the customer groups as demand in one of the market segments increases or if costs of operation increase. They separately consider the cases of fixed non-binding and binding capacity, as well as the possibility of capacity adjustment. They find that in the case of non-binding capacity constraints changes in the marginal cost of operation directly affect prices and optimal quantities. In the case of binding capacity constraints, as long as marginal cost is below the point of intersection of the marginal revenue curves, changes in marginal cost do not affect the optimal capacity allocation; demand changes for one group however influence the optimal quantity and price of the other group. If the firm also optimally chooses capacity in the long run, prices will only rise or fall in the long run if marginal cost changes as capacity is adjusted. With my comparative statics result, I can easily address a variety of combinations of the above mentioned scenarios.

Consider an airline that price-discriminates between two groups of customers, i.e. business and leisure travelers. Since the number of seats on a plane is limited, the airline faces the following constrained profit maximization problem:

$$\max \quad \pi = p_L(q_L, \phi_L) \cdot q_L + p_B(q_B, \phi_B) \cdot q_B - C_L(q_L, \omega_L) - C_B(q_B, \omega_B)$$

$$\text{subject to } q_L + q_B \leq \bar{q}$$

Since we are interested in comparative statics, the demand functions and the cost function have been parametrized.  $\phi_L$  and  $\phi_B$  capture exogenous demand shocks such as holiday travel

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<sup>12</sup>This multiple fare class problem is also briefly discussed in Belobaba (1989).

or vacation time with  $\frac{\partial p_i}{\partial \phi_i} \geq 0$  for  $i = L, B$ . The cost function parameters  $\omega_L$  and  $\omega_B$  account for changes in transportation cost inputs such as fuel prices and assume marginal cost of transportation is nonincreasing in  $\omega_L$  and  $\omega_B$ . Moreover, assume linear demand functions with  $\frac{\partial p_i}{\partial q_j} \geq 0$ ,  $i = L, B$  and constant marginal costs of transportation.

A straightforward comparative statics question is how the firms optimal allocation between business and leisure travelers changes during peak travel season compared to normal traffic. In anticipation of higher demand, the airline increases its capacity by assigning larger planes to popular routes. Additionally, I can add a decrease in input costs, such as lower kerosene prices to the scenario. Then,  $t' = (\phi'_L, \phi'_B, \omega'_L, \omega'_B, \bar{q}') \geq t$ .

Once again, since the constraint sets are not lattices, the comparative statics result by Milgrom and Shannon (1994) does not apply here. The result in Quah (2007), while allowing for non-lattice constraint sets, does not address parameter changes in the objective function, while Theorem 1 on the other hand is able to handle these types of problems.

Clearly, the constraint set at higher capacity  $S(\bar{q}')$  dominates  $S(\bar{q})$  in the  $C_i$ -flexible set order. Under the above assumptions on the objective function,  $\pi$  is  $C_i$ -quasisupermodular and satisfies the  $i$ -SCP. Thus, by Theorem 1, the airline's optimal number of seats allotted to either leisure and business customers is nondecreasing.

When comparing these results to those in Reece and Sobel (2000), we see that when solely focusing on changes in marginal cost my comparative statics theorem yields the same conclusions in the cases of non-binding capacity constraints and when considering capacity adjustments.

Demand changes in the case of fixed and binding capacity are one aspect that cannot be addressed by my result, since at capacity, the marginal cost for an extra seat for either one of the market segments is the marginal revenue of the other. So if as in the above example demand for leisure travel increases during holidays,  $MR_L$  and thus  $MC_B$  increase. For my result to apply, we need  $t' \geq t$ , which is not consistent with demand in one market increasing and marginal cost of transportation in the other market increasing.

Besides this special case of binding and fixed capacity, my framework can address comparative

statics questions of a more general nature in this model, as it allows for other factors like demand shocks occurring in conjunction with transportation cost and capacity changes. Moreover, unlike the generally used comparative statics approach that requires uniqueness of solution, this result also applies in the case of multiple optimizers, as it could occur for piecewise profit functions that have a linear part.

### 1.4.6 Production Regulation by Efficiency Standards

Another area of application with a variety of examples is production regulation, particularly in environmental economics. Production can either be regulated by direct restrictions on production such as quotas or indirectly through efficiency standards, value restrictions, etc.

Production quotas directly impose a limit on the quantity a firm may produce to restrict supply and maintain a certain price level. The value of this limit is set by some regulatory agency, hence it depends on the strictness of the regulator. In the model, the rigidity of regulation is captured by the parameter  $\theta$ . Additionally, the firm's profit function  $\pi$  will also be parametrized by  $\phi$  and  $\omega$  to capture shifts in demand and changes to the firm's costs.

A very general version of the firm's constrained optimization problem can then be written as follows:

$$\begin{aligned} \max \quad & \pi = V(x_R, x_{UR}, \phi) - C(x_R, x_{UR}, \omega_R, \omega_{UR}) \\ \text{subject to } & x_R \leq \bar{q}(\theta), \end{aligned}$$

where  $x_R$  denotes the regulated commodities the firm produces and  $x_{UR}$  is the vector of all unregulated goods the firm produces.

Naturally, a question of interest is how policy changes, economic shocks or technological progress affect the firm's optimal output, that is what impact changes of the parameter vector  $t = (\phi, \omega_R, \omega_{UR}, \theta)$  have on the optimal solutions. Denote the constraint set depending on the parameter vector  $t$  by  $S(t) = \{x \in \mathbb{R}^l \mid x_R \leq \bar{q}(\theta)\}$ . It can easily be seen that for  $t' \geq t$  with  $\theta' \geq \theta$ ,

$S(t')$  dominates  $S(t)$  in the strong set order. This type of problem can be addressed using the result by Milgrom and Shannon (1994).

Instead of imposing a direct limit on the quantity that a firm may produce of a certain good, an alternative policy to monitor output is to establish efficiency standards. For an efficiency standard, a weighted sum of all commodities with efficiency based weights needs to lie below an upper bound determined by the regulator. The constraint set depending on the parameter vector in this case can be written as  $S(t) = \{x \in \mathbb{R}^l \mid \alpha \cdot x \leq \bar{\theta}\}$ , where  $\alpha$  represents either the vector of weights or prices. Notice that these constraint sets are not lattices like in the previous case of a production quota, hence the set at a higher parameter  $t'$  does not dominate the initial one at  $t$  in the strong set order and Milgrom and Shannon's result does not apply here. However,  $S(t')$  dominates  $S(t)$  in the  $C_i$ -flexible set order. These types of problems have first been addressed in Quah (2007), but his result does not consider parameter changes in the objective function. Therefore, the above described comparative statics problem that involves parameter changes in the objective function and a non-lattice constraint set cannot be addressed by existing results.

My comparative statics theorem however does apply in this case, given that the firm's objective function satisfies  $C_i$ -quasisupermodularity and the  $i$ -Single Crossing Property. For example, this is the case for simple linear demand and cost functions where demand for goods  $i$  and  $j$  is unrelated and marginal cost of good  $i$  is nonincreasing in the parameter  $\omega_i$  and marginal cost of good  $j$  is independent of  $\omega_i$ .<sup>13</sup>

**Example 6.** Consider a car producer that produces two types of cars, one with high fuel efficiency and one with low fuel efficiency. The produced quantities of each type are denoted  $x_H$  and  $x_L$ . Let  $\phi_H$ ,  $\phi_L$  and  $\theta$  be parameters that capture changes in demand for high and low efficiency cars and the strictness of regulation for the production of the fuel-inefficient model. Inverse demand  $p_i(x_i)$  for either car models increases in  $\phi_i$ , so  $\frac{\partial p_i}{\partial \phi_i} > 0$ . Moreover, let low values of  $\theta$  imply a "green" mindset, which results in stricter regulation and lower production limits.

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<sup>13</sup>See Appendix for a more detailed discussion of  $C_i$ -quasisupermodularity and the  $i$ -SCP for profit functions.

In the case of an efficiency standard, the optimization problem of the firm can be written as

$$\begin{aligned} \max \quad & \pi = p_H(\phi_H)x_H + p_L(\phi_L)x_L - C(x_H, x_L) \\ \text{subject to} \quad & \alpha_H x_H + \alpha_L x_L \leq \eta(\theta), \end{aligned}$$

where  $\alpha_H$  and  $\alpha_L$  are weights based on energy consumption and  $\eta(\theta)$  is an upper bound.

A question of interest now is how an increase in all parameters from  $t = (\phi_H, \phi_L, \theta)$  to  $t' > t$ , that represents a change to a generally less environmentally conscious attitude in society, affects the firm's optimal output for both types of cars. It seems straightforward that less emphasis on the environment leads to laxer regulation standards for inefficient cars. Also, a less "green" attitude by society increases the demand for cars. Demand for low efficiency cars is higher because people are not willing to buy the more expensive, but also more efficient cars. On the other hand, people that would not buy cars at all in the greener mindset and rely solely on public transportation, bicycles and walking may now buy high efficiency cars.

First of all, notice that the new constraint set for  $\theta' > \theta$  dominates the previous one in the  $C_i$ -flexible set order as illustrated in Figure 1.3, while these constraint sets cannot be ranked in the strong set order.

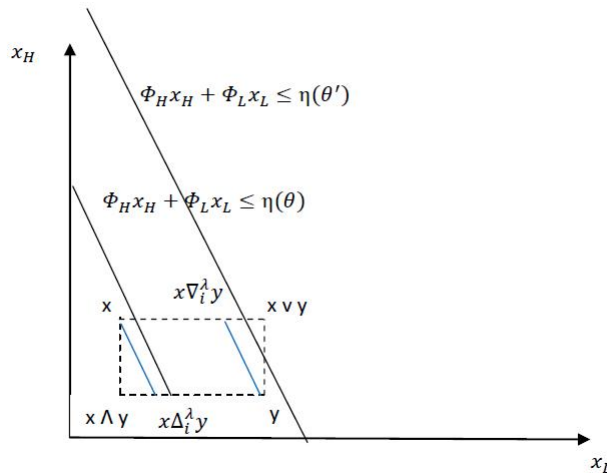


Figure 1.3: Efficiency Standard Constraint Sets



As previously pointed out, the profit function  $\pi$  is  $C_i$ -quasisupermodular if the demand and cost function are linear and if either demand of good  $i$  and good  $j$  are unrelated or if  $\frac{\partial p_i}{\partial x_j} \geq 0$  and  $\frac{\partial p_j}{\partial x_i} \geq 0$ . Clearly, this is the case if both markets are perfectly competitive and therefore prices are determined by the market and costs are linear.

Moreover, the profit function also needs to satisfy the  $i$ -Single Crossing Property. The conditions under which the profit function has the  $i$ -SCP are  $\frac{\partial^2 \pi}{\partial x_L \partial \phi_L} = \frac{\partial p_L}{\partial \phi_L} \geq 0$ ,  $\frac{\partial^2 \pi}{\partial x_H \partial \phi_L} = 0$ ,  $\frac{\partial^2 \pi}{\partial x_H \partial \phi_H} = \frac{\partial p_H}{\partial \phi_H} \geq 0$  and  $\frac{\partial^2 \pi}{\partial x_L \partial \phi_H} = 0$ . These conditions are consistent with the idea of the model that demand is increasing in the parameters  $\phi_L$  and  $\phi_H$  and independent of the other product's parameter.

Hence the conditions for Theorem 1 are satisfied and we have nondecreasing optimal solutions. Thus production of either high and low efficiency cars is nondecreasing if society cares less about reducing emissions and the environment.

Other possible interpretations of changes in the parameters  $\phi_L$  and  $\phi_H$  in this example are economic conditions such as changes in household wealth or more or less favorable interest rates that lead to increases or decreases in the demand for cars. Moreover, we can parametrize the cost function by  $\omega_H$  and  $\omega_L$  as well to account for changes in production technology or government taxes and subsidies for either type.

### 1.4.7 Emissions Standards and Production Decisions

**Example 7.** Helfand (1991) examines the effect of various different forms of emissions standards on a firm's optimal output decision. Some of their findings can be replicated and extended using my result. For example, Helfand (1991) finds that the direction of input adjustments after the introduction of a standard depends on the sign of the cross-partial of the production function, which is the change in the marginal product of one input as another one changes. As will be shown in the following, a nonnegative cross-partial of the production function is a sufficient condition for  $C_i$ -quasisupermodularity, which needs to be satisfied for my result to apply.

Consider a firm with a production function  $f(x_1, x_2)$  and output is nondecreasing with nonincreasing marginal returns for each input ( $\frac{\partial f}{\partial x_i} \geq 0$ ,  $\frac{\partial^2 f}{\partial x_i^2} \leq 0$  for all  $i$ ). During the production process

the firm also produces pollution  $A(x_1, x_2)$ . First consider the case where both inputs contribute to pollution, so  $\frac{\partial A}{\partial x_i} > 0$ . Furthermore, suppose the firm has a linear cost function and that the market for this good is perfectly competitive. Without any regulatory constraints the firm maximizes its profit

$$\pi = p \cdot f(x_1, x_2) - \omega_1 x_1 - \omega_2 x_2.$$

One type of standard discussed in Helfand (1991) is the case of a set amount of a specific input. Here, the regulator puts a limit on how much of a polluting input can be used in the production process or requires a minimum amount of a pollution-abating input. Suppose input 1 increases pollution, while input 2 reduces emissions.

First, suppose the regulator restricts the use of the polluting input. So the standard takes the form  $x_1 \leq \bar{A}_1$ . For  $\bar{A}_1' \geq \bar{A}_1$ , the constraint set  $S(\bar{A}_1')$  dominates  $S(\bar{A}_1)$  in the strong set order. Alternatively, a minimum amount of the pollution-reducing input could be required, so  $x_2 \geq \bar{A}_2$ . Again, for  $\bar{A}_2' \geq \bar{A}_2$ ,  $S(\bar{A}_2')$  dominates  $S(\bar{A}_2)$  in the strong set order.

While these types of standards can be addressed using the result of Milgrom and Shannon (1994), there are other ones that do not fall into this framework. For example, Helfand (1991) also considers the case where the regulator introduces an emissions standard  $\bar{A}$  that limits the total amount of allowed pollution by this firm in a given period of time. Moreover, assume that pollution is proportional to the amount of input used in the production process. We can then write this constraint as  $a_1 x_1 + a_2 x_2 \leq \bar{A}$ . As both inputs cause pollution,  $a_1, a_2 > 0$ .

With regard to comparative statics, the natural question to ask is how changes in the emissions standard affect the firm's input decision. With the comparative statics result in Theorem 1 I can go a little bit further and examine how changes in the parameter vector  $t = (p, -\omega_1, -\omega_2, \bar{A})$  affect the optimal input decision of the firm. Suppose  $t$  increases to  $t' = (p', -\omega_1', -\omega_2', \bar{A}') \geq t$ , that is the price for the finished product increases, input prices decrease and regulation is loosened. As mentioned above, problems with non-lattice constraint sets like this don't fall into the framework of Milgrom and Shannon (1994). Quah (2007) can also not be applied here, as his result does not account for parameter changes in the objective function.

To see that my comparative statics result applies to this case, we need to verify the assumptions of Theorem 1. Clearly, the new constraint set  $S(\bar{A}')$  dominates  $S(\bar{A})$  in the  $C_i$ -flexible set order. For Theorem 1 to apply, the objective function needs to satisfy  $C_i$ -quasisupermodularity and the  $i$ -SCP. For the above profit function, this will be the case given that  $\frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0, i \neq j$ .<sup>14</sup> Then the firm's optimal input quantities are nondecreasing in  $t$ .

Hence, when limiting attention to changes in the emissions standard, I find the same results as Helfand (1991). If  $\frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0$  and the Hessian is negative semi-definite, which in my approach is a sufficient condition for  $C_i$ -quasisupermodularity, more polluting inputs and less of emissions-abating inputs are used when standards are looser (or non-existent) than under stricter regulation. Moreover, my comparative statics theorem can address a more general version of these constrained optimization problems. Price and technology parameters can be included in the objective function and my result addresses changes in optimal inputs for changes in the parameter vector consisting of price, technology and emissions standard.

In addition to that, Helfand (1991) restricts attention to unique optimal solutions of the firm's constrained optimization problem by assuming that the profit function is strictly concave. My result goes beyond that, as it does not require strict concavity and provides comparative statics results also in the case of multiple solutions.

### 1.4.8 Emissions Standards and Technological Innovation

**Example 8.** Another example in the context of production regulation is technological innovation under emissions standards. Bruneau (2004) compares incentives for innovation in emissions-reducing technology for emissions and performance standards. In the course of the discussion, the paper also addresses changes in output and emissions abatement due to technical innovation.

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<sup>14</sup> $\pi$  is  $C_i$ -quasisupermodular if it is supermodular and  $i$ -concave. The former is the case if  $\frac{\partial^2 \pi}{\partial x_i \partial x_j} = p \cdot \frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0$ . By assumption,  $\frac{\partial^2 f}{\partial x_i^2} \leq 0$ ; thus  $\frac{\partial^2 \pi}{\partial x_i^2} = p \cdot \frac{\partial^2 f}{\partial x_i^2} \leq 0$  for all  $i$  and  $\pi$  is  $i$ -concave for all  $i$  in the 2 goods case. In the general case, the Hessian  $H_{-i}$  (see Appendix A) needs to be negative semi-definite. Moreover,  $\frac{\partial^2 \pi}{\partial x_i \partial p} = \frac{\partial f}{\partial x_i} \geq 0$ ,  $\frac{\partial^2 \pi}{\partial x_i \partial (-\omega_j)} = 0$  for  $i \neq j$  and  $\frac{\partial^2 \pi}{\partial x_i \partial (-\omega_i)} = 1 \geq 0$ ,  $\frac{\partial^2 \pi}{\partial x_j \partial (-\omega_i)} = 0$  for all  $i$ . Hence the  $i$ -SCP holds.

Consider the case of an emissions standard, that restricts emissions by a firm to  $\bar{e}$ . Suppose units are scaled such that emissions  $e$  rise and abatement  $a$  falls one-to-one with output  $q$ , so  $e = q - a$ . Then a firm, when choosing optimal output and abatement levels faces the constraint  $q - a \leq \bar{e}$ .

Abatement however is costly, but these costs can be lowered by technological innovation. This is represented by the abatement cost  $kC(a)$ , where  $k \leq 1$  is a technology parameter. Moreover, assume that the market price  $P$  depends on industry output  $Q$ .

The firm then solves the following constraint profit maximization problem:

$$\max_{q,a} \pi = P \cdot q - C(q) - k \cdot C(a)$$

$$\text{subject to } q - a \leq \bar{e}.$$

Bruneau (2004) uses quadratic specifications for both cost functions, namely  $C(q) = \frac{1}{2}cq^2$  and  $kC(a) = \frac{1}{2}ka^2$ , and first considers the simplified case where production costs are equal to zero. He finds that in both cases, technological progress from  $k = 1$  to  $k < 1$  leads to increases in output and abatement.

