Central Limit Theorems for Some Symmetric Stochastic Integrals

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Abstract

The problem of stochastic integration with respect to fractional Brownian motion (fBm) with $H < 1/2$ and other 'rough path' Gaussian processes is considered. We use a Riemann sum approach to construct stochastic integrals. It is known, for example, that a Midpoint Riemann sum converges in probability to a stable integral for fBm with $H > 1/4$, but not in general if $H \leq 1/4$. We consider four different types of Riemann sums and their associated critical values: Midpoint (2 types), Trapezoidal, and Simpson's rule. At the critical value ($H = 1/4, 1/6$, and $1/10$, respectively), the sums converge only in distribution. Convergence in distribution is proved by means of theorems and techniques of Malliavin calculus. We consider asymptotic behavior of a specific stochastic integral with respect to fBm with $H > 1/2$. This result approximates an fBm version of Spitzer's theorem for planar Brownian motion.

Contents

1 Introduction **1**
- 1.1 Stochastic calculus with respect to Gaussian processes. 2
- 1.2 On an approximation of Spitzer's theorem for fBm 5
- 1.3 Malliavin calculus 5

2 Theoretical background **7**
- 2.1 Definitions and notation 7
- 2.2 Elements of Malliavin calculus 7
- 2.3 Convergence theorems 10
- 2.4 Stochastic calculus for a specific Gaussian process 13

3 Stochastic integration and fBm **14**
- 3.1 Cases with $H < 1/2$ 17
- 3.2 The case $H > 1/2$ 26

4 CLT for a Midpoint Stochastic Integral **29**
- 4.1 Introduction 29
- 4.2 Central limit theorem for the Midpoint (type 1) integral 31
- 4.3 Examples 39
  - 4.3.1 Bifractional Brownian Motion 39
  - 4.3.2 A Gaussian process with differentiable covariance function 45
  - 4.3.3 Empirical quantiles of independent Brownian motions 53
- 4.4 Proof of the technical Lemmas 60
  - 4.4.1 Proof of Lemma 4.4 62
  - 4.4.2 Proof of Lemma 4.5 65
  - 4.4.3 Proof of Lemma 4.9 67

5 Two constructions with critical value $H = 1/6$ **75**
- 5.1 Introduction 75
- 5.2 Weak convergence of the trapezoidal and midpoint (type 2) sums 76
  - 5.2.1 Covariance conditions 76
  - 5.2.2 Taylor expansion of $f(X_t)$ 79
  - 5.2.3 Malliavin calculus representation of 3rd order term 82
  - 5.2.4 Weak convergence of non-trivial part of 3rd order term 85
  - 5.2.5 Relative compactness of the sequences 90

5.2.6 Proof of Theorem 5.1 94
5.3 Examples of suitable processes 95
5.3.1 Bifractional Brownian motion 95
5.3.2 Extended bifractional Brownian motion 101
5.3.3 Sub-fractional Brownian motion 103
5.4 Proof of Technical Lemmas 106
5.4.1 Proof of Lemma 5.3 106
5.4.2 Proof of Lemma 5.6 108
5.4.3 Proof of Lemma 5.10 110

6 Simpson's rule and fBm with $H = 1/10$ 116
6.1 Introduction 116
6.2 Results 117
6.2.1 Malliavin calculus representation. 118
6.2.2 Conditions of Theorem 2.3. 120
6.2.3 Proof of Theorem 6.1. 125
6.3 Proof of Technical Lemmas 125
6.3.1 Proof of Lemma 6.4. 125
6.3.2 Proof of Lemma 6.5. 128

7 CLT for an iterated integral with respect to fBm with $H > 1/2$ 132
7.1 Introduction 132
7.2 Main result 133
7.2.1 Convergence of the covariance function 134
7.2.2 Conditions for weak convergence of $\{X_k(t)\}$ 137
7.2.3 Proof of Theorem 7.2 139
7.2.4 Rate of convergence 140
7.3 A technical lemma 141

Chapter 1

Introduction

The main topic of this dissertation is a study of different constructions of the stochastic integral, where the integration is with respect to continuous Gaussian process. More specifically, we are interested in Gaussian processes that are characterized as by trajectories that are 'rougher' than standard Brownian motion.

The roughness of a stochastic process can be characterized by the p-variation, defined as follows. Let $X = \{X_t, t \in [0,1]\}$ be a Gaussian process. For real $p \geq 1$ (usually an integer) the p-variation is defined as

$$\sum_{j=0}^{n-1} \left(X_{\frac{j+1}{n}} - X_{\frac{j}{n}} \right)^p .$$

In the case of Brownian motion, it is well known that the 2-variation, usually called quadratic variation, we have

$$\mathbb{E}\left[\left(B_{\frac{j+1}{n}} - B_{\frac{j}{n}} \right)^2 \right] = \frac{1}{n},$$

hence the expectation of the quadratic variation is 1 over the interval $[0,1]$. Compared to this result, a simple definition for a rougher-path process is one for which the quadratic variation diverges.

We will consider a variety of Gaussian processes in Chapters 4 and 5, but the best known example of a rough path process is the fractional Brownian motion (fBm). Many aspects of this process have been studied elsewhere [5, 13, 22, 27]. Let $B^H = \left\{ B_t^H, t \geq \right\}$ denote fBm with Hurst parameter H. For our purposes, it is enough to note that fBm is a Guassian process with continuous trajectories and covariance given by

$$R_H(s,t) := \mathbb{E}\left[B_s^H B_t^H \right] = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H} \right).$$

It follows that $\mathbb{E}\left[\left(B^H_{(j+1)/n} - B^H_{j/n} \right)^2 \right] = 1/n^{2H}$, hence if $H > 1/2$, then the quadratic variation tends to zero in probability for large n, but the quadratic variation diverges if $H < 1/2$. In this way, we say that fBm is smoother than standard Brownian motion when $H > 1/2$, and rougher for $H < 1/2$.

1.1 Stochastic calculus with respect to Gaussian processes.

We refer first to the traditional Itô calculus as presented by, for example, Durrett [15], Øksendal [30] or Shreve [35]. In this setting, the Itô stochastic integral for a suitable function f is defined as the limit of a forward Riemann sum:

$$\int_0^t f(B_s)\, dB_s := \lim_{n\to\infty} \sum_{j=0}^{\lfloor nt\rfloor -1} f(B_{\frac{j}{n}}) \left(B_{\frac{j+1}{n}} - B_{\frac{j}{n}} \right).$$

A consequence of the forward construction is the Itô formula:

$$f(B_t) = f(0) + \int_0^t f'(B_s)\, dB_s + \frac{1}{2}\int_0^t f''(B_s)\, ds.$$

A sketch of the derivation of this formula is as follows: On a nonnegative interval $[0,T]$, Let $f = f(t,\omega)$ be a $\mathscr{C}^2$ function satisfying

$$\mathbb{E}\left[\int_0^T f^2 du\right] < \infty.$$

For some $0 < t \le T$, we consider a standard Brownian motion $B = \{B_s, s \in [0,t]\}$ and a uniform partition of the interval $[0,t]$ given by $\{j/n, 0 \le j \le \lfloor nt\rfloor\}$. Then by a Taylor formula we have

$$f(B_{\frac{j+1}{n}}) = f(B_{\frac{j}{n}}) + f'(B_{\frac{j}{n}}) \left(B_{\frac{j+1}{n}}) - B_{\frac{j}{n}} \right) + \frac{1}{2} f''(\xi_j) \left(B_{\frac{j+1}{n}}) - B_{\frac{j}{n}} \right)^2,$$

for some intermediate value ξ_j between $B_{j/n}$ and $B_{(j+1)/n}$. Hence, we can write the Riemann sum

$$f(B_{\frac{\lfloor nt\rfloor}{n}}) = f(0) + \sum_{j=0}^{\lfloor nt\rfloor -1} f'(B_{\frac{j}{n}})\Delta B_{\frac{j}{n}} + \frac{1}{2}\sum_{j=0}^{\lfloor nt\rfloor -1} f''(\xi_j)\Delta B^2_{\frac{j}{n}}, \tag{1.1}$$

where $\Delta B_{j/n} = B_{(j+1)/n} - B_{j/n}$. Taking $n \to \infty$, by definition the first sum converges to $\int_0^t f'(B_s)dB_s$. Then it can be shown (see [15], section 2.7), that the term

$$\frac{1}{2}\sum_{j=0}^{\lfloor nt\rfloor -1} f''(\xi_j)\Delta B^2_{\frac{j}{n}} \xrightarrow{\mathscr{P}} \int_0^t f''(B_s)\, ds,$$

where this integral is a standard Lebesgue integral. From (1.1), we can see that if B is replaced with a fBm B^H with $H < 1/2$, then the Itô integral is unsuitable, since the quadratic variation term in general will not converge.

Our approach to this problem is choose an alternate construction of the stochastic integral, each arising from a different type or Riemann sum. Four different constructions are considered.

- *Midpoint (type 1) integral.* This construction uses a Riemann sum of the form:

$$S_n^{M1}(t) = \sum_{j=1}^{\lfloor \frac{nt}{2}\rfloor} f'(B^H_{\frac{2j-1}{n}}) \left(B^H_{\frac{2j}{n}} - B^H_{\frac{2j-2}{n}} \right).$$

It was shown in [8] and [25] that this sum converges in probability when $H > 1/4$, and can diverge when $H < 1/4$. For the case $H = 1/4$, [8] and [25] proved independently that $S_n^{M1}(t)$ converges weakly as $n \to \infty$, so that

$$S_n^{M1}(t) \xrightarrow{\mathscr{L}} f(B_t^H) - f(0) - \frac{1}{2}\int_0^t f''(B_s^H)dW_s,$$

where W is a scaled Brownian motion, independent of B^H. This weak convergence results in the Itô-like change-of-variable formula

$$f(B_t^H) \stackrel{\mathscr{L}}{=} f(0) + \int_0^t f'(B_s^H)d^{M1}B_s^H + \frac{1}{2}\int_0^t f''(B_s^H)dW_s.$$

It was subsequently shown in [37] that a similar weak limit (the scaling for W is different) holds when B^H is replaced with a Gaussian process essentially similar to bifractional Brownian motion with parameters $H = K = 1/2$. In Chapter 4 (which follows [17]) we prove a similar theorem for a generalized Gaussian process meeting certain conditions. In particular this generalized family of processes includes the above B^H with $H = 1/4$ and the bifractional Brownian motion with $H = K = 1/2$, in fact it is extended to the bifractional family with $H \leq 1/2,\ HK = 1/4$. This theorem also extends the results of [8, 25, 37] in that it proves convergence in the Skorohod space $\mathbf{D}[0,\infty)$.

- *Trapezoidal rule integral*. This has the form

$$S_n^T(t) = \sum_{j=0}^{\lfloor nt\rfloor-1} \frac{1}{2}\left(f'(B_{\frac{j}{n}}^H + f'(B_{\frac{j+1}{n}}^H)\right)\left(B_{\frac{j+1}{n}}^H - B_{\frac{j}{n}}^H\right).$$

The stochastic integral arising from this sum is also known as the Stratonovich integral, and this form has been studied for many years. In the case of fBm with $H < 1/2$, it is proved independently in [10] and [16] that $S_n^T(t)$ converges in probability if and only if $H > 1/6$. For the case $H = 1/6$, $S_n^T(t)$ converges weakly, and as in the Midpoint (type 1) case above, we have a weak change-of-variable formula:

$$f(B_t^H) \stackrel{\mathscr{L}}{=} f + \int_0^t f'(B_s^H)d^\circ B_s^H + \gamma\int_0^t f^{(3)}(B_s^H)dW_s,$$

where γ is a known constant and again W is a scaled Brownian motion, independent of B^H. This weak convergence was first proved in [26]. In Chapter 5, we prove a more general version, that applies to a class of Gaussian processes which includes fBm with $H = 1/6$, as well as other known processes.

- *Midpoint (type 2) integral*. This has the form

$$S_n^{M2}(t) = \sum_{j=0}^{\lfloor nt\rfloor-1} f'\left(\frac{1}{2}\left(B_{\frac{j}{n}}^H + B_{\frac{j+1}{n}}^H\right)\right)\left(B_{\frac{j+1}{n}}^H - B_{\frac{j}{n}}^H\right).$$

It is proved in [16] that $S_n^{M2}(t)$ converges in probability when $H > 1/6$, and indeed this sum behaves very similarly to the Trapezoidal sum above. In Chapter 5, we prove that for the case $H = 1/6$, we have the change-of-variable formula,

$$f(B_t^H) \stackrel{\mathscr{L}}{=} f + \int_0^t f'(B_s^H) d^{M2} B_s^H + \alpha \int_0^t f^{(3)}(B_s^H) dW_s,$$

that is, the result only differs from the Trapezoidal case by the scaling factor for the variance term.

- *Simpson's rule integral.* This has the form

$$S_n^S(t) = \sum_{j=0}^{\lfloor nt \rfloor - 1} \frac{1}{6} \left(f'(B^H_{\frac{j}{n}} + 4f' \left(\frac{B^H_{\frac{j}{n}} + B^H_{\frac{j+1}{n}}}{2} \right) + f'(B_{\frac{j+1}{n}}) \right) \left(B^H_{\frac{j+1}{n}} - B^H_{\frac{j}{n}} \right).$$

  In [16] and Chapter 3, it is proved that $S_n^S(t)$ converges in probability when $H > 1/10$. In Chapter 6, we prove that $S_n^S(t)$ converges weakly in the case $H = 1/10$, with a result similar to the above cases:

$$f(B_t^H) \stackrel{\mathscr{L}}{=} f + \int_0^t f'(B_s^H) d^S B_s^H + \beta \int_0^t f^{(3)}(B_s^H) dW_s,$$

  where β is a known constant and W is a Brownian motion, independent of B^H.

The above sums $S_n^T(t)$, $S_n^{M2}(t)$, $S_n^S(t)$ are described more generally in [16], as cases of an object they define as the $(\nu, 1)$-integral:

$$\int_0^t g(X_u)\, d^{\nu,1} X_u := \mathbb{P} \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t du (X_{u+\varepsilon} - X_u) \int_0^1 g(X_u + \alpha(X_{u+\varepsilon} - X_u)) \nu(d\alpha),$$

where g is a locally bounded function, X is a stochastic process, and ν is a probability measure. Chapters 5 and 6 are essentially dedicated to three of the 'critical value' cases for this $(\nu, 1)$-integral, where we consider the end point cases for which convergence in probability does not hold, but weak convergence holds under certain circumstances.

A common property of the sums $S_n^T(t)$, $S_n^{M2}(t)$, $S_n^S(t)$ and the $\nu, 1$ integral in general is that the integral depends on the values of $B^H_{j/n}$ and $B^H_{(j+1)/n}$, but not on any process values inside the subinterval $(j/n, (j+1)/n)$. In numerical analysis, we have the general rule that more sample points yield a better estimate on the approximate integral. This explains why, for example, that Simpson's rule has a smaller error term than the Trapezoidal rule. In the stochastic case, one might expect a similar result, that more sample points allows control over a rougher path, but this is not exactly the case. Note here that $S_n^S(t)$ is stable for rougher paths (down to $H > 1/10$ rather than $H > 1/6$) compared to the Trapezoidal and Midpoint (type 2) rules, but when n is fixed, all sums use the same set of process sample points.

1.2 On an approximation of Spitzer's theorem for fBm

In Chapter 7 we consider a stochastic integral with respect to fBm with $H > 1/2$. The goal of this chapter is not to study the construction of the integral, indeed, integration when $H > 1/2$ is relatively well developed (see, for example [13, 27]). In Chapter 7, the goal is to study the asymptotics of a particular integrand.

Let $W_t = W_t^1 + iW_t^2$ denote a standard Brownian motion in the complex plane, where we assume $W_0 = 1$. From complex analysis, the integral

$$\theta_t := \operatorname{Im} \int_0^t \frac{dW_s}{W_s}$$

gives the swept angle, or windings, of the trajectory of W_t. There is a famous theorem by Spitzer [36] about this integral, namely that as $t \to \infty$, the scaled random variable $2\theta_t / \log t$ converges in distribution to a Cauchy random variable with parameter 1. The proof of Spitzer's result uses the time-change property of Brownian motion, and hence is not applicable to fBm with $H \neq 1/2$.

To our knowledge, there is no comparable fBm version of Spitzer's theorem. In Chapter 7, we consider the asymptotic behavior of an approximation to the windings,

$$\operatorname{Im} \int_1^{k^t} \frac{dB_s^H}{s^{2H}},$$

where B^H is a complex fBm with Hurst parameter $H > 1/2$. We actually study a generalization, which is a stochastic integral of the form

$$\int_1^{k^t} \int_1^{s_q} \cdots \int_1^{s_2} \frac{1}{s^{2H}} dB_{s_1}^1 \dots dB_{s_{q-1}}^{q-1} dB_{s_q}^q,$$

where each $B_{s_i}^i$ is an independent fBm. For technical reasons, this integral is not the generalized Weiner-Itô integral, but a symmetric integral in the sense of Russo and Vallois. In Chapter 7, it is proved that when scaled by $(\log k)^{-\frac{1}{2}}$, the integral converges in distribution to a Gaussian random variable. This result follows a previously published result in [19].

1.3 Malliavin calculus

The results discussed in Sections 1.1 and 1.2, and proved in Chapters 3 - 7 are central limit theorems, that is, theorems showing that a sequence of random variables converge in distribution to a random variable with Gaussian law, which may be univariate or multivariate Gaussian. To show convergence, we use the techniques of Malliavin calculus.

Malliavin calculus, also called the stochastic calculus of variations, is a differential calculus on the space generated by a Gaussian stochastic process. It was introduced in the 1970s as a method to investigate the probability laws of solutions to stochastic differential equations driven by Brownian motion. Its scope has since been expanded. In particular, Malliavin calculus gives a way to extend the Itô calculus from Brownian motion to non-adapted stochastic processes. Areas of application include mathematical finance [12], and statistics of stochastic processes [11]. A thorough treatment of the subject can be found in [27]. The first chapters of [24] give a gentle introduction.

The basic object of study is an *isonormal Gaussian process* on a Hilbert space $\mathscr{H}$. That is, $\{X(h), h \in \mathscr{H}\}$ is a family of mean-zero Gaussian random variables, such that $\mathbb{E}[X(h)X(g)] = \langle h, g\rangle_{\mathscr{H}}$ for all $h, g \in \mathscr{H}$. For example, a Brownian motion $\{B_t, t \in [0,T]\}$ can be extended to an isonormal Gaussian process on the Hilbert space $L^2([0,T])$. Here we identify B_t with the indicator function $\mathbf{1}_{[0,t]}$, and for an arbitrary h in $\mathscr{H}$, we define $B(h)$ by the Wiener-Itô integral $\int_0^T h(s)\, dB_s$.

Recently, researchers have used Malliavin calculus to prove central limit theorems for functionals of Gaussian processes. Most notable is the Fourth Moment Theorem of Nualart and Peccati [29], which gives conditions under which a sequence of random variables in the form of divergence integrals will converge in distribution to a Gaussian random variable. As described in [24], there is a natural connection between Malliavin calculus and Stein's Lemma, which gives a way to measure the distance in law between a random variable Z and a $\mathscr{N}(0,1)$ random variable.

Some necessary definitions and identities of the Malliavin calculus are presented in Chapter 2. Also in Chapter 2, we provide the two convergence theorems that are the main theoretical machinery for Chapters 4 - 7.

Chapter 2

Theoretical background

2.1 Definitions and notation

Let $f : \mathbb{R} \to \mathbb{R}$ be a function and N be a Gaussian random variable with mean zero and variance σ^2. We say that f satisfies *moderate growth conditions* if there exist constants $A, B > 0$, and a constant $\alpha < 2$ such that $|f(x)| \leq Ae^{B|x|^\alpha}$. Note that this implies $\mathbb{E}[|f(N)|^p] < \infty$ for all $p \geq 1$. We use the symbol $\mathbf{1}_{\mathscr{A}}$ to denote the indicator function for a set $\mathscr{A}$. The symbol C denotes a generic positive constant, which may vary from line to line. In general, the value of C will depend on and the growth conditions of a test function f and the properties of a stochastic process. Unless otherwise specified, we will use the symbols X, W, and Z to denote a generic, Gaussian stochastic process. The symbols B, B^H will denote fractional Brownian motion, which may include standard Brownian motion. For a process X indexed by a real interval $[0, T]$, we will use the notation X_t and $X(t)$ interchangeably.

2.2 Elements of Malliavin calculus

Following is a brief description of some identities that will be used. The reader may refer to [27] for detailed coverage of this topic. Let $Z = \{Z(h), h \in \mathscr{H}\}$ be an *isonormal Gaussian process* on a probability space $(\Omega, \mathscr{F}, P)$, and indexed by a real separable Hilbert space $\mathscr{H}$. That is, Z is a family of Gaussian random variables such that $\mathbb{E}[Z(h)] = 0$ and $\mathbb{E}[Z(h)Z(g)] = \langle h, g \rangle_{\mathscr{H}}$ for all $h, g \in \mathscr{H}$.

For integers $q \geq 1$, let $\mathscr{H}^{\otimes q}$ denote the q^{th} tensor product of $\mathscr{H}$, and $\mathscr{H}^{\odot q}$ denote the subspace of symmetric elements of $\mathscr{H}^{\otimes q}$. We will also use the notation $\bigotimes_{i=1}^r h_i$ to denote an arbitrary tensor product, with the convention that $\bigotimes_{i=1}^0$ is the empty set. Given a real function $f \in \mathscr{H}^{\otimes q}$, we define the symmetrization $\tilde{f} \in \mathscr{H}^{\odot q}$ as

$$\tilde{f}(x_1, \ldots, x_q) = \frac{1}{q!} \sum_{\sigma} f(x_{\sigma(1)}, \ldots, x_{\sigma(q)}), \tag{2.1}$$

where σ includes all permutations of $\{1, \ldots, q\}$.

Let $\{e_n, n \geq 1\}$ be a complete orthormal system in $\mathscr{H}$. For functions $f, g \in \mathscr{H}^{\odot q}$ and $p \in$

$\{0,\dots,q\}$, we define the p^{th}-order contraction of f and g as that element of $\mathscr{H}^{\otimes 2(q-p)}$ given by

$$f \otimes_p g = \sum_{i_1,\dots,i_p=1}^{\infty} \left\langle f, e_{i_1} \otimes \cdots \otimes e_{i_p} \right\rangle_{\mathscr{H}^{\otimes p}} \otimes \left\langle g, e_{i_1} \otimes \cdots \otimes e_{i_p} \right\rangle_{\mathscr{H}^{\otimes p}} \tag{2.2}$$

where $f \otimes_0 g = f \otimes g$ and, if $f,g \in \mathscr{H}^{\odot q}$, $f \otimes_q g = \langle f,g \rangle_{\mathscr{H}^{\otimes q}}$. While f,g are symmetric, the contraction $f \otimes_q g$ may not be. We denote its symmetrization by $f \tilde{\otimes}_q g$.

Let $\mathscr{H}_q$ be the q^{th} Wiener chaos of Z, that is, the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_q(Z(h)), h \in \mathscr{H}, \|h\|_{\mathscr{H}} = 1\}$, where $H_q(x)$ is the q^{th} Hermite polynomial, defined as

$$H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q} e^{-\frac{x^2}{2}},$$

and we follow the convention of Hermite polynomials with unity as a leading coefficient. Equivalently, it can be shown (see [24]) that the Hermite polynomials can be defined recursively by

$$H_0(x) = 1, \;\; H_1(x) = x, \;\; \text{and} \;\; H_{n+1}(x) = xH_n(x) - nH_{n-1}(x) \quad \text{for } n \geq 2. \tag{2.3}$$

For $q \geq 1$, it is known that the map

$$I_q(h^{\otimes q}) = H_q(Z(h)) \tag{2.4}$$

provides a linear isometry between $\mathscr{H}^{\odot q}$ (equipped with the modified norm $\sqrt{q!}\|\cdot\|_{\mathscr{H}^{\otimes q}}$) and $\mathscr{H}_q$, where $I_q(\cdot)$ is the Wiener-Itô stochastic integral. By convention, $\mathscr{H}_0 = \mathbb{R}$ and $I_0(x) = x$. It follows from (2.4) and the properties of the Hermite polynomials that for $f \in \mathscr{H}^{\odot p}$, $g \in \mathscr{H}^{\odot q}$ we have

$$\mathbb{E}\left[I_p(f) I_q(g)\right] = \begin{cases} p! \langle f,g \rangle_{\mathscr{H}^{\otimes p}} & if \ p = q \\ 0 & otherwise \end{cases}. \tag{2.5}$$

Let $\mathscr{S}$ be the set of all smooth and cylindrical random variables of the form $F = g(Z(\phi_1),\dots,Z(\phi_n))$, where $n \geq 1$; $g : \mathbb{R}^n \to \mathbb{R}$ is an infinitely differentiable function with compact support, and $\phi_i \in \mathscr{H}$. The Malliavin derivative of F with respect to Z is the element of $L^2(\Omega;\mathscr{H})$ defined as

$$DF = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(Z(\phi_1),\dots,Z(\phi_n))\phi_i.$$

By iteration, for any integer $q > 1$ we can define the q^{th} derivative $D^q F$, which is an element of $L^2(\Omega;\mathscr{H}^{\odot q})$.

We let $\mathbb{D}^{q,2}$ denote the closure of $\mathscr{S}$ with respect to the norm $\|\cdot\|_{\mathbb{D}^{q,2}}$ defined as

$$\|F\|^2_{\mathbb{D}^{q,2}} = \mathbb{E}\left[F^2\right] + \sum_{i=1}^{q} \mathbb{E}\left[\|D^i F\|^2_{\mathscr{H}^{\otimes i}}\right].$$

More generally, let $\mathbb{D}^{k,p}(\mathfrak{H}^{\otimes k})$ denote the corresponding Sobolev space of $\mathfrak{H}^{\otimes k}$-valued random variables.

We denote by δ the Skorohod integral, which is defined as the adjoint of the operator D. A random element $u \in L^2(\Omega;\mathscr{H})$ belongs to the domain of δ, Dom δ, if and only if,

$$|\mathbb{E}\left[\langle DF,u \rangle_{\mathscr{H}}\right]| \leq c_u \|F\|_{L^2(\Omega)}$$

for any $F \in \mathbb{D}^{1,2}$, where c_u is a constant which depends only on u. If $u \in \text{Dom}\ \delta$, then the random variable $\delta(u) \in L^2(\Omega)$ is defined for all $F \in \mathbb{D}^{1,2}$ by the duality relationship,

$$\mathbb{E}[F\delta(u)] = \mathbb{E}[\langle DF, u\rangle_{\mathscr{H}}].$$

This is sometimes called the Malliavin integration by parts formula. We iteratively define the multiple Skorohod integral for $q \geq 1$ as $\delta(\delta^{q-1}(u))$, with $\delta^0(u) = u$. For this definition we have,

$$\mathbb{E}[F\delta^q(u)] = \mathbb{E}[\langle D^q F, u\rangle_{\mathscr{H}^{\otimes q}}], \tag{2.6}$$

where $u \in \text{Dom}\ \delta^q$ and $F \in \mathbb{D}^{q,2}$. The adjoint operator δ^q is an integral in the sense that for a (non-random) $h \in \mathscr{H}^{\odot q}$, we have $\delta^q(h) = I_q(h)$.

The following results will be used extensively in this paper. The reader may refer to [23] and [27] for proofs and details.

Lemma 2.1. *Let $q \geq 1$ be an integer, and $r, j, k > 0$ be integers.*

(a) Assume $F \in \mathbb{D}^{q,2}$, u is a symmetric element of Dom δ^q*, and $\langle D^r F, \delta^j(u)\rangle_{\mathscr{H}^{\otimes r}} \in L^2(\Omega; \mathscr{H}^{\otimes q-r-j})$ for all $0 \leq r+j \leq q$. Then $\langle D^r F, u\rangle_{\mathscr{H}^{\otimes r}} \in$* Dom δ^r *and*

$$F\delta^q(u) = \sum_{r=0}^{q} \binom{q}{r} \delta^{q-r}(\langle D^r F, u\rangle_{\mathscr{H}^{\otimes r}}).$$

(b) Suppose that u is a symmetric element of $\mathbb{D}^{j+k,2}(\mathscr{H}^{\otimes j})$. Then we have,

$$D^k \delta^j(u) = \sum_{i=0}^{j\wedge k} i! \binom{k}{i} \binom{j}{i} \delta^{j-i}\left(D^{k-i} u\right).$$

(c) Meyer inequality*: for $p \geq 1$ and integers $k \geq q \geq 1$, we have,*

$$\|\delta^q(u)\|_{\mathbb{D}^{k-q,p}} \leq c_{k,p} \|u\|_{\mathbb{D}^{k,p}(\mathfrak{H}^{\otimes q})} \tag{2.7}$$

for all $u \in \mathbb{D}^{k,p}(\mathfrak{H}^{\otimes k})$ and some constant $c_{k,p}$.

(d) Let $u \in \mathscr{H}^{\odot p}$ and $v \in \mathscr{H}^{\odot q}$. Then

$$\delta^p(u)\delta^q(v) = \sum_{z=0}^{p\wedge q} z! \binom{p}{z} \binom{q}{z} \delta^{p+q-2z}(u \otimes_z v),$$

where $\otimes_z$ is the contraction operator defined in (2.2).

We will use the following hypercontractivity property of iterated integrals (see [29], Theorem 2.7.2, or [27], Sec. 1.4.3 for complete details). Let $f \in \mathscr{H}^{\odot q}$ and $p \geq 2$. Then there exists a positive constant $C_{p,q} < \infty$, depending only on p and q, such that

$$\mathbb{E}\left[|I_q(f)|^p\right] \leq C_{p,q} \left(\mathbb{E}\left[I_q(f)^2\right]\right)^{\frac{p}{2}}. \tag{2.8}$$

2.3 Convergence theorems

We begin with an illustrative example, which shows how the Malliavin duality may be exploited to to establish a central limit theorem. Suppose $\{F_n, n \geq 1\}$ is a bounded sequence of $\mathbb{R}$-valued random variables of the form $F_n = \delta(u_n)$ for a sequence $\{u_n\} \subset \mathscr{H}$. For a real value t, consider the function

$$\phi_n(t) = \mathbb{E}\left[e^{itF_n}\right].$$

We have the derivative

$$\phi_n'(t) = \mathbb{E}\left[iF_n e^{itF_n}\right] = i\mathbb{E}\left[e^{itF_n}\delta(u_n)\right].$$

By the Malliavin duality, this is

$$i\mathbb{E}\left\langle De^{itF_n}, u_n\right\rangle_{\mathscr{H}} = -\mathbb{E}\left\langle te^{itF_n}DF_n, u_n\right\rangle_{\mathscr{H}} = -t\mathbb{E}\left[e^{itF_n}\left\langle DF_n, u_n\right\rangle_{\mathscr{H}}\right].$$

Now, if it happens that the term $\langle DF_n, u_n\rangle_{\mathscr{H}}$ converges in probability to a real number $\sigma^2 \geq 0$, then as $n \to \infty$ we have for large n that $\phi_n'(t) \stackrel{\mathscr{P}}{=} -t\sigma^2\phi_n(t)$, that is, $\phi_n(t)$ converges in probability to a function satisfying $\phi' = -t\sigma^2\phi$, which is to be recognized as the Gaussian characteristic function. Hence, the Malliavin duality allows us to express the characteristic function as a differential equation. Theorem 2.3, below, is a vector-valued, multiple integral version of the above example.

The first version of the following central limit theorem appeared in [23]. In [17], we extended this to a multi-dimensional version, where the sequence was a vector of d components all in the same Wiener chaos. For this version, we lay out conditions for stable convergence of a sequence of vectors, where the vector components are not necessarily in the same Wiener chaos. This theorem will be the main theoretical tool of Chapters 4 - 6. We begin with a definition of a form of weak convergence. Note that this definition implies the usual convergence in distribution.

Definition 2.2. Assume F_n is a sequence of $d-$dimensional random variables defined on a probability space $(\Omega, \mathscr{F}, P)$, and F is a $d-$dimensional random variable defined on $(\Omega, \mathscr{G}, P)$, where $\mathscr{F} \subset \mathscr{G}$. We say that F_n *converges stably* to F as $n \to \infty$, if, for any continuous and bounded function $f : \mathbb{R}^d \to \mathbb{R}$ and $\mathbb{R}$-valued, $\mathscr{F}-$measurable random variable M, we have

$$\lim_{n\to\infty} \mathbb{E}\left(f(F_n)M\right) = \mathbb{E}\left(f(F)M\right).$$

Theorem 2.3. *Let $d \geq 1$ be an integer, and $q_1, \ldots, q_d$ be positive integers with $q^* = \max\{q_1, \ldots, q_d\}$. Suppose that F_n is a sequence of random variables in $\mathbb{R}^d$ of the form $F_n = \left(\delta^{q_1}(u_n^1), \ldots, \delta^{q_d}(u_n^d)\right)$, where each u_n^i is a $\mathbb{R}-$valued symmetric function in $\mathbb{D}^{2q^*, 2q_i}(\mathscr{H}^{\otimes q_i})$. Suppose that the sequence F_n is bounded in $L^1(\Omega)$ and that:*

(a) $\left\langle u_n^j, \bigotimes_{\ell=1}^m (D^{a_\ell}F_n^{j_\ell}) \otimes h\right\rangle_{\mathscr{H}^{\otimes q}}$ converges to zero in $L^1(\Omega)$ for all integers $1 \leq j, j_\ell \leq d$, all integers $1 \leq a_1, \ldots, a_m, r \leq q_j - 1$ such that $a_1 + \cdots + a_m + r = q_j$; and all $h \in \mathscr{H}^{\otimes r}$.

(b) For each $1 \leq i, j \leq d$, $\langle u_n^i, D^{q_i}F_n^i\rangle_{\mathscr{H}^{\otimes q_i}}$ converges in $L^1(\Omega)$ to a nonnegative random variable s_i^2, and for $i \neq j$, $\left\langle u_n^i, D^{q_i}F_n^j\right\rangle_{\mathscr{H}^{\otimes q_i}}$ converges to zero in $L^1(\Omega)$.

Then F_n converges stably to a random vector in $\mathbb{R}^d$, whose components each have independent Gaussian law $\mathscr{N}(0, s_i^2)$ given Z.

Proof. We use the conditional characteristic function. Given any $h_1, \dots h_m \in \mathscr{H}$, we want to show that the sequence

$$\xi_n = \left(F_n^1, \dots, F_n^d, Z(h_1), \dots, Z(h_m)\right)$$

converges in distribution to a vector $\left(F_\infty^1, \dots F_\infty^d, Z(h_1), \dots, Z(h_m)\right)$, where, for any vector $\lambda \in \mathbb{R}^d$, F_∞ satisfies

$$\mathbb{E}\left(e^{i\lambda^T F_\infty} | Z(h_1), \dots, Z(h_m)\right) = \exp\left(-\frac{1}{2}\lambda^T S \lambda\right), \tag{2.9}$$

where S is the diagonal $d \times d$ matrix with entries s_i^2.

Since F_n is bounded in $L^1(\Omega)$, the sequence ξ_n is tight in the sense that for any $\varepsilon > 0$, there is a $K > 0$ such that $P\left(F_n \in [-K,K]^d\right) > 1 - \varepsilon$, which follows from Chebyshev inequality. Dropping to a subsequence if necessary, we may assume that ξ_n converges in distribution to a limit $\left(F_\infty^1, \dots F_\infty^d, Z(h_1), \dots Z(h_m)\right)$. Let $Y := g\left(Z(h_1), \dots, Z(h_m)\right)$, where $g \in \mathscr{C}_b^\infty(\mathbb{R}^m)$, and consider $\phi_n(\lambda) = \phi(\lambda, \xi_n) := \mathbb{E}\left(e^{i\lambda^T F_n} Y\right)$ for $\lambda \in \mathbb{R}^d$. The convergence in law of ξ_n implies that for each $1 \le j \le d$:

$$\lim_{n\to\infty} \frac{\partial \phi_n}{\partial \lambda_j} = \lim_{n\to\infty} i\mathbb{E}\left(F_n^j e^{i\lambda^T F_n} Y\right) = i\mathbb{E}\left(F_\infty^j e^{i\lambda^T F_\infty} Y\right), \tag{2.10}$$

where convergence in distribution follows from a truncation argument applied to F_n^j.

On the other hand, using the duality property of the Skorohod integral and the Malliavin derivative:

$$\begin{aligned}
\frac{\partial \phi_n}{\partial \lambda_j} &= i\mathbb{E}\left(\delta^{q_j}(u_n^j) e^{i\lambda^T F_n} Y\right) = i\mathbb{E}\left(\left\langle u_n^j, D^{q_j}\left(e^{i\lambda^T F_n} Y\right)\right\rangle_{\mathfrak{H}^{\otimes q_j}}\right) \\
&= i\sum_{a=0}^{q_j} \binom{q_j}{a} \mathbb{E}\left(\left\langle u_n^j, D^a\left(e^{i\lambda^T F_n}\right) \widetilde{\otimes} D^{q_j - a} Y\right\rangle_{\mathfrak{H}^{\otimes q_j}}\right)
\end{aligned}$$

$$= i\left\{\mathbb{E}\left\langle u_n^j, Y D^{q_j} e^{i\lambda^T F_n}\right\rangle_{\mathscr{H}^{\otimes q_j}} + \sum_{a=0}^{q_j - 1} \binom{q_j}{a} \mathbb{E}\left\langle u_n^j, D^a e^{i\lambda^T F_n} \widetilde{\otimes} D^{q_j - a} Y\right\rangle_{\mathscr{H}^{\otimes q_j}}\right\} \tag{2.11}$$

By condition (a), we have that $\left\langle u_n^j, D^a e^{i\lambda^T F_n} \widetilde{\otimes} D^{q_j - a} Y\right\rangle_{\mathscr{H}^{\otimes q_j}}$ converges to zero in $L^1(\Omega)$ when $a < q_j$, so the sum term vanishes as $n \to \infty$, and this leaves

$$\begin{aligned}
\lim_{n\to\infty} i\mathbb{E}\left\langle u_n^j, Y D^q e^{i\lambda^T F_n}\right\rangle_{\mathscr{H}^{\otimes q_j}} &= \lim_{n\to\infty} i\sum_{k=1}^{d} \mathbb{E}\left(i\lambda_k e^{i\lambda^T F_n}\left\langle u_n^j, Y D^{q_j} F_n^k\right\rangle_{\mathscr{H}^{\otimes q_j}}\right) \\
&= -\mathbb{E}\left(\lambda_j e^{i\lambda^T F_\infty} s_j^2 Y\right)
\end{aligned}$$

because the lower-order derivatives in $D^{q_j} e^{i\lambda^T F_n}$ also vanish by condition (a), and cross terms $(j \ne k)$ terms vanish by condition (b). Combining this with (2.10), we obtain:

$$i\mathbb{E}\left(F_\infty^j e^{i\lambda \cdot F_\infty} Y\right) = -\lambda_j \mathbb{E}\left(e^{i\lambda \cdot F_\infty} s_j^2 Y\right).$$

This leads to the PDE system:

$$\frac{\partial}{\partial \lambda_j}\mathbb{E}\left(e^{i\lambda^T F_\infty}|Z(h_1),\ldots,Z(h_m)\right) = -\lambda_j s_j^2 \mathbb{E}\left(e^{i\lambda^T F_\infty}|Z(h_1),\ldots,Z(h_m)\right)$$

which has unique solution (2.9). □

Remark 2.4. It suffices to impose condition (a) for $h \in \mathscr{S}_0$, where $\mathscr{S}_0$ is a total subset of $\mathscr{H}^{\otimes r}$.

If it can be shown that F_n also satisfies a relative compactness condition, then we can prove convergence in the Skorohod space $\mathbf{D}[0,\infty)$.

Corollary 2.5. *Suppose $\{G_n(t), t \geq 0\}$ is a sequence of $\mathbb{R}$-valued processes of the form $G_n(t) = \delta^q(u_n(t))$, where $u_n(t)$ is a sequence of symmetric functions in $\mathbb{D}^{2q,2q}(\mathfrak{H}^{\otimes q})$. Assume that for any finite set of times $\{0 = t_0 < t_1 < \cdots < t_d\}$, the sequence*

$$(G_n(t_1) - G_n(t_0), \ldots, G_n(t_d) - G_n(t_{d-1}))$$

satisfies Theorem 2.3; where the $d \times d$ matrix Σ is diagonal with entries $s^2(t_i) - s^2(t_{i-1})$. Suppose further that there exist real numbers $C > 0$, $\gamma > 0$, and $\beta > 1$ such that for each n and for any $0 \leq t_1 < t < t_2$, we have

$$\mathbb{E}\left[|G_n(t) - G_n(t_1)|^\gamma |G_n(t_2) - G_n(t)|^\gamma\right] \leq C\left(\frac{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor}{n}\right)^\beta.$$

Then the family of stochastic processes $\{G_n, n \geq 1\}$ converges as $n \to \infty$ to the process $G = \{G_t, t \geq 0\}$, where $G(t)$ is a Gaussian random variable with mean zero and variance $s^2(t)$. Equivalently, we can say that $G_n(t) \xrightarrow{\mathscr{L}} \sqrt{s^2(t)}\, Z$ as $n \to \infty$, where $Z \sim \mathscr{N}(0,1)$.

This convergence criteria in $\mathbb{D}$ is well known (see, *e.g*, Theorem 13.5 of Billingsley [6]).

In Chapter 7 we will use a version of the Fourth Moment Theorem, which is stated below. This theorem, first published in 2005, has inspired an extensive body of literature, and provided solution techniques to a new class of problems. This first version (which was 1-dimensional) of this theorem was proved in [29]. Since then, other equivalent conditions have been added [24, 28]. The multi-dimensional version stated above was proved by Peccati and Tudor [31]. A key advantage of this theorem is that, unlike the standard method of moments, it is not necessary to know about moments of any order higher than four.

Theorem 2.6. *Fix integers $n \geq 2$ and $d \geq 1$. Let $\left\{\left(f_1^{(k)}, \ldots, f_d^{(k)}\right), k \geq 1\right\}$ be a sequence of vectors such that $f_i^{(k)} \in \mathscr{H}^{\odot n}$ for each k and $i = 1, \ldots, d$; and*

$$\lim_{k\to\infty} n!\|f_i^{(k)}\|^2_{\mathscr{H}^{\otimes n}} = \lim_{k\to\infty}\left\|I_n\left(f_i^{(k)}\right)\right\|^2_{L^2(\Omega)} = C_{ii},\ \forall i = 1,\ldots,d;$$

$$\lim_{k\to\infty}\mathbb{E}\left[I_n\left(f_i^{(k)}\right) I_n\left(f_j^{(k)}\right)\right] = C_{ij},\ \forall 1 \leq i < j \leq d.$$

Then the following are equivalent:

(i) As $k \to \infty$, the vector $\left(I_n(f_1^{(k)}),\dots,I_n(f_d^{(k)})\right)$ converges in distribution to a d-dimensional Gaussian vector with distribution $\mathscr{N}(0,\mathbf{C}_d)$, where $\mathbf{C}_d$ is a symmetric, $d \times d$ matrix with entries C_{ij};

(ii) For each $i = 1,\dots,d$, $I_n(f_i^{(k)})$ converges in distribution to N_i, where N_i is a centered Gaussian random variable with variance C_{ii};

(iii) For each $i = 1,\dots,d$,

$$\lim_{k\to\infty} \mathbb{E}\left[I_n\left(f_i^{(k)}\right)^4\right] = 3C_{ii}^2;$$

(iv) For each $i = 1,\dots,d$, and each integer $1 \le p \le n-1$, $\lim_{k\to\infty} \left\| f_i^{(k)} \otimes_p f_i^{(k)} \right\|_{\mathscr{H}^{\otimes 2(n-p)}} = 0$.

2.4 Stochastic calculus for a specific Gaussian process

For some $T > 0$, let $X = \{X_t, 0 \le t \le T\}$ be a centered Gaussian process with covariance

$$\mathbb{E}[X_s X_t] = R(s,t) \tag{2.12}$$

for $s,t \in [0,T]$. Let $\mathscr{E}$ denote the set of $\mathbb{R}$-valued step functions on $[0,T]$. We then let $\mathfrak{H}$ be the Hilbert space defined as the closure of $\mathscr{E}$ with respect to the inner product

$$\left\langle \mathbf{1}_{[0,s]}, \mathbf{1}_{[0,t]} \right\rangle_{\mathfrak{H}} = R(s,t).$$

The mapping $\mathbf{1}_{[0,t]} \mapsto X_t$ can be extended to a linear isometry between $\mathfrak{H}$ and the Gaussian space spanned by X. In this way, $\{X(h), h \in \mathfrak{H}\}$ is an isonormal Gaussian process as in Section 2.2.

For an integer $n \ge 2$, we consider a uniform partition of $[0,\infty)$ given by $\{j/n, j \ge 1\}$. Define the following notation:

- $\Delta X_{\frac{j}{n}} = X_{\frac{j+1}{n}} - X_{\frac{j}{n}}$, $\widetilde{X}_{\frac{j}{n}} = \frac{1}{2}\left(X_{\frac{j}{n}} + X_{\frac{j+1}{n}}\right)$, and $\hat{X}_{\frac{j}{n}} = X_{\frac{2j+1}{2n}}$
- $\partial_{\frac{j}{n}} = \mathbf{1}_{[\frac{j}{n},\frac{j+1}{n}]}$, $\varepsilon_t = \mathbf{1}_{[0,t]}$
- $\widetilde{\varepsilon}_{\frac{j}{n}} = \frac{1}{2}\left(\mathbf{1}_{[0,\frac{j}{n}]} + \mathbf{1}_{[0,\frac{j+1}{n}]}\right)$, and $\hat{\varepsilon}_{\frac{j}{n}} = \mathbf{1}_{[0,\frac{2j+1}{2n}]}$

This notation will be used extensively in the chapters to follow.

Chapter 3

Stochastic integration and fBm

Let $B^H = \{B_t^H, t \geq 0\}$ be a fractional Brownian motion with Hurst parameter H. That is, B^H is a centered Gaussian process with covariance given by

$$R_H(s,t) = t^{2H} + s^{2H} - |t-s|^{2H}, \tag{3.1}$$

for $s,t \geq 0$ and some value $H \in (0,1)$. FBm is a well-known process that generalizes the standard Brownian motion, indeed, it can be seen from the form of R_H that $H = 1/2$ corresponds to standard Brownian motion. It is also known that for parameter values $1/2 < H < 1$, the trajectories are 'smoother' that standard Bm, while the paths are 'rougher' for $H < 1/2$. In Chapters 4 and 5, we consider a generalized Gaussian process, for which fBm can be considered the prototype, and the results of those theorems hold for fBm with an appropriate values of H. In Chapter 7, we provide more specific details about stochastic calculus based on fBm with $H > 1/2$.

The following fBm properties follow from (3.1).

(B.1) $\mathbb{E}\left[(\Delta B^H_{\frac{j}{n}})^2\right] = \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{j}{n}} \right\rangle_{\mathfrak{H}} = \frac{1}{n^{2H}}$.

(B.2) $\mathbb{E}\left[\Delta B^H_{\frac{j}{n}} \Delta B^H_{\frac{j+1}{n}}\right] = \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{j+1}{n}} \right\rangle_{\mathfrak{H}} = (2^{2H}-2)/2n^{2H}$.

(B.3) If $|k-j| \geq 2$, $\left|\mathbb{E}\left[\Delta B^H_{\frac{j}{n}} \Delta B^H_{\frac{k}{n}}\right]\right| = \left|\left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}}\right| \leq Cn^{-2H}|j-k|^{2H-2}$, where the constant C does not depend on j.

(B.4) For any $t \in [0,T]$ and integer $1 \leq j$,

$$\left|\mathbb{E}\left[\Delta B^H_{\frac{j}{n}} B^H_t\right]\right| = \left|\left\langle \partial_{\frac{j}{n}}, \varepsilon_t \right\rangle_{\mathfrak{H}}\right| \leq Cn^{-2H}\left(j^{2H-1} + |j-nt|^{2H-1}\right).$$

In particular, $\sup_{[0,T]}\left|\mathbb{E}\left[\Delta B^H_{j/n} B^H_t\right]\right| \leq 2n^{-2H}$, and if $|k-j| \geq 2$,

$$\left|\mathbb{E}\left[\Delta B^H_{\frac{j}{n}} \widetilde{B}^H_{\frac{k}{n}}\right]\right| = \left|\left\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{k}{n}} \right\rangle_{\mathfrak{H}}\right| \leq n^{-2H}\left(j^{2H-1} + |j-k|^{2H-1}\right).$$

(B.5) For any integer $1 \leq j$, $\left|\mathbb{E}\left[\Delta B^H_{\frac{j}{n}} \widetilde{B}^H_{\frac{j}{n}}\right]\right| = \left|\left\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{j}{n}} \right\rangle_{\mathfrak{H}}\right| \leq n^{-2H} j^{2H-1}$.

As a result of properties (B.1) - (B.5), we have the following technical results.

