The Relationship between Abstract Reasoning and Performance in High School Algebra

BY

Kirk W. Drager

Submitted to the graduate degree program in Psychology and Research in Education and the graduate faculty of the University of Kansas in partial fulfillment of the degree of Master of Science

Chairperson Bruce Frey

Susan Gay

David Hansen

Date Defended: July 24, 2014

The Thesis Committee for Kirk W. Drager certifies that this is the approved version of the following thesis:

The Relationship Between Abstract Reasoning and Performance in High School Algebra

____

Chairperson Bruce Frey

Date Approved: July 24, 2014

The Relationship between Abstract Reasoning and Performance in High School Algebra

ABSTRACT: This study examined abstract reasoning ability as a predictor of success in high school algebra controlling for age, motivation, and previous math achievement. A valid and reliable instrument, the *Abstract Reasoning Assessment* (ARA), a matrix completion instrument based upon a protocol by Embretson (1998), was developed for the study. Motivation was measured using the Personal Achievement Goal Orientations scale (Midgeley, 2000). Previous math achievement was measured using the course grade from the previous year's math course. Success in algebra was measured by the final exam grade from a first year high school algebra course. 220 ninth grade students took part in the study. A multiple regression analysis found that abstract reasoning ability explained a significant proportion of the variance in high school algebra performance beyond that explained by previous math achievement, motivation and age. Further, based on effect size, abstract reasoning was a better predictor than previous math achievement.

Table of Contents

Introduction

The following story may sound familiar to many people. "As someone who struggled hard with math in school until I was 15, and then got it all at once, I never believed the math-gene theory. What made the difference for me was that everything suddenly made sense, perfect, simple, elegant sense" (Devlin, 2000, p. 63).

Teaching Algebra to all students in eighth grade has been the subject of much debate in mathematics education. McKibben (2009) points out that advanced math requirements don't necessarily result in higher math achievement; for example in 2007 the District of Columbia had the highest percent of students in advanced math courses, but they had the lowest math scores on the National Assessment of Educational Progress (NAEP). Viadero (2010) found that as Chicago Public Schools initiated a policy to have more students complete Algebra I by 9^{th} grade, their failure rates increased, grades decreased, assessment scores failed to rise, and students' college attendance rates did not increase. On the other hand a study by Spielhagen (2006) indicates that policies providing algebra instruction in the eighth grade can assist in closing SES achievement gaps in school populations and increase the likelihood of those students taking higher level mathematics courses in high school. Using 2005 NAEP data Loveless (2008) found that about 120,000 eighth grade students are misplaced in Algebra I because they possessed second grade math skills. At least that many more have skills only slightly better (Bracey, 2008). Loveless' (2008) findings also show that many of the states with the highest percentages of eighth graders in algebra classes have the lowest NAEP math scores, and many states with low percentages of eighth grade algebra students had the highest NAEP scores. Cavanaugh (2008) found that California had almost 60 percent of eighth grade students enrolled Algebra 1 or above, but scored an average of 270 on the NAEP, one of the country's lowest average scores. In contrast Vermont had only 26 percent of eighth graders taking Algebra or above, yet Vermont students earned an average NAEP score of 298, one of the highest in the nation. McClure (2009) notes that according to the National Mathematics Advisory Panel's final report in 2008 there is a "sharp falloff in mathematics achievement" as students begin study in algebra toward the end of middle school.

One concern is that there may be a developmental component related to abstract thinking capacity that may inhibit some students from learning algebraic concepts during adolescence. The question is not whether algebra should be taught in eighth grade, but rather who should, and who should not take algebra in eighth grade. This study will specifically explore the relationship between abstract thought and success in learning Algebra I. *The hypothesis for this study is that performance on an algebra final exam is related to abstract reasoning after controlling for age, motivation, and previous math achievement.*

Abstraction In Algebra

Algebra, of course, is filled with abstraction. Several Algebra concepts are highlighted as the source of abstraction difficulties among algebra learners. To fully understand algebra each of these abstraction difficulties must be overcome. Sfard and Linchevski (1994) point out that the expression $3(x + 5) - 1$ can have different meanings depending on the level of abstraction. At a simple level of abstraction, the above expression can represent a computational process. At a deeper level of abstraction, the expression might represent a single numerical value. With a little more complexity, the expression can be interpreted as a function. If the three is replaced by the letter a to form the expression $a(x + 5) - 1$, the expression can be expanded to an entire family of functions. Another common abstraction that students don't always understand is the concept of the equal sign (McNiel, 2008; Rojano & Martinez, 2009; Sfard, & Linchevski, 1994). Goodson-Espy's (1998) study of university students' methods of solving problems about automobile rental situations involving the concept of inequalities showed that students operating at the lower levels of reflective abstraction typically held weak conceptions of variable and equality. When students could not solve the problems arithmetically, they often resorted to creating tables to find a solution.

Students often have trouble with the transition from arithmetic problem solving to algebraic problem solving. Many studies found that students from middle school to university use arithmetic methods to solve problems that can be solved algebraically (Hershkowitz, 2001; MacGregor & Stacey, 1997; Nathan & Koedinger, 2000; Stacey & MacGregor, 1997). "Students' prior experiences with solving problems in arithmetic gives them a compulsion to calculate which is manifested in

1. the meaning they give to 'the unknown';
2. their interpretation of what an equation is;
3. the methods they choose to solve equations" (Stacey & MacGregor, 1999, p. 149).

The differences between arithmetic and Algebraic problem solving are summarized in Table 1.

Table 1. Arithmetic vs Algebraic Problem Solving

Arithmetic Problem Solving	**Algebraic Problem Solving**
Work from knowns to unknowns	Working with and on unknowns throughout
Unknowns change through problem	Unknown fixed
Equation as formula to produce an answer	Equation as description of relationship
Chains of successive calculations	Chains of successive equalities
Guess and check equation solving	"Do the same to both sides" equation solving
Intermediate results can be interpreted in problem situation	Intermediate results are not interpreted in problem situation
Doing operations one by one	Undoing operations one by one

(Stacey, 1997)

Capraro and Joffrion (2006) found that the percentages of seventh and eighth grade students correctly answering three questions (two multiple choice and one free response) involving translating words into algebraic symbols were 54.7%, 33.5%, and 43.1%, respectively, and only 9% of the students answered all three questions correctly. This demonstrates a lack of ability to use simple abstraction among seventh and eighth graders. Dede (2004) found that eighth grade students do not have the correct concept of a variable. They instead hold a simpler view that a literal symbol is an unknown value rather than a variable quantity. Stacey (1997) found that student concepts of "the unknown" had different and sometimes shifting understandings: x refers to the value currently being calculated, x refers to different quantities in one equation, x refers to different quantities at different stages, x is a general label for any unknown quantity or a combination of quantities.

Piagetian theory suggests that the capacity for formal reasoning begins around age eleven or twelve. Bitner-Corvin (1987) found that in seventh grade 96% of students are still in the concrete operational phase, but by tenth grade only 50% remain in the concrete stage, and 22% have reached formal operations. While there is no set definition of what abstract thought is, there is much agreement that it involves symbolic representation that becomes progressively abstract as an increasing number of symbolic abstractions are added to the system, and that adolescents will progress through these levels in a predictable manner (Marini & Case, 1994). Marini and Case went on to find that there was indeed a continuum of abstraction among adolescents that increased with age throughout the teen years.

Review of Literature

Abstract Reasoning is also known as fluid intelligence (Cattell, 1963) or analytic intelligence. "Fluid intelligence is reasoning ability in its most abstract and purest form. It is the ability to analyse novel problems, identify the patterns and relationships that underpin these problems and extrapolate from these using logic. This ability is central to all logical problem solving and is crucial for solving scientific, technical and mathematical problems" (ART Technical Manual, 2006, p. 5), "We use the term analytic intelligence to refer to the ability to reason and solve problems involving new information, without relying extensively on an explicit base of declarative knowledge derived from either schooling or previous experience" (Carpenter, Just, and Shell, 1990, p. 404). Carpenter, Just and Shell assert that the Raven Progressive Matrices test is an appropriate test for measuring analytic intelligence. The Raven Matrix Tests were originally designed to be a non-verbal measure of Spearman's general intelligence, *g* (1927).

Hershkowitz, Schwarz, and Dreyfus (2001) view "*abstraction* as a process in which students vertically reorganize previously constructed mathematics into a new mathematical structure" (p.195). They go on to say that for most mathematics educators, "abstraction proceeds from a set of mathematical objects (or processes) and consists of focusing on some distinguishing properties and relationships of these objects rather than on the objects themselves" (p. 196).

Reification Theory

Reification describes the nature of a student's understanding of a mathematical concept. Goodson-Espy (1998) summarizes Sfard's (1991) theory of reification, "The theory of reification posits the existence of three stages of concept formation: (a) interiorization; (b) condensation; and (c) reification" (p. 220). Goodson-Espy (1998) later clarifies, "The process is said to have been interiorized when the learner no longer has to perform the operation in order to think about the process... Condensation was described as the stage where a complicated process is condensed into a form that becomes easier to use and think about... Reification is termed as the stage where the solver can conceive of the mathematical concept as a complete object with characteristics of its own" (p. 223).

Reflective Abstraction

In contrast Cifarelli (as cited in Godson-Espy, 1998) describes reflective abstraction as a learning process used to characterize problem solving activities. There are four levels of reflective abstraction, (1) *Recognition*, (2) *Re-presentation*, (3) *Structural Abstraction*, and (4) *Structural Awareness*.

> Cifarelli described the *Recognition* level as the ability to recognize characteristics of a previously solved problem in a new situation and to believe that one can do again what one did before. ...
>
> *Re-presentation* was described as the level where a student becomes able to run through a problem mentally and is able to anticipate potential sources of difficulty and promise. ...
>
> *Structural Abstraction* was said to occur when the student evaluates solution prospects based on mental run-throughs of potential methods as well as methods that have been used previously. ...
>
> A solver operating at [the *Structural Awareness*] level is able to anticipate the results of potential activity without having to complete a mental run-through of the solution activity" (Goodson-Espy, 1998, pp. 224-225).

Aiding Abstraction in the Classroom

Hazzan and Zazkis (2005) found that students' mistakes in problem solving are often an attempt on the part of the student to reduce the level of abstraction, and identified three different interpretations for levels of abstraction discussed in the literature: (a) abstraction level as the quality of the relationships between the object of thought and the thinking person, (b) abstraction level as reflection of the process-object duality, and (c) abstraction level as the degree of complexity of the concept of thought.

In the classroom, teachers need methods to scaffold students' ability to form the abstractions needed to solve problems. The common theme of these methods is helping students construct abstractions using an understanding of reification theory, and reflective abstraction. Using reification teachers can interiorize a concept by beginning the discussion of the concept. Teachers are very good at helping consolidation by summarizing processes by a variety of methods. One method is asking students about characteristics of a concept can aid reification. Teachers can use the steps of reflective abstraction to help students bridge the gaps in abstract constructions. Teachers can help recognition by helping students remember similar problems or processes from prior learning. They can suggest re-presentation by prompting students to attempt a mental run through of the problem. Asking students to evaluate whether they think a method will work can facilitate students' structural abstraction and may eventually lead to structural awareness. Specific methods found in the literature for addressing abstraction include use of computer simulations (Rojano & Martinez, 2009), creating tables to see patterns (Goodson-Espy, 1998), avoid using trivial problems when illustrating algebraic problem solving (Stacey, 1997), having students construct their own rules (Demby, 1997), encouraging metacognition by asking "why" questions (Clements & Sarama, 2004), distributing abstraction among multiple participants (Hershkowitz, et. al., 2001), reorganizing curriculum historically to facilitate abstraction, and highlighting interconnections between new and prior learning (Sfard & Linchevski, 1994).

Abstract Reasoning Correlations

Bird (2010) compared abstract reasoning to academic performance in university chemistry using the Group Assessment of Logical Thinking (GALT) (Roadranga, 1986). The GALT is a twelve item, paper and pencil test designed to measure logical thought in students from sixth grade to college. The GALT measures six logical operations: conservation, proportional reasoning, controlling variables, combinatorial reasoning, probabilistic reasoning, and correlational reasoning, and classifies students as concrete, transitional or formal operational. The GALT has proven to be a reliable and valid measure of logical thinking ability (Bird, 2010; Roadranga, 1986). The GALT was not used for this study because it measures a subject's ability to use formal logic, rather that the ability to use abstraction.

Abstract reasoning is generally considered to be the use of symbolic and logical processes to solve problems, and Embretson (1998) developed a protocol for constructing and analyzing matrix completion items like those in thethe Raven matrix tests. Embretson (1995) found in a study of 728 air force recruits that scores on her Abstract Reasoning Test (ART) correlated most strongly with the recruits' Arithmetic Reasoning and Math Knowledge as measured by the Armed Services Vocational Aptitude Battery (ASVAB).

Methods

The hypothesis of this study is that performance on an algebra final exam is related to abstract reasoning after controlling for age, motivation, and previous math achievement.