My comparative statics result can be applied in this context to replicate the comparative statics results in Bruneau (2004) and provides conditions on the profit function that yields nondecreasing optimal solutions. Technical innovation means a decrease in  $k$  (or increase in  $-k$ ). As in the previous example, the result in Milgrom and Shannon (1994) cannot be applied to this problem, as the constraint set is not a lattice. Moreover, it cannot be addressed by the main theorem in Quah (2007), as his result does not account for parameter changes in the objective function. For my comparative statics theorem to apply, the profit function needs to satisfy  $C_i$ -quasisupermodularity and the  $i$ -Single Crossing Property.<sup>15</sup> We can easily check that  $\pi$  is  $C_i$ -quasisupermodular, as it is supermodular ( $\frac{\partial^2 \pi}{\partial q \partial a} = \frac{\partial^2 \pi}{\partial a \partial q} = 0$ ) and  $i$ -concave ( $\frac{\partial^2 \pi}{\partial q^2} \leq 0$  for linear demand functions,  $\frac{\partial^2 \pi}{\partial a^2} \leq 0$ ). It also has the  $i$ -SCP, since  $\frac{\partial^2 \pi}{\partial q \partial (-k)} = 0$  and  $\frac{\partial^2 \pi}{\partial a \partial (-k)} = a \geq 0$ . Thus the profit function satisfies the assumptions for Theorem 1 and therefore yields nondecreasing optimal solutions for abatement.

<sup>15</sup>As the constraint set remains unchanged, I do not have to check if the sets can be ranked by the  $C_i$ -flexible set order.

With my result, I can easily extend the results in Bruneau (2004) by considering comparative statics with respect to parameter changes such as demand shocks or production costs with simultaneous changes in the constraint set due to regulatory adjustments to the emissions standard  $\bar{e}$ .

Rewrite the profit function as  $\pi = P(\alpha) \cdot q - \frac{1}{2}cq^2 - \frac{1}{2}ka^2$  and let  $t = (\alpha, -c, -k, \bar{e})$ . An increase in the parameter vector to  $t'$  can be interpreted as a positive demand shock and technological progress that lowers production and abatement costs, while the regulator loosens the emissions standard. Theorem 1 yields nondecreasing optimal solutions for abatement, since we can check that the  $i$ -SCP for the additional parameters is satisfied and the constraint set at  $\bar{e}'$  dominates the one at  $\bar{e}$  in the  $C_i$ -flexible set order.

### 1.4.9 Ethanol Quota

**Example 9.** In the Energy Independence and Security Act of 2007, the Renewable Fuel Standard (RFS) requires the blenders to use increasing amounts of renewable fuel, which in practice mainly means ethanol made from corn. The policy aims to reduce dependence on oil for gasoline production and replace petroleum gasoline by renewable fuels such as ethanol. For conventional cars, fuel can be blended with up to 10% to 15% of ethanol, flex-fuel vehicles can also run on 85% ethanol.

In this example adapted from Zhang, Qiu and Wetzstein (2010), consider the gasoline blending sector. Blenders provide two types of fuel, E85 (85% ethanol and 15% petroleum gasoline) suitable for flex-fuel vehicles and  $E_\gamma$ , which contains  $\gamma$  % ethanol and  $(1 - \gamma)$ % petroleum gasoline, for regular cars. Currently the maximum blend is regulated at 15%. To consider possible adjustments of this “blend wall”, let  $0.1 \leq \gamma \leq 0.2$ . Let  $p_{85}(E_{85}, \phi)$  and  $p_\gamma(E_\gamma, \phi)$  denote inverse market demand for E85 and  $E_\gamma$ , with  $\phi$  being a parameter that captures changes in demand. The blender uses  $e_{85}$  and  $g_{85}$  in ethanol and petroleum gasoline in the blending process, which is proportional. Hence,  $\frac{e_{85}}{e_{85} + g_{85}} = 0.85$  or after rearranging terms,  $g_{85} = \frac{3}{17}e_{85}$ . Similarly for  $E_\gamma$ ,  $g_\gamma = \frac{1-\gamma}{\gamma}e_\gamma$ . The production technologies  $y_{85}(e_{85})$  and  $y_\gamma(e_\gamma)$  of blended fuel can be expressed only in terms of amount of ethanol used because of the proportional production process and output is assumed to be increasing with diminishing marginal returns.

Moreover, under the RFS, each blender has to meet his mandated ethanol quota of  $e_0$ , that is his total quantity of ethanol  $e = e_{85} + e_\gamma$  must exceed  $e_0$ . The blenders profit maximization problem can then be written as follows.

$$\begin{aligned} \max \quad \pi &= p_{85}(E_{85}, \phi) \cdot y_{85}(e_{85}) + p_\gamma(E_\gamma, \phi) \cdot y_\gamma(e_\gamma) - c_G\left(\frac{3}{17}e_{85} + \frac{1-\gamma}{\gamma}e_\gamma\right) - c_e \cdot (e_{85} + e_\gamma) \\ &\text{subject to } e_{85} + e_\gamma \geq e_0. \end{aligned} \quad ^{16}$$

Zhang, Qiu and Wetzstein (2010) finds that under certain conditions on demand elasticities, an increase in the “blend wall”  $\gamma$  likely delays the shift towards flex-fuel vehicles and leads to an increase in  $E_\gamma$  supply and a decrease in  $E_{85}$  provided.

For the same reasons as in the previous examples, while existing comparative statics theorems cannot be applied in this case, my result allows us to consider the impact of raising the “blend wall”  $\gamma$  and increasing the ethanol quota. The constraint set  $S(e'_0)$  under the new policy dominates the original one,  $S(e_0)$  in the  $C_i$ -flexible set order. As discussed in the appendix, the profit function is  $C_i$ -quasisupermodular for linear demand and cost functions and unrelated demand. The  $i$ -Single Crossing Property is satisfied for  $p_{85}$  and  $p_\gamma$  increasing in  $\phi$  and I can check that  $\frac{\partial^2 \pi}{\partial e_\gamma \partial \gamma} \geq 0$  and  $\frac{\partial^2 \pi}{\partial e_\gamma \partial \gamma} = 0$ .

Thus, Theorem 1 applies and yields nondecreasing optimal solutions for  $e_\gamma$ ; hence the policy change favoring the use of renewable fuel leads to an increase in regular fuel. My result is the same as in Zhang, Qiu and Wetzstein (2010) for regular fuel and also yields the conclusion that these policies promoting renewable fuels don't necessarily accelerate the transition to flex-fuel vehicles. The advantages of my approach lie in the ability to easily consider a more general problem with additional parameters and changes in the constraint set as well as providing assumptions on the objective function instead of demand elasticities.

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<sup>16</sup>Unlike Zhang, Qiu and Wetzstein (2010), this example considers the case of a constrained optimization problem with an ethanol quota without the option to buy and sell permits, which eliminates the constraint. The unconstrained problem can be addressed by existing comparative statics result, this type of constrained optimization problem however cannot.

### 1.4.10 Cost-Efficient Emissions Regulation

**Example 10.** Another application in environmental economics is the problem of cost-efficient emissions regulation. The goal of the regulator is to limit the amount of total emissions of a given pollutant. As the costs to reduce emissions vary among producers, the regulator's goal is to achieve the desired emissions limit at the lowest total abatement cost. Theoretical models for efficient emissions allocation can first be found in Baumol and Oates (1971) or Montgomery (1972) and continue to serve as a baseline model as for example in Muller and Mendelsohn (2009). Cost-effectiveness has been the main criterion for the design of permit markets and therefore the cost-efficient allocation continues to be of interest as the baseline comparison even as regulation is moving more and more towards market-based approaches and away from the traditional command-and-control regulation.

The design of regulatory standards depends on the class of pollutants. First, consider the class of uniformly mixed assimilative pollutants, such as greenhouse gases. These types of pollutants do not accumulate in the atmosphere over time and their concentration only depends on the total amount of emissions regardless the source and its location. In the following example, consider a version of this model adapted from Montgomery (1972) and Tietenberg (2006).

Suppose there are  $J$  producers that emit a uniformly mixed assimilative pollutant, for example carbon dioxide. The regulator sets the target emissions limit at  $\bar{A}$ . The relationship between firms' emissions and total pollution can be described as

$$A = a + b \sum_{j=1}^J (\bar{e}_j - r_j),$$

where  $A$  is pollution per year,  $\bar{e}_j$  denotes the uncontrolled emission rate of firm  $j$  and  $r_j$  is firm  $j$ 's emissions reduction. The parameters  $a$  and  $b$  represent "background pollution" from other sources and the degree of proportionality between emissions and total pollution. Moreover, assume that each firm's cost function  $C_j(r_j, \omega)$  is continuous and that the marginal cost of emissions reduction is increasing. The parameter  $\omega_j$  captures technological change for firm  $j$ .

The regulators problem can then be written as

$$\begin{aligned} \max \quad & \tilde{C} = -\sum_{j=1}^J C_j(r_j, \omega_j) \\ \text{subject to} \quad & a + b\sum_{j=1}^J (\bar{e}_j - r_j) \leq \bar{A}, r_j \geq 0 \end{aligned}$$

An interesting comparative statics question in this context is the effect of more environmentally conscious mindset in society. On the firm's side, this manifests itself in new emissions reducing technology, on the government's side greener policy means stricter emissions standards. Now consider such an increase in the parameter vector  $t = (\omega, -\bar{A})$  caused by new emissions reducing technology and a reduction of the pollution target by the regulator.

Because of the parameter change in the objective function in a problem with a non-lattice constraint set, the existing monotone comparative statics results in Milgrom and Shannon (1994) and Quah (2007) do not apply to this particular problem. My result however is able to address this type of constrained optimization problem, as I can verify that all conditions in Theorem 1 are satisfied.

The constraint set  $S(-\bar{A}')$  dominates  $S(-\bar{A})$  by the  $C_i$ -flexible set order and the objective function  $\tilde{C}$  is supermodular ( $\frac{\partial^2 \tilde{C}}{\partial r_i \partial r_j} = 0$ , for all  $i, j$ ) and  $i$ -concave (in the two firm case,  $\frac{\partial^2 \tilde{C}}{\partial r_j^2} = -\frac{\partial MC_j}{\partial r_j} < 0$  since marginal abatement cost is increasing; in the general case, we need to verify that the Hessian  $H_{-i}$  is negative semi-definite). Moreover,  $\frac{\partial^2 \tilde{C}}{\partial r_i \partial \omega_i} = -\frac{\partial MC_i}{\partial \omega_i} \geq 0$ ,  $\frac{\partial^2 \tilde{C}}{\partial r_i \partial \omega_j} = -\frac{\partial MC_i}{\partial \omega_j} = 0$  as technological progress decreases firm  $i$ 's marginal cost of emissions reduction and hence  $\tilde{C}$  exhibits the  $i$ -Single Crossing Property. Therefore, all conditions of Theorem 1 are satisfied and the cost-effective, optimal amounts of emissions reduction are nondecreasing for firm  $i$ .

My result can be used in this setting to illustrate the effect of regulation regime changes and technological progress in emissions abatement on the cost-efficient emissions allocation. Under natural assumptions on the primitive, my theorem provides comparative statics results for a common baseline model in emissions regulation. As this is the allocation that other regulatory systems are designed to attain as well, it also suggests how these changes affect the outcome in today's market-based approaches.



Unlike the standard comparative statics approach that requires uniqueness of solution, this result can also be applied in cases with multiple optimizers. This could occur in the case of a piecewise cost function that is partly linear.

# Chapter 2

## Comparative Statics and the Gross Substitutes Property

### 2.1 Introduction

Besides the question of normality of goods, the effect of a price change on demand is one of the basic questions in consumer theory. The question of whether two goods are substitutes or complements is not only of interest to analyze consumers' reactions to price changes, it also can be useful in other economic models. For example, in the standard Arrow-Debreu model, the gross substitutes property of demand provides insights regarding the competitive equilibrium. If excess demands for all goods display this property, we can obtain very strong results with regard to stability and uniqueness. Its usefulness in these matters entails the question what assumptions will guarantee gross substitutes or complements. Most attempts to derive these properties focus on characteristics of the demand curve. However, it has turned out to be rather difficult to guarantee gross substitutability and complementarity by conditions on the primitive, the utility function, instead of the resulting demand function.

In this chapter, I will present easy conditions on the utility function that will yield the desired gross substitutes (complements) property. The approach taken has its foundation in the standard

decomposition of the total effect of a price change on demand into a substitution and an income effect. The sign of the substitution effect can easily be determined by basic economic theory in the case of two goods, but requires additional assumptions on the marginal rate of substitution in the general case. With regard to the income effect, there are various recent works on comparative statics of demand that have focused on conditions that yield normal demand. I will use the assumptions on the utility function provided in Quah (2007) to sign the income effect and then introduce a simple and intuitive assumption on the elasticity of marginal rate of substitution, which will allow us to determine the direction of the overall effect of a price change.

The question of gross substitutability and complementarity has been addressed in the literature in various different ways. Fisher (1972) finds that two goods are gross substitutes if the Allen-Uzawa elasticity of substitution between two commodities exceeds the maximum of the respective income elasticities of demand. As defined in Uzawa (1962), the partial elasticity of substitution between two commodities depends on the expenditure function. Since the expenditure function and demand functions are results of optimization problems, Fisher's condition is not on the primitives. The intuition behind his condition is that indifference curves, that only have little curvature will yield gross substitutes; in the limiting case of perfect substitutes the indifference curves are straight lines. For gross complements, the opposite has to be true; indifference curves need to be sufficiently curved, with the right-angled indifference curves of perfect complements as the extreme case. Later papers that establish results in accordance with this intuition, but impose conditions directly on the utility function, are Mirman and Ruble (2003), Maks (2006) and Quah (2004). For the two good case, Mirman and Ruble (2003) find an assumption on the marginal rate of substitution that yields gross substitutes (complements). When extending this to the case of many goods, they find a similar condition that in combination with the assumption that the marginal rate of substitution between two goods is independent of all other goods that guarantees gross substitutes (complements). Maks (2006) derives a condition on the elasticity of marginal utility with the same intuition for utility functions based on Walras' utility concept that are strongly separable or additive in each commodity. Quah (2004) also provides a result of gross substitutability in the special

case of additive utility functions.

Many of the recent works on comparative statics theorems for constrained optimization problems take a lattice-theoretic approach, as first developed by Vives (1990) based on Topkis (1978). Milgrom and Roberts (1992) also focuses on monotone comparative statics results under cardinal assumptions (supermodular games), which is extended by Milgrom and Shannon (1994) to the ordinal case (quasisupermodular games). These results however cannot be easily applied to optimization problems where the constraint sets are not sublattices and comparable in the strong set order, such as budget sets at two levels of income. Quah (2007) expands Milgrom and Shannon's work by making it applicable to these types of optimization problems by using a weaker order on the constraint sets ( $C_i$ -flexible set order). His comparative statics theorem provides easily verifiable conditions on the utility function that yield normal demand, namely supermodularity and a form of concavity. Antoniadou (2007), which is largely based on Antoniadou (2004a, 2004b), and Mirman and Ruble (2008) also address the question of normality of demand, but their approach introduces a direct value order to construct the relevant order-lattice structure. While Quah (2007) and Mirman and Ruble (2008) solely focus on income changes, Antoniadou (2007) also provides the relevant order-lattice structure for price changes in the case of two goods.

In this chapter, I will provide conditions that yield gross substitutes and complements under assumptions on the primitives. I approach the problem by decomposing the effect of a price change in substitution and income effect. Existing results based on lattice theory can be used to determine the direction of the latter, while basic economic theory provides the sign of the former. Even though the conditions for gross substitutes and complements are closely related to existing literature, my approach is helpful in providing the economic intuition behind these rather technical results.

The next section introduces the theoretical framework for the lattice-based approach and defines the gross substitutes property. Section 3 gives results for the two goods case, section 4 addresses the extension to many goods. In section 5, the results will be applied to more general optimization problems, section 6 considers net substitutes and complements.

## 2.2 Preliminaries

Quah (2007) introduces the following concepts to extend the comparative statics results from Milgrom and Shannon (1994) to problems where the constraint sets are not lattices. This case is easily encountered even in simple optimization problems like the consumers maximizing utility subject to the budget constraint. The main result in Quah (2007) is the following comparative statics theorem.

### **Theorem 3.** (*Quah*)

*The function  $f : X \rightarrow R$  is  $C_i$ -quasisupermodular ( $C$ -quasisupermodular) if and only if  $\operatorname{argmax}_{x \in S'} f(x) \geq_i \operatorname{argmax}_{x \in S} f(x)$  (for all  $i$ ) whenever  $S' \geq_i (\geq) S$ .*

This chapter focuses on comparative statics results for price changes. In this context, the following standard definitions will be used to characterize the effect of price changes on other goods.

### **Definition 6.** Gross Substitutes and Gross Complements

Good  $j$  is a gross substitute (complement) of good  $i$  if (Marshallian) demand  $x_j(p', w) \geq (\leq) x_j(p, w)$  whenever  $p' > p$  with  $p'_i > p_i$  and  $p'_k = p_k$  for all  $k \neq i$ .

### **Definition 7.** Net Substitutes and Net Complements

Good  $j$  is a net substitute (complement) of good  $i$  if (Hicksian) demand  $h_j(p', u) \geq (\leq) h_j(p, u)$  whenever  $p' > p$  with  $p'_i > p_i$  and  $p'_k = p_k$  for all  $k \neq i$ .

Providing assumptions on the primitives to establish the gross substitutes property is one of the goals of this chapter. Generally, two goods are considered gross substitutes if a price increase of one of the goods results in nondecreasing demand for the other good. Moreover, a demand function is said to satisfy the gross substitutes property if an increase of the price of good  $i$  leads to nondecreasing demand for all goods  $j \neq i$ .

### **Definition 8.** Gross Substitutes Property of Demand

A (Marshallian) demand function  $x(p, w)$  satisfies the gross substitutes property if  $x_j(p', w) \geq x_j(p, w)$  whenever  $p' > p$  with  $p'_i > p_i$  and  $p'_j = p_j$  for all  $j \neq i$ .

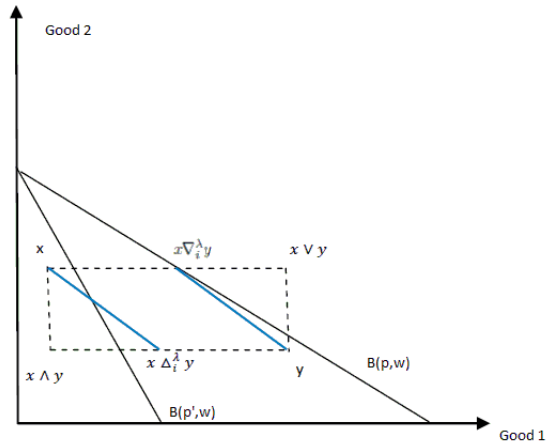


Figure 2.1: Violation of  $C_i$ -flexible set order by budget sets before and after a price change

A stronger version of this property can be defined by replacing the weak inequality by a strict one, hence requiring demand of good  $j$  to be strictly increasing given an increase of the price of good  $i$ . I refer to this case as strict gross substitutes.

## 2.3 Price effects with two goods

Quah's new approach, the  $C_i$ -flexible set order, works well in cases like budget sets before and after an income change, where both sets have the same slope. However, there are fairly simple constraint sets, like budget sets with different slopes, that cannot be ordered in this way. For example, consider a price change of good 1. The boundary of the smaller budget set will be steeper than the one of the larger set, which allows the conclusion that the bigger set dominates the smaller one in the  $C_1$ -flexible set order, but not in the  $C_2$ -flexible set order as shown in Figure 2.1. Similarly, for price changes of good 2, the conclusion is reversed and we can only order the sets in the  $C_2$ -flexible set order. Hence this framework only gives comparative statics results for changes of a good's own price, but cannot provide insight into whether goods are gross substitutes or complements.