Lemma 3.1. *Let $H < 1/2$ and $0 < t \le T$, and let $n \ge 2$ be an integer. Then*

(a) For fixed $0 \le s \le T$ and integer $r \ge 1$,

$$\sum_{j=0}^{\lfloor nt \rfloor -1} \left| \left\langle \partial_{\frac{j}{n}}, \varepsilon_s \right\rangle_{\mathfrak{H}}^r \right| \le C n^{-2(r-1)H}.$$

(b) For integer $r \ge 1$,

$$\sum_{j=0}^{\lfloor nt \rfloor -1} \left| \left\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{j}{n}} \right\rangle_{\mathfrak{H}}^r \right| \le C n^{-2(r-1)H}.$$

(c) For integers $r \ge 1$ and $0 \le k \le \lfloor nt \rfloor$,

$$\sum_{j=0}^{\lfloor nt \rfloor -1} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}}^r \right| \le C n^{-2rH},$$

and consequently

$$\sum_{j,k=0}^{\lfloor nt \rfloor -1} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}}^r \right| \le C \lfloor nt \rfloor n^{-2rH}.$$

Proof. For (a), first note that we have $\left|\langle \partial_0, \varepsilon_t \rangle_{\mathfrak{H}}\right| \le T^H n^{-H}$ by (B.1) and Cauchy-Schwarz. Further, if $\left|\frac{j}{n} - s\right| < \frac{2}{n}$, then by (B.4) we have $\left|\left\langle \partial_{\frac{j}{n}}, \varepsilon_s \right\rangle_{\mathfrak{H}}\right| \le C n^{-2H}$. Let $\mathscr{J} = \{1 \le j \le \lfloor nt \rfloor, |j - ns| > 1\}$; and note that $|\mathscr{J}^c| \le 2$. Then for the case $r = 1$ we have

$$\begin{aligned}
\sum_{j=0}^{\lfloor nt \rfloor -1} \left| \left\langle \partial_{\frac{j}{n}}, \varepsilon_s \right\rangle_{\mathfrak{H}} \right| &= |\langle \partial_0, \varepsilon_t \rangle_{\mathfrak{H}}| + \sum_{j \in \mathscr{J}^c} \left| \left\langle \partial_{\frac{j}{n}}, \varepsilon_s \right\rangle_{\mathfrak{H}} \right| + \sum_{j \in \mathscr{J}} \left| \left\langle \partial_{\frac{j}{n}}, \varepsilon_s \right\rangle_{\mathfrak{H}} \right| \\
&\le T^H n^{-H} + C n^{-2H} + C n^{-2H} \sum_{j=1}^{\lfloor nt \rfloor -1} j^{2H-1} + |j - ns|^{2H-1} \\
&\le C \lfloor nt \rfloor^{2H} n^{-2H} \le C.
\end{aligned}$$

For the case $r > 1$, we have by (B.4)

$$\sum_{j=0}^{\lfloor nt \rfloor -1} \left| \left\langle \partial_{\frac{j}{n}}, \varepsilon_s \right\rangle_{\mathfrak{H}}^r \right| \le \sup_{0 \le j \le \lfloor nt \rfloor} \left| \left\langle \partial_{\frac{j}{n}}, \varepsilon_s \right\rangle_{\mathfrak{H}}^{r-1} \right| \sum_{j=0}^{\lfloor nt \rfloor -1} \left| \left\langle \partial_{\frac{j}{n}}, \varepsilon_s \right\rangle_{\mathfrak{H}} \right| \le C n^{-2(r-1)H}.$$

For (b), we have by (B.4) and (3.1)

$$\sum_{j=0}^{\lfloor nt\rfloor-1}\left|\left\langle \partial_{\frac{j}{n}},\widetilde{\varepsilon}_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}^{r}\right| \le \sup_{0\le j\le\lfloor nt\rfloor}\left|\left\langle \partial_{\frac{j}{n}},\widetilde{\varepsilon}_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}^{r-1}\right|\sum_{j=0}^{\lfloor nt\rfloor-1}\left|\left\langle \partial_{\frac{j}{n}},\widetilde{\varepsilon}_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}\right|$$
$$\le Cn^{-2(r-1)H}\sum_{j=0}^{\lfloor nt\rfloor-1}\frac{1}{2}\left|\mathbb{E}\left[\Delta B_{\frac{j}{n}}\left(B_{\frac{j}{n}}+B_{\frac{j+1}{n}}\right)\right]\right|$$
$$= Cn^{-2(r-1)H}\sum_{j=0}^{\lfloor nt\rfloor-1}\frac{1}{2}\left|\mathbb{E}\left[B^2_{\frac{j+1}{n}}-B^2_{\frac{j}{n}}\right]\right|$$
$$= Cn^{-2(r-1)H}\sum_{j=0}^{\lfloor nt\rfloor-1}\frac{1}{2}\left[\left(\frac{j+1}{n}\right)^{2H}-\left(\frac{j}{n}\right)^{2H}\right]$$
$$\le Cn^{-2(r-1)H}\frac{\lfloor nt\rfloor}{n}\le Cn^{-2(r-1)H}.$$

For (c), we note that $\left|\left\langle \partial_{j/n},\partial_0\right\rangle_{\mathfrak{H}}\right| = \left|\left\langle \partial_{j/n},\varepsilon_{1/n}\right\rangle_{\mathfrak{H}}\right| \le n^{-2H}$. Also note that by (B.1) and Cauchy-Schwarz we have $\left|\left\langle \partial_{j/n},\partial_{k/n}\right\rangle_{\mathfrak{H}}\right| \le n^{-2H}$ for any $1\le j,k\le\lfloor nt\rfloor$. To begin the proof, we consider the case when $1\le k\le\lfloor nt\rfloor-1$ is fixed. Then

$$\sum_{j=0}^{\lfloor nt\rfloor-1}\left|\left\langle \partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{r}\right| \le \sup_{0\le æ\le\lfloor nt\rfloor}\left\{\sup_{0\le k\le\lfloor nt\rfloor}\left|\left\langle \partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{r-1}\right|\right\}\sum_{j=0}^{\lfloor nt\rfloor-1}\left|\left\langle \partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|$$
$$\le n^{-2(r-1)H}\left(n^{-2H}+\sum_{j=1}^{k-2}\left|\left\langle \partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|+\sum_{j=k-1}^{k+1}\left|\left\langle \partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|+\sum_{j=k+2}^{\lfloor nt\rfloor-1}\left|\left\langle \partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|\right)$$

Then we use (B.2) and (B.3) to write

$$n^{-2(r-1)H}\left(n^{-2H}+\sum_{j=1}^{k-2}\left|\left\langle \partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|+\sum_{j=k-1}^{k+1}\left|\left\langle \partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|+\sum_{j=k+2}^{\lfloor nt\rfloor-1}\left|\left\langle \partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|\right)$$
$$\le n^{2(r-1)H}\left(n^{-2H}+Cn^{-2H}\sum_{j=1}^{k-2}(k-j)^{2H-2}+\sum_{j=k-1}^{k+1}n^{-2H}+Cn^{-2H}\sum_{j=k+2}^{\lfloor nt\rfloor-1}(j-k)^{2H-2}\right)$$
$$\le Cn^{-2rH}\left(4+2\sum_{m=1}^{\infty}m^{2H-2}\right)\le Cn^{-2rH},$$

where we note the sum is finite because $H<1/2$. For the double sum result we have

$$\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left|\left\langle \partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{r}\right| \le \sum_{k=0}^{\lfloor nt\rfloor-1}\sup_{0\le k\le\lfloor nt\rfloor}\left\{\sum_{j=0}^{\lfloor nt\rfloor-1}\left|\left\langle \partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{r}\right|\right\}\le C\lfloor nt\rfloor n^{-2rH}.$$

□

In order to identify the 'critical' cases, we will use the following theorem that was first proved by Nualart and Ortiz-Latorre [28]:

Theorem 3.2. *Fix $H < 1/2$ and an odd integer $k \geq 1$. For integers $n \geq 2$ define*

$$Z_t^{(n)} = n^{kH-\frac{1}{2}} \sum_{j=0}^{\lfloor nt \rfloor -1} \left(\Delta B^H_{\frac{j}{n}}\right)^k.$$

Then as $n \to \infty$, the two-dimensional process $(B^H, Z^{(n)})$ converges in distribution in the Skorohod space $\mathbf{D}([0,T])^2$ to $(B^H, cB^{\frac{1}{2}})$, where $B^{\frac{1}{2}}$ is a standard Brownian motion independent of B^H, and

$$c^2 = \mathbb{E}\left[X_1^{2k}\right] + 2\sum_{j=1}^{\infty} \mathbb{E}\left[(X_1 X_{1=j})^k\right], \quad \textit{for } X_j = B^H_j - B^H_{j-1}.$$

3.1 Cases with $H < 1/2$

In this section we consider the different integral constructions discussed in Chapter 1, and show how each has an associated critical case, for which the given Riemann sum does not converge in general. The following proposition summarizes some known results about stochastic integrals with respect to fBm, when the integrals arise from a Riemann sum construction. A comprehensive treatment can be found in an important paper by Gradinaru, Nourdin, Russo & Vallois [16].

Proposition 3.3. *Let $g \in \mathscr{C}^\infty(\mathbb{R})$, such that g and its derivatives have moderate growth. The following Riemann sums converge in probability as $n \to \infty$ to $g(B_t) - g(0)$ for the given ranges of H:*

(a) Midpoint (type 2) rule: for $1/6 < H < 1/2$,

$$\sum_{j=0}^{\lfloor nt \rfloor -1} g'(\widetilde{B}_{\frac{j}{n}})\Delta B_{\frac{j}{n}},$$

where $\widetilde{B}_{\frac{j}{n}} = \frac{1}{2}\left(B_{\frac{j}{n}} + B_{\frac{j+1}{n}}\right)$.

(b) Trapezoidal rule: For $1/6 < H < 1/2$,

$$\sum_{j=0}^{\lfloor nt \rfloor -1} \frac{1}{2}\left(g'(B_{\frac{j}{n}}) + g'(B_{\frac{j+1}{n}})\right)\Delta B_{\frac{j}{n}}.$$

(c) Simpson's rule: For $1/10 < H < 1/2$,

$$\sum_{j=0}^{\lfloor nt \rfloor -1} \frac{1}{6}\left(g'(B_{\frac{j}{n}}) + 4g'(\widetilde{B}_{\frac{j}{n}}) + g'(B_{\frac{j+1}{n}})\right)\Delta B_{\frac{j}{n}}.$$

(d) Milne's rule: For $1/14 < H < 1/2$,

$$\sum_{j=0}^{\lfloor nt\rfloor -1} \frac{1}{90}\left(7g'(B_{\frac{j}{n}}) + 32g'(B_{\frac{j}{n}} + \frac{1}{4}\Delta B_{\frac{j}{n}}) + 12g'(\widetilde{B}_{\frac{j}{n}}) + 32g'(B_{\frac{j}{n}} + \frac{3}{4}\Delta B_{\frac{j}{n}}) + 7g'(B_{\frac{j+1}{n}})\right)\Delta B_{\frac{j}{n}}.$$

All of these results follow from Theorem 4.4 of [16], in fact they are also proved there for $H \geq 1/2$. However, here we give a different proof of part (c). By similar techniques, results (a), (b) and (d) could also be done in this way. This proof will contain some results that will be used in Chapter 6, and help set up the proof of the main result. We begin with a technical result.

Lemma 3.4. *Let $r = 1,3,5,\ldots$ and $n \geq 2$ be an integer. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a $\mathscr{C}^{2r}$ function such that ϕ and all derivatives up to order $2r$ have moderate growth, and let $\{B_t, t \geq 0\}$ be fBm with Hurst parameter H. Then for each r, there is a constant $C > 0$ such that*

$$\mathbb{E}\left[\left(\sum_{j=0}^{\lfloor nt\rfloor -1} \phi(\widetilde{B}_{\frac{j}{n}})\Delta B^r_{\frac{j}{n}}\right)^2\right] \leq C \sup_{0\leq j\leq \lfloor nt\rfloor} \left\|\phi(\widetilde{B}_{\frac{j}{n}})\right\|^2_{\mathbb{D}^{2r,2}} \lfloor nt\rfloor n^{-2rH},$$

where C depends on r and H.

Proof. To simplify notation, let $Y_j := \phi(\widetilde{B}_{\frac{j}{n}})$. Note that by (B.1), we have $\|\Delta B_{\frac{j}{n}}\|_{L^2(\Omega)} = \|\partial_{\frac{j}{n}}\|_{\mathfrak{H}} = n^{-H}$. For Hermite polynomials $H_r(x)$, $r \geq 1$, it can be shown by induction on the relation $H_{q+1}(x) = xH_q(x) - qH_{q-1}(x)$ that

$$x^r = \sum_{p=0}^{\lfloor \frac{r}{2}\rfloor} C(r,p)H_{r-2p}(x),$$

where each $C(r,p)$ is an integer constant. From Section 2.1, we use (2.4) with $x = \Delta B_{\frac{j}{n}}/\|\Delta B_{\frac{j}{n}}\|_{L^2(\Omega)} = n^H\Delta B_{\frac{j}{n}}$ to write

$$H_r\left(n^H \Delta B_{\frac{j}{n}}\right) = \delta^r\left(n^{rH}\partial_{\frac{j}{n}}^{\otimes r}\right).$$

It follows that

$$n^{rH}\Delta B^r_{\frac{j}{n}} = \sum_{p=0}^{\lfloor \frac{r}{2}\rfloor} C(r,p)H_{r-2p}(n^H\Delta B_{\frac{j}{n}}) = \sum_{p=0}^{\lfloor \frac{r}{2}\rfloor} C(r,p)\delta^{r-2p}\left(n^{(r-2p)H}\partial_{\frac{j}{n}}^{\otimes r-2p}\right),$$

which implies

$$\Delta B^r_{\frac{j}{n}} = \sum_{p=0}^{\lfloor \frac{r}{2}\rfloor} C(r,p)n^{-2pH}\delta^{r-2p}\left(\partial_{\frac{j}{n}}^{\otimes r-2p}\right).$$

With this representation for $\Delta B^r_{j/n}$, we then have

$$\mathbb{E}\left[\left(\sum_{j=0}^{\lfloor nt\rfloor-1} Y_j \Delta B^r_{\frac{j}{n}}\right)^2\right]$$

$$= \sum_{p,p'=0}^{\lfloor \frac{r}{2}\rfloor} C(r,p)C(r,p')n^{-2H(p+p')} \sum_{j,k=0}^{\lfloor nt\rfloor-1} \mathbb{E}\left[Y_j Y_k \delta^{r-2p}\left(\partial_{\frac{j}{n}}^{\otimes r-2p}\right)\delta^{r-2p'}\left(\partial_{\frac{k}{n}}^{\otimes r-2p'}\right)\right]$$

$$\leq \sum_{p,p'=0}^{\lfloor \frac{r}{2}\rfloor} |C(r,p)C(r,p')| n^{-2H(p+p')} \sum_{j,k=0}^{\lfloor nt\rfloor-1} \left|\mathbb{E}\left[Y_j Y_k \delta^{r-2p}\left(\partial_{\frac{j}{n}}^{\otimes r-2p}\right)\delta^{r-2p'}\left(\partial_{\frac{k}{n}}^{\otimes r-2p'}\right)\right]\right|. \quad (3.2)$$

By Lemma 2.1.d, the product

$$\delta^{r-2p}\left(\partial_{\frac{j}{n}}^{\otimes r-2p}\right)\delta^{r-2p'}\left(\partial_{\frac{k}{n}}^{\otimes r-2p'}\right)$$

consists of terms of the form

$$C\delta^{2r-2(p+p')-2z}\left(\partial_{\frac{j}{n}}^{\otimes r-2p-z} \otimes \partial_{\frac{k}{n}}^{\otimes r-2p'-z}\right)\left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^z, \quad (3.3)$$

where $z \geq 0$ is an integer satisfying $2r - 2(p+p') - 2z \geq 0$. Using (3.3), we can write that (3.2) consists of nonnegative terms of the form

$$Cn^{-2H(p+p')} \sum_{j,k=0}^{\lfloor nt\rfloor-1} \left|\mathbb{E}\left[Y_j Y_k \delta^{2r-2(p+p')-2z}\left(\partial_{\frac{j}{n}}^{\otimes r-2p-z} \otimes \partial_{\frac{k}{n}}^{\otimes r-2p'-z}\right)\left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^z\right]\right|. \quad (3.4)$$

To address terms of this type, suppose first that $z \geq 1$. Lemma 2.1.c implies that

$$\left\|\delta^{2r-2(p+p')-2z}\left(\partial_{\frac{j}{n}}^{\otimes r-2p-z} \otimes \partial_{\frac{k}{n}}^{\otimes r-2p'-z}\right)\right\|_{L^2(\Omega)} \leq C\left(\|\partial_{\frac{j}{n}}\|_{\mathfrak{H}}^{r-2p-z}\|\partial_{\frac{k}{n}}\|_{\mathfrak{H}}^{r-2p'-z}\right)$$

$$\leq C\left\|\partial_{\frac{1}{n}}\right\|_{\mathfrak{H}}^{2r-2(p+p')-2z} = Cn^{-2H(r-p-p'-z)}.$$

Hence, for $z \geq 1$, (3.4) is bounded by

$$Cn^{-2H(p+p')} \sup_{0\leq j\leq \lfloor nt\rfloor} \left\|Y_j\right\|_{L^2(\Omega)}^2 \left\|\partial_{\frac{1}{n}}\right\|_{\mathfrak{H}}^{2r-2(p+p')-2z} \sum_{j,k=0}^{\lfloor nt\rfloor-1} \left|\left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^z\right|$$

$$\leq C \sup_{0\leq j\leq \lfloor nt\rfloor} \left\|Y_j\right\|_{\mathbb{D}^{2r,2}}^2 \lfloor nt\rfloor n^{-2rH},$$

which follows from Lemma 3.1.c.

On the other hand, for the terms with $z = 0$, by (2.6) we have

$$\mathbb{E}\left[Y_j Y_k \delta^{2r-2(p+p')}\left(\partial_{\frac{j}{n}}^{\otimes r-2p} \otimes \partial_{\frac{k}{n}}^{\otimes r-2p'}\right)\right]$$

$$= \mathbb{E}\left\langle D^{2r-2(p+p')} Y_j Y_k, \partial_{\frac{j}{n}}^{\otimes r-2p} \otimes \partial_{\frac{k}{n}}^{\otimes r-2p'}\right\rangle_{\mathfrak{H}^{\otimes 2r-2(p+p')}}. \quad (3.5)$$

By definition of the Malliavin derivative and Leibniz rule, $D^{2r-2(p+p')}Y_jY_k$ consists of terms of the form $D^aY_j \otimes D^bY_k$, where $a+b = 2r-2(p+p')$. Without loss of generality, we may assume $b \geq 1$. By assumptions on ϕ and the definition of the Malliavin derivative, we know that $D^bY_k = \phi^{(b)}(\widetilde{B}_{k/n})\widetilde{\varepsilon}_{k/n}^{\otimes b}$, and we know that for each $b \leq 2r$, $D^bY_k \in L^2(\Omega;\mathfrak{H}^{\otimes b})$. It follows that we can write,

$$\left|\mathbb{E}\left\langle D^aY_j \otimes D^bY_k, \partial_{\frac{j}{n}}^{\otimes r-2p} \otimes \partial_{\frac{k}{n}}^{\otimes r-2p'}\right\rangle_{\mathfrak{H}^{\otimes a+b}}\right| \leq C\|Y_j\|_{\mathbb{D}^{2r,2}}\|Y_k\|_{\mathbb{D}^{2r,2}}\left|\left\langle \widetilde{\varepsilon}_{\frac{j}{n}}, \partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}^{\phi}\right| \left|\left\langle \widetilde{\varepsilon}_{\frac{j}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{a-\phi}\right| \times \left|\left\langle \widetilde{\varepsilon}_{\frac{k}{n}}, \partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}^{\psi}\right| \left|\left\langle \widetilde{\varepsilon}_{\frac{k}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{b-\psi}\right|,$$

for integers $0 \leq \phi \leq a$, $0 \leq \psi \leq b$. Without loss of generality, we may assume $\psi \geq 1$, and by implication $b \geq 1$. Then using (B.4),

$$\left|\mathbb{E}\left\langle D^aY_jD^bY_k, \partial_{\frac{j}{n}}^{\otimes r-2p} \otimes \partial_{\frac{k}{n}}^{\otimes r-2p'}\right\rangle_{\mathfrak{H}^{\otimes a+b}}\right| \leq C \sup_{0\leq j\leq \lfloor nt\rfloor} \|Y_j\|_{\mathbb{D}^{2r,2}}^2 n^{-2H(a+b-1)} \left|\left\langle \widetilde{\varepsilon}_{\frac{k}{n}}, \partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}\right|.$$

Thus, for each pair (a,b), the corresponding term of (3.4) is bounded by

$$\begin{aligned} Cn^{-2H(p+p')}\sum_{j,k=0}^{\lfloor nt\rfloor -1}&\left|\mathbb{E}\left[Y_jY_k\delta^{2r-2(p+p')}\left(\partial_{\frac{j}{n}}^{\otimes r-2p} \otimes \partial_{\frac{k}{n}}^{\otimes r-2p'}\right)\right]\right| \\ &\leq Cn^{-2H(p+p'+a+b-1)} \sup_{0\leq j\leq \lfloor nt\rfloor} \|Y_j\|_{\mathbb{D}^{2r,2}}^2 \sum_{j,k=0}^{\lfloor nt\rfloor -1} \left|\left\langle \widetilde{\varepsilon}_{\frac{k}{n}}, \partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}\right| \\ &\leq Cn^{-2H(p+p'+a+b-1)} \sup_{0\leq j\leq \lfloor nt\rfloor} \|Y_j\|_{\mathbb{D}^{2r,2}}^2 \sum_{j,k=0}^{\lfloor nt\rfloor -1} \left|\left\langle \widetilde{\varepsilon}_{\frac{k}{n}}, \partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}\right|. \end{aligned}$$

By Lemma 3.1.a,

$$\sum_{j=0}^{\lfloor nt\rfloor -1} \left|\left\langle \widetilde{\varepsilon}_{\frac{k}{n}}, \partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}\right| \leq C\lfloor nt\rfloor^{2H} n^{-2H} \leq C$$

for all $0 \leq k \leq \lfloor nt\rfloor$, so that

$$\begin{aligned} Cn^{-2H(p+p'+a+b-1)} \sup_{0\leq j\leq \lfloor nt\rfloor} \|Y_j\|_{\mathbb{D}^{2r,2}}^2 \sum_{k=0}^{\lfloor nt\rfloor -1}&\left\{\sup_{0\leq k\leq \lfloor nt\rfloor} \sum_{j=0}^{\lfloor nt\rfloor -1} \left|\left\langle \widetilde{\varepsilon}_{\frac{k}{n}}, \partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}\right|\right\} \\ &\leq C \sup_{0\leq j\leq \lfloor nt\rfloor} \|Y_j\|_{\mathbb{D}^{2r,2}}^2 \lfloor nt\rfloor n^{-2H(p+p'+a+b-1)}, \end{aligned}$$

where $p+p'+a+b-1 = 2r-(p+p')-1 \geq r$, since $p+p'+1 \leq 2\left\lfloor \frac{r}{2}\right\rfloor + 1 \leq r$, for odd integer r. This concludes the proof. $\square$

Now for the convergence of the Simpson's rule sum. We begin with some elementary results from the calculus of deterministic functions. For $x,h \in \mathbb{R}$ and a $\mathscr{C}^\infty$ function g, we have the following integral form for the Simpson's rule sum:

$$
\begin{aligned}
g(x+h)-g(x-h) &= \int_{-h}^{h} g'(x+u)\,du \\
&= \frac{h}{3}\left(g'(x-h)+4g'(x)+g'(x+h)\right)+\frac{1}{6}\int_0^h \left(g^{(4)}(x-u)-g^{(4)}(x+u)\right)\,u(h-u)^2du.
\end{aligned}
$$

See Talman [40] for a nice discussion of the Simpson's rule error term. Next, we consider a Taylor expansion of order 7 for $g^{(4)}$:

$$
\begin{aligned}
g^{(4)}(x+u)-g^{(4)}(x) &= \sum_{\ell=1}^{6}\frac{g^{(4+\ell)}(x)}{\ell!}u^{\ell}\;+\;\frac{g^{(11)}(\xi)}{7!}u^7;\text{ and} \\
g^{(4)}(x)-g^{(4)}(x-u) &= \sum_{\ell=1}^{6}\frac{(-1)^{\ell+1}g^{(4+\ell)}(x)}{\ell!}u^{\ell}\;+\;\frac{g^{(11)}(\eta)}{7!}u^7
\end{aligned}
$$

Adding the above equations, we obtain

$$
g^{(4)}(x+u)-g^{(4)}(x-u)=2\sum_{\nu=1}^{3}\frac{g^{(4+2\nu-1)}(x)}{(2\nu-1)!}u^{2\nu-1}\;+\;\frac{g^{(11)}(\xi)+g^{(11)}(\eta)}{7!}u^7.
$$

It follows that we can write

$$
\begin{aligned}
g(x+h)-g(x-h) &= \frac{h}{3}\left(g'(x-h)+4g'(x)+g'(x+h)\right)-\frac{1}{3}\sum_{\nu=1}^{3}\frac{g^{(4+2\nu-1)}(x)}{(2\nu-1)!}\int_0^h u^{2\nu}(h-u)^2du \\
&\qquad -\frac{g^{(11)}(\xi)+g^{(11)}(\eta)}{(6)(7!)}\int_0^h u^8(h-u)^2du \\
&= \frac{h}{3}\left(g'(x-h)+4g'(x)+g'(x+h)\right)-\frac{g^{5)}(x)}{90}h^5-A_7g^{(7)}(x)h^7-A_9g^{(9)}(x)h^9 \\
&\qquad -\frac{1}{6(7!)}\int_0^h\left[g^{(11)}(\xi)+g^{(11)}(\eta)\right]u^8(h-u)^2du, \qquad (*)
\end{aligned}
$$

where A_7, A_9 are positive constants, and $\xi=\xi(u)\in[x-h,x+h]$, with similar for η. With this relation, we now return to Proposition 3.3.c. We begin with the telescoping series,

$$
\begin{aligned}
g(B_t)-g(0) &= \sum_{j=0}^{\lfloor nt\rfloor-1}\left(g(B_{\frac{j+1}{n}})-g(B_{\frac{j}{n}})\right)+\left(g(B_t)-g(B_{\frac{\lfloor nt\rfloor}{n}})\right) \\
&= \sum_{j=0}^{\lfloor nt\rfloor-1}\int_{B_{j/n}}^{B_{(j+1)/n}} g'(u)\,du+\left(g(B_t)-g(B_{\frac{\lfloor nt\rfloor}{n}})\right).
\end{aligned}
$$

By continuity, the term $\left(g(B_t)-g(B_{\lfloor nt\rfloor/n})\right)$ tends to zero uniformly on compacts in probability (ucp) as $n\to\infty$, and may be neglected. For each integral term, we use (*) with $x=\widetilde{B}_{j/n}$ and

$h = \frac{1}{2}\Delta B_{j/n}$ to obtain

$$\sum_{j=0}^{\lfloor nt\rfloor -1}\int_{B_{j/n}}^{B_{(j+1)/n}} g'(u)\,du = \sum_{j=0}^{\lfloor nt\rfloor -1}\frac{1}{6}\left(g'(B_{\frac{j}{n}}) + 4g'(\widetilde{B}_{\frac{j}{n}}) + g'(B_{\frac{j+1}{n}})\right) - \frac{1}{2^5\,90}\sum_{j=0}^{\lfloor nt\rfloor -1} g^{(5)}(\widetilde{B}_{\frac{j}{n}})\Delta B_{\frac{j}{n}}^5$$
$$- A_7\sum_{j=0}^{\lfloor nt\rfloor -1} g^{(7)}(\widetilde{B}_{\frac{j}{n}})\Delta B_{\frac{j}{n}}^7 - A_9\sum_{j=0}^{\lfloor nt\rfloor -1} g^{(9)}(\widetilde{B}_{\frac{j}{n}})\Delta B_{\frac{j}{n}}^9$$
$$- \frac{1}{6(7!)}\sum_{j=0}^{\lfloor nt\rfloor -1}\int_0^{\Delta B_{j/n}}\left(g^{(11)}(\xi) + g^{(11)}(\eta)\right)u^8(\Delta B_{\frac{j}{n}} - u)^2 du. \quad (3.6)$$

By Lemma 3.4, the terms

$$\sum_{j=0}^{\lfloor nt\rfloor -1}\frac{g^{(5)}(\widetilde{B}_{\frac{j}{n}})}{2880}\Delta B_{\frac{j}{n}}^5, \quad A_7\sum_{j=0}^{\lfloor nt\rfloor -1} g^{(7)}(\widetilde{B}_{\frac{j}{n}})\Delta B_{\frac{j}{n}}^7, \quad A_9\sum_{j=0}^{\lfloor nt\rfloor -1} g^{(9)}(\widetilde{B}_{\frac{j}{n}})\Delta B_{\frac{j}{n}}^9$$

all tend to zero in $L^2(\Omega)$ as $n\to\infty$. For the last term, we have the $L^2(\Omega)$ estimate

$$\mathbb{E}\left[\left(\sum_{j=0}^{\lfloor nt\rfloor -1}\int_0^{\Delta B_{j/n}}\left[g^{(11)}(\xi) + g^{(11)}(\eta)\right]u^8(\Delta B_{\frac{j}{n}} - u)^2 du\right)^2\right]$$
$$\le C\left(\mathbb{E}\left[\sup_{s\in[0,t]}|g^{(11)}(B_s)^4|\right]\right)^{\frac{1}{2}}\left(\sum_{j=0}^{\lfloor nt\rfloor -1}\|\Delta B_{\frac{j}{n}}^{11}\|_{L^4(\Omega)}\right)^2 \le C\lfloor nt\rfloor^2 n^{-22H} \le Cn^{-2H},$$

because $\|\Delta B_{j/n}^{11}\|_{L^4(\Omega)} \le C\left(\mathbb{E}|\Delta B_{j/n}^2|\right)^{\frac{11}{2}} \le Cn^{-11H}$ by (B.1) and the Gaussian moments formula. Thus, we have

$$\mathbb{P}\lim_{n\to\infty}\sum_{j=0}^{\lfloor nt\rfloor -1}\frac{1}{6}\left(g'(B_{\frac{j}{n}}) + 4g'(\widetilde{B}_{\frac{j}{n}}) + g'(B_{\frac{j+1}{n}})\right)\Delta B_{\frac{j}{n}} = f(B_t) - f(0),$$

when $H > 1/10$, and Proposition 3.3.c is proved. □

As a converse to Proposition 3.3.c (and parts (a), (b) and (d) by similar computation), let $g(x) = f(x)$ be a polynomial such that $g^{(5)} = f^{(5)} = 1$. Then

$$S_n^S(t) = f(B_{\frac{\lfloor nt\rfloor}{n}}) - f(0) + \frac{1}{2880}\sum_{j=0}^{\lfloor nt\rfloor -1}\Delta B_{\frac{j}{n}}^5.$$

By Theorem 3.2, the sequence $\left(B_t, \sum_{j=0}^{\lfloor nt\rfloor -1}\Delta B_{j/n}^5\right)$ converges in distribution to (B_t, W), where W is a Gaussian random variable, independent of B. It follows that $S_n^S(t)$ does not, in general, converge in probability when $H \le 1/10$.

Since Proposition 3.3 is restricted to odd powers, we need a different proof to address the Midpoint (type 1) case.

Proposition 3.5. *Let $t > 0$, let g be a $\mathscr{C}^6(\mathbb{R})$ function such that g and its first 6 derivatives satisfy moderate growth conditions, and let $\{B_t^H, t \geq 0\}$ be a fractional Brownian motion with Hurst parameter H. Then if $H > 1/4$, we have*

$$\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} g'(B^H_{\frac{2j-1}{n}}) \left(B^H_{\frac{2j}{n}} - B^H_{\frac{2j-2}{n}} \right) \xrightarrow{\mathscr{P}} g(B_t^H) - g(0),$$

as $n \to \infty$. The sum does not, in general, converge if $H < 1/4$.

Proof. In this proof, we write B_t instead of B_t^H to simplify notation. For each $1 \leq j \leq \lfloor \frac{nt}{2} \rfloor$, we consider two Taylor expansions of order 4:

$$g(B_{\frac{2j}{n}}) = g(B_{\frac{2j-1}{n}}) + \sum_{r=1}^{3} \frac{1}{r!} g^{(r)}(B_{\frac{2j-1}{n}}) \Delta B^r_{\frac{2j-1}{n}} + \frac{1}{24} g^{(4)}(\zeta_{2j-1}) \Delta B^4_{\frac{2j-1}{n}}$$

$$g(B_{\frac{2j-2}{n}}) = g(B_{\frac{2j-1}{n}}) \sum_{r=1}^{3} \frac{(-1)^{r+1}}{r!} g^{(r)}(B_{\frac{2j-1}{n}}) \Delta B^r_{\frac{2j-2}{n}} - \frac{1}{24} g^{(3)}(\eta_{2j-2}) \Delta B^4_{\frac{2j-2}{n}},$$

for intermediate values $\zeta_{2j-1}, \eta_{2j-2}$. Subtracting the above, we obtain

$$\begin{aligned} g(B_{\frac{2j}{n}}) - g(B_{\frac{2j-2}{n}}) &= g'(B_{\frac{2j-1}{n}})(B_{\frac{2j}{n}} - B_{\frac{2j-2}{n}}) + \frac{1}{2} g''(B_{\frac{2j-1}{n}}) \left(\Delta B^2_{\frac{2j-1}{n}} - \Delta B^2_{\frac{2j-2}{n}} \right) \\ &\quad + \frac{1}{6} g^{(3)}(B_{2j-1}) \left(\Delta B^3_{\frac{2j-1}{n}} + \Delta B^3_{\frac{2j-2}{n}} \right) + \frac{1}{24} g^{(4)}(\zeta_{2j-1}) \Delta B^4_{\frac{2j-1}{n}} \\ &\quad - \frac{1}{24} g^{(4)}(\eta_{2j-2}) \Delta B^4_{\frac{2j-2}{n}}, \end{aligned}$$

so that

$$\begin{aligned} \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} g'(B^H_{\frac{2j-1}{n}}) \left(B^H_{\frac{2j}{n}} - B^H_{\frac{2j-2}{n}} \right) &= g(B_{\frac{2}{n}\lfloor \frac{nt}{2} \rfloor}) - g(0) \\ - \frac{1}{2} \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} g''(B_{\frac{2j-1}{n}}) \left(\Delta B^2_{\frac{2j-1}{n}} - \Delta B^2_{\frac{2j-2}{n}} \right) &- \frac{1}{6} \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} g^{(3)}(B_{\frac{2j-1}{n}}) \Delta B^3_{\frac{2j-1}{n}} - \frac{1}{6} \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} g^{(3)}(B_{\frac{2j-2}{n}}) \Delta B^3_{\frac{2j-2}{n}} \\ &- \frac{1}{24} \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} g^{(4)}(\zeta_{2j-1}) \Delta B^4_{\frac{2j-1}{n}} + \frac{1}{24} \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} g^{(4)}(\eta_{2j-2}) \Delta B^4_{\frac{2j-2}{n}}. \end{aligned}$$

We want to show that the 3rd and 4th order terms vanish in probability. We can write

$$\begin{aligned} \frac{1}{6} \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} g^{(3)}(B_{\frac{2j-1}{n}}) \Delta B^3_{\frac{2j-1}{n}} &= \frac{1}{6} \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} g^{(3)} \left(\frac{1}{2} \left(B_{\frac{2j-2}{n}} + B_{\frac{2j}{n}} \right) \right) \Delta B^3_{\frac{2j-1}{n}} \\ &\quad + \frac{1}{6} \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} \left[g^{(3)}(B_{\frac{2j-1}{n}}) - g^{(3)} \left(\frac{1}{2} \left(B_{\frac{2j-2}{n}} + B_{\frac{2j}{n}} \right) \right) \right] \Delta B^3_{\frac{2j-1}{n}}. \end{aligned}$$

By Lemma 3.4,

$$\mathbb{E}\left[\left(\sum_{j=1}^{\lfloor \frac{nt}{2}\rfloor} g^{(3)}\left(\frac{1}{2}\left(B_{\frac{2j-2}{n}}+B_{\frac{2j}{n}}\right)\right)\Delta B^3_{\frac{2j-1}{n}}\right)^2\right] \le C\lfloor nt\rfloor n^{-6H},$$

which vanishes for $H > 1/4$. For the other term, we have by Mean Value theorem and (B.1)

$$\mathbb{E}\left|g^{(3)}(B_{\frac{2j-1}{n}})-g^{(3)}\left(\frac{1}{2}\left(B_{\frac{2j-2}{n}}+B_{\frac{2j}{n}}\right)\right)\right| \le \mathbb{E}\left|\sup_{s\in[0,t]} g^{(4)}(B_s)\left[\frac{1}{2}\left(B_{\frac{2j-2}{n}}+B_{\frac{2j}{n}}\right)-\Delta B_{\frac{2j-1}{n}}\right]\right| \le Cn^{-H}.$$

Moreover, $\sup_{0\le j\le\lfloor nt/2\rfloor}\mathbb{E}\left|\Delta B^3_{(2j-1)/n}\right| \le Cn^{-3H}$ by (B.1) and Hölder inequality, hence we can write

$$\mathbb{E}\sum_{j=1}^{\lfloor \frac{nt}{2}\rfloor}\left|g^{(3)}(B_{\frac{2j-1}{n}})-g^{(3)}\left(\frac{1}{2}\left(B_{\frac{2j-2}{n}}+B_{\frac{2j}{n}}\right)\right)\right| \le C\sum_{j=1}^{\lfloor \frac{nt}{2}\rfloor} n^{-4H} \le \lfloor nt\rfloor n^{-4H},$$

which tends to zero if $H > 1/4$. The computation for

$$\frac{1}{6}\sum_{j=1}^{\lfloor \frac{nt}{2}\rfloor} g^{(3)}(B_{\frac{2j-1}{n}})\Delta B^3_{\frac{2j-2}{n}}$$

is similar. Next, for the 4th order terms we have the estimate

$$\mathbb{E}\left|g^{(4)}(\zeta_{2j-1})\Delta B^4_{\frac{2j-1}{n}}\right| \le C\mathbb{E}\left[\sup_{s\in[0,t]}|g^{(4)}(B_s)|\right]n^{-4H},$$

so that

$$\frac{1}{24}\sum_{j=1}^{\lfloor \frac{nt}{2}\rfloor}\mathbb{E}\left|g^{(4)}(\zeta_{2j-1})\Delta B^4_{\frac{2j-1}{n}}\right| \le C\lfloor nt\rfloor n^{-4H},$$

with a similar estimate for the term with $g^{(4)}(\eta_{2j-2})\Delta B^4_{(2j-2)/n}$, so these terms tend to zero. Hence, it is enough to study the term

$$\sum_{j=1}^{\lfloor \frac{nt}{2}\rfloor} g''(B_{\frac{2j-1}{n}})\left(\Delta B^2_{\frac{2j-1}{n}}-\Delta B^2_{\frac{2j-2}{n}}\right). \tag{3.7}$$

As in the proof of Lemma 3.4, we use the Hermite polynomials, in this case $H_2(x)$:

$$n^{2H}\Delta B^2_{\frac{2j-1}{n}}-n^{2H}\Delta B_{\frac{2j-2}{n}} = H_2(n^H\Delta B_{\frac{2j-1}{n}})-H_2(n^H\Delta B_{\frac{2j-2}{n}}).$$

Using (2.4), this equals $n^{2H}\delta^2\left(\partial^{\otimes 2}_{\frac{2j-1}{n}}\right)-n^{2H}\delta^2\left(\partial^{\otimes 2}_{\frac{2j-2}{n}}\right)$, so that we can write (3.7) as

$$\sum_{j=1}^{\lfloor \frac{nt}{2}\rfloor} g''(B_{\frac{2j-1}{n}})\left(\delta^2\left(\partial^{\otimes 2}_{\frac{2j-1}{n}}\right)-\delta^2\left(\partial^{\otimes 2}_{\frac{2j-2}{n}}\right)\right).$$

To prove the result when $H > 1/4$, it is enough to prove that

$$\mathbb{E}\left|\sum_{j=0}^{\lfloor nt\rfloor-1} g''(B_{t_j})\delta^2\left(\partial_{\frac{j}{n}}^{\otimes 2}\right)\right|^2 \le C\lfloor nt\rfloor n^{-4H},$$

where $j/n \le t_j \le (j+1)/n$ for each j. By Lemma 2.1.d,

$$\begin{aligned}
\mathbb{E}\left|\sum_{j=0}^{\lfloor nt\rfloor-1} g''(B_{t_j})\delta^2\left(\partial_{\frac{j}{n}}^{\otimes 2}\right)\right|^2 &\le C\lfloor nt\rfloor n^{-4H} = \sum_{j,k=0}^{\lfloor nt\rfloor-1} \mathbb{E}\left[g''(B_{t_j})g(B_{t_k})\delta^2\left(\partial_{\frac{j}{n}}^{\otimes 2}\right)\delta^2\left(\partial_{\frac{k}{n}}^{\otimes 2}\right)\right] \\
&= \sum_{j,k=0}^{\lfloor nt\rfloor-1} \mathbb{E}\left[g''(B_{t_j})g(B_{t_k})\left(\delta^4\left(\partial_{\frac{j}{n}}^{\otimes 2}\otimes\partial_{\frac{k}{n}}^{\otimes 2}\right)+4\delta^2\left(\partial_{\frac{j}{n}}\otimes\partial_{\frac{k}{n}}\right)\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}+2\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^2\right)\right] \\
&\le \sum_{j,k=0}^{\lfloor nt\rfloor-1} \mathbb{E}\left[g''(B_{t_j})g(B_{t_k})\delta^4\left(\partial_{\frac{j}{n}}^{\otimes 2}\otimes\partial_{\frac{k}{n}}^{\otimes 2}\right)\right] \\
&\quad +4\sup_{0\le j\le\lfloor nt\rfloor}\left\|g''(B_{t_j})^2\right\|_{L^2(\Omega)}\sup_{0\le j,k\le\lfloor nt\rfloor}\left\|\delta^2(\partial_{\frac{j}{n}}\otimes\partial_{\frac{k}{n}})\right\|_{L^2(\Omega)}\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right| \\
&\quad +2\sup_{0\le j\le\lfloor nt\rfloor}\left\|g''(B_{t_j})^2\right\|_{L^2(\Omega)}\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^2 \\
&\le \sum_{j,k=0}^{\lfloor nt\rfloor-1}\mathbb{E}\left\langle D^4\left(g''(B_{t_j})g(B_{t_k})\right),\partial_{\frac{j}{n}}^{\otimes 2}\otimes\partial_{\frac{k}{n}}^{\otimes 2}\right\rangle_{\mathfrak{H}^{\otimes 4}}+C\|\partial_{\frac{1}{n}}\|_{\mathfrak{H}}^2\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right| \\
&\qquad +C\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^2,
\end{aligned}$$

where we used Lemma 2.1.c for the estimate $\left\|\delta^2(\partial_{\frac{j}{n}}\otimes\partial_{\frac{k}{n}})\right\|_{L^2(\Omega)} \le C\|\partial_{\frac{1}{n}}\|_{\mathfrak{H}}^2$. By (B.1) and Lemma 3.1.b,

$$\|\partial_{\frac{1}{n}}\|_{\mathfrak{H}}^2\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|+\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^2 \le C\lfloor nt\rfloor n^{-4H}.$$

For the first term we have

$$\sum_{j,k=0}^{\lfloor nt\rfloor-1}\mathbb{E}\left\langle D^4\left(g''(B_{t_j})g(B_{t_k})\right),\partial_{\frac{j}{n}}^{\otimes 2}\otimes\partial_{\frac{k}{n}}^{\otimes 2}\right\rangle_{\mathfrak{H}^{\otimes 4}} \le C\lfloor nt\rfloor n^{-4H}$$

by the argument applied to (3.5) in the proof of Proposition 3.3. Hence, we have that (3.7) tends to zero in $L^2(\Omega)$ if $H > 1/4$.

Conversely, suppose $g(x) = x$, then

$$\mathbb{E}\left|\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} g'(B_{\frac{2j-1}{n}})\left(B_{\frac{2j}{n}} - B_{\frac{2j-2}{n}}\right)\right|^2 = \sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \mathbb{E}\left[(B_{\frac{2j}{n}} - B_{\frac{2j-2}{n}})(B_{\frac{2k}{n}} - B_{\frac{2k-2}{n}})\right]$$

$$= \frac{1}{2n^{2H}} \sum_{j,k=0}^{\lfloor \frac{nt}{2} \rfloor} \left(|2j-2k+2|^{2H} - 2|2j-2k| + |2j-2k-2|^{2H}\right),$$

which diverges if $H < 1/4$. □

3.2 The case $H > 1/2$

In this section, we provide some background material which will be used in Chapter 7. Here, we assume throughout that $H > 1/2$. We work with a multi-dimensional process, and consider integration in the Russo-Vallois sense rather than based on the Riemann sum approach of Section 3.1.

Let $F = g(B(\phi_1), \dots, B(\phi_n))$, where $n \geq 1$, $g : \mathbb{R}^n \to \mathbb{R}$, $\phi_i \in \mathfrak{H}_d$, and g is a smooth function. The Malliavin derivative of F is an element of $\mathfrak{H}_d$ (which is isomorphic to the product space $(\mathfrak{H}_1)^d$), and we can write $D = (D^{(1)}, \dots, D^{(d)})$, where

$$D_t^{(i)} F = \sum_{j=1}^{n} \frac{\partial g}{\partial x_j}(B(\phi_1), \dots, B(\phi_n))\, \phi_j(t,i),$$

where we use the notation $D_t^{(i)} F = D^{(i)} F(t)$. We define the 'component integral' $\delta^{(i)}$ as the adjoint of $D^{(i)}$, and use the notation

$$\delta^{(i)}(u) = \int_0^T u_t \delta B_t^i; \quad \text{and} \tag{3.8}$$

$$\delta(u) = \int_0^T u_t \delta B_t = \sum_{i=1}^{d} \delta^{(i)}(u).$$

where $u \in \text{Dom}\ \delta^{(i)} \subset L^2(\Omega, \mathfrak{H}_1)$ for every $i = 1, \dots, d$ implies $u \in \text{Dom}\ \delta \subset L^2(\Omega, \mathfrak{H}_d)$.

Fix $T > 0$ and an integer $d \geq 1$. Let $B = \{B_t, 0 \leq t \leq T\} = (B_t^1, \dots, B_t^d)$ be a d-dimensional fBm, that is, each B_t^i is an independent, centered Gaussian process with $B_0^i = 0$ and covariance

$$\mathbb{E}\left[B_s^i B_t^i\right] := R(s,t) = \frac{1}{2}\left(s^{2H} + t^{2H} - |s-t|^{2H}\right)$$

for $t, s \geq 0$. We will use the following elementary properties of $R_H(s,t)$:

(R.1) $R_H(s,t) = R_H(t,s)$; and for any $\varepsilon > 0$, $R_H(s+\varepsilon, t) \geq R_H(s,t)$.

(R.2) There are constants $1 \leq c_0 < c_1 \leq 2$ such that $c_0(st)^H \leq R_H(s,t) \leq c_1(st)^H$.

(R.3) As an alternate bound, if $s \leq t$ then the Mean Value Theorem implies

$$R_H(s,t) \leq s^{2H} + t^{2H} - (t-s)^{2H} \leq s^{2H} + 2Hst^{2H-1}.$$

Let $\mathscr{E}$ denote the set of $\mathbb{R}-$valued step functions on $[0,T]\times\{1,\dots,d\}$. Note that any $f=f(t,i)\in\mathscr{E}$ may be written as a linear combination of elementary functions $e_t^k=\mathbf{1}_{[0,t]\times\{k\}}$. Let $\mathfrak{H}_d$ be the Hilbert space defined as the closure of $\mathscr{E}$ with respect to the inner product

$$\left\langle e_s^k, e_t^j\right\rangle_{\mathfrak{H}_d}=\mathbb{E}[B_s^k B_t^j]=R_H(s,t)\delta_{kj},$$

where δ_{kj} is the Kronecker delta. The mapping $e_t^k\mapsto B^k(t)$ can be extended to a linear isometry between $\mathfrak{H}_d$ and the Gaussian space spanned by B. In this way, $\{B(h),h\in\mathfrak{H}_d\}$ is an isonormal Gaussian process.