Participants

Permission was sought and received from the University of Kansas' Institutional Review Board to conduct this study. The sample students attend school in a public school district on the outskirts of a medium sized Mid-western city. The district contains one large high school and has a population consisting of urban, suburban and rural students from varied socio-economic classes, as well as a number of students living in group homes.

During the Spring 2014 semester about 320 total students completed the *Abstract Reasoning Assessment* consisting of 25 matrix completion items with increasing difficulty. The subjects were enrolled in Math 1 as high school students. The curriculum of Math 1 in this district corresponds to that of a traditional Algebra I course with the addition of a few topics from Geometry. Only those students whose previous mathematics course was 8th grade algebra were used for the analysis. Ultimately, data for all variables were available for 216 qualifying students. Of these students, 47% were female and the mean age was 15.32 years.

Instrument

Data on five variables were collected from this sample: abstract reasoning ability, motivation for class work, algebra ability, previous math achievement, and age.

An instrument to assess abstract reasoning ability was constructed based on Embertson's (1998) protocol for developing abstract reasoning tests. Items consist of a three by three matrix containing entries in the form of simple graphics in each cell except for the lower right cell. There are patterns among the entries that determine the rules for completing the matrix with one multiple choice solution and seven distracters. Items were designed using the Taxonomy of Rules in the Raven Test as described by Carpenter et al. (1990) There is a hierarchy of five rules used in Raven's Matrix tests and other tests of its kind. The first rule is the Constant in a Row, or Identity, rule. The Identity rule refers to entries that remain unchanged across a row of the matrix. The second rule is Quantitative Pairwise Progression, which means that there is a natural progression among the entries of a row or column. Third is Figure Addition or Subtraction in which the figures in two elements are combined to produce a third element. Fourth is the Distribution of Three Values rule where each row or column contains each three elements or attributes exactly once. The fifth rule is Distribution of Two Values- it is like distribution of three values, but the third attribute is null. The figures in each element are typically simple geometric figures. Attributes of the figures include size, orientation, shading, outlining, and distortion. Figures in each entry may be juxtaposed, superimposed, or combined together to make one object.

Embertson (1995) found that the best way to operationalize the difficulty level of a matrix item is to calculate its Working Memory Load (WML), by using the number of rules used. Carpenter et al. (1990) found the five rules to be hierarchical, meaning that subjects try the rules in order from simplest (Identity) to most complex (Distribution of Two). WML is a sum of the rules tried, so if a question uses Identity, Pairwise Progression, and Distribution of Three, its Working Memory Load is the sum of one for Identity, plus two for pairwise progression, plus four for Distribution of Three. So the question's WML = 1 + 2 + 4 = 7.

The researcher developed the *Abstract Reasoning Assessment* (ARA) to measure abstract reasoning for this study using Embretson's protocol for developing valid matrix completion tests. The test developed for this study consists of 25 matrix completion questions with difficulties from Working Memory Load (WML) = 1 to WML = 24. Questions on the test are arranged from least difficult to most difficult primarily based on WML. The test is to be completed in 20 minutes. The ARA contains 25 matrix completion items, each having eight multiple-choice responses and is shown in the Appendix.

Because motivation toward classroom work likely contributes to classroom achievement, a six-question scale using a five-point Likert-type set of answer options (1 = Not at all true to 5 = Very true), the *Personal Achievement Goal Orientations* (PAGO) scale from the *Patterns of Adaptive Learning Scales* (PALS), Midgeley (2000) was included. The PALS assessment has proven to be a valid and reliable instrument for evaluating student motivation (Muis et al., 2009). The PAGO questions are shown in the Appendix.

Algebra ability was measured using the Math 1 Final Exam. This exam is a common summative assessment given to every student who takes the Math 1 course in this school district. It is 50 questions long containing multiple-choice items covering the content of the course. The questions were developed by a team of math teachers from the high school, and was designed to test the topics covered in the course based upon the Common Core Curriculum Standards adopted by the school district.

Previous Math Achievement (PMA) was determined based upon the last math course the student took prior to taking Math 1. For all students included in the analysis, this was Eighth Grade Math. The PMA score was calculated as an average of each student's quarter grades based on a modified 4.0 scale where, for example, B- = 2.7, B = 3.0, and B+ = 3.3.

Procedure

The *Abstract Reasoning Assessment* (ARA) and the Personal Achievement Goal Orientations (PAGO) scale were administered to the algebra classes of eight math teachers at the high school. The teachers gave their students the ARA during the week before the final exam. They told the students to solve as many of the puzzles as they could in twenty minutes. After twenty minutes the students were instructed to turn to the six questions at the end that comprise the PAGO scale used for assessing motivation. The booklets were then collected and returned to the researcher. School records were accessed to find each student's age, final exam score, and previous year's math grades.

Analysis

A multiple linear regression was conducted. The final exam score was the criterion variable and scores on the *Abstract Reasoning Assessment*, the Personal Achievement Goal Orientation (PAGO) scale, previous math score, and age were used as predictor variables. Age, PAGO and Previous Math Achievement (PMA) were entered first in the analysis as control variables.

Results

The hypothesis of the study is that performance on an algebra final exam is related to abstract reasoning after controlling for age, motivation, and previous math achievement.

Data were collected and analyzed from 216 students who had valid data for all five variables: *Abstract Reasoning Assessment* (ARA), the Personal Achievement Goal Orientations scale (PAGO), eighth grade math grades, age, and Math 1 final exam score. Item responses for the ARA were coded for each respondent and then recoded as correct or incorrect. Unanswered questions were counted incorrect because it was a timed test to see how many matrix completion items a student could correctly answer in twenty minutes. Students were given an ARA score equal to the number of correct answers. Student responses to the Personal Achievement Goal Orientations scale (PAGO) were coded 1-5 from the Likert scale. The PAGO score was the mean of their responses. Students who answered more than half of the PAGO questions were included for analysis. Algebra ability was measured using scores from the Spring Semester Final Exam for the Math 1 class. Descriptive statistics for the sample are shown in Table 2.

Table 2

Descriptive Statistics

	Mean	Std. Deviation	Min.	Max
Final	71.96	13.81	20	100
age	15.32	0.39	14.18	16.56
PAGO	2.93	0.76	1.0	4.5
PMA	3.29	0.73	0.50	4.15
ARA	14.72	3.52	3	22

Correlations among the variables are displayed in Table 3.

Table 3

Correlations

		Final	age	Mot	LY	ARA
Pearson Correlation	Final	1.000	-.119	.221	.441	.516
	age	-.119	1.000	.020	-.025	-.046
	Mot	.221	.020	1.000	.104	.246
	LY	.441	-.025	.104	1.000	.302
	ARA	.516	-.046	.246	.302	1.000

Validity and Reliability of the *Abstract Reasoning Assessment*

The *Abstract Reasoning Assessment* (ARA) was analyzed to determine its validity and reliability. In terms of validity, the ARA was developed following the Embretson protocols for generating valid tests containing matrix completion items.

Embretson's hypothesis is that question difficulty depends on Working Memory Load (WML), so one would expect a positive correlation between WML and the proportion of students that answer an item incorrectly. The correlation between WML and the proportion of incorrect responses was found to be $R = 0.56$. For comparison, double histogram comparing the WML of each item, scaled as a percent of the highest WML value of 24, to percent of students who missed each item is shown in Figure 1.

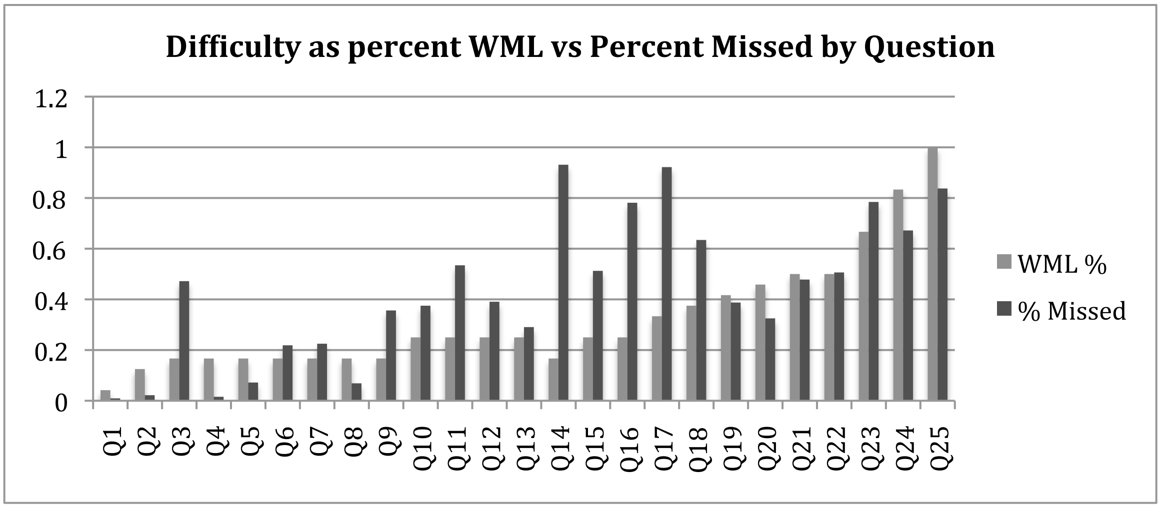


Figure 1

The pattern was generally as expected, with some exceptions. A few questions were more difficult than their Working Memory Load (WML) would suggest. Question 14 used a different kind of patterning than other items on the test so it was not placed in strict order of difficulty because the researcher anticipated that it would be more difficult than its WML suggested. It turned out to be even more difficult than anticipated. Question 16's reasoning involves figures that cancel each other out analogous to positive and negative numbers which should perhaps add an additional layer of WML to the Embretson model. Question 17 used a pattern that is difficult to see because the two objects in question look identical and are moving in the same field, and can sometimes be superimposed. It is interesting to note that questions 3 and 17 both involve rotating figures and both were more difficult than anticipated. Another anomaly of the test was that question 6 inadvertently had the correct answer listed twice among its answer choices. When scoring the tests both correct responses were accepted. This item proved to be reliable in all reliability analyses so it was retained.

As further validity evidence, item discrimination indices were calculated for each question on the *Abstract Reasoning Assessment* (ARA) by correlating each item to the Total ARA score. As an indication that an item is measuring the same construct as the entire test, one would expect a positive correlation between each item and the total test score. All questions had positive correlations to the ARA score, so all questions were retained for analysis. To determine reliability, Cronbach's Alpha was calculated. The value of Cronbach's Alpha for all 25 items was 0.70, indicating adequate reliability. Further analysis suggested that, if 9 items were removed, the alpha level could be increased to 0.73. The researcher decided to retain all 25 items, because the increase in reliability gained by removing those items was small, and all items in question were deemed important in distinguishing the very highest and very lowest performers on the ARA.

The results of the validity and reliability analyses suggested that the *Abstract Reasoning Assessment* is valid and reliable. No changes were made to the test before performing further analyses.

Analysis

A multiple linear regression was conducted. The final exam score was the criterion variable and scores on the *Abstract Reasoning Assessment* (ARA), the Personal Achievement Goal Orientations scale (PAGO), Previous Math Acheivement (PMA), and age were used as predictor variables. Age, motivation as measured by PAGO, and PMA were entered first in the analysis as control variables. The first analysis used all four variables. The regression shows a significant multiple correlation of $R = 0.61$, R-squared $= 0.37$, $F = 31.38$, $p < .005$. However, in this model age did not have a significant correlation, R-square change = .01, F change = 3.05 and PAGO had a small correlation R-square change = 0.05, F change = 11.35. Results are summarized in Table 4

Table 4

Model Summary

Model	R	R Square	Adjusted R Square	Std. Error of the Estimate	Change Statistics				
					R Square Change	F Change	df1	df2	Sig. F Change
1	.12[a]	.01	.01	13.74	.01	3.05	1	214	.08
2	.25[b]	.06	.06	13.42	.05	11.35	1	213	.00
3	.49[c]	.24	.23	12.13	.17	48.56	1	212	.00
4	.61[d]	.37	.36	11.04	.14	45.31	1	211	.00

a. Predictors: (Constant), age

b. Predictors: (Constant), age, PAGO

c. Predictors: (Constant), age, PAGO, PMA

d. Predictors: (Constant), age, PAGO, PMA, ARA

Table 4 illustrates that when adding ARA to the model the R-square change = 0.14, and $p < .005$, indicating that ARA is a significant predictor of algebra achievement. Table 5 displays the coefficients for the four variable multiple regression model.

Table 5

Coefficients[a]

Model		Unstandardized Coefficients		Standardized Coefficients	t	Sig.
		B	Std. Error	Beta		
4	(Constant)	76.69	30.41		2.52	.01
	age	-0.01	0.01	-0.10	-1.73	.09
	PAGO	1.70	1.03	0.09	1.66	.10
	PMA	5.85	1.08	0.31	5.42	.00
	ARA	1.55	0.23	0.40	6.73	.00

a. Dependent Variable: Final

Table 5 shows that the effect size indicated by the standardized coefficient of the model indicates that ARA is the largest significant contributor in predicting algebra achievement, Beta = 0.40, $p < .005$. PMA is also a significant predictor, Beta = .31, $p < .005$. Note that the effect sizes indicated by the standardized coefficients are small for both age and PAGO Beta = -0.10, and Beta = 0.09 respectively and neither is significant, $p = 0.09$, and $p = .10$ respectively.