I take the approach to consider the substitution and income effect of a price change separately to obtain results for this class of comparative statics problems. For simplicity, to begin with consider

the consumer's utility maximization problem in the case when there are only two goods and assume that the utility function  $u$  is locally nonsatiated.

Let  $x^*$  be a maximizer for the initial budget set  $B(p, w) = \{x \in X \mid p \cdot x \leq w\}$ . By local non-satiation, the budget constraint is binding at optimum, thus  $p \cdot x^* = w$ . Now suppose the price of good  $i$  ( $i = 1, 2$ ) increases to  $p'_i$ . As we know, this price change will affect the consumer's demand in two ways. On the one hand, there is a substitution effect due to the change of the relative prices when the consumer's income will be adjusted in such a way, that the old preferred bundle  $x^*$  is still affordable at new prices. On the other hand, the income effect will capture the change in demand when the consumer's income adjustment is reversed.

**Proposition 4.** *If  $u$  is locally nonsatiated, the substitution effect on demand for good  $j$  will be non-negative (non-positive) when price of good  $i$  increases (decreases).*

*Proof.* The proof will focus on the unbracketed claim, the bracketed version can be shown in similar fashion. Suppose price for good  $i$  increases from  $p_i$  to  $p'_i$ . Because the budget constraint was binding at the old optimal bundle  $x^*$ , the consumer will no longer be able to afford this bundle at the new prices. Now suppose the consumer is given a higher income  $w' = p'_i x_i^* + p_j x_j^* > w$  to compensate him for the price increase. Let  $\hat{x}$  be the optimal bundle at  $(p', w')$ . Then, by local non-satiation,  $p_i \hat{x}_i + p_j \hat{x}_j \geq p_i x_i^* + p_j x_j^*$ . This inequality can be rewritten as

$$\hat{x}_i - x_i^* \geq -\frac{p_j}{p_i}(\hat{x}_j - x_j^*) \quad (2.1)$$

At the compensated level of income  $w'$  both bundles are affordable and  $w' = p'_i x_i^* + p_j x_j^* = p'_i \hat{x}_i + p_j \hat{x}_j$ . Rearranging terms yields

$$\hat{x}_i - x_i^* = -\frac{p_j}{p'_i}(\hat{x}_j - x_j^*) \quad (2.2)$$

It follows from 2.1 and 2.2 that  $-\frac{p_j}{p_i}(\hat{x}_j - x_j^*) \geq -\frac{p_j}{p'_i}(\hat{x}_j - x_j^*)$  or equivalently

$\frac{p_j(p'_i - p_i)}{p_i p'_i}(\hat{x}_j - x_j^*) \geq 0$ . Since the first term is positive, this implies  $\hat{x}_j \geq x_j^*$ . □

The other effect of a price change is the income effect, which captures the impact on demand when the consumer's income adjustment for the price change is reversed. In other words, the income effect compares the optimal consumption bundles after the price change at income levels  $w$  and  $w'$ . Since this is a comparative statics problem concerning an income change and the budget sets are comparable in the  $C_i$ -flexible set order, Quah's result for normality of demand can be applied here.

**Proposition 5.** *If  $u$  is  $C_j$ -quasisupermodular, the income effect of a price increase (decrease) of good  $i$  is non-positive (non-negative) on the demand for good  $j$ .*

*Proof.* This result follows immediately from Proposition 1 from Quah (2007). □

Since the substitution effect and income effect are opposite in sign for normal goods, it is not possible to tell whether the goods are gross substitutes or gross complements. For the former, the substitution effect needs to dominate the income effect and vice versa for the latter. In order to compare the income and substitution effect additional assumption on the utility function are needed.

The question of interest now is under what assumptions on the primitives two goods will be gross substitutes or gross complements. Intuitively, the needed condition has to ensure that the indifference curves only have little curvature for gross substitutes; in the limiting case of perfect substitutes the indifference curves are straight lines. For gross complements, the opposite has to be true; indifference curves need to be sufficiently curved, with the right-angled indifference curves of perfect complements as the extreme case. In other words, if the indifference curves are relatively straight, the substitution effect of a price change will outweigh the income effect and hence the two goods will be gross substitutes. Figure 2.2 depicts the two cases.

In the following, a property that will yield the desired shape of the indifference curves will be introduced. It is similar to, but more general than the condition in Maks (2006) and the approach



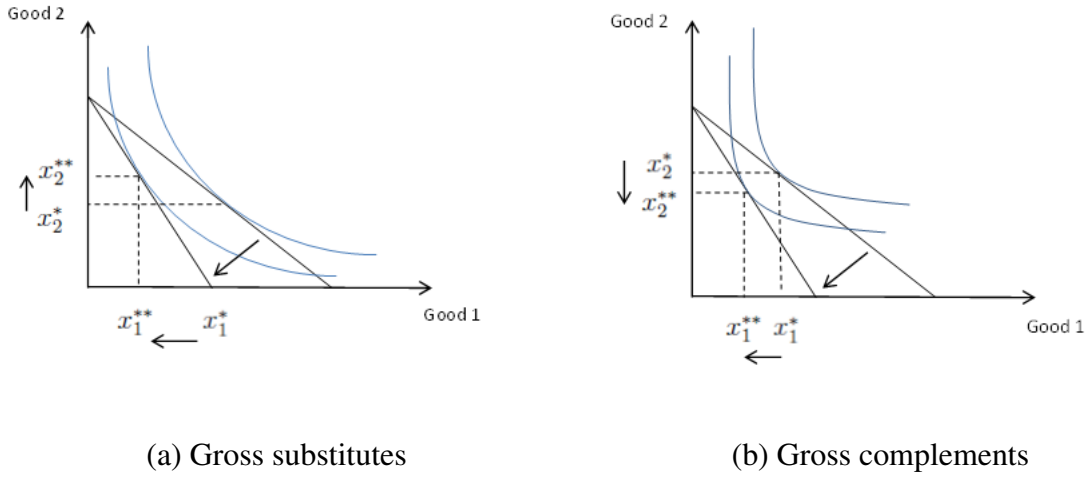


Figure 2.2: Effect of a price change for different curvatures of indifference curves

is more intuitive than the one proposed in Mirman and Ruble (2003). The idea here is, that indifference curves will only have little curvature if the elasticity of the marginal rate of substitution between goods  $i$  and  $j$  with respect to good  $i$  is small. In order for two goods to be gross substitutes (complements), the absolute value of the marginal rate of substitution elasticity has to be less (greater) than 1. This condition extends the one in Maks (2006) to a wider class of utility functions, since there is no restriction that marginal utilities only depend on one commodity. It is easy to show that this more general condition reduces to the one discussed in Maks (2006) when only considering strongly separable utility functions.<sup>17</sup> My condition is equivalent to the one proposed in Mirman and Ruble (2003) for well-behaved preferences in the 2 goods case (see Appendix B), but the decomposition in substitution and income effect provides more economic intuition than their value-order approach and shows how the shape of the indifference curves determines the size of the substitution effect.

For simplicity, we still consider the case with only two goods and assume that the utility function is differentiable. To begin with, consider only cases where the utility maximization problem has an interior solution. To guarantee this, we need to impose regularity conditions on the utility

<sup>17</sup>  $\frac{\partial MRS_{ij}}{\partial x_i} \cdot \frac{x_i}{MRS_{ij}} = \frac{\partial}{\partial x_i} \left( \frac{MU_i}{MU_j} \right) \cdot \frac{x_i}{MRS_{ij}} = \frac{\partial MU_i}{\partial x_i} \cdot \frac{x_i}{MU_i} - \frac{\partial MU_j}{\partial x_i} \cdot \frac{x_i}{MU_i}$

For strongly separable utility functions  $MU_j$  only depends on  $x_j$ , therefore  $\frac{\partial MU_j}{\partial x_i} = 0$ . As a result, marginal rate of substitution elasticity and marginal utility elasticity are equivalent.

function, i.e.  $u$  is differentiable,  $\frac{\partial u}{\partial x_i} > 0$ ,  $\lim_{x_i \rightarrow 0} \frac{\partial u}{\partial x_i} = \infty$ ,  $\lim_{x_i \rightarrow \infty} \frac{\partial u}{\partial x_i} = 0$ . Given these assumptions, the utility function does not cross the axes and therefore corner solutions are ruled out.

**Theorem 4.** *If  $u$  is locally nonsatiated, satisfies regularity conditions and  $\left| \frac{\partial MRS_{ij}}{\partial x_i} \cdot \frac{x_i}{MRS_{ij}} \right| < (>) 1$  and both goods are normal, then good  $j$  is a gross substitute (complement) of good  $i$ .*

*Proof.* The following proof will focus on the gross substitutes. The bracketed version for complements can be shown accordingly. Now consider the case when  $p_i$  increases. From Proposition 4 and normality of the goods we know that  $\hat{x}_j \geq x_j^*$  and  $\hat{x}_j \geq x_j^{**}$ , where  $x^*$  is the original optimal bundle before the price change,  $\hat{x}$  is the optimal bundle at new prices and compensated income and  $x^{**}$  is the new maximizer after the price change.

Since the budget constraint is binding at the old and new optimal bundle, we know that

$$p_i x_i^* + p_j x_j^* = w = p'_i x_i^{**} + p_j x_j^{**},$$

which can be rewritten as  $p_j(x_j^{**} - x_j^*) = p_i x_i^* - p'_i x_i^{**}$ . The total change of expenditures on each of the goods can be decomposed in the change resulting from the substitution and the income effect, resulting in  $p_j(\hat{x}_j - x_j^*) + p_j(x_j^{**} - \hat{x}_j) = p_i x_i^* - p'_i \hat{x}_i + p'_i \hat{x}_i - p'_i x_i^{**}$ . First consider only the change in expenditures attributed to the income effect. Since  $\hat{x}_i \geq x_i^{**}$  and  $\hat{x}_j \geq x_j^{**}$ ,  $p'_i(x_i^{**} - \hat{x}_i) \leq 0$  and  $p_j(x_j^{**} - \hat{x}_j) \leq 0$ . Hence expenditures on both goods decrease as a result of the income effect.

Since  $\hat{x}_j \geq x_j^*$ , the expenditures on good  $j$  will increase when considering the substitution effect only. Thus we cannot draw any conclusion on the total change of expenditures on good  $j$ , because the income and substitution effect will cause them to go in opposite directions.

The impact of the substitution effect on good  $i$  on the other hand is not as obvious. Using the Slutsky compensation, it can be written as

$$p'_i \hat{x}_i - p_i x_i^* = w' - w - p_j(\hat{x}_j - x_j^*) = (p'_i - p_i)x_i^* - p_j(\hat{x}_j - x_j^*) \quad (2.3)$$

To use the assumption that  $\left| \frac{\partial MRS_{ij}}{\partial x_i} \cdot \frac{x_i}{MRS_{ij}} \right| < 1$ , we first approximate the derivative  $\frac{\partial MRS_{ij}}{\partial x_i}$  by  $\frac{\Delta MRS_{ij}}{\Delta x_i}$  using a linear Taylor approximation.  $\hat{x}_i$  is a continuous function of  $p_i$ , so for sufficiently

small price changes  $\frac{\Delta MRS_{ij}}{\Delta x_i} = \frac{MRS_{ij}(\hat{x}) - MRS_{ij}(x^*)}{\hat{x}_i - x_i^*}$ . Assuming that the solution is interior, we have  $\frac{\Delta MRS_{ij}}{\Delta x_i} = \frac{MRS_{ij}(\hat{x}) - MRS_{ij}(x^*)}{\hat{x}_i - x_i^*} = \frac{\frac{p'_i - p_i}{p_j - p_i}}{\hat{x}_i - x_i^*}$ . Since  $|\frac{\partial MRS_{ij}}{\partial x_i} \cdot \frac{x_i}{MRS_{ij}}| < 1$ , there exists some  $\varepsilon > 0$ , such that  $|\frac{\partial MRS_{ij}}{\partial x_i} \cdot \frac{x_i}{MRS_{ij}}| + \varepsilon \leq 1$ . Moreover, if we suppose that the price changes are sufficiently small,  $|\frac{\Delta MRS_{ij}}{\Delta x_i} \cdot \frac{x_i}{MRS_{ij}}| - |\frac{\partial MRS_{ij}}{\partial x_i} \cdot \frac{x_i}{MRS_{ij}}| < \varepsilon$ . Therefore,  $|\frac{\Delta MRS_{ij}}{\Delta x_i} \cdot \frac{x_i}{MRS_{ij}}| = \frac{\frac{p'_i - p_i}{p_j - p_i}}{x_i^* - \hat{x}_i} \cdot \frac{x_i^*}{p_j} \leq 1$ , which implies  $\frac{\frac{p'_i - p_i}{p_j - p_i}}{x_i^* - \hat{x}_i} \cdot \frac{x_i^*}{p_j} \leq 1$ .

Again using the Slutsky compensation, this can be rewritten as  $p_j(\hat{x}_j - x_j^*) \geq (p'_i - p_i)x_i^* = w' - w$ .

This condition and 2.3 imply

$$p'_i \hat{x}_i - p_i x_i^* = w' - w - p_j(\hat{x}_j - x_j^*) = (p'_i - p_i)x_i^* - p_j(\hat{x}_j - x_j^*) \leq 0.$$

Therefore, both the income and the substitution effect lead to a decrease of expenditures on good  $i$ , so  $p'_i x_i^{**} \leq p_i x_i^*$ . Consequently,  $p_j x_j^{**} \geq p_j x_j^*$  which is equivalent to  $x_j^{**} \geq x_j^*$ .  $\square$

This proof nicely shows how the curvature of the indifference curves, quantified by the elasticity of the marginal rate of substitution, determines the size of the substitution effect and hence whether in consequence expenditures on good  $i$  increase or decrease. In the case of gross substitutes, the sufficiently small amount of curvature of the indifference curves leads to a large enough decrease in demand for commodity  $i$  to result in lower expenditures on this good despite its price increase. Conversely, this means a large enough increase in expenditures on good  $j$  due to the substitution effect to offset a negative income effect. Hence the strength of my approach lies in the fact that it provides the economic intuition behind the complex and technical results in Mirman and Ruble (2003).

**Corollary 1.** *If  $u$  is locally nonsatiated, is  $C_i$ - and  $C_j$ -quasisupermodular, satisfies regularity conditions and  $|\frac{\partial MRS_{ij}}{\partial x_i} \cdot \frac{x_i}{MRS_{ij}}| < (>)1$ , then good  $j$  is a gross substitute (complement) of good  $i$ .*

*Proof.* The only change that needs to be made to the proof of Theorem 4 is using Proposition 5 to establish normality of goods  $i$  and  $j$ .  $\square$

**Example 11. CES Utility**

Consider the CES utility function  $u(x_1, x_2) = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}$  with  $\rho \leq 1$ . Since the marginal utilities for both goods are non-negative,<sup>18</sup>  $u$  is monotone and locally non-satiated. For  $\rho \leq 1$ , the CES utility function is concave<sup>19</sup> and therefore  $u$  is 1,2-concave as well. Moreover, the utility function is supermodular for  $\rho \leq 1$ ,<sup>20</sup> which in combination with 1- and 2- concavity implies that  $u$  is  $C_1$ - and  $C_2$ -supermodular (see Proposition 2 in Quah (2007)) and therefore also  $C_1$ - and  $C_2$ -quasisupermodular. The regularity conditions that guarantee an interior solution are satisfied for  $\rho < 1$ , and for  $0 \leq \rho \leq 1$  ( $\rho \leq 0$ ) the absolute value of the elasticity of the marginal rate of substitution with respect to  $x_1$  is less (greater) than or equal to 1.<sup>21</sup> Thus suppose  $0 < \rho < 1$  ( $\rho < 0$ ) and then the above comparative statics theorem applies and the goods are gross substitutes (complements).

Moreover, notice that for the CES utility function Mirman and Ruble's condition coincides with the one proposed here. Using either condition, goods will be gross substitutes (complements) if  $0 \leq \rho \leq 1$  ( $\rho \leq 0$ ).<sup>22</sup>

Now consider cases where the solution to the utility maximization problem is not interior, that is one the consumer only consumes one of the two goods and nothing of the other one. First, there is the trivial case of perfect substitutes. For these preferences, the marginal rate of substitution is constant along the utility curve. The consumer chooses  $x^* = (x_1^*, 0)$  if  $MRS_{12} > \frac{p_1}{p_2}$  and  $x^* = (0, x_2^*)$

<sup>18</sup>  $\frac{\partial u}{\partial x_1} = (x_1^\rho + x_2^\rho)^{\frac{1-\rho}{\rho}} x_1^{\rho-1} \geq 0$ ,  $\frac{\partial u}{\partial x_2} = (x_1^\rho + x_2^\rho)^{\frac{1-\rho}{\rho}} x_2^{\rho-1} \geq 0$

<sup>19</sup>  $\frac{\partial^2 u}{\partial x_1^2} = (\rho - 1)(x_1^\rho + x_2^\rho)^{\frac{1-2\rho}{\rho}} x_1^{\rho-2} x_2^\rho \leq 0$ ,  $\frac{\partial^2 u}{\partial x_2^2} = (\rho - 1)(x_1^\rho + x_2^\rho)^{\frac{1-2\rho}{\rho}} x_1^\rho x_2^{\rho-2} \leq 0$  for  $\rho \leq 1$

<sup>20</sup>  $\frac{\partial^2 u}{\partial x_1 \partial x_2} = \frac{\partial^2 u}{\partial x_2 \partial x_1} = (1 - \rho)(x_1^\rho + x_2^\rho)^{\frac{1-2\rho}{\rho}} x_1^{\rho-1} x_2^{\rho-1} \geq 0$  for  $\rho \leq 1$

<sup>21</sup>  $|\frac{\partial MRS_{12}}{\partial x_1} \cdot \frac{x_1}{MRS_{12}}| = |\rho - 1| = \begin{cases} 1 - \rho & \text{for } \rho \leq 1 \\ \rho - 1 & \text{for } \rho > 1 \end{cases}$

Thus,  $|\frac{\partial MRS_{12}}{\partial x_1} \cdot \frac{x_1}{MRS_{12}}| = 1 - \rho$ , which is  $> 1$  for  $\rho \leq 0$  and  $< 1$  for  $0 < \rho \leq 1$ .

<sup>22</sup>  $\frac{\partial(\frac{MRS_{12}}{x_2})}{\partial x_2} = -\rho \frac{x_1^{\rho-1}}{x_2^{\rho+1}} \leq (\geq) 0 \iff \rho \geq (\leq) 0$

if  $MRS_{12} < \frac{p_1}{p_2}$ . So as long as the relationship between the price ratio and the  $MRS$  stays the same after a price change of either good, the optimal consumption bundle will remain unchanged. If the price change of good  $i$  changes the inequality between the marginal rate of substitution and the price ratio, then the optimal quantity of good  $j$  will increase, i.e. for a price increase of good 1 and  $\frac{p'_1}{p_2} > MRS_{12} > \frac{p_1}{p_2}$ , the consumer will move from consuming no good 2 at all to only consuming good 2. Therefore, in the case of perfect substitutes the demand for good  $j$  is nondecreasing in the price of good  $i$ .