Let $\alpha_H=H(2H-1)$. It is well known that we can write

$$R_H(s,t)=\alpha_H\int_0^s\int_0^t|\eta-\theta|^{2H-2}d\eta\, d\theta. \tag{3.9}$$

Consequently, for $f,g\in\mathscr{E}$ we can write

$$\langle f,g\rangle_{\mathfrak{H}_d}=\alpha_H\sum_{i=1}^d\int_0^T\int_0^T f(s,i)g(t,i)|t-s|^{2H-2}ds\, dt. \tag{3.10}$$

We recall (see [27], Sec. 5.1.3) that $\mathfrak{H}_d$ contains the linear subspace of measurable, $\mathbb{R}$-valued functions φ on $[0,T]\times\{1,\dots,d\}$ such that

$$\sum_{i=1}^d\int_0^T\int_0^T|\varphi(s,i)|\,|\varphi(t,i)|\,|t-s|^{2H-2}ds\, dt<\infty.$$

We denote this space by $|\mathfrak{H}_d|$. Let $|\mathfrak{H}_d^{q,s}|$ be the space of symmetric functions $f:([0,T]\times\{1,\dots,d\})^q\to\mathbb{R}$ such that

$$\sum_{i_1,\dots,i_q=1}^d\int_{[0,T]^{2q}}|f\left((\eta_1,i_1),\dots,(\eta_q,i_q)\right)|\,|f\left((\theta_1,i_1),\dots,(\theta_q,i_q)\right)|\,\prod_{j=1}^p|\eta_j-\theta_j|^{2H-2}d\eta\, d\theta\ <\infty.$$

Then $|\mathfrak{H}_d^{q,s}|\subset\mathfrak{H}^{\odot q}$, and for $f,g\in|\mathfrak{H}_d^{q,s}|$ we can write (2.2) as

$$f\otimes_p g=\alpha_H^p\sum_{k=1}^d\int_{[0,T]^{2p}}f\left((\eta,k),(\mathbf{t}_1,\mathbf{i}_1)\right)g\left((\theta,k),(\mathbf{t}_2,\mathbf{i}_2)\right)\prod_{j=1}^p|\eta_j-\theta_j|^{2H-2}d\eta\, d\theta, \tag{3.11}$$

where

$(\eta,k)=(\eta_1,k),\dots,(\eta_p,k)$; $(\theta,k)=(\theta_1,k),\dots,(\theta_p,k)$; $(\mathbf{t}_1,\mathbf{i}_1)=(t_1,i_1),\dots,(t_{q-p},i_{q-p})$; and $(\mathbf{t}_2,\mathbf{i}_2)=(t_{q-p+1},i_{q-p+1}),\dots,(t_{2(q-p)},i_{2(q-p)})$.

The pathwise stochastic integral with respect to fBm with $H>1/2$ has been studied extensively [1, 13, 27]. For our purposes, we will use the symmetric Stratonovich integral discussed by Russo and Vallois [34]:

Definition 3.6. For some $T > 0$, let $u = \{u_t, 0 \le t \le T\}$ be a stochastic process with integrable trajectories. The symmetric integral with respect to the fBm B is defined as

$$\int_0^t u_s dB_s = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t u_s \left(B_{(s+\varepsilon)\wedge t} - B_{(s-\varepsilon)\vee 0} \right) ds,$$

where the limit exists in probability.

This theorem was first proved in [1].

Theorem 3.7. *Let $u = \{u_t, t \ge 0\}$ be a stochastic process in $\mathbb{D}^{1,2}(\mathfrak{H}_1)$ such that, for some $T > 0$,*

$$\mathbb{E}\left[\int_0^T \int_0^T |u_t|\, |u_s|\, |t-s|^{2H-2}\, ds\, dt \right] < \infty;$$
$$\mathbb{E}\left[\int_{[0,T]^4} |D_t u_\theta|\, |D_s u_\eta|\, |t-s|^{2H-2}\, |\theta - \eta|^{2H-2} du\, dt\, d\theta\, d\eta \right] < \infty;$$
$$\text{and} \quad \int_0^T \int_0^T |D_s u_t|\, |t-s|^{2H-2}\, ds\, dt < \infty \text{ a.s.}$$

Then the limit of definition 2.1 exists in probability, and we have

$$\int_0^T u_t dB_t = \int_0^T u_t \delta B_t + \alpha_H \int_0^T \int_0^T D_s u_t |t-s|^{2H-2} ds\, dt,$$

where $\alpha_H = H(2H-1)$.

Chapter 4

CLT for a Midpoint Stochastic Integral

4.1 Introduction

In this chapter, we consider the Midpoint (type 1) integral of Proposition 3.5. Most of the material in this chapter is nearly identical to that published in [17]. The aim is to obtain a change-of-variable formula in distribution for a class of Gaussian stochastic processes $W = \{W_t, t \geq 0\}$ under certain conditions on the covariance function. The model for this generalized process is fBm with $H = 1/4$, but it will be shown that this is not the only suitable process. For the process and a suitable function f we study the behavior of the 'type 1' midpoint Riemann sum

$$S_n^{M1}(t) := \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} f'(W_{\frac{2j-1}{n}})(W_{\frac{2j}{n}} - W_{\frac{2j-2}{n}}).$$

The limit of this sum as n tends to infinity is the midpoint (type 1) integral, denoted by

$$(MP1)\int_0^t f'(W_s)dW_s.$$

We show that the couple of processes $\left\{\left(W_t, S_n^{M1}(t)\right), t \geq 0\right\}$ converges in distribution in the Skorohod space $(\mathbb{D}[0,\infty))^2$ to $\{(W_t, \Phi(t)), t \geq 0\}$, where

$$\Phi(t) = f(W_t) - f(W_0) - \frac{1}{2}\int_0^t f''(W_s)dB_s$$

and $B = \{B_t, t \geq 0\}$ is a Gaussian martingale independent of W with variance $\eta(t)$, depending on the covariance properties of W. This limit theorem can be reformulated by saying that the following Itô formula in distribution holds

$$f(W_t) \stackrel{\mathscr{L}}{=} f(W_0) + \int_0^t f'(W_s)d^{M1}W_s + \frac{1}{2}\int_0^t f''(W_s)dB_s. \tag{4.1}$$

The above mentioned convergence is proven by showing the stable convergence of a random vector $(S_n^{M1}(t_1), \ldots, S_n^{M1}(t_d))$ and a tightness argument. Convergence in law of the finite dimensional distributions follows from Theorem 2.3, once we verify that $S_n^{M1}(t)$ satisfies the conditions.

Recent papers by Swanson [37], Nourdin and Réveillac [25], and Burdzy and Swanson [8] presented results comparable to (4.1) for a specific stochastic process. In [37], a change-of-variable form was found for a process equivalent to the bifractional Brownian motion with parameters $H = K = 1/2$, arising as the solution to the one-dimensional stochastic heat equation with an additive space-time white noise. This result was proven mostly by martingale methods. In [8] and [25], the respective authors considered fractional Brownian motion with Hurst parameter 1/4. In [8], the authors covered integrands of the form $f(t, W_t)$, which can be applied to fBm on $[\varepsilon, \infty)$. The authors of [25] proved a change-of-variable formula that holds on $[0, \infty)$ in the sense of marginal distributions. The proof in [25] uses Malliavin calculus; several similar methods were used in the present chapter.

It happens that the conditions on the process W are satisfied by a bifractional Brownian motion with parameters $H \leq 1/2$, $HK = 1/4$. In this case $\eta(t) = Ct$ and the process B is a Brownian motion. This includes both cases studied in [25] and [37], and extends to a larger class of processes. For another example, we consider a class of centered Gaussian processes with twice-differentiable covariance function of the form

$$\mathbb{E}[W_r W_t] = r\phi\left(\frac{t}{r}\right), \quad t \geq r,$$

where ϕ is a bounded function on $[1, \infty)$ such that

$$\phi'(x) = \frac{\kappa}{\sqrt{x-1}} + \frac{\psi(x)}{\sqrt{x}},$$

and ψ is bounded, differentiable and $|\psi'(x)| \leq C(x-1)^{-\frac{1}{2}}$. This class of Gaussian processes includes the process arising as the limit of the median of a system of independent Brownian motions studied by Swanson in [38]. For this process,

$$\phi(x) = \sqrt{x}\ \arctan\left(\frac{1}{\sqrt{x-1}}\right).$$

It is surprising to remark that in this case $\eta(t) = Ct^2$. This is related to the fact that the variance of the increments of W on the interval $[t-s, t]$ behaves as $C\sqrt{s}$, when s is small, although the variance of $W(t)$ behaves as Ct. Our third example is another Gaussian process studied by Swanson in [39]. This process also arises from the empirical quantiles of a system of independent Brownian motions. Let $B = \{B(t), t \geq 0\}$ be a Brownian motion, where $B(0)$ is a random variable with density $f \in \mathscr{C}^\infty$. Given certain growth conditions on f, Swanson proves there is a Gaussian process $F = \{F(t), t \geq 0\}$ with covariance given by

$$\mathbb{E}[F(r)F(t)] = \rho(r,t) = \frac{\mathbb{P}(B(r) \leq q(r), B(t) \leq q(t)) - \alpha^2}{u(q(r), r)\ u(q(t), t)},$$

where $\alpha \in (0,1)$ and $q(t)$ are defined by $\mathbb{P}(B(t) \leq q(t)) = \alpha$. It is shown that this family of processes satisfies the required conditions, where $\eta(t)$ is determined by f and α.

4.2 Central limit theorem for the Midpoint (type 1) integral

Suppose that $W = \{W_t, t \geq 0\}$ is a centered Gaussian process, as in Chapter 2, that meets conditions (M.1) through (M.5), below, for any $T > 0$, where the constants C_i may depend on T.

(M.1) For any $0 < s \leq t \leq T$, there is a constant C_1 such that

$$\mathbb{E}\left[(W_t - W_{t-s})^2\right] \leq C_1 s^{\frac{1}{2}}.$$

(M.2) For any $s > 0$ and $2s \leq r,t \leq T$ with $|t-r| \geq 2s$,

$$|\mathbb{E}\left[(W_t - W_{t-s})(W_r - W_{r-s})\right]| \leq C_1 s^2 |t-r|^{-\alpha}(t \wedge r - s)^{-\beta} + s^2|t-r|^{-\frac{3}{2}};$$

for positive constants α, β, γ, such that $1 < \alpha \leq \frac{3}{2}$ and $\alpha + \beta = \frac{3}{2}$.

(M.3) For $0 < t \leq T$ and $0 < s \leq r \leq T$,

$$|\mathbb{E}\left[W_t(W_{r+s} - 2W_r + W_{r-s})\right]| \leq \begin{cases} C_2 s^{\frac{1}{2}} & if\ r < 2s\ or\ |t-r| < 2s \\ C_2 s^2\left((r-s)^{-\frac{3}{2}} + |t-r|^{-\frac{3}{2}}\right) & if\ r \geq 2s\ and\ |t-r| \geq 2s \end{cases}$$

for some positive constant C_2.

(M.4) For any $0 < s \leq t \leq T - s$

$$|\mathbb{E}\left[W_t(W_{t+s} - W_{t-s})\right]| \leq \begin{cases} C_3 s^{\frac{1}{2}} & if\ t < 2s \\ C_3 s(t-s)^{-\frac{1}{2}} & if\ t \geq 2s \end{cases}$$

and for each $0 < s \leq r \leq T$,

$$|\mathbb{E}\left[W_r(W_{t+s} - W_{t-s})\right]| \leq \begin{cases} C_3 s^{\frac{1}{2}} & if\ t < 2s\ or\ |t-r| < 2s \\ C_3 s(t-s)^{-\frac{1}{2}} + C_3 s|t-r|^{-\frac{1}{2}} & if\ t \geq 2s\ and\ |t-r| \geq 2s \end{cases}$$

for some positive constant C_3. In addition, for $t > 2s$,

$$|\mathbb{E}\left[W_s(W_t - W_{t-s})\right]| \leq C_3 s^{\frac{1}{2}+\gamma}(t-2s)^{-\gamma}$$

for some $\gamma > 0$.

(M.5) Consider a uniform partition of $[0,\infty)$ with increment length $1/n$. Define for integers $j,k \geq 0$ and $n \geq 1$:

$$\beta_n(j,k) = \mathbb{E}\left[\left(W_{\frac{j+1}{n}} - W_{\frac{j}{n}}\right)\left(W_{\frac{k+1}{n}} - W_{\frac{k}{n}}\right)\right].$$

Next, define

$$\eta_n^+(t) = \sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \beta_n(2j-1, 2k-1)^2 + \beta_n(2j-2, 2k-2)^2;$$

$$\eta_n^-(t) = \sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \beta_n(2j-2,2k-1)^2 + \beta_n(2j-1,2k-2)^2.$$

Then for each $t \geq 0$,

$$\lim_{n\to\infty} \eta_n^+(t) = \eta^+(t) \quad \text{and} \quad \lim_{n\to\infty} \eta_n^-(t) = \eta^-(t)$$

both exist, where $\eta^+(t), \eta^-(t)$ are nonnegative and nondecreasing functions.

Consider a real-valued function $f \in \mathscr{C}^9(\mathbb{R})$, such that f and all its derivatives up to order 9 satisfy moderate growth conditions, as defined in Section 2.1. We will refer to this as Condition (M.0).

In the following, the term C represents a generic positive constant, which may change from line to line. The constant C may depend on T and the constants in conditions (M.0) - (M.5) listed above.

The results of the next lemma follow from conditions (M.1) and (M.2).

Lemma 4.1. *Using the notation described above, for integers $0 \leq a < b$ and integers $r,n \geq 1$, we have the estimate,*

$$\sum_{j,k=a}^{b} |\beta_n(j,k)|^r \leq C(b-a+1)n^{-\frac{r}{2}}.$$

Proof. Suppose first that $r=1$. Let $I = \{(j,k) : a \leq j,k \leq b, |k-j| \geq 2, j \wedge k \geq 2\}$, and $J = \{(j,k) : a \leq j,k \leq b, (j,k) \notin I\}$. Consider the decomposition

$$\sum_{j,k=a}^{b} |\beta_n(j,k)| = \sum_{(j,k)\in I} |\beta_n(j,k)| + \sum_{(j,k)\in J} |\beta_n(j,k)|.$$

Then by condition (M.2), the first sum is bounded by

$$\sum_{(j,k)\in I} n^{-\frac{1}{2}} |j-k|^{-\alpha} \leq Cn^{-\frac{1}{2}}(b-a+1),$$

and the second sum, using condition (M.1) and Cauchy-Schwarz, is bounded by $Cn^{-\frac{1}{2}}(b-a+1)$. For the case $r>1$, condition (M.1) implies $|\beta_n(j,k)| \leq C_1 n^{-\frac{1}{2}}$ for all j,k. It follows that we can write,

$$\sum_{j,k=a}^{b} |\beta_n(j,k)|^r \leq C_1 n^{-\frac{r-1}{2}} \sum_{j,k=a}^{b} |\beta_n(j,k)| \leq C(b-a+1)n^{-\frac{r}{2}}.$$

□

Corollary 4.2. *Using the notation of Lemma 4.1, for each integer $r \geq 1$,*

$$\sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \left(|\beta_n(2j-1,2k-1)|^r + |\beta_n(2j-1,2k-2)|^r + |\beta_n(2j-2,2k-1)|^r + |\beta_n(2j-2,2k-2)|^r\right)$$
$$\leq C \left\lfloor \frac{nt}{2} \right\rfloor n^{-\frac{r}{2}}.$$

Proof. Note that

$$\sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \left(|\beta_n(2j-1,2k-1)|^r + |\beta_n(2j-1,2k-2)|^r + |\beta_n(2j-2,2k-1)|^r + |\beta_n(2j-2,2k-2)|^r\right)$$
$$= \sum_{j,k=0}^{2\lfloor \frac{nt}{2} \rfloor -1} |\beta_n(j,k)|^r .$$

□

Consider a uniform partition of $[0,\infty)$ with increment length $1/n$. The Stratonovich midpoint integral of $f'(W)$ will be defined as the limit in distribution of the sequence (see [37]):

$$S_n^{M1}(t) := \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} f'(W_{\frac{2j-1}{n}})(W_{\frac{2j}{n}} - W_{\frac{2j-2}{n}}). \tag{4.2}$$

Recall the notation of Section 2.4: $\varepsilon_t := \mathbf{1}_{[0,t]}$; and $\partial_{\frac{j}{n}} := \mathbf{1}_{\left[\frac{j}{n},\frac{j+1}{n}\right]}$.

The following is the major result of this section.

Theorem 4.3. *Let f be a real function satisfying condition (M.0), and let $W = \{W_t, t \geq 0\}$ be a Gaussian process satisfying conditions (M.1) through (M.5). Then:*

$$\left(W_t, S_n^{M1}(t)\right) \xrightarrow{\mathscr{L}} \left(W_t, f(W_t) - f(W_0) - \frac{1}{2}\int_0^t f''(W_s)\, dB_s\right)$$

as $n \to \infty$ in the Skorohod space $(\mathbb{D}[0,\infty))^2$, where $\eta(t) = \eta^+(t) - \eta^-(t)$ for the functions defined in condition (v); and $B = \{B_t, t \geq 0\}$ is scaled Brownian motion, independent of W, and with variance $\mathbb{E}\left[B_t^2\right] = \eta(t)$.

The rest of this section consists of the proof of Theorem 4.3, and is presented in a series of lemmas. The proofs of Lemmas 4.4, 4.5, and 4.9, which are rather technical, are deferred to Section 4.4. We begin with an expansion of $f(W_t)$, following the methodology used in [37]. Consider the telescoping series

$$f(W_t) = f(W_0) + \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} \left[f(W_{\frac{2j}{n}}) - f(W_{\frac{2j-2}{n}})\right] + f(W_t) - f(W_{\frac{2}{n}\lfloor \frac{nt}{2} \rfloor}),$$

where the sum is zero by convention if $\lfloor \frac{nt}{2} \rfloor = 0$. Using a Taylor series expansion of order 2, we obtain

$$\Phi_n(t) = f(W_t) - f(W_0) - \frac{1}{2}\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} f''(W_{\frac{2j-1}{n}})\left(\Delta W_{\frac{2j}{n}}^2 - \Delta W_{\frac{2j-1}{n}}^2\right)$$
$$- \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} R_0(W_{\frac{2j}{n}}) + \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} R_1(W_{\frac{2j-2}{n}}) - \left(f(W_t) - f(W_{\frac{2}{n}\lfloor \frac{nt}{2} \rfloor})\right),$$

where R_0, R_1 represent the third-order remainder terms in the Taylor expansion, and can be expressed in integral form as:

$$R_0(W_{\frac{2j}{n}}) = \frac{1}{2}\int_{W_{\frac{2j-1}{n}}}^{W_{\frac{2j}{n}}} (W_{\frac{2j}{n}} - u)^2 f^{(3)}(u)\,du; \text{ and} \tag{4.3}$$

$$R_1(W_{\frac{2j-2}{n}}) = -\frac{1}{2}\int_{W_{\frac{2j-2}{n}}}^{W_{\frac{2j-1}{n}}} (W_{\frac{2j-2}{n}} - u)^2 f^{(3)}(u)\,du. \tag{4.4}$$

By condition (0) we have for any $T > 0$ that

$$\lim_{n\to\infty} \mathbb{E} \sup_{0\le t\le T} \left| f(W_t) - f(W_{\frac{2}{n}\lfloor\frac{nt}{2}\rfloor}) \right| = 0,$$

so this term vanishes uniformly on compacts in probability (ucp), and may be neglected. Therefore, it is sufficient to work with the term

$$\Delta_n(t) := f(W_t) - f(W_0) - \frac{1}{2}\Psi_n(t) + R_n(t), \tag{4.5}$$

where

$$\Psi_n(t) = \sum_{j=1}^{\lfloor\frac{nt}{2}\rfloor} f''(W_{\frac{2j-1}{n}}) \left(\Delta W^2_{\frac{2j}{n}} - \Delta W^2_{\frac{2j-1}{n}} \right); \text{ and}$$

$$R_n(t) = \sum_{j=1}^{\lfloor\frac{nt}{2}\rfloor} \left(R_1(W_{\frac{2j-2}{n}}) - R_0(W_{\frac{2j}{n}}) \right).$$

We will first decompose the term $\Psi_n(t)$, using a Skorohod integral representation. Using (2.4) and the second Hermite polynomial, one can write $\Delta W^2(h) = 2H_2(W(h)) + 1 = \delta^2(h^{\otimes 2}) + 1$ for any $h \in \mathfrak{H}$ with $\|h\|_{\mathfrak{H}} = 1$. It follows that,

$$\Psi_n(t) = \sum_{j=1}^{\lfloor\frac{nt}{2}\rfloor} f''(W_{\frac{2j-1}{n}})\delta^2 \left(\partial^{\otimes 2}_{\frac{2j-1}{n}} - \partial^{\otimes 2}_{\frac{2j-2}{n}} \right).$$

From Lemma 2.1.a, we have for random variables u, F

$$F\delta^2(u) = \delta^2(Fu) + 2\delta\left(\langle DF, u\rangle_{\mathfrak{H}}\right) + \left\langle D^2F, u\right\rangle_{\mathfrak{H}^{\otimes 2}},$$

so we can write:

$$\begin{aligned}
\Psi_n(t) = &\sum_{j=1}^{\lfloor\frac{nt}{2}\rfloor} \delta^2 \left(f''(W_{\frac{2j-1}{n}}) \left(\partial^{\otimes 2}_{\frac{2j-1}{n}} - \partial^{\otimes 2}_{\frac{2j-2}{n}} \right) \right) \\
&+ \sum_{j=1}^{\lfloor\frac{nt}{2}\rfloor} 2\delta \left(f^{(3)}(W_{\frac{2j-1}{n}}) \left\langle \varepsilon_{\frac{2j-1}{n}}, \partial^{\otimes 2}_{\frac{2j-1}{n}} - \partial^{\otimes 2}_{\frac{2j-2}{n}} \right\rangle_{\mathfrak{H}} \right) \\
&+ \sum_{j=1}^{\lfloor\frac{nt}{2}\rfloor} f^{(4)}(W_{\frac{2j-1}{n}}) \left(\left\langle \varepsilon_{\frac{2j-1}{n}}, \partial_{\frac{2j-1}{n}} \right\rangle^2_{\mathfrak{H}} - \left\langle \varepsilon_{\frac{2j-1}{n}}, \partial_{\frac{2j-2}{n}} \right\rangle^2_{\mathfrak{H}} \right)
\end{aligned}$$

$$:= F_n(t) + B_n(t) + C_n(t).$$

Hence, we have $\Delta_n(t) = f(W_t) - f(W_0) - \frac{1}{2}(F_n(t) + B_n(t) + C_n(t)) + R_n(t)$. In the next two lemmas, we show that the terms $B_n(t), C_n(t)$, and $R_n(t)$ converge to zero in probability as $n \to \infty$. The proofs of these lemmas are deferred to Section 4.4.

Lemma 4.4. *Let $0 \le r < t \le T$. Using the notation defined above,*

$$\mathbb{E}\left[(R_n(t) - R_n(r))^2\right] \le C\left(\left\lfloor \frac{nt}{2} \right\rfloor - \left\lfloor \frac{nr}{2} \right\rfloor\right) n^{-\frac{3}{2}}$$

for some positive constant C, which may depend on T. It follows that for any $0 \le t \le T$, $R_n(t)$ converges to zero in probability as $n \to \infty$.

Lemma 4.5. *Let $0 \le r < t \le T$. Using the above notation, there exist constants C_B, C_C such that*

$$\mathbb{E}\left[(B_n(t) - B_n(r))^2\right] \le C_B\left(\left\lfloor \frac{nt}{2} \right\rfloor - \left\lfloor \frac{nr}{2} \right\rfloor\right) n^{-\frac{3}{2}}; \text{ and}$$
$$\mathbb{E}\left[(C_n(t) - C_n(r))^2\right] \le C_C\left(\left\lfloor \frac{nt}{2} \right\rfloor - \left\lfloor \frac{nr}{2} \right\rfloor\right) n^{-\frac{3}{2}}.$$

It follows that for any $0 \le t \le T$, $B_n(t)$ and $C_n(t)$ converge to zero in probability as $n \to \infty$.

Corollary 4.6. *Let $Z_n(t) := R_n(t) - \frac{1}{2}B_n(t) - \frac{1}{2}C_n(t)$. Then given $0 \le t_1 < t < t_2 \le T$, there exists a positive constant C such that*

$$\mathbb{E}\left[|Z_n(t) - Z_n(t_1)|\ |Z_n(t_2) - Z_n(t)|\right] \le C(t_2 - t_1)^{\frac{3}{2}}.$$

Proof. By Lemmas 4.4 and 4.5,

$$\begin{aligned}
\mathbb{E}\left[(Z_n(t_2) - Z_n(t_1))^2\right] &\le 3\mathbb{E}\left[(R_n(t_2) - R_n(t_1))^2\right] + 2\mathbb{E}\left[(B_n(t_2) - B_n(t_1))^2\right] \\
&\quad + 2\mathbb{E}\left[(C_n(t_2) - C_n(t_1))^2\right] \\
&\le C\left(\left\lfloor \frac{nt_2}{2} \right\rfloor - \left\lfloor \frac{nt_1}{2} \right\rfloor\right) n^{-\frac{3}{2}}.
\end{aligned}$$

Then by Cauchy-Schwarz inequality,

$$\begin{aligned}
\mathbb{E}\left[|Z_n(t) - Z_n(t_1)|\ |Z_n(t_2) - Z_n(t)|\right] &\le \left(\mathbb{E}\left[(Z_n(t) - Z_n(t_1))^2\right]\ \mathbb{E}\left[(Z_n(t) - Z_n(t_1))^2\right]\right)^{\frac{1}{2}} \\
&\le C\left(\left\lfloor \frac{nt_2}{2} \right\rfloor - \left\lfloor \frac{nt_1}{2} \right\rfloor\right)^{\frac{3}{2}} n^{-\frac{3}{2}}.
\end{aligned}$$

This estimate implies the required bound $C(t_2 - t_1)^{\frac{3}{2}}$, see, for example [6], p. 156. □

Next, we will develop a comparable estimate for differences of the form $F_n(t) - F_n(r)$. In order to prove this estimate, we need a technical lemma which will be used here and also in Section 4.4.

Lemma 4.7. *Suppose a,b are nonnegative integers such that $a+b\le 9$. For fixed $T>0$ and interval $[t_1,t_2]\subset[0,T]$, let*

$$g_a=\sum_{\ell=\lfloor\frac{nt_1}{2}\rfloor+1}^{\lfloor\frac{nt_2}{2}\rfloor} f^{(a)}(W_{\frac{2\ell-1}{n}})\left(\partial_{\frac{2\ell-1}{n}}^{\otimes 2}-\partial_{\frac{2\ell-2}{n}}^{\otimes 2}\right).$$

Then we have for $1\le p<\infty$

$$\mathbb{E}\left[\|D^b g_a\|_{\mathfrak{H}^{\otimes 2+b}}^p\right]\le C\left(\left\lfloor\frac{nt_2}{2}\right\rfloor-\left\lfloor\frac{nt_1}{2}\right\rfloor\right)^{\frac{p}{2}} n^{-\frac{p}{2}}.$$

Proof. We may assume $t_1=0$ with $t_2\le T$. For each b we can write

$$\begin{aligned}
&\mathbb{E}\left[\left(\|D^b g_a\|_{\mathfrak{H}^{\otimes 2+b}}^2\right)^{\frac{p}{2}}\right]\\
&=\mathbb{E}\left[\left(\sum_{\ell,m=1}^{\lfloor\frac{nt_2}{2}\rfloor} f^{(a+b)}(W_{\frac{2\ell-1}{n}})f^{(a+b)}(W_{\frac{2m-1}{n}})\left\langle\varepsilon_{\frac{2\ell-1}{n}}^{\otimes b},\varepsilon_{\frac{2m-1}{n}}^{\otimes b}\right\rangle_{\mathfrak{H}^{\otimes b}}\left\langle\partial_{\frac{2\ell-1}{n}}^{\otimes 2}-\partial_{\frac{2\ell-2}{n}}^{\otimes 2},\partial_{\frac{2m-1}{n}}^{\otimes 2}-\partial_{\frac{2m-2}{n}}^{\otimes 2}\right\rangle_{\mathfrak{H}^{\otimes 2}}\right)^{\frac{p}{2}}\right]\\
&\le\mathbb{E}\left[\sup_{0\le s\le t}\left|f^{(a+b)}(W_s)\right|^p\right]\left(\sup_{\ell,m}\left|\left\langle\varepsilon_{\frac{2\ell-1}{n}},\varepsilon_{\frac{2m-1}{n}}\right\rangle_{\mathfrak{H}}\right|^b\right)^{\frac{p}{2}}\left(\sum_{\ell,m=1}^{\lfloor\frac{nt_2}{2}\rfloor}\left|\left\langle\partial_{\frac{2\ell-1}{n}}^{\otimes 2}-\partial_{\frac{2\ell-2}{n}}^{\otimes 2},\partial_{\frac{2m-1}{n}}^{\otimes 2}-\partial_{\frac{2m-2}{n}}^{\otimes 2}\right\rangle_{\mathfrak{H}^{\otimes 2}}\right|\right)^{\frac{p}{2}}.
\end{aligned}$$

Recall that (M.0) covers f and its first 9 derivatives, so the first two terms are bounded. For the last term, note that by Corollary 4.2 with $r=2$,

$$\begin{aligned}
&\sum_{\ell,m=1}^{\lfloor\frac{nt_2}{2}\rfloor}\left|\left\langle\partial_{\frac{2\ell-1}{n}}^{\otimes 2}-\partial_{\frac{2\ell-2}{n}}^{\otimes 2},\partial_{\frac{2m-1}{n}}^{\otimes 2}-\partial_{\frac{2m-2}{n}}^{\otimes 2}\right\rangle_{\mathfrak{H}^{\otimes 2}}\right|\\
&\quad=\sum_{\ell,m=1}^{\lfloor\frac{nt_2}{2}\rfloor}\left|\beta_n(2\ell-1,2m-1)^2-\beta_n(2\ell-1,2m-2)^2-\beta_n(2\ell-2,2m-1)^2+\beta_n(2\ell-2,2m-2)^2\right|\\
&\quad\le C\left\lfloor\frac{nt_2}{2}\right\rfloor n^{-1}.
\end{aligned}$$

□

Lemma 4.8. *For $0\le s<t\le T$, write*

$$F_n(t)-F_n(s)=\sum_{j=\lfloor\frac{ns}{2}\rfloor+1}^{\lfloor\frac{nt}{2}\rfloor}\delta^2\left(f''(W_{\frac{2j-1}{n}})\left(\partial_{\frac{2j-1}{2}}^{\otimes 2}-\partial_{\frac{2j-2}{n}}^{\otimes 2}\right)\right)$$

Then given $0\le t_1<t<t_2\le T$, there exists a positive constant C such that

$$\mathbb{E}\left[|F_n(t)-F_n(t_1)|^2|F_n(t_2)-F_n(t)|^2\right]\le C(t_2-t_1)^2. \tag{4.6}$$

Proof. First, for each $n \geq 1$, we want to show that there is a C such that,

$$\mathbb{E}\left[(F_n(t_2) - F_n(t_1))^4\right] \leq C\left(\left\lfloor \frac{nt_2}{2} \right\rfloor - \left\lfloor \frac{nt_1}{2} \right\rfloor\right)^2 n^{-2}.$$

By the Meyer inequality (6.4) there exists a constant $c_{2,4}$ such that

$$\mathbb{E}\left|\left(\delta^2(u_n)\right)^4\right| \leq c_{2,4}\|u_n\|^4_{\mathbb{D}^{2,4}(\mathfrak{H}^{\otimes 2})},$$

where in this case,

$$u_n = \sum_{j=\lfloor \frac{nt_1}{2} \rfloor+1}^{\lfloor \frac{nt_2}{2} \rfloor} f''(W_{\frac{2j-1}{n}})\left(\partial^{\otimes 2}_{\frac{2j-1}{n}} - \partial^{\otimes 2}_{\frac{2j-2}{n}}\right)$$

and

$$\|u_n\|^4_{\mathbb{D}^{2,4}(\mathfrak{H}^{\otimes 2})} = \mathbb{E}\|u_n\|^4_{\mathfrak{H}^{\otimes 2}} + \mathbb{E}\|Du_n\|^4_{\mathfrak{H}^{\otimes 3}} + \mathbb{E}\|D^2u_n\|^4_{\mathfrak{H}^{\otimes 4}}.$$

From Lemma 4.7 we have $\mathbb{E}\|u_n\|^4_{\mathfrak{H}^{\otimes 2}}, \mathbb{E}\|Du_n\|^4_{\mathfrak{H}^{\otimes 3}}, \mathbb{E}\|D^2u_n\|^4_{\mathfrak{H}^{\otimes 4}} \leq C\left(\lfloor \frac{nt_2}{2} \rfloor - \lfloor \frac{nt_1}{2} \rfloor\right)^2 n^{-2}$, and so it follows that,

$$\mathbb{E}\left[\left(\delta^2(u_n)\right)^4\right] \leq C\left(\left\lfloor \frac{nt_2}{2} \right\rfloor - \left\lfloor \frac{nt_1}{2} \right\rfloor\right)^2 n^{-2}.$$

From this result, given $0 \leq t_1 < t < t_2$, it follows from the Hölder inequality that

$$\begin{aligned}\mathbb{E}\left[|F_n(t) - F_n(t_1)|^2|F_n(t_2) - F_n(t)|^2\right] &\leq \left(\mathbb{E}\left[|F_n(t) - F_n(t_1)|^4\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[|F_n(t_2) - F_n(t)|^4\right]\right)^{\frac{1}{2}} \\ &\leq C\left(\left\lfloor \frac{nt_2}{2} \right\rfloor - \left\lfloor \frac{nt_1}{2} \right\rfloor\right)^2 n^{-2}.\end{aligned}$$

As in Corollary 4.6, this implies the required bound $C(t_2 - t_1)^2$. □

By Corollary 4.6 and Lemma 4.8, it follows that $\Delta_n(t) = f(W_t) - f(W_0) - \frac{1}{2}F_n(t) + Z_n(t)$ is tight, since both sequential parts $F_n(t), Z_n(t)$ are tight. Further, we have that $Z_n(t)$ tends to zero in probability, and $F_n(t)$ is in a form suitable for Theorem 2.3. In the next lemma, we show that the conditions of Theorem 2.3 are satisfied by $F_n(t)$ evaluated at a finite set of points.

Lemma 4.9. *Fix $0 = t_0 < t_1 < t_2 < \cdots < t_d$. Set $F_n^i = F_n(t_i) - F_n(t_{i-1})$ for $i = 1, \ldots d$, and let $F_n = (F_n^i, \ldots, F_n^d)$. Then under conditions (M.0) - (M.5), F_n satisfies conditions (a) and (b) of Theorem 2.3, and so given W, F_n converges stably as $n \to \infty$ to a random variable $\xi = (\xi_1, \ldots, \xi_d)$ with distribution $\mathcal{N}(0, \Sigma)$, where Σ is a diagonal $d \times d$ matrix with entries:*

$$s_i^2 = \int_{t_{i-1}}^{t_i} f''(W_s)^2 \eta(ds),$$

where $\eta(t) = \eta^+(n) - \eta^-(t)$ is as defined in condition (v).

Remark 4.10. As we will see later, $\eta(t)$ is continuous, nonnegative, and nondecreasing.

It follows from the structure of Σ that, given W, F_n converges stably to a d-dimensional vector with conditionally independent components of the form

$$F_\infty^i = \zeta_i \sqrt{\int_{t_{i-1}}^{t_i} f''(W_s)^2 \eta(ds)},$$

where each $\zeta_i \sim \mathcal{N}(0,1)$. Thus, we may conclude that for each i,

$$F_n^i \xrightarrow{\mathscr{L}} \int_{t_{i-1}}^{t_i} f''(W_s)\, dB_s$$

for a scaled Brownian motion $B = \{B_t, t \geq 0\}$ that is independent of W_t, with $\mathbb{E}\left[B_t^2\right] = \eta(t)$.

Proof of Theorem 4.3 It is enough to show that for any finite set of times $0 = t_0 < t_1 < t_2 < \cdots < t_d$ we have

$$(\Delta_n(t_1), \Delta_n(t_2) - \Delta_n(t_1), \ldots, \Delta_n(t_d) - \Delta_n(t_{d-1})) \xrightarrow{\mathscr{L}} (\Delta(t_1), \Delta(t_2) - \Delta(t_1), \ldots, \Delta(t_d) - \Delta(t_{d-1}))$$

as $n \to \infty$; and that $\Delta_n(t)$ satisfies the tightness condition

$$\mathbb{E}\left[|\Delta_n(t) - \Delta_n(t_1)|^\gamma \, |\Delta_n(t_2) - \Delta_n(t)|^\gamma\right] \leq C(t_2 - t_1)^\alpha \tag{4.7}$$

for $0 \leq t_1 < t < t_2 < \infty$, $\gamma > 0$, and $\alpha > 1$.

For $\Delta_n(t) = f(W_t) - f(W_0) - \frac{1}{2}F_n(t) + Z_n(t)$, we have shown in Lemmas 4.4 and 4.5 that

$$Z_n(t) = R_n(t) - \frac{1}{2}\left(B_n(t) + C_n(t)\right) \xrightarrow{\mathscr{P}} 0$$

for each $0 \leq t \leq T$, and hence $Z_n(t_i) - Z_n(t_{i-1}) \xrightarrow{\mathscr{P}} 0$ for each t_i, $1 \leq i \leq d$. By Lemma 4.9, the pair (W, F_n) converges in law to (W, F_∞), where F_∞ is a $d-$dimensional random vector with conditional Gaussian law and whose covariance matrix is diagonal with entries

$$s_i^2 = \int_{t_{i-1}}^{t_i} f''(W_s)^2 \eta(ds).$$

It follows that, conditioned on W, each component may be expressed as an independent Gaussian random variable, equivalent in law to

$$\int_{t_{i-1}}^{t_i} f''(W_s)\, dB_s,$$

where $B = \{B_t, t \geq 0\}$ is a scaled Brownian motion independent of W with $\mathbb{E}\left[B_t^2\right] = \eta(t)$. Finally, tightness follows from Lemma 4.8 and Corollary 4.6. Theorem 4.3 is proved. □

4.3 Examples

4.3.1 Bifractional Brownian Motion

The bifractional Brownian motion is a generalization of fractional Brownian motion, first introduced by Houdré and Villa [20]. It is defined as a centered Gaussian process $B^{H,K} = \{B^{H,K}(t), t \geq 0\}$,with covariance defined by,

$$\mathbb{E}[B_t^{H,K} B_s^{H,K}] = \frac{1}{2^K}\left(t^{2H} + s^{2H}\right)^K + \frac{1}{2^K}|t-s|^{2HK},$$

where $H \in (0,1)$, $K \in (0,1]$ (Note that the case $K = 1$ corresponds to fractional Brownian motion with Hurst parameter H). The reader may refer to [21] and its references for further discussion of properties.

In this section, we show that the results of Section 4.2 are valid for bifractional Brownian motion with parameter values H, K such that $H \leq 1/2$ and $2HK = 1/2$. In particular, this includes the end point cases $H = 1/4$, $K = 1$ studied in [25], and $H = 1/2$, $K = 1/2$ studied in [37].

Proposition 4.11. *Let* $\left\{B_t^{H,K}, t \geq 0\right\}$ *denote a bifractional Brownian motion. The covariance conditions (M.1) - (M.4) of Section 4 are satisfied for values of* $0 < H \leq 1/2$ *and* $0 < K \leq 1$ *such that* $2HK = 1/2$.

Proof. Condition (M.1).

$$\begin{aligned}
\mathbb{E}\left[\left(B_t^{H,K} - B_{t-s}^{H,K}\right)^2\right] &= t^{2HK} + \frac{2}{2^K}(t-s)^{2HK} - \left[t^{2H} + (t-s)^{2H}\right]^K - \frac{2}{2^K}s^{2HK} \\
&\leq \left[\left|\sqrt{t} - \frac{1}{2^K}\left(t^{2H} + (t-s)^{2H}\right)^K\right| + \left|\sqrt{t-s} - \frac{1}{2^K}\left(t^{2H} + (t-s)^{2H}\right)^K\right| + \frac{1}{2^K}s^{\frac{1}{2}}\right] \\
&\leq Cs^{\frac{1}{2}},
\end{aligned}$$

where we used the inequality $a^m - b^m \leq (a-b)^m$ for $a > b > 0$ and $m < 1$.

Condition (M.2).

$$\begin{aligned}
&\mathbb{E}\left[(B_t^{H,K} - B_{t-s}^{H,K})(B_r^{H,K} - B_{r-s}^{H,K})\right] \\
&= \frac{1}{2^K}\left(\left[t^{2H} + r^{2H}\right]^K - \left[t^{2H} + (r-s)^{2H}\right]^K - \left[(t-s)^{2H} + r^{2H}\right]^K + \left[(t-s)^{2H} + (r-s)^{2H}\right]^K\right) \\
&\quad + \frac{1}{2^K}\left(|t-r+s|^{2HK} - 2|t-r|^{2HK} + |t-r-s|^{2HK}\right).
\end{aligned}$$

This can be interpreted as the sum of a position term, $\frac{1}{2^K}\varphi(t,r,s)$, and a distance term, $\frac{1}{2^K}\psi(t-r,s)$, where

$$\varphi(t,r,s) = \left[t^{2H} + r^{2H}\right]^K - \left[t^{2H} + (r-s)^{2H}\right]^K - \left[(t-s)^{2H} + r^{2H}\right]^K + \left[(t-s)^{2H} + (r-s)^{2H}\right]^K; \text{ and}$$

$$\psi(t-r,s) = |t-r+s|^{2HK} - 2|t-r|^{2HK} + |t-r-s|^{2HK}.$$

We begin with the position term. Note that if $K=1$, then $\varphi(t,r,s)=0$, so we may assume $K<1$ and $H>\frac{1}{4}$. Assume $0<s\le r\le t$, and let $p:=t-r$. By Fundamental Theorem of Calculus, we can write $\varphi(t,t-p,s)$ as

$$\begin{aligned}
&2HK\int_0^s \left(\left[t^{2H}+(t-p-\xi)^{2H}\right]^{K-1}-\left[(t-s)^{2H}+(t-p-\xi)^{2H}\right]^{K-1}\right)(t-p-\xi)^{2H-1}d\xi\\
&=\int_0^s\int_0^s 4H^2K(1-K)\left[(t-\eta)^{2H}+(t-p-\xi)^{2H}\right]^{K-2}(t-\eta)^{2H-1}(t-p-\xi)^{2H-1}\,d\xi\,d\eta\\
&\le 4H^2K(1-K)s^2\left[(t-r)^{2H}+(r-s)^{2H}\right]^{K-2}(t-r)^{2H-1}(r-s)^{2H-1}\\
&\le Cs^2\,(t-r)^{2HK-2H-1}(r-s)^{2H-1}.
\end{aligned}$$

This implies condition (M.2) for the position term taking $\alpha=\frac{1}{2}+2H>1$ and $\beta=1-2H$.

Next, consider the distance term $\psi(t-r,s)$. Without loss of generality, assume $r<t$. Again using an integral representation, we have

$$\begin{aligned}
\psi(t-r,s)&=|t-r+s|^{2HK}-2|t-r|^{2HK}+|t-r-s|^{2HK}\\
&=\int_0^s 2HK\left[(t-r+\xi)^{2HK-1}-(t-r-\xi)^{2HK-1}\right]\,d\xi\\
&=\int_0^s\int_{-\xi}^{\xi} 2HK(2HK-1)\,[t-r+\eta]^{2HK-2}d\eta\,d\xi\\
&\le Cs^2(t-r-s)^{2HK-2}\le Cs^2|t-r|^{-\frac{3}{2}},
\end{aligned}$$

since $|t-r|\ge 2s$ implies $(t-r-s)^{-\frac{3}{2}}\le 2^{\frac{3}{2}}|t-r|^{-\frac{3}{2}}$.

Condition (M.3).

$$\begin{aligned}
&\left|\mathbb{E}\left[B_t^{H,K}(B_{r+s}^{H,K}-2B_r^{H,K}+B_{r-s}^{H,K})\right]\right|\\
&\quad=\frac{1}{2^K}|[t^{2H}+(r+s)^{2H}]^K-2[t^{2H}+r^{2H}]^K+[t^{2H}+(r-s)^{2H}]^K\\
&\qquad-\frac{1}{2^K}\left[|t-r+s|^{2HK}-2|t-r|^{2HK}+|t-r-s|^{2HK}\right]|.
\end{aligned}$$

Take first the term, $\varphi(t,r,s)$. If $r<2s$, then

$$|[t^{2H}+(r+s)^{2H}]^K-2[t^{2H}+r^{2H}]^K+[t^{2H}+(r-s)^{2H}]^K|\le Cs^{2HK}=Cs^{\frac{1}{2}},$$

based on the inequality $a^K-b^K\le(a-b)^K$ for $a>b>0$ and $K<1$. Hence, we will assume $r\ge 2s$. If $K=1$, then $H=\frac{1}{4}$, and we have

$$\begin{aligned}
\left|\sqrt{r+s}-2\sqrt{r}+\sqrt{r-s}\right|&=\left|\int_0^s\frac{1}{2\sqrt{r+x}}\,dx-\int_0^s\frac{1}{2\sqrt{r-s+x}}\,dx\right|\\
&=\frac{1}{4}\int_0^s\int_0^s\frac{1}{(r-s+x+y)^{\frac{3}{2}}}\,dy\,dx\\
&\le\frac{1}{4}s^2(r-s)^{-\frac{3}{2}};
\end{aligned}$$

and if $K<1$,

$$
\begin{aligned}
&|\varphi(t,r,s)|\\
&=\left|\int_0^s 2HK[t^{2H}+(r+x)^{2H}]^{K-1}(r+x)^{2H-1}dx-\int_0^s 2HK[t^{2H}+(r-s+x)^{2H}]^{K-1}(r-s+x)^{2H-1}dx\right|\\
&\quad\le\left|\int_0^s\int_0^s 4H^2K(K-1)[t^{2H}+(r-s+x+y)^{2H}]^{K-2}(r-s+x+y)^{4H-2}\,dy\,dx\right|\\
&\qquad+\left|\int_0^s\int_0^s 2H(2H-1)K[t^{2H}+(r-s+x+y)^{2H}]^{K-1}(r-s+x+y)^{2H-2}\,dy\,dx\right|\\
&\quad\le 4H^2K(1-K)s^2(r-s)^{2HK-2}+2H(1-2H)Ks^2(r-s)^{2HK-2}\le Cs^2(r-s)^{-\frac{3}{2}}.
\end{aligned}
$$

This bound for $\varphi(t,r,s)$ also holds in the case $|t-r|<2s$, so the bound of $Cs^{\frac{1}{2}}$ is valid for this case. Next for the second term. Note that if $|t-r|<2s$, then

$$
\left|\frac{1}{2^K}\left(|t-r+s|^{2HK}-2|t-r|^{2HK}+|t-r-s|^{2HK}\right)\right|\le 2(3s)^{2HK}\le Cs^{\frac{1}{2}}.
$$

If $|t-r|\ge 2s$, then we have

$$
\begin{aligned}
\left|\sqrt{|t-r|+s}-2\sqrt{|t-r|}+\sqrt{|t-r|-s}\right|&=\left|\int_0^s\frac{1}{2\sqrt{|t-r|+x}}dx-\int_0^s\frac{1}{2\sqrt{|t-r|-s+x}}dx\right|\\
&=\int_0^s\int_0^s\frac{1}{(|t-r|-s+x+y}\,dy\,dx\\
&\le\frac{s^2}{4(|t-r|-s)^{\frac{3}{2}}}\le\frac{s^2}{2|t-r|^{\frac{3}{2}}},
\end{aligned}
$$

using the inequality $\frac{1}{|t-r|-s}\le\frac{2}{|t-r|}$ for $|t-r|\ge 2s$. This bound for $\psi(t-r,s)$ holds even in the case $r<2s$, so the bound of $Cs^{\frac{1}{2}}$ when $r<2s$ is verified as well.

Condition (M.4).

For the first part, we have for all $t\ge s$,

$$
\left|\mathbb{E}\left[B_t^{H,K}\left(B_{t+s}^{H,K}-B_{t-s}^{H,K}\right)\right]\right|=\left|\frac{1}{2^K}\left[t^{2H}+(t+s)^{2H}\right]^K-\frac{1}{2^K}\left[t^{2H}+(t-s)^{2H}\right]^K\right|.
$$

This is bounded by $Cs^{\frac{1}{2}}$ if $t<2s$. On the other hand, if $t\ge 2s$,

$$
\begin{aligned}
\left|\frac{1}{2^K}\left[t^{2H}+(t+s)^{2H}\right]^K-\frac{1}{2^K}\left[t^{2H}+(t-s)^{2H}\right]^K\right|&=\left|\frac{1}{2^K}\int_{-s}^s 2HK\left[t^{2H}+(t+x)^{2H}\right]^{K-1}(t+x)^{2H-1}\,dx\right|\\
&\le Cs(t-s)^{2HK-1}\;=\;Cs(t-s)^{-\frac{1}{2}}.
\end{aligned}
$$

For $0<s\le r\le T$ with $t\ge 2s$ and $|t-r|\ge 2s$,

$$
\begin{aligned}
\left|\mathbb{E}\left[B_r^{H,K}\left(B_{t+s}^{H,K}-B_{t-s}^{H,K}\right)\right]\right|&\le\left|\frac{1}{2^K}\left[r^{2H}+(t+s)^{2H}\right]^K-\frac{1}{2^K}\left[r^{2H}+(t-s)^{2H}\right]^K\right|\\
&\quad+\left|\frac{1}{2^K}|r-t+s|^{2HK}-\frac{1}{2^K}|r-t-s|^{2HK}\right|\\
&\le Cs(t-s)^{-\frac{1}{2}}+Cs|r-t|^{-\frac{1}{2}}.
\end{aligned}
$$

If $t<2s$ or $|t-r|<2s$, then we have an upper bound of $Cs^{\frac{1}{2}}$ by condition (M.1) and Cauchy-Schwarz.