For the sake of parsimony, a final regression was performed with only the significant predictors of Previous Math Achievement (PMA), and the *Abstract Reasoning Assessment* (ARA). 220 students had valid data for this analysis. The regression shows a significant multiple correlation of R = 0.60, R-squared = 0.36, F = 60.25, $p < .005$. The results are shown in Table 6.

Table 6

Model Summary

Model	R	R Square	Adjusted R Square	Std. Error of the Estimate	Change Statistics				
					R Square Change	F Change	df1	df2	Sig. F Change
1	.44[a]	.20	.19	12.44	.20	52.74	1	218	.00
2	.60[b]	.36	.35	11.14	.16	54.76	1	217	.00

a. Predictors: (Constant), PMA

b. Predictors: (Constant), PMA, ARA

The model containing Previous Math Achievement and the *Abstract Reasoning Assessment* scores significantly predicts algebra achievement with a multiple correlation of R = 0.60, R-squared = 0.36, F = 60.25, $p < .001$. Abstract reasoning ability has, an R-squared change = 0.16 which shows that 16.2% of the variation in algebra achievement can be explained by abstract reasoning ability. The multiple linear regression coefficients are shown in Table 7.

Table 7

Coefficients[a]

Model		Unstandardized Coefficients B	Unstandardized Coefficients Std. Error	Standardized Coefficients Beta	t	Sig.
2	(Constant)	28.18	4.13		6.82	.00
	PMA	5.98	1.08	.32	5.52	.00
	ARA	1.70	0.22	.42	7.40	.00

a. Dependent Variable: Final

The regression model has a standardized coefficient for ARA of Beta = 0.42. The coefficient for PMA is Beta = 0.32. This coincides with the result from the earlier regression showing a larger effect size for ARA compared to PMA, indicating the importance of abstract reasoning in algebra achievement.

Discussion

Results from these analyses indicate that abstract reasoning ability is the most important among the variables studied in predicting success in high school algebra. Scores on the *Abstract Reasoning Assessment* predicted performance in a high school algebra course even after controlling for general mathematics ability.

The findings suggest strongly that abstract reasoning ability is critical for success in algebra. The test developed for this study using Embretson's protocol is completely devoid of any algebraic or mathematical content other than pattern recognition, and the patterns used are non-mathematical. This gives credence to the famous definition of mathematics by Walter Warwick Sawyer (1955), "Mathematics is the classification and study of all possible patterns" (p. 12).

One interesting finding of the study is that age is not a significant predictor of algebra proficiency. In fact the not-quite-significant correlation was negative. That may be because this study's regression was done using only the students whose previous class was Eighth Grade Math, so there was not a great deal of variation in the ages among those included in the analysis. The direction of the correlation in this sample could be due to the fact that some students may be behind their age cohorts due to ability selection. Preliminary analyses performed before the formal analysis began using all data showed an even higher negative correlation because the students whose previous math class was not Eighth grade math were enrolled in Math 1 classes because they either failed Algebra I previously or had demonstrated a need for further skill-building or remediation before taking Math 1.

Another interesting finding is that motivation was not shown to be a significant predictor of algebra achievement. This could be because the effect of motivation as measured by the Personal Achievement Goal Orientations scale is already accounted for in the other significant predictor, previous math achievement.

Implications

Identifying the factors leading to success in high school algebra is critical in planning course placement for students. When students are placed in classes before they are sufficiently equipped, those students have a greater chance of failure. For instance, over one third of eighth grade algebra students in California had to repeat an algebra course in ninth grade (EdSource, 2009). Some feel that algebra instruction at the eighth grade level is not as rigorous and not focused and taught in a way that facilitates understanding for subsequent high school math classes (Loveless, 2008; Stephany, 2011).

Stephany goes on to point out that succeeding in algebra is more important than the grade level in which it is taken. This study presents another factor leading to higher failure rates, an underdeveloped abstract reasoning capability.

This study's finding that abstract thinking ability is related to success in learning algebra could potentially be used as an aid in proper placement of students into a high school algebra class at the proper time. Of course, this study does not establish whether an innate ability to reason abstractly leads to increased algebra achievement, or that practicing algebra leads to increased ability to reason abstractly. While most research associates abstract reasoning to Piaget's formal operations, it is also possible that abstract reasoning is developed through experience rather than as a developmental stage.

Limitations

One limiting factor of the study is that the spring semester's Math 1 curriculum in the study's district includes a few topics traditionally taught in Geometry. Another drawback is that the eighth grade Math 1 students were not included in the study, because the eighth grade curriculum and final exam were not identical to that of the high school.

The fact that motivation was not a significant factor in this study could be due to other considerations regarding the Personal Achievement Goal Orientations scale and how it was used. Although it is a commonly used measure, it could be that it does not capture motivation for achievement on an algebra final exam. Another concern is the timing of measuring motivation at the end of the semester may not accurately reflect students' motivation level as they were learning the material. It could also be that using the PAGO scale in conjunction with and at the end of the ARA may have influenced its result.

Future Study

There are many potential directions for further study. Firstly, the study could be repeated in schools with differing demographics and curricula. The study could also be focused on specific areas of the curriculum such as linear functions, quadratic equations, or systems of equations. It could also be used for other levels of mathematics such as pre-algebra or geometry.

A particularly interesting and practical application of this study could be for educators to explore the possibility of using the *Abstract Reasoning Assessment* as a diagnostic measure to determine a student's placement into an algebra class now that a link between abstract thinking and success in algebra has been suggested. Because the *Abstract Reasoning Assessment* is reliable, quick, and easy to administer, it would make an ideal tool as a predictor for success in high school algebra. The use of this tool could help many students be more successful in algebra and beyond when used to place students in algebra at the proper time, when their skill set indicates they are ready for the abstraction inherent in high school algebra.

References

ART Technical Manual (2006). Psychometrics Ltd.

Bird, L. (2010). Logical Reasoning ability and student performance in general chemistry. *Journal of Chemical Education, 87*(5), 541-546.

Bitner-Corvin, B. (1987). The GALT: A Measure of Logical Thinking Ability of 7th through 12th Grade Students. Paper presented at the Annual Meeting of the National Association for Research in Science Teaching, Washington, DC, April 23-25, 1987

Bracey, G. (2009). The algebra hoax. *Phi Delta Kappan, 90*(4), 306-307.

Carpenter, P., Just, M., & Schell, P. (1990). What one intelligence test measures: A theoretical account of the processing in the Raven's progressive matrices test. *Psychological Review*, 97, 404 – 431.

Case, R., & Marini, Z. (1994). The development of abstract reasoning about the physical and social world. *Child Development*, *65*, 147-159.

Cattell, R. B. (1963). Theory of fluid and crystallized intelligence: A critical experiment. *Journal of Educational Psychology*, *54*(1), 1-22.

Cavenaugh, S. (2008). Low performers unready to take algebra. *Education Week*, 28(5), 1-13.

Capraro, M., & Joffrion, H. (2006). Can middle-school students meaningfully translate from words to mathematical symbols? *Reading Psychology*, *27*, 147-164.

Clements, D., & Sarama, J. (2004). Building abstract thinking through math. *Scholastic Early Childhood Today, 18* (5), 34-40.

Dede, Y. (2004). The concept of variable and identification its learning difficulties. *Educational Sciences: Theory and Practice*, *4*(1), 50-56.

Demby, A. (1997). Algebraic procedures used by 13-to-15-year-olds. *Educational Studies in Mathematics*, *33*, 45-70.

Devlin, K. (2000). Finding your inner mathematician: As the abstraction turns. *The Education Digest, 66*(4), 63-66.

EdSource. (2008). *Algebra policy in California: Great expectations and serious challenges.* Mountain View, CA: EdSource Inc.

Embretson, S. E. (1995). The role of working memory capacity and general control processes. *Intelligence*, *20*, 169-189.

Embretson, S. E. (1998). A cognitive design system approach to generating valid tests: Application to abstract reasoning. *Psychological Methods*, *3*(3), 380-396.

Goodson-Espy, T. (1998). The roles of reification and reflective abstraction in the development of abstract thought: Transitions from arithmetic to algebra. *Educational Studies in Mathematics, 36,* 219-245.

Hazzan, O., & Zazkis, R. (2005). Reducing abstraction: The case of school mathematics. *Educational Studies in Mathematics, 58* (1), 101-119.

Hershkowitz, R., Schwarz, B. & Dreyfus, T. (2001). Abstraction in context: Epistemic actions, *Journal for Research in Mathematics Education, 32*(2), 195-222.

Loveless, T. (2008). The misplaced math student: Lost in eighth-grade algebra. *Brown Center Report on American Education, 9.*

MacGregor, M. & Stacey, K. (1997). Ideas about symbolism that students bring to algebra. *The Mathematics Teacher*, *90*(2), 110–113.

McClure, C. (2009). Algebraic thinking: What it is and why it matters. *District Administration*, *45*(4), 44,45.

McKibben, S. (2009). Should all 8th graders take algebra? *Education Digest, 74*(7), 62-64.

McNiel, N. (2008). Limitations to teaching Children 2 + 2 = 4: Typical arithmetic problems can hinder learning of mathematical equivalence. *Child Development, 79*(5). 1524-1537

Marini, Z., & Case, R. (1994). The development of abstract reasoning about the physical and social world. *Child Development*, *65*, 147-159.

Midgley, C, Maehr, M., Hruda, L., Anderman, E, Anderman, L., Freeman, K, Gheen, M., Kaplan, A., Kumar, R., Middleton, M., Nelson, J., Roeser, R, & Urdan, T. (2000) *Manual for Patterns of Adaptive Learning Scales*, University of Michigan

Monaghan, J. & Ozmantar, M. (2006) Abstraction and consolidation. *Educational Studies in Mathematics, 62* (3), 233-58

Muís, K., Philip H., Winne, P., & Ordene V. (2009) Modern psychometrics for assessing achievement goal orientation: A Rasch analysis. *British Journal of Educational Psychology*, *79*, 547-576.

Nathan, M., & Koedinger, K. (2000). Teachers' and researchers' beliefs about the development of algebraic reasoning. *Journal for Research in Mathematics Education, 31*(2), 168-190.

Roadranga, V. (1986). *The construction of a group assessment of logical thinking*

(GALT). Retrieved from http://ebookbrowse.com/the-construction-of-a-group-assessment-of-logical-thinking-galt-pdf-d65698268

Rojano, T., & Martinez, M. (2009). From concrete modeling to algebraic syntax: Learning to solve linear equations with a virtual balance. *Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University. 5*, 235-243.

Sawyer, W. (1955) *Prelude to mathematics*. Harmondsworth, Middlesex, England; Penguin Books.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics 22*, 1–36.

Sfard, A., Linchevski, L. (1994). The gains and pitfalls of reification – The case of algebra. *Educational Studies in Mathematics* 26, 191–228.

Spearman, C. (1927). *The abilities of man.* London: Macmillan.

Spielhagen, F. (2006). Closing the achievement gap in math: Considering eighth grade algebra for all students. *American Secondary Education, 34*(3), 29-42.

Stacey, K. (1997). The transition from arithmetic thinking to algebraic thinking. Retrieved from http://www.mathhouse.org/files/filebox/File/IMECstaceyALGEBRA.doc.

Stacey, K., & MacGregor, M. (1997). Multiple referents and shifting meanings of unknowns in students' use of algebra. In E. Pehkonen (Ed.), *Proceedings of the 21st Conference of the International group for the Psychology of Mathematics*

Education. University of Helsinki, Finland. *Vol 4*. (pp.190–197).

Stacey, K., & MacGregor, M. (1999). Learning the algebraic method of solving problems. *The Journal of Mathematical Behavior, 18*(2), 149-167.