The other class of preferences where corner solutions often occur is quasilinear utility. In this case, interior and corner solutions are both possible. Assuming the optimal bundle before the price change was a corner solution on the  $x_1$ -axis with  $MRS_{12} > \frac{p_1}{p_2}$ , the optimal consumption bundle will remain the same if the price ratio decreases, e.g.  $p_1$  goes down or  $p_2$  goes up. If the price change causes the price ratio to increase such that it exceeds the marginal rate of substitution at the original optimal bundle, the consumer will move to an interior solution where  $x_j$  will move in the same direction as  $p_i$ . Similarly, if the initial corner solution lies on the  $x_2$ -axis and  $MRS_{12} < \frac{p_1}{p_2}$ , a price change that increases the price ratio will not change the optimal choice; a change that will decrease the relative price below the  $MRS$  at  $x^*$  will shift the consumer's utility maximizing bundle to an interior solution and the change in  $x_j$  will have the same sign as the price change. Thus, also in this case the demand of good  $j$  is non-decreasing in the price of good  $i$ .

**Example 12.** Substitutes and Complements in Input

Another application of Theorem 4 can be found in the firm's production decision. Suppose the firm's production function is given by  $F : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ . Standard assumptions assure the unique existence of a cost minimizing input bundle  $x(p, q) = \underset{\{x \in \mathbb{R}_+^2 \mid F(x) \geq q\}}{\operatorname{argmin}} p \cdot x$  given strictly positive input prices  $p$  and output  $q$ . Alternatively,  $x(p, q)$  can be found by maximizing output while limiting expenditures to  $p \cdot x(p, q)$ , hence  $x(p, q) = \underset{x \in B(p, p \cdot x(p, q))}{\operatorname{argmax}} F(x)$ .

Quah's comparative statics theorem allows us to make statements about how  $x_i(p, q)$  changes with respect to output or its own price. Theorem 4 takes it a step further, because it yields the

answer to the question how optimal inputs react to price changes of other inputs. In other words, it allows us to examine the effect of a change of  $p_i$  on  $x_j(p, q)$ . Under the assumptions that  $F$  is monotone,  $C_i$ - and  $C_j$ -quasisupermodular (which is the case if it is supermodular and  $i$ -,  $j$ -concave) and  $|\frac{\partial MRTS_{ij}}{\partial x_i} \cdot \frac{x_i}{MRTS_{ij}}| < (>) 1$ , the optimal amount of input  $j$  will increase (decrease) as the price for input  $i$  goes up.

Consider for example a CES production function,  $F(x_1, x_2) = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}$  with  $\rho \leq 1$ . It is easy to check that  $F$  is monotone,<sup>23</sup> 1- and 2-concave,<sup>24</sup> and supermodular.<sup>25</sup> Moreover, we see that  $|\frac{\partial MRTS_{12}}{\partial x_1} \cdot \frac{x_1}{MRTS_{12}}| < (>) 1$  for  $0 < \rho \leq 1$  ( $\rho < 0$ ),<sup>26</sup> therefore the 2 inputs are gross substitutes (complements).

Other functional forms that will yield gross substitutes are production functions with a root polynomial or logarithmic specification.

### Example 13. Linear PPF in General Equilibrium Model

Consider an open economy that produces 2 goods using the inputs capital and labor. Assume that the production functions for both goods  $f_i : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ ,  $i = 1, 2$  are linear. As a result, the PPF of these two goods will also be linear. On the consumption side, suppose the representative consumer has a locally non-satiated, 1- and 2-concave and supermodular utility function  $u$ . The markets for both goods are perfectly competitive and in equilibrium  $MRS_{12} = \frac{p_1}{p_2} = MRTS_{12}$ .

Now suppose there is technological progress in the production process of one of the goods, i.e. good 2. The new PPF will be steeper than the original one, thus the relative price between the goods and the marginal rate of substitution need to increase for the economy to return to equilibrium. To determine the effect of this change of technology on the optimal quantity of the

$$^{23} \frac{\partial F}{\partial x_1} = (x_1^\rho + x_2^\rho)^{\frac{1-\rho}{\rho}} x_1^{\rho-1} \geq 0, \frac{\partial F}{\partial x_2} = (x_1^\rho + x_2^\rho)^{\frac{1-\rho}{\rho}} x_2^{\rho-1} \geq 0$$

$$^{24} \frac{\partial^2 F}{\partial x_1^2} = (\rho - 1)(x_1^\rho + x_2^\rho)^{\frac{1-2\rho}{\rho}} x_1^{\rho-2} x_2^\rho \leq 0, \frac{\partial^2 F}{\partial x_2^2} = (\rho - 1)(x_1^\rho + x_2^\rho)^{\frac{1-2\rho}{\rho}} x_1^\rho x_2^{\rho-2} \leq 0 \text{ for } \rho \leq 1$$

$$^{25} \frac{\partial^2 F}{\partial x_1 \partial x_2} = \frac{\partial^2 F}{\partial x_2 \partial x_1} = (1 - \rho)(x_1^\rho + x_2^\rho)^{\frac{1-2\rho}{\rho}} x_1^{\rho-1} x_2^{\rho-1} \geq 0 \text{ for } \rho \leq 1$$

$$^{26} \left| \frac{\partial MRTS_{12}}{\partial x_1} \cdot \frac{x_1}{MRTS_{12}} \right| = |\rho - 1| = \begin{cases} 1 - \rho & \text{for } \rho \leq 1 \\ \rho - 1 & \text{for } \rho > 1 \end{cases}$$

Thus,  $|\frac{\partial MRTS_{12}}{\partial x_1} \cdot \frac{x_1}{MRTS_{12}}| = 1 - \rho$ , which is  $> 1$  for  $\rho < 0$  and  $< 1$  for  $0 < \rho \leq 1$ .

other good, Theorem 4 can be applied.

Consider for example the negative exponential utility function  $u(x_1, x_2) = 1 - e^{-x_1} - e^{-x_2}$ .  $u$  is monotone,<sup>27</sup> 1- and 2-concave<sup>28</sup> as well as supermodular.<sup>29</sup> The elasticity of the marginal rate of substitution with respect to good 1 is equal to  $x_1$ ,<sup>30</sup> therefore the goods will be gross substitutes if  $x_1 < 1$  and gross complements for  $x_1 > 1$ . In this context, the optimal choice of good 1 will increase if  $x_1 > 1$  and decrease if  $x_1 < 1$  if technological progress occurs in the production process of good 2.

## 2.4 Price effects with $n$ goods

### 2.4.1 Gross Substitutes (Complements) Property

The next question to be addressed is how this result can be generalized to the case of  $n$  goods. As previously, at first the utility function is assumed to satisfy the regularity conditions that guarantee an interior solution and the substitution effect will be identified using the Slutsky compensation.

The identification of the substitution effects sign is no longer as straightforward as in the previous case with two goods. It is a well-established result in microeconomic theory that the substitution effect of a price change of good  $i$  is negative for that good in the general case with  $n$  goods as well. However, the sign of the substitution effect for the other  $n - 1$  goods is no longer clear. Obviously, if the substitution effect leads to a decrease in  $x_i$ , there has to be at least one good among the remaining  $n - 1$  goods for which the substitution effect goes in the opposite direction of good  $i$  because of the Slutsky compensation.<sup>31</sup> Therefore, additional properties need to be imposed on the objective function to guarantee a non-negative substitution effect for all goods other than  $i$ .

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<sup>27</sup>  $\frac{\partial u}{\partial x_i} = e^{-x_i} > 0$  for  $i = 1, 2$

<sup>28</sup>  $\frac{\partial^2 u}{\partial x_i^2} = -e^{-x_i} < 0$  for  $i = 1, 2$

<sup>29</sup>  $\frac{\partial^2 u}{\partial x_i \partial x_j} = 0$  for  $i \neq j, i = 1, 2$

<sup>30</sup>  $MRS_{12} = \frac{e^{-x_1}}{e^{-x_2}}, \left| \frac{\partial MRS_{12}}{\partial x_1} \cdot \frac{x_1}{MRS_{12}} \right| = \left| -\frac{e^{-x_1}}{e^{-x_2}} \cdot \frac{x_1}{\frac{e^{-x_1}}{e^{-x_2}}} \right| = x_1$

<sup>31</sup>  $p_i'x_i^* + \sum_{i \neq j} p_j x_j^* = w' = p_i' \hat{x}_i + \sum_{i \neq j} p_j \hat{x}_j$

**Proposition 6.** *If  $u$  is locally nonsatiated and  $MRS_{lm}$  is independent of all goods other than  $l$  and  $m$  for all  $l, m$ , the substitution effect on demand for good  $j$  will be positive (negative) when price of good  $i$  increases (decreases).*

*Proof.* The proof will focus on the unbracketed claim, the bracketed version can be shown in similar fashion. Suppose price for good  $i$  increases from  $p_i$  to  $p'_i$ . Because the budget constraint was binding at the old optimal bundle  $x^*$ , the consumer will no longer be able to afford this bundle at the new prices. Now suppose the consumer is given a higher income  $w' = p'_i x_i^* + \sum_{j \neq i} p_j x_j^* > w$  to compensate him for the price increase. Let  $\hat{x}$  be the optimal bundle at  $(p'_i, p_{-i}, w')$ . Then,  $p_i \hat{x}_i + \sum_{j \neq i} p_j \hat{x}_j = w' > w = p_i x_i^* + \sum_{j \neq i} p_j x_j^*$ . This inequality can be rewritten as

$$\hat{x}_i - x_i^* > \sum_{j \neq i} -\frac{p_j}{p_i} (\hat{x}_j - x_j^*) \quad (2.4)$$

At the compensated level of income  $w'$  both bundles are affordable and  $p'_i \hat{x}_i + \sum_{j \neq i} p_j \hat{x}_j = w' = p'_i x_i^* + \sum_{j \neq i} p_j x_j^*$ . Rearranging terms yields

$$\hat{x}_i - x_i^* = \sum_{j \neq i} -\frac{p_j}{p'_i} (\hat{x}_j - x_j^*) \quad (2.5)$$

It follows from 2.4 and 2.5 that  $\sum_{j \neq i} -\frac{p_j}{p_i} (\hat{x}_j - x_j^*) > \sum_{j \neq i} -\frac{p_j}{p'_i} (\hat{x}_j - x_j^*)$ , or equivalently

$\sum_{j \neq i} \frac{p_j (p'_i - p_i)}{p_i p'_i} (\hat{x}_j - x_j^*) > 0$ . Since the first term is always positive, this implies that there exists  $j$  such that  $\hat{x}_j > x_j^*$ . Moreover, given that  $p_i$  increases and that the solution is interior,  $MRS_{ij}$  increases since  $MRS_{ij} = \frac{p_i}{p_j}$  for  $i \neq j$ . The marginal rate between any two goods other than  $i$ ,  $MRS_{jk} = \frac{p_j}{p_k}$  for  $j \neq k, j, k \neq i$ , however remains unchanged. In other words,

$$dMRS_{jk} = \frac{\partial MRS_{jk}}{\partial x_j} dx_j + \frac{\partial MRS_{jk}}{\partial x_k} dx_k = 0,$$

given that  $MRS_{jk}$  is independent of all goods other than  $j$  and  $k$  for all  $i, j$ .  $MRS_{jk}$  is of opposite sign in good  $j$  and good  $k$ , hence it is easy to see that  $dx_j$  and  $dx_k$  need to have the same sign for the above equation to hold. Consequently,  $x_j$  and  $x_k$  need to move in the same direction. Furthermore, the changes in  $x_i$  and  $x_j$  need to be of opposite signs because of the Slutsky compensation



$(p'_i x_i^* + \sum_{i \neq j} p_j x_j^* = w' = p'_i \hat{x}_i + \sum_{i \neq j} p_j \hat{x}_j)$ . Therefore,  $x_i$  decreases and all  $x_j$  increase.  $\square$

Hence, the substitution effect can be generalized for  $n$  goods under additional assumptions on the marginal rate of substitution between any two goods. The result for the income effect obtained earlier carries over to this case.

As already observed in the above discussion with two goods, the substitution and income effect of a good other than  $i$  go in opposite directions and therefore the net effect is unclear. Once again the idea is to find assumptions on the utility function that will yield gross substitutes (complements).

The condition on the utility function that will allow us to do this again focuses on the elasticity of the marginal rate of substitution between the good  $i$  and another good  $k$  with respect to  $x_i$ . If the absolute value of this elasticity is less than 1, the marginal rate of substitution increases relatively slow as  $x_i$  decreases, therefore the indifference curves are characterized by relatively small curvature. On the other hand, if the absolute value of the marginal rate of substitution elasticity exceeds 1, the marginal rate of substitution reacts relatively strong to changes in  $x_i$  and hence the indifference curves display more curvature.

This condition applies to a wider class of utility functions than the one proposed in Maks (2006), which only considers strongly separable utility functions. It will also be used to establish the gross substitutes property of demand in a more intuitive manner than Mirman and Ruble (2003), who use complex value orders.

The first step towards the gross substitutes property of demand is the existence of a gross substitute of good  $i$  among the other  $n - 1$  goods.

**Proposition 7.** *If  $u$  is locally nonsatiated, satisfies regularity conditions,  $|\frac{\partial MRS_{ik}}{\partial x_i} \cdot \frac{x_i}{MRS_{ik}}| < (>) 1$  for some  $k \neq i$  and all goods are normal, then there exists a gross substitute (complement) among the other goods for sufficiently small price changes of good  $i$ .*

*Proof.* The following proof will focus on the gross substitutes. The bracketed version for com-

plements can be shown accordingly. Now consider the case when  $p_i$  increases. From above we know that  $\hat{x}_j \geq x_j^*$  and  $\hat{x}_j \geq x_j^{**}$  for all  $j \neq i$ , where  $x^*$  is the original optimal bundle before the price change,  $\hat{x}$  is the optimal bundle at new prices and compensated income and  $x^{**}$  is the new maximizer after the price change.

Since the budget constraint is binding at both the old and new optimal bundle, we know that  $p_i x_i^* + \sum_{j \neq i} p_j x_j^* = w = p_i' x_i^{**} + \sum_{j \neq i} p_j x_j^{**}$ , which can be rewritten as  $\sum_{j \neq i} p_j (x_j^{**} - x_j^*) = p_i x_i^* - p_i' x_i^{**}$ . The total change of expenditures on each of the goods can be decomposed in the change resulting from the substitution and the income effect, resulting in  $\sum_{j \neq i} [p_j (\hat{x}_j - x_j^*) + p_j (x_j^{**} - \hat{x}_j)] = p_i x_i^* - p_i' \hat{x}_i + p_i' \hat{x}_i - p_i' x_i^{**}$ . First consider only the change in expenditures attributed to the income effect. Since  $\hat{x}_i \geq x_i^{**}$  and  $\hat{x}_j \geq x_j^{**}$  for each  $j$ ,  $p_i' (x_i^{**} - \hat{x}_i) < 0$  and  $\sum_{j \neq i} p_j (x_j^{**} - \hat{x}_j) < 0$ . Hence expenditures on both good decrease as a result of the income effect.

The impact of the substitution effect on good  $i$  on the other hand is not as obvious. Using the Slutsky compensation, it can be written as

$$p_i' \hat{x}_i - p_i x_i^* = w' - w - \sum_{j \neq i} p_j (\hat{x}_j - x_j^*) = (p_i' - p_i) x_i^* - \sum_{j \neq i} p_j (\hat{x}_j - x_j^*) \quad (2.6)$$

To use the assumption that  $|\frac{\partial MRS_{ij}}{\partial x_i} \cdot \frac{x_i}{MRS_{ij}}| < 1$ , we first approximate the derivative  $\frac{\partial MRS_{ij}}{\partial x_i}$  by  $\frac{\Delta MRS_{ij}}{\Delta x_i}$  using a linear Taylor approximation.  $\hat{x}_i$  is a continuous function of  $p_i$ , so for sufficiently small price changes  $\frac{\Delta MRS_{ij}}{\Delta x_i} = \frac{MRS_{ij}(\hat{x}) - MRS_{ij}(x^*)}{\hat{x}_i - x_i^*}$ . Assuming that the solution is interior,  $\frac{\Delta MRS_{ij}}{\Delta x_i} = \frac{MRS_{ij}(\hat{x}) - MRS_{ij}(x^*)}{\hat{x}_i - x_i^*} = \frac{p_i' - p_i}{p_j - p_j}$ . Since  $|\frac{\partial MRS_{ij}}{\partial x_i} \cdot \frac{x_i}{MRS_{ij}}| < 1$ , there exists some  $\varepsilon > 0$ , such that  $|\frac{\partial MRS_{ij}}{\partial x_i} \cdot \frac{x_i}{MRS_{ij}}| + \varepsilon \leq 1$ . For sufficiently small price changes,  $|\frac{\Delta MRS_{ij}}{\Delta x_i} \cdot \frac{x_i}{MRS_{ij}} - \frac{\partial MRS_{ij}}{\partial x_i} \cdot \frac{x_i}{MRS_{ij}}| < \varepsilon$ . Therefore,  $|\frac{\Delta MRS_{ij}}{\Delta x_i} \cdot \frac{x_i}{MRS_{ij}}| = \frac{p_i' - p_i}{p_j - p_j} \cdot \frac{x_i^*}{x_i^* - \hat{x}_i} \leq 1$ , which implies  $\frac{p_i' - p_i}{p_j - p_j} \cdot \frac{x_i^*}{x_i^* - \hat{x}_i} \leq 1$ . Again using the Slutsky compensation, this can be rewritten as  $\sum_{j \neq i} p_j (\hat{x}_j - x_j^*) \geq (p_i' - p_i) x_i^* = w' - w$ . This condition and 2.6 imply  $p_i' \hat{x}_i - p_i x_i^* \leq 0$ .

Therefore both the income and the substitution effect lead to a decrease of expenditures on good  $i$ , so  $\sum_{j \neq i} p_j (x_j^{**} - x_j^*) \geq 0$ . Consequently,  $x_{j_0}^{**} \geq x_{j_0}^*$  for some good  $j_0$ .  $\square$

Proposition 7 only guarantees the existence of a gross substitute (complement) among the  $n - 1$  goods for good  $i$ . However, it fails to provide insight in the effect of a price change on each of these  $n - 1$  goods individually. As for the positive substitution effect for all goods other than good  $i$ , the additional assumption, that the marginal rate of substitution between any two goods only depends on the quantities of these two goods, will allow us to draw wider-ranging conclusions in comparative statics problems with price changes.