For the third bound, if $t>2s$,

$$\begin{aligned}\left|\mathbb{E}\left[B_s^{H,K}\left(B_t^{H,K}-B_{t-s}^{H,K}\right)\right]\right| &\le \left|\frac{1}{2^K}\left[s^{2H}+t^{2H}\right]^K-\frac{1}{2^K}\left[s^{2H}+(t-s)^{2H}\right]^K\right| \\ &\quad+\left|\frac{1}{2^K}(t-s)^{2HK}-\frac{1}{2^K}(t-2s)^{2HK}\right| \\ &\le \frac{2}{2^K}\int_0^s HK\left[s^{2H}+(t-s+x)^{2H}\right]^K(t-s+x)^{2H-1}\,dx \\ &\quad+\frac{1}{2^{K+1}}\int_0^s (t-2s+x)^{-\frac{1}{2}}dx \\ &\le Cs(t-2s)^{-\frac{1}{2}}=Cs^{\frac{1}{2}+\gamma}(t-2s)^{-\gamma}\end{aligned}$$

for $\gamma=\frac{1}{2}$.

□

Proposition 4.12. *Let $B^{H,K}$ be a bifractional Brownian motion with parameters $H\le 1/2$ and $HK=1/4$. Then Condition (M.5) holds, with the functions $\eta^+(t)=2C_K^+t$ and $\eta^-(t)=2C_K^-t$, where*

$$C_K^+=\frac{1}{4^K}\left(2+\sum_{m=1}^{\infty}\left(\sqrt{2m+1}-2\sqrt{2m}+\sqrt{2m-1}\right)^2\right),$$
$$C_K^-=\frac{(2-\sqrt{2})^2}{2^{2K+1}}+\frac{1}{4^K}\sum_{m=1}^{\infty}\left(\sqrt{2m+2}-2\sqrt{2m+1}+\sqrt{2m}\right)^2.$$

Proof. As in Proposition 4.11, we use the decomposition,

$$\beta_n(j,k)=\frac{1}{2^K}\varphi\left(\frac{j}{n},\frac{k}{n},\frac{1}{n}\right)+\frac{1}{2^K}\psi\left(\frac{j-k}{n},\frac{1}{n}\right)=2^{-K}n^{-\frac{1}{2}}\varphi(j,k,1)+2^{-K}n^{-\frac{1}{2}}\psi(j-k,1).$$

The first task is to show that

$$\lim_{n\to\infty}\sum_{j,k=1}^{\lfloor nt\rfloor} n^{-1}\varphi(j,k,1)^2 \;=\; 0. \tag{4.8}$$

Proof of (4.8). We consider two cases, based on the value of H. First, assume $H<\frac{1}{2}$. Then

$$\begin{aligned}\varphi(j,k,1)&=\left[(j+1)^{2H}+(k+1)^{2H}\right]^K-\left[(j+1)^{2H}+k^{2H}\right]^K \\ &\quad-\left[j^{2H}+(k+1)^{2H}\right]^K+\left[j^{2H}+k^{2H}\right]^K \\ &=\int_0^1 2HK\left[(j+1)^{2H}+(k+x)^{2H}\right]^{K-1}(k+x)^{2H-1}dx \\ &\quad-\int_0^1 2HK\left[j^{2H}+(k+x)^{2H}\right]^{K-1}(k+x)^{2H-1}dx \\ &=\int_0^1\int_0^1 4H^2K(1-K)\left[(j+y)^{2H}+(k+x)^{2H}\right]^{K-2}(k+x)^{2H-1}(j+y)^{2H-1}\,dy\,dx \\ &\le Ck^{2HK-2H-1}j^{2H-1}=Ck^{-\frac{1}{2}-2H}j^{2H-1}.\end{aligned}$$

With this bound, it follows that

$$\frac{1}{n}\sum_{j,k=1}^{\lfloor nt\rfloor}\varphi(j,k,1)^2 \le \frac{C}{n}\sum_{j=1}^{\lfloor nt\rfloor} j^{4H-2}\sum_{k=1}^{\infty}k^{-1-4H}$$
$$\le \frac{C}{n}\lfloor nt\rfloor^{4H-1} \le Ctn^{4H-2},$$

which tends to zero as $n\to\infty$ because $H<\frac{1}{2}$.

Next, the case $H=\frac{1}{2}$. Note that this implies $K=\frac{1}{2}$, and we have

$$|\varphi(j,k,1)| = \left|\sqrt{j+k+2}-2\sqrt{j+k+1}+\sqrt{j+k}\right| \le C(j+k)^{-\frac{3}{2}}.$$

So with this bound,

$$\sum_{j,k=1}^{\lfloor nt\rfloor} n^{-1}\varphi(j,k,1)^2 \le \frac{C}{n}\sum_{j,k=1}^{\lfloor nt\rfloor}(j+k)^{-3}$$
$$\le \frac{C}{n}\sum_{j=1}^{\lfloor nt\rfloor}\sum_{m=j+1}^{\infty}m^{-3} \le \frac{C}{n}\sum_{j=1}^{\lfloor nt\rfloor}j^{-2}$$

which tends to zero as $n\to\infty$ because j^{-2} is summable. Hence, (4.8) is proved.

From (4.8), it follows that to investigate the limit behavior of $\eta_n^+(t),\eta_n^-(t)$, it is enough to consider

$$\frac{1}{n}\sum_{j,k=1}^{\lfloor\frac{nt}{2}\rfloor}\psi(2j-2k,1)^2+\psi(2j-2k,1)^2 = \frac{2}{n}\sum_{j,k=1}^{\lfloor\frac{nt}{2}\rfloor}\psi(2j-2k,1)^2;\text{ and}$$
$$\frac{1}{n}\sum_{j,k=1}^{\lfloor\frac{nt}{2}\rfloor}\psi(2j-2k+1,1)^2+\psi(2j-2k-1,1)^2 = \frac{2}{n}\sum_{j,k=1}^{\lfloor\frac{nt}{2}\rfloor}\psi(2j-2k+1,1)^2;$$

since the sums of $\psi(2j-2k+1,1)^2$ and $\psi(2j-2k-1,1)^2$ are equal by symmetry. We start with

$$\begin{aligned}
&\frac{1}{n}\sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor}\psi(2j-2k,1)^2 \\
&= \frac{1}{4^K n}\sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor}\left(\sqrt{|2j-2k+1|}-2\sqrt{|2j-2k|}+\sqrt{|2j-2k-1|}\right)^2 \\
&= \frac{1}{4^K n}\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor}4+\frac{2}{4^K n}\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor}\sum_{k=1}^{j-1}\left(\sqrt{2j-2k+1}-2\sqrt{2j-2k}+\sqrt{2j-2k-1}\right)^2 \\
&= \frac{4\lfloor \frac{nt}{2} \rfloor}{4^K n}+\frac{2}{4^K n}\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor}\sum_{m=1}^{j-1}\left(\sqrt{2m+1}-2\sqrt{2m}+\sqrt{2m-1}\right)^2 \\
&= \frac{4\lfloor \frac{nt}{2} \rfloor}{4^K n}+\frac{2}{4^K n}\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor}\sum_{m=1}^{\infty}\left(\sqrt{2m+1}-2\sqrt{2m}+\sqrt{2m-1}\right)^2 \\
&\quad -\frac{2}{4^K n}\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor}\sum_{m=j}^{\infty}\left(\sqrt{2m+1}-2\sqrt{2m}+\sqrt{2m-1}\right)^2,
\end{aligned}$$

where the last term tends to zero since

$$\sum_{m=j}^{\infty}\left(\sqrt{2m+1}-2\sqrt{2m}+\sqrt{2m-1}\right)^2 \leq \sum_{m=j}^{\infty}(2m-1)^{-3} \leq C(2j-1)^{-2},$$

and,

$$\frac{C}{n}\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor}(2j-1)^{-2} \longrightarrow 0$$

as $n \to \infty$. We therefore conclude that,

$$\begin{aligned}
\eta^+(t) &= \lim_{n\to\infty}\sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor}\left(\beta_n(2j-1,2k-1)^2+\beta_n(2j-2,2k-2)^2\right) \\
&= \lim_{n\to\infty}\frac{2}{n}\sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor}\psi(2j-2k,1)^2 = 2C_K^+ t,
\end{aligned}$$

where

$$C_K^+ = \frac{1}{4^K}\left(2+\sum_{m=1}^{\infty}\left(\sqrt{2m+1}-2\sqrt{2m}+\sqrt{2m-1}\right)^2\right).$$

For the other term,

$$\frac{1}{n}\sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \psi(2j-2k+1,1)^2$$
$$= \frac{1}{4^K n}\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor}(2-\sqrt{2})^2 + \frac{2}{4^K n}\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor}\sum_{k=1}^{j-1}\left(\sqrt{2j-2k+2}-2\sqrt{2j-2k+1}-\sqrt{2j-2k}\right)^2.$$

Hence, by a similar computation,

$$\eta^-(t) = \lim_{n\to\infty}\sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \beta_n(2j-1,2k-2)^2 + \beta_n(2j-2,2k-1)^2 = 2C_K^- t,$$

where

$$C_K^- = \frac{(2-\sqrt{2})^2}{2^{2K+1}} + \frac{1}{4^K}\sum_{m=1}^{\infty}\left(\sqrt{2m+2}-2\sqrt{2m+1}+\sqrt{2m}\right)^2.$$

□

As a concluding remark, it is easy to show that $C_K^+ > C_K^-$, and in general we have $\eta^+(t) \geq \eta^-(t)$.

4.3.2 A Gaussian process with differentiable covariance function

Consider the following class of Gaussian processes. Let $\{F_t, 0 \leq t \leq T\}$ be a mean-zero Gaussian process with covariance defined by,

$$\mathbb{E}[F_r F_t] = r\phi\left(\frac{t}{r}\right), \quad t \geq r \tag{4.9}$$

where $\phi : [1,\infty) \to \mathbb{R}$ is twice-differentiable on $(1,\infty)$ and satisfies the following:

(ϕ.1) $\|\phi\|_\infty := \sup_{x\geq 1}|\phi(x)| \leq c_{\phi,0} < \infty$.

(ϕ.2) For $1 < x < \infty$,

$$|\phi'(x)| \leq \frac{c_{\phi,1}}{\sqrt{x-1}}.$$

(ϕ.3) For $1 < x < \infty$,

$$|\phi''(x)| \leq c_{\phi,2}x^{-\frac{1}{2}}(x-1)^{-\frac{3}{2}}.$$

where $c_{\phi,j}$, $j = 0,1,2$ are nonnegative constants.

Proposition 4.13. *The process $\{F_t, 0 \leq t \leq T\}$ described above satisfies conditions (M.1) - (M.4).*

Proof. Condition (M.1). By Conditions (ϕ.1) and (ϕ.2),

$$\begin{aligned}\mathbb{E}\left[(F_t-F_{t-s})^2\right] &= t\phi(1)+(t-s)\phi(1)-2(t-s)\phi\left(1+\frac{s}{t-s}\right)\\ &\le 2(t-s)\left|\phi\left(1+\frac{s}{t-s}\right)-\phi(1)\right|+s\,|\phi(1)|\\ &\le 2(t-s)\left|\int_1^{1+\frac{s}{t-s}}\phi'(x)\,dx\right|+s\|\phi\|_\infty\\ &\le 2(t-s)\int_1^{1+\frac{s}{t-s}}\frac{c_{\phi,1}}{\sqrt{x-1}}dx+s\|\phi\|_\infty\\ &\le Cs^{\frac{1}{2}}\sqrt{t-s}+s\|\phi\|_\infty\\ &\le Cs^{\frac{1}{2}},\end{aligned}$$

where the constant C depends on $\max\left\{\sqrt{T},\|\phi\|_\infty\right\}$.

Condition (M.2). For $2s\le r\le t-2s$ we have by the Mean Value Theorem,

$$\begin{aligned}|\mathbb{E}\left[F_tF_r-F_{t-s}F_r-F_tF_{r-s}+F_{t-s}F_{r-s}\right]| &= \left|r\,\left[\phi\left(\frac{t}{r}\right)-\phi\left(\frac{t-s}{r}\right)\right]-(r-s)\,\left[\phi\left(\frac{t}{r-s}\right)-\phi\left(\frac{t-s}{r-s}\right)\right]\right|\\ &\le s\sup_{\left[\frac{t-s}{r},\frac{t}{r-s}\right]}\left|\phi''(x)\right|\left(\frac{t}{r-s}-\frac{t-s}{r}\right)\\ &\le c_{\phi,2}s\left(\frac{t-s}{r}\right)^{-\frac{1}{2}}\left(\frac{t-s}{r}-1\right)^{-\frac{3}{2}}\left(\frac{ts}{r(r-s)}\right)\\ &\le \frac{C\sqrt{T}\,s^2}{(t-r)^{\frac{3}{2}}}=C\sqrt{T}\,s^2|t-r|^{-\frac{3}{2}}.\end{aligned}$$

Condition (M.3). By symmetry we can assume $r\le t$. Consider the following cases: First, suppose $2s\le r\le t-2s$. Then we have

$$\begin{aligned}|\mathbb{E}\left[F_t(F_{r+s}-2F_r+F_{r-s})\right]| &= \left|(r+s)\phi\left(\frac{t}{r+s}\right)-2r\phi\left(\frac{t}{r}\right)+(r-s)\phi\left(\frac{t}{r-s}\right)\right|\\ &= \left|(r+s)\left[\phi\left(\frac{t}{r+s}\right)-\phi\left(\frac{t}{r}\right)\right]-(r-s)\left[\phi\left(\frac{t}{r}\right)-\phi\left(\frac{t}{r-s}\right)\right]\right|\\ &\le \frac{st}{r}\sup_{\left[\frac{t}{r+s},\frac{t}{r-s}\right]}\left|\phi''(x)\right|\left(\frac{t}{r-s}-\frac{t}{r+s}\right)\\ &\le \frac{2s^2t^2c_{\phi,2}}{r(r-s)(r+s)}\left(\frac{r+s}{t}\right)^{\frac{1}{2}}\left(\frac{r+s}{t-r-s}\right)^{\frac{3}{2}}\\ &\le \frac{Cs^2t^{\frac{3}{2}}}{r(t-r)^{\frac{3}{2}}}.\end{aligned}$$

There are two possibilities, depending on the value of r. If $r \geq \frac{t}{2}$, then $\frac{t}{r} \leq 2$, and we have a bound of

$$Cs^2\left(\frac{t}{r}\right)\left(\frac{\sqrt{T}}{(t-r)^{\frac{3}{2}}}\right) \leq 2C\sqrt{T}\, s^2|t-r|^{-\frac{3}{2}}.$$

on the other hand, if $r < \frac{t}{2}$, then $\frac{t}{t-r} \leq 2$ and $r < t-r$. Then the bound is

$$Cs^2\left(\frac{t}{t-r}\right)\left(\frac{\sqrt{T}}{r\sqrt{t-r}}\right) \leq 2C\sqrt{T}\, s^2\left[(r-s)^{-\frac{3}{2}} + |t-r|^{-\frac{3}{2}}\right].$$

For the case $|t-r| < 2s$, assume that $t = r+ks$ for some $0 \leq k < 2$. Then

$$\begin{aligned}
&|\mathbb{E}\left[F_t\left(F_{r+s} - 2F_r + F_{r-s}\right)\right]| \\
&\quad = \left|(t\wedge(r_s))\,\phi\left(\frac{t\vee(r+s)}{t\wedge(r+s)}\right) - 2r\phi\left(\frac{t}{r}\right) + (r-s)\phi\left(\frac{t}{r-s}\right)\right| \\
&\quad = \left|(t\wedge(r_s))\,\phi\left(\frac{t\vee(r+s)}{t\wedge(r+s)}\right) - (r+s)\phi(1) - 2r\phi\left(\frac{t}{r}\right) + 2r\phi(1) + (r-s)\phi\left(\frac{t}{r-s}\right) - (r-s)\phi(1)\right| \\
&\quad \leq 3(r+s)\left|\phi\left(1+\frac{(k+1)s}{r-s}\right) - \phi(1)\right| \;\leq\; 3(r+s)\left|\int_1^{1+\frac{(k+1)s}{r-s}} \phi'(x)\,dx\right| \\
&\quad \leq 3(r+s)\int_1^{1+\frac{(k+1)s}{r-s}} \frac{c_{\phi,1}}{\sqrt{x-1}}dx \;\leq\; C\sqrt{T}s^{\frac{1}{2}}.
\end{aligned}$$

For the last case, note that if $t \wedge r < 2s$, then we have an upper bound of $Cs^{\frac{1}{2}}$, since $\mathbb{E}\left[F_sF_t\right] \leq s\|\phi\|_\infty$.

Condition (M.4). Take first the bound for $\mathbb{E}\left[F_t(F_{t+s} - F_{t-s})\right]$. Note that if $t < 2s$, then an upper bound of $Cs^{\frac{1}{2}}$ is clear, so we will assume $t \geq 2s$. We have

$$\begin{aligned}
|\mathbb{E}\left[F_tF_{t+s} - F_tF_{t-s}\right]| &= \left|t\phi\left(\frac{t+s}{t}\right) - (t-s)\phi\left(\frac{t}{t-s}\right)\right| \\
&\leq (t-s)\sup_{\left[\frac{t+s}{t},\frac{t}{t-s}\right]}\left|\phi'(x)\right|\left|\frac{t+s}{t} - \frac{t}{t-s}\right| + s\left|\phi\left(\frac{t+s}{t}\right)\right| \\
&\leq c_{\phi,1}\,\frac{s^2}{t}\sqrt{\frac{t}{t+s}}\sqrt{\frac{t}{s}} + c_{\phi,0}\,s\frac{\sqrt{T}}{\sqrt{t-s}} \\
&\leq Cs\sqrt{T}\,(t-s)^{-\frac{1}{2}}.
\end{aligned}$$

For the case $r \neq t$, first assume $r \leq t-2s$. By condition $(\phi.2)$,

$$\begin{aligned}
|\mathbb{E}\left[F_rF_{t+s} - F_rF_{t-s}\right]| &= \left|r\phi\left(\frac{t+s}{r}\right) - r\phi\left(\frac{t-s}{r}\right)\right| \leq 2s\sup_{\left[\frac{t-s}{r},\frac{t+s}{r}\right]}|\phi'(x)| \\
&\leq \frac{2s\sqrt{r}\,c_{\phi,1}}{\sqrt{t-r-s}} \leq \frac{C\sqrt{T}\,s}{\sqrt{t-r}}.
\end{aligned}$$

If $r \geq t+2s$, then

$$\begin{aligned}
|\mathbb{E}[F_rF_{t+s} - F_rF_{t-s}]| &= \left|(t+s)\phi\left(\frac{r}{t+s}\right) - (t-s)\phi\left(\frac{r}{t-s}\right)\right| \\
&\leq t\int_0^{2s}\left|\phi'\left(\frac{r}{t-s+x}\right)\right| dx + 2s\|\phi\|_\infty \\
&\leq \frac{2stc_{\phi,1}\sqrt{t+s}}{\sqrt{r-t}} + \frac{2sc_{\phi,0}\sqrt{T}}{\sqrt{t-s}} \\
&\leq Cs(r-t)^{-\frac{1}{2}} + Cs(t-s)^{-\frac{1}{2}}.
\end{aligned}$$

For the case $t < 2s$ or $|r-t| < 2s$, the bound follows from condition (M.1) and Cauchy-Schwarz.

For the third part of condition (M.4), we have for $t > 2s$,

$$\begin{aligned}
\mathbb{E}[F_sF_t - F_sF_{t-s}] &= s\phi\left(\frac{t}{s}\right) - s\phi\left(\frac{t-s}{s}\right) \\
&\leq s\sup_{\left[\frac{t-s}{s},\frac{t}{s}\right]}|\phi'(x)|\left(\frac{t}{s} - \frac{t-s}{s}\right) \\
&\leq \frac{c_{\phi,1}s}{\sqrt{\frac{t-s}{s}-1}} \\
&\leq Cs^{\frac{3}{2}}(t-2s)^{-\frac{1}{2}} \\
&= Cs^{\frac{1}{2}+\gamma}(t-2s)^{-\gamma}
\end{aligned}$$

where $\gamma = \frac{1}{2}$. □

Proposition 4.14. *Suppose $\phi(x)$ satisfies conditions $(\phi.1)$, $(\phi.3)$ and in addition $\phi(x)$ satisfies:*

$$(\phi.4): \quad \phi'(x) = \frac{\kappa}{\sqrt{x-1}} + \frac{\psi(x)}{\sqrt{x}},$$

where $\kappa \in \mathbb{R}$ and $\psi : (1,\infty) \to \mathbb{R}$ is a bounded differentiable function satisfying $|\psi'(1+x)| \leq C_\psi x^{-\frac{1}{2}}$ for some positive constant C_ψ. Then Condition (v) of Section 4 is satisfied, with $\eta^+(t) = C_\beta^+ t^2$, and $\eta^-(t) = C_\beta^- t^2$ for positive constants C_β^+, C_β^-.

Remark 4.15. Observe that condition $(\phi.4)$ implies $(\phi.2)$ but not $(\phi.3)$.

Proof. We want to show

$$\sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \beta_n(2j-1,2k-1)^2 \longrightarrow C_{\beta,1}t^2; \tag{4.10}$$

$$\sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \beta_n(2j-2,2k-2)^2 \longrightarrow C_{\beta,2}t^2; \text{ and} \tag{4.11}$$

$$\sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \beta_n(2j-1,2k-2)^2 \longrightarrow C_{\beta,3}t^2; \tag{4.12}$$

so that $C_\beta^+ = C_{\beta,1} + C_{\beta,2}$, and $C_\beta^- = 2C_{\beta,3}$. We will show computations for (4.10), with the others being similar. As in Prop. 5.2,

$$\sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \beta_n(2j-1,2k-1)^2 = \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} \beta_n(2j-1,2j-1)^2 + 2\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} \sum_{k=1}^{j-1} \beta_n(2j-1,2k-1)^2,$$

so it is enough to show

$$\lim_{n\to\infty} \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} \sum_{k=1}^{j-1} \beta_n(2j-1,2k-1)^2 = C_1 t^2; \text{ and} \tag{4.13}$$

$$\lim_{n\to\infty} \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} \beta_n(2j-1,2j-1)^2 = C_2 t^2. \tag{4.14}$$

Proof of (4.13). For $1 \le k \le j-1$ we have

$$\begin{aligned}\beta_n(2j-1,2k-1) &= \frac{2k}{n}\left(\phi\left(\frac{2j}{2k}\right) - \phi\left(\frac{2j-1}{2k}\right)\right) - \frac{2k-1}{n}\left(\phi\left(\frac{2j}{2k-1}\right) - \phi\left(\frac{2j-1}{2k-1}\right)\right)\\ &= \frac{2k}{n}\int_{\frac{2j-1}{2k}}^{\frac{2j}{2k}} \phi'(x)\,dx - \frac{2k-1}{n}\int_{\frac{2j-1}{2k-1}}^{\frac{2j}{2k-1}} \phi'(x)\,dx.\end{aligned}$$

Using the change of index $j = k+m$ and a change of variable for the two integrals, this becomes,

$$\beta_n(2j-1,2k-1) = \frac{1}{n}\int_{2m-1}^{2m} \phi'\left(1+\frac{y}{2k}\right)\,dy - \frac{1}{n}\int_{2m}^{2m+1} \phi'\left(1+\frac{y}{2k-1}\right)\,dy. \tag{4.15}$$

With the decomposition of (ϕ.4), we will address (4.15) in two parts. Using the first term, we have

$$\begin{aligned}&\frac{\kappa}{n}\int_{2m-1}^{2m}\sqrt{\frac{2k}{y}}\,dy - \frac{\kappa}{n}\int_{2m}^{2m+1}\sqrt{\frac{2k-1}{y}}\,dy\\ &= \frac{2\kappa}{n}\left[\sqrt{2k}\left(\sqrt{2m}-\sqrt{2m-1}\right) - \sqrt{2k-1}\left(\sqrt{2m+1}-\sqrt{2m}\right)\right].\end{aligned}$$

We are interested in the sum,

$$\sum_{k=1}^{\lfloor \frac{nt}{2} \rfloor} \sum_{m=1}^{\lfloor \frac{nt}{2} \rfloor - k} \frac{4\kappa^2}{n^2}\left[\sqrt{2k}\left(\sqrt{2m}-\sqrt{2m-1}\right) - \sqrt{2k-1}\left(\sqrt{2m+1}-\sqrt{2m}\right)\right]^2. \tag{4.16}$$

We can write

$$\begin{aligned}&\sqrt{2k}\left(\sqrt{2m}-\sqrt{2m-1}\right) - \sqrt{2k-1}\left(\sqrt{2m+1}-\sqrt{2m}\right)\\ &= -\sqrt{2k-1}\left(\sqrt{2m+1}-2\sqrt{2m}+\sqrt{2m-1}\right) + \left(\sqrt{2k}-\sqrt{2k-1}\right)\left(\sqrt{2m}-\sqrt{2m-1}\right).\end{aligned}$$

Observe that

$$\left[\left(\sqrt{2k}-\sqrt{2k-1}\right)\left(\sqrt{2m}-\sqrt{2m-1}\right)\right]^2 \le \frac{1}{(2k-1)(2m-1)},$$

and so

$$\frac{4\kappa^2}{n^2}\sum_{k=1}^{\lfloor\frac{nt}{2}\rfloor}\sum_{m=1}^{\lfloor\frac{nt}{2}\rfloor-k}\frac{1}{(2k-1)(2m-1)} \le \frac{4\kappa^2}{n^2}\left(\sum_{k=1}^{\lfloor\frac{nt}{2}\rfloor}\frac{1}{2k-1}\right)^2 \le \frac{C\log(nt)^2}{n^2}.$$

Therefore the contribution of this term is zero, and it follows by Cauchy-Schwarz that the only significant term is

$$\begin{aligned}
&\frac{4\kappa^2}{n^2}\sum_{k=1}^{\lfloor\frac{nt}{2}\rfloor}\sum_{m=1}^{\lfloor\frac{nt}{2}\rfloor-k}(2k-1)\left(\sqrt{2m+1}-2\sqrt{2m}+\sqrt{2m-1}\right)^2\\
&\quad=4\kappa^2\sum_{m=1}^{\lfloor\frac{nt}{2}\rfloor}\left(\sqrt{2m+1}-2\sqrt{2m}+\sqrt{2m-1}\right)^2\sum_{k=1}^{\lfloor\frac{nt}{2}\rfloor-m}\frac{2k-1}{n^2}\\
&\quad=4\kappa^2\sum_{m=1}^{\lfloor\frac{nt}{2}\rfloor}\left(\sqrt{2m+1}-2\sqrt{2m}+\sqrt{2m-1}\right)^2\frac{\left(\lfloor\frac{nt}{2}\rfloor-m\right)^2}{n^2},
\end{aligned}$$

which converges as $n\to\infty$ to

$$\kappa^2t^2\sum_{m=1}^{\infty}\left(\sqrt{2m+1}-2\sqrt{2m}+\sqrt{2m-1}\right)^2.$$

Next, we consider the term $\frac{1}{\sqrt{x}}\psi(x)$. The contribution of this term to (4.15) is

$$\frac{1}{n}\int_{2m-1}^{2m}\sqrt{\frac{2k}{2k+y}}\,\psi\left(1+\frac{y}{2k}\right)\,dy-\frac{1}{n}\int_{2m}^{2m+1}\sqrt{\frac{2k-1}{2k-1+y}}\,\psi\left(1+\frac{y}{2k-1}\right)\,dy. \tag{4.17}$$

We can bound (4.17) by

$$\begin{aligned}
&\frac{1}{n}\left|\int_{2m-1}^{2m}\sqrt{\frac{2k}{2k+y}}\,\psi\left(1+\frac{y}{2k}\right)\,dy-\int_{2m}^{2m+1}\sqrt{\frac{2k-1}{2k-1+y}}\,\psi\left(1+\frac{y}{2k-1}\right)\,dy\right|\\
&\quad\le\frac{1}{n}\sup_{(1,\infty)}|\psi(x)|\frac{\sqrt{2k}-\sqrt{2k-1}}{\sqrt{2k+2m-1}}\\
&\qquad+\left[\sqrt{\frac{2k}{2k+2m-1}}\left|\int_{2m-1}^{2m}\psi\left(1+\frac{y}{2k}\right)\,dy-\int_{2m}^{2m+1}\psi\left(1+\frac{y}{2k-1}\right)\,dy\right|\right]\\
&\quad=\frac{1}{n}\left(A_{k,m}+B_{k,m}\right).
\end{aligned}$$

Since $|\psi(x)|$ is bounded, we have

$$A_{k,m}\le\frac{C}{\sqrt{2k-1}\sqrt{2k+2m-1}}\le\frac{C}{\sqrt{2k-1}\,\sqrt{2m-1}}\,. \tag{4.18}$$

For $B_{k,m}$ using that $|\psi'(x+1)| \le Cx^{-\frac{1}{2}}$,

$$\begin{aligned}
&\left|\int_{2m-1}^{2m} \psi\left(1+\frac{y}{2k}\right)\, dy - \int_{2m}^{2m+1} \psi\left(1+\frac{y}{2k-1}\right)\, dy\right| \\
&= \left|\int_{2m-1}^{2m} \psi\left(1+\frac{u}{2k}\right) - \psi\left(1+\frac{u+1}{2k-1}\right)\, du\right| \\
&\le \int_{2m-1}^{2m} \left|\int_{\frac{u+1}{2k-1}}^{\frac{u}{2k}} \psi'(1+v)dv\right| du \\
&\le C\int_{2m-1}^{2m} \int_{\frac{u}{2k}}^{\frac{u+1}{2k-1}} v^{-\frac{1}{2}}dv\, du \le \frac{C}{\sqrt{2k-1}}\left(\sqrt{2m+1}-\sqrt{2m}\right) \\
&\le \frac{C}{\sqrt{2k-1}\sqrt{2m-1}}
\end{aligned}$$

so that

$$B_{k,m} \le \sqrt{\frac{2k}{2k+2m-1}} \cdot \frac{C}{\sqrt{2k-1}\sqrt{2m-1}} \le \frac{C}{\sqrt{2k-1}\sqrt{2m-1}}. \tag{4.19}$$

Hence, from (4.18) and (4.19), we obtain

$$\begin{aligned}
&\sum_{k=1}^{\lfloor \frac{nt}{2} \rfloor} \sum_{m=1}^{\lfloor \frac{nt}{2} \rfloor - k} \frac{C}{n^2}\left(\frac{1}{\sqrt{2k-1}\sqrt{2m-1}}\right)^2 \\
&\qquad \le \frac{C}{n^2} \sum_{k,m=1}^{\lfloor \frac{nt}{2} \rfloor} \frac{1}{(2m-1)(2k-1)} \ \le\ \frac{C\log(n)^2}{n^2}
\end{aligned}$$

so the portion represented by (4.17) tends to zero as $n \to \infty$. Since this term is not significant, it follows by Cauchy-Schwarz that the behavior of

$$\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} \sum_{k=1}^{j-1} \beta_n(2j-1,2k-1)^2$$

is dominated by eq. (4.16), and we have the result (4.13), with

$$C_1 = \kappa^2 \sum_{m=1}^{\infty} \left(\sqrt{2m+1} - 2\sqrt{2m} + \sqrt{2m-1}\right)^2.$$

Proof of (4.14). For each j,

$$\begin{aligned}
\beta_n(2j-1,2j-1)^2 &= \left(\frac{2j}{n}\phi(1) - 2\frac{2j-1}{n}\phi\left(\frac{2j}{2j-1}\right) + \frac{2j-1}{n}\phi(1)\right)^2 \\
&= \frac{1}{n^2}\left[\phi(1) + (4j-2)\left(\phi(1) - \phi\left(1+\frac{1}{2j-1}\right)\right)\right]^2 \\
&= \frac{\phi(1)^2}{n^2} + \frac{4(2j-1)\phi(1)}{n^2}\left(\phi(1) - \phi\left(1+\frac{1}{2j-1}\right)\right) \\
&= \qquad + \frac{4(2j-1)^2}{n^2}\left(\phi(1) - \phi\left(1+\frac{1}{2j-1}\right)\right)^2.
\end{aligned}$$

Since $\left|\phi(1)-\phi\left(1+\frac{1}{2j-1}\right)\right| \le \frac{c_{\phi,3}}{\sqrt{2j-1}}$ by $(\phi.3)$, we see that

$$\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} \left[\frac{\phi(1)^2}{n^2} + \frac{4(2j-1)\phi(1)}{n^2} \left| \phi(1) - \phi\left(1+\frac{1}{2j-1}\right) \right| \right] \le Cn^{-\frac{1}{2}};$$

which implies only the last term is significant in the limit. Again we use $(\phi.4)$ to obtain:

$$\begin{aligned}
\phi(1)-\phi\left(1+\frac{1}{2j-1}\right) &= -\int_1^{1+\frac{1}{2j-1}} \phi'(x)\,dx \\
&= -\kappa \int_1^{1+\frac{1}{2j-1}} \frac{1}{\sqrt{x-1}}\,dx - \int_1^{1+\frac{1}{2j-1}} \frac{1}{\sqrt{x}}\psi(x)\,dx \\
&= -\frac{2\kappa}{\sqrt{2j-1}} + O\left(\frac{1}{2j-1}\right);
\end{aligned}$$

hence

$$\frac{4(2j-1)^2}{n^2}\left(\phi(1)-\phi\left(1+\frac{1}{2j}\right)\right)^2 = \frac{16\kappa^2(2j-1)^2}{n^2(2j-1)} + O\left(\frac{j^{\frac{1}{2}}}{n^2}\right),$$

and taking $n \to \infty$,

$$\lim_{n\to\infty} \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} \frac{16\kappa^2(2j-1)}{n^2} + O\left(\frac{j^{\frac{1}{2}}}{n^2}\right) = 4\kappa^2 t^2,$$

which gives (4.14). Thus (4.10) is proved with

$$C_{\beta,1} = 4\kappa^2 + 2\kappa^2 \sum_{m=1}^{\infty} \left(\sqrt{2m+1} - 2\sqrt{2m} + \sqrt{2m-1}\right)^2.$$

By similar computations,

$$\begin{aligned}
C_{\beta,2} &= 4\kappa^2 + 2\kappa^2 \sum_{m=1}^{\infty} \left(\sqrt{2m+1} - 2\sqrt{2m} + \sqrt{2m-1}\right)^2; \text{ and} \\
C_{\beta,3} &= 4\kappa^2 + 2\kappa^2 \sum_{m=1}^{\infty} \left(\sqrt{2m+2} - 2\sqrt{2m+1} + \sqrt{2m}\right)^2;
\end{aligned}$$

and so

$$\begin{aligned}
C_\beta^+ = C_{\beta,1} + C_{\beta,2} &= 8\kappa^2 + 4\kappa^2 \sum_{m=1}^{\infty} \left(\sqrt{2m+1} - 2\sqrt{2m} + \sqrt{2m-1}\right)^2, \\
C_\beta^- = 2C_{\beta,3} &= 8\kappa^2 + 4\kappa^2 \sum_{m=1}^{\infty} \left(\sqrt{2m+2} - 2\sqrt{2m+1} + \sqrt{2m}\right)^2.
\end{aligned}$$

Note that $C_\beta^+ \ge C_\beta^-$, and it follows that $\eta(t) = \eta^+(t) - \eta^-(t)$ is nonnegative, and strictly positive if $\kappa \ne 0$. □

For a particular example, we consider a mean-zero Gaussian process $\{F_t, t \geq 0\}$, with covariance given by

$$\mathbb{E}\left[F_r F_t\right] = \sqrt{rt}\ \sin^{-1}\left(\frac{r \wedge t}{\sqrt{rt}}\right).$$

This process was studied by Jason Swanson in a 2007 paper [38], and it appears in the limit of normalized empirical quantiles of a system of independent Brownian motions.

Corollary 4.16. *The process $\{F_t, 0 \leq t \leq T\}$ with covariance described above satisfies conditions (M.1) - (M.5), with $\eta(t) = \left(C_\beta^+ - C_\beta^-\right)t^2$, where C_β^+, C_β^- are as given in Proposition 4.15, with $\kappa^2 = 1/4$.*

Proof. Assume $0 \leq r < t \leq T$. We can write,

$$\sqrt{rt}\ \sin^{-1}\left(\sqrt{\frac{r}{t}}\right) = \sqrt{rt}\ \tan^{-1}\left(\sqrt{\frac{r}{t-r}}\right) = r\phi\left(\frac{t}{r}\right),$$

where

$$\phi(x) = \begin{cases} \sqrt{x}\tan^{-1}\left(\frac{1}{\sqrt{x-1}}\right), & if\ \ x > 1 \\ \frac{\pi}{2}, & if\ \ x = 1 \end{cases}. \tag{4.20}$$

Condition $(\phi.1)$ is clear by continuity and L'Hôpital. Conditions $(\phi.2)$ and $(\phi.3)$ are easily verified by differentiation. For $(\phi.4)$ we can write,

$$\phi'(x) = -\frac{1}{2\sqrt{x-1}} + \frac{1}{2\sqrt{x}}\left(\frac{\sqrt{x}-1}{\sqrt{x-1}} - \tan^{-1}\left(\frac{1}{\sqrt{x-1}}\right)\right),$$

so that $\kappa = -1/2$, and

$$\psi(x) = \frac{1}{2}\left(\frac{\sqrt{x}-1}{\sqrt{x-1}} - \tan^{-1}\left(\frac{1}{\sqrt{x-1}}\right)\right)$$

satisfies $(\phi.4)$. □

4.3.3 Empirical quantiles of independent Brownian motions

For our last example, we consider a family of processes studied by Jason Swanson in [39]. Like [38], this Gaussian family arises from the empirical quantiles of independent Brownian motions, but this case is more general, and does not have a covariance representation (4.9).

Let $B = \{B(t), t \geq 0\}$ be a Brownian motion with random initial position. Assume $B(0)$ has a density function $f \in \mathscr{C}^\infty(\mathbb{R})$ such that

$$\sup_{x\in\mathbb{R}}(1 + |x|^m)|f^{(n)}(x)| < \infty$$

for all nonnegative integers m and n. It follows that for $t > 0$, B has density

$$u(x,t) = \int_{\mathbb{R}} f(y)p(t, x-y)\ dy,$$

where $p(t,x) = (2\pi t)^{-\frac{1}{2}} e^{-\frac{x^2}{2t}}$. For fixed $\alpha \in (0,1)$, define the α-quantile $q(t)$ by

$$\int_{-\infty}^{q(t)} u(x,t)\, dx = \alpha,$$

where we assume $f(q(0)) > 0$. It is proved in [39] (Theorem 1.4) that there exists a continuous, centered Gaussian process $\{F(t), t \geq 0\}$ with covariance

$$\mathbb{E}[F_r F_t] = \rho(r,t) = \frac{\mathbb{P}(B(r) \leq q(r), B(t) \leq q(t)) - \alpha^2}{u(q(r),r)\, u(q(t),t)}. \tag{4.21}$$

In [39], the properties of ρ are studied in detail, and we follow the notation and proof methods given in Section 3 of that paper. Swanson defines the following factors:

$$\tilde{\rho}(r,t) = \mathbb{P}(B(r) \leq q(r), B(t) \leq q(t)) - \alpha^2; \text{ and} \qquad \theta(t) = (u(q(t),t))^{-1};$$

so that $\rho(r,t) = \theta(r)\theta(t)\tilde{\rho}(r,t)$. For fixed $T > 0$ and $0 < r < t \leq T$, the first partial derivatives of $\widetilde{\rho}$ are calculated in [39](see eqs. (3.4), (3.7)):

$$\begin{aligned} \frac{\partial}{\partial t}\tilde{\rho}(r,t) = q'(t) \int_{-\infty}^{q(r)} p(t-r, x-q(t))\, u(x,r)\, dy\, dx \\ -\frac{1}{2} p(t-r, q(r)-q(t)) u(q(r),r) + u(q(r),r) q'(r) \int_{-\infty}^{q(t)} p(t-r, q(r)-y)\, dy \\ + \frac{1}{2} \int_{-\infty}^{q(t)} \int_{-\infty}^{q(r)} p(t-r, x-y) \frac{\partial^2}{\partial x^2} u(x,r)\, dx\, dy; \end{aligned} \tag{4.22}$$

$$\frac{\partial}{\partial r}\tilde{\rho}(r,t) = \frac{1}{2} p(t-r, q(t)-q(r))\, u(q(r),r). \tag{4.23}$$

Lemma 4.17. *Let $0 < T$, and $0 < r < t \leq T$. Then there exist constants C_i, $i = 1,2,3,4$, such that:*

(a)

$$\left| \frac{\partial}{\partial r} \rho(r,t) \right| \leq C_1 |t-r|^{-\frac{1}{2}}$$

(b)

$$\left| \frac{\partial^2}{\partial r^2} \rho(r,t) \right| \leq C_2 |t-r|^{-\frac{3}{2}}$$

(c)

$$\left| \frac{\partial}{\partial t} \rho(r,t) \right| \leq C_3 |t-r|^{-\frac{1}{2}}$$

(d)

$$\left| \frac{\partial^2}{\partial t^2} \rho(r,t) \right| \leq C_4 |t-r|^{-\frac{3}{2}}.$$

Proof. Results (a) and (c) are proved in Theorem 3.1 of [39]. Bounds for (b) and (d) follow by differentiating the expressions for $\partial_r \rho(r,t)$ and $\partial_t \rho(r,t)$ given in the proof of that theorem. □

Proposition 4.18. *Let $T > 0$, $0 < s < T \wedge 1$, and $s \leq r \leq t \leq T$. Then $\rho(r,t)$ satisfies conditions (M.1) - (M.4).*

Proof. Conditions (M.1) and (M.2) are proved in [39] (Corollaries 3.2, 3.5 and Remark 3.6). For condition (M.3), there are several cases to consider.
Case 1: $s \leq r \leq t - 2s$. Using Lemma 5.7(a),

$$\begin{aligned}
|\mathbb{E}\left[F_t(F_{r+s} - 2F_r + F_{r-s})\right]| &\leq |\rho(r+s,t) - \rho(r,t)| + |\rho(r,t) - \rho(r-s,t)| \\
&\leq \int_0^s \left| \frac{\partial}{\partial r}\rho(r+x,t) \right| \, dx \; + \; \int_{-s}^0 \left| \frac{\partial}{\partial r}\rho(r+y,t) \right| \, dy \\
&\leq 2 \int_0^s C_1 |t-r-x|^{-\frac{1}{2}} \, dx \; \leq \; C s^{\frac{1}{2}}.
\end{aligned}$$

Case 2: If $|t-r| < 2s$, the computation is similar to Case 1, where we use the fact that

$$\int_0^s x^{-\frac{1}{2}} \, dx = 2s^{\frac{1}{2}}.$$

Case 3: For r, $t \geq 2s$ and $|t-r| \geq 2s$, the results follow from Lemma 5.7 (b) and (d) for $r < t$ and $r > t$, respectively.