Stephany, N. (2011). *The predictors and consequences of eight grade algebra success.* (Doctoral Dissertation). Retrieved from Pro Quest Dissertation and Theses. (3466108)

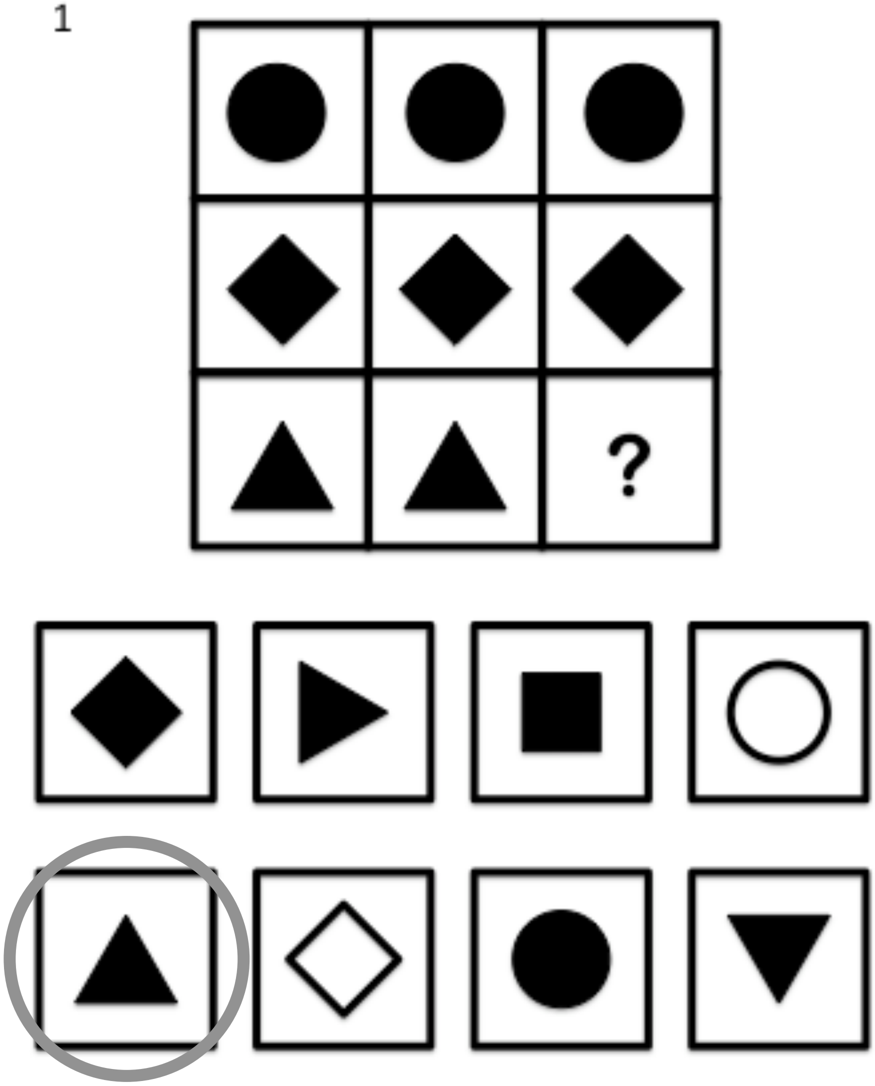
Viadero, D. (2010). Early algebra push seen to be flawed. *Education Week, 29*(21), 1-14.

Appendix

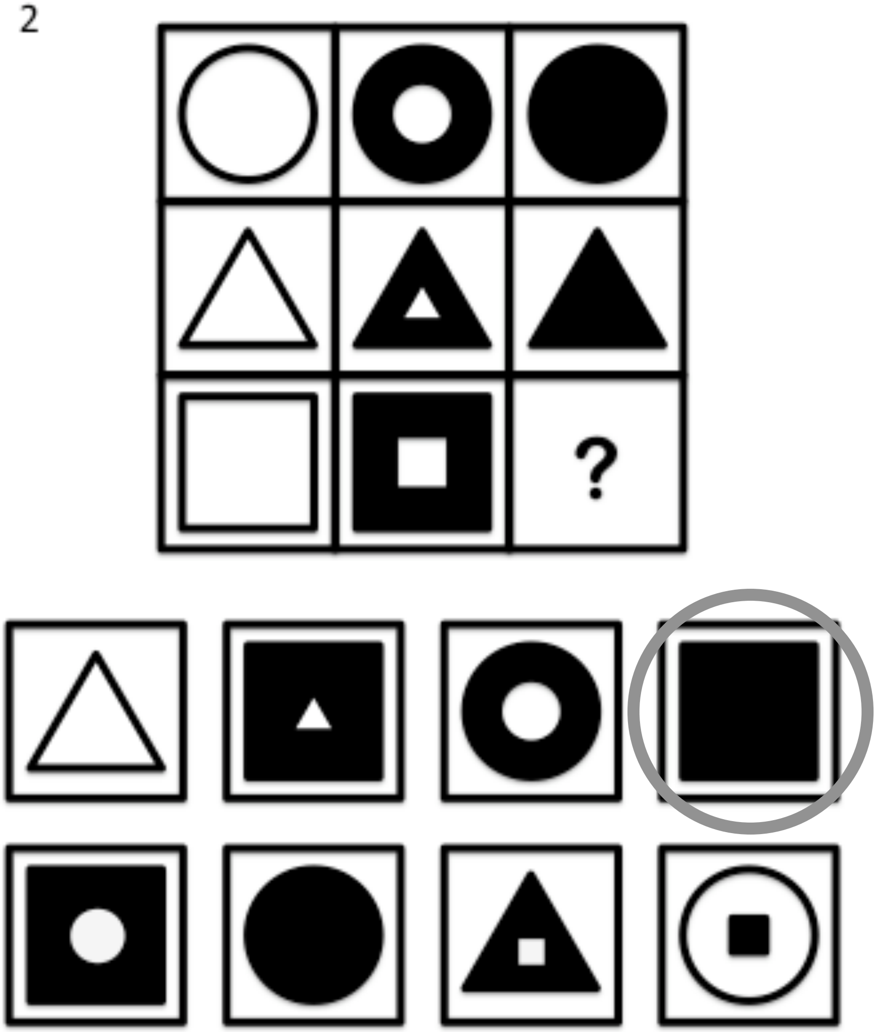
Abstract Reasoning Assessment

Look at the pattern in the three by three grid. Circle the choice below that best completes the pattern