**Theorem 5.** *If  $u$  is locally nonsatiated, satisfies regularity conditions,  $MRS_{lm}$  is independent of all goods other than  $l$  and  $m$  for all  $l, m$ ,  $|\frac{\partial MRS_{ik}}{\partial x_i} \cdot \frac{x_i}{MRS_{ik}}| < (>) 1$  for some  $k \neq i$  and all goods are normal, then all goods  $j \neq i$  are gross substitutes (complements) of good  $i$ .*

*Proof.* As previously, the proof will focus on the case of gross substitutes. From Proposition 7 we know that there exists a substitute good of good  $i$  among the other goods. At an interior solution,  $MRS_{ij} = \frac{p_i}{p_j}$  for all  $j$  and  $MRS_{jk} = \frac{p_j}{p_k}$  for all  $j \neq k, j, k \neq i$ . As  $p_i$  increases, so does  $MRS_{ij}$ . Since  $MRS_{jk}$  is independent of all goods other than  $j$  and  $k$ , increasing in good  $k$  and  $\frac{p_j}{p_k}$  remains unchanged at the new optimal solution, the marginal utilities of all goods other than  $i$  need to move in the same direction. Because both the substitution effect and the income effect have led to a decrease in the optimal quantity of good  $i$ ,  $MU_i$  has increased. Moreover, the existence of a substitute yields that  $MU_j$  needs to decrease and  $x_j^{**} \geq x_j^*$  for all  $j \neq i$ .  $\square$

**Corollary 2.** *If  $u$  is locally nonsatiated,  $C_j$ -quasisupermodular for all  $j = 1, \dots, n$ , satisfies regularity conditions,  $MRS_{lm}$  is independent of all goods other than  $l$  and  $m$  for all  $l, m$ ,  $|\frac{\partial MRS_{ik}}{\partial x_i} \cdot \frac{x_i}{MRS_{ik}}| < (>) 1$  for some  $k \neq i$  and all goods are normal, then all goods  $j \neq i$  are gross substitutes (complements) of good  $i$ .*

*Proof.* This corollary follows from Theorem 5 by using Quah's result for normality.  $\square$

**Example 14.** CES utility (3 goods case)

Consider the CES utility function  $u(x_1, x_2, x_3) = (x_1^\rho + x_2^\rho + x_3^\rho)^{\frac{1}{\rho}}$ . Like in the previously mentioned 2 goods case,  $u$  is monotone and locally nonsatiated. Furthermore, it is quasiconcave and  $C_1$ -,  $C_2$ - and  $C_3$ -quasisupermodular if  $\rho \leq 1$ . All marginal rates of substitution  $MRS_{ij}$  only depend only goods  $i$  and  $j$  and are decreasing in good  $j$ .<sup>32</sup> Moreover,  $|\frac{\partial MRS_{ij}}{\partial x_i} \cdot \frac{x_i}{MRS_{ij}}| = |\rho - 1| < 1$  for all  $i \neq j$  and  $0 < \rho \leq 1$ .

Therefore, if we consider a price increase of good 1, by Proposition 7 optimal consumption of either good 2 or good 3 has to increase. Theorem 5 goes even further and says that more of both goods will be consumed after the price change. It can easily be seen in footnote 32 that if either  $x_2$  or  $x_3$  goes up, the other one has to increase equally for  $MRS_{23}$  to remain unchanged. This needs to be the case at an interior solution where one of the first order condition requires  $MRS_{23} = \frac{p_2}{p_3}$ , since the price ratio between these two goods has not changed.

The following example is an application of Theorem 5. Milgrom and Shannon (1994) show how results for games with strategic complements can be applied to a general equilibrium model with gross substitutes.

**Example 15.** General Equilibrium Model with Gross Substitutes

Let there be  $L + 1$  goods in an economy with good 0 as the numéraire. For each other good  $n$ , a market maker announces a price  $p_n$  and his objective is to minimize excess market demand  $d_n(p_n, p_{-n}) = \sum_{i \in I} x_{in}(p_n, p_{-n}) - e_n$  for that good, where  $x_{in}$  denotes the demand of consumer  $i$  for good  $n$  and  $e_n$  is the total endowment of it.

So his payoff can be written as  $\pi_n(p_n, p_{-n}) = - |d_n(p_n, p_{-n})|$ .

In Milgrom and Shannon, a game with strategic complementarities is defined as follows.  $N$  players each have a strategy set  $S_n$ , partially ordered by  $\geq_n$ , and a payoff function  $\pi_n(x_n, x_{-n})$ .

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<sup>32</sup> $MRS_{12} = \frac{x_1^{\rho-1}}{x_2^{\rho-1}}$ ,  $MRS_{13} = \frac{x_1^{\rho-1}}{x_3^{\rho-1}}$ ,  $MRS_{23} = \frac{x_2^{\rho-1}}{x_3^{\rho-1}}$

Moreover, the following conditions need to be satisfied for all  $n$  :

1.  $S_n$  is a compact lattice;
2.  $\pi_n$  is upper semi-continuous in  $x_n$  for  $x_{-n}$  fixed and continuous in  $x_{-n}$  for fixed  $x_n$ ;
3.  $\pi_n$  is quasisupermodular in  $x_n$  and satisfies the single crossing property in  $(x_n; x_{-n})$ .

The previously described fictional game with a market maker for each good  $n$  satisfies the conditions for a game with strategic complementarities (GSC) if the economy exhibits gross substitutes. This is the case if  $d_n$  is continuous and decreasing in  $p_n$  and continuous and monotone nondecreasing in  $p_{-n}$  for all  $n$ . The above results can be used to guarantee these properties making assumptions directly on the utility functions instead of the excess demand function. If the individuals' uncompensated demand functions  $x_{in}(p_n, p_{-n}, w_i)$  has the gross substitutes property, then the individual excess demand functions  $x_{in}(p_n, p_{-n}, p \cdot e_i) - e_{in}$  also have the gross substitutes property. In case of a price increase of  $p_{-n}$ , uncompensated demand for good  $n$  increases and the wealth effect goes in the same direction, assuming that the good is normal. The gross substitutes property of aggregate excess demand follows immediately from the individuals' excess demands. Thus  $d_n$  has the gross substitutes property if all individual Marshallian demand functions have this property.

Hence this game is a GSC if the consumers' utility functions are monotone, locally nonsatiated, quasiconcave,  $C_j$ -quasisupermodular for all  $j = 1, \dots, L + 1$ , satisfies regularity conditions,  $MRS_{ij}$  is independent of all goods  $l \neq i, j$  and  $|\frac{\partial MRS_{ik}}{\partial x_i} \cdot \frac{x_i}{MRS_{ik}}| < 1$  for some  $k \neq i$ .

Examples of utility functions that satisfy these conditions include as previously mentioned CES utility for  $\rho \leq 1$  and monotone logarithmic or trans-log utility functions.

## 2.4.2 Gross Substitutability between 2 Goods

If we do not want to limit our attention to utility functions where the marginal rate of substitution between two goods is independent of all other goods, we can still provide conditions under which two goods are gross substitutes. Without the marginal rate of substitution's independence of other

goods, the sign of the substitution effect of a change of  $p_i$  is undetermined for all goods other than good  $i$  and as shown in Proposition 7 we can only guarantee the existence of a gross substitute under the provided restriction on the marginal rate of substitution elasticity. Therefore, an additional assumption will be needed to establish gross substitutability between goods  $i$  and  $k$ .

The following result is based on the previous observation that given that  $|\frac{\partial MRS_{ik}}{\partial x_i} \cdot \frac{x_i}{MRS_{ik}}| < 1$  for some  $k \neq i$ , expenditures on good  $i$  decrease as  $p_i$  goes up (see proof of Proposition 7). The change in expenditures on good  $l$  is the difference between the decrease in expenditures on good  $i$  and the change of all commodities other than  $i$  and  $l$  combined, thus

$$p_l(x_l^{**} - x_l^*) = p_i x_i^* - p_i' x_i^{**} - \sum_{j \neq i, l} p_j (x_j^{**} - x_j^*) \quad (2.7)$$

From this, we see that the demand for good  $l$  is increasing in  $p_i$  if the expenditures on all other goods decrease. From above we already know under what condition expenditures on good  $i$  will decrease. Hence, gross substitutability between good  $i$  and  $l$  can be established if we can provide a condition on  $u$  that yields decreasing aggregate expenditures of all goods other than  $i$  and  $l$ .

**Theorem 6.** *If  $u$  is locally nonsatiated,  $C_j$ -quasisupermodular for all  $j=1, \dots, n$ , satisfies regularity conditions,  $|\frac{\partial MRS_{ik}}{\partial x_i} \cdot \frac{x_i}{MRS_{ik}}| < 1$  for some  $k \neq i$  and*

1.  $\sum_{j \neq i, l} \frac{1}{\frac{\partial MRS_{ij}}{\partial x_j}} < 0$  or
2.  $0 < \frac{\partial MRS_{il}}{\partial x_l} \cdot x_l < 1$ ,

*then good  $l$  is a gross substitute of good  $i$ .*

*Proof.* First, suppose (1) holds. From the proof of Proposition 7, we have  $p_i x_i^* - p_i' x_i^{**} \geq 0$ . Approximate the derivative  $\frac{\partial MRS_{ij}}{\partial x_j} \approx \frac{\Delta MRS_{ij}}{\Delta x_j}$  using a linear Taylor approximation.  $\hat{x}_j$  is a continuous function of  $p_i$ , so for sufficiently small price changes  $\frac{\Delta MRS_{ij}}{\Delta x_j} = \frac{MRS_{ij}(\hat{x}) - MRS_{ij}(x^*)}{\hat{x}_j - x_j^*}$ . Assuming that the solution is interior,  $\frac{\Delta MRS_{ij}}{\Delta x_j} = \frac{MRS_{ij}(\hat{x}) - MRS_{ij}(x^*)}{\hat{x}_j - x_j^*} = \frac{p_i' - p_i}{\hat{x}_j - x_j^*}$ . Since  $\sum_{j \neq i, l} \frac{1}{\frac{\partial MRS_{ij}}{\partial x_j}} < 0$ , there exists some

$\varepsilon > 0$ , such that  $\sum_{j \neq i, l} \frac{1}{\frac{\partial MRS_{ij}}{\partial x_j}} + \varepsilon \leq 0$ . For sufficiently small price changes,  $\sum_{j \neq i, l} \frac{1}{\frac{\Delta MRS_{ij}}{\Delta x_j}} - \sum_{j \neq i, l} \frac{1}{\frac{\partial MRS_{ij}}{\partial x_j}} < \varepsilon$ . Therefore,  $\sum_{j \neq i, l} \frac{1}{\frac{\Delta MRS_{ij}}{\Delta x_j}} = \sum_{j \neq i, l} \frac{p_j}{p'_i - p_i} (\hat{x}_j - x_j^*) = \frac{1}{p'_i - p_i} \sum_{j \neq i, l} p_j (\hat{x}_j - x_j^*) \leq 0$ , which implies

$$\sum_{j \neq i, l} p_j (\hat{x}_j - x_j^*) \leq 0 \quad (2.8)$$

If we now decompose the total change in expenditures for goods  $j \neq i, l$  in the part that stems from the substitution and the part resulting from the income effect, we easily see from equation 2.8 and Proposition 5 that  $\sum_{j \neq i, l} p_j (x_j^{**} - x_j^*) = \sum_{j \neq i, l} p_j (\hat{x}_j - x_j^*) + \sum_{j \neq i, l} p_j (x_j^{**} - \hat{x}_j) \leq 0$ . This together with equation 2.7 implies  $x_l^{**} \geq x_l^*$ .

Now suppose (2) holds. Following the same approximation procedure as above,  $0 < \frac{\partial MRS_{il}}{\partial x_l} \cdot x_l < 1$  yields  $0 \leq \frac{(p'_i - p_i)x_l^*}{p_l(\hat{x}_l - x_l^*)} \leq 1$ . We can rewrite this as  $p_l(\hat{x}_l - x_l^*) \geq (p'_i - p_i)x_l^* = w' - w \geq 0$ . Because all goods are normal, the expenditure decrease due to the income effect on each good  $j$  is at most the amount of compensated income  $w' - w$  ( $\hat{x}_j - x_j^{**} \leq w' - w$  for all  $j$ ). Using these two inequalities, we obtain that  $p_l(x_l^{**} - x_l^*) = p_l(\hat{x}_l - x_l^*) - p_l(\hat{x}_l - x_l^{**}) \geq 0$ . Hence  $x_l^{**} \geq x_l^*$ .  $\square$

To give some intuition behind this result, first notice that in the first additional condition requires that the aggregate of all goods other than  $i$  and  $l$  is a net complement of good  $i$ . In combination with the negative sign of the income effect, the aggregate of all other goods is also a gross complement of good  $i$ . As the condition on the absolute elasticity of MRS with respect to good  $k$  yields existence of a gross substitute, it has to be good  $l$ .

The second possible additional condition first of all requires good  $l$  to have a positive substitution effect, which necessarily needs to be the case for a gross substitute in order to outweigh the negative income effect for normal goods. Second, this condition results in a very strong substitution effect and even if the entire compensating amount that is being taken away in the income effect were to come from that particular good, overall expenditures on good  $l$  would still have increased and therefore also demand.

### Example 16. CES Utility

Consider the CES utility function  $u(x_1, \dots, x_n) = \left( \sum_{i=1}^n x_i^\rho \right)^{\frac{1}{\rho}}$  and suppose  $p_i$  increases to  $p'_i$ . Like in the previously mentioned examples,  $u$  is monotone and locally nonsatiated. Furthermore, it is quasiconcave and  $C_i$ -quasisupermodular for all  $i$  if  $\rho \leq 1$ . The marginal rate of substitution between good  $i$  and  $l$  is equal to  $MRS_{il} = \frac{x_i^{\rho-1}}{x_l^{\rho-1}}$ . Thus,  $\frac{\partial MRS_{il}}{\partial x_l} \cdot x_i = (1 - \rho) \frac{x_i^\rho}{x_l^\rho} \geq 0$  for  $\rho \leq 1$ .

It is not straightforward to see when  $\frac{\partial MRS_{il}}{\partial x_l} \cdot x_i < 1$ , as this will depend on the ratio between  $x_i$  and  $x_l$ . At optimal bundles, this ratio is  $\frac{x_i^*}{x_l^*} = \left( \frac{p_l}{p_i} \right)^{\frac{1}{1-\rho}}$ . Then,  $\frac{\partial MRS_{il}}{\partial x_l} \cdot x_i = (1 - \rho) \left( \frac{p_l}{p_i} \right)^{\frac{1}{1-\rho}}$ . Hence the value of  $\rho$  for which this expression will be less than 1 depends on the price ratio. When solving this equation for  $\rho$ , we get the transcendental equation  $(1 - \rho)^{1-\rho} < \frac{p_i}{p_l}$ .

## 2.5 Hicksian Effects

When considering income and substitution effect of a price change, there are two different approaches regarding the consumer's compensation. The previously used Slutsky compensation provides the consumer with enough additional income to afford the initial optimal bundle at new prices. Under Hicksian compensation, the consumer's income gets adjusted so that his utility level does not change. The substitution effect then captures how Hicksian demand changes with prices and the sign of the substitution effect classifies goods as net substitutes or complements.

By Shephard's Lemma, we know that  $h_j(p, u) = \frac{\partial e(p, u)}{\partial p_j}$ , where  $h_j(p, u)$  denotes the Hicksian demand for good  $j$  at price  $p$  and utility level  $u$ ,  $e(p, u)$  is the respective expenditure function. We note that net substitutability (complementarity) are equivalent to super- (sub-) modularity of the expenditure function, since  $\frac{\partial h_j(p, u)}{\partial p_i} = \frac{\partial^2 e(p, u)}{\partial p_j \partial p_i}$ .

Again, a question of interest is what conditions on the primitives, namely the utility function, will yield Hicksian substitutes or complements. Allen (1934) provides a characterization of complementary and substitute goods based on the marginal rate of substitution responds to changes in both commodities. The following proposition uses a similar notion for Hicksian demand.



**Proposition 8.** *Given any utility level  $u$ , if  $MRS_{ij}$  is increasing (decreasing) in  $x_j$ , then goods  $i$  and  $j$  are net substitutes (complements).*

*Proof.* Consider a price increase of good  $i$  for the substitutes case. Given that the solution to the consumer's utility maximization problem is interior, at optimum  $MRS_{ij} = \frac{p_i}{p_j}$ . Let  $x^* = h(p, u)$  be the optimal bundle at old prices and  $\hat{x} = h(p', u)$  the optimizer after the price change when the consumer is compensated to maintain the initial level of utility. Then  $MRS_{ij}(\hat{x}) = \frac{p'_i}{p_j} > \frac{p_i}{p_j} = MRS_{ij}(x^*)$ . Hence  $\hat{x}_j > x^*_j$ .  $\square$

The following corollary states the result for the differentiable case.

**Corollary 3.** *If  $\frac{\partial MRS_{ij}}{\partial x_j} > (<) 0$ , goods  $i$  and  $j$  are net substitutes (complements).*

*Proof.* The proof focuses on the net substitutes case. The bracketed version can be shown in the same fashion. As above, approximate  $\frac{\partial MRS_{ij}}{\partial x_j} \approx \frac{\Delta MRS_{ij}}{\Delta x_j}$ . For sufficiently small price changes of  $p_i$ ,  $\frac{\Delta MRS_{ij}}{\Delta x_j} = \frac{MRS_{ij}(\hat{x}) - MRS_{ij}(x^*)}{\hat{x}_j - x^*_j}$ . Assuming that the solution is interior,  $\frac{\Delta MRS_{ij}}{\Delta x_j} = \frac{MRS_{ij}(\hat{x}) - MRS_{ij}(x^*)}{\hat{x}_j - x^*_j} = \frac{\frac{p'_i}{p_j} - \frac{p_i}{p_j}}{\hat{x}_j - x^*_j} = \frac{\frac{p'_i - p_i}{p_j}}{x_j(p', e(p', u)) - x_j(p, e(p, u))} = \frac{\frac{p'_i - p_i}{p_j}}{h_j(p', u) - h_j(p, u)}$ . Since  $\frac{\partial MRS_{ij}}{\partial x_j} > 0$ , there exists some  $\varepsilon > 0$ , such that  $\frac{\partial MRS_{ij}}{\partial x_j} - \varepsilon \geq 0$ . For sufficiently small price changes,  $|\frac{\Delta MRS_{ij}}{\Delta x_j} - \frac{\partial MRS_{ij}}{\partial x_j}| < \varepsilon$ . Therefore,  $\frac{\Delta MRS_{ij}}{\Delta x_j} = \frac{\frac{p'_i - p_i}{p_j}}{h_j(p', u) - h_j(p, u)} \geq 0$ , which implies  $h_j(p', u) - h_j(p, u) \geq 0$ .  $\square$

Necessary conditions are provided in the following proposition and corollary.

**Proposition 9.** *If goods  $i$  and  $j$  are net substitutes (complements), then  $MRS_{ij}$  is nondecreasing (nonincreasing) in  $x_j$ .*

*Proof.* Suppose good  $i$  and good  $j$  are net substitutes, that is  $h_j(p', u) \geq h_j(p, u)$  for a price increase of good  $i$  from  $p_i$  to  $p'_i$ . Then  $\frac{\frac{p'_i - p_i}{p_j}}{h_j(p', u) - h_j(p, u)} \geq 0$ . Assuming that the solution is interior,  $\frac{\frac{p'_i - p_i}{p_j}}{h_j(p', u) - h_j(p, u)} = \frac{\frac{p'_i - p_i}{p_j}}{x_j(p', e(p', u)) - x_j(p, e(p, u))} = \frac{\frac{p'_i - p_i}{p_j}}{\hat{x}_j - x^*_j} = \frac{MRS_{ij}(\hat{x}) - MRS_{ij}(x^*)}{\hat{x}_j - x^*_j} = \frac{\Delta MRS_{ij}}{\Delta x_j} \geq 0$ . Since the denominator is nonnegative, this implies  $MRS_{ij}(\hat{x}) \geq MRS_{ij}(x^*)$ .  $\square$

**Corollary 4.** *If goods  $i$  and  $j$  are strict net substitutes (complements), then  $\frac{\partial MRS_{ij}}{\partial x_j} \geq (\leq) 0$ .*

*Proof.* Suppose good  $i$  and good  $j$  are strict net substitutes, that is  $h_j(p', u) > h_j(p, u)$  for a price

increase of good  $i$  from  $p_i$  to  $p'_i$ . Then  $\frac{\frac{p'_i - p_i}{p_j}}{h_j(p', u) - h_j(p, u)} > 0$ . Assuming that the solution is interior,

$\frac{\frac{p'_i - p_i}{p_j}}{h_j(p', u) - h_j(p, u)} = \frac{\frac{p'_i - p_i}{p_j}}{x_j(p', e(p', u)) - x_j(p, e(p, u))} = \frac{\frac{p'_i - p_i}{p_j}}{\hat{x}_j - x_j^*} = \frac{MRS_{ij}(\hat{x}) - MRS_{ij}(x^*)}{\hat{x}_j - x_j^*} = \frac{\Delta MRS_{ij}}{\Delta x_j} > 0$ . For sufficiently small price changes,  $\exists \varepsilon > 0$  such that  $\frac{\Delta MRS_{ij}}{\Delta x_j} + \varepsilon \geq 0$  and  $|\frac{\Delta MRS_{ij}}{\Delta x_j} - \frac{\partial MRS_{ij}}{\partial x_j}| < \varepsilon$ .