Now to condition (M.4). For the first part, we first assume $t \geq 2s$. Then using the above decomposition,

$$\begin{aligned}
\mathbb{E}\left[F_t(F_{t+s} - F_{t-s})\right] &= \rho(t,t+s) - \rho(t,t-s) \\
&= \theta(t)\left[\theta(t+s)\widetilde{\rho}(t,t+s) - \theta(t-s)\widetilde{\rho}(t,t-s)\right] \\
&= \theta(t)\left[\widetilde{\rho}(t,t+s)\Delta\theta + \theta(t-s)\Delta\widetilde{\rho}\right],
\end{aligned}$$

where $\Delta\theta = \theta(t) - \theta(t-s)$ and $\Delta\widetilde{\rho} = \widetilde{\rho}(t,t+s) - \widetilde{\rho}(t,t-s)$. First, note that

$$\left|u'(q(t),t)\right| = \left| \frac{\partial}{\partial x}u(q(t),t)q'(t) + \frac{\partial}{\partial t}u(q(t),t) \right| \leq C,$$

where we used that $q'(t)$ is bounded (see Lemma 1.1 of [39]). Since $u(q(t),t)$ is continuous and strictly positive on $[0,T]$, it follows that $\theta(t)$ is bounded and

$$\left|\theta'(t)\right| = \frac{|u'(q(t),t)|}{u^2(q(t),t)} \leq C, \tag{4.24}$$

hence,

$$|\Delta\theta| \leq \int_{-s}^s |\theta'(t+x)| \, dx \leq Cs.$$

For $\Delta\widetilde{\rho}$ we have

$$\begin{aligned}
|\Delta\widetilde{\rho}| &= |\mathbb{P}(B(t)\le q(t), B(t+s)\le q(t+s)) - \mathbb{P}(B(t)\le q(t), B(t-s)\le q(t-s))| \\
&= \int_{-\infty}^{q(t)}\int_{-\infty}^{q(t+s)} p(s,x-y)u(x,t)\,dy\,dx - \int_{-\infty}^{q(t-s)}\int_{-\infty}^{q(t)} p(s,x-y)u(x,t-s)\,dy\,dx \\
&\le \left|\int_{-\infty}^{q(t-s)}\int_{-\infty}^{q(t)} p(s,x-y)u(x,t) - p(s,x-y)u(x,t-s)\,dy\,dx\right| + Cs \\
&\le \int_{-\infty}^{q(t-s)} |u(x,t-s)-u(x,t)|\,dx + Cs \\
&\le \int_{-\infty}^{\infty}\left|\int_{t-s}^{t}\frac{\partial}{\partial r}u(x,r)dr\right|\,dx + Cs = \frac{1}{2}\int_{-\infty}^{\infty}\left|\int_{t-s}^{t}\frac{\partial^2}{\partial x^2}u(x,r)dr\right|\,dx + Cs \\
&\le \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{t-s}^{t}\left|f''(y)\right| p(r,x-y)\,dr\,dy\,dx + Cs \le Cs.
\end{aligned}$$

When $t<2s$, we write

$$\begin{aligned}
|\mathbb{E}[F_t(F_{t+s}-F_{t-s})]| &\le |\rho(t,t+s)-\rho(t,t)| + |\rho(t,t)-\rho(t-s,t)| \\
&\le \int_0^s \left|\frac{\partial}{\partial t}\rho(t,t+x)\right|\,dx + \int_{-s}^{0}\left|\frac{\partial}{\partial r}\rho(t+y,t)\right|\,dy \\
&\le Cs^{\frac{1}{2}},
\end{aligned}$$

using Lemma 4.17 and the fact that

$$\int_0^s x^{-\frac{1}{2}}\,dx = 2s^{\frac{1}{2}}.$$

For the second part of condition (M.4), we consider

$$|\mathbb{E}[F_r(F_{t+s}-F_{t-s})]| \quad\text{and}\quad |\mathbb{E}[F_s(F_t-F_{t-s})]|.$$

When $r<t-s$ (including $r=s$), an upper bound of $Cs|t-r|^{-\frac{1}{2}}$ is proved in [39] (see Corollary 3.4 and Remark 3.6). When $r\ge t+2s$, or $|t-r|<2s$, the bounds follow from Lemma 4.17. □

The rest of this section is dedicated to verifying condition (M.5). We start with two useful estimates. As in Proposition 5.8, suppose $0<s\le r\le t\le T$. It follows from Lemma 1.1 of [39] that for some positive constant C,

$$|q(t)-q(r)| \le C(t-r). \tag{4.25}$$

Using this estimate and the fact that $e^{-a}-e^{-b}\le b-a$ for $0\le a\le b$, we obtain

$$\left|e^{-\frac{(q(t)-q(r))^2}{2(t-r)}} - e^{-\frac{(q(t)-q(r-s))^2}{2(t-r+s)}}\right| \le Cs \le 1. \tag{4.26}$$

Recalling the definitions in condition (M.5), we can write for $t \in [0,T]$

$$\eta_n^+(t) - \eta_n^-(t) = \sum_{\ell=1}^{2\lfloor \frac{nt}{2} \rfloor} \beta_n(\ell-1,\ell-1)^2 + 2\sum_{k\le j-1} \beta_n(2k-1,2j-1)^2 + 2\sum_{k\le j-1} \beta_n(2k-2,2j-2)^2$$
$$- 2\sum_{k\le j-1} \beta_n(2k-2,2j-1)^2 - 2\sum_{k\le j-1} \beta_n(2k-1,2j-2)^2.$$

For the first sum, since $F_{\frac{\ell}{n}} - F_{\frac{\ell-1}{n}}$ is Gaussian, we have

$$\beta_n(\ell-1,\ell-1)^2 = \left(\mathbb{E}\left[\left(F_{\frac{\ell}{n}} - F_{\frac{\ell-1}{n}}\right)^2\right]\right)^2 = \frac{1}{3}\mathbb{E}\left[\left(F_{\frac{\ell}{n}} - F_{\frac{\ell-1}{n}}\right)^4\right].$$

By Theorem 3.7 of [39],

$$\sum_{\ell=1}^{\lfloor nt \rfloor} \left(F_{\frac{\ell}{n}} - F_{\frac{\ell-1}{n}}\right)^4 \longrightarrow \frac{6}{\pi}\int_0^t (u(q(s),s))^{-2}\, ds$$

in L^2 as $n \to \infty$. For the second sum, assume $1 \le k < j$, and we study the term

$$\begin{aligned}\beta_n(2k-1,2j-1) &= \rho\left(\frac{2k}{n},\frac{2j}{n}\right) - \rho\left(\frac{2k-1}{n},\frac{2j}{n}\right) - \rho\left(\frac{2k}{n},\frac{2j-1}{n}\right) + \rho\left(\frac{2k-1}{n},\frac{2j-1}{n}\right)\\ &= \theta\left(\frac{2j}{n}\right)\int_{\frac{2k-1}{n}}^{\frac{2k}{n}} \left[\theta'(r)\widetilde{\rho}\left(r,\frac{2j}{n}\right) + \theta(r)\partial_r\widetilde{\rho}\left(r,\frac{2j}{n}\right)\right] dr\\ &\quad - \theta\left(\frac{2j-1}{n}\right)\int_{\frac{2k-1}{n}}^{\frac{2k}{n}} \left[\theta'(r)\widetilde{\rho}\left(r,\frac{2j-1}{n}\right) + \theta(r)\partial_r\widetilde{\rho}\left(r,\frac{2j-1}{n}\right)\right] dr.\end{aligned}$$

We can write this as

$$\theta\left(\frac{2j}{n}\right)\int_{\frac{2k-1}{n}}^{\frac{2k}{n}} \theta(r)\left(\partial_r\widetilde{\rho}\left(r,\frac{2j}{n}\right) - \partial_r\widetilde{\rho}\left(r,\frac{2j-1}{n}\right)\right) dr \tag{4.27}$$

$$+ \left[\theta\left(\frac{2j}{n}\right) - \theta\left(\frac{2j-1}{n}\right)\right]\int_{\frac{2k-1}{n}}^{\frac{2k}{n}} \theta(r)\left(\partial_r\widetilde{\rho}\left(r,\frac{2j-1}{n}\right)\right) dr \tag{4.28}$$

$$+ \int_{\frac{2k-1}{n}}^{\frac{2k}{n}} \theta'(r)\left[\theta\left(\frac{2j}{n}\right)\widetilde{\rho}\left(r,\frac{2j}{n}\right) - \theta\left(\frac{2j-1}{n}\right)\widetilde{\rho}\left(r,\frac{2j-1}{n}\right)\right] dr. \tag{4.29}$$

The first task is to show that components (4.28) and (4.29) have a negligible contribution to $\eta(t)$. For (4.28), it follows from (4.24) that

$$\left|\theta\left(\frac{2j}{n}\right) - \theta\left(\frac{2j-1}{n}\right)\right| \le Cn^{-1}, \tag{4.30}$$

and using (4.23), we have

$$\int_{\frac{2k-1}{n}}^{\frac{2k}{n}} \theta(r)\partial_r\widetilde{\rho}\left(r,\frac{2j-1}{n}\right) dr = \int_{\frac{2k-1}{n}}^{\frac{2k}{n}} p\left(\frac{2j-1}{n} - r, q\left(\frac{2j-1}{n}\right) - q(r)\right) dr \le Cn^{-\frac{1}{2}}.$$

Hence, the contribution of (4.28) to the sum of $\beta_n(2k-1,2j-1)^2$ is bounded by $C\left(n^{-\frac{3}{2}}\right)^2\cdot n^2\leq Cn^{-1}$. We can write component (4.29) as

$$\int_{\frac{2k-1}{n}}^{\frac{2k}{n}}\theta'(r)\left[\theta\left(\frac{2j}{n}\right)\left(\widetilde{\rho}\left(r,\frac{2j}{n}\right)-\widetilde{\rho}\left(r,\frac{2j-1}{n}\right)\right)+\left(\theta\left(\frac{2j}{n}\right)-\theta\left(\frac{2j-1}{n}\right)\right)\widetilde{\rho}\left(r,\frac{2j-1}{n}\right)\right]dr.$$

Using (4.23), we have for each $r\in\left[\frac{2k-1}{n},\frac{2k}{n}\right]$,

$$\left|\widetilde{\rho}\left(r,\frac{2j}{n}\right)-\widetilde{\rho}\left(r,\frac{2j-1}{n}\right)\right|\leq Cn^{-\frac{1}{2}}(2j-2k-1)^{-\frac{1}{2}}.$$

Then, using (4.30) and (4.24), we have (4.29) bounded by

$$C\left[n^{-\frac{1}{2}}(2j-2k-1)^{-\frac{1}{2}}+n^{-1}\right]n^{-1}.$$

Hence, the contribution of (4.29) to the sum of $\beta_n(2k-1,2j-1)^2$ is bounded by

$$Cn^{-2}\sum_{j=1}^{\lfloor\frac{nt}{2}\rfloor}\sum_{k=1}^{j-1}\left[n^{-1}(2j-2k-1)^{-1}+n^{-2}\right]\leq Cn^{-1}.$$

We now turn to component (4.27). By (4.23),

$$\theta(r)\frac{\partial}{\partial r}\widetilde{\rho}\left(r,\frac{2j}{n}\right)=\frac{1}{2}p\left(\frac{2j}{n}-r,q\left(\frac{2j}{n}\right)-q(r)\right).$$

To simplify notation, define

$$\psi_n(j,r)=e^{-\frac{(q(\frac{j}{n})-q(r))^2}{2(\frac{j}{n}-r)}}.$$

By (4.25), we have for the interval $I_{2k}=\left[\frac{2k-1}{n},\frac{2k}{n}\right]$,

$$\sup_{r\in I_{2k}}\left\{\frac{\left((q\left(\frac{2j}{n}\right)-q(r)\right)^2}{2(\frac{2j}{n}-r)}\right\}\leq\frac{C(2j-2k+1)}{n}.$$

This implies that $\inf\{\psi_n(2j,r),r\in I_{2k}\}\geq e^{-C\frac{2j-2k+1}{n}}$, hence, when j,k are small compared to n, $|\psi|$ is close to unity. We can write,

$$\int_{\frac{2k-1}{n}}^{\frac{2k}{n}}\theta(r)\left(\partial_r\widetilde{\rho}\left(r,\frac{2j}{n}\right)-\partial_r\widetilde{\rho}\left(r,\frac{2j-1}{n}\right)\right)dr=\frac{1}{2\sqrt{2\pi}}\int_{\frac{2k-1}{n}}^{\frac{2k}{n}}\frac{1}{\sqrt{\frac{2j}{n}-r}}-\frac{1}{\sqrt{\frac{2j-1}{n}-r}}dr \tag{4.31}$$

$$-\frac{1}{2\sqrt{2\pi}}\int_{\frac{2k-1}{n}}^{\frac{2k}{n}}(1-\psi_n(2j-1,r))\left(\frac{1}{\sqrt{\frac{2j}{n}-r}}-\frac{1}{\sqrt{\frac{2j-1}{n}-r}}\right)dr \tag{4.32}$$

$$+\frac{1}{2\sqrt{2\pi}}\int_{\frac{2k-1}{n}}^{\frac{2k}{n}}\frac{\psi_n(2j,r)-\psi_n(2j-1,r)}{\sqrt{\frac{2j}{n}-r}}dr. \tag{4.33}$$

For component (4.32), by the above estimate for $\inf\{\psi_n(2j,r), r\in I_{2k}\}$ we have

$$\sup_{r\in I_{2k}} |1-\psi(2j,r)| \le Cn^{-1}(2j-2k+1) \le 1,$$

hence (4.32) is bounded by

$$Cn^{-\frac{3}{2}}(2j-2k+1)\left(\sqrt{2j-2k+1}-2\sqrt{2j-2k}+\sqrt{2j-2k-1}\right).$$

Given $\varepsilon > 0$, we can find an $M > 1$ such that

$$\sum_{m=M}^{\infty}\left(\sqrt{2m+1}-2\sqrt{2m}+\sqrt{2m-1}\right)^2 < \varepsilon.$$

The contribution of (4.32) to the sum of $\beta_n(2k-1,2j-1)^2$ is thus bounded by,

$$\begin{aligned}
&(2\pi n)^{-1}\sum_{j=1}^{\lfloor \frac{nt}{2}\rfloor}\theta^2\left(\frac{2j}{n}\right)\sum_{k=1}^{j-1}\sup_{r\in I_{2k}}(1-\psi_n(2j,r))^2\left(\sqrt{2j-2k+1}-2\sqrt{2j-2k}+\sqrt{2j-2k-1}\right)^2\\
&\qquad\le Cn^{-1}\sum_{j=1}^{\lfloor \frac{nt}{2}\rfloor}\sum_{k=1}^{j-M-1}\left(\sqrt{2j-2k+1}-2\sqrt{2j-2k}+\sqrt{2j-2k-1}\right)^2\\
&\qquad\quad+Cn^{-1}\sum_{j=1}^{\lfloor \frac{nt}{2}\rfloor}\sum_{k=j-M}^{j-1}Cn^{-1}(2j-2k+1)\left(\sqrt{2j-2k+1}-2\sqrt{2j-2k}+\sqrt{2j-2k-1}\right)^2\\
&\qquad\le Cn^{-1}\sum_{j=1}^{\lfloor \frac{nt}{2}\rfloor}\varepsilon+Cn^{-1}\sum_{j=1}^{\lfloor \frac{nt}{2}\rfloor}\frac{M^2}{n^2},
\end{aligned}$$

which is less than $C\varepsilon$ as $n\to\infty$, since $\theta(t)$ is bounded.

For (4.33), by we have $\sup\{|\psi_n(2j,r)-\psi_n(2j-1,r)|, r\in I_{2k}\}\le Cn^{-1}$, hence (4.33) is bounded by $Cn^{-\frac{3}{2}}(2j-2k-1)^{-\frac{1}{2}}$. Therefore the contribution of the term including (4.33) to the sum of $\beta_n(2k-1,2j-1)^2$ is bounded by

$$Cn^{-3}\sum_{j=1}^{\lfloor \frac{nt}{2}\rfloor}\sum_{k=1}^{j-1}(2j-2k-1)^{-1}\le Cn^{-2}\log(nt),$$

because $\theta(t)$ is bounded.

It follows that the sum of $\beta_n(2k-1,2j-1)^2$ is dominated by (4.27), and the significant term in (4.27) is given by (4.31). Hence, it is enough to consider

$$\frac{2}{n\pi}\sum_{j\le k-1}\theta^2\left(\frac{2j}{n}\right)\left(\sqrt{2j-2k+1}-2\sqrt{2j-2k}+\sqrt{2j-2k-1}\right)^2.$$

Using the change of index $j=k+m$, this is

$$\frac{2}{n\pi}\sum_{j=1}^{\lfloor \frac{nt}{2}\rfloor}\theta^2\left(\frac{2j}{n}\right)\sum_{m=1}^{j-1}\left(\sqrt{2m+1}-2\sqrt{2m}+\sqrt{2m-1}\right)^2.$$

Taking $n \to \infty$, this behaves like

$$\frac{a}{\pi} \int_0^t \theta^2(s)\, ds,$$

where

$$a = \sum_{m=1}^{\infty} \left(\sqrt{2m+1} - 2\sqrt{2m} + \sqrt{2m-1} \right)^2.$$

By similar computation,

$$\sum_{k \le j-1} \beta_n(2k-2, 2j-2)^2 \longrightarrow \frac{a}{\pi} \int_0^t \theta^2(s)\, ds,$$
$$\sum_{k \le j-1} \beta_n(2k-2, 2j-1)^2 \longrightarrow \frac{b_1}{\pi} \int_0^t \theta^2(s)\, ds, \text{ and}$$
$$\sum_{k \le j-1} \beta_n(2k-1, 2j-2)^2 \longrightarrow \frac{b_2}{\pi} \int_0^t \theta^2(s)\, ds,$$

where,

$$b_1 = \sum_{m=1}^{\infty} \left(\sqrt{2m+2} - 2\sqrt{2m+1} + \sqrt{2m} \right)^2,$$
$$b_2 = \sum_{m=1}^{\infty} \left(\sqrt{2m} - 2\sqrt{2m-1} + \sqrt{2m-2} \right)^2.$$

We have proved the following result:

Proposition 4.19. *Under the above assumptions, $\rho(r,t)$ satisfies condition (M.5), where*

$$\eta(t) = \frac{2 + 4a - 2b_1 - 2b_2}{\pi} \int_0^t (u(q(s),s))^{-2}\, ds.$$

The coefficient $2 + 4a - 2b_1 - 2b_2$ is approximately 1.3437, while $u(q(t),t)$ depends on f and α.

4.4 Proof of the technical Lemmas

We begin with two technical lemmas. The first is a version of Corollary 4.2 with disjoint intervals.

Lemma 4.20. *For $0 \le t_0 < t_1 \le t_2 < t_3 \le T$,*

$$\lim_{n\to\infty} \sum_{j=\left\lfloor \frac{nt_0}{2} \right\rfloor + 1}^{\left\lfloor \frac{nt_1}{2} \right\rfloor} \sum_{k=\left\lfloor \frac{nt_2}{2} \right\rfloor + 1}^{\left\lfloor \frac{nt_3}{2} \right\rfloor} \left| \left\langle \partial_{\frac{2j-1}{n}}^{\otimes 2} - \partial_{\frac{2j-2}{n}}^{\otimes 2}, \partial_{\frac{2k-1}{n}}^{\otimes 2} - \partial_{\frac{2k-2}{n}}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}} \right| = 0.$$

Proof. We may assume $t_0 = 0$ and $t_1 = t_2$. Observe that

$$\left\langle \partial^{\otimes 2}_{\frac{2j-1}{n}} - \partial^{\otimes 2}_{\frac{2j-2}{n}}, \partial^{\otimes 2}_{\frac{2k-1}{n}} - \partial^{\otimes 2}_{\frac{2k-2}{n}} \right\rangle_{\mathfrak{H}^{\otimes 2}}$$

$$= \beta_n(2j-1,2k-1)^2 - \beta_n(2j-1,2k-2)^2 - \beta_n(2j-2,2k-1)^2 + \beta_n(2j-2,2k-2)^2.$$

Therefore, it is enough to show that,

$$\sum_{j=0}^{\lfloor nt_2 \rfloor} \sum_{k=\lfloor nt_2 \rfloor+1}^{\lfloor nt_3 \rfloor} \beta_n(j,k)^2 \leq Cn^{-\varepsilon} \tag{4.34}$$

for some $\varepsilon > 0$. We can decompose the sum in (4.34) as:

$$\sum_{k=\lfloor nt_2 \rfloor+1}^{\lfloor nt_3 \rfloor} \beta_n(0,k)^2 + \sum_{k=\lfloor nt_2 \rfloor+1}^{\lfloor nt_3 \rfloor} \beta_n(\lfloor nt_2 \rfloor,k)^2 + \sum_{j=1}^{\lfloor nt_2 \rfloor-1} \sum_{k=\lfloor nt_2 \rfloor+1}^{\lfloor nt_3 \rfloor} \beta_n(j,k)^2.$$

By condition (M.4), for some $\gamma > 0$ we have

$$\begin{aligned}\sum_{k=\lfloor nt_2 \rfloor+1}^{\lfloor nt_3 \rfloor} \beta_n(0,k)^2 &\leq \sup_{1\leq j\leq \lfloor nt_3 \rfloor} |\beta_n(0,k)| \sum_{k=\lfloor nt_2 \rfloor+1}^{\lfloor nt_3 \rfloor} |\beta_n(0,k)| \\ &\leq Cn^{-1} \sum_{k=\lfloor nt_2 \rfloor+2}^{\lfloor nt_3 \rfloor} (k-1)^{-\gamma} + Cn^{-1} \leq Cn^{-\gamma}.\end{aligned}$$

By condition (M.2), for some $1 < \alpha \leq \frac{3}{2}$,

$$\begin{aligned}\sum_{k=\lfloor nt_2 \rfloor+1}^{\lfloor nt_3 \rfloor} \beta_n(\lfloor nt_2 \rfloor,k)^2 &\leq \beta_n(\lfloor nt_2 \rfloor, \lfloor nt_2 \rfloor+1)^2 + Cn^{-1} \sum_{k=\lfloor nt_2 \rfloor+2}^{\lfloor nt_3 \rfloor} \beta_n(\lfloor nt_2 \rfloor,k) \\ &\leq Cn^{-1} + Cn^{-1} \sum_{k=\lfloor nt_2 \rfloor+1}^{\lfloor nt_3 \rfloor} (k-\lfloor nt_2 \rfloor)^{-\alpha} \leq Cn^{-1},\end{aligned}$$

and again by condition (M.2), for $\beta = \frac{3}{2} - \alpha$,

$$\begin{aligned}\sum_{j=1}^{\lfloor nt_2 \rfloor-1} \sum_{k=\lfloor nt_2 \rfloor+1}^{\lfloor nt_3 \rfloor} \beta_n(j,k)^2 &\leq Cn^{-1} \sum_{j=1}^{\lfloor nt_2 \rfloor-1} \sum_{k=\lfloor nt_2 \rfloor+1}^{\lfloor nt_3 \rfloor} \left[(k-\lfloor nt_2 \rfloor)^{-\alpha} j^{-\beta} + (k-j)^{-\frac{3}{2}} \right] \\ &\leq Cn^{-1} \left(\sum_{k=1}^{\lfloor nt_3 \rfloor} k^{-\alpha} \right) \left(\sum_{j=1}^{\lfloor nt_2 \rfloor} j^{-\beta} \right) + Cn^{-1} \sum_{j=1}^{\lfloor nt_2 \rfloor} (\lfloor nt_2 \rfloor - j)^{-\frac{1}{2}} \\ &\leq Cn^{-\beta} + Cn^{-\frac{1}{2}};\end{aligned}$$

hence the sum is bounded by $Cn^{-\varepsilon}$ for $\varepsilon = \min\left\{\beta, \gamma, \frac{1}{2}\right\}$. □

Lemma 4.21. *For $0 \le t \le T$ and integer $j \ge 1$,*

$$\left|\left\langle \varepsilon_t, \partial_{\frac{j}{n}} \right\rangle_{\mathfrak{H}}\right| \le Cn^{-\frac{1}{2}}$$

for a positive constant C which depends on T.

Proof. By conditions (M.1) and (M.2), we have for $j \ge 1$ and $t > 0$,

$$\begin{aligned}\left|\left\langle \varepsilon_t, \partial_{\frac{j}{n}} \right\rangle_{\mathfrak{H}}\right| &\le \sum_{k=0}^{\lfloor nt \rfloor -1} \left|\left\langle \partial_{\frac{k}{n}}, \partial_{\frac{j}{n}} \right\rangle_{\mathfrak{H}}\right| + \left|\left\langle \varepsilon_t - \varepsilon_{\lfloor nt \rfloor}, \partial_{\frac{j}{n}} \right\rangle_{\mathfrak{H}}\right| \\ &\le C \sum_{k=0}^{\infty} n^{-\frac{1}{2}} \left(|j-k|^{-\alpha} \wedge 1\right) + O(n^{-\frac{1}{2}}) \le Cn^{-\frac{1}{2}}. \end{aligned} \tag{4.35}$$

□

4.4.1 Proof of Lemma 4.4

By the Lagrange theorem for the Taylor expansion remainder, the terms $R_0(W_{\frac{2j}{n}}), R_1(W_{\frac{2j-2}{n}})$ can be expressed in integral form:

$$R_0(W_{\frac{2j}{n}}) = \frac{1}{2} \int_{W_{\frac{2j-1}{n}}}^{W_{\frac{2j}{n}}} (W_{\frac{2j}{n}} - u)^2 f^{(3)}(u)\, du; \text{ and}$$

$$R_1(W_{\frac{2j-2}{n}}) = -\frac{1}{2} \int_{W_{\frac{2j-2}{n}}}^{W_{\frac{2j-1}{n}}} (W_{\frac{2j-2}{n}} - u)^2 f^{(3)}(u)\, du.$$

After a change of variables, we obtain

$$R_0(W_{\frac{2j}{n}}) = \frac{1}{2}(W_{\frac{2j}{n}} - W_{\frac{2j-1}{n}})^3 \int_0^1 v^2 f^{(3)}(vW_{\frac{2j-1}{n}} + (1-v)W_{\frac{2j}{n}})\, dv;$$

and

$$R_1(W_{\frac{2j-2}{n}}) = \frac{1}{2}(W_{\frac{2j-2}{n}} - W_{\frac{2j-1}{n}})^3 \int_0^1 v^2 f^{(3)}(vW_{\frac{2j-1}{n}} + (1-v)W_{\frac{2j-2}{n}})\, dv.$$

Define

$$G_0(2j) = \frac{1}{2} \int_0^1 v^2 f^{(3)}(vW_{\frac{2j-1}{n}} + (1-v)W_{\frac{2j}{n}})\, dv;$$

and

$$G_1(2j-2) = \frac{1}{2} \int_0^1 v^2 f^{(3)}(vW_{\frac{2j-1}{n}} + (1-v)W_{\frac{2j-2}{n}})\, dv.$$

We may assume $r = 0$. Define $\Delta W_{\frac{\ell}{n}} = W_{\frac{\ell+1}{n}} - W_{\frac{\ell}{n}}$. We want to show that

$$\mathbb{E}\left[\left(\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} \left\{ G_0(2j)\Delta W^3_{\frac{2j-1}{n}} + G_1(2j-2)\Delta W^3_{\frac{2j-2}{n}} \right\}\right)^2\right] \le C \left\lfloor \frac{nt}{2} \right\rfloor n^{-\frac{3}{2}}. \tag{4.36}$$

This part of the proof was inspired by a computation in [25] (see Lemma 4.2). Consider the Hermite polynomial identity $x^3 = H_3(x) + 3H_1(x)$. We use (2.4) for $h \in \mathfrak{H}$ with $\|h\|_{\mathfrak{H}} = 1$. For each j, let $w_j := \|\Delta W_{\frac{j}{n}}\|_{\mathfrak{H}}$, and note that condition (M.1) implies $w_j \le Cn^{-\frac{1}{4}}$ for all j. Then

$$\frac{\Delta W_{\frac{j}{n}}^3}{w_j^3} = H_3\left(\frac{\Delta W_{\frac{j}{n}}}{w_j}\right) + 3H_1\left(\frac{\Delta W_{\frac{j}{n}}}{w_j}\right) = \delta^3\left(\frac{\partial_{\frac{j}{n}}^{\otimes 3}}{w_j^3}\right) + 3\delta\left(\frac{\partial_{\frac{j}{n}}}{w_j}\right)$$

so that

$$\Delta W_{\frac{j}{n}}^3 = \frac{1}{2}\delta^3(\partial_{\frac{j}{n}}^{\otimes 3}) + w_j^2\delta(\partial_{\frac{j}{n}}).$$

It follows that we can write,

$$\begin{aligned}
&G_0(2j)\Delta W_{\frac{2j-1}{n}}^3 - G_1(2j-2)\Delta W_{\frac{2j-2}{n}}^3 \\
&\quad = G_0(2j)\delta^3(\partial_{\frac{2j-1}{n}}^{\otimes 3}) - G_1(2j-2)\delta^3(\partial_{\frac{2j-2}{n}}^{\otimes 3}) \\
&\quad + 3w_{2j}^2 G_0(2j)\delta(\partial_{\frac{2j-1}{n}}) - 3w_{2j-1}^2 G_1(2j-2)\delta(\partial_{\frac{2j-2}{n}}).
\end{aligned}$$

It is enough to verify the individual inequalities

$$\mathbb{E}\left[\left|\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} G_0(2j)\delta^3(\partial_{\frac{2j-1}{n}}^{\otimes 3})\right|^2\right] \le C\left\lfloor \frac{nt}{2} \right\rfloor n^{-\frac{3}{2}}, \tag{4.37}$$

$$\mathbb{E}\left[\left|\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} G_1(2j-2)\delta^3(\partial_{\frac{2j-2}{n}}^{\otimes 3})\right|^2\right] \le C\left\lfloor \frac{nt}{2} \right\rfloor n^{-\frac{3}{2}}, \tag{4.38}$$

$$\mathbb{E}\left[\left|\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} w_{2j}^2 G_0(2j)\delta(\partial_{\frac{2j-1}{n}})\right|^2\right] \le C\left\lfloor \frac{nt}{2} \right\rfloor n^{-\frac{3}{2}}, \tag{4.39}$$

and

$$\mathbb{E}\left[\left|\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} w_{2j-1}^2 G_1(2j-2)\delta(\partial_{\frac{2j-2}{n}})\right|^2\right] \le C\left\lfloor \frac{nt}{2} \right\rfloor n^{-\frac{3}{2}}. \tag{4.40}$$

We will show (4.37) and (4.39), with (4.38) and (4.40) essentially similar.

Proof of (4.37). Using Lemma 2.1.d and the duality property,

$$\mathbb{E}\left[\left(\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} G_0(2j)\delta^3(\partial_{\frac{2j-1}{n}}^{\otimes 3})\right)^2\right]$$

$$= \mathbb{E}\sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor}\left[G_0(2j)G_0(2k)\left(\sum_{r=0}^{3}\delta^{6-2r}(\partial_{\frac{2j-1}{n}}^{\otimes 3-r}\otimes\partial_{\frac{2k-1}{n}}^{\otimes 3-r})\left\langle\partial_{\frac{2j-1}{n}},\partial_{\frac{2k-1}{n}}\right\rangle_{\mathfrak{H}}^{r}\right)\right]$$

$$\leq \sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor}\sum_{r=0}^{3}\left|\left\langle\partial_{\frac{2j-1}{n}},\partial_{\frac{2k-1}{n}}\right\rangle_{\mathfrak{H}}^{r}\right|\mathbb{E}\left[\left|\left\langle D^{6-2r}\left(G_0(2j)G_0(2k)\right),\partial_{\frac{2j-1}{n}}^{\otimes 3-r}\otimes\partial_{\frac{2k-1}{n}}^{\otimes 3-r}\right\rangle_{\mathfrak{H}^{\otimes 6-2r}}\right|\right].$$

For integers $r \geq 0$, we have

$$D^r G_0(2j) = D^r \int_0^1 \frac{1}{2}v^2 f^{(3)}\left(vW_{\frac{2j-1}{n}}+(1-v)W_{\frac{2j}{n}}\right)dv$$

$$= \frac{1}{2}\int_0^1 v^2 f^{(3+r)}\left(vW_{\frac{2j-1}{n}}+(1-v)W_{\frac{2j}{n}}\right)\left(v\varepsilon_{\frac{2j-1}{n}}^{\otimes r}+(1-v)\varepsilon_{\frac{2j}{n}}^{\otimes r}\right)dv. \tag{4.41}$$

By product rule and (4.41) we have

$$\mathbb{E}\left[\left|\left\langle D^{6-2r}\left(G_0(2j)G_0(2k)\right),\partial_{\frac{2j-1}{n}}^{\otimes 3-r}\otimes\partial_{\frac{2k-1}{n}}^{\otimes 3-r}\right\rangle_{\mathfrak{H}^{\otimes 6-2r}}\right|\right] \tag{4.42}$$

$$\leq C\sum_{a+b=6-2r}\mathbb{E}\left[\sup_{0\leq v,w\leq 1}\left|f^{(a)}(vW_{\frac{2j-1}{n}}+(1-v)W_{\frac{2j-2}{n}})f^{(b)}(wW_{\frac{2k-1}{n}}+(1-w)W_{\frac{2k-2}{n}})\right|\right]$$
$$\times\int_0^1\int_0^1\left|\left\langle\left(v\varepsilon_{\frac{2j-1}{n}}^{\otimes a}+(1-v)\varepsilon_{\frac{2j}{n}}^{\otimes a}\right)\otimes\left(w\varepsilon_{\frac{2k-1}{n}}^{\otimes b}+(1-w)\varepsilon_{\frac{2k}{n}}^{\otimes b}\right),\partial_{\frac{2j-1}{n}}^{\otimes 3-r}\otimes\partial_{\frac{2k-1}{n}}^{\otimes 3-r}\right\rangle_{\mathfrak{H}^{\otimes 6-2r}}\right|dv\,dw.$$

Notice that by condition (0), $\mathbb{E}\left[\sup\left|f^{(3+r)}(\xi)\right|^p\right]<\infty$, where the supremum is taken over the random variables $\{\xi = vW_{s_1}+(1-v)W_{s_2}, 0\leq v\leq 1, 0\leq s_1,s_2\leq T\}$. From Lemma 4.21, for integers a and b with $a+b=6-2r$, we have the following estimate

$$\int_0^1\int_0^1\left|\left\langle\left(v\varepsilon_{\frac{2j-1}{n}}^{\otimes a}+(1-v)\varepsilon_{\frac{2j}{n}}^{\otimes a}\right)\otimes\left(w\varepsilon_{\frac{2k-1}{n}}^{\otimes b}+(1-w)\varepsilon_{\frac{2k}{n}}^{\otimes b}\right),\partial_{\frac{2j-1}{n}}^{\otimes 3-r}\otimes\partial_{\frac{2k-1}{n}}^{\otimes 3-r}\right\rangle_{\mathfrak{H}^{\otimes 6-2r}}\right|dv\,dw$$
$$\leq Cn^{-(3-r)}. \tag{4.43}$$

It follows that if $r\neq 0$, then by Lemma 4.1, Equation (4.42), and Equation (4.43)

$$C\sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor}\left|\left\langle\partial_{\frac{2j-1}{n}},\partial_{\frac{2k-1}{n}}\right\rangle_{\mathfrak{H}}^{r}\right|\mathbb{E}\left[\left|\left\langle D^{6-r}\left(G_0(2j)G_0(2k)\right),\partial_{\frac{2j-1}{n}}^{\otimes 3-r}\otimes\partial_{\frac{2k-1}{n}}^{\otimes 3-r}\right\rangle_{\mathfrak{H}^{\otimes 6-2r}}\right|\right]$$
$$\leq Cn^{r-3}\sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor}\left|\left\langle\partial_{\frac{2j-1}{n}},\partial_{\frac{2k-1}{n}}\right\rangle_{\mathfrak{H}}^{r}\right|$$
$$\leq C\left\lfloor\frac{nt}{2}\right\rfloor n^{\frac{r}{2}-3},$$

which satisfies (4.36) because $\frac{r}{2} - 3 \le -\frac{3}{2}$. On the other hand, if $r = 0$, then

$$\sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} Cn^{-3} \le C \left\lfloor \frac{nt}{2} \right\rfloor n^{-2},$$

and we are done with (4.37).

Proof of (4.39). Proceeding along the same lines as above,

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} w_{2j}^2 G_0(2j)\delta\left(\partial_{\frac{2j-1}{n}}\right)\right)^2\right] \\
&\qquad = \mathbb{E}\left[\sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} w_{2j}^2 w_{2k}^2 G_0(2j)G_0(2k)\left\{\delta^2\left(\partial_{\frac{2j-1}{n}} \otimes \partial_{\frac{2k-1}{n}}\right) + \left\langle \partial_{\frac{2j-1}{n}}, \partial_{\frac{2k-1}{n}} \right\rangle_{\mathfrak{H}}\right\}\right] \\
&\qquad \le Cn^{-1} \sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \mathbb{E}\left[\mathbb{E} \sup_{0 \le \ell \le \lfloor \frac{nt}{2} \rfloor} |G_0(\ell)|^2 \left| \left\langle \partial_{\frac{2j-1}{n}}, \partial_{\frac{2k-1}{n}} \right\rangle_{\mathfrak{H}} \right|\right] \\
&\qquad\quad + Cn^{-1} \sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \mathbb{E}\left[\sum_{a+b=2} \mathbb{E}\left| \left\langle D^a G_0(2j) D^b G_0(2k), \delta^2\left(\partial_{\frac{2j-1}{n}} \otimes \partial_{\frac{2k-1}{n}}\right) \right\rangle_{\mathfrak{H}^{\otimes 2}} \right|\right].
\end{aligned}$$

By Lemma 4.1 we have an estimate for the second term:

$$Cn^{-1} \sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \left| \left\langle \partial_{\frac{2j-1}{n}}, \partial_{\frac{2k-1}{n}} \right\rangle_{\mathfrak{H}} \right| \le C \left\lfloor \frac{nt}{2} \right\rfloor n^{-\frac{3}{2}}.$$

Then the first term has the same estimate as (4.42) when $r = 2$, which proves (4.39) and the lemma.

4.4.2 Proof of Lemma 4.5

As in Lemma 4.4, we may assume $r = 0$. Start with $B_n(t)$. Define

$$\begin{aligned}
\gamma_n(t) &:= \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} f^{(3)}(W_{\frac{2j-1}{n}}) \left\langle \varepsilon_{\frac{2j-1}{n}}, \partial_{\frac{2j-1}{n}}^{\otimes 2} - \partial_{\frac{2j-2}{n}}^{\otimes 2} \right\rangle_{\mathfrak{H}} \\
&= \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} f^{(3)}(W_{\frac{2j-1}{n}}) \left\langle \varepsilon_{\frac{2j-1}{n}}, \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}} \right\rangle_{\mathfrak{H}} \left(\partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}}\right),
\end{aligned}$$

so that $B_n(t) = 2\delta(\gamma_n(t))$. By Lemma 2.1.c, we have $\|\delta(\gamma_n(t))\|^2_{L^2(\Omega)} \le \mathbb{E}\|\gamma_n(t)\|^2_{\mathfrak{H}} + \mathbb{E}\|D\gamma_n(t)\|^2_{\mathfrak{H}^{\otimes 2}}$. We can write

$$\begin{aligned}
\|\gamma_n(t)\|^2_{\mathfrak{H}} &= \sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} f^{(3)}(W_{\frac{2j-1}{n}}) f^{(3)}(W_{\frac{2k-1}{n}}) \left\langle \varepsilon_{\frac{2j-1}{n}}, \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}} \right\rangle_{\mathfrak{H}} \\
&\quad \times \left\langle \varepsilon_{\frac{2k-1}{n}}, \partial_{\frac{2k-1}{n}} - \partial_{\frac{2k-2}{n}} \right\rangle_{\mathfrak{H}} \left\langle \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}}, \partial_{\frac{2k-1}{n}} - \partial_{\frac{2k-2}{n}} \right\rangle_{\mathfrak{H}} \\
&\le \sup_{0\le s\le t} \left| f^{(3)}(W_s) \right|^2 \sup_{1\le j\le \lfloor nt \rfloor} \left\langle \varepsilon_{\frac{2j-1}{n}}, \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}} \right\rangle^2_{\mathfrak{H}} \sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \left| \left\langle \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}}, \partial_{\frac{2k-1}{n}} - \partial_{\frac{2k-2}{n}} \right\rangle_{\mathfrak{H}} \right|.
\end{aligned}$$

Note that $\mathbb{E}\left[\sup_{0\le s\le t} |f^{(3)}(W_s)|^2\right] = C$ by (M.0), and by Lemma 4.21, $\left| \left\langle \varepsilon_t, \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}} \right\rangle_{\mathfrak{H}} \right| \le C_2 n^{-\frac{1}{2}}$ for all j,t. By Corollary 4.2 we know,

$$\sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \left| \left\langle \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}}, \partial_{\frac{2k-1}{n}} - \partial_{\frac{2k-2}{n}} \right\rangle_{\mathfrak{H}} \right| \le C \left\lfloor \frac{nt}{2} \right\rfloor n^{-\frac{1}{2}}.$$

Hence, it follows that $\mathbb{E}\|\gamma_n(t)\|^2_{\mathfrak{H}} \le C \left\lfloor \frac{nt}{2} \right\rfloor n^{-1} n^{-\frac{1}{2}} \le C \left\lfloor \frac{nt}{2} \right\rfloor n^{-\frac{3}{2}}$. Next,

$$D\gamma_n(t) = \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} f^{(4)}(W_{\frac{2j-1}{n}}) \left\langle \varepsilon_{\frac{2j-1}{n}}, \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}} \right\rangle_{\mathfrak{H}} \left(\varepsilon_{\frac{2j-1}{n}} \otimes \left(\partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}} \right) \right)$$

and this implies

$$\begin{aligned}
\|D\gamma_n(t)\|^2_{\mathfrak{H}^{\otimes 2}} &\le \sup_{0\le s\le t} \left| f^{(4)}(W_s) \right|^2 \left| \sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \left\langle \varepsilon_{\frac{2j-1}{n}}, \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}} \right\rangle_{\mathfrak{H}} \left\langle \varepsilon_{\frac{2k-1}{n}}, \partial_{\frac{2k-1}{n}} - \partial_{\frac{2k-2}{n}} \right\rangle_{\mathfrak{H}} \right| \\
&\quad \times \left| \left\langle \varepsilon_{\frac{2j-1}{n}} \otimes \left(\partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}} \right), \varepsilon_{\frac{2k-1}{n}} \otimes \left(\partial_{\frac{2k-1}{n}} - \partial_{\frac{2k-2}{n}} \right) \right\rangle_{\mathfrak{H}^{\otimes 2}} \right| \\
&\le \sup_{0\le s\le t} \left| f^{(4)}(W_t) \right|^2 \left(\sup_j \left\langle \varepsilon_{\frac{2j-1}{n}}, \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}} \right\rangle^2_{\mathfrak{H}} \right) \\
&\quad \times \sup_{0\le s,r\le t} \left| \langle \varepsilon_s, \varepsilon_r \rangle_{\mathfrak{H}} \right| \sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \left| \left\langle \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}}, \partial_{\frac{2k-1}{n}} - \partial_{\frac{2k-2}{n}} \right\rangle_{\mathfrak{H}} \right|.
\end{aligned}$$

By condition (M.0), $\mathbb{E}\left[\sup_{0\le s\le t} |f^{(4)}(W_s)|\right]$ is bounded, and $\sup_{0\le s,r\le t} |\langle \varepsilon_r, \varepsilon_s \rangle_{\mathfrak{H}}|$ is bounded. Hence, it can be seen that $\mathbb{E}\|D\gamma_n(t)\|^2_{\mathfrak{H}^{\otimes 2}}$ gives the same estimate as $\gamma_n(t)$.

For $C_n(t)$, using condition (M.0) and the identity $a^2 - b^2 = (a-b)(a+b)$, we can write

$$\mathbb{E}\left[C_n(t)^2\right] \le \mathbb{E}\left[\sup_{0\le s\le t} |f^{(4)}(W_s)|^2\right] \left(\sup_{1\le j\le \frac{nt}{2}} \left| \left\langle \varepsilon_{\frac{2j-1}{n}}, \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}} \right\rangle_{\mathfrak{H}} \right| \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} \left| \left\langle \varepsilon_{\frac{2j-1}{n}}, \partial_{\frac{2j-1}{n}} + \partial_{\frac{2j-2}{n}} \right\rangle_{\mathfrak{H}} \right| \right)^2$$

By Lemma 4.21, $\left|\left\langle \varepsilon_{\frac{2j-1}{n}}, \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}} \right\rangle_{\mathfrak{H}}\right| \leq C_2 n^{-\frac{1}{2}}$, and by condition (M.4),

$$\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} \left|\left\langle \varepsilon_{\frac{2j-1}{n}}, \mathbf{1}_{[\frac{2j-2}{n}, \frac{2j}{n})} \right\rangle_{\mathfrak{H}}\right| \leq Cn^{-\frac{1}{2}} + Cn^{-\frac{1}{2}} \sum_{j=2}^{\lfloor \frac{nt}{2} \rfloor} (2j-2)^{-\frac{1}{2}} \leq C \left\lfloor \frac{nt}{2} \right\rfloor^{\frac{1}{2}} n^{-\frac{1}{2}}.$$

Hence it follows that $\mathbb{E}\left[C_n(t)^2\right] \leq C \left\lfloor \frac{nt}{2} \right\rfloor n^{-2}$ for some constant C, and the lemma is proved.

4.4.3 Proof of Lemma 4.9

For $i = 1, \ldots, d$, set

$$u_n^i = \sum_{j=\lfloor \frac{nt_{i-1}}{2} \rfloor + 1}^{\lfloor \frac{nt_i}{2} \rfloor} f''(W_{\frac{2j-1}{n}}) \left(\partial_{\frac{2j-1}{n}}^{\otimes 2} - \partial_{\frac{2j-2}{n}}^{\otimes 2} \right),$$

and recall that $F_n^i = \delta^2(u_n^i)$. We want to show:
Condition (a). For each $1 \leq i \leq d$, the following converge to zero in $L^1(\Omega)$:

(a.1) $\langle u_n^i, h_1 \otimes h_2 \rangle_{\mathfrak{H}^{\otimes 2}}$ for all $h_1, h_2 \in \mathfrak{H}$ of the form ε_τ (see Remark 2.4).

(a.2) $\left\langle u_n^i, DF_n^j \otimes h \right\rangle_{\mathfrak{H}^{\otimes 2}}$ for each $1 \leq j \leq d$ and $h \in \mathfrak{H}$.

(a.3) $\left\langle u_n^i, DF_n^j \otimes DF_n^k \right\rangle_{\mathfrak{H}^{\otimes 2}}$ for each $1 \leq j, k \leq d$.

Condition (b).

(b.1) $\left\langle u_n^i, D^2 F_n^j \right\rangle_{\mathfrak{H}^{\otimes 2}} \longrightarrow 0$ in L^1 if $i \neq j$.

(b.2) $\langle u_n^i, D^2 F_n^i \rangle_{\mathfrak{H}^{\otimes 2}}$ converges in L^1 to a random variable of the form

$$F_\infty^j = c \int_{t_{i-1}}^{t_i} f''(W_s)^2 \eta(ds).$$

The proofs of (a.1) and (a.2) are essentially the same as given in [23] (see Theorem 5.2) but the proof of (a.3) is new.

Proof of (a.1). We may assume $i = 1$. Let $h_1 \otimes h_2 = \varepsilon_s \otimes \varepsilon_\tau \in \mathfrak{H}^{\otimes 2}$ for some values $s, \tau \in [0, t]$. Then

$$\langle u_n^1, h_1 \otimes h_2 \rangle_{\mathfrak{H}^{\otimes 2}} = \sum_{j=1}^{\lfloor \frac{nt_1}{2} \rfloor} f''(W_{\frac{2j-1}{n}}) \left\langle \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}}, \varepsilon_s \right\rangle_{\mathfrak{H}} \left\langle \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}}, \varepsilon_\tau \right\rangle_{\mathfrak{H}};$$

so that

$$\left|\langle u_n^1, h_1 \otimes h_2 \rangle_{\mathfrak{H}^{\otimes 2}}\right| \leq \sup_{0 \leq s \leq t} |f''(W_s)| \sup_{1 \leq j \leq \lfloor \frac{nt_1}{2} \rfloor} \sup_{0 \leq s \leq t_1} \left|\left\langle \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}}, \varepsilon_s \right\rangle_{\mathfrak{H}}\right| \sum_{j=1}^{\lfloor \frac{nt_1}{2} \rfloor} \left|\left\langle \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}}, \varepsilon_\tau \right\rangle_{\mathfrak{H}}\right|.$$

It follows from condition (M.3) that for fixed $\tau \geq 0$

$$\begin{aligned}
\sum_{j=1}^{\lfloor \frac{nt_1}{2} \rfloor} \left| \left\langle \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}}, \varepsilon_\tau \right\rangle_{\mathfrak{H}} \right| &= \sum_{j=1}^{\lfloor \frac{nt_1}{2} \rfloor} \left| \mathbb{E}\left[W_\tau (W_{\frac{2j}{n}} - 2W_{\frac{2j-1}{n}} + W_{\frac{2j-2}{n}}) \right] \right| \\
&\leq Cn^{-\frac{1}{2}} + Cn^{-\frac{1}{2}} \sum_{j=2}^{\lfloor \frac{nt_1}{2} \rfloor} \left((2j-2)^{-\frac{3}{2}} + |\tau - 2j|^{-\frac{3}{2}} \wedge 1 \right) \\
&\leq Cn^{-\frac{1}{2}} \qquad (4.44)
\end{aligned}$$

and Lemma 4.21 implies,

$$\sup_{1 \leq j \leq \lfloor \frac{nt_1}{2} \rfloor} \sup_{0 \leq s \leq t} \left| \left\langle \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}}, \varepsilon_s \right\rangle_{\mathfrak{H}} \right| \leq Cn^{-\frac{1}{2}}$$

so that

$$\mathbb{E}\left(\left| \langle u_n^1, h_1 \otimes h_2 \rangle_{\mathfrak{H}^{\otimes 2}} \right| \right) \leq Ct_1 n^{-1} \longrightarrow 0.$$

Proof of (a.2). As in (a.1), assume $i = 1$. Using the same technique as in (a.1), we can write $DF_n^j \otimes h$ as $DF_n^j \otimes \varepsilon_\tau$ for some $\tau \in [0, T]$. By Lemma 2.1.b, $DF_n^j = D\delta^2(u_n^j) = \delta^2(Du_n^j) + \delta(u_n^j)$, which gives

$$\langle u_n^1, DF_n^j \otimes \varepsilon_\tau \rangle_{\mathfrak{H}^{\otimes 2}} = \langle u_n^1, \delta^2(Du_n^j) \otimes \varepsilon_\tau \rangle_{\mathfrak{H}^{\otimes 2}} + \langle u_n^1, \delta(u_n^j) \otimes \varepsilon_\tau \rangle_{\mathfrak{H}^{\otimes 2}}.$$

For the first term,

$$\begin{aligned}
\mathbb{E}\left| \langle u_n^1, \delta^2(Du_n^j) \otimes \varepsilon_\tau \rangle_{\mathfrak{H}^{\otimes 2}} \right| &= \sum_{\ell=1}^{\lfloor \frac{nt_1}{2} \rfloor} \mathbb{E}\left| f''(W_{\frac{2\ell-1}{n}}) \left\langle \partial_{\frac{2\ell-1}{n}} - \partial_{\frac{2\ell-2}{n}}, \delta^2(Du_n^j) \right\rangle_{\mathfrak{H}} \left\langle \partial_{\frac{2\ell-1}{n}} - \partial_{\frac{2\ell-2}{n}}, \varepsilon_\tau \right\rangle_{\mathfrak{H}} \right| \\
&\leq 2\mathbb{E}\left[\sup_{0 \leq s \leq t_1} |f''(W_s)| \right] \mathbb{E}\left[\sup_{1 \leq \ell \leq \lfloor \frac{nt_1}{2} \rfloor} \left| \left\langle \partial_{\frac{\ell}{n}}, \delta^2(Du_n^j) \right\rangle_{\mathfrak{H}} \right| \right] \\
&\quad \times \sum_{\ell=1}^{\lfloor \frac{nt_1}{2} \rfloor} \left| \left\langle \partial_{\frac{2\ell-1}{n}} - \partial_{\frac{2\ell-2}{n}}, \varepsilon_\tau \right\rangle_{\mathfrak{H}} \right|.
\end{aligned}$$

By (4.44), the sum has estimate $Cn^{-\frac{1}{2}}$, and for the second term we can take

$$\left| \left\langle \partial_{\frac{\ell}{n}}, \delta^2(Du_n^j) \right\rangle_{\mathfrak{H}} \right| \leq \sup_{\ell} \|\partial_{\frac{\ell}{n}}\|_{\mathfrak{H}} \|\delta^2(Du_n^j)\|_{\mathfrak{H}}.$$

It follows from condition (M.1) that $\|\partial_{\frac{\ell}{n}}\|_{\mathfrak{H}} \leq Cn^{-\frac{1}{4}}$. This leaves the $\|\delta^2(Du_n^j)\|_{\mathfrak{H}}$ term. By the Meyer inequality for a process taking values in $\mathfrak{H}$,

$$\mathbb{E}\left[\|\delta^2(Du_n^j)\|_{\mathfrak{H}}^2 \right] \leq c_1 \mathbb{E}\|Du_n^j\|_{\mathfrak{H}^{\otimes 3}}^2 + c_2 \mathbb{E}\|D^2 u_n^j\|_{\mathfrak{H}^{\otimes 4}}^2 + c_3 \mathbb{E}\|D^3 u_n^j\|_{\mathfrak{H}^{\otimes 5}}^2, \qquad (4.45)$$

so that by Lemma 4.7, $\mathbb{E}\left[\|\delta^2(Du)\|_{\mathfrak{H}}^2\right] \le C$, and we have

$$\mathbb{E}\left|\left\langle u_n^1, \delta^2(Du_n^j) \otimes \varepsilon_\tau \right\rangle_{\mathfrak{H}^{\otimes 2}}\right| \le Cn^{-\frac{3}{4}}.$$

Then similarly,

$$\left|\left\langle u_n^1, \delta(u_n^j) \otimes \varepsilon_t \right\rangle_{\mathfrak{H}^{\otimes 2}}\right| \le 2\left[\sup_{0\le s\le t_1} |f''(W_s)| \sup_{\ell} \left|\left\langle \partial_{\frac{\ell}{n}}, \delta(u_n^j) \right\rangle_{\mathfrak{H}}\right| \sum_{\ell} \left|\left\langle \partial_{\frac{2\ell-1}{n}} - \partial_{\frac{2\ell-2}{n}}, \varepsilon_t \right\rangle_{\mathfrak{H}}\right|\right].$$

Similar to the above case, for each $1 \le \ell \le \left\lfloor \frac{nt_1}{2} \right\rfloor$,

$$\begin{aligned}\mathbb{E}\left[\left|\left\langle \partial_{\frac{\ell}{n}}, \delta(u_n^j) \right\rangle_{\mathfrak{H}}\right|\right] &\le \mathbb{E}\left[\|\partial_{\frac{\ell}{n}}\|_{\mathfrak{H}} \|\delta(u_n^j)\|_{\mathfrak{H}}\right] \\ &\le Cn^{-\frac{1}{4}}\left(\mathbb{E}\|u_n^j\|_{\mathfrak{H}^{\otimes 2}} + \mathbb{E}\|Du_n^j\|_{\mathfrak{H}^{\otimes 3}}\right) \le Cn^{-\frac{1}{4}},\end{aligned}$$

hence with (4.44) we have

$$\mathbb{E}\left[\left|\left\langle u_n^1, \delta(u_n^j) \otimes \varepsilon_\tau \right\rangle_{\mathfrak{H}^{\otimes 2}}\right|\right] \le Cn^{-\frac{3}{4}}.$$