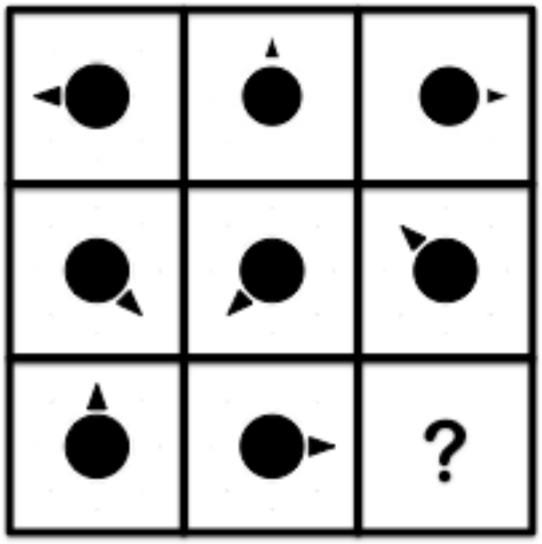
1

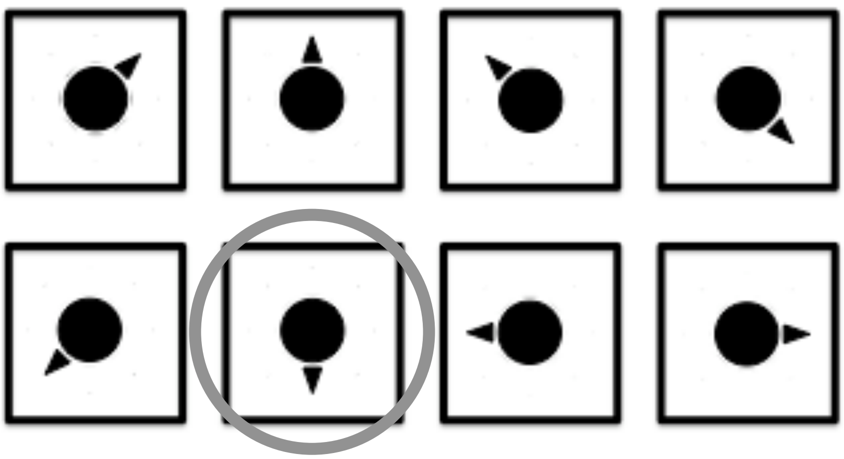


2

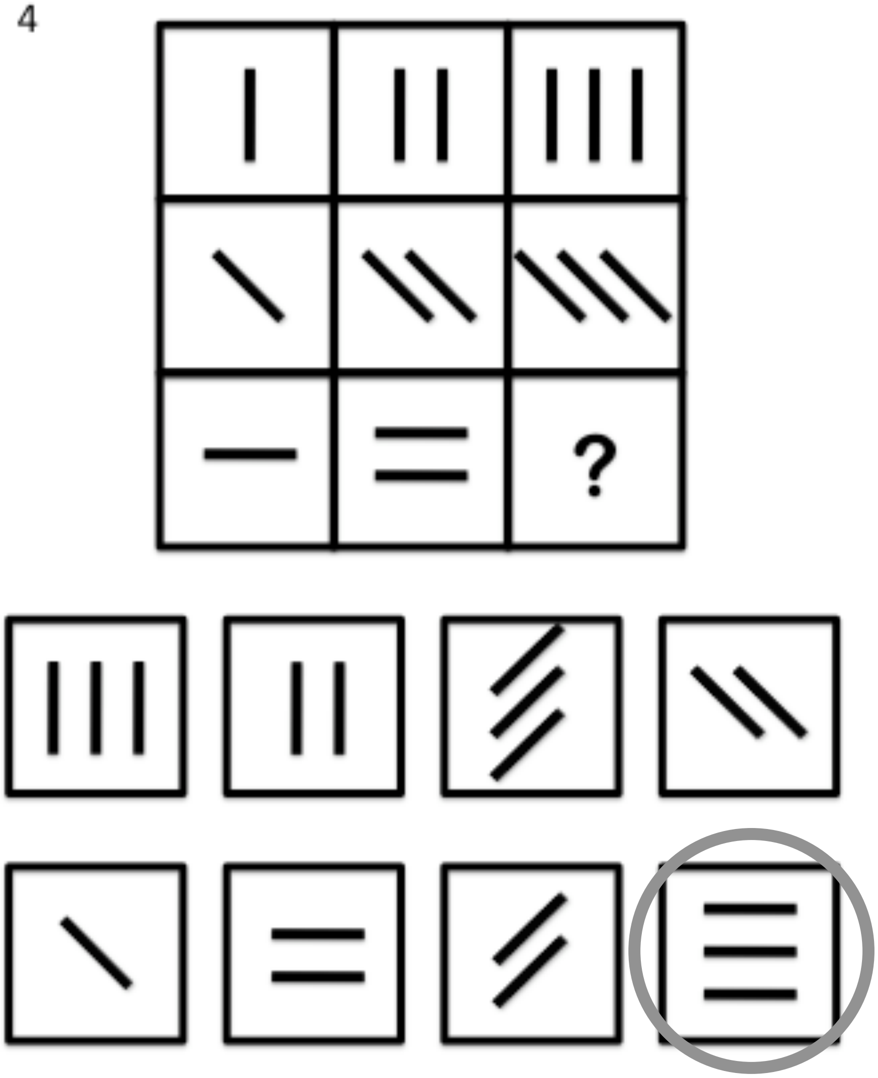


3

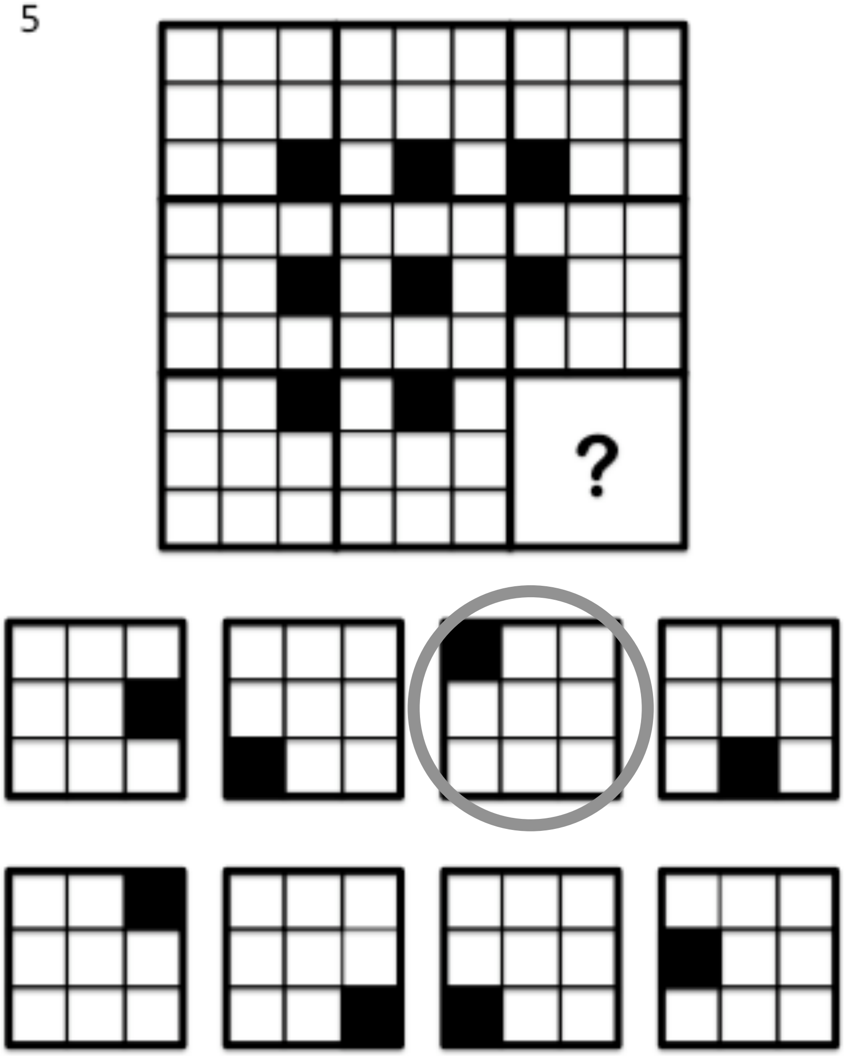




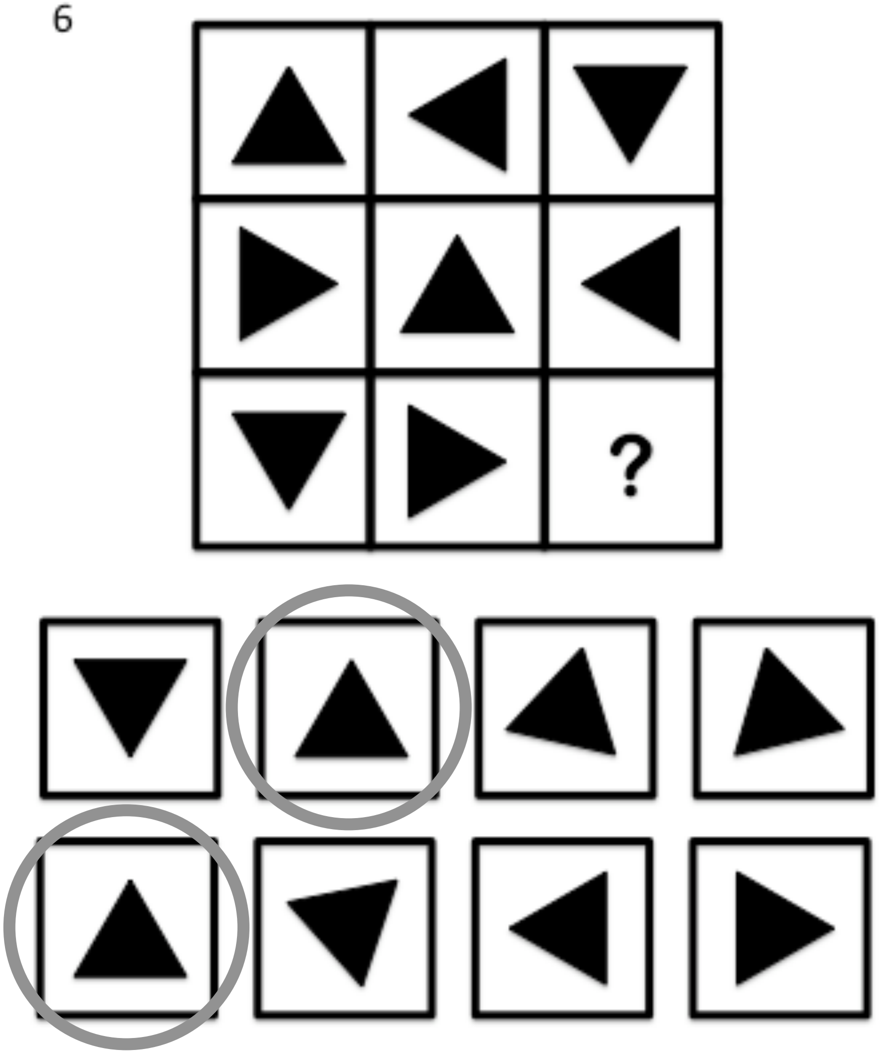
4



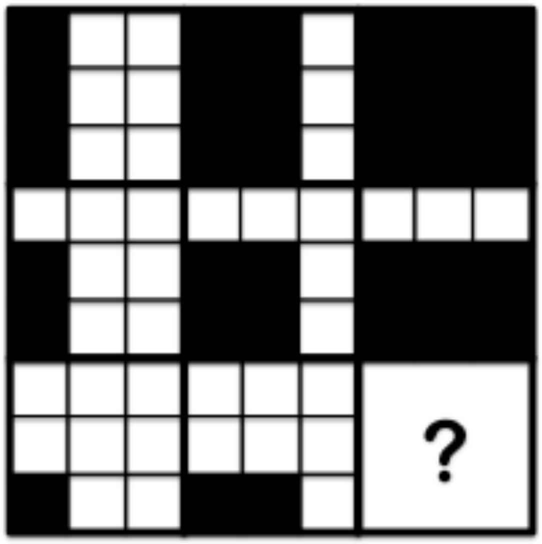
5

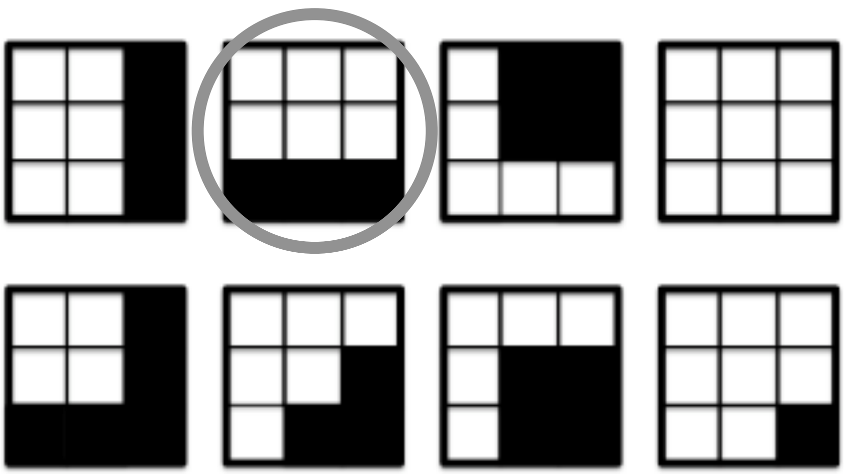


6

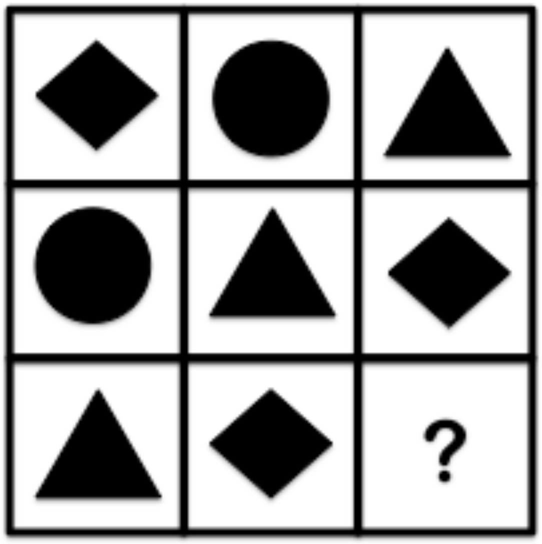


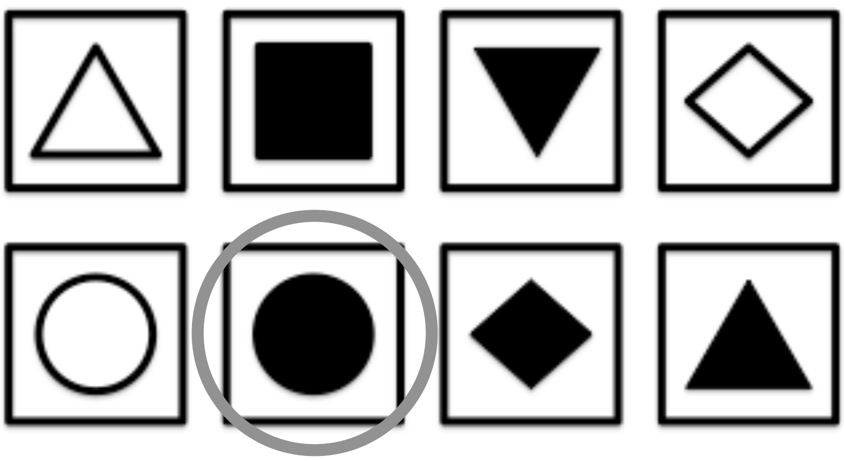
7



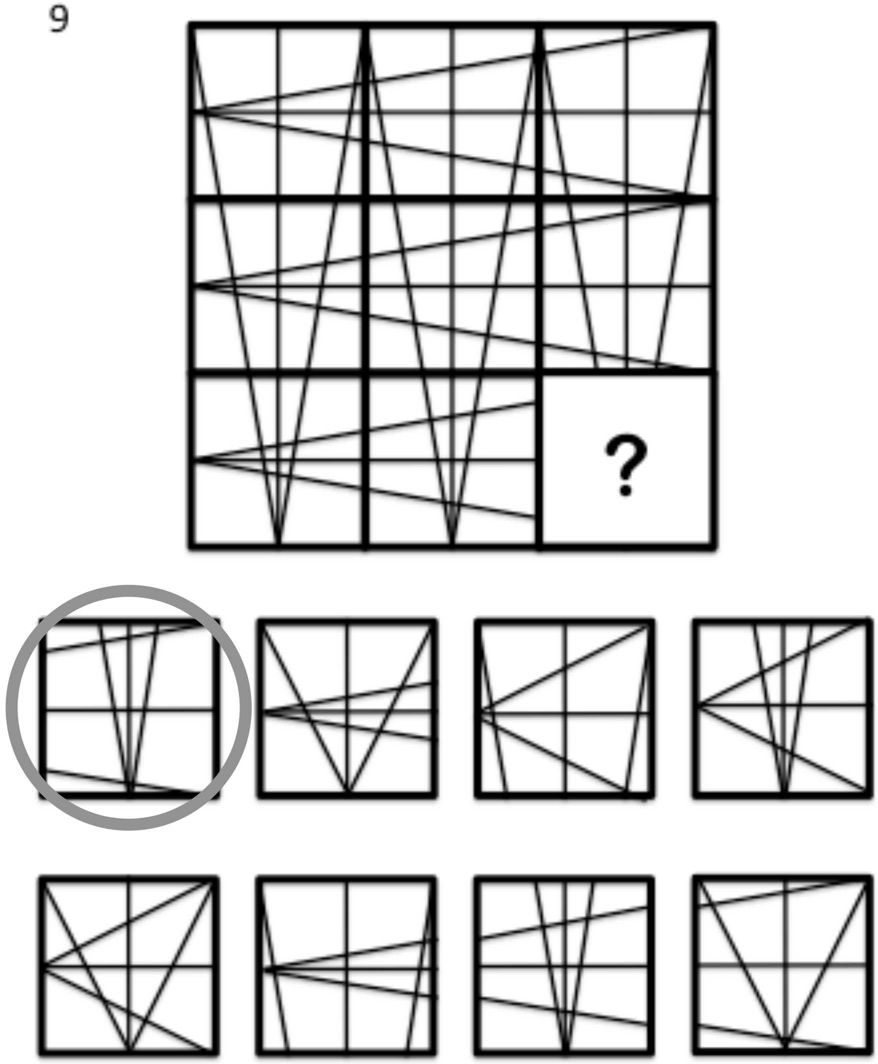


8

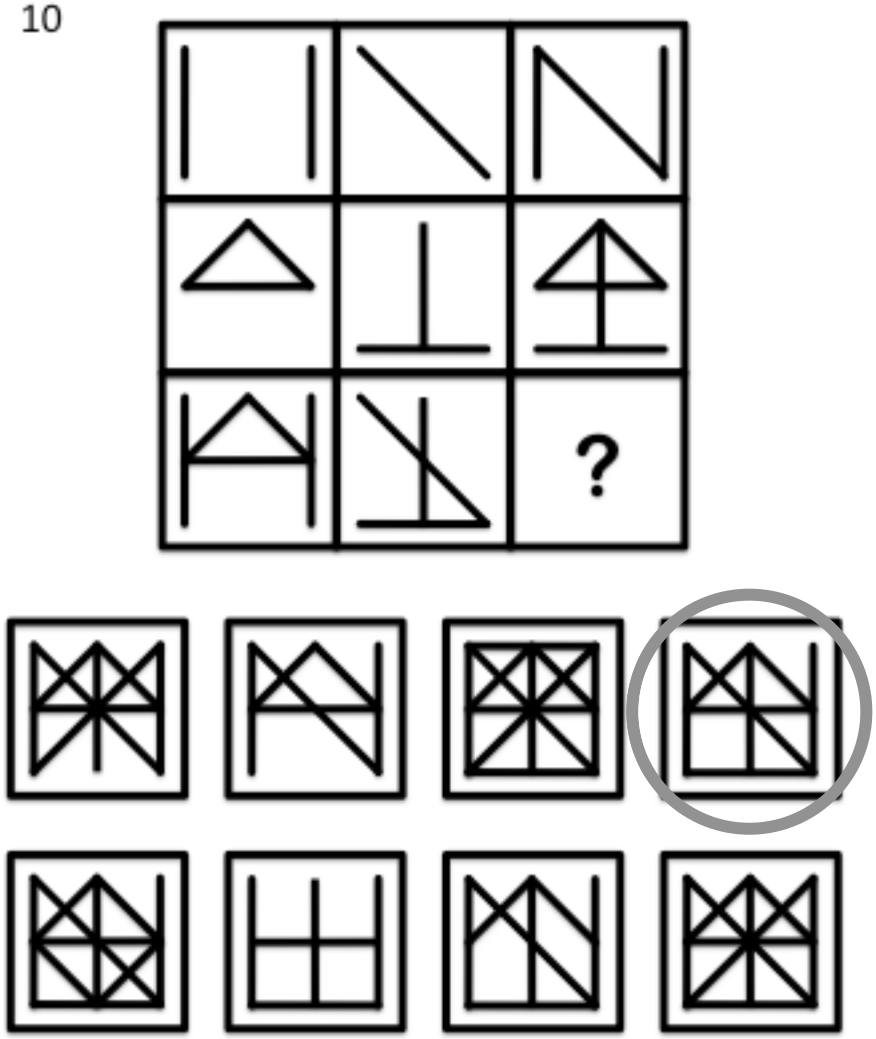




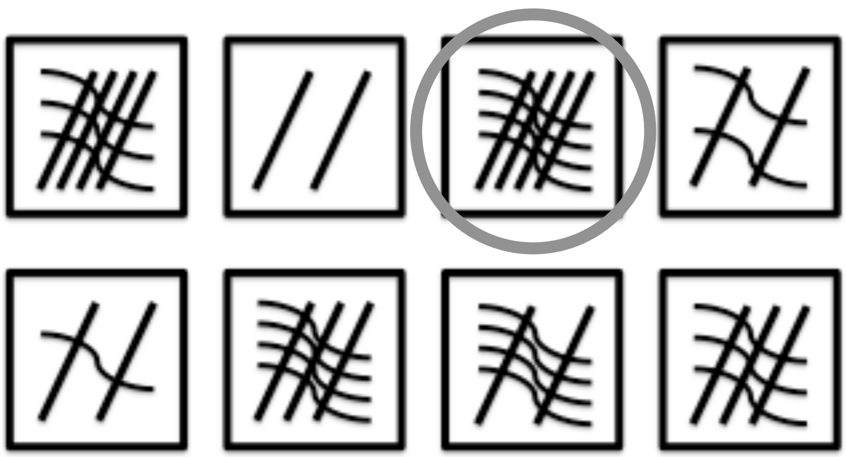
9



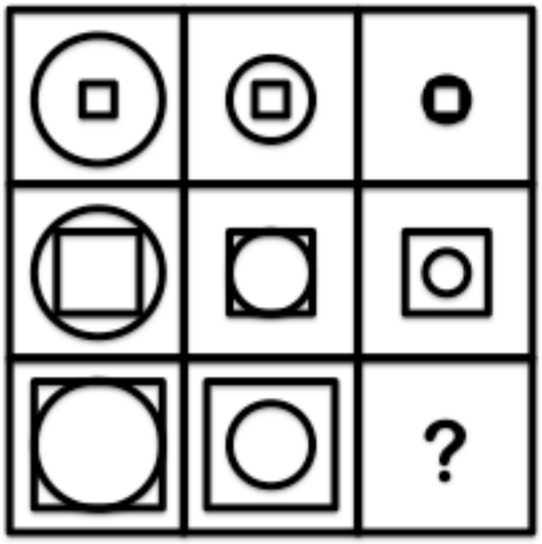
10