Hence  $\frac{\partial MRS_{ij}}{\partial x_j} \geq 0$ . □

#### **Example 17.** Cobb-Douglas Utility

Consider a standard Cobb-Douglas utility function  $u(x_1, \dots, x_n) = \prod_{i=1}^n x_i^{\alpha_i}$ , where  $\alpha_i > 0$  for all  $i$ .

Instead of computing the Hicksian, we can apply the above results and it can easily be seen that

$MRS_{ij} = \frac{\alpha_i}{\alpha_j} \cdot \frac{x_j}{x_i}$  is increasing in  $x_j$ . Therefore goods  $i$  and  $j$  are net substitutes.

Ultimately however we are interested in providing conditions under which goods are gross substitutes or complements, taking into account demand changes due to the income effect as well.

From the above proposition and Quah's normality result, we can immediately establish gross complementarity between two goods if the utility function satisfies the following properties.

**Corollary 5.** *If  $u$  is  $C_j$ -quasisupermodular and  $\frac{\partial MRS_{ij}}{\partial x_j} < 0$ , then good  $j$  is a gross complement of good  $i$ .*

*Proof.* From Proposition 7 it follows that the substitution effect of a price increase of good  $i$  is negative, as is the income effect by Proposition 5. Hence the overall effect is negative and the goods are gross complements. □

Since the above result requires normality and net complementarity, these assumptions are rather strict and don't include the case when two goods are net substitutes, but gross complements because of the income effect. In the previous sections, when using Slutsky instead of Hicksian compensation, the proofs of Theorem 4 and 7 made use of the direction of the change in expenditures of good  $i$  as its price changed. These results can be shown in similar fashion using Hicksian compensation.<sup>33</sup>

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<sup>33</sup>Consider the gross substitutes case. In the proof of Theorem 4, we see that the elasticity condition eventually leads to the implication that expenditures on good  $i$  decrease. If we only consider the condition before using the Slutsky compensation, it implies  $\frac{p'_i - p_i}{x_i^* - \hat{x}_i} \cdot \frac{x_i^*}{p_i} \leq 1$ , which can be rewritten as  $\frac{x_i^* - \hat{x}_i}{p'_i - p_i} \geq \frac{x_i^*}{p_i}$ . Alternatively, consider the expenditure function for good  $i$ ,  $e_i(p, u) = p_i h_i(p, u)$ . An expenditure decrease on good  $i$  as a result of the substitution effect,  $p'_i h_i(p', u) - p_i h_i(p, u) \leq 0$  can be expressed as  $-\frac{h_i(p', u) - h_i(p, u)}{p'_i - p_i} = \frac{x_i^* - \hat{x}_i}{p'_i - p_i} \geq \frac{h_i(p, u)}{p_i} = \frac{x_i^*}{p_i}$ . This is equivalent to what we obtained from the elasticity condition. The results from Theorems 4 and 7 follow from here.

# Chapter 3

## Environmental Regulation and R&D:

### A Monotone Comparative Statics Approach

#### 3.1 Introduction

Traditionally, economists held the view that restricting firms' choices by requiring them to reduce pollution in the production process would have a negative effect on their profits. Porter (1991) challenges this belief and suggests that stricter environmental regulation encourages innovation in more efficient technology and can give the firm a competitive advantage, which has since become known as the "Porter hypothesis". While Porter was not the first to point out a positive relationship between pollution regulation and incentives for innovation, his work has been the foundation for much of the subsequent research on this topic.

Besides cost-efficiency, the question how different policy instruments such as pollution standards, taxes or market-based permit systems "spur new technology toward the efficient conservation of the environment" is an important criterion when evaluating their performance, as Kneese and Schulze (1975) point out.

This chapter distinguishes itself from the existing literature on the question how environmental regulation affects innovation through the use of a monotone comparative statics approach that

relies on lattice-theoretic concepts. These methods have mainly been developed in Topkis (1978), Vives (1990), Milgrom and Roberts (1990) and Milgrom and Shannon (1994). Of particular interest throughout this chapter is the monotonicity theorem in Milgrom and Shannon (1994), which provides necessary and sufficient conditions on the primitives for nondecreasing solutions in the parameters of the constrained optimization problem.

Imposing this minimal monotonicity structure on the model in the form of complementarity assumptions on the primitives allows me to derive comparative statics results with respect to changes in environmental policy variables. Besides less restrictive assumptions, an advantage of this approach is that it uncovers some economic intuition behind the relationship between regulation and innovation.

A large part of the literature on environmental regulation and R&D<sup>34</sup> focuses on comparing investment incentives under different regulation approaches (e.g., see Wenders (1975), Downing and White (1986), Milliman and Prince (1989), Jung et al. (1996) or Bruneau (2004)).

Montero (2002a, 2002b) look beyond perfectly competitive environments and compare investment R&D incentives of different policy instruments in oligopolistic markets. Not only are real-world markets rarely perfectly competitive and this extension to imperfect competition can provide policy implications regarding environmental regulation, but also the industrial organization literature has shown that strategic interaction in markets can significantly affect investment decisions such as cost-reducing R&D (see Brander and Spencer (1983), Spence (1984), Fudenberg and Tirole (1984) or Bulow et al. (1985)).<sup>35</sup> Simpson and Bradford (1996) also considers the case of imperfect competition and shows that governments can provide a strategic advantage to domestic industry through stricter environmental regulation (effluent taxes). They also study the effect of taxes on innovation expenditures, but find that its direction and magnitude are difficult to analyze and providing general solutions seems impossible.

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<sup>34</sup>For an overview of the theoretical literature, see Requate (2005). A survey on the empirical literature can be found in Jaffe et al. (2002).

<sup>35</sup>A survey on strategic interaction in oligopoly markets can be found in Shapiro (1989) and Tirole (1988).

Other papers focus on the effect of increasing tightness of regulation on R&D incentives. Oates et al. (1993) shows that increasing pollution tax rates increases the firm's incentive to invest in pollution abatement technology in perfectly competitive industries. Biglaiser and Horowitz (1994) on the other hand find that stricter technology adoption standards decreases the number of firms undertaking research.

It is important to point out that investment incentives here can mean either adoption of already existing new technology or R&D in new technology. In the survey by Requate (2005), innovation (rather than adoption) occurs when there is uncertainty about the R&D success, patents are granted for innovations, if technology spillovers occur or when imitation is possible. Papers that fall in the adoption category are for example Downing and White (1986), Milliman and Prince (1989), Jung et al. (1996), Montero (2002a) and Bruneau (2004). Biglaiser and Horowitz (1994) and Montero (2002b) are examples for models of innovation and diffusion. This distinction between adoption and innovation is not always sharp. Phaneuf and Requate (2002), Petrakis and Xepapadeas (2003), or Gersbach and Requate (2004), for example, assume cost functions that depend on the amount of investment into abatement equipment or R&D effort.

In this chapter, I provide monotone comparative statics results for the effect of more stringent environmental regulation on innovation spending. I consider three different regulatory instruments (emissions standard, effluent tax and effluent tax in combination with a technology adoption standard) in an imperfectly competitive output market, where two firms compete in Cournot fashion. I find that a form of weak complementarity between environmental R&D investment and the policy variable plays a crucial role in the case when tighter regulation has a positive effect on innovation.

Moreover, I provide assumptions on demand, Cournot output and cost functions that yield such profit functions. I find that profit functions for which the Porter hypothesis holds consist of concave demand and convex abatement and production costs. Furthermore, the marginal effect of investment in innovation on the other firm's Cournot output needs to be decreasing in the policy variable and the size of spillover effects from investment in innovation by one firm on the other firm needs to be limited, that is the effects of investment and policy variable on the price through

the investing firm's quantity is greater than the spillover through the other firm.

These results provide some economic intuition under what circumstances we should expect the Porter hypothesis to hold. The monotone comparative statics results apply in a more general framework than the existing literature as they require less restrictive assumptions on the objective function and R&D production function.

The remainder of the chapter is organized as follows: Section 2 introduces the lattice-theoretic background, sections 3 through 5 provide monotone comparative statics results for emissions standards, effluent taxes and technology adoption standards and section 6 concludes.

## 3.2 Preliminaries

To obtain monotone comparative statics results with respect to changes in the constraint set we need to be able to order these sets. Milgrom and Shannon (1994) use the strong set order (see Definition 1).

Besides an order on the constraint sets, we also need assumptions on the objective function for monotone comparative statics. The monotonicity theorem in Milgrom and Shannon (1994) requires quasisupermodularity, which can be interpreted as weak complementarity between the choice variables.

### **Definition 9.** Quasisupermodularity

Let  $X$  be a lattice. A function  $f : X \rightarrow \mathbb{R}$  is quasisupermodular if

1.  $f(x) \geq f(x \wedge y) \implies f(x \vee y) \geq f(y)$
2.  $f(x) > f(x \wedge y) \implies f(x \vee y) > f(y)$

Milgrom and Shannon (1994) also include comparative statics with respect to parameter changes in the objective function. Therefore, their result also requires the objective function to satisfy the (standard) Single Crossing Property (see Definition 5).

With these definitions in place we can state the monotone comparative statics theorem that will be used throughout the chapter.

**Theorem 7.** *Monotonicity Theorem, Milgrom and Shannon (1994)*

*Let  $f : X \times T \rightarrow \mathbb{R}$ , where  $X$  is a lattice,  $T$  is a partially ordered set and  $S \subset X$ . Then  $\operatorname{argmax}_{x \in S} f(x, t)$  is monotone nondecreasing in  $(t, S)$  if and only if  $f$  is quasisupermodular in  $x$  and satisfies the Single Crossing Property in  $(x; t)$ .*

*Proof.* See Milgrom and Shannon (1994). □

### 3.3 Emissions Standards in Imperfectly Competitive Markets

While a large part of the literature on environmental regulation and R&D incentives assumes perfectly competitive output markets, Montero (2002a) is one paper that addresses the case of imperfect competition, focusing on oligopoly settings in particular. Real world markets are rarely perfectly competitive and hence the question how environmental regulation affects firms' investment in innovation has important policy implications. The main objective of Montero's paper is to compare R&D incentives under different types of regulation, namely emissions and performance standards as well as tradeable permits and auctioned permits, and finds a variety of situations where investment incentives are greater under standards than permits.

As the goal of this chapter is to approach the Porter hypothesis using monotone comparative statics results, a question of interest is how stricter environmental regulation affects a firm's investment in R&D in oligopoly markets. As Montero (2002a) points out, most markets in the real world are not perfectly competitive which makes the oligopoly case an interesting and policy-relevant one. The monotone comparative statics approach to this problem provides assumptions for the Porter hypothesis to hold directly on the firm's objective function.



### 3.3.1 Model

Following the model in Montero (2002a), consider 2 firms in a market that is subject to environmental regulation. Firm  $i$  produces output  $q_i$  at a cost of  $\tilde{C}_i(q_i)$ , which differs from the assumption in Montero (2002a) that firms produce at no cost. Inverse demand is given as  $P(Q)$ , where  $Q$  denotes industry output  $Q = q_1 + q_2$ . If unregulated the firm produces  $q_i$  units of emission; abatement costs are  $C_i(q_i - e_i)$  with  $C' \geq 0$ <sup>36</sup>, where  $e_i$  is the amount of emission after abatement. Investments of  $K_i$  in environmental R&D lower the firm's abatement costs from  $C_i(q_i - e_i)$  to  $k_i C_i(q_i - e_i)$  where  $k_i = f_i(K_i)$ , with  $f_i$  decreasing in  $K_i$ . Note that these assumptions on the R&D production function are weaker than in Montero (2002a), who assumes  $f(0) = 1$ ,  $f(\infty) > 0$ ,  $f' < 0$ ,  $f'' > 0$  and  $f''' < 0$ .

The regulator's goal is to limit total emissions by both firms to  $\bar{E} = e_1 + e_2$ . One of the policy instruments considered in Montero (2002a) is an emissions standard that is instituted before R&D spending by the firms, but the regulator will not adjust the standard based on firms' investments.<sup>37</sup>

The model involves two stages. In the first stage, the firms decide on their investment in environmental R&D  $K_i$  or indirectly through the resulting  $k_i$  on their abatement cost reduction. In the second stage, when  $K_i$  and  $K_j$  are known to all firms, they choose output  $q_i$  and  $q_j$  as well as emissions levels  $e_i$  and  $e_j$ . As in Brander and Spencer (1983), for example, I assume that firms have complete information and in the first stage correctly anticipate the Cournot-Nash output for any given level of investments.

Firm  $i$ 's profit maximization problem in stage 2 can be written as

$$\begin{aligned} \max \quad & \pi_i = P(Q) \cdot q_i - \tilde{C}_i(q_i) - k_i \cdot C_i(q_i - e_i) \\ & \text{subject to } e_i \leq \bar{e}_i, \end{aligned}$$

where  $\bar{e}_i$  is the emissions standard for firm  $i$ .

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<sup>36</sup>Montero (2002a) assumes strictly increasing and convex abatement costs. For my results, I can weaken those assumptions to nondecreasing and convex abatement costs.

<sup>37</sup>As pointed out in Montero (2002a), alternatively one could assume the regulator cannot observe R&D expenditures or observes them after the fact.

Hence, in the first stage, the firm chooses  $k_i$  to maximize

$$\pi_i(q_i^*, q_j^*, k_i) = P(q_i^*, q_j^*) \cdot q_i^* - \tilde{C}_i(q_i^*) - k_i \cdot C_i(q_i^* - e_i^*),$$

where  $q_i^* = q_i^*(k_i, k_j, e_i^*, e_j^*)$  and  $q_j^* = q_j^*(k_i, k_j, e_i^*, e_j^*)$ .

### 3.3.2 Monotone Comparative Statics

The main question that this chapter is trying to address is how tighter environmental regulation affects R&D investment. To analyze this monotone comparative statics problem we need to look at the firm's profit function in the first stage of the game, when the firm makes the investment decision based on the expected Cournot outcome in the last period. Firm  $i$ 's objective function thus becomes

$$\pi_i(q_i^*, q_j^*, k_i) = P(q_i^*, q_j^*) \cdot q_i^* - \tilde{C}_i(q_i^*) - k_i \cdot C_i(q_i^* - e_i^*),$$

where  $q_i^* = q_i^*(k_i, k_j, e_i^*, e_j^*)$  and  $q_j^* = q_j^*(k_i, k_j, e_i^*, e_j^*)$ .

As profits are strictly increasing in  $e_i$ , the firm will choose a level of emissions  $e_i^*$  equal to their allowance  $\bar{e}_i$  in the second stage. Hence we can rewrite the first stage objective function as

$$\pi_i(q_i^*, q_j^*, k_i) = P(q_i^*, q_j^*) \cdot q_i^* - \tilde{C}_i(q_i^*) - k_i \cdot C_i(q_i^* - \bar{e}_i)$$

with  $q_i^* = q_i^*(k_i, k_j, \bar{e}_i, \bar{e}_j)$  and  $q_j^* = q_j^*(k_i, k_j, \bar{e}_i, \bar{e}_j)$ .

So we can view the firm's first stage objective function as parametrized by  $\bar{e}_i$  and apply existing monotone comparative statics results to address the question how changes in this parameter, such as stricter environmental regulation, affect investment in R&D.

**Proposition 10.** Let firm  $i$ 's profit function during the investment stage be  $\pi_i(q_i^*(\bar{e}_i), q_j^*(\bar{e}_i), k_i) : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ . Then

1. stricter environmental regulation leads to nondecreasing investment in R&D if and only if  $\pi_i$  satisfies the Single Crossing Property in  $(k_i; \bar{e}_i)$ .
2.  $\pi_i$  satisfies the Single Crossing Property in  $(k_i; \bar{e}_i)$  if

$$(a) \frac{\partial^2 P}{\partial q_i^2} \leq 0, \frac{\partial^2 P}{\partial q_j^2} \leq 0 \ (i \neq j)$$

$$(b) \frac{\partial^2 q_j^*}{\partial k_i \partial \bar{e}_i} \leq 0$$

$$(c) \frac{\partial P}{\partial q_i} \cdot \frac{\partial q_i^*}{\partial k_i} \geq -\frac{\partial P}{\partial q_j} \cdot \frac{\partial q_j^*}{\partial k_i}$$

$$(d) \frac{\partial P}{\partial q_i} \cdot \frac{\partial q_i^*}{\partial \bar{e}_i} \geq -\frac{\partial P}{\partial q_j} \cdot \frac{\partial q_j^*}{\partial \bar{e}_i}$$

$$(e) C_i'(q_i - e_i) \geq 0$$

$$(f) \frac{\partial C_i}{\partial q_i} \geq 0, \frac{\partial^2 C_i}{\partial q_i^2} \geq 0$$

$$(g) \frac{\partial \tilde{C}_i}{\partial q_i} \geq 0, \frac{\partial^2 \tilde{C}_i}{\partial q_i^2} \geq 0$$

*Proof.* (1) also follows almost immediately from the Monotonicity Theorem (Theorem 4) in Milgrom and Shannon (1994). As there is only one variable of choice,  $k_i$ , the profit function satisfies supermodularity and thus also quasisupermodularity. Consequently,  $\pi_i$  satisfies the SCP in  $(k_i; \bar{e}_i)$  if and only if  $k_i^*$  is nondecreasing in  $\bar{e}_i$ , that is  $k_i^* \leq k_i^{*'}$  for  $\bar{e}_i \leq \bar{e}_i'$ . Recall that  $k_i$  is the result of investment spending  $K_i$  by  $k_i = f(K_i)$  with  $f' < 0$ . Thus  $K_i^{*' \leq K_i^*$ , which means that investment in R&D is higher when the environmental standard is stricter.