Proof of (a.3). For this term we consider the product $\left\langle u_n^i, DF_n^j \otimes DF_n^k \right\rangle_{\mathfrak{H}^{\otimes 2}}$. Lemma 4.20 shows that scalar products of this kind are small in absolute value when the time intervals are disjoint, therefore it is enough to consider the worst case $\langle u_n^1, DF_n^1 \otimes DF_n^1 \rangle_{\mathfrak{H}^{\otimes 2}}$, and assume $t_1 = t$. We have

$$\begin{aligned}\mathbb{E}\left[\left|\left\langle u_n^1, DF_n^1 \otimes DF_n^1 \right\rangle_{\mathfrak{H}^{\otimes 2}}\right|\right] &\le \sum_{\ell=1}^{\left\lfloor \frac{nt}{2} \right\rfloor} \left|\mathbb{E}\left[\left\langle f''(W_{\frac{2\ell-1}{n}})\left(\partial_{\frac{2\ell-1}{n}}^{\otimes 2} - \partial_{\frac{2\ell-2}{n}}^{\otimes 2}\right), DF_n^1 \otimes DF_n^1 \right\rangle_{\mathfrak{H}^{\otimes 2}}\right]\right| \\ &\le C\sum_{\ell=1}^{\left\lfloor \frac{nt}{2} \right\rfloor} \mathbb{E}\left[\left|\left\langle \partial_{\frac{2\ell-1}{n}}, DF_n^1 \right\rangle_{\mathfrak{H}}^2 - \left\langle \partial_{\frac{2\ell-2}{n}}, DF_n^1 \right\rangle_{\mathfrak{H}}^2\right|\right] \\ &\le C\sum_{\ell=1}^{\left\lfloor \frac{nt}{2} \right\rfloor} \mathbb{E}\left[\left|\left\langle \partial_{\frac{2\ell-1}{n}} - \partial_{\frac{2\ell-2}{n}}, DF_n^1 \right\rangle_{\mathfrak{H}}\right| \left|\left\langle \mathbf{1}_{[\frac{2\ell-2}{n}, \frac{2\ell}{n}]}, DF_n^1 \right\rangle_{\mathfrak{H}}\right|\right].\end{aligned}$$

Using the decomposition $DF_n^1 = \delta^2(Du_n^1) + \delta(u_n^1)$, the above summand expands into four terms:

$$\begin{aligned}&(1) \quad \left|\left\langle \partial_{\frac{2\ell-1}{n}} - \partial_{\frac{2\ell-2}{n}}, \delta^2(Du_n^1) \right\rangle_{\mathfrak{H}}\right| \left|\left\langle \mathbf{1}_{[\frac{2\ell-2}{n}, \frac{2\ell}{n}]}, \delta^2(Du_n^1) \right\rangle_{\mathfrak{H}}\right| \\ &(2) \quad \left|\left\langle \partial_{\frac{2\ell-1}{n}} - \partial_{\frac{2\ell-2}{n}}, \delta^2(Du_n^1) \right\rangle_{\mathfrak{H}}\right| \left|\left\langle \mathbf{1}_{[\frac{2\ell-2}{n}, \frac{2\ell}{n}]}, \delta(u_n^1) \right\rangle_{\mathfrak{H}}\right| \\ &(3) \quad \left|\left\langle \partial_{\frac{2\ell-1}{n}} - \partial_{\frac{2\ell-2}{n}}, \delta(u_n^1) \right\rangle_{\mathfrak{H}}\right| \left|\left\langle \mathbf{1}_{[\frac{2\ell-2}{n}, \frac{2\ell}{n}]}, \delta^2(Du_n^1) \right\rangle_{\mathfrak{H}}\right| \\ &(4) \quad \left|\left\langle \partial_{\frac{2\ell-1}{n}} - \partial_{\frac{2\ell-2}{n}}, \delta(u_n^1) \right\rangle_{\mathfrak{H}}\right| \left|\left\langle \mathbf{1}_{[\frac{2\ell-2}{n}, \frac{2\ell}{n}]}, \delta(u_n^1) \right\rangle_{\mathfrak{H}}\right|.\end{aligned}$$

We will show computations for the terms (1) and (4) only, with the others similar. For (1) we have

$$
\begin{aligned}
&C\sum_{\ell=1}^{\lfloor\frac{nt}{2}\rfloor}\mathbb{E}\left[\left|\left\langle\partial_{\frac{2\ell-1}{n}}-\partial_{\frac{2\ell-2}{n}},\delta^2(Du_n^1)\right\rangle_{\mathfrak{H}}\right|\,\left|\left\langle\mathbf{1}_{[\frac{2\ell-2}{n},\frac{2\ell}{n}]},\delta^2(Du_n^1)\right\rangle_{\mathfrak{H}}\right|\right]\\
&=C\sum_{\ell,m,m'=1}^{\lfloor\frac{nt}{2}\rfloor}\mathbb{E}\left|\left\langle\partial_{\frac{2\ell-1}{n}}-\partial_{\frac{2\ell-2}{n}},\delta^2\left(f^{(3)}(W_{\frac{2m-1}{n}})\varepsilon_{\frac{2m-1}{n}}\left(\partial^{\otimes 2}_{\frac{2m-1}{n}}-\partial^{\otimes 2}_{\frac{2m-2}{n}}\right)\right)\right\rangle_{\mathfrak{H}}\right|\\
&\quad\times\left|\left\langle\mathbf{1}_{[\frac{2\ell-2}{n},\frac{2\ell}{n}]},\delta^2\left(f^{(3)}(W_{\frac{2m'-1}{n}})\varepsilon_{\frac{2m'-1}{n}}\left(\partial^{\otimes 2}_{\frac{2m'-1}{n}}-\partial^{\otimes 2}_{\frac{2m'-2}{n}}\right)\right)\right\rangle_{\mathfrak{H}}\right|\\
&\le C\sup_{1\le k\le\lfloor\frac{nt}{2}\rfloor}\left(\mathbb{E}\left[\left\|\delta^2\left(f^{(3)}(W_{\frac{2k-1}{n}})\left(\partial^{\otimes 2}_{\frac{2k-1}{n}}-\partial^{\otimes 2}_{\frac{2k-2}{n}}\right)\right)\right\|_{\mathfrak{H}^{\otimes 2}}\right]\right)^2\\
&\quad\times\sum_{\ell,m,m'=1}^{\lfloor\frac{nt}{2}\rfloor}\left|\left\langle\partial_{\frac{2\ell-1}{n}}-\partial_{\frac{2\ell-2}{n}},\varepsilon_{\frac{2m-1}{n}}\right\rangle_{\mathfrak{H}}\right|\,\left|\left\langle\mathbf{1}_{[\frac{2\ell-2}{n},\frac{2\ell}{n}]},\varepsilon_{\frac{2m'-1}{n}}\right\rangle_{\mathfrak{H}}\right|.
\end{aligned}
$$

By Lemmas 2.1.c and 4.7, the Skorohod integral term is bounded by $Cn^{-\frac{1}{2}}$, and we use conditions (M.3) and (M.4) for the scalar products to obtain an estimate of the form

$$
Cn^{-2}\sum_{\ell,m,m'=1}^{\lfloor\frac{nt}{2}\rfloor}\left((2m-1)^{-\frac{3}{2}}+|2\ell-2m|^{-\frac{3}{2}}\right)\,\left((2\ell-2)^{-\frac{1}{2}}+|2\ell-2m'|^{-\frac{1}{2}}\right)\le Cn^{-\frac{1}{2}}.
$$

For term (4), we have by a computation similar to the proof of Lemma 4.7,

$$
\mathbb{E}\left[\left\|\delta\left(f^{(3)}(W_{\frac{2k-1}{n}})\left(\partial_{\frac{2k-1}{n}}-\partial_{\frac{2k-2}{n}}\right)\right)\right\|_{\mathfrak{H}}\right]\le Cn^{-\frac{1}{4}},
$$

and by conditions (M.1) and (M.2) we have

$$
\begin{aligned}
&Cn^{-\frac{3}{2}}\sum_{\ell,m,m'=1}^{\lfloor\frac{nt}{2}\rfloor}\left|\left\langle\partial_{\frac{2\ell-1}{n}}-\partial_{\frac{2\ell-2}{n}},\partial_{\frac{2m-1}{n}}-\partial_{\frac{2m-2}{n}}\right\rangle_{\mathfrak{H}}\right|\,\left|\left\langle\mathbf{1}_{[\frac{2\ell-2}{n},\frac{2\ell}{n}]},\partial_{\frac{2m'-1}{n}}-\partial_{\frac{2m'-2}{n}}\right\rangle_{\mathfrak{H}}\right|\\
&\le Cn^{-\frac{3}{2}}\sum_{\ell,m,m'=1}^{\lfloor\frac{nt}{2}\rfloor}\left(|2\ell-2m|^{-\alpha}\right)\,\left(|2\ell-2m'|^{-\alpha}\right)\\
&\le Cn^{-\frac{1}{2}}.
\end{aligned}
$$

Proof of (b.1). By Lemma 2.1.b, we can expand D^2F_n as follows:

$$
\langle u_n^i,D^2F_n^j\rangle_{\mathfrak{H}^{\otimes 2}}=\langle u_n^i,\delta^2(D^2u_n^j)\rangle_{\mathfrak{H}^{\otimes 2}}+4\langle u_n^i,\delta(Du_n^j)\rangle_{\mathfrak{H}^{\otimes 2}}+2\langle u_n^i,u_n^j\rangle_{\mathfrak{H}^{\otimes 2}} \tag{4.46}
$$

Without loss of generality, we may assume that u_n^i is defined on $[0,t_1]$ and F_n^j is defined on $[t_1,t_2]$ for $t_1<t_2$, so that the sums are over

$$
u_n^i=\sum_{\ell=1}^{\lfloor\frac{nt_1}{2}\rfloor}f''(W_{\frac{2\ell-1}{n}})\left(\partial^{\otimes 2}_{\frac{2\ell-1}{n}}-\partial^{\otimes 2}_{\frac{2\ell-2}{n}}\right);\quad\text{and}\quad u_n^j=\sum_{m=\lfloor\frac{nt_1}{2}\rfloor+1}^{\lfloor\frac{nt_2}{2}\rfloor}f''(W_{\frac{2m-1}{n}})\left(\partial^{\otimes 2}_{\frac{2m-1}{n}}-\partial^{\otimes 2}_{\frac{2m-2}{n}}\right).
$$

First term

$$
\begin{aligned}
&\mathbb{E}\left|\left\langle u_n^i, \delta^2(D^2u_n^j)\right\rangle_{\mathfrak{H}^{\otimes 2}}\right| \\
&= \mathbb{E}\left|\left\langle \sum_{\ell=1}^{\lfloor \frac{nt_1}{2}\rfloor} f''(W_{\frac{2\ell-1}{n}})\left(\partial_{\frac{2\ell-1}{n}}^{\otimes 2} - \partial_{\frac{2\ell-2}{n}}^{\otimes 2}\right), \delta^2\left(\sum_{m=\lfloor \frac{nt_1}{2}\rfloor+1}^{\lfloor \frac{nt_2}{2}\rfloor} f^{(4)}(W_{\frac{2m-1}{n}})\varepsilon_{\frac{2m-1}{n}}^{\otimes 2}\otimes\left(\partial_{\frac{2m-1}{n}}^{\otimes 2} - \partial_{\frac{2m-2}{n}}^{\otimes 2}\right)\right)\right\rangle_{\mathfrak{H}^{\otimes 2}}\right| \\
&\le \mathbb{E}\left[\sup_{0\le s\le t}|f''(W_s)|\right]\mathbb{E}\left[\sum_\ell\sum_m\left|\left\langle \varepsilon_{\frac{2m-1}{n}}^{\otimes 2}, \partial_{\frac{2\ell-1}{n}}^{\otimes 2} - \partial_{\frac{2\ell-2}{n}}^{\otimes 2}\right\rangle_{\mathfrak{H}^{\otimes 2}}\right|\left|\delta^2\left(f^{(4)}(W_{\frac{2m-1}{n}})\left(\partial_{\frac{2m-1}{n}}^{\otimes 2} - \partial_{\frac{2m-2}{n}}^{\otimes 2}\right)\right)\right|\right] \\
&\le \mathbb{E}\left[\sup_{0\le s\le t}|f''(W_s)|\right]\sup_m\|\delta^2(g_4)\|_{L^2(\Omega)}\sum_{\ell=1}^{\lfloor \frac{nt_2}{2}\rfloor}\sum_{m=1}^{\lfloor \frac{nt_2}{2}\rfloor}\left[\left\langle \varepsilon_{\frac{2m-1}{n}}, \partial_{\frac{2\ell-1}{n}}\right\rangle_{\mathfrak{H}}^2 - \left\langle \varepsilon_{\frac{2m-1}{n}}, \partial_{\frac{2\ell-2}{n}}\right\rangle_{\mathfrak{H}}^2\right]
\end{aligned}
$$

First we need an estimate for the $\delta^2(g_4)$ term, where in the notation of Lemma 4.7,

$$g_4 := f^{(4)}(W_{\frac{2m-1}{n}})\left(\partial_{\frac{2m-1}{n}}^{\otimes 2} - \partial_{\frac{2m-2}{n}}^{\otimes 2}\right).$$

By Lemma 2.1.c, $\|\delta^2(g_4)\|_{L^2(\Omega)} \le c_1\mathbb{E}\|g_4\|_{\mathfrak{H}^{\otimes 2}} + c_2\mathbb{E}\|Dg_4\|_{\mathfrak{H}^{\otimes 3}} + c_3\mathbb{E}\|D^2g_4\|_{\mathfrak{H}^{\otimes 4}}$, and so

$$\|\delta^2(g_4)\|_{L^2(\Omega)} \le Cn^{-\frac{1}{2}}$$

for each $\lfloor \frac{nt_1}{2}\rfloor < m \le \lfloor \frac{nt_2}{2}\rfloor$. We can write,

$$\mathbb{E}\left|\left\langle u_n^i, \delta^2(D^2u_n^i)\right\rangle_{\mathfrak{H}^{\otimes 2}}\right| \le Cn^{-\frac{1}{2}}\sum_{\ell,m=1}^{\lfloor \frac{nt_2}{2}\rfloor}\left|\left\langle \varepsilon_{\frac{2m-1}{n}}, \partial_{\frac{2\ell-1}{n}}\right\rangle_{\mathfrak{H}}^2 - \left\langle \varepsilon_{\frac{2m-1}{n}}, \partial_{\frac{2\ell-2}{n}}\right\rangle_{\mathfrak{H}}^2\right|$$

We need an estimate for the double sum. We have by condition (M.3),

$$
\begin{aligned}
&\sum_{\ell,m=1}^{\lfloor \frac{nt_2}{2}\rfloor}\left[\left|\left\langle \varepsilon_{\frac{2m-1}{n}}, \partial_{\frac{2\ell-1}{n}}\right\rangle_{\mathfrak{H}}^2 - \left\langle \varepsilon_{\frac{2m-1}{n}}, \partial_{\frac{2\ell-2}{n}}\right\rangle_{\mathfrak{H}}^2\right|\right] \\
&\qquad\le \sup_{\ell,m}\left|\left\langle \varepsilon_{\frac{2m-1}{n}}, \mathbf{1}_{[\frac{2\ell-2}{n},\frac{2\ell}{n}]}\right\rangle_{\mathfrak{H}}\right|\sum_{\ell,m=1}^{\lfloor \frac{nt_2}{2}\rfloor}\left|\left\langle \varepsilon_{\frac{2m-1}{n}}, \partial_{\frac{2\ell-1}{n}} - \partial_{\frac{2\ell-2}{n}}\right\rangle_{\mathfrak{H}}\right| \\
&\qquad\le Cn^{-\frac{1}{2}}\sum_{\ell,m=1}^{\lfloor \frac{nt_2}{2}\rfloor}C_2n^{-\frac{1}{2}}\left[\left(|\ell-m|^{-\frac{3}{2}} + (\ell-1)^{-\frac{3}{2}}\right)\wedge 1\right] \\
&\qquad\le Cn^{-1}\sum_{\ell=1}^{\lfloor \frac{nt_2}{2}\rfloor}\sum_{p=1}^{\infty}p^{-\frac{3}{2}} \le C
\end{aligned}
$$

This provides an upper bound for the double sum, hence the first term of (4.46) is $O(n^{-\frac{1}{2}})$. Note that in the above estimate the double sum is taken over $1 \le \ell, m \le \lfloor \frac{nt_2}{2}\rfloor$. It follows that this estimate also holds for the case $i = j$, that is, $\mathbb{E}\left|\left\langle u_n^i, \delta^2(D^2u_n^i)\right\rangle_{\mathfrak{H}^{\otimes 2}}\right| \le Cn^{-\frac{1}{2}}$.

Second Term

Using $t_1 < t_2$ as above,

$$\mathbb{E}\left|\left\langle u_n^i, \delta(Du_n^j)\right\rangle_{\mathfrak{H}^{\otimes 2}}\right|$$

$$= \mathbb{E}\left|\left\langle \sum_{j=1}^{\lfloor \frac{nt_1}{2} \rfloor} f''(W_{\frac{2j-1}{n}})\left(\partial_{\frac{2j-1}{n}}^{\otimes 2} - \partial_{\frac{2j-2}{n}}^{\otimes 2}\right), \delta\left(\sum_{k=\lfloor \frac{nt_1}{2} \rfloor}^{\lfloor \frac{nt_2}{2} \rfloor} f^{(3)}(W_{\frac{2k-1}{n}})\varepsilon_{\frac{2k-1}{n}} \otimes \left(\partial_{\frac{2k-1}{n}}^{\otimes 2} - \partial_{\frac{2k-2}{n}}^{\otimes 2}\right)\right)\right\rangle_{\mathfrak{H}^{\otimes 2}}\right|$$

$$= \mathbb{E}\left|\sum_{j,k} f''(W_{\frac{2j-1}{n}})\left\langle \varepsilon_{\frac{2k-1}{n}}, \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}}\right\rangle_{\mathfrak{H}}\left\langle \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}}, \partial_{\frac{2k-1}{n}} - \partial_{\frac{2k-2}{n}}\right\rangle_{\mathfrak{H}}\right|$$
$$\times\left|\delta\left(f^{(3)}(W_{\frac{2k-1}{n}})\left(\partial_{\frac{2k-1}{n}} - \partial_{\frac{2k-2}{n}}\right)\right)\right|$$
$$\le C\,\mathbb{E}\left[\sup_{0\le s\le t}|f''(W_s)|\right]\left(\sup_{s,j}\left|\left\langle \varepsilon_s, \partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}\right|\right)\left(\sup_k \|\delta(g_3)\|_{L^2(\Omega)}\right)\sum_{j=0}^{\lfloor nt_2 \rfloor}\sum_{k=0}^{\lfloor nt_2 \rfloor}\left|\left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|,$$

where in this case, g_3 corresponds to the term including $f^{(3)}(W_t)$. It follows from Lemma 4.21 that $\sup|\left\langle \varepsilon_s, \partial_{k/n}\right\rangle_{\mathfrak{H}}| \le Cn^{-\frac{1}{2}}$; and the double sum is bounded by $Cn^{\frac{1}{2}}$ by Corollary 4.2. This leaves an estimate for $\|\delta(g_3)\|_{L^2(\Omega)}$. By Lemma 2.1, $\|\delta(g_3)\|_{L^2(\Omega)} \le c_1\|g_3\|_{\mathfrak{H}} + c_2\|Dg_3\|_{\mathfrak{H}^{\otimes 2}}$. For this case,

$$\|g_3\|_{\mathfrak{H}}^2 \le \mathbb{E}\left[\sup_{0\le s\le t}|f^{(3)}(W_s)|^2\right]\left\|\partial_{\frac{2k-1}{n}} - \partial_{\frac{2k-2}{n}}\right\|_{\mathfrak{H}}^2 \le Cn^{-\frac{1}{2}},$$

hence $\|g_3\|_{\mathfrak{H}} \le Cn^{-\frac{1}{4}}$. Similarly,

$$\|Dg_3\|_{\mathfrak{H}^{\otimes 2}} \le \mathbb{E}\left[\sup_{0\le s\le t}|f^{(4)}(W_s)|\right]\sup_{0\le s\le t}\|\varepsilon_s\|_{\mathfrak{H}}\left\|\partial_{\frac{2k-1}{n}} - \partial_{\frac{2k-2}{n}}\right\|_{\mathfrak{H}} \le Cn^{-\frac{1}{4}},$$

hence the second term is $O(n^{-\frac{1}{4}})$. As in the first term, the double sum estimate shows that this result also holds for $\left\langle u_n^i, \delta(DF_n^i)\right\rangle_{\mathfrak{H}^{\otimes 2}}$.

Third Term

We can write

$$\left|\left\langle u_n^i, u_n^j\right\rangle_{\mathfrak{H}^{\otimes 2}}\right| \le \sup_{0\le s\le t}|f''(W_s)|^2 \sum_{\ell=1}^{\lfloor \frac{nt_1}{2} \rfloor}\sum_{m=\lfloor \frac{nt_1}{2} \rfloor+1}^{\lfloor \frac{nt_2}{2} \rfloor}\left|\left\langle \partial_{\frac{2\ell-1}{n}}^{\otimes 2} - \partial_{\frac{2\ell-2}{n}}^{\otimes 2}, \partial_{\frac{2m-1}{n}}^{\otimes 2} - \partial_{\frac{2m-2}{n}}^{\otimes 2}\right\rangle_{\mathfrak{H}^{\otimes 2}}\right|$$

and it follows from Lemma 4.20 that $\mathbb{E}\left|\left\langle u_n^i, u_n^j\right\rangle_{\mathfrak{H}^{\otimes 2}}\right| \le Cn^{-\varepsilon}$, for some $\varepsilon > 0$.

Proof of (b.2). As in case (b.1), this has the expansion (4.46). From remarks in the proof of (b.1), the first two terms have the same estimate as the $i \neq j$ case, hence only the term $\left\langle u_n^i, u_n^i\right\rangle_{\mathfrak{H}^{\otimes 2}}$ is significant.

Third Term

Assume for the summation terms that the indices run over $\lfloor \frac{nt_{i-1}}{2} \rfloor + 1 \le j,k \le \lfloor \frac{nt_i}{2} \rfloor$. We have

$$\langle u_n^i, u_n^i \rangle_{\mathfrak{H}^{\otimes 2}} = \sum_{j,k} f''(W_{\frac{2j-1}{n}}) f''(W_{\frac{2k-1}{n}}) \left\langle \partial^{\otimes 2}_{\frac{2j-1}{n}} - \partial^{\otimes 2}_{\frac{2j-2}{n}}, \partial^{\otimes 2}_{\frac{2k-1}{n}} - \partial^{\otimes 2}_{\frac{2k-2}{n}} \right\rangle_{\mathfrak{H}^{\otimes 2}}.$$

Expanding the product, observe that,

$$\begin{aligned} \left\langle \partial^{\otimes 2}_{\frac{2j-1}{n}} - \partial^{\otimes 2}_{\frac{2j-2}{n}}, \partial^{\otimes 2}_{\frac{2k-1}{n}} - \partial^{\otimes 2}_{\frac{2k-2}{n}} \right\rangle_{\mathfrak{H}^{\otimes 2}} &= \beta_n(2j-1,2k-1)^2 - \beta_n(2j-1,2k-2)^2 \\ &\quad - \beta_n(2j-2,2k-1)^2 + \beta_n(2j-2,2k-2)^2, \end{aligned}$$

where $\beta_n(\ell,m)$ is as defined for condition (M.5). For each n, define discrete measures on $\{1,2,\dots\}^{\otimes 2}$ by

$$\mu_n^+ := \sum_{j,k=1}^{\infty} \beta_n(2j-1,2k-1)^2 + \beta_n(2j-2,2k-2)^2 \delta_{jk};$$

$$\mu_n^- := \sum_{j,k=1}^{\infty} \beta_n(2j-1,2k-2)^2 + \beta_n(2j-2,2k-1)^2 \delta_{jk}.$$

where in this case δ_{jk} denotes the Kronecker delta. In the following, we show only η_n^+, with η_n^- being similar. It follows from condition (M.5) that for each $t > 0$,

$$\begin{aligned} \mu^+\left([0,t]^2\right) &:= \lim_{n\to\infty} \mu_n \left(\left\lfloor \frac{nt}{2} \right\rfloor, \left\lfloor \frac{nt}{2} \right\rfloor \right) \\ &= \lim_n \sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \beta_n(2j-1,2k-1)^2 + \beta_n(2j-2,2k-2)^2 = \eta^+(t). \end{aligned}$$

Moreover, if $0 < s < t$ then

$$\mu_n \left(\left\lfloor \frac{ns}{2} \right\rfloor, \left\lfloor \frac{nt}{2} \right\rfloor \right) = \mu_n \left(\left\lfloor \frac{ns}{2} \right\rfloor, \left\lfloor \frac{ns}{2} \right\rfloor \right) + \sum_{j=1}^{\lfloor \frac{ns}{2} \rfloor} \sum_{k=\lfloor \frac{ns}{2} \rfloor + 1}^{\lfloor \frac{nt}{2} \rfloor} \beta_n(2j-1,2k-1)^2 + \beta_n(2j-2,2k-2)^2$$

which converges to $\mu^+([0,s]^2)$ because the disjoint sum vanishes by Lemma 4.21. Hence, we can conclude that μ_n converges weakly to the measure given by $\mu^+([0,s]\times[0,t]) = \eta^+(s\wedge t)$. It follows by continuity of $f''(W_t)$ and Portmanteau Theorem that

$$\begin{aligned} &\sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} f''(W_{\frac{2j-1}{n}}) f''(W_{\frac{2k-1}{n}}) \left(\beta_n(2j-1,2k-1)^2 + \beta_n(2j-2,2k-2)^2 \right) \\ &\qquad = \int_{\mathbb{R}^2} f''(W_s) f''(W_u) \mathbf{1}_{s<t} \mathbf{1}_{u<t} \mu_n^+(ds,du) \end{aligned}$$

converges to

$$\int_0^t f''(W_s)^2 \eta^+(ds).$$

Combining this result with a similar integral defined for μ^-, we have for $t > 0$,

$$\lim_{n\to\infty} \sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} f''(W_{\frac{2j-1}{n}}) f''(W_{\frac{2k-1}{n}}) \left\langle \partial^{\otimes 2}_{\frac{2j-1}{n}} - \partial^{\otimes 2}_{\frac{2j-2}{n}}, \partial^{\otimes 2}_{\frac{2k-1}{n}} - \partial^{\otimes 2}_{\frac{2k-2}{n}} \right\rangle_{\mathfrak{H}^{\otimes 2}}$$
$$= \int_0^t f''(W_s)\, \mu^+(ds) - \int_0^t f''(W_s)\, \mu^-(ds) = \int_0^t f''(W_s)\, \eta(ds)$$

where we define $\eta(t) = \eta^+(t) - \eta^-(t)$. It follows that on the subinterval $[t_{i-1}, t_i]$ we have the result

$$\langle u_n^i, u_n^i \rangle_{\mathfrak{H}^{\otimes 2}} \longrightarrow \int_{t_{i-1}}^{t_i} f''(W_s)^2 \eta(ds)$$

in $L^1(\Omega)$ as $n \to \infty$. $\square$

Chapter 5

Two constructions with critical value $H = 1/6$

5.1 Introduction

In this chapter, we consider two Riemann sum constructions that have similar characteristics, namely the Trapezoidal sum,

$$S_n^T(t) := \frac{1}{2} \sum_{j=0}^{\lfloor nt \rfloor - 1} \left(f'(X_{\frac{j+1}{n}}) + f'(X_{\frac{j}{n}}) \right) \left(X_{\frac{j+1}{n}} - X_{\frac{j}{n}} \right) \tag{5.1}$$

and the Midpoint (type 2) sum,

$$S_n^{M2}(t) := \sum_{j=0}^{\lfloor nt \rfloor - 1} f'(\widetilde{X}_{\frac{j}{n}}) \left(X_{\frac{j+1}{n}} - X_{\frac{j}{n}} \right), \tag{5.2}$$

where we assume a uniform partition of $[0,\infty)$ of increment length $1/n$, and we recall the notation $\widetilde{X}_{j/n} = \frac{1}{2}(X_{j/n} + X_{(j+1)/n})$ from Section 2.4. As discussed in Chapter 3, in the case of fBm both sums have a critical value of $H = 1/6$. It will be shown that for the critical case $H = 1/6$, both sums converge weakly to a similar random variable, where the correction terms differ only by a factor of 2.

The fractional Brownian motion case with $H = 1/6$ was first studied in [26] for the trapezoidal case, which coincides with the classical Stratonovich integral. That paper was followed by [18], where, as in Chapter 4, we used a version of Theorem 2.3 to show the Riemann sum converges in law to a Gaussian random variable. The Midpoint (type 2) was not considered in the papers [18, 26], but the proof is quite similar. As in Chapter 4 and [17], the result of [18] was proved for a generalized Gaussian process $X = \{X_t, t \geq 0\}$, which includes fBm as well as others. In fact, the outlines of Chapters 4 and 5 are essentially the same, but the technical details of the respective proofs are so different that nearly all of the results must be re-created for this chapter.

The outline of this paper essentially follows Sections 3-5 of [18], with the exception that we expanded the main theorem to include the Midpoint (type 2) case as well. This had very little impact on most of the supporting lemmas.

5.2 Weak convergence of the trapezoidal and midpoint (type 2) sums

5.2.1 Covariance conditions

Consider a Gaussian stochastic process $X := \{X_t, t \geq 0\}$ with covariance function $\mathbb{E}[X_s X_t] = R(s,t)$. Assume $R(s,t)$ satisfies the following bounds: for any $T > 0$, $0 < s \leq 1$, and $s \leq r,t \leq T$:

(T.1) $\mathbb{E}\left[(X_t - X_{t-s})^2\right] \leq C_1 s^{\frac{1}{3}}$, for a positive constant C_1.

(T.2) If $t > s$,

$$\left|\mathbb{E}\left[X_t^2 - X_{t-s}^2\right]\right| \leq C_2 s^{\frac{1}{3}+\theta}(t-s)^{-\theta}$$

for some C_2 and $1/2 < \theta < 1$.

(T.3) For $t \geq 4s$,

$$\left|\mathbb{E}\left[(X_t - X_{t-s})^2 - (X_{t-s} - X_{t-2s})^2\right]\right| \leq C_3 s^{\frac{1}{3}+\nu}(t-2s)^{-\nu}$$

for some constants C_3 and $\nu > 1$.

(T.4) There is a constant C_4 and a real number $\lambda \in (\frac{1}{6}, \frac{1}{3}]$ such that

$$|\mathbb{E}[X_r(X_t - X_{t-s})]| \leq \begin{cases} C_4 s\left((t-s)^{\lambda-1} + |t-r|^{\lambda-1}\right) & \text{if } |t-r| \geq 2s \text{ and } t \geq 2s \\ C_4 s^{\lambda} & \text{otherwise} \end{cases}$$

(T.5) There is a constant C_5 and a real number $\gamma > 1$ such that for $t \wedge r \geq 2s$ and $|t-r| \geq 2s$,

$$|\mathbb{E}[(X_t - X_{t-s})(X_r - X_{r-s})]| \leq C_5 s^{\frac{1}{3}+\gamma}|t-r|^{-\gamma}.$$

(T.6) For integers $n > 0$ and integers $0 \leq j,k \leq nT$, define $\beta_n(j,k) := \mathbb{E}\left[(X_{\frac{j+1}{n}} - X_{\frac{j}{n}})(X_{\frac{k+1}{n}} - X_{\frac{k}{n}})\right]$. Then for each real number $0 \leq t \leq T$,

$$\lim_{n\to\infty} \sum_{j,k=0}^{\lfloor nt \rfloor - 1} \beta_n(j,k)^3 = \eta(t), \tag{5.3}$$

where $\eta(t)$ is a continuous and nondecreasing function with $\eta(0) = 0$. As we will see, $\eta(t)$ is comparable to the 'cubic variation' $[X,X,X]_t$ discussed in [16] and [26]. As described in [26], these terms are related by Theorem 10 of [28].

In particular, it can be shown that the above conditions are satisfied by fBm with Hurst parameter $H = 1/6$. In Section 5.3 we show additional examples.

In addition to conditions (T.1) - (T.6) on X, we will also assume the following condition (T.0) on the test function f:

(T.0) Let $f : \mathbb{R} \to \mathbb{R}$ be a $\mathscr{C}^\infty$ function, such that f and all its derivatives satisfy moderate growth conditions.

The $\mathscr{C}^\infty$ condition is stronger than necessary, though derivatives of order higher than 15 appear in the proofs.

The following is the major result of this section.

Theorem 5.1. *Let f be a real function satisfying condition (T.0), and let $X = \{X_t, t \geq 0\}$ be a Gaussian process satisfying conditions (T.1) through (T.6). Then:*

$$\left(X_t, S_n^T(t)\right) \xrightarrow{\mathscr{L}} \left(X_t, f(X_t) - f(X_0) + \frac{\sqrt{6}}{12}\int_0^t f^{(3)}(X_s)\, dB_s\right)$$

and

$$\left(X_t, S_n^{M2}(t)\right) \xrightarrow{\mathscr{L}} \left(X_t, f(X_t) - f(X_0) - \frac{\sqrt{6}}{24}\int_0^t f^{(3)}(X_s)\, dB_s\right)$$

as $n \to \infty$ in the Skorohod space $\mathbf{D}[0,\infty)$, where $B = \{B_t, t \geq 0\}$ is a scaled Brownian motion, independent of X, and with variance $\mathbb{E}\left[B_t^2\right] = \eta(t)$ for the function η defined in condition (T.6).

The proof follows from Theorem 2.3 and Corollary 2.5; and is given in a series of lemmas. Following is an outline of the proof. After a preliminary technical lemma, we use a Taylor expansion to decompose

$$S_n^T(t) = f(X_t) - f(X_0) + \frac{1}{12}\sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(3)}(\widetilde{X}_{\frac{j}{n}})\Delta X_{\frac{j}{n}}^3 - \Delta_n^T(t),$$

and

$$S_n^{M2}(t) = f(X_t) - f(X_0) - \frac{1}{24}\sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(3)}(\widetilde{X}_{\frac{j}{n}})\Delta X_{\frac{j}{n}}^3 - \Delta_n^{M2}(t).$$

We first show that $|\Delta_n^T(t)| + |\Delta_n^{M2}(t)| \xrightarrow{\mathscr{P}} 0$ as $n \to \infty$; then we show that the sum $\sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(3)}(\widetilde{X}_{\frac{j}{n}})\Delta X_{\frac{j}{n}}^3$ satisfies Theorem 2.3. Next we show that the sequences are relatively compact in the sense of Corollary 2.5, and the results follow.

We begin with the following technical results, which follow from conditions (T.1) through (T.5).

Lemma 5.2. *Let $T > 0$, and assume $\{X_t, 0 \leq t \leq T\}$ satisfies conditions (T.1), (T.2), (T.4) and (T.5). For integers $n \geq 1$, $r \geq 1$ and integers $0 \leq a < b < c \leq \lfloor nT \rfloor$, there exists a constant $C > 0$, which does not depend on a, b, c or r, such that:*

(a)

$$\sup_{0 \leq j,k \leq \lfloor nT \rfloor}\left|\left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right| \leq Cn^{-\frac{1}{3}}; \quad \text{and} \quad \sup_{0 \leq u \leq T}\sup_{0 \leq j \leq \lfloor nT \rfloor}\left|\left\langle \varepsilon_u, \partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}\right| \leq Cn^{-\lambda}.$$

(b)

$$\sum_{j=a}^{b}\left|\left\langle \widetilde{\varepsilon}_{\frac{j}{n}}, \partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}\right| \leq Cn^{-\frac{1}{3}}(b-a+1)^{1-\theta}; \text{ and} \tag{5.4}$$

$$\sum_{j=a}^{b}\left|\left\langle \widetilde{\varepsilon}_{\frac{j}{n}}, \partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}\right|^r \leq Cn^{-\frac{r}{3}} \text{ for } r > 1. \tag{5.5}$$

(c) For $0 \leq u, v \leq T$,

$$\sum_{j=a}^{b} \left| \left\langle \varepsilon_u, \partial_{\frac{j}{n}} \right\rangle_{\mathfrak{H}} \right| \leq C; \;\; \textit{and} \tag{5.6}$$

$$\sum_{j=a}^{b} \left| \left\langle \varepsilon_u, \partial_{\frac{j}{n}} \right\rangle_{\mathfrak{H}} \left\langle \varepsilon_v, \partial_{\frac{j}{n}} \right\rangle_{\mathfrak{H}} \right| \leq C n^{-2\lambda}. \tag{5.7}$$

(d)

$$\sum_{j,k=a}^{b} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}} \right|^r \leq C(b-a+1) n^{-\frac{r}{3}}. \tag{5.8}$$

(e)

$$\sum_{k=b+1}^{c} \sum_{j=a}^{b} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}} \right|^r \leq C(c-b)^{\varepsilon} n^{-\frac{r}{3}} \tag{5.9}$$

where $\varepsilon = \max\{1-\theta, 2-\gamma\}$.

Proof. We may assume $a = 0$. For part (a), the first inequality follows immediately from condition (T.1) and Cauchy-Schwarz; and the second inequality is just a restatement of condition (T.4). For (b), applying condition (T.1) for $j = 0$ and condition (T.2) for $j \geq 1$, we have:

$$\begin{aligned}
\sum_{j=0}^{b} \left| \mathbb{E}\left[X^2_{\frac{j+1}{n}} - X^2_{\frac{j}{n}} \right] \right| &\leq C n^{-\frac{1}{3}} \sum_{j=1}^{b} j^{-\theta} + C n^{-\frac{1}{3}} \\
&\leq C n^{-\frac{1}{3}} \int_0^b u^{-\theta} du + C n^{-\frac{1}{3}} \\
&\leq C n^{-\frac{1}{3}} (b+1)^{1-\theta}.
\end{aligned}$$

Then if $r \geq 2$,

$$\begin{aligned}
\sum_{j=0}^{b} \left| \mathbb{E}\left[X^2_{\frac{j+1}{n}} - X^2_{\frac{j}{n}} \right] \right|^r &\leq C n^{-\frac{r}{3}} \sum_{j=1}^{b} j^{-r\theta} + C n^{-\frac{r}{3}} \\
&\leq C n^{-\frac{r}{3}}
\end{aligned}$$

because $\theta > 1/2$ implies $j^{-r\theta}$ is summable.

For (c), define the set $J_c = \{j : 0 \leq j \leq b,\, j = 0 \textit{ or } |j - nu| < 2 \textit{ or } |j - nv| < 2\}$, and note that $|J_c| \leq 7$. Then we have by (a) and condition (T.4),

$$\begin{aligned}
\sum_{j=0}^{b} \left| \left\langle \varepsilon_u, \partial_{\frac{j}{n}} \right\rangle_{\mathfrak{H}} \right| &\leq \sum_{j \in J_c} C n^{-\lambda} + C n^{-\lambda} \sum_{j \notin J_c} \left(j^{\lambda-1} + |j - nu|^{\lambda-1} \right) \\
&\leq C n^{-\lambda} + C n^{-\lambda} (b+1)^{\lambda} \leq C,
\end{aligned}$$

and

$$\sum_{j=0}^{b}\left|\left\langle \varepsilon_u, \partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}\left\langle \varepsilon_v, \partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}\right| \le \sum_{j\in J_c} Cn^{-2\lambda} + Cn^{-2\lambda}\sum_{j\notin J_c}\left(j^{2\lambda-2}+|j-nu|^{2\lambda-2}+|j-nv|^{2\lambda-2}\right)$$
$$\le Cn^{-2\lambda}+Cn^{-2\lambda}\sum_{p=1}^{\infty}p^{2\lambda-2}$$
$$\le Cn^{-2\lambda}$$

because $\lambda \le 1/3$.

For (d), define the set: $J_d = \{j,k : j\wedge k < 1 \; or \; |j-k|<2\}$, and note that $|J_d| \le 6(b+1)$. Then we have by (a) and condition (T.5)

$$\sum_{j,k=0}^{b}\left|\left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|^r \le \sup_{j,k}\left|\left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}}\right\rangle\right|^{r-1}\sum_{j,k=0}^{b}\left|\left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right| \le Cn^{-\frac{r-1}{3}}\sum_{j,k=0}^{b}\left|\left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|$$
$$\le Cn^{-\frac{r-1}{3}}\left(\sum_{(j,k)\in J_d} n^{-\frac{1}{3}} + n^{-\frac{1}{3}}\sum_{(j,k)\notin J_d}|j-k|^{-\gamma}\right)$$
$$\le C(b+1)n^{-\frac{r}{3}}.$$

In particular, if $r=3$ and $b=\lfloor nt\rfloor -1$ (as in condition (T.6)), the sum converges absolutely, and the sum vanishes if $r>3$.

For (e), we consider the maximal case, which occurs when $a=0$:

$$\sum_{k=b+1}^{c}\sum_{j=0}^{b}\left|\left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|^r = \sum_{k=b+1}^{c}\left|\left\langle \partial_{\frac{0}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|^r + \sum_{k=b+1}^{c}\sum_{j=1}^{b-1}\left|\left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|^r + \sum_{k=b+1}^{c}\left|\left\langle \partial_{\frac{b}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|^r.$$

Note that $\partial_{\frac{0}{n}} = \varepsilon_{\frac{1}{n}}$. By part (c) and condition (T.5), respectively, this is

$$\le \sum_{k=b+1}^{c}\left|\left\langle \varepsilon_{\frac{1}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|^r + Cn^{-\frac{r}{3}}\sum_{k=b+1}^{c}\sum_{j=1}^{b-1}(k-j)^{-\gamma} + Cn^{-\frac{r}{3}}\sum_{k=b+1}^{c}(k-b)^{-\gamma}$$
$$\le Cn^{-\frac{r}{3}}(c-b)^{1-\theta} + Cn^{-\frac{r}{3}}(c-b)^{2-\gamma} + Cn^{-\frac{r}{3}}$$
$$\le Cn^{-\frac{r}{3}}(c-b)^{\varepsilon},$$

where $\varepsilon = \max\{1-\theta, 2-\gamma\} < 1$. □

5.2.2 Taylor expansion of $f(X_t)$

The details of this expansion were mainly inspired by Lemma 5.2 of [26] in the trapezoidal case, and Theorem 4.4 of [16] in the Midpoint (type 2) case. We begin with the telescoping series,

$$f(X_t) = f(0) + f(X_t) - f(X_{\frac{\lfloor nt\rfloor}{n}}) + \sum_{j=0}^{\lfloor nt\rfloor -1}\left[f(X_{\frac{j+1}{n}}) - f(X_{\frac{j}{n}})\right].$$

By continuity of f and X, we know that for large n, $f(X_t)-f(X_{\frac{\lfloor nt\rfloor}{n}})\to 0$ uniformly on compacts in probability (ucp), so this term may be neglected. For each j, we use a Taylor expansion of order 6 with residual term. Let $h_j := \frac{1}{2}\left[X_{\frac{j+1}{n}} - X_{\frac{j}{n}}\right]$. Then:

$$\begin{aligned}
f(X_{\frac{j+1}{n}})-f(X_{\frac{j}{n}}) &= \Big(f(\widetilde{X}_{\frac{j}{n}}+h_j)-f(\widetilde{X}_{\frac{j}{n}})\Big)-\Big(f(\widetilde{X}_{\frac{j}{n}}-h_j)-f(\widetilde{X}_{\frac{j}{n}})\Big)\\
&= \sum_{k=1}^{6} f^{(k)}(\widetilde{X}_{\frac{j}{n}})\frac{h_j^k}{k!}+R_n^+(j)-\left(\sum_{k=1}^{6}(-1)^k f^{(k)}(\widetilde{X}_{\frac{j}{n}})\frac{h_j^k}{k!}+R_n^-(j)\right)\\
&= f'(\widetilde{X}_{\frac{j}{n}})\Delta X_{\frac{j}{n}}+\frac{1}{24}f^{(3)}(\widetilde{X}_{\frac{j}{n}})\Delta X^3_{\frac{j}{n}}+\frac{1}{2^4 5!}f^{(5)}(\widetilde{X}_{\frac{j}{n}})\Delta X^5_{\frac{j}{n}}+R_n^+(j)-R_n^-(j),
\end{aligned}$$

where $\Delta X_{\frac{j}{n}} = 2h_j$; and $R_n^+(j), R_n^-(j)$ are Taylor series remainder terms of order 7. From this we get the result

$$\begin{aligned}
S_n^{M2}(t) &= \sum_{j=0}^{\lfloor nt\rfloor-1}\Big(f(X_{\frac{j+1}{n}})-f(X_{\frac{j}{n}})\Big)-\frac{1}{24}\sum_{j=0}^{\lfloor nt\rfloor-1} f^{(3)}(\widetilde{X}_{\frac{j}{n}})\Delta X^3_{\frac{j}{n}}\\
&\qquad -\frac{1}{2^5 5!}\sum_{j=0}^{\lfloor nt\rfloor-1} f^{(5)}(\widetilde{X}_{\frac{j}{n}})\Delta X^5_{\frac{j}{n}}-\sum_{j=0}^{\lfloor nt\rfloor-1}\big(R_n^+(j)-R_N^-(j)\big)\\
&= f(X_t)-f(X_0)-\frac{1}{24}\sum_{j=0}^{\lfloor nt\rfloor-1} f^{(3)}(\widetilde{X}_{\frac{j}{n}})\Delta X^3_{\frac{j}{n}}-\Delta_n^{M2}(t),
\end{aligned}$$

where $\Delta_n^{M2}(t) = (2^5 5!)^{-1}\sum_{j=0}^{\lfloor nt\rfloor-1} f^{(5)}(\widetilde{X}_{\frac{j}{n}})\Delta X^5_{\frac{j}{n}}+\sum_{j=0}^{\lfloor nt\rfloor-1}\big(R_n^+(j)-R_N^-(j)\big)-f(X_{\lfloor nt\rfloor/n})+f(X_t)$. On the other hand, for the trapezoidal case we use 6th order Taylor expansions of $f'(X_{j/n})$, $f'(X_{(j+1)/n})$ to write:

$$\begin{aligned}
\frac{f'(X_{\frac{j+1}{n}})+f'(X_{\frac{j}{n}})}{2}-f'(\widetilde{X}_{\frac{j}{n}}) &= \frac{1}{2}\Big(f'(\widetilde{X}_{\frac{j}{n}}+h_j)-f'(\widetilde{X}_{\frac{j}{n}})\Big)+\frac{1}{2}\Big(f'(\widetilde{X}_{\frac{j}{n}}-h_j)-f'(\widetilde{X}_{\frac{j}{n}})\Big)\\
&= \frac{1}{2}\sum_{k=1}^{5} f^{(1+k)}(\widetilde{X}_{\frac{j}{n}})\frac{h_j^k}{k!}+K_n^+(j)+\frac{1}{2}\sum_{k=1}^{5}(-1)^k f^{(1+k)}(\widetilde{X}_{\frac{j}{n}})\frac{h_j^k}{k!}+K_n^-(j)\\
&= \frac{1}{8}f^{(3)}(\widetilde{X}_{\frac{j}{n}})\Delta X^2_{\frac{j}{n}}+\frac{1}{2^4 4!}f^{(5)}(\widetilde{X}_{\frac{j}{n}})\Delta X^4_{\frac{j}{n}}+\frac{1}{2}\Big(K_j^+ + K_j^-\Big),
\end{aligned}$$

where $K_n^+(j), K_n^-(j)$ are remainder terms of order 6. Combining the two expansions, we obtain

$$\begin{aligned}
f(X_{\frac{j+1}{n}})-f(X_{\frac{j}{n}}) &= \frac{f'(X_{\frac{j+1}{n}})+f'(X_{\frac{j}{n}})}{2}\Delta X_{\frac{j}{n}}-\frac{1}{12}f^{(3)}(\hat{X}_j)\Delta X^3_{\frac{j}{n}}-\frac{4}{2^5 5!}f^{(5)}(\hat{X}_j)\Delta X^5_{\frac{j}{n}}\\
&\quad +R_n^+(j)-R_n^-(j)-\frac{1}{4}\left[K_n^+(j)+K_n^-(j)\right]\Delta X_{\frac{j}{n}}.
\end{aligned}$$

so that

$$
\begin{aligned}
S_n^T(t) = &\sum_{j=0}^{\lfloor nt\rfloor-1}\Big(f(X_{\frac{j+1}{n}})-f(X_{\frac{j}{n}})\Big)-\frac{1}{12}\sum_{j=0}^{\lfloor nt\rfloor-1} f^{(3)}(\widetilde{X}_{\frac{j}{n}})\Delta X_{\frac{j}{n}}^3-\frac{4}{2^5 5!}\sum_{j=0}^{\lfloor nt\rfloor-1} f^{(5)}(\widetilde{X}_{\frac{j}{n}})\Delta X_{\frac{j}{n}}^5\\
&-\sum_{j=0}^{\lfloor nt\rfloor-1}\left(R_n^+(j)-R_N^-(j)\right)+\frac{1}{4}\sum_{j=0}^{\lfloor nt\rfloor-1}\left(K_n^+(t)+K_n^-(t)\right)\Delta X_{\frac{j}{n}}\\
= &f(X_t)-f(X_0)-\frac{1}{24}\sum_{j=0}^{\lfloor nt\rfloor-1} f^{(3)}(\widetilde{X}_{\frac{j}{n}})\Delta X_{\frac{j}{n}}^3-\Delta_n^T(t),
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta_n^T(t) = \frac{4}{2^5 5!}\sum_{j=0}^{\lfloor nt\rfloor-1} f^{(5)}(\widetilde{X}_{\frac{j}{n}})\Delta X_{\frac{j}{n}}^5+\sum_{j=0}^{\lfloor nt\rfloor-1}\left(R_n^+(j)-R_N^-(j)\right)-\frac{1}{4}\sum_{j=0}^{\lfloor nt\rfloor-1}\left(K_n^+(t)+K_n^-(t)\right)\Delta X_{\frac{j}{n}}\\
+f(X_{\frac{\lfloor nt\rfloor}{n}})-f(X_t).
\end{aligned}
$$

Our first task is to show that the terms $\Delta_n^T(t)$, $\Delta_n^{M2}(t)$ vanish in probability for each t. We do this by checking each component in Lemmas 5.3, 5.4.