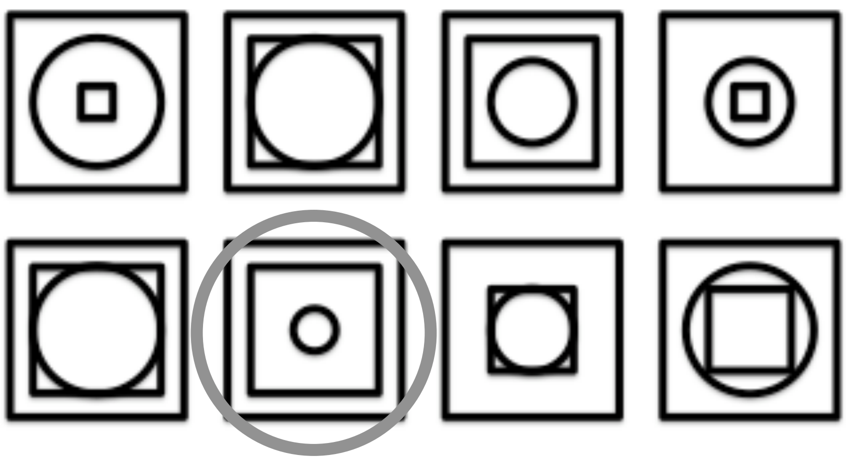


11

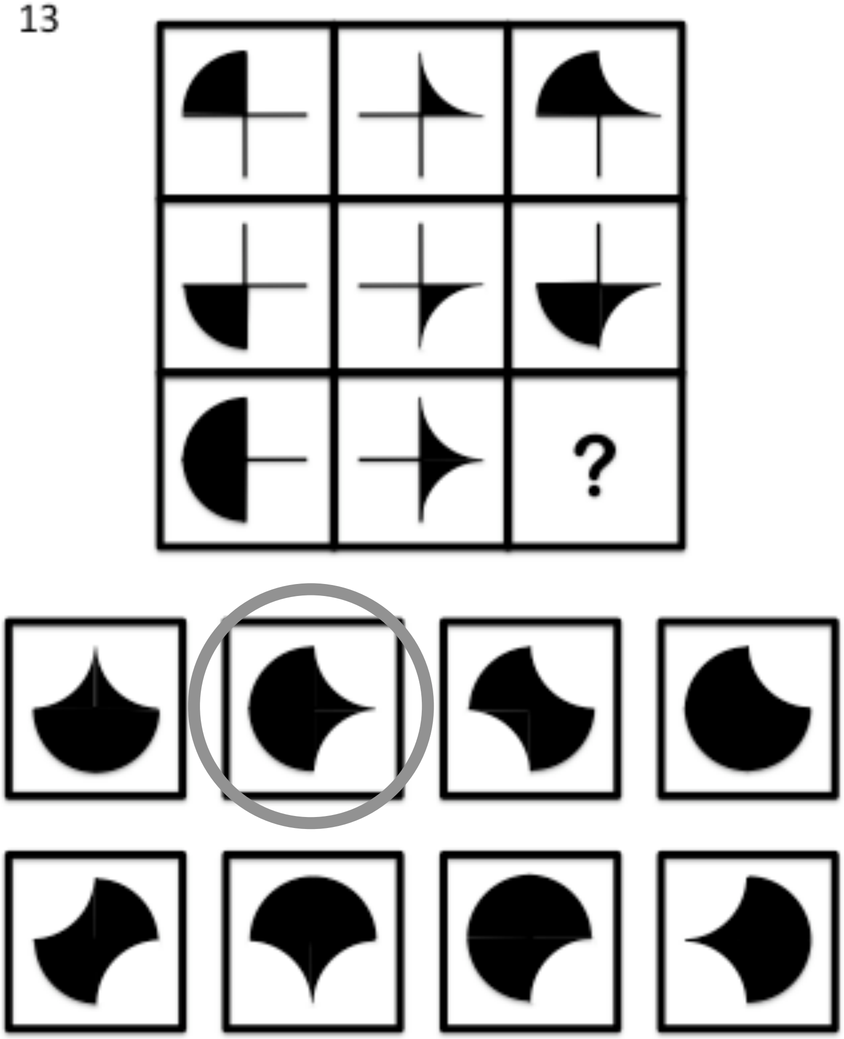


12

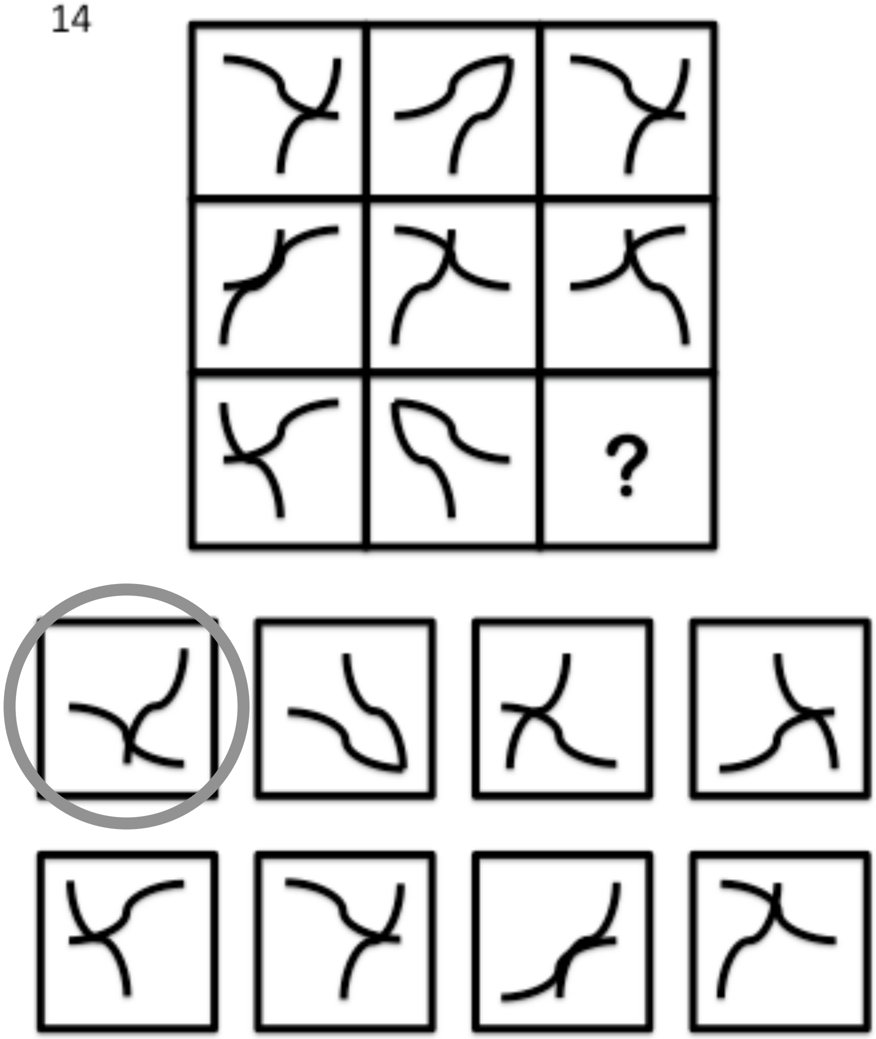




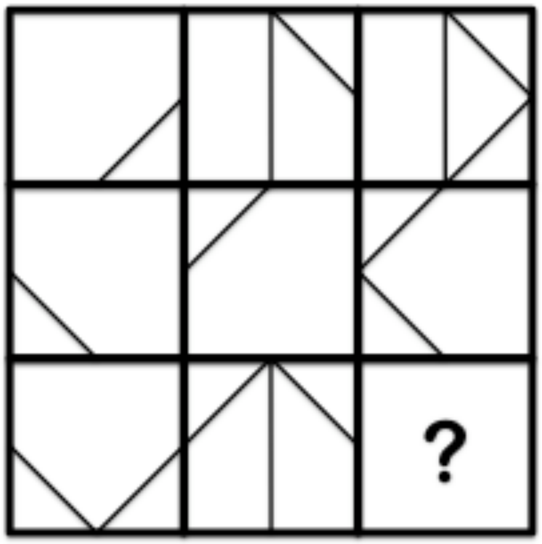
13

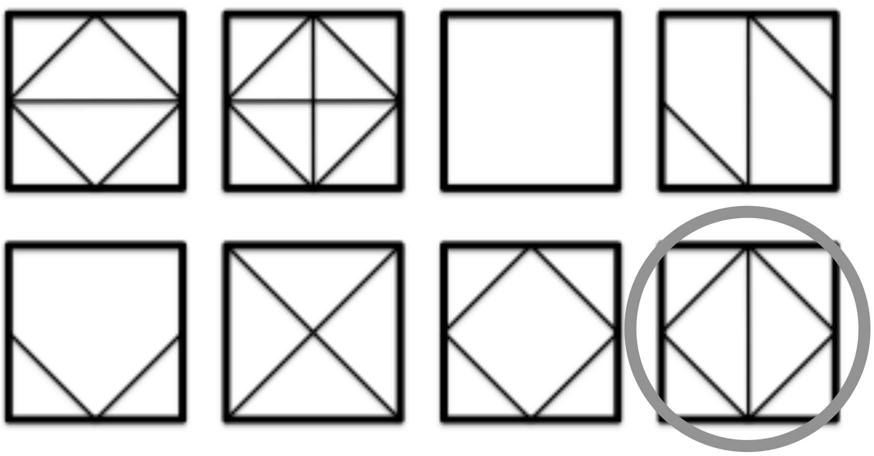


14

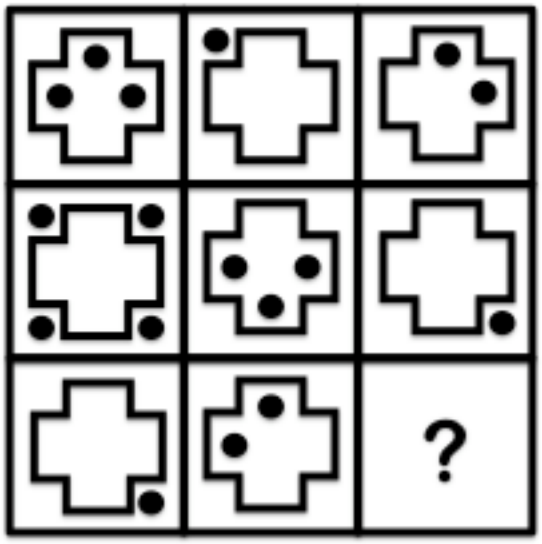


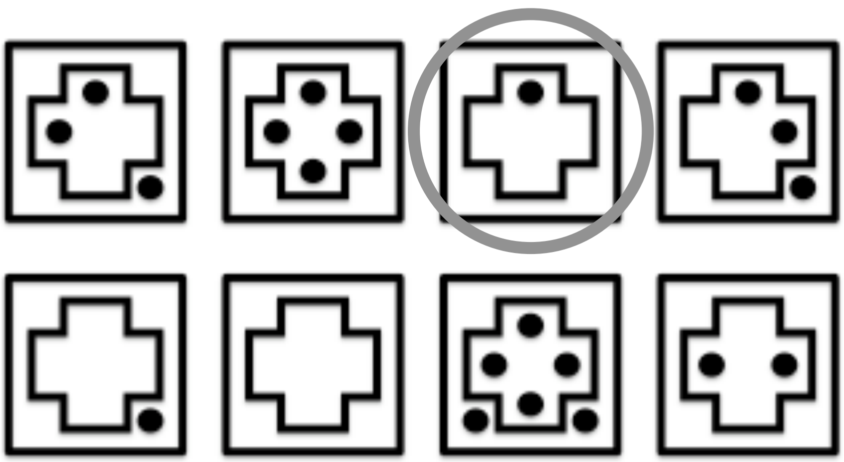
15



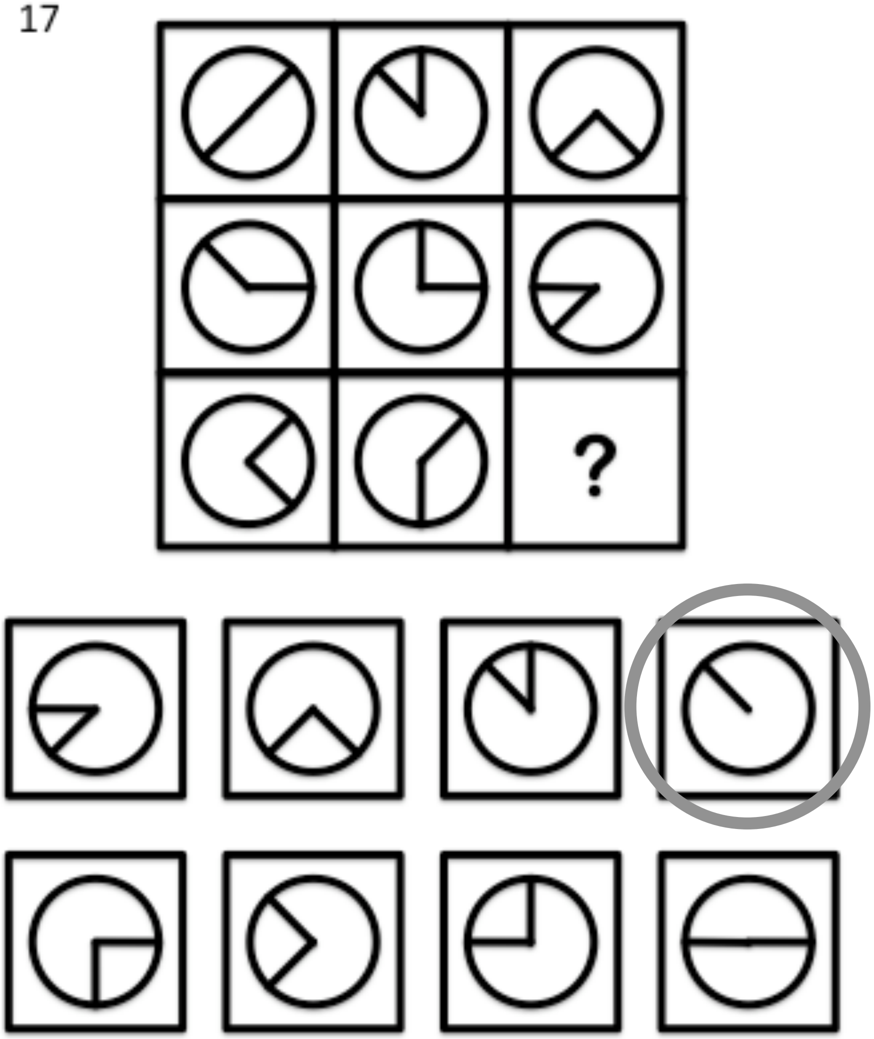


16

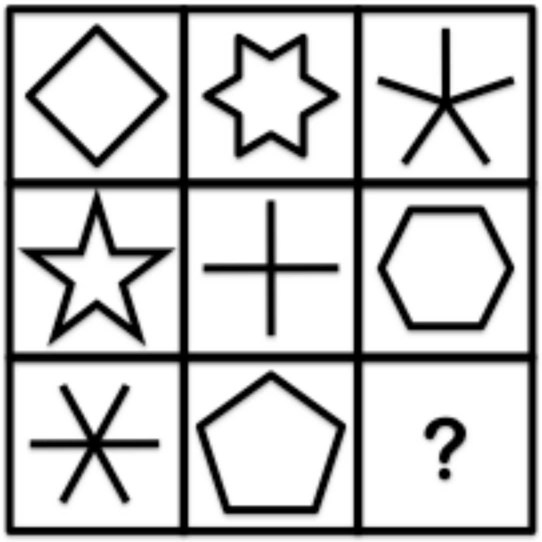


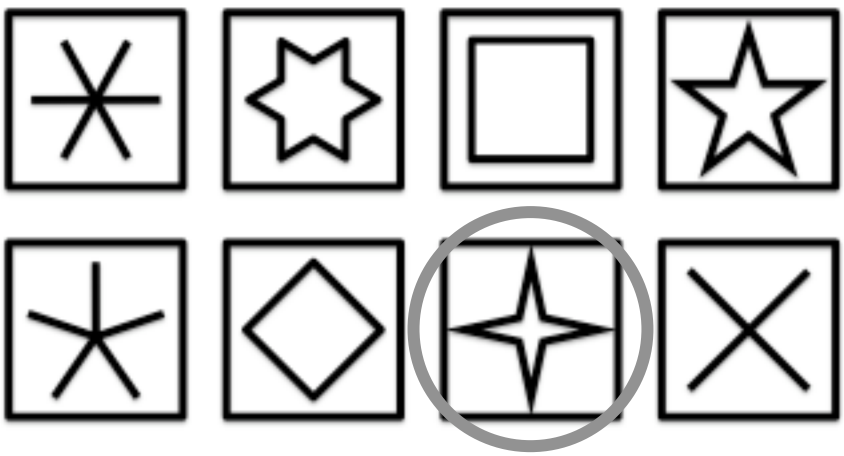


17

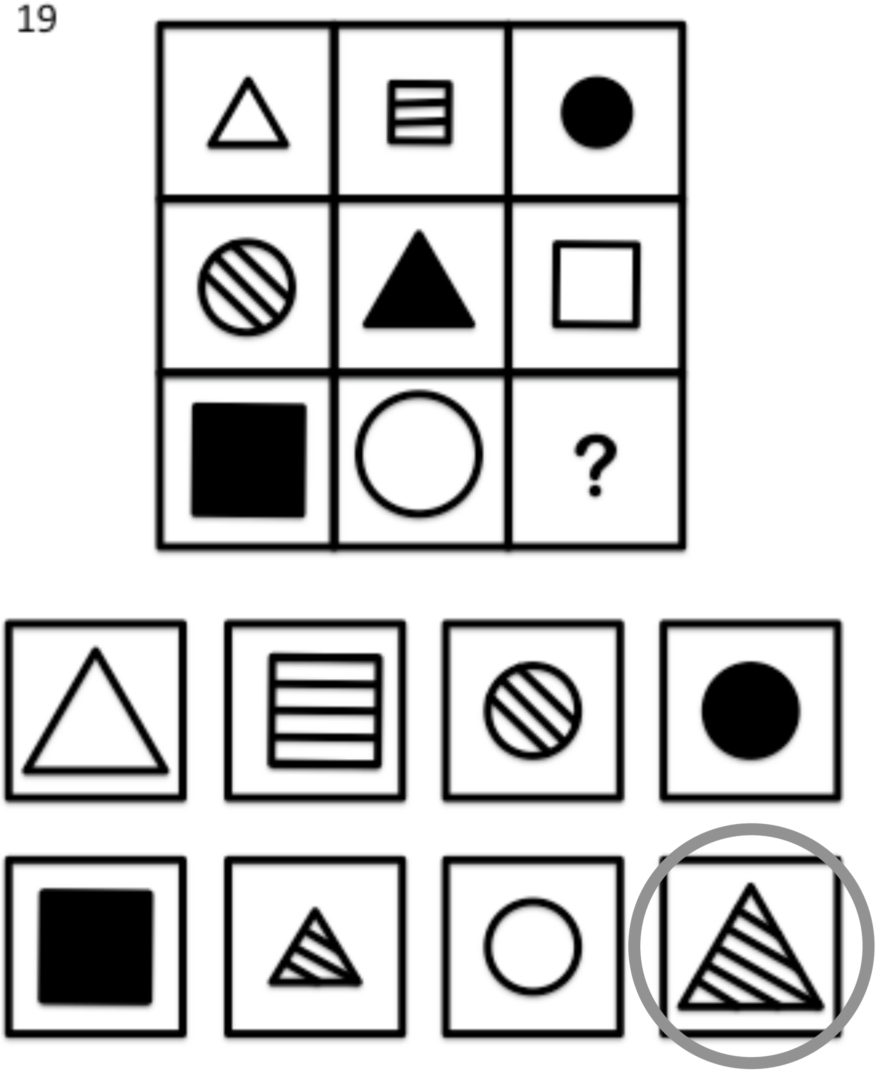


18

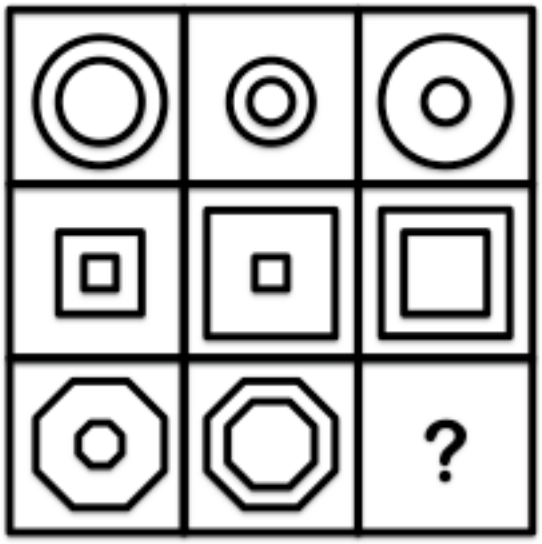


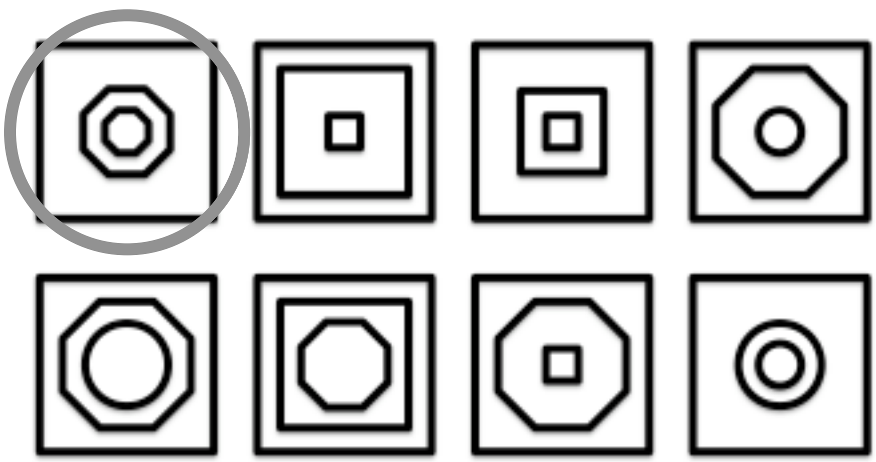