(2) The Single Crossing Property is implied by increasing differences, which for twice differentiable profit functions in this case is equivalent to  $\frac{\partial^2 \pi_i}{\partial k_i \partial \bar{e}_i} \geq 0$ . Calculation of this cross-partial yields

$$\begin{aligned}
\frac{\partial^2 \pi_i}{\partial k_i \partial \bar{e}_i} &= \left( \frac{\partial^2 P}{\partial q_i^2} \cdot \frac{\partial q_i^*}{\partial \bar{e}_i} \cdot \frac{\partial q_i^*}{\partial k_i} + \frac{\partial^2 P}{\partial q_j^2} \cdot \frac{\partial q_j^*}{\partial \bar{e}_i} \cdot \frac{\partial q_j^*}{\partial k_i} + \frac{\partial P}{\partial q_j} \cdot \frac{\partial^2 q_j^*}{\partial k_i \partial \bar{e}_i} \right) \cdot q_i^* \\
&+ \left( \frac{\partial P}{\partial q_i} \cdot q_i^* + P - \frac{\partial \tilde{C}_i}{\partial q_i} - k_i \cdot \frac{\partial C_i}{\partial q_i} \right) \cdot \frac{\partial^2 q_i^*}{\partial k_i \partial \bar{e}_i} + \left( \frac{\partial P}{\partial q_i} \cdot \frac{\partial q_i^*}{\partial \bar{e}_i} + \frac{\partial P}{\partial q_j} \cdot \frac{\partial q_j^*}{\partial \bar{e}_i} \right) \cdot \frac{\partial q_i^*}{\partial k_i} \\
&+ \left( \frac{\partial P}{\partial q_i} \cdot \frac{\partial q_i^*}{\partial k_i} + \frac{\partial P}{\partial q_j} \cdot \frac{\partial q_j^*}{\partial k_i} \right) \cdot \frac{\partial q_i^*}{\partial \bar{e}_i} - \frac{\partial^2 \tilde{C}_i}{\partial q_i^2} \cdot \frac{\partial q_i^*}{\partial \bar{e}_i} \cdot \frac{\partial q_i^*}{\partial k_i} \\
&- \frac{\partial \tilde{C}_i}{\partial q_i} \cdot \frac{\partial^2 q_i^*}{\partial k_i \partial \bar{e}_i} - C_i'(q_i^* - \bar{e}_i) \cdot \left( \frac{\partial q_i^*}{\partial \bar{e}_i} - 1 \right) - k_i \cdot \frac{\partial^2 C_i}{\partial q_i^2} \cdot \frac{\partial q_i^*}{\partial \bar{e}_i} \cdot \frac{\partial q_i^*}{\partial k_i}
\end{aligned}$$

Innovation expenditures and a higher (less strict) emissions standard are assumed to decrease the firm's marginal cost. From the standard Cournot model we know that the optimal output is decreasing in the firm's own marginal cost, hence  $\frac{\partial q_i^*}{\partial k_i} < 0$ <sup>38</sup> and  $\frac{\partial q_i^*}{\partial \bar{e}_i} > 0$ . As the other firm's cost reduction from the spillover is smaller than that for the firm investing in R&D, the effect of firm  $i$ 's lower marginal cost on firm  $j$ 's output dominates and  $\frac{\partial q_j^*}{\partial k_i} > 0$ . As usual, assume inverse demand is downward sloping. Note that  $\frac{\partial P}{\partial q_i} \cdot q_i^* + P$  is equal to marginal revenue, hence the term  $\left( \frac{\partial P}{\partial q_i} \cdot q_i^* + P - \frac{\partial \tilde{C}_i}{\partial q_i} - k_i \cdot \frac{\partial C_i}{\partial q_i} \right)$  is equal to zero at output level  $q_i^*$ . Lastly, we need to take a closer look at the last term,  $\left( \frac{\partial q_i^*}{\partial \bar{e}_i} - 1 \right)$ . The model assumes that without abatement each unit of output creates one unit of emissions. Thus  $q_i^*$  can increase at most by one unit for each unit of additional permissible emissions and  $\frac{\partial q_i^*}{\partial \bar{e}_i} \leq 1$  which makes the bracketed term less than or equal to 0. Now using assumptions (a)-(g), we can conclude that  $\frac{\partial^2 \pi_i}{\partial k_i \partial \bar{e}_i} \geq 0$ , which implies the SCP.  $\square$

While the above condition looks rather intricate, the signs of most of the terms are rather straightforward and economically intuitive. In the first assumption,  $\frac{\partial^2 P}{\partial q_j^2}$  and  $\frac{\partial^2 P}{\partial q_i^2}$ , describe the curvature of the inverse demand curve. For the commonly used linear inverse demand, these terms are equal to zero. The literature on Cournot competition<sup>39</sup> states that some form of concavity of inverse demand is a sufficient condition for the existence and uniqueness of a Nash equilibrium,

<sup>38</sup>Recall that higher innovation expenditures decrease  $k_i$  and thus marginal cost. Therefore increases in  $k_i$  are increases in marginal cost, which result in decreasing Cournot output.

<sup>39</sup>see Rosen (1965) and Szidarovszky and Yakowitz (1977)

hence it may be reasonable to assume that  $\frac{\partial^2 P}{\partial q_i^2} \leq 0$  and  $\frac{\partial^2 P}{\partial q_j^2} \leq 0$ .

The next assumption is on the term  $\frac{\partial^2 q_j^*}{\partial k_i \partial \bar{e}_i}$ , the change in the marginal effect of  $k_i$  on  $q_j^*$  as environmental standards change. It seems reasonable to expect that the effect of changes in the standard on the decrease of  $q_j^*$  with respect to  $k_i$  is smaller when regulation is strict, because in that case firm  $i$ 's advantage of lower costs is partially offset by stricter regulation. Thus,  $\frac{\partial^2 q_j^*}{\partial k_i \partial \bar{e}_i} \leq 0$ .

The third assumption requires the effect of  $k_i$  on the price through  $q_i$  to be larger than its impact through  $q_j$ . In Cournot competition, where  $q_i$  and  $q_j$  are perfect substitutes this is reasonable to assume, since the magnitude of the direct effect of R&D expenditures on  $q_i$  is expected to exceed the absolute value of the spillover effect on  $q_j$ . Similarly, (d) assumes that the absolute value of the effect of the environmental standard on the price through  $q_i$  is larger than through  $q_j$ , which also seems reasonable to assume as we can expect the direct effect of the change in the standard on the firm it is imposed on to be greater than the spillover to the other firm.

By earlier assumption,  $C' \geq 0$  as abatement costs are generally nondecreasing. The last two assumptions are increasing and convex abatement and production costs.

To summarize, the condition in (2) is nonnegative under not very restrictive assumptions that lie within the modeling framework or are economically intuitive on the profit function. In comparison with Montero (2002a), this result replicates the finding that there are investment incentives when emissions regulation is tightened. The result is obtained in a more general framework that includes costs of production and constant marginal abatement costs, while making economically reasonable assumptions regarding demand, costs and firm  $j$ 's Cournot output.

Overall, it provides assumptions on the demand and cost functions as well as the optimal second period Cournot output that if satisfied support the Porter hypothesis.

## 3.4 Effluent Taxes in Imperfectly Competitive Markets

The Porter hypothesis in the case of effluent taxes in an imperfectly competitive has been addressed in Simpson and Bradford (1996). They use a strategic trade model similar to Brander and Spencer (1985), where the government imposes an effluent tax on the domestic firm as an incentive to invest in cost-reducing technology and hence having a competitive advantage over its foreign rival. They find that the effect of higher effluent taxes on innovation expenditures is difficult to analyze and provide some specific examples when the Porter hypothesis holds.

### 3.4.1 Model

A domestic and a foreign firm are the only producers of a good and produce quantities  $q_D$  and  $q_F$  respectively. The goods are perfect substitutes, thus inverse demand  $P(Q)$  is a function of the sum of  $q_D$  and  $q_F$ . The production process leads to environmental pollution and the pollutant is taxed at rate  $t$  by the country's government. The firms can invest  $x_i$  ( $i = D, F$ ) in cost-reducing R&D and thus their cost function,  $C_D$  or  $C_F$ , depends on their own output, their investment in innovation and their rival's R&D spending. The latter allows for innovation spillovers, but we assume that the cost-reducing effect is smaller than for the innovating firm. The emissions tax  $t$  only directly influences the domestic firm's costs.

In the first stage of the game, the government chooses the tax rate  $t$ , then the firms decide on how much to invest in R&D and in the final stage they choose  $q_D$  and  $q_F$ . As in the previous section, I will focus on the firms' choices in the last two stages.

In the last stage, the firms compete in Cournot fashion and maximize  $P(Q) \cdot q_i - C_i$  ( $i = D, F$ ). In the innovation decision stage, the firms choose  $x_i$  to maximize

$$P(q_i^*, q_j^*) \cdot q_i^* - C_i - x_i, \quad i, j = D, F, \quad i \neq j,$$

where  $q_i^* = q_i^*(x_i, x_j, t)$  and  $q_j^* = q_j^*(x_i, x_j, t)$  denote the Cournot quantities from the final stage.

### 3.4.2 Monotone Comparative Statics

Simpson and Bradford (1994) points out that a general answer to the question how investment in R&D is influenced by the emissions tax is impossible and that the analysis becomes very complex. Therefore, they consider specific examples where stricter environmental regulation leads to an increase in innovation expenditures. The goal of this chapter is to provide assumptions for the general case that are consistent with the Porter hypothesis.

First we consider the general case where technology spillovers may occur.

**Proposition 11.** *Let  $\pi_i(q_i^*(t), q_j^*(t))$  be firm  $i$ 's profit function in the investment stage. Then*

1. *stricter environmental regulation leads to nondecreasing investment in R&D if and only if  $\pi_i$  satisfies the Single Crossing Property in  $(x_i; t)$ .*
2.  *$\pi_i$  satisfies the Single Crossing Property in  $(x_i; t)$  if*

$$(a) \quad \frac{\partial^2 P}{\partial q_i^2} \leq 0, \quad \frac{\partial^2 P}{\partial q_j^2} \leq 0 \quad (i, j = D, F, i \neq j)$$

$$(b) \quad \frac{\partial^2 q_j^*}{\partial x_i \partial t} \leq 0$$

$$(c) \quad -\frac{\partial P}{\partial q_i} \cdot \frac{\partial q_i^*}{\partial x_i} \geq \frac{\partial P}{\partial q_j} \cdot \frac{\partial q_j^*}{\partial x_i}$$

$$(d) \quad \frac{\partial P}{\partial q_i} \cdot \frac{\partial q_i^*}{\partial t} \geq -\frac{\partial P}{\partial q_j} \cdot \frac{\partial q_j^*}{\partial t}$$

$$(e) \quad \frac{\partial^2 C_i}{\partial x_i \partial t} \leq 0$$

$$(f) \quad \frac{\partial C_i}{\partial q_i} \geq 0, \quad \frac{\partial^2 C_i}{\partial q_i^2} \geq 0$$

*Proof.* (1) follows almost immediately from the Monotonicity Theorem (Theorem 4) in Milgrom and Shannon (1994). As there is only one variable of choice,  $x_i$ , the profit function satisfies supermodularity and thus also quasisupermodularity. Hence,  $\pi_i$  satisfies the SCP in  $(x_i; t)$  if and only if  $x_i^*$  is nondecreasing in  $t$ .

(2) The Single Crossing Property is implied by increasing differences, that is  $\frac{\partial^2 \pi_i}{\partial x_i \partial t} \geq 0$  for twice continuously profit functions. Differentiating the profit function with respect to  $x_i$  and  $t$  yields

$$\begin{aligned}
\frac{\partial^2 \pi_i}{\partial x_i \partial t} &= \left( \frac{\partial^2 P}{\partial q_i^2} \cdot \frac{\partial q_i^*}{\partial t} \cdot \frac{\partial q_i^*}{\partial x_i} + \frac{\partial^2 P}{\partial q_j^2} \cdot \frac{\partial q_j^*}{\partial t} \cdot \frac{\partial q_j^*}{\partial x_i} + \frac{\partial P}{\partial q_j} \cdot \frac{\partial^2 q_j^*}{\partial x_i \partial t} \right) \cdot q_i^* \\
&+ \left( \frac{\partial P}{\partial q_i} \cdot q_i^* + P - \frac{\partial C_i}{\partial q_i} \right) \cdot \frac{\partial^2 q_i^*}{\partial x_i \partial t} + \left( \frac{\partial P}{\partial q_i} \cdot \frac{\partial q_i^*}{\partial t} + \frac{\partial P}{\partial q_j} \cdot \frac{\partial q_j^*}{\partial t} \right) \cdot \frac{\partial q_i^*}{\partial x_i} \\
&+ \left( \frac{\partial P}{\partial q_i} \cdot \frac{\partial q_i^*}{\partial x_i} + \frac{\partial P}{\partial q_j} \cdot \frac{\partial q_j^*}{\partial x_i} \right) \cdot \frac{\partial q_i^*}{\partial t} - \frac{\partial^2 C_i}{\partial q_i^2} \cdot \frac{\partial q_i^*}{\partial t} \cdot \frac{\partial q_i^*}{\partial x_i} - \frac{\partial^2 C_i}{\partial x_i \partial t}
\end{aligned}$$

As innovation expenditures are assumed to decrease the firm's marginal cost while the emissions tax on the other hand has the opposite effect. From the standard Cournot model we know that the optimal output is decreasing in the firm's own marginal cost, hence  $\frac{\partial q_i^*}{\partial x_i} > 0$  and  $\frac{\partial q_i^*}{\partial t} < 0$ . As the other firm's cost reduction from the spillover is smaller than that for the firm investing in R&D, the effect of firm  $i$ 's lower marginal cost on firm  $j$ 's output dominates and  $\frac{\partial q_j^*}{\partial x_i} < 0$ . As usual, we also assume inverse demand is downward sloping. Note that  $\frac{\partial P}{\partial q_i} \cdot q_i^* + P$  is equal to marginal revenue, hence this term is equal to zero at output level  $q_i^*$ . Now using assumptions (a)-(f), we can conclude that  $\frac{\partial^2 \pi_i}{\partial x_i \partial t} \geq 0$ , which implies the SCP.  $\square$

While this general solution requires a couple of assumptions on demand, the cost function and the optimal Cournot output, these conditions are economically quite intuitive. Assumption (a) is concavity of inverse demand in both  $q_i$  and  $q_j$ . As concavity is a sufficient condition for the existence and uniqueness of the Cournot equilibrium, this seems to be a reasonable assumption. Condition (b) requires the marginal effect of innovation expenditures  $x_i$  on  $q_j^*$  to decrease in  $t$ . This means that firm  $i$ 's cost advantage through investment, which leads to a negative effect on  $q_j$ , gets partially offset by higher tax rates. Therefore, the decrease in  $q_j$  when firm  $i$  increases investment is larger when taxes are low. The third assumption requires the effect of  $x_i$  on the price through  $q_i$  to be larger than its impact through  $q_j$ . In Cournot competition, where  $q_i$  and  $q_j$  are perfect substitutes this is reasonable to assume, since the magnitude of the direct effect of R&D expenditures on  $q_i$  is expected to exceed the absolute value of the spillover effect on  $q_j$ . Similarly,



assumption (d) assumes that the effect of the tax on the price through  $q_i$  is larger than through  $q_j$ , which also seems reasonable to assume as we can expect the direct effect of the tax on the domestic firm to be greater than the spillover to the foreign firm. Assumption (e) says that the marginal effect of R&D spending on abatement cost decreases with the effluent tax rate. Lastly, I assume that marginal abatement cost is increasing and convex in output.

Thus the application of a monotone comparative statics result yields economically plausible assumptions that support the Porter hypothesis in a much more general framework than considered in Simpson and Bradford (1994).

### **3.5 Technology Adoption Standards**

Besides effluent taxes and emissions standards, another type of environmental regulation uses technology adoption standards. Regulators often require firms to use the “best available technology” (BAT), “best practicable control technology currently available” (BPT) or “best conventional pollutant control technology” (BCT). In contrast to other policy instruments, few papers in the literature on the effects of environmental regulation on innovation incentives address technology adoption standards. An example is Biglaiser and Horowitz (1994), who study technology adoption standards as one component of optimal regulation and point out that these standards should be used in addition to effluent taxes or tradeable emissions permits, but not instead of these traditional regulations. They consider a model with a competitive polluting industry where an exogenously given number of firms compete in research in pollution-control innovations. The size of the innovation and R&D success are stochastic and drawn from an identical distribution for all firms. Successful firms are granted patents and the new technology can then be licensed to other firms. One finding of their paper is that stricter technology adoption standards have a negative effect on the number of researching firms.

### 3.5.1 Model

Now we add a technology adoption standard to the model used in Simpson and Bradford (1996). The firms are required to use technology better than or equal to  $\bar{\theta}$ , the BAT, BPT or BCT. The level of technology a firm possesses depends on the level of innovation spending  $x_i$ , either on adoption of already existing technology or R&D.<sup>40</sup> In the production process, each firm emits a pollutant  $e_i$ . The level of emissions increases and is convex in output  $q_i$  and decreases in advances of technology  $\theta_i$ . The government taxes emissions at rate  $t$ , hence the cost of emissions can be written as  $t e(q_i, \theta_i(x_i))$ .<sup>41</sup> Besides emissions costs, the firm also incurs the standard increasing costs of production denoted by  $C(q_i)$ .

Like in the model of Simpson and Bradford (1994), a domestic and a foreign firm are the only producers of a good and produce quantities  $q_D$  and  $q_F$  respectively. The goods are perfect substitutes, thus inverse demand  $P(Q)$  is a function of the sum of  $q_D$  and  $q_F$ .

In the first stage, the government chooses the tax rate  $t$  and technology standard  $\bar{\theta}$ , then the firms decide on how much to invest in R&D (and hence their level of technology) and in the final stage they choose  $q_D$  and  $q_F$ . As in the previous sections, we will focus on the firms' choices in the last two stages.

Finally, in the last stage of the game, the firms compete in Cournot fashion and maximize  $P(Q) \cdot q_i - C_i(q_i) - t \cdot e(q_i, \theta_i(x_i))$  ( $i = D, F$ ). In the innovation decision stage, the firm  $i$  chooses  $x_i$  to maximize

$$\pi_i(q_i^*, q_j^*) = P(q_i^*, q_j^*) \cdot q_i^* - C_i(q_i^*) - t \cdot e(q_i^*, \theta(x_i)) - x_i$$

$$\text{subject to } \theta_i(x_i) \geq \bar{\theta},$$

where  $q_i^* = q_i^*(x_i, x_j, t)$  and  $q_j^* = q_j^*(x_i, x_j, t)$  denote the Cournot quantities from the final stage.

<sup>40</sup>Assume that innovation is firm specific (and hence a private good). In this case, the firm's abatement cost can be modeled as a function of emissions and investment level, with costs increasing in emissions and decreasing in investment. In contrast to Biglaiser and Horowitz (1994), R&D success is treated as deterministic and the level of technology is determined by investment in innovation.

<sup>41</sup>This specification of emissions costs is adapted from Biglaiser and Horowitz (1994).

### 3.5.2 Monotone Comparative Statics

As previously, we want to analyze the effect of stricter environmental regulation on innovation. Here, the tighter regulation takes the form of higher taxes and higher technology adoption standards. In this case, we have parameter changes in both the objective function and the constraint set of the firm's innovation decision problem.