Lemma 5.3. *For each integer $n \geq 1$ and real numbers $0 \leq t_1 < t_2 \leq T$,*

$$
\mathbb{E}\left[\left(\sum_{j=\lfloor nt_1\rfloor}^{\lfloor nt_2\rfloor-1} f^{(5)}(\widetilde{X}_{\frac{j}{n}})\Delta X_{\frac{j}{n}}^5\right)^2\right] \leq Cn^{-\frac{4}{3}}\left(\lfloor nt_2\rfloor-\lfloor nt_1\rfloor\right). \tag{5.10}
$$

The proof of this lemma is technical, and is deferred to Section 5.4.

Lemma 5.4. *For integers $n \geq 1$, let*

$$
Z_n(t) = \sum_{j=0}^{\lfloor nt\rfloor-1}\left[R_n^+(j)-R_n^-(j)+\frac{1}{4}\left(K_n^+(j)+K_n^-(j)\right)\Delta X_{\frac{j}{n}}\right].
$$

Then for real numbers $0 \leq t_1 < t_2 \leq T$, we have

$$
\mathbb{E}\left[\left(Z_n(t_2)-Z_n(t_1)\right)^2\right] \leq Cn^{-\frac{7}{3}}\left(\lfloor nt_2\rfloor-\lfloor nt_1\rfloor\right)^2. \tag{5.11}
$$

Proof. We may assume $t_1 = 0$. Observe that each term in the sum $Z_n(t)$ has the form

$$
Cf^{(7)}(\xi_j)\Delta X_{\frac{j}{n}}^7,
$$

where ξ_j is an intermediate value between $X_{\frac{j}{n}}$ and $X_{\frac{j+1}{n}}$. Using the Hölder inequality, for each $0 \leq j,k < \lfloor nt_2\rfloor$ we have

$$
\mathbb{E}\left[f^{(7)}(\xi_j)f^{(7)}(\xi_k)\Delta X_{\frac{j}{n}}^7\Delta X_{\frac{k}{n}}^7\right]
$$

$$\leq \left(\sup_{0<u<1} \mathbb{E}\left[f^{(7)}(uX_{\frac{j}{n}} + (1-u)X_{\frac{j+1}{n}})^4 \right] \sup_{0<v<1} \mathbb{E}\left[f^{(7)}(vX_{\frac{k}{n}} + (1-v)X_{\frac{k+1}{n}})^4 \right] \mathbb{E}\left[\left| \Delta X_{\frac{j}{n}}^7 \right|^4 \right] \mathbb{E}\left[\left| \Delta X_{\frac{k}{n}}^7 \right|^4 \right] \right)^{\frac{1}{4}}.$$

By condition (T.0), the first two terms are bounded. By condition (T.1), $\mathbb{E}\left[\Delta X_{\frac{j}{n}}^2\right] \leq C_1 n^{-\frac{1}{3}}$; and we have by the Gaussian moments formula that

$$\mathbb{E}\left[\Delta X_{\frac{j}{n}}^{28}\right] \leq 27!! \left(C_1 n^{-\frac{1}{3}}\right)^{14},$$

hence it follows that

$$C \sum_{j,k=0}^{\lfloor nt_2 \rfloor - 1} \mathbb{E}\left[f^{(7)}(\xi_j) f^{(7)}(\xi_k) \Delta X_{\frac{j}{n}}^7 \Delta X_{\frac{k}{n}}^7 \right] \leq C \sum_{j,k=0}^{\lfloor nt_2 \rfloor - 1} n^{-\frac{7}{3}} \leq C \lfloor nt_2 \rfloor^2 n^{-\frac{7}{3}}.$$

□

Since $|f(X_{\lfloor nt \rfloor/n}) - f(X_t)| \to 0$ ucp as $n \to \infty$, it follows from Lemmas 5.3 and 5.4 that $\Delta_n^T(t)$, $\Delta_n^{M2}(t)$ both tend to zero in $L^2(\Omega)$, where we take $K_n^+(t), K_n^-(t) = 0$ in Lemma 5.4 when applied to $\Delta_n^{M2}(t)$.

5.2.3 Malliavin calculus representation of 3rd order term

From Lemmas 5.3 and 5.4, we see that proof of both results of Theorem 5.1 depend on the behavior of the term

$$\sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(3)}(\widetilde{X}_{\frac{j}{n}}) \Delta X_{\frac{j}{n}}^3.$$

It may happen that the upper bound of condition (T.5) is such that

$$|\eta(t)| \leq \lim_{n \to \infty} \sum_{j,k=0}^{\lfloor nt \rfloor - 1} \left| \beta_n(j,k)^3 \right| = 0$$

for all t, which implies

$$\lim_n \mathbb{E}\left[\left(\sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(3)}(\widetilde{X}_{\frac{j}{n}}) \Delta X_{\frac{j}{n}}^3 \right)^2 \right] = 0$$

for any function f satisfying condition (T.0). This is a generalization of the fBm case $H > 1/6$ and the case of zero cubic variation in [16]. As in Proposition 3.3, the sums $S_n^T(t)$, $S_n^{M2}(t)$ will then converge in probability to $f(X_t) - f(X_0)$, and we can say that the stochastic integrals

$$\int_0^t f'(X_s)\, d^\circ X_s, \quad \int_0^t f'(X_s)\, d^{M2} X_s$$

exist in probability. In the rest of this section, we will assume that $\eta(t)$ is non-trivial.

Consider the 3^{rd} Hermite polynomial $H_3(x) = x^3 - 3x$. For $x = \frac{\Delta X}{\|\Delta X\|_{L^2}}$, it follows that

$$\sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(3)}(\widetilde{X}_{\frac{j}{n}}) \Delta X_{\frac{j}{n}}^3 = \sum_{j=0}^{\lfloor nt \rfloor - 1} \|\Delta X_{\frac{j}{n}}\|_{L^2}^3 f^{(3)}(\widetilde{X}_{\frac{j}{n}}) H_3\left(\frac{\Delta X_{\frac{j}{n}}}{\|\Delta X_{\frac{j}{n}}\|_{L^2}}\right) + 3\sum_{j=0}^{\lfloor nt \rfloor - 1} \|\Delta X_{\frac{j}{n}}\|_{L^2}^2 f^{(3)}(\widetilde{X}_{\frac{j}{n}}) \Delta X_{\frac{j}{n}}$$

The second term is dealt with in the next lemma. The proof is technical, and is deferred to Section 5.4.

Lemma 5.5. *For integers $n \geq 1$ and integers $0 \leq a < b \leq nT$,*

$$\mathbb{E}\left|\sum_{j=a}^{b-1} \|\Delta X_{\frac{j}{n}}\|_{L^2}^2 f^{(3)}(\widetilde{X}_{\frac{j}{n}}) \Delta X_{\frac{j}{n}}\right| \leq C n^{-\frac{1}{3}}.$$

Next, we consider the H_3 term. By (2.4) and Lemma 2.1.a we have

$$\begin{aligned}
\sum_{j=0}^{\lfloor nt \rfloor - 1} \|\Delta X_{\frac{j}{n}}\|_{L^2}^3 f^{(3)}(\widetilde{X}_{\frac{j}{n}}) H_3\left(\frac{\Delta X_{\frac{j}{n}}}{\|\Delta X_{\frac{j}{n}}\|_{L^2}}\right) &= \sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(3)}(\widetilde{X}_{\frac{j}{n}}) \delta^3\left(\partial_{\frac{j}{n}}^{\otimes 3}\right) \\
&= \sum_{j=0}^{\lfloor nt \rfloor - 1} \delta^3\left(f^{(3)}(\widetilde{X}_{\frac{j}{n}}) \partial_{\frac{j}{n}}^{\otimes 3}\right) + 3\delta^2\left(\left\langle D f^{(3)}(\widetilde{X}_{\frac{j}{n}}), \partial_{\frac{j}{n}}^{\otimes 3}\right\rangle_{\mathfrak{H}}\right) \\
&\quad + 3\delta\left(\left\langle D^2 f^{(3)}(\widetilde{X}_{\frac{j}{n}}), \partial_{\frac{j}{n}}^{\otimes 3}\right\rangle_{\mathfrak{H}^{\otimes 2}}\right) + \left\langle D^3 f^{(3)}(\widetilde{X}_{\frac{j}{n}}), \partial_{\frac{j}{n}}^{\otimes 3}\right\rangle_{\mathfrak{H}^{\otimes 3}} \\
&= \sum_{j=0}^{\lfloor nt \rfloor - 1} \delta^3\left(f^{(3)}(\widetilde{X}_{\frac{j}{n}}) \partial_{\frac{j}{n}}^{\otimes 3}\right) + P_n(t).
\end{aligned}$$

As $n \to \infty$, we show that the term $P_n(t)$ vanishes in probability.

Lemma 5.6. *For integers $n \geq 1$ and real numbers $0 \leq t_1 < t_2 \leq T$,*

$$\mathbb{E}\left[P_n(t)^2\right] \leq C\left(\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor\right) n^{-\frac{4}{3}}.$$

Proof. We may assume $t_1 = 0$. We want to show

$$\mathbb{E}\left[\left(\delta^2\left(\sum_{j=0}^{\lfloor nt_2 \rfloor - 1} \left\langle D f^{(3)}(\widetilde{X}_{\frac{j}{n}}), \partial_{\frac{j}{n}}^{\otimes 3}\right\rangle_{\mathfrak{H}}\right)\right)^2\right] \leq C \lfloor nt_2 \rfloor n^{-\frac{4}{3}}; \tag{5.12}$$

$$\mathbb{E}\left[\left(\delta\left(\sum_{j=0}^{\lfloor nt_2 \rfloor - 1} \left\langle D^2 f^{(3)}(\widetilde{X}_{\frac{j}{n}}), \partial_{\frac{j}{n}}^{\otimes 3}\right\rangle_{\mathfrak{H}^{\otimes 2}}\right)\right)^2\right] \leq C \lfloor nt_2 \rfloor n^{-\frac{1}{5}\lambda}; \tag{5.13}$$

and

$$\mathbb{E}\left[\left(\sum_{j=0}^{\lfloor nt_2\rfloor-1}\left\langle D^3 f^{(3)}(\widetilde{X}_{\frac{j}{n}}),\partial_{\frac{j}{n}}^{\otimes 3}\right\rangle_{\mathfrak{H}}\right)^2\right]\leq C\lfloor nt_2\rfloor n^{-2}. \tag{5.14}$$

Proof of (5.12). By Lemma 2.1.d we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\delta^2\left(\sum_{j=\lfloor nt_1\rfloor}^{\lfloor nt_2\rfloor-1}\left\langle Df^{(3)}(\widetilde{X}_{\frac{j}{n}}),\partial_{\frac{j}{n}}^{\otimes 3}\right\rangle_{\mathfrak{H}}\right)\right)^2\right]\\
&\quad\leq \mathbb{E}\left[\sum_{j,k=0}^{\lfloor nt_2\rfloor-1}\left\langle\left\langle Df^{(3)}(\widetilde{X}_{\frac{j}{n}}),\partial_{\frac{j}{n}}^{\otimes 3}\right\rangle_{\mathfrak{H}},\left\langle Df^{(3)}(\widetilde{X}_{\frac{k}{n}}),\partial_{\frac{k}{n}}^{\otimes 3}\right\rangle_{\mathfrak{H}}\right\rangle_{\mathfrak{H}^{\otimes 2}}\right]\\
&\quad+2\mathbb{E}\left[\sum_{j,k=0}^{\lfloor nt_2\rfloor-1}\left\langle\left\langle D^2f^{(3)}(\widetilde{X}_{\frac{j}{n}}),\partial_{\frac{j}{n}}^{\otimes 3}\right\rangle_{\mathfrak{H}^{\otimes 2}},\left\langle D^2f^{(3)}(\widetilde{X}_{\frac{k}{n}}),\partial_{\frac{k}{n}}^{\otimes 3}\right\rangle_{\mathfrak{H}^{\otimes 2}}\right\rangle_{\mathfrak{H}}\right]\\
&\quad+\mathbb{E}\left[\sum_{j,k=0}^{\lfloor nt_2\rfloor-1}\left\langle D^3f^{(3)}(\widetilde{X}_{\frac{j}{n}}),\partial_{\frac{j}{n}}^{\otimes 3}\right\rangle_{\mathfrak{H}^{\otimes 3}}\cdot\left\langle D^3f^{(3)}(\widetilde{X}_{\frac{k}{n}}),\partial_{\frac{k}{n}}^{\otimes 3}\right\rangle_{\mathfrak{H}^{\otimes 3}}\right]\\
&\leq \sup_{0\leq j<\lfloor nt_2\rfloor}\mathbb{E}\left[f^{(4)}(\widetilde{X}_{\frac{j}{n}})^2\right]\sup_{0\leq j<\lfloor nt_2\rfloor}\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}^2\sum_{j,k=0}^{\lfloor nt_2\rfloor-1}\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^2\\
&\quad+\sup_{0\leq j<\lfloor nt_2\rfloor}\mathbb{E}\left[f^{(5)}(\widetilde{X}_{\frac{j}{n}})^2\right]\sup_{0\leq j<\lfloor nt_2\rfloor}\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}^4\sum_{j,k=0}^{\lfloor nt_2\rfloor-1}\left|\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|\\
&\quad+\sup_{0\leq j<\lfloor nt_2\rfloor}\mathbb{E}\left[f^{(6)}(\widetilde{X}_{\frac{j}{n}})^2\right]\sum_{j,k=0}^{\lfloor nt_2\rfloor-1}\left|\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}^3\right|\left|\left\langle\widetilde{\varepsilon}_{\frac{k}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^3\right|.
\end{aligned}$$

By condition (T.0) and Lemma 5.2.a and 5.2.d,

$$\sup_{0\leq j<\lfloor nt_2\rfloor}\mathbb{E}\left[f^{(4)}(\widetilde{X}_{\frac{j}{n}})^2\right]\sup_{0\leq j<\lfloor nt_2\rfloor}\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}^2\sum_{j,k=0}^{\lfloor nt_2\rfloor-1}\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^2\leq Cn^{-\frac{4}{3}}\lfloor nt_2\rfloor;\ \text{ and}$$

$$\sup_{0\leq j<\lfloor nt_2\rfloor}\mathbb{E}\left[f^{(5)}(\widetilde{X}_{\frac{j}{n}})^2\right]\sup_{0\leq j<\lfloor nt_2\rfloor}\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}^4\sum_{j,k=0}^{\lfloor nt_2\rfloor-1}\left|\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|\leq Cn^{-\frac{5}{3}}\lfloor nt_2\rfloor.$$

Then by condition (T.0), Lemma 5.2.a and 5.2.b,

$$\begin{aligned}
&\sup_{0\leq j<\lfloor nt_2\rfloor}\mathbb{E}\left[f^{(6)}(\widetilde{X}_{\frac{j}{n}})^2\right]\sum_{j,k=0}^{\lfloor nt_2\rfloor-1}\left|\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}^3\right|\left|\left\langle\widetilde{\varepsilon}_{\frac{k}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^3\right|\\
&\qquad\leq C\sup_{0\leq j\leq\lfloor nt_2\rfloor}\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}^2\left(\sum_{j=0}^{\lfloor nt_2\rfloor-1}\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}^2\right)^2\leq Cn^{-2}.
\end{aligned}$$

Proof of (5.13) *and* (5.14). The same estimates apply for the other terms, since by Lemma 2.1.c,

$$\begin{aligned}
&\mathbb{E}\left[\left(\delta\left(\sum_{j=0}^{\lfloor nt_2\rfloor-1}\left\langle D^2 f^{(3)}(\widetilde{X}_{\frac{j}{n}}),\partial_{\frac{j}{n}}^{\otimes 3}\right\rangle_{\mathfrak{H}^{\otimes 2}}\right)\right)^2\right]\\
&\quad\le\left|\mathbb{E}\left[\sum_{j,k=0}^{\lfloor nt_2\rfloor-1}\left\langle\left\langle D^2 f^{(3)}(\widetilde{X}_{\frac{j}{n}}),\partial_{\frac{j}{n}}^{\otimes 3}\right\rangle_{\mathfrak{H}^{\otimes 2}},\left\langle D^2 f^{(3)}(\widetilde{X}_{\frac{k}{n}}),\partial_{\frac{k}{n}}^{\otimes 3}\right\rangle_{\mathfrak{H}^{\otimes 2}}\right\rangle_{\mathfrak{H}}\right]\right|\\
&\qquad+\left|\mathbb{E}\left[\sum_{j,k=0}^{\lfloor nt_2\rfloor-1}\left\langle D^3 f^{(3)}(\widetilde{X}_{\frac{j}{n}}),\partial_{\frac{j}{n}}^{\otimes 3}\right\rangle_{\mathfrak{H}^{\otimes 3}}\cdot\left\langle D^3 f^{(3)}(\widetilde{X}_{\frac{k}{n}}),\partial_{\frac{k}{n}}^{\otimes 3}\right\rangle_{\mathfrak{H}^{\otimes 3}}\right]\right|
\end{aligned}$$

and (5.14) is bounded in the above computation as well. □

5.2.4 Weak convergence of non-trivial part of 3rd order term

We are now ready to apply Theorem 2.3 to the term

$$\sum_{j=0}^{\lfloor nt\rfloor-1}\delta^3\left(f^{(3)}(\widetilde{X}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 3}\right).$$

Let $0=t_0<t_1<\cdots<t_d\le T$ be a finite set of real numbers. For $i=1,\dots,d$ define

$$u_n^i=\sum_{j=\lfloor nt_{i-1}\rfloor}^{\lfloor nt_i\rfloor-1}f^{(3)}(\widetilde{X}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 3};$$

and define the $d-$dimensional vector $F_n=(F_n^1,\dots,F_n^d)$, where each $F_n^i=\delta^3(u_n^i)$.

To satisfy the conditions of Theorem 2.3, we must deal with terms of the following forms:

1. $\langle u_n^i,h\rangle_{\mathfrak{H}^{\otimes 3}}$ for $h\in\mathfrak{H}^{\otimes 3}$,
2. $\left\langle u_n^i,DF_n^j\otimes h\right\rangle_{\mathfrak{H}^{\otimes 3}}$ for $h\in\mathfrak{H}^{\otimes 2}$,
3. $\left\langle u_n^i,D^2F_n^j\otimes h_1\right\rangle_{\mathfrak{H}^{\otimes 3}}+\left\langle u_n^i,DF_n^j\otimes DF_n^k\otimes h_2\right\rangle_{\mathfrak{H}^{\otimes 3}}$ for $h_1,\ h_2\in\mathfrak{H}$, and
4. $\langle u_n^i,D^3F_n^i\rangle_{\mathfrak{H}^{\otimes 3}}+\left\langle u_n^i,D^3F_n^j\right\rangle_{\mathfrak{H}^{\otimes 3}}$
$$+\left\langle u_n^i,D^2F_n^j\otimes DF_n^k\right\rangle_{\mathfrak{H}^{\otimes 3}}+\left\langle u_n^i,DF_n^j\otimes DF_n^k\otimes DF_n^\ell\right\rangle_{\mathfrak{H}^{\otimes 3}}.$$

We must show that all terms converge to zero *except* for the terms $\langle u_n^i,D^3F_n^i\rangle_{\mathfrak{H}^{\otimes 3}}$, $i=1,\dots,d$, which will converge stably to a Gaussian random vector (Lemma 5.11).

Lemma 5.7. *For each $i,j,k,\ell=1,\dots,d$, the following terms vanish in $L^1(\Omega)$ as $n\to\infty$:*

(a) $\langle u_n^i,h\rangle_{\mathfrak{H}^{\otimes 3}}$ *for each $h\in\mathfrak{H}^{\otimes 3}$.*

(b) $\left\langle u_n^i, DF_n^j \otimes h \right\rangle_{\mathfrak{H}^{\otimes 3}}$ *for each* $h \in \mathfrak{H}^{\otimes 2}$.

(c) $\left\langle u_n^i, D^2F_n^j \otimes h \right\rangle_{\mathfrak{H}^{\otimes 3}} + \left\langle u_n^i, DF_n^j \otimes DF_n^k \otimes h \right\rangle_{\mathfrak{H}^{\otimes 3}}$ *for* $h \in \mathfrak{H}$.

Proof. We begin with two estimates that will be needed. For each $1 \le i \le d$,

$$\mathbb{E}\left\|DF_n^i\right\|_{\mathfrak{H}}^2 < C; \text{ and} \tag{5.15}$$

$$\mathbb{E}\left\|D^2F_n^i\right\|_{\mathfrak{H}^{\otimes 2}}^2 < C. \tag{5.16}$$

Proof of (5.15). Let $a_i = \lfloor nt_{i-1} \rfloor$ and $b_i = \lfloor nt_i \rfloor$. By Lemma 2.1.b,

$$DF_n^i = \delta^3(Du_n^i) + 3\delta^2(u_n^i).$$

Hence, using Lemma 2.1.c,

$$\begin{aligned}
\mathbb{E}\left\|DF_n^i\right\|_{\mathfrak{H}}^2 \le\, & 2\sum_{j,k=a_i}^{b_i-1} \mathbb{E}\left[\delta^3\left(f^{(4)}(\widetilde{X}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 3}\right)\delta^3\left(f^{(4)}(\widetilde{X}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 3}\right)\right]\left\langle \widetilde{\varepsilon}_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}} \\
&+ 18\sum_{j,k=a_i}^{b_i-1} \mathbb{E}\left[\delta^2\left(f^{(3)}(\widetilde{X}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 2}\right)\delta^2\left(f^{(3)}(\widetilde{X}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 2}\right)\right]\left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}} \\
=\, & 2\sum_{j,k=a_i}^{b_i-1}\sum_{\ell=0}^{3}\binom{3}{\ell}^2 \mathbb{E}\left[f^{(7-\ell)}(\widetilde{X}_{\frac{j}{n}})f^{(7-\ell)}(\widetilde{X}_{\frac{k}{n}})\right]\left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^3\left\langle \widetilde{\varepsilon}_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{\ell} \\
&+ 18\sum_{j,k=a_i}^{b_i-1}\sum_{\ell=0}^{2}\binom{2}{\ell}^2 \mathbb{E}\left[f^{(5-\ell)}(\widetilde{X}_{\frac{j}{n}})f^{(5-\ell)}(\widetilde{X}_{\frac{k}{n}})\right]\left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^3\left\langle \widetilde{\varepsilon}_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{\ell} \\
\le\, & C
\end{aligned}$$

by (T.0) and Lemma 5.2.d. The proof of (5.16) follows the same lines, using Lemma 2.1.b to obtain

$$D^2F_n^i = \delta^3(D^2u_n^i) + 6\delta^2(Du_n^i) + 6\delta(u_n^i).$$

Now for the main proof. Without loss of generality, we may assume that each $h \in \mathfrak{H}$ is of the form ε_τ for some $0 \le \tau \le T$ (see Remark 2.4). Then for (a) we have:

$$\begin{aligned}
\mathbb{E}\left|\langle u_n^i, h\rangle_{\mathfrak{H}^{\otimes 3}}\right| &= \mathbb{E}\left|\sum_{m=a_i}^{b_i-1}\left\langle f^{(3)}(\widetilde{X}_{\frac{m}{n}})\partial_{\frac{m}{n}}^{\otimes 3}, \varepsilon_\tau \otimes \varepsilon_u \otimes \varepsilon_v\right\rangle_{\mathfrak{H}^{\otimes 3}}\right| \\
&\le \sup_{a_i \le m \le b_i}\left\{\mathbb{E}\left|f^{(3)}(\widetilde{X}_{\frac{m}{n}})\right|\right\}\sup_{\tau,m}\left\{\left\langle \varepsilon_\tau, \partial_{\frac{m}{n}}\right\rangle_{\mathfrak{H}}^2\right\}\sum_{m=a_i}^{b_i-1}\left|\left\langle \varepsilon_\tau, \partial_{\frac{m}{n}}\right\rangle_{\mathfrak{H}}\right| \\
&\le Cn^{-2\lambda},
\end{aligned}$$

where we used Lemma 5.2.a and Lemma 5.2.c. For (b),

$$\begin{aligned}
\mathbb{E}\left|\langle u_n^i, DF_n^j \otimes \varepsilon_\tau \otimes \varepsilon_u\rangle_{\mathfrak{H}^{\otimes 3}}\right| &\le \sqrt{\sup_{a_i \le m \le b_i}\mathbb{E}\left|f^{(3)}(\widetilde{X}_{\frac{m}{n}})\right|^2}\sum_{m=a_i}^{b_i-1}\left(\mathbb{E}\left\langle \partial_{\frac{m}{n}}^{\otimes 3}, DF_n^j \otimes \varepsilon_\tau \otimes \varepsilon_u\right\rangle_{\mathfrak{H}^{\otimes 3}}^2\right)^{\frac{1}{2}} \\
&\le C\sup_m\left\|\partial_{\frac{m}{n}}\right\|_{\mathfrak{H}}\sqrt{\mathbb{E}\|DF_n^j\|_{\mathfrak{H}}^2}\sum_{m=a_i}^{b_i-1}\left|\left\langle \partial_{\frac{m}{n}}, \varepsilon_\tau\right\rangle_{\mathfrak{H}}\left\langle \partial_{\frac{m}{n}}, \varepsilon_u\right\rangle_{\mathfrak{H}}\right|.
\end{aligned}$$

By condition (T.1), $\|\partial_{\frac{m}{n}}\|_{\mathfrak{H}} \leq Cn^{-\frac{1}{6}}$, and so by (5.15) and Lemma 5.2.c we have an upper bound of $Cn^{-\frac{1}{6}-2\lambda}$. For (c), by similar reasoning along with Lemma 5.2.c and (5.16),

$$\begin{aligned}\mathbb{E}\left|\left\langle u_n^i, D^2F_n^j \otimes \varepsilon_\tau\right\rangle_{\mathfrak{H}^{\otimes 3}}\right| &\leq \sqrt{\sup_{a_i\leq m\leq b_i} \mathbb{E}\left|f^{(3)}(\widetilde{X}_{\frac{m}{n}})\right|^2} \sum_{m=a_i}^{b_i-1}\left(\mathbb{E}\left\langle \partial_{\frac{m}{n}}^{\otimes 3}, D^2F_n^j\otimes\varepsilon_\tau\right\rangle_{\mathfrak{H}^{\otimes 3}}^2\right)^{\frac{1}{2}} \\ &\leq C\sup_m \left\|\partial_{\frac{m}{n}}\right\|_{\mathfrak{H}} \sqrt{\mathbb{E}\|D^2F_n^j\|_{\mathfrak{H}^{\otimes 2}}^2} \sum_{m=a_i}^{b_i-1}\left|\left\langle \partial_{\frac{m}{n}}, \varepsilon_\tau\right\rangle_{\mathfrak{H}}\right| \\ &\leq Cn^{-\frac{1}{6}}.\end{aligned}$$

The estimate is similar for the term $\mathbb{E}\left|\left\langle u_n^i, DF_n^j\otimes DF_n^k\otimes\varepsilon_t\right\rangle_{\mathfrak{H}^{\otimes 3}}\right|$, and Lemma 5.8 is proved. $\square$

Now we focus on the terms $\left\langle u_n^i, D^3F_n^j\right\rangle_{\mathfrak{H}^{\otimes 3}}$. By Lemma 2.1.b,

$$D^3F_n^j = \delta^3(D^3u_n^j) + 9\delta^2(D^2u_n^j) + 18\delta(Du_n^j) + 6u_n^j,$$

so that $\left\langle u_n^i, D^3F_n^j\right\rangle_{\mathfrak{H}^{\otimes 3}}$ can be written as,

$$\begin{aligned}&\sum_{\ell,m} f^{(3)}(\widetilde{X}_{\frac{m}{n}})\delta^3\left(f^{(6)}(\widetilde{X}_{\frac{\ell}{n}})\partial_{\frac{\ell}{n}}^{\otimes 3}\right)\left\langle \partial_{\frac{m}{n}}, \widetilde{\varepsilon}_{\frac{\ell}{n}}\right\rangle_{\mathfrak{H}}^3 \\ &\quad + 9\sum_{\ell,m} f^{(3)}(\widetilde{X}_{\frac{m}{n}})\delta^2\left(f^{(5)}(\widetilde{X}_{\frac{\ell}{n}})\partial_{\frac{\ell}{n}}^{\otimes 3}\right)\left\langle \partial_{\frac{m}{n}}, \widetilde{\varepsilon}_{\frac{\ell}{n}}\right\rangle_{\mathfrak{H}}^2\left\langle \partial_{\frac{m}{n}}, \partial_{\frac{\ell}{n}}\right\rangle_{\mathfrak{H}} \\ &\quad + 18\sum_{\ell,m} f^{(3)}(\widetilde{X}_{\frac{m}{n}})\delta\left(f^{(4)}(\widetilde{X}_{\frac{\ell}{n}})\partial_{\frac{\ell}{n}}\right)\left\langle \partial_{\frac{m}{n}}, \widetilde{\varepsilon}_{\frac{\ell}{n}}\right\rangle_{\mathfrak{H}}\left\langle \partial_{\frac{m}{n}}, \partial_{\frac{\ell}{n}}\right\rangle_{\mathfrak{H}}^2 + 6\left\langle u_n^i, u_n^j\right\rangle_{\mathfrak{H}^{\otimes 3}} \\ &:= A_n(i,j) + B_n(i,j) + C_n(i,j) + 6\left\langle u_n^i, u_n^j\right\rangle_{\mathfrak{H}^{\otimes 3}},\end{aligned}$$

where m, ℓ are the indices for $u_n^i, D^3F_n^j$, respectively, with $\lfloor nt_{i-1}\rfloor \leq m \leq \lfloor nt_i\rfloor$, and $\lfloor nt_{j-1}\rfloor \leq \ell \leq \lfloor nt_j\rfloor$.

Lemma 5.8. *For each $1 \leq i, j \leq d$ we have*

$$\left|\left\langle u_n^i, D^3F_n^j\right\rangle_{\mathfrak{H}^{\otimes 3}} - 6\ \delta_{ij}\left\langle u_n^i, u_n^j\right\rangle_{\mathfrak{H}^{\otimes 3}}\right| \xrightarrow{\mathbb{P}} 0$$

as $n \to \infty$, where δ_{ij} is the Kronecker delta.

Proof. We will show that for each $1 \leq i, j \leq d$

$$\lim_{n\to\infty} \mathbb{E}\left|A_n(i,j)\right| = \lim_{n\to\infty} \mathbb{E}\left|B_n(i,j)\right| = \lim_{n\to\infty} \mathbb{E}\left|C_n(i,j)\right| = 0.$$

and moreover, if $i \neq j$ then $\lim_{n\to\infty}\mathbb{E}\left|\left\langle u_n^i, u_n^j\right\rangle_{\mathfrak{H}^{\otimes 3}}\right| = 0$.

To begin with, observe that if $g(x)$ is a function satisfying (T.0), then it follows from condition (T.1) and Lemma 2.1.c that that for $q = 1,2,3$,

$$\sup_j \left\| \delta^q \left(g(\widetilde{X}_{\frac{j}{n}}) \partial_{\frac{j}{n}}^{\otimes q} \right) \right\|_{L^2} \leq C \sup_j \left\| \partial_{\frac{j}{n}} \right\|_{\mathfrak{H}}^q \leq C n^{-\frac{q}{6}}. \tag{5.17}$$

For the terms $A_n(i,j), B_n(i,j), C_n(i,j)$ we include the case $i = j$. We have

$$\mathbb{E}\left|A_n(i,j)\right| \leq \sup_m \left\| f^{(3)}(\widetilde{X}_{\frac{m}{n}}) \right\|_{L^2} \sup_\ell \left\| \delta^3 \left(f^{(6)}(\widetilde{X}_{\frac{\ell}{n}}) \partial_{\frac{\ell}{n}}^{\otimes 3} \right) \right\|_{L^2} \sup_{\ell,m} \left| \left\langle \partial_{\frac{m}{n}}, \widetilde{\varepsilon}_{\frac{\ell}{n}} \right\rangle_{\mathfrak{H}} \right| \sum_{\ell,m} \left\langle \partial_{\frac{m}{n}}, \widetilde{\varepsilon}_{\frac{\ell}{n}} \right\rangle_{\mathfrak{H}}^2.$$

Using (T.0), (5.17), and Lemma 5.2.a, respectively, we have

$$\mathbb{E}\left|A_n(i,j)\right| \leq C n^{-\frac{1}{2}-\lambda} \sum_{\ell,m} \left\langle \partial_{\frac{m}{n}}, \widetilde{\varepsilon}_{\frac{\ell}{n}} \right\rangle_{\mathfrak{H}}^2,$$

so that Lemma 5.2.c gives $\mathbb{E}\left|A_n(i,j)\right| \leq C n^{-\frac{1}{2}-3\lambda}$. Next, using (T.0), (5.17) and Lemma 5.2.a,

$$\begin{aligned}\mathbb{E}\left|B_n(i,j)\right| &\leq 9 \sup_m \left\| f^{(3)}(\widetilde{X}_{\frac{m}{n}}) \right\|_{L^2} \sup_\ell \left\| \delta^2 \left(f^{(5)}(\widetilde{X}_{\frac{\ell}{n}}) \partial_{\frac{\ell}{n}}^{\otimes 2} \right) \right\|_{L^2} \sup_{\ell,m} \left\langle \partial_{\frac{m}{n}}, \widetilde{\varepsilon}_{\frac{\ell}{n}} \right\rangle_{\mathfrak{H}}^2 \sum_{\ell,m} \left| \left\langle \partial_{\frac{m}{n}}, \partial_{\frac{\ell}{n}} \right\rangle_{\mathfrak{H}} \right| \\ &\leq C n^{-\frac{1}{3}-2\lambda} \sum_{\ell,m} \left| \left\langle \partial_{\frac{\ell}{n}}, \partial_{\frac{m}{n}} \right\rangle_{\mathfrak{H}} \right|;\end{aligned}$$

and so by Lemma 5.2.d,

$$\mathbb{E}\left|B_n(i,j)\right| \leq C n^{-\frac{2}{3}-2\lambda} \max\{\lfloor nt_i \rfloor, \lfloor nt_j \rfloor\},$$

which converges to zero since $2\lambda > 1/3$. Similarly for $C_n(i,j)$ using Lemma 5.2.d,

$$\begin{aligned}\mathbb{E}\left|C_n(i,j)\right| &\leq 18 \sup_m \left\| f^{(3)}(\widetilde{X}_{\frac{m}{n}}) \right\|_{L^2} \sup_\ell \left\| \delta \left(f^{(4)}(\widetilde{X}_{\frac{\ell}{n}}) \partial_{\frac{\ell}{n}} \right) \right\|_{L^2} \sup_{\ell,m} \left| \left\langle \partial_{\frac{m}{n}}, \widetilde{\varepsilon}_{\frac{\ell}{n}} \right\rangle_{\mathfrak{H}} \right| \sum_{\ell,m} \left| \left\langle \partial_{\frac{m}{n}}, \partial_{\frac{\ell}{n}} \right\rangle_{\mathfrak{H}}^2 \right| \\ &\leq C n^{-\frac{1}{6}-\lambda} \sum_{\ell,m} \left\langle \partial_{\frac{\ell}{n}}, \partial_{\frac{m}{n}} \right\rangle_{\mathfrak{H}}^2 \leq C n^{-\frac{5}{6}-\lambda} \max\{\lfloor nt_i \rfloor, \lfloor nt_j \rfloor\} \leq C n^{-\lambda+\frac{1}{6}}.\end{aligned}$$

For the second part, we may assume $i < j$. Using Lemma 5.2.e,

$$\begin{aligned}\mathbb{E}\left| \langle u_n^i, u_n^j \rangle_{\mathfrak{H}^{\otimes 3}} \right| &\leq \sup_m \left\| f^{(3)}(\widetilde{X}_{\frac{m}{n}}) \right\|_{L^2}^2 \sum_{\ell,m} \left| \left\langle \partial_{\frac{m}{n}}, \partial_{\frac{\ell}{n}} \right\rangle_{\mathfrak{H}}^3 \right| \\ &\leq C n^{-1} \lfloor nt_j \rfloor^{\varepsilon},\end{aligned}$$

which converges to zero because $\varepsilon < 1$. □

Lemma 5.9. *Using the above notation, for each $1 \leq i,j,k,l \leq d$ we have*

$$\lim_{n\to\infty} \mathbb{E}\left[\left\langle u_n^i, D^2 F_n^j \otimes D F_n^k \right\rangle_{\mathfrak{H}^{\otimes 3}}^2 \right] = 0, \text{ and} \tag{5.18}$$

$$\lim_{n\to\infty} \mathbb{E}\left[\left\langle u_n^i, D F_n^j \otimes D F_n^k \otimes D F_n^\ell \right\rangle_{\mathfrak{H}^{\otimes 3}}^2 \right] = 0. \tag{5.19}$$

The proof of this lemma is deferred to Section 5.4.

Lemmas 5.8, 5.9 and 5.10 show that condition (a) of Theorem 2.3 is satisfied, and moreover that the only non-trivial terms are of the form $6\left\langle u_n^i, u_n^i \right\rangle_{\mathfrak{H}^{\otimes 3}}$. It remains to establish the convergence of these terms to a non-negative random variable $6s_i^2$. With this result, it follows from Theorem 2.3 that the couple (X, F_n) converges stably to (X, ζ), where $\zeta = (\zeta_1, \ldots, \zeta_d)$ is a vector whose components are conditionally independent Gaussian random variables with mean zero and variance $6s_i^2$.

Lemma 5.10. *For each $1 \le i \le d$, conditioned on X,*

$$\lim_{n\to\infty} \left\langle u_n^i, u_n^i \right\rangle_{\mathfrak{H}^{\otimes 3}} = s_i^2,$$

where each s_i^2 has the form

$$s_i^2 = s(t_i)^2 - s(t_{i-1})^2 = \int_{t_{i-1}}^{t_i} f^{(3)}(X_s)^2 \eta(ds).$$

It follows that on the subinterval $[t_{i-1}, t_i]$ we have the conditional result

$$6\left\langle u_n^i, u_n^i \right\rangle_{\mathfrak{H}^{\otimes 3}} \longrightarrow 6\int_{t_{i-1}}^{t_i} f^{(3)}(X_s)^2 \eta(ds),$$

almost surely as $n \to \infty$, which implies

$$F_n^i \xrightarrow{\mathscr{L}} \sqrt{6}\int_{t_{i-1}}^{t_i} f^{(3)}(X_s) dB_s, \tag{5.20}$$

where $\{B_t, t \ge 0\}$ is a Brownian motion, independent of X, with variance $\eta(t)$.

Proof. Let $a = \lfloor nt_{i-1} \rfloor$ and $b = \lfloor nt_i \rfloor$, and recall $\beta_n(j,k) = \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}}$, from condition (T.6). We have

$$\left\langle u_n^i, u_n^i \right\rangle_{\mathfrak{H}^{\otimes 3}} = \sum_{j,k=a}^{b-1} f^{(3)}(\widetilde{X}_{\frac{j}{n}}) f^{(3)}(\widetilde{X}_{\frac{k}{n}}) \beta_n(j,k)^3.$$

For each n, define a discrete measure on $\{1, 2, \ldots\}^{\otimes 2}$ by

$$\mu_n := \sum_{j,k=0}^{\infty} \beta_n(j,k)^3 \delta_{jk},$$

where δ_{jk} denotes the Kronecker delta. It follows from condition (T.6) that for each $t > 0$,

$$\begin{aligned} \mu\left([0,t]^2\right) &:= \lim_{n\to\infty} \mu_n\left(\lfloor nt \rfloor, \lfloor nt \rfloor\right) \\ &= \lim_{n} \sum_{j,k=0}^{\lfloor nt \rfloor - 1} \beta_n(j,k)^3 = \eta(t). \end{aligned}$$

Moreover, if $0 < s < t$ then

$$\mu_n(\lfloor ns \rfloor, \lfloor nt \rfloor) = \mu_n(\lfloor ns \rfloor, \lfloor ns \rfloor) + \sum_{j=0}^{\lfloor ns \rfloor - 1} \sum_{k=\lfloor ns \rfloor}^{\lfloor nt \rfloor - 1} \beta_n(j,k)^3,$$

which converges to zero because the disjoint sum vanishes by Lemma 5.2.e. Hence we can conclude that μ_n converges weakly to the measure given by $\mu([0,s] \times [0,t]) = \eta(s \wedge t)$. It follows by continuity of $f^{(3)}(X_t)$ and Portmanteau theorem that

$$\sum_{j,k=0}^{\lfloor nt \rfloor - 1} f^{(3)}(\widetilde{X}_{\frac{j}{n}}) f^{(3)}(\widetilde{X}_{\frac{k}{n}}) \beta_n(j,k)^3 = \int_{\mathbb{R}^2} f^{(3)}(X_s) f^{(3)}(X_u) \mathbf{1}_{s<t} \mathbf{1}_{u<t} \, \mu_n(ds, du)$$

converges in $L^1(\Omega, \mathfrak{H})$ to

$$\int_0^t f^{(3)}(X_s)^2 \eta(ds).$$

It follows that on the subinterval $[t_{i-1}, t_i]$ we have the result

$$\langle u_n^i, u_n^i \rangle_{\mathfrak{H}^{\otimes 3}} \longrightarrow \int_{t_{i-1}}^{t_i} f^{(3)}(X_s)^2 \eta(ds)$$

in $L^1(\Omega, \mathfrak{H})$ as $n \to \infty$. Using the Itô isometry for the above integral, we conclude (5.20). □

5.2.5 Relative compactness of the sequences

To establish convergence of $S_n^T(t)$ and $S_n^{M2}(t)$ in $\mathbf{D}[0,\infty)$, we need to show that $\{S_n^*(t)\}$ is relatively compact in the sense of Corollary 2.5, where $S_n^*(t)$ denotes either $S_n^T(t)$ or $S_n^{M2}(t)$ as appropriate. For this, it is enough to show that there exist real numbers $\alpha > 0$, $\beta > 1$ such that for each $T > 0$ and any $0 \le t_1 < t < t_2 \le T$ we have,

$$\mathbb{E}\left[|S_n^*(t) - S_n^*(t_1)|^\alpha \, |S_n^*(t_2) - S_n^*(t)|^\alpha \right] \le C \left(\frac{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor}{n} \right)^\beta .$$

We will do this in several parts. From the preceding section we have,

$$S_n^T(t) = f(X_{\frac{\lfloor nt \rfloor}{n}}) - f(X_0) + \frac{1}{12} \sum_{j=0}^{\lfloor nt \rfloor - 1} \delta^3 \left(f^{(3)}(\widetilde{X}_{\frac{j}{n}}) \partial_{\frac{j}{n}}^{\otimes 3} \right) + \frac{3}{12} \sum_{j=0}^{\lfloor nt \rfloor - 1} \left\| \Delta X_{\frac{j}{n}} \right\|_{L^2}^2 f^{(3)}(\widetilde{X}_{\frac{j}{n}}) \delta(\partial_{\frac{j}{n}}) + \frac{1}{12} P_n(t) - \Delta_n^T(t),$$

and

$$S_n^{M2}(t) = f(X_{\frac{\lfloor nt \rfloor}{n}}) - f(X_0) - \frac{1}{24} \sum_{j=0}^{\lfloor nt \rfloor - 1} \delta^3 \left(f^{(3)}(\widetilde{X}_{\frac{j}{n}}) \partial_{\frac{j}{n}}^{\otimes 3} \right) - \frac{3}{24} \sum_{j=0}^{\lfloor nt \rfloor - 1} \left\| \Delta X_{\frac{j}{n}} \right\|_{L^2}^2 f^{(3)}(\widetilde{X}_{\frac{j}{n}}) \delta(\partial_{\frac{j}{n}}) - \frac{1}{24} P_n(t) - \Delta_n^{M2}(t).$$

By Lemmas 5.3, 5.4, and 5.7 we have

$$\mathbb{E}\left[\left(\sum_{j=\lfloor nt_1\rfloor}^{\lfloor nt_2\rfloor-1} f^{(5)}(\widetilde{X}_{\frac{j}{n}})\Delta X_{\frac{j}{n}}^5\right)^2\right] \leq Cn^{-\frac{4}{3}}\left(\lfloor nt_2\rfloor - \lfloor nt_1\rfloor\right);$$
$$\mathbb{E}\left[\left(Z_n(t_2)-Z_n(t_1)\right)^2\right] \leq Cn^{-\frac{7}{3}}\left(\lfloor nt_2\rfloor - \lfloor nt_1\rfloor\right)^2; \text{ and}$$
$$\mathbb{E}\left[P_n(t)^2\right] \leq Cn^{-\frac{4}{3}}\left(\lfloor nt_2\rfloor - \lfloor nt_1\rfloor\right),$$

where we recall the first two items are components of $\Delta_n^*(t)$. Each of these estimates has the form

$$\mathbb{E}\left[\left(U_n(t_2)-U_n(t_1)\right)^2\right] \leq Cn^{-\beta}\left(\lfloor nt_2\rfloor - \lfloor nt_1\rfloor\right)^{\zeta} \leq C\left(\frac{\lfloor nt_2\rfloor - \lfloor nt_1\rfloor}{n}\right)^{\beta},$$

where $\zeta < \beta$ and $\beta > 1$, hence it follows by Cauchy-Schwarz that for $t_1 < t < t_2$ we have

$$\mathbb{E}\left[\left|U_n(t)-U_n(t_1)\right|\,\left|U_n(t_2)-U_n(t)\right|\right] \leq C\left(\frac{\lfloor nt_2\rfloor - \lfloor nt_1\rfloor}{n}\right)^{\beta},$$

so each of these individual sequences is relatively compact. For the term,

$$Y_n(t) = \sum_{j=0}^{\lfloor nt\rfloor-1} \left\|\Delta X_{\frac{j}{n}}\right\|_{L^2}^2 f^{(3)}(\widetilde{X}_{\frac{j}{n}})\delta(\partial_{\frac{j}{n}}),$$

we have by Lemma 5.6 that $Y_n(t)$ vanishes in probability. However, to show relative compactness we need a different estimate.

Lemma 5.11. *For $0 \leq t_1 < t_2 \leq T$ such that $\lfloor nt_2\rfloor - \lfloor nt_1\rfloor \geq 1$, we have*

$$\mathbb{E}\left[\left(Y_n(t_2)-Y_n(t_1)\right)^4\right] \leq Cn^{-2}\left(\lfloor nt_2\rfloor - \lfloor nt_1\rfloor\right)^2 + Cn^{-\frac{4}{3}-4\lambda}\left(\lfloor nt_2\rfloor - \lfloor nt_1\rfloor\right)^{\frac{4}{3}+4\lambda}.$$

It follows that the sequence $\{Y_n(t)\}$ is relatively compact.