19

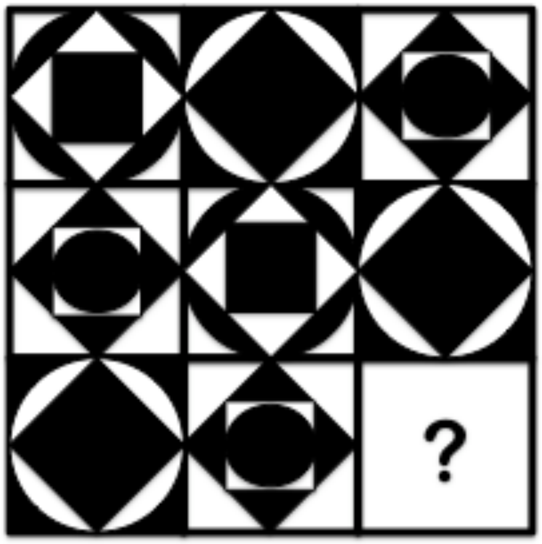


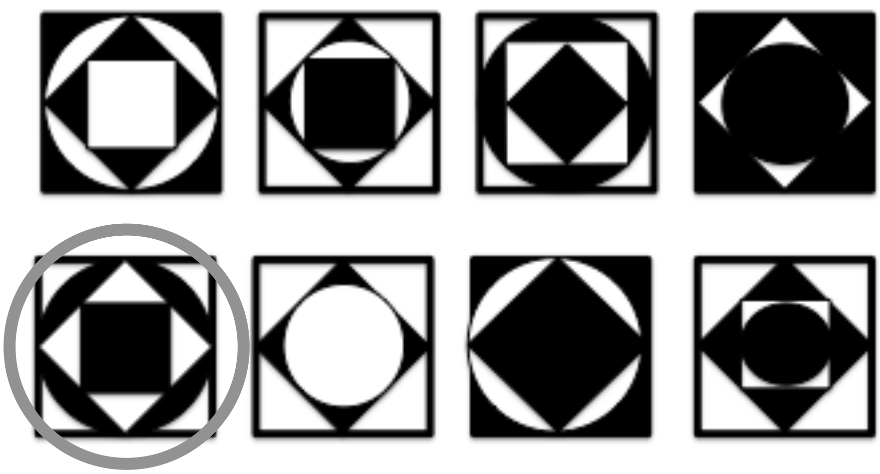
20



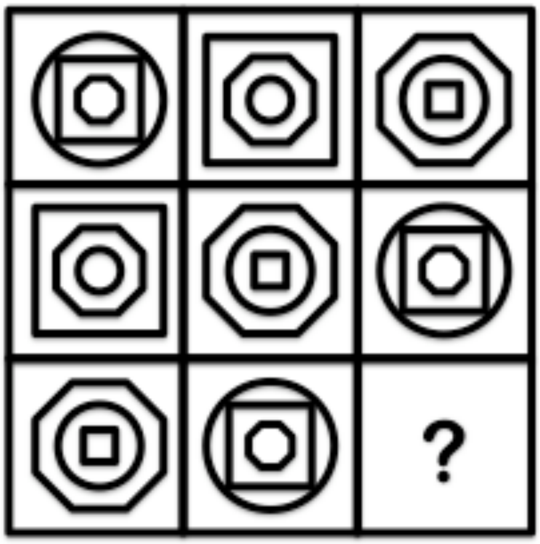


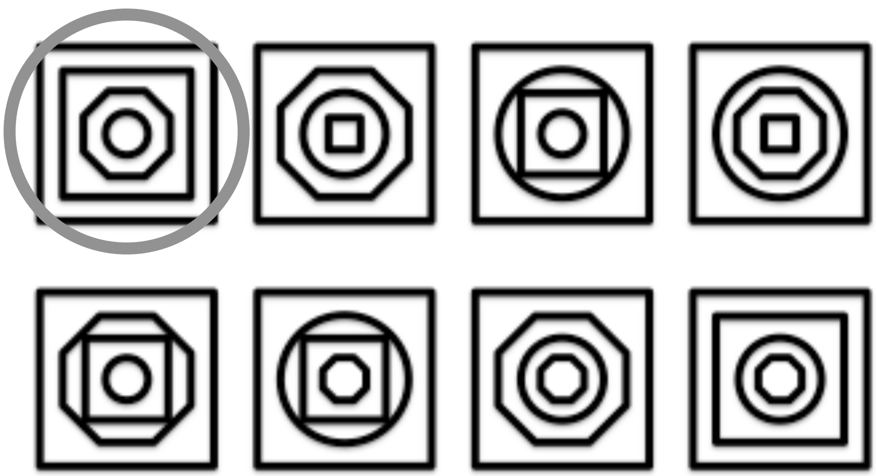
21



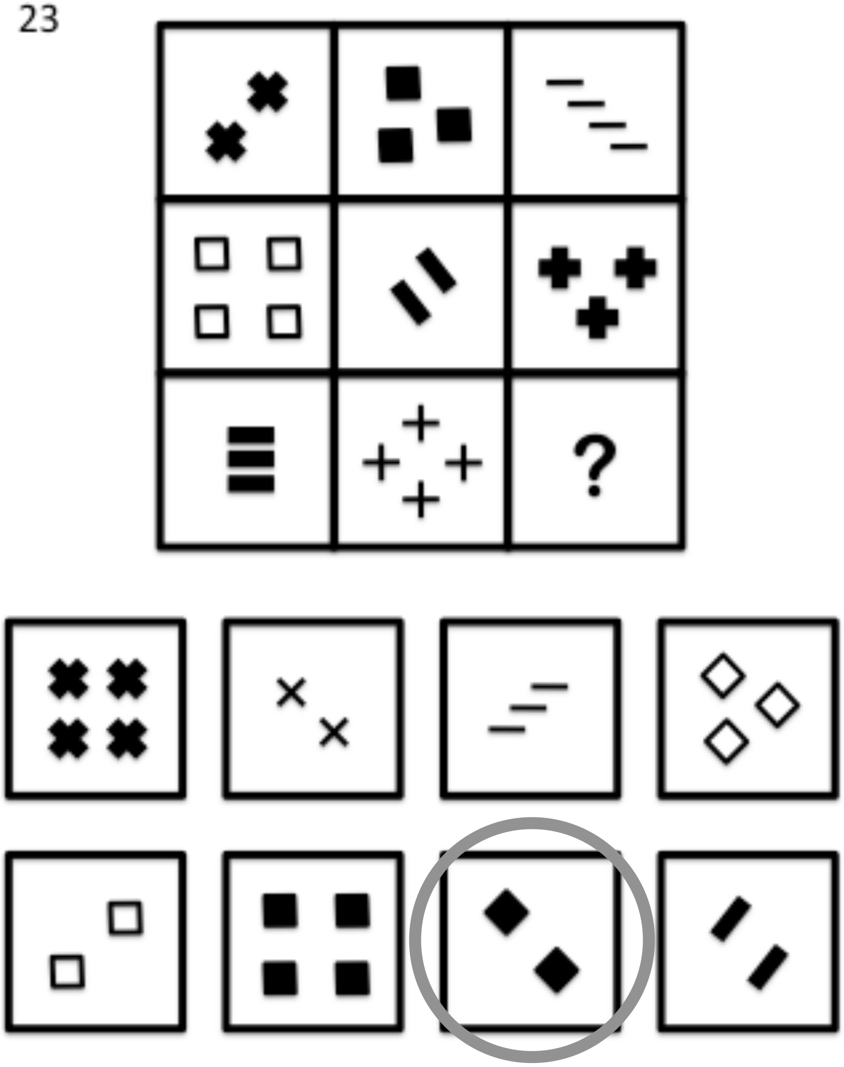


22

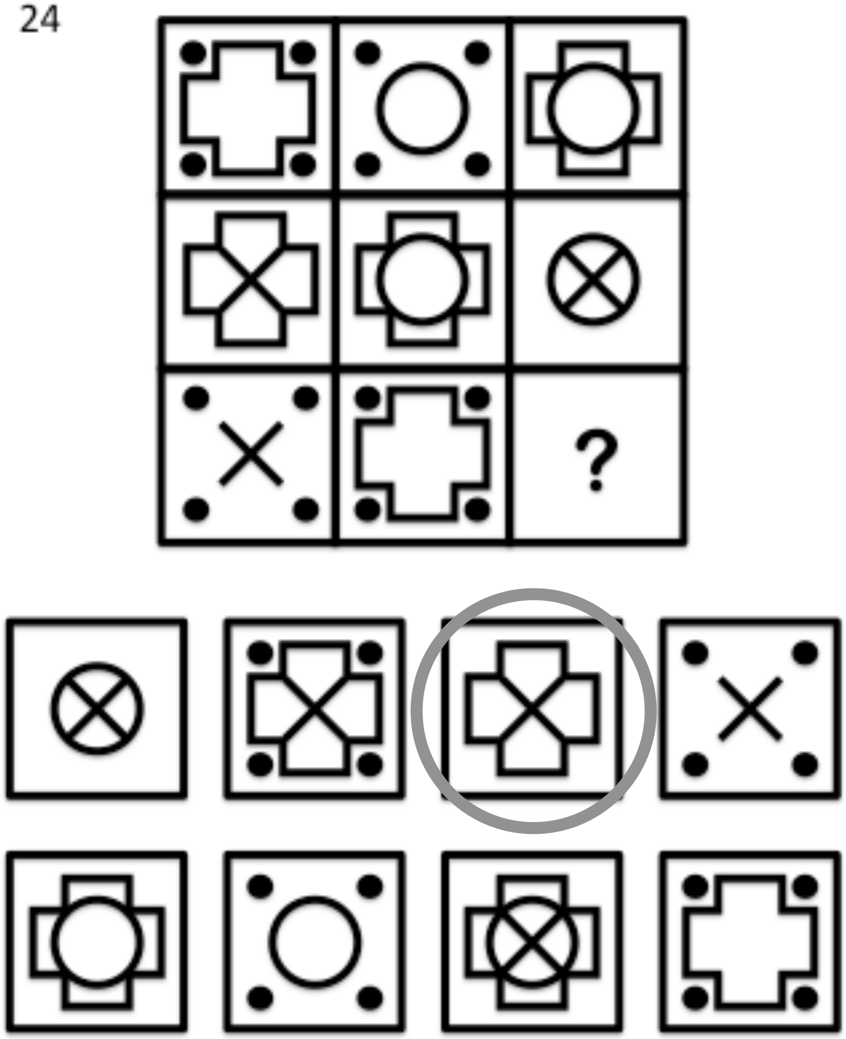




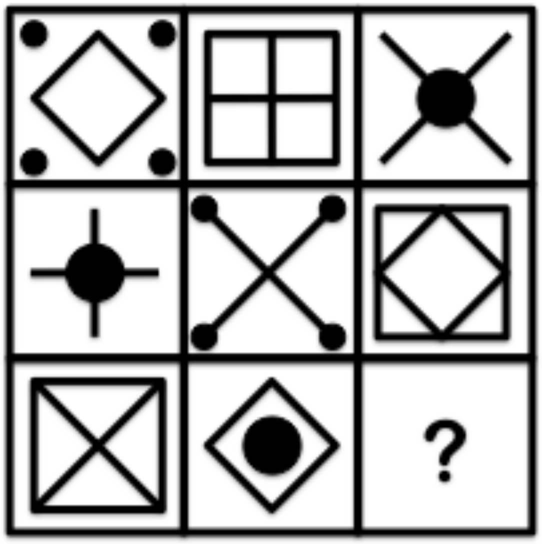
23

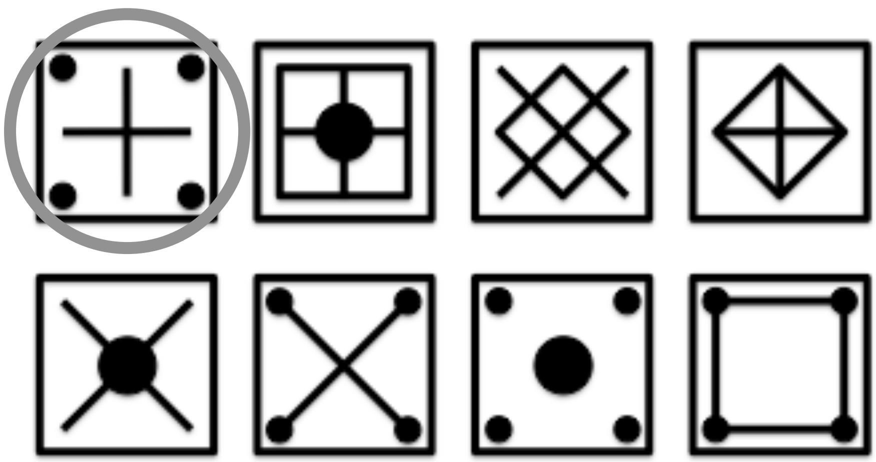


24



25





Personal Achievement Goal Orientation Items

1. I like class work that I'll learn from even if I make a lot of mistakes.

Not at all true		Somewhat true		Very true
1	2	3	4	5

2. An important reason why I do my class work is because I like to learn new things.

Not at all true		Somewhat true		Very true
1	2	3	4	5

3. I like class work best when it really makes me think.

Not at all true		Somewhat true		Very true
1	2	3	4	5

4. An important reason why I do my work in class is because I want to get better at it.

Not at all true		Somewhat true		Very true
1	2	3	4	5

5. An important reason I do my class work is because I enjoy it.

Not at all true		Somewhat true		Very true
1	2	3	4	5

6. I do my class work because I'm interested in it.

Not at all true		Somewhat true		Very true
1	2	3	4	5