**Proposition 12.** *Let  $x_i^* = \operatorname{argmax}_{x_i \in S} \pi_i(q_i^*(t), q_j^*(t))$  with  $S = \{x_i \in \mathbb{R} \mid \theta_i(x_i) \geq \bar{\theta}\}$ . Then*

1. *stricter environmental regulation  $(t', \bar{\theta}') \geq (t, \bar{\theta})$  leads to nondecreasing investment in R&D  $x_i^*$  if and only if  $\pi_i$  satisfies the Single Crossing Property in  $(x_i; t)$ .*
2.  *$\pi_i$  satisfies the Single Crossing Property in  $(x_i; t)$  if*

$$(a) \quad \frac{\partial^2 P}{\partial q_i^2} \leq 0, \quad \frac{\partial^2 P}{\partial q_j^2} \leq 0 \quad (i, j = D, F, i \neq j)$$

$$(b) \quad \frac{\partial^2 q_j^*}{\partial x_i \partial t} \leq 0$$

$$(c) \quad -\frac{\partial P}{\partial q_i} \cdot \frac{\partial q_i^*}{\partial x_i} \geq \frac{\partial P}{\partial q_j} \cdot \frac{\partial q_j^*}{\partial x_i}$$

$$(d) \quad \frac{\partial P}{\partial q_i} \cdot \frac{\partial q_i^*}{\partial t} \geq -\frac{\partial P}{\partial q_j} \cdot \frac{\partial q_j^*}{\partial t}$$

$$(e) \quad \frac{\partial e}{\partial q_i} \cdot \frac{\partial q_i^*}{\partial x_i} \leq -\frac{\partial e}{\partial \theta_i} \cdot \frac{\partial \theta_i}{\partial x_i}$$

$$(f) \quad \frac{\partial C_i}{\partial q_i} \geq 0, \quad \frac{\partial^2 C_i}{\partial q_i^2} \geq 0$$

*Proof.* (1) follows almost immediately from the Monotonicity Theorem (Theorem 4) in Milgrom and Shannon (1994). As there is only one variable of choice,  $x_i$ , the profit function satisfies supermodularity and thus also quasisupermodularity.

Moreover, the constraint set  $S' = \{x_i \in \mathbb{R} \mid \theta_i(x_i) \geq \bar{\theta}'\}$ ,  $\bar{\theta}' \geq \bar{\theta}$ , dominates  $S$  in the strong set order. Hence,  $\pi_i$  satisfies the SCP in  $(x_i; t)$  if and only if  $x_i^*$  is nondecreasing in  $(t, \bar{\theta})$ .

(2) The Single Crossing Property is implied by increasing differences, that is  $\frac{\partial^2 \pi_i}{\partial x_i \partial t} \geq 0$  for twice continuously profit functions. Differentiating the profit function with respect to  $x_i$  and  $t$  yields

$$\begin{aligned}
\frac{\partial^2 \pi_i}{\partial x_i \partial t} &= \left( \frac{\partial^2 P}{\partial q_i^2} \cdot \frac{\partial q_i^*}{\partial t} \cdot \frac{\partial q_i^*}{\partial x_i} + \frac{\partial^2 P}{\partial q_j^2} \cdot \frac{\partial q_j^*}{\partial t} \cdot \frac{\partial q_j^*}{\partial x_i} + \frac{\partial P}{\partial q_j} \cdot \frac{\partial^2 q_j^*}{\partial x_i \partial t} \right) \cdot q_i^* \\
&+ \left( \frac{\partial P}{\partial q_i} \cdot q_i^* + P - \frac{\partial C_i}{\partial q_i} - t \cdot \frac{\partial e}{\partial q_i} \right) \cdot \frac{\partial^2 q_i^*}{\partial x_i \partial t} + \left( \frac{\partial P}{\partial q_i} \cdot \frac{\partial q_i^*}{\partial t} + \frac{\partial P}{\partial q_j} \cdot \frac{\partial q_j^*}{\partial t} \right) \cdot \frac{\partial q_i^*}{\partial x_i} \\
&+ \left( \frac{\partial P}{\partial q_i} \cdot \frac{\partial q_i^*}{\partial x_i} + \frac{\partial P}{\partial q_j} \cdot \frac{\partial q_j^*}{\partial x_i} \right) \cdot \frac{\partial q_i^*}{\partial t} - \frac{\partial^2 C_i}{\partial q_i^2} \cdot \frac{\partial q_i^*}{\partial t} \cdot \frac{\partial q_i^*}{\partial x_i} \\
&- \left( \frac{\partial e}{\partial q_i} \cdot \frac{\partial q_i^*}{\partial x_i} + \frac{\partial e}{\partial \theta_i} \cdot \frac{\partial \theta_i}{\partial x_i} \right) - t \cdot \frac{\partial^2 e}{\partial q_i^2} \cdot \frac{\partial q_i^*}{\partial t} \cdot \frac{\partial q_i^*}{\partial x_i}
\end{aligned}$$

As innovation expenditures are assumed to decrease the firm's marginal cost while the emissions tax on the other hand has the opposite effect. From the standard Cournot model we know that the optimal output is decreasing in the firm's own marginal cost, hence  $\frac{\partial q_i^*}{\partial x_i} > 0$  and  $\frac{\partial q_i^*}{\partial t} < 0$ . As the other firm's cost reduction from the spillover is smaller than that for the firm investing in R&D, the effect of firm  $i$ 's lower marginal cost on firm  $j$ 's output dominates and  $\frac{\partial q_j^*}{\partial x_i} < 0$ . As usual, assume inverse demand is downward sloping. Note that  $\frac{\partial P}{\partial q_i} \cdot q_i^* + P$  is equal to marginal revenue, hence this term is equal to zero at output level  $q_i^*$ . Now using assumptions (a)-(f), we can conclude that  $\frac{\partial^2 \pi_i}{\partial x_i \partial t} \geq 0$ , which implies the SCP.  $\square$

When taking a closer look at this characterization of the SCP, notice that the revenue in the profit function remained the same as in the previous section in the Simpson and Bradford model. Therefore assumptions (a) through (d) are the same as above. Assumption (e) guarantees that the overall effect of increased investment leads to a reduction in emissions; that is, the increasing effect of  $x_i$  on emissions through  $q_i$  is dominated by the emissions reduction of  $x_i$  through technology  $\theta_i$ . Lastly, (f) assumes that costs of production are increasing and convex.

## 3.6 Conclusion

After the analysis of the effect of different environmental policy instruments on innovation in imperfectly competitive markets, my monotone comparative statics approach to this problem shows that the question whether or not the firm's profit function satisfies the Single Crossing Property in the investment and policy variables plays a crucial role. The Single Crossing Property can be interpreted as weak complementarity between investment and the regulatory variable. I provide properties of profit functions that satisfy the Single Crossing Property for each of the considered policy instruments. The results vary only slightly across the three different types of regulations.

In all three models, demand is assumed to be concave in both firms' quantities. We also see that the marginal effect of investment in innovation on the other firm's Cournot output needs to be decreasing in the policy variable. Moreover, it is required that the size of spillover effects from investment in innovation by one firm on the other firm needs to be limited, that is the effects of investment and policy variable on the price through firm  $i$ 's quantity is greater than the spillover through firm  $j$ . Lastly, convexity of production and abatement costs is another property of profit functions that satisfy the SCP. This suggests that we should not expect the Porter hypothesis to hold in markets where firms experience economies of scale.

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# Appendix A

## Appendix Chapter 1

### A.1 Proof of Theorem 2

( $\Rightarrow$ ) Suppose  $S' \geq_{(\Delta, \nabla)} S$  for  $t' \geq t$  and let  $x \in M(t, S)$ ,  $y \in M(t', S')$ .

Since  $x \in M(t, S)$  and  $S \leq_{(\Delta, \nabla)} S'$ ,  $f(x, t) \geq f(x\Delta y, t)$ . By  $(\Delta, \nabla)$ -quasisupermodularity of  $f$ , this implies  $f(x\nabla y, t) \geq f(y, t)$ . Since  $f$  also has the  $(\Delta, \nabla)$ -Single Crossing Property in  $(x, t)$ ,

$f(x\nabla y, t) \geq f(y, t) \Rightarrow f(x\nabla y, t') \geq f(y, t')$  for  $t' \geq t$ . As  $y \in M(t', S')$ ,  $x\nabla y \in M(t', S')$ .

Now suppose  $x\Delta y \notin M(t, S)$  and hence  $f(x, t) > f(x\Delta y, t)$ .  $(\Delta, \nabla)$ -quasisupermodularity of  $f$  implies  $f(x\nabla y, t) > f(y, t)$  and by the  $(\Delta, \nabla)$ -Single Crossing Property,  $f(x\nabla y, t') > f(y, t')$  for any  $t' \geq t$ . This contradicts the assumption that  $y \in M(t', S')$ . Therefore,  $x\Delta y \in M(t, S)$  and  $M(t, S) \leq_{(\Delta, \nabla)} M(t', S')$ .

( $\Leftarrow$ ) Fix  $t$ . Let  $x$  and  $y$  be two elements in  $X$  and suppose that  $f$  is not  $(\Delta, \nabla)$ -quasisupermodular.

The only case I need to look at is when  $x$  and  $y$  are unordered. Also,  $x\Delta y \neq x$  and  $x\nabla y \neq y$ . Let

$S = \{x, x\Delta y\}$  and  $S' = \{y, x\nabla y\}$ . Then  $S' \geq_{(\Delta, \nabla)} S$ .

$(\Delta, \nabla)$ -quasisupermodularity of  $f$  can be violated in the following two ways.

First, suppose  $f(x, t) \geq f(x\Delta y, t)$ , but  $f(x\nabla y, t) < f(y, t)$ . In this case  $x$  is a maximizer of  $f$  in  $S$  and  $y$  maximizes  $f$  uniquely in  $S'$ , which violates  $M(t, S') \geq_{(\Delta, \nabla)} M(t, S)$  (since  $x$  and  $y$  are unordered).

Alternatively, suppose  $f(x, t) > f(x\Delta y, t)$ , but  $f(x\nabla y, t) = f(y, t)$ . Now  $y$  maximizes  $f$  in  $S'$  while

$x$  is the unique maximizer in  $S$ . This again contradicts  $M(t, S') \geq_{(\Delta, \nabla)} M(t, S)$ . So  $f$  is  $(\Delta, \nabla)$ -quasisupermodular.

Now let  $S \equiv \{x, \bar{x}\}$  with  $x \leq_{(\Delta, \nabla)} \bar{x}$ . Then  $f(\bar{x}, t) - f(x, t) \geq 0$  implies  $\bar{x} \in M(t, S)$ . Moreover since we have  $M(t, S) \leq_{(\Delta, \nabla)} M(\bar{t}, S)$  for  $\bar{t} \geq t$  it follows that  $f(\bar{x}, \bar{t}) - f(x, \bar{t}) \geq 0$  for all  $\bar{t} \geq t$ .

Moreover,  $f(\bar{x}, t) - f(x, t) > 0$  implies that  $\bar{x} \in M(t, S)$  and is unique. Since  $M(t, S) \leq_{(\Delta, \nabla)} M(\bar{t}, S)$  for  $\bar{t} \geq t$  it follows that  $f(\bar{x}, \bar{t}) - f(x, \bar{t}) > 0$  for all  $\bar{t} \geq t$ . Thus  $f$  has the  $(\Delta, \nabla)$ -Single Crossing Property.

## A.2 $C_i$ -Quasisupermodularity and the $i$ -SCP of Profit Functions

The objective function is  $C_i$ -quasisupermodular if it is supermodular and  $i$ -concave (Proposition 2, Quah (2007)).  $\pi$  is supermodular if  $\frac{\partial^2 \pi}{\partial x_i \partial x_j} \geq 0$  for all  $i, j$  and  $i \neq j$ . This is equivalent to  $\frac{\partial^2 V}{\partial x_i \partial x_j} - \frac{\partial^2 C}{\partial x_i \partial x_j} \geq 0$ . Thus the profit function is supermodular if marginal revenue of good  $i$  is nondecreasing in  $x_j$  and marginal cost of good  $i$  is nonincreasing in  $x_j$ . We can decompose the total revenue in two parts, the revenue from regulated goods (R) and revenue (UR) from unregulated goods.

$$\text{So, } V(x_R, x_{UR}, \phi) = p_R(x_R, x_{UR}, \phi) \cdot x_R + p_{UR}(x_R, x_{UR}, \phi) \cdot x_{UR}.$$

Hence,

$$\frac{\partial^2 V}{\partial x_i \partial x_j} = \frac{\partial^2 p_i}{\partial x_i \partial x_j} \cdot x_i + \frac{\partial p_i}{\partial x_j} + \frac{\partial^2 p_j}{\partial x_i \partial x_j} \cdot x_j + \frac{\partial p_j}{\partial x_i}.$$

For linear demand,  $\frac{\partial^2 p_i}{\partial x_i \partial x_j} = \frac{\partial^2 p_j}{\partial x_i \partial x_j} = 0$ . Therefore, marginal revenue of good  $i$  is nondecreasing in  $x_j$  for linear demand if either demand of good  $i$  and good  $j$  are unrelated or if  $\frac{\partial p_i}{\partial x_j} \geq 0$  and  $\frac{\partial p_j}{\partial x_i} \geq 0$ .

By Definition 4, the objective function is  $i$ -concave if it is concave in all variables other than  $i$ .

So for the firm's profit function, the Hessian,

$$H_{-i} = \begin{bmatrix} \pi_{11} & \cdots & \pi_{1,i-1} & \pi_{1,i+1} & \cdots & \pi_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \pi_{i-1,1} & \cdots & \pi_{i-1,i-1} & \pi_{i-1,i+1} & \cdots & \pi_{i-1,n} \\ \pi_{i+1,1} & \cdots & \pi_{i+1,i-1} & \pi_{i+1,i+1} & \cdots & \pi_{i+1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \pi_{n,1} & \cdots & \pi_{i-1,n} & \pi_{i+1,n} & \cdots & \pi_{n,n} \end{bmatrix}$$

needs to be negative semi-definite.

In the simple case of only 2 commodities and linear cost and demand functions, we only need to check that  $\frac{\partial^2 \pi}{\partial x_j^2} \leq 0$  for all  $j \neq i$ .

For linear cost and demand functions, this expression becomes

$$\frac{\partial^2 \pi}{\partial x_j^2} = \frac{\partial^2 V}{\partial x_j^2} - \frac{\partial^2 C}{\partial x_j^2} = \frac{\partial^2 V}{\partial x_j^2} = \frac{\partial^2 p_j}{\partial x_j^2} \cdot x_j + 2 \frac{\partial p_j}{\partial x_j} + \frac{\partial^2 p_i}{\partial x_j^2} \cdot x_i = 2 \frac{\partial p_j}{\partial x_j} \leq 0.$$

Hence in this simple case,  $\pi$  is  $C_i$ -quasisupermodular if the demand and cost function are linear and if either demand of good  $i$  and good  $j$  are unrelated or if  $\frac{\partial p_i}{\partial x_j} \geq 0$  and  $\frac{\partial p_j}{\partial x_i} \geq 0$ .

The  $i$ -Single Crossing Property is satisfied if  $\frac{\partial^2 \pi}{\partial x_i \partial \phi_i} \geq 0$ ,  $\frac{\partial^2 \pi}{\partial x_i \partial \omega} \geq 0$  and  $\frac{\partial^2 \pi}{\partial x_i \partial \theta} \geq 0$ . Again assuming a linear demand function,

$$\frac{\partial^2 \pi}{\partial x_i \partial \phi_i} = \frac{\partial^2 V}{\partial x_i \partial \phi_i} = \frac{\partial^2 p_i}{\partial x_i \partial \phi_i} \cdot x_i + \frac{\partial p_i}{\partial \phi_i} = \frac{\partial p_i}{\partial \phi_i},$$

$$\frac{\partial^2 \pi}{\partial x_i \partial \omega} = -\frac{\partial^2 C}{\partial x_i \partial \omega} = -\frac{\partial MC_i}{\partial \omega} \text{ and}$$

$$\frac{\partial^2 \pi}{\partial x_i \partial \theta} = 0.$$

Thus, we need  $\frac{\partial p_i}{\partial \phi_i} \geq 0$  and marginal cost of good  $i$  to be nonincreasing in  $\omega$  for the  $i$ -Single Crossing Property to hold.

# Appendix B

## Appendix Chapter 2

### B.1 Relation to Mirman and Ruble (2003) in the 2 goods case

Consider a price increase of good 2 in the 2 goods case.

Mirman and Ruble's condition for good 1 to be a substitute of good 2,

$$\frac{\partial}{\partial x_2} \left( \frac{MRS_{12}}{x_2} \right) < 0,$$

can be rewritten as

$$\frac{x_2 \frac{\partial MRS_{12}}{\partial x_2} - MRS_{12}}{x_2^2} < 0 \text{ or } \frac{\partial MRS_{12}}{\partial x_2} < \frac{MRS_{12}}{x_2}.$$

This is equivalent to

$$\epsilon_{MRS_{12}, x_2} = \frac{\partial MRS_{12}}{\partial x_2} \cdot \frac{x_2}{MRS_{12}} < 1.$$

Our condition for good 1 to be a substitute of good 2 in the case of a price increase of good 2 is

$$|\epsilon_{MRS_{21}, x_2}| < 1.$$

Notice that

$$\begin{aligned}\mathcal{E}_{MRS_{12},x_2} &= \frac{\partial MRS_{12}}{\partial x_2} \cdot \frac{x_2}{MRS_{12}} = \frac{\partial}{\partial x_2} \left( \frac{MU_1}{MU_2} \right) \cdot \frac{x_2}{\frac{MU_1}{MU_2}} = \frac{MU_2 \frac{\partial MU_1}{\partial x_2} - MU_1 \frac{\partial MU_2}{\partial x_2}}{MU_2^2} \cdot \frac{x_2}{\frac{MU_1}{MU_2}} = \frac{(MU_2 \frac{\partial MU_1}{\partial x_2} - MU_1 \frac{\partial MU_2}{\partial x_2}) \cdot x_2}{MU_2 MU_1} \text{ and} \\ \mathcal{E}_{MRS_{21},x_2} &= \frac{\partial}{\partial x_2} \left( \frac{MU_2}{MU_1} \right) \cdot \frac{x_2}{\frac{MU_2}{MU_1}} = \frac{(MU_1 \frac{\partial MU_2}{\partial x_2} - MU_2 \frac{\partial MU_1}{\partial x_2}) \cdot x_2}{MU_2 MU_1} = -\mathcal{E}_{MRS_{12},x_2}.\end{aligned}$$

Hence for well-behaved preferences, when  $\frac{\partial MU_1}{\partial x_2} > 0$  and  $\frac{\partial MU_2}{\partial x_2} < 0$ , the two conditions are equivalent.

## B.2 Relation to Mirman and Ruble (2003) in the $n$ goods case

Now consider a price increase of good  $n$  in the  $n$  goods case.

Mirman and Ruble's condition provides conditions when good 1 is a gross substitute of good  $n$ , whereas my condition yields the gross substitutes property of demand, where all goods  $j \neq n$  are substitutes of good  $n$ .

For example, let  $n = 3$ . According to Mirman and Ruble, good 1 is a gross substitute of good 3 if  $MRS_{12}$  is independent of  $x_3$ ,  $MRS_{23}$  is independent of  $x_1$  and  $\frac{MRS_{23}}{x_3}$  is independent of  $x_1$  and decreasing in  $x_2$  and  $x_3$ .

Using my condition, all goods are gross substitutes of good 3 if  $MRS_{21}$  is independent of  $x_3$ ,  $MRS_{31}$  is independent of  $x_2$ ,  $MRS_{32}$  is independent of  $x_1$  and  $|\frac{\partial MRS_{3k}}{\partial x_3} \cdot \frac{x_3}{MRS_{3k}}| < 1$  for some  $k \neq 3$ .

Thus the independence assumptions are the same in both papers, the assumptions on the elasticity of the marginal rate of substitution differ slightly. Mirman and Ruble specifically impose this condition on one specific elasticity, whereas my version only requires it to hold for one out of all the elasticities involving good 3.



Because of the assumption that requires the marginal rates of substitution to be independent of other goods, the utility function needs to be symmetric with regard to the marginal utilities. As a consequence, the elasticities involving good 3 are going to be equal. Hence my condition and Mirman and Ruble's apply to the same classes of utility functions.