Proof. Let $\Phi_n := \Phi_n(j_1, j_2, j_3, j_4) = \prod_{i=1}^{4} f^{(3)}(\widetilde{X}_{\frac{j_i}{n}})$, and let $a = \lfloor nt_1 \rfloor$, $b = \lfloor nt_2 \rfloor$. We have

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{j=a}^{b-1}\left\|\Delta X_{\frac{j}{n}}\right\|_{L^2}^2 f^{(3)}(\widetilde{X}_{\frac{j}{n}})\delta(\partial_{\frac{j}{n}})\right)^4\right] \\
&\quad\le \sup_{a\le j<b}\left\|\Delta X_{\frac{j}{n}}\right\|_{L^2}^8 \sum_{j_1,j_2,j_3,j_3}\left|\mathbb{E}\left[\Phi_n(j_1,j_2,j_3,j_4)\delta(\partial_{\frac{j_1}{n}})\delta(\partial_{\frac{j_2}{n}})\delta(\partial_{\frac{j_3}{n}})\delta(\partial_{\frac{j_4}{n}})\right]\right| \\
&\quad\le Cn^{-\frac{4}{3}}\sum_{j_1,j_2,j_3,j_4}\left|\mathbb{E}\left[\left\langle D\left[\Phi_n\delta(\partial_{\frac{j_1}{n}})\delta(\partial_{\frac{j_2}{n}})\delta(\partial_{\frac{j_3}{n}})\right],\partial_{\frac{j_4}{n}}\right\rangle_{\mathfrak{H}}\right]\right| \\
&\quad\le Cn^{-\frac{4}{3}}\sum_{j_1,j_2,j_3,j_4}\sum_{r=1}^{4}\left|\mathbb{E}\left[\Phi_n^{(r)}\delta(\partial_{\frac{j_1}{n}})\delta(\partial_{\frac{j_2}{n}})\delta(\partial_{\frac{j_3}{n}})\right]\left\langle\widetilde{\varepsilon}_{\frac{j_r}{n}},\partial_{\frac{j_4}{n}}\right\rangle_{\mathfrak{H}}\right| \\
&\qquad+3Cn^{-\frac{4}{3}}\sum_{j_1,j_2,j_3,j_4}\left|\mathbb{E}\left[\Phi_n\delta(\partial_{\frac{j_1}{n}})\delta(\partial_{\frac{j_2}{n}})\right]\left\langle\partial_{\frac{j_3}{n}},\partial_{\frac{j_4}{n}}\right\rangle_{\mathfrak{H}}\right| \\
&\quad= Cn^{-\frac{4}{3}}\sum_{j_1,j_2,j_3,j_4}\sum_{r=1}^{4}\left|\mathbb{E}\left[\left\langle D\left[\Phi_n^{(r)}\delta(\partial_{\frac{j_1}{n}})\delta(\partial_{\frac{j_2}{n}})\right],\partial_{\frac{j_3}{n}}\right\rangle_{\mathfrak{H}}\right]\left\langle\widetilde{\varepsilon}_{\frac{j_r}{n}},\partial_{\frac{j_4}{n}}\right\rangle_{\mathfrak{H}}\right| \\
&\qquad+3Cn^{-\frac{4}{3}}\sum_{j_1,j_2,j_3,j_4}\left|\mathbb{E}\left[\left\langle D\left[\Phi_n\delta(\partial_{\frac{j_1}{n}})\right],\partial_{\frac{j_2}{n}}\right\rangle_{\mathfrak{H}}\right]\left\langle\partial_{\frac{j_3}{n}},\partial_{\frac{j_4}{n}}\right\rangle_{\mathfrak{H}}\right|
\end{aligned}$$

where

$$\Phi_n^{(r)} = f^{(4)}(\widetilde{X}_{\frac{j_r}{n}})\prod_{\substack{i=1\\ i\neq r}}^{4} f^{(3)}(\widetilde{X}_{\frac{j_i}{n}}).$$

Continuing this process, we obtain terms of the form:

$$\begin{aligned}
&Cn^{-\frac{4}{3}}\sum_{j_1,j_2,j_3,j_4}\left|\mathbb{E}\left[\Phi_n\right]\left\langle\partial_{\frac{j_1}{n}},\partial_{\frac{j_2}{n}}\right\rangle_{\mathfrak{H}}\left\langle\partial_{\frac{j_3}{n}},\partial_{\frac{j_4}{n}}\right\rangle_{\mathfrak{H}}\right|, \\
&Cn^{-\frac{4}{3}}\sum_{j_1,j_2,j_3,j_4}\left|\mathbb{E}\left[\partial^2\Phi_n\right]\left\langle\widetilde{\varepsilon}_{\frac{j_a}{n}},\partial_{\frac{j_1}{n}}\right\rangle_{\mathfrak{H}}\left\langle\widetilde{\varepsilon}_{\frac{j_b}{n}},\partial_{\frac{j_2}{n}}\right\rangle_{\mathfrak{H}}\left\langle\partial_{\frac{j_3}{n}},\partial_{\frac{j_4}{n}}\right\rangle_{\mathfrak{H}}\right|, \text{ and} \\
&Cn^{-\frac{4}{3}}\sum_{j_1,j_2,j_3,j_4}\left|\mathbb{E}\left[\partial^4\Phi_n\right]\left\langle\widetilde{\varepsilon}_{\frac{j_a}{n}},\partial_{\frac{j_1}{n}}\right\rangle_{\mathfrak{H}}\left\langle\widetilde{\varepsilon}_{\frac{j_b}{n}},\partial_{\frac{j_2}{n}}\right\rangle_{\mathfrak{H}}\left\langle\widetilde{\varepsilon}_{\frac{j_c}{n}},\partial_{\frac{j_3}{n}}\right\rangle_{\mathfrak{H}}\left\langle\widetilde{\varepsilon}_{\frac{j_d}{n}},\partial_{\frac{j_4}{n}}\right\rangle_{\mathfrak{H}}\right|,
\end{aligned}$$

where $\partial^k\Phi_n$ represents the appropriate k^{th} derivative of Φ_n. By Lemma 5.2.c and 5.2.d, the sums of each type have, respectively, upper bounds of the form

$$Cn^{-2}(b-a)^{-2} + Cn^{-\frac{5}{3}}(b-a) + Cn^{-\frac{4}{3}-4\lambda}(b-a)^{4\lambda},$$

hence we conclude that

$$\mathbb{E}\left[(Y_n(t_2)-Y_n(t_1))^4\right] \le C\left(n^{-2}(\lfloor nt_2\rfloor-\lfloor nt_1\rfloor)^2 + n^{-\frac{4}{3}-4\lambda}(\lfloor nt_2\rfloor-\lfloor nt_1\rfloor)^{\frac{4}{3}+4\lambda}\right),$$

As for above terms, it follows by Cauchy-Schwarz that

$$\mathbb{E}\left[|Y_n(t)-Y_n(t_1)|^2\,|Y_n(t_2)-Y_n(t)|^2\right] \le C\left(n^{-2}(\lfloor nt_2\rfloor - \lfloor nt_1\rfloor)^2 + n^{-\frac{4}{3}-4\lambda}(\lfloor nt_2\rfloor - \lfloor nt_1\rfloor)^{\frac{4}{3}+4\lambda}\right),$$

and thus $\{Y_n(t)\}$ is relatively compact. □

Tightness of F_n.

To conclude the proof of Theorem 5.1, we want to show that the sequence $\{F_n(t)\}$ satisfies the relative compactness condition.

Lemma 5.12. *For $0 \le t_1 < t_2 \le T$, write*

$$F_n(t_2) - F_n(t_1) = \sum_{j=\lfloor nt_1\rfloor+1}^{\lfloor nt_2\rfloor-1} \delta^3\left(f^{(3)}(\widetilde{X}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 3}\right)$$

Then given $0 \le t_1 < t < t_2 \le T$, there exists a positive constant C such that

$$\mathbb{E}\left[|F_n(t)-F_n(t_1)|^2|F_n(t_2)-F_n(t)|^2\right] \le C\left(\frac{\lfloor nt_2\rfloor - \lfloor nt_1\rfloor}{n}\right)^2. \tag{5.21}$$

Proof. We begin with a general claim about the norm of DF_n. Suppose a,b are nonnegative integers. Let

$$g_a = \sum_{j=\lfloor nt_1\rfloor}^{\lfloor nt_2\rfloor-1} f^{(a)}(\widetilde{X}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 3}.$$

Then we have

$$\mathbb{E}\left[\|D^b g_a\|^4_{\mathfrak{H}^{\otimes 3+b}}\right] \le C\left(\frac{\lfloor nt_2\rfloor - \lfloor nt_1\rfloor}{n}\right)^2. \tag{5.22}$$

Proof of (5.22). For each b we can write

$$\begin{aligned}
&\mathbb{E}\left[\left(\|D^b g_a\|^2_{\mathfrak{H}^{\otimes 3+b}}\right)^2\right] \\
&\qquad = \mathbb{E}\left[\left(\sum_{j,k=\lfloor nt_1\rfloor}^{\lfloor nt_2\rfloor-1} f^{(a+b)}(\widetilde{X}_{\frac{j}{n}})f^{(a+b)}(\widetilde{X}_{\frac{k}{n}})\left\langle \widetilde{\varepsilon}_{\frac{j}{n}}^{\otimes b}, \widetilde{\varepsilon}_{\frac{k}{n}}^{\otimes b}\right\rangle_{\mathfrak{H}^{\otimes b}}\left\langle \partial_{\frac{j}{n}}^{\otimes 3}, \partial_{\frac{k}{n}}^{\otimes 3}\right\rangle_{\mathfrak{H}^{\otimes 3}}\right)^2\right] \\
&\qquad \le \sup_{\lfloor nt_1\rfloor \le j < \lfloor nt_2\rfloor}\left(\mathbb{E}\left|f^{(a+b)}(\widetilde{X}_{\frac{j}{n}})\right|^4\right)^{\frac{1}{2}}\left(\sup_{j,k}\left|\left\langle \widetilde{\varepsilon}_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|^{2b}\right)\left(\sum_{j,k=\lfloor nt_1\rfloor}^{\lfloor nt_2\rfloor-1}\left|\left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|^3\right)^2 \\
&\qquad \le Cn^{-2}\left(\lfloor nt_2\rfloor - \lfloor nt_1\rfloor\right)^2,
\end{aligned}$$

by Lemma 5.2.d.

Proof of (5.21). By the Meyer inequality (6.4) there exists a constant $c_{2,4}$ such that

$$\mathbb{E}\left|\left(\delta^3(u_n)\right)^4\right| \le c_{3,4}\|u_n\|^4_{\mathbb{D}^{3,4}(\mathfrak{H}^{\otimes 3})},$$

where in this case,

$$u_n = \sum_{j=\lfloor nt_1 \rfloor}^{\lfloor nt_2 \rfloor - 1} f^{(3)}(\widetilde{X}_{\frac{j}{n}}) \partial_{\frac{j}{n}}^{\otimes 3}$$

and

$$\|u_n\|^4_{\mathbb{D}^{3,4}(\mathfrak{H}^{\otimes 3})} = \mathbb{E}\|u_n\|^4_{\mathfrak{H}^{\otimes 3}} + \mathbb{E}\|Du_n\|^4_{\mathfrak{H}^{\otimes 4}} + \mathbb{E}\|D^2 u_n\|^4_{\mathfrak{H}^{\otimes 5}} + \mathbb{E}\|D^3 u_n\|^4_{\mathfrak{H}^{\otimes 6}}.$$

From (5.22) we have $\mathbb{E}\|D^b u_n\|^4_{\mathfrak{H}^{\otimes 3+b}} \leq C_b n^{-2} \left(\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor\right)^2$ for $b = 0, 1, 2, 3$. From this result, given $0 \leq t_1 < t < t_2$, it follows from the Hölder inequality that

$$\mathbb{E}\left[|F_n(t) - F_n(t_1)|^2 |F_n(t_2) - F_n(t)|^2\right] \leq C \left(\frac{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor}{n}\right)^2.$$

□

5.2.6 Proof of Theorem 5.1

Here we give a brief summary of the preceding lemmas. For $S_n^T(t)$ we have

$$S_n^T(t) = f(X_t) - f(X_0) + \frac{1}{12} \sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(3)}(\widetilde{X}_{\frac{j}{n}}) \Delta X^3_{\frac{j}{n}} - \Delta_n^T(t),$$

where we can express

$$\sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(3)}(\widetilde{X}_{\frac{j}{n}}) \Delta X^3_{\frac{j}{n}} = \sum_{j=0}^{\lfloor nt \rfloor - 1} \delta^3 \left(f^{(3)}(\widetilde{X}_{\frac{j}{n}}) \partial_{\frac{j}{n}}^{\otimes 3} \right) + P_n(t) + 3 \sum_{j=0}^{\lfloor nt \rfloor - 1} \|\Delta X_{\frac{j}{n}}\|^2_{L^2(\Omega)} f^{(3)}(\widetilde{X}_{\frac{j}{n}}) \Delta X_{\frac{j}{n}}.$$

From Lemmas 5.3 - 5.6 and 5.11, we have that the terms $\sum_{j=0}^{\lfloor nt \rfloor - 1} \|\Delta X_{j/n}\|^2_{L^2(\Omega)} f^{(3)}(\widetilde{X}_{j/n}) \Delta X_{j/n}$, $\Delta_n^T(t)$, and $P_n(t)$ tend to zero in $L^1(\Omega)$ for each t, and moreover these terms satisfy the tightness condition of Corollary 2.5. By Lemmas 5.7 - 5.10, the random vector $F_n = (F_n^1, \ldots, F_n^d)$ satisfies the conditions of Theorem 2.3, where

$$F_n^i = \sum_{j=\lfloor nt_{i-1} \rfloor}^{\lfloor nt_i \rfloor - 1} \delta^3 \left(f^{(3)}(\widetilde{X}_{\frac{j}{n}}) \partial_{\frac{j}{n}}^{\otimes 3} \right),$$

and Theorem 5.1 follows from Theorem 2.3 and Corollary 2.5. For $S_n^{M2}(t)$, the proof is the same except that

$$S_n^{M2}(t) = f(X_t) - f(X_0) - \frac{1}{24} \sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(3)}(\widetilde{X}_{\frac{j}{n}}) \Delta X^3_{\frac{j}{n}} - \Delta_n^{M2}(t),$$

and this concludes the proof of Theorem 5.1.

5.3 Examples of suitable processes

5.3.1 Bifractional Brownian motion

The bifractional Brownian motion is a generalization of fractional Brownian motion, first introduced by Houdré and Villa [20]. It is defined as a centered Gaussian process $B^{H,K} = \{B_t^{H,K}, t \geq 0\}$,with covariance given by,

$$\mathbb{E}[B_t^{H,K} B_s^{H,K}] = \frac{1}{2^K}\left(t^{2H}+s^{2H}\right)^K - \frac{1}{2^K}|t-s|^{2HK},$$

where $H \in (0,1)$, $K \in (0,1]$ (Note that the case $K=1$ corresponds to fractional Brownian motion with Hurst parameter H). The reader may refer to [33] and [21] for further discussion of properties.

In this section, we show that the results of Section 5.2 are valid for bifractional Brownian motion with parameter values H, K such that $HK \geq 1/6$.

Proposition 5.13. *Let $B_t = \{B_t^{H,K}, t \geq 0\}$ be a bifractional Brownian motion with parameters H, K satisfying $HK = 1/6$. Then conditions (T.1) - (T.5) are satisfied, with $\theta = 2/3$; $\lambda = 1/3$;*

$$\nu = \begin{cases} 5/3 & \text{if } H < 1/2 \\ 4H - \frac{1}{3} & \text{if } H \geq 1/2 \end{cases}; \qquad \text{and} \qquad \gamma = \begin{cases} 2/3 + 2H & \text{if } H \leq 1/2 \text{ and } K < 1 \\ 5/3 & \text{otherwise} \end{cases}.$$

Proof. Condition (T.1). From Proposition 3.1 of [20] we have

$$\mathbb{E}\left[(B_t - B_{t-s})^2\right] \leq Cs^{2HK} = Cs^{\frac{1}{3}}.$$

Condition (T.2). By Fundamental Theorem of Calculus,

$$\left|\mathbb{E}\left[B_t^2 - B_{t-s}^2\right]\right| = t^{2HK} - (t-s)^{2HK} = \int_{-s}^{0} 2HK(t+\xi)^{2HK-1} d\xi \leq Cs(t-s)^{-\frac{2}{3}}.$$

Condition (T.3).

$$\begin{aligned}\mathbb{E}\left[(B_t - B_{t-s})^2 - (B_{t-s} - B_{t-2s})^2\right] &= \mathbb{E}\left[(B_t - B_{t-2s})(B_t - 2B_{t-s} + B_{t-2s})\right] \\ &= t^{2HK} - \frac{2}{2^K}\left[t^{2H} + (t-s)^{2H}\right]^K + \frac{1}{2^K}\left[t^{2H} + (t-2s)^{2H}\right]^K \\ &\quad - \frac{1}{2^K}\left[t^{2H} + (t-2s)^{2H}\right]^K + \frac{2}{2^K}\left[(t-s)^{2H} + (t-2s)^{2H}\right]^K.\end{aligned}$$

In absolute value, this is bounded by

$$\begin{aligned}&\frac{1}{2^K}\left|\left[t^{2H} + t^{2H}\right]^K - 2\left[t^{2H} + (t-s)^{2H}\right]^K + \left[t^{2H} + (t-2s)^{2H}\right]^K\right| \\ &\qquad + \frac{1}{2^K}\left|\left[t^{2H} + (t-2s)^{2H}\right]^K - 2\left[(t-s)^{2H} + (t-2s)^{2H}\right]^K + \left[(t-2s)^{2H} + (t-2s)^{2H}\right]^K\right|.\end{aligned}$$

Both terms have the form

$$2^{-K}\left|g(t) - 2g(t-s) + g(t-2s)\right| \leq Cs^2 \sup_{x \in [t-2s, t]} |g''(x)|,$$

for $g(x) = (c + x^{2H})^K$. We show an upper bound for the first term $g''(x)$, with the other one similar. We have

$$\sup_{x \in [t-2s,t]} |g''(x)| \leq 4H^2 K(1-K) \left[t^{2H} + x^{2H}\right]^{K-2} x^{4H-2} + 2HK|2H-1| \left[t^{2H} + x^{2H}\right]^{K-1} x^{2H-2}.$$

For the above values of H, K we have

$$\sup_{x \in [t-2s,t]} 2HK|2H-1| \left[t^{2H} + x^{2H}\right]^{K-1} x^{2H-2} \leq C(t-2s)^{2HK-2}.$$

For the first term, if $H < 1/2$ then

$$\begin{aligned}\sup_{x \in [t-2s,t]} 4H^2 K(1-K) \left[t^{2H} + x^{2H}\right]^{K-2} x^{4H-2} &\leq C(t-2s)^{2HK-4H+4H-2} \\ &= C(t-2s)^{2HK-2}.\end{aligned}$$

On the other hand, if $H \geq 1/2$, then $t \geq 4s$ implies $t \geq 2(t-2s)$, hence

$$\begin{aligned}\sup_{x \in [t-2s,t]} 4H^2 K(1-K) \left[t^{2H} + x^{2H}\right]^{K-2} x^{4H-2} &\leq 4H^2 K(1-K) 3^{K-2} (t-2s)^{2H(K-2)} x^{4H-2} \\ &\leq C(t-2s)^{2HK-4H} = C(t-2s)^{\frac{1}{3}-4H}.\end{aligned}$$

Condition (T.4). First, for the case $|t-r| < 2s$ or $t < 2s$, we have

$$\begin{aligned}|\mathbb{E}[B_r(B_t - B_{t-s})]| &= |\mathbb{E}[B_r B_t - B_r B_{t-s}]| \\ &\leq \frac{1}{2^K}\left(\left[r^{2H} + t^{2H}\right]^K - \left[r^{2H} + (t-s)^{2H}\right]^K\right) + \frac{1}{2^K}\left||r-t+s|^{2HK} - |r-t|^{2HK}\right| \\ &\leq Cs^{2HK} = Cs^{\frac{1}{3}}\end{aligned}$$

using the inequality $a^r - b^r \leq (a-b)^r$ for $0 < r < 1$. For $|t-r| \geq 2s$, $t \geq 2s$, we consider two cases. First, assume $r \geq t + 2s$.

$$\begin{aligned}|\mathbb{E}[B_r(B_t - B_{t-s})]| &= |\mathbb{E}[B_r B_t - B_r B_{t-s}]| \\ &\leq \frac{1}{2^K}\left(\left[r^{2H} + t^{2H}\right]^K - \left[r^{2H} + (t-s)^{2H}\right]^K\right) + \frac{1}{2^K}\left||r-t+s|^{2HK} - |r-t|^{2HK}\right| \\ &= \frac{1}{2^K}\int_{-s}^{0} 2HK\left[r^{2H} + (t+\xi)^{2H}\right]^{K-1}(t+\xi)^{2H-1} d\xi \\ &\quad + \frac{1}{2^K}\int_0^s 2HK(r-t+\eta)^{2HK-1} d\eta \\ &\leq 2^{1-K} HKs(t-s)^{-\frac{2}{3}} + 2^{1-K} HKs(r-t)^{-\frac{2}{3}},\end{aligned}$$

where we used the fact that $r - t \geq 2s$ implies $r - t \geq 2(r-t-s)$. On the other hand, if $r \leq t - 2s$, then the estimate for

$$\frac{1}{2^K}\left(\left[r^{2H} + t^{2H}\right]^K - \left[r^{2H} + (t-s)^{2H}\right]^K\right)$$

is the same, and for the other term we have,

$$\frac{1}{2^K}\left||r-t+s|^{2HK}-|r-t|^{2HK}\right| \le \frac{1}{2^K}\int_{-s}^{0} 2HK(t-r-\xi)^{2HK-1}d\xi$$
$$\le 2^{1-K}HKs(t-r-s)^{2HK-1} \le 2^{\frac{5}{3}-K}HKs(t-r)^{-\frac{2}{3}},$$

hence for either case we have an upper bound of $Cs\left((t-s)^{\lambda-1}+|t-r|^{\lambda-1}\right)$ for $\lambda=\frac{1}{3}$.

Condition (T.5). Assume $t\wedge r \ge 2s$ and $|t-r|\ge 2s$. We have

$$\begin{aligned}
&\mathbb{E}\left[(B_t-B_{t-s})(B_r-B_{r-s})\right]\\
&\quad=\frac{1}{2^K}\left(\left[t^{2H}+r^{2H}\right]^K-\left[t^{2H}+(r-s)^{2H}\right]^K-\left[(t-s)^{2H}+r^{2H}\right]^K+\left[(t-s)^{2H}+(r-s)^{2H}\right]^K\right)\\
&\qquad+\frac{1}{2^K}\left(|t-r+s|^{2HK}-2|t-r|^{2HK}+|t-r-s|^{2HK}\right).
\end{aligned}$$

This can be interpreted as the sum of a position term, $\frac{1}{2^K}\varphi(t,r,s)$, and a distance term, $\frac{1}{2^K}\psi(t-r,s)$, where

$$\varphi(t,r,s)=\left[t^{2H}+r^{2H}\right]^K-\left[t^{2H}+(r-s)^{2H}\right]^K-\left[(t-s)^{2H}+r^{2H}\right]^K+\left[(t-s)^{2H}+(r-s)^{2H}\right]^K;\ \text{ and}$$

$$\psi(t-r,s)=|t-r+s|^{2HK}-2|t-r|^{2HK}+|t-r-s|^{2HK}.$$

We begin with the position term. Note that if $K=1$, then $\varphi(t,r,s)=0$, so we may assume $K<1$ and $H>\frac{1}{6}$. Without loss of generality, assume $0<2s\le r\le t$. We can write $\varphi(t,r,s)$ as

$$\begin{aligned}
&2HK\int_0^s\left(\left[t^{2H}+(r-\xi)^{2H}\right]^{K-1}(r-\xi)^{2H-1}-\left[(t-s)^{2H}+(r-\xi)^{2H}\right]^{K-1}(r-\xi)^{2H-1}\right)d\xi\\
&\quad=\int_0^s\int_0^s 4H^2K(1-K)\left[(t-\eta)^{2H}+(r-\xi)^{2H}\right]^{K-2}(t-\eta)^{2H-1}(r-\xi)^{2H-1}\,d\xi\,d\eta,
\end{aligned}$$

so that

$$|\varphi(t,r,s)|\le 4H^2K(1-K)s^2\left[(t-s)^{2H}+(r-s)^{2H}\right]^{K-2}(t-s)^{2H-1}(r-s)^{2H-1}. \tag{5.23}$$

Using (5.23), there are 3 cases to consider:

- If $H<1/2$, then for $2s\le r\le t-2s$, we have $t-r<t-s$ and

  $$\begin{aligned}
  Cs^2\left[(t-s)^{2H}+(r-s)^{2H}\right]^{K-2}(t-s)^{2H-1}(r-s)^{2H-1} &\le Cs^2(t-r)^{2HK-2H-1}(r-s)^{2H-1}\\
  &=C\left(\frac{s}{r-s}\right)^{1-2H}s^{\frac{1}{3}+\frac{2}{3}+2H}(t-r)^{-\frac{2}{3}-2H}\\
  &\le Cs^{\frac{1}{3}+\gamma}|t-r|^{-\gamma},
  \end{aligned}$$

  where $\gamma=\frac{2}{3}+2H>1$.

- If $H=1/2$, then $K=1/3$ and for $2s\le r\le t-2s$

  $$s^2\left[(t-s)^{2H}+(r-s)^{2H}\right]^{K-2}(t-s)^{2H-1}(r-s)^{2H-1}\le s^2|t-r|^{-\frac{5}{3}}.$$

- If $H > 1/2$, then note that for $2s \le r \le t-2s$

$$s^2\left[(t-s)^{2H}+(r-s)^{2H}\right]^{K-2}(t-s)^{2H-1}(r-s)^{2H-1} \le s^2(t-s)^{2HK-2} \le s^2|t-r|^{-\frac{5}{3}}.$$

Next, consider the distance term $\psi(t-r,s)$. Without loss of generality, assume $2s \le r \le t-2s$. We have

$$\begin{aligned}|\psi(t-r,s)| &= \left|\ |t-r+s|^{2HK}-2|t-r|^{2HK}+|t-r-s|^{2HK}\ \right|\\ &= \left|\int_0^s\int_{-\xi}^{\xi} 2HK(2HK-1)\left[t-r+\eta\right]^{2HK-2}d\eta\ d\xi\right|\\ &\le Cs^2(t-r-s)^{2HK-2} \le Cs^2|t-r|^{-\frac{5}{3}},\end{aligned}$$

since $|t-r| \ge 2s$ implies $(t-r-s)^{-\frac{5}{3}} \le 2^{\frac{5}{3}}|t-r|^{-\frac{5}{3}}$. Note that when $K<1$, then $H<1/2$ implies $\gamma \le 5/3$, so the upper bound is controlled by $\varphi(t,r,s)$ in this $K=1$ case. □

Proposition 5.14. *Let $\{B_t^{H,K}, t \ge 0\}$ be a bifractional Brownian motion with parameters $H \le 1/2$ and $HK = 1/6$. Then Condition (T.6) holds, with the function $\eta(t) = C_K t$, where*

$$C_K = \frac{1}{8^K}\left(8+2\sum_{m=1}^{\infty}\left((m+1)^{\frac{1}{3}}-2m^{\frac{1}{3}}+(m-1)^{\frac{1}{3}}\right)^3\right).$$

Proof. First of all, we write

$$\sum_{j,k=0}^{\lfloor nt\rfloor-1}\beta_n(j,k)^3 = 2\sum_{j=0}^{\lfloor nt\rfloor-1}\beta_n(j,0)^3 \ + \sum_{j,k=1}^{\lfloor nt\rfloor-1}\beta_n(j,k)^3.$$

When $j \ge 2$, we have

$$\begin{aligned}|\beta_n(j,0)| &= \left|\mathbb{E}\left[X_{\frac{1}{n}}(X_{\frac{j+1}{n}}-X_{\frac{j}{n}})\right]\right|\\ &\le \frac{1}{2^K n^{\frac{1}{3}}}\left(\left[1+(j+1)^{2H}\right]^K-\left[1+j^{2H}\right]^K\right)+\frac{1}{2^K n^{\frac{1}{3}}}\left|(j-1)^{2HK}-j^{2HK}\right|\\ &\le \frac{1}{2^K n^{\frac{1}{3}}}\int_0^1 2HK\left[1+(j+x)^{2H}\right]^{K-1}(j+x)^{2H-1}dx+\frac{1}{2^K n^{\frac{1}{3}}}\int_0^1 2HK(j-1+y)^{2HK-1}dy\\ &\le Cn^{-\frac{1}{3}}(j-1)^{-\frac{2}{3}}.\end{aligned}$$

Therefore, using Lemma 5.2.a for $\beta_n(0,0)$ and $\beta_n(1,0)$,

$$\sum_{j=0}^{\lfloor nt\rfloor-1}|\beta_n(j,0)^3| \le 2Cn^{-1}+\sum_{j=2}^{\lfloor nt\rfloor-1}Cn^{-1}(j-1)^{-2} \le Cn^{-1};$$

and in the rest of the proof we will always assume $j,k \ge 1$.

As in Proposition 5.14, we use the decomposition,

$$\beta_n(j,k) = \frac{1}{2^K}\varphi\left(\frac{j+1}{n},\frac{k+1}{n},\frac{1}{n}\right) + \frac{1}{2^K}\psi\left(\frac{j-k}{n},\frac{1}{n}\right)$$
$$= 2^{-K}n^{-\frac{1}{3}}\varphi(j+1,k+1,1) + 2^{-K}n^{-\frac{1}{3}}\psi(j-k,1),$$

which gives

$$\beta_n(j,k)^3 = \frac{1}{8^K n}\left(\varphi^3 + 3\varphi^2\psi + 3\varphi\psi^2 + \psi^3\right).$$

To begin, we want to show that

$$\lim_{n\to\infty}\sum_{j,k=1}^{\lfloor nt\rfloor-1} n^{-1}|\varphi(j+1,k+1,1)| \;=\; 0. \tag{5.24}$$

Proof of (5.24). Note that $\varphi = 0$ if $K = 1$, so we may assume $K < 1$ and $H > 1/6$. From (5.23), when $t \wedge r \geq 2s$ and $|t-r| \geq 2s$ we have

$$|\varphi(t,r,s)| \leq 4H^2K(1-K)s^2\left[(t-s)^{2H} + (r-s)^{2H}\right]^{K-2}(t-s)^{2H-1}(r-s)^{2H-1}$$
$$\leq Cs^2(t-s)^{HK-1}(r-s)^{HK-1},$$

so that

$$|\varphi(j+1,k+1,1)| \leq Cn^{-2HK}j^{HK-1}k^{HK-1}.$$

Recalling the notation J_d from Lemma 5.2.d, we have

$$n^{-1}\sum_{j,k=0}^{\lfloor nt\rfloor-1}|\varphi(j+1,k+1,1)| = n^{-1}\sum_{(j,k)\in J_d}|\varphi(j+1,k+1,1)| + n^{-1}\sum_{(j,k)\notin J_d}|\varphi(j+1,k+1,1)|$$
$$\leq C\lfloor nt\rfloor n^{-\frac{4}{3}} + Cn^{-\frac{4}{3}}\left(\sum_{j=2}^{\lfloor nt\rfloor-1} j^{HK-1}\right)^2$$
$$\leq C\lfloor nt\rfloor n^{\frac{4}{3}} + C\lfloor nt\rfloor^{2HK}n^{-\frac{4}{3}} \leq Cn^{-\frac{1}{3}},$$

where we used the fact (which follows from Lemma 5.2.a and the definition of φ and ψ) that $|\varphi(j+1,k+1,1)|$ is bounded. Hence, (5.24) is proved. It follows from (5.24) that

$$\frac{1}{8^K n}\sum_{j,k=1}^{\lfloor nt\rfloor-1}\left|\varphi^3 + 3\varphi^2\psi + 3\varphi\psi^2\right| \to_n 0, \tag{5.25}$$

since φ and ψ are both bounded. Hence, it is enough to consider

$$\eta(t) = \lim_{n\to\infty}\frac{1}{8^K n}\sum_{j,k=0}^{\lfloor nt\rfloor-1}\psi(j-k,1)^3. \tag{5.26}$$

To evaluate (5.26), we have

$$\begin{aligned}
&\frac{1}{8^K n}\sum_{j,k=0}^{\lfloor nt\rfloor-1}\psi(j-k,1)^3\\
&=\frac{1}{8^K n}\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left(|j-k+1|^{\frac{1}{3}}-2|j-k|^{\frac{1}{3}}+|j-k-1|^{\frac{1}{3}}\right)^3\\
&=\frac{1}{8^K n}\sum_{j=0}^{\lfloor nt\rfloor-1}2^3+\frac{2}{8^K n}\sum_{j=0}^{\lfloor nt\rfloor-1}\sum_{k=0}^{j-1}\left((j-k+1)^{\frac{1}{3}}-2(j-k)^{\frac{1}{3}}+(j-k-1)^{\frac{1}{3}}\right)^3\\
&=\frac{8\lfloor nt\rfloor}{8^K n}+\frac{2}{8^K n}\sum_{j=1}^{\lfloor nt\rfloor-1}\sum_{m=1}^{j-1}\left((m+1)^{\frac{1}{3}}-2m^{\frac{1}{3}}+(m-1)^{\frac{1}{3}}\right)^3\\
&=\frac{8\lfloor nt\rfloor}{8^K n}+\frac{2}{8^K n}\sum_{j=1}^{\lfloor nt\rfloor-1}\sum_{m=1}^{\infty}\left((m+1)^{\frac{1}{3}}-2m^{\frac{1}{3}}+(m-1)^{\frac{1}{3}}\right)^3\\
&\quad-\frac{2}{8^K n}\sum_{j=1}^{\lfloor nt\rfloor-1}\sum_{m=j}^{\infty}\left((m+1)^{\frac{1}{3}}-2m^{\frac{1}{3}}+(m-1)^{\frac{1}{3}}\right)^3,
\end{aligned}$$

where the last term tends to zero since

$$Cn^{-1}\sum_{j=1}^{\lfloor nt\rfloor-1}\sum_{m=j}^{\infty}\left((m+1)^{\frac{1}{3}}-2m^{\frac{1}{3}}+(m-1)^{\frac{1}{3}}\right)^3\le Cn^{-1}\sum_{j=1}^{\infty}j^{-4}\to_n 0.$$

We therefore conclude that $\eta(t)=C_K t$, where

$$C_K=\frac{1}{8^K}\left(8+2\sum_{m=1}^{\infty}\left((m+1)^{\frac{1}{3}}-2m^{\frac{1}{3}}+(m-1)^{\frac{1}{3}}\right)^3\right).$$

This number is approximately $\frac{7.188}{8^K}$. □

As an immediate consequence of our proof of Theorem 5.1, we have an alternate proof and extension of previous results in Gradinaru *et al.* In [16], it was proved that $S_n^*(t)$ converges in probability for any fractional Brownian motion with $H>1/6$, that is, the correction term vanishes. Following Remark 3.5, we may conclude the following:

Corollary 5.15. *Let $B_t=\{B_t^{H,K},t\ge 0\}$ be a bifractional Brownian motion with parameters $1/6<HK<1$. Then on a fixed interval $[0,T]$ and for $0<s\le 1$, B satisfies Corollary 3.4.*

Proof. Notice that $s\le 1$ implies $s^{2HK}\le s^{\frac{1}{3}}$. With small modifications to the proof of Proposition 5.14, it is easy to verify that conditions (T.1) - (T.5) are satisfied when $HK>1/6$. We want to show that

$$\lim_{n\to\infty}\sum_{j,k=0}^{\lfloor nt\rfloor-1}|\beta_n(j,k)^3|=0.$$

We may assume $K < 1$. From Proposition 3.1 of [20], we have that

$$\mathbb{E}\left[(B_t - B_{t-s})^2\right] \leq Cs^{2HK}.$$

Recalling the notation J_d from Lemma 5.2.d, Cauchy-Schwarz implies for $(j,k) \in J_d$, we have $|\beta_n(j,k)| \leq Cn^{-2HK}$. For $(j,k) \notin J_d$, by (5.23) we have

$$\begin{aligned}|\varphi(j+1,k+1,1)| &\leq 4H^2K(1-K)\left[j^{2H}+k^{2H}\right]^{K-2} j^{2H-1}k^{2H-1}\\ &\leq C|j-k|^{-1-2H(1-K)},\end{aligned}$$

and similar to Proposition 5.14, we have

$$|\psi(j-k,1)| \leq C|j-k|^{2HK-2};$$

hence for $(j,k) \notin J_d$ we have $|\beta_n(j,k)| \leq Cn^{-2HK}|j-k|^{-\gamma}$ for $\gamma = \min\{1+2H(1-K), 2HK-2\}$. It follows that

$$\begin{aligned}\left|\sum_{j,k=0}^{\lfloor nt\rfloor -1} \beta_n(j,k)^3\right| &\leq \sum_{(j,k)\in J_d} |\beta_n(j,k)^3| + \sum_{(j,k)\notin J_d} |\beta_n(j,k)^3|\\ &\leq \sum_{(j,k)\in J_d} Cn^{-2HK} + Cn^{-6HK} \sum_{(j,k)\notin J_d} |j-k|^{-\gamma}\\ &\leq Cn^{-6HK}\lfloor nt\rfloor\end{aligned}$$

so $|\eta(t)| = 0$ because $HK > 1/6$. □

5.3.2 Extended bifractional Brownian motion

This process is discussed in a recent paper by Bardina and Es-Sebaiy [2]. The covariance has the same formula as standard bBm, but it is 'extended' in the sense that $1 < K < 2$, with H restricted to satisfy $0 < HK < 1$. Within the context of this paper, this allows us to consider values of $1/12 < H < 1/6$. As in section 5.3.1, we show computations only for the case $HK = 1/6$. A result similar to Corollary 5.16 can also be shown by modification to the proposition below.

Proposition 5.16. *Let $Y = \{Y_t^{H,K}, t \geq 0\}$ be an extended bifractional Brownian motion with parameters $1 < K < 2$, $HK = 1/6$. Then Y satisfies conditions (T.1) - (T.6), with $\theta = 2/3$, $\lambda = (2H) \wedge \frac{1}{3}$, and with ν, γ and $\eta(t)$ as given in Proposition 5.14.*

Proof. Conditions (T.2) and (T.5) are the same as for standard bBm, as shown in Proposition 5.14. In particular, the decomposition into $\phi(t,r,s)$ and $\psi(t-r,s)$ for condition (T.5) is the same, so it follows that $\eta(t)$ of condition (T.6) has the same form. The proofs for conditions (T.1), (T.3) and (T.4) require some modifications to accept the case $K > 1$.

Condition (T.1). From Prop. 3 of [2] we have

$$\left|\mathbb{E}\left[(Y_t - Y_{t-s})^2\right]\right| \leq s^{2HK} = s^{\frac{1}{3}}.$$

Condition (T.3). First, we have

$$\begin{aligned}
&\mathbb{E}\left[(Y_t - Y_{t-s})^2 - (Y_{t-s} - Y_{t-2s})^2\right] \\
&\quad = t^{2HK} - \frac{2}{2^K}\left[t^{2H} + (t-s)^{2H}\right]^K + \frac{2}{2^K}\left[(t-s)^{2H} + (t-2s)^{2H}\right]^K - \frac{2}{2^K}(t-2s)^{2HK} \\
&\quad \le 2t^{2HK} - \frac{2}{2^K}\left[t^{2H} + (t-s)^{2H}\right]^K - 2(t-2s)^{2HK} + \frac{2}{2^K}\left[(t-s)^{2H} + (t-2s)^{2H}\right]^K \\
&\quad = 4HK\int_{-s}^{0} (t+\xi)^{2HK-1} - (t-s+\xi)^{2HK-1}\, d\xi \\
&\quad \le Cs^2(t-2s)^{2HK-2}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&t^{2HK} - \frac{2}{2^K}\left[t^{2H} + (t-s)^{2H}\right]^K + \frac{2}{2^K}\left[(t-s)^{2H} + (t-2s)^{2H}\right]^K - \frac{2}{2^K}(t-2s)^{2HK} \\
&\quad \ge 2(t-s)^{2HK} - \frac{2}{2^K}\left[t^{2H} + t^{2H}\right]^K + \frac{2}{2^K}\left[(t-2s)^{2H} + (t-2s)^{2H}\right]^K - 2(t-s)^{2HK} \\
&\quad = -4HK\int_0^s (t-s+\eta)^{2HK-1} - (t-2s+\eta)^{2HK-1}\, d\eta \\
&\quad \ge -Cs^2(t-2s)^{2HK-2},
\end{aligned}$$

hence the term is bounded in absolute value as required, with $\nu = 2 - 2HK = 5/3$.

Condition (T.4).

$$\begin{aligned}
|\mathbb{E}\left[Y_r(Y_t - Y_{t-s})\right]| &= \frac{1}{2^K}\left(\left[r^{2H} + t^{2H}\right]^K - \left[r^{2H} + (t-s)^{2H}\right]^K\right) + \frac{1}{2^K}\left(|r-t+s|^{2HK} - |r-t|^{2HK}\right) \\
&\le \frac{1}{2^K}\left(\left[r^{2H} + t^{2H}\right]^K - \left[r^{2H} + (t-s)^{2H}\right]^K\right) + \frac{1}{2^K}\left((|r-t| + s)^{2HK} - |r-t|^{2HK}\right).
\end{aligned}$$

We consider two cases for the first term. If $t < 2s$, then by Fundamental Theorem of Calculus,

$$\begin{aligned}
\frac{1}{2^K}\left(\left[r^{2H} + t^{2H}\right]^K - \left[r^{2H} + (t-s)^{2H}\right]^K\right) &\le \frac{1}{2^K}\left(\left[r^{2H} + (2s)^{2H}\right]^K - r^{2HK}\right) \\
&= \frac{K}{2^K}\int_0^{(2s)^{2H}} \left[r^{2H} + u\right]^{K-1} du \le Cs^{2H}.
\end{aligned}$$

If $t \ge 2s$, then

$$\begin{aligned}
&\frac{1}{2^K}\left(\left[r^{2H} + t^{2H}\right]^K - \left[r^{2H} + (t-s)^{2H}\right]^K\right) = 2HK\int_{-s}^{0} \left[r^{2H} + (t+u)^{2H}\right]^{K-1} (t+u)^{2H-1} du \\
&\le CsT^{2H(K-1)}(t-s)^{2H-1} \le Cs(t-s)^{2H-1}.
\end{aligned}$$

In particular, if $|r-t| < 2s$ then this is bounded by

$$Cs^{2H}\left(\frac{s}{t-s}\right) \le Cs^{2H}.$$

For the second term, if $|r-t| < 2s$, then it easily follows that

$$\frac{1}{2^K}\left((|r-t|+s)^{2HK} - |r-t|^{2HK}\right) \leq Cs^{2HK} \leq Cs^{2H};$$

and if $|r-t| \geq 2s$, then by Mean Value

$$\frac{1}{2^K}\left((|r-t|+s)^{2HK} - |r-t|^{2HK}\right) \leq Cs|r-t|^{2HK-1} \leq CsT^{2H(K-1)}|r-t|^{2H-1} \leq Cs|r-t|^{2H-1}.$$

In particular, if $t < 2s$ then

$$Cs|r-t|^{2H-1} = Cs^{2H}\left(\frac{s}{|r-t|}\right) \leq Cs^{2H}.$$

Hence, we have shown that

$$|\mathbb{E}\left[Y_r(Y_t - Y_{t-s})\right]| \leq \begin{cases} Cs\left[(t-s)^{2H-1} + |r-t|^{2H-1}\right] & \text{if } T \geq 2s \text{ and } |r-t| \geq 2s \\ Cs^{2H} & \text{otherwise} \end{cases}$$

and so condition (T.4) is satisfied by taking $\lambda = \min\{2H, \frac{1}{3}\}$, where $K \in (1,2)$ implies $\lambda > 1/6$. □

5.3.3 Sub-fractional Brownian motion

Another variant on fBm is the process known as sub-fractional Brownian motion (sfBm). This is a centered Gaussian process $\{Z_t, t \geq 0\}$, with covariance defined by:

$$R_h(s,t) = s^h + t^h - \frac{1}{2}\left[(s+t)^h + |s-t|^h\right], \tag{5.27}$$

with real parameter $h \in (0,2)$. Some properties of sfBm are given in [7] and [9]. Note that $h = 1$ is a standard Brownian motion, and also note the similarity of $R_h(t,s)$ to the covariance of fBm with $H = h/2$. Indeed, in [9] it is shown that sfBm may be decomposed into an fBm with $H = h/2$ and another centered Gaussian process.

Similar to Section 5.3.1, we discuss only the case $h = 1/3$. For $h > 1/3$, it can be shown that conditions (T.1)-(T.6) are satisfied with $\eta(t) = 0$, hence $S_n^*(t)$ converges in probability.

Proposition 5.17. *Let $Z = \{Z_t, t \geq 0\}$ be a sub-fractional Brownian motion with covariance* (5.27) *and parameter $h = 1/3$. Then Z satisfies conditions (T.1) - (T.6) of Section 3; hence Theorem 3.1 holds. For condition (T.6) we have $\eta(t) = C_h t$, where*

$$C_h = 1 + \frac{1}{4}\sum_{m=1}^{\infty}\left((m+1)^{\frac{1}{3}} - 2m^{\frac{1}{3}} + (m-1)^{\frac{1}{3}}\right)^3.$$

Proof. Condition (T.1). We have

$$\begin{aligned}\mathbb{E}\left[(Z_t - Z_{t-s})^2\right] &= R_h(t,t) + R_h(t-s,t-s) - 2R_h(t,t-s)\\ &= -\frac{2^h}{2}t^h + \frac{1}{2}(2t-s)^h - \frac{2^h}{2}(t-s)^h + \frac{1}{2}(2t-s)^h + s^h\\ &= -\frac{1}{2}\left[(2t)^h - (2t-s)^h\right] - \frac{1}{2}\left[(2t-2s)^h - (2t-s)^h\right] + s^h.\end{aligned}$$

This is bounded in absolute value by Cs^h, using the inequality $a^h - b^h \leq (a-b)^h$.

Condition (T.2).

$$\begin{aligned}\left|\mathbb{E}\left[Z_t^2 - Z_{t-s}^2\right]\right| &= \left|2t^h - \frac{2^h}{2}t^h - 2(t-s)^h + \frac{2^h}{2}(t-s)^h\right|\\ &= \frac{|4-2^h|}{2}\left[t^h - (t-s)^h\right].\end{aligned}$$

By Mean Value this is bounded by

$$Cs(t-s)^{h-1} = Cs(t-s)^{-\frac{2}{3}},$$

which implies (T.2) with $\theta = 2/3$.

Condition (T.3).

$$\begin{aligned}\mathbb{E}\left[(Z_t - Z_{t-s})^2 - (Z_{t-s} - Z_{t-2s})^2\right] &= R_h(t,t) - 2R_h(t,t-s) + 2R_h(t-s,t-2s) - R_h(t-2s,t-2s)\\ &= -\frac{(2t)^h}{2} + (2t-s)^h - (2t-3s)^h + \frac{1}{2}(2t-4s)^h\\ &= -\frac{1}{2}\left[(2t)^h - 2(2t-s)^h + (2t-2s)^h\right]\\ &\quad + \frac{1}{2}\left[(2t-2s)^h - 2(2t-3s)^h + (2t-4s)^h\right].\end{aligned}$$

By Mean Value, these terms are bounded in absolute value by

$$Cs^2(2t-4s)^{h-2} \leq Cs^{\frac{1}{3}+\nu}(t-s)^{-\nu}$$

for $\nu = 5/3$.

Condition (T.4).

$$\begin{aligned}\left|\mathbb{E}\left[Z_r(Z_t - Z_{t-s})\right]\right| &= \left|R_h(r,t) - R_h(r,t-s)\right|\\ &= \left|t^h - (t-s)^h - \frac{1}{2}\left[(r+t)^h - (r+t-s)^h\right] + \frac{1}{2}\left(|r-t+s|^h - |r-t|^h\right)\right|\end{aligned}$$

Note that the above expression is always bounded by Cs^h by the inequality $a^h - b^h \leq (a-b)^h$. Hence, the bound is satisfied for the cases $t < 2s$ or $|t-r| < 2s$. Assuming $t \geq 2s$, $|r-t| \geq 2s$, we have

$$\left|t^h-(t-s)^h-\frac{1}{2}\left[(r+t)^h-(r+t-s)^h\right]+\frac{1}{2}\left(|r-t+s|^h-|r-t|^h\right)\right|$$
$$\leq h\int_{-s}^{0}(t+u)^{h-1}du+\frac{h}{2}\int_{-s}^{0}(r+t+u)^{h-1}du+\frac{h}{2}\int_{0}^{s}(|r-t|+u)^{h-1}du$$
$$\leq Cs(t-s)^{-\frac{2}{3}}+Cs(|r-t|-s)^{-\frac{2}{3}}.$$

For $|r-t|\geq 2s$, we have $(|r-t|-s)\geq\frac{1}{2}|r-t|$, so

$$Cs(t-s)^{-\frac{2}{3}}+Cs(|r-t|-s)^{-\frac{2}{3}}\leq Cs(t-s)^{\lambda-1}+Cs|r-t|^{\lambda-1}$$

for $\lambda=1/3$.

Condition (T.5).

$$\begin{aligned}\mathbb{E}\left[(Z_t-Z_{t-s})(Z_r-Z_{r-s})\right]&=R_h(t,r)-R_h(t-s,r)-R_h(t,r-s)+R_h(t-s,r-s)\\&=-\frac{1}{2}\left[(t+r)^h-2(t+r-s)^h+(t+r-2s)^h\right]\\&\quad+\frac{1}{2}\left[|t-r+s|^h-2|t-r|+|t-r-s|^h\right].\end{aligned}$$

Assuming that $|t-r|\geq 2s$, by Mean Value this is bounded in absolute value by

$$Cs^2|t-r-s|^{h-2}\leq Cs^2|t-r|^{h-2}$$

since $|t-r|\geq 2s$ implies $|t-r-s|\geq\frac{1}{2}|t-r|$. If $h<1$, then we take $\gamma=2-h=5/3$, and we have an upper bound of

$$Cs^{h+2-h}|t-r|^{h-2}=Cs^{h+\gamma}|t-r|^{-\gamma}.$$

Condition (T.6). First assume $h=1/3$. Referring to condition (T.5) above, we can decompose $\beta_n(j,k)$ as

$$\beta_n(j,k)=\frac{1}{2n^h}\omega(j,k,1)+\frac{1}{2n^h}\psi(j-k,1),$$

where $\omega(j,k,1)=-(j+k+2)^h+2(j+k+1)^h-(j+k)^h$ and $\psi(j-k,1)=|j-k+1|^h-2|j-k|^h+|j-k-1|^h$. Note that $\psi(j-k,1)$ is identical to the ψ used in Proposition 5.14, where in this case $h=2HK$. Following the proof of Proposition 5.15, it is enough to show

$$\lim_{n\to\infty}n^{-1}\sum_{j,k=0}^{\lfloor nt\rfloor-1}|\omega(j,k,1)|=0, \tag{5.28}$$

so that, similar to (5.26) in the proof of Proposition 5.15, we have

$$\eta(t)=\lim_{n\to\infty}\frac{1}{8n^{3h}}\sum_{j,k=0}^{\lfloor nt\rfloor-1}\psi(j-k,1)^3=C_ht,$$

where

$$C_h = 1 + \frac{1}{4}\sum_{m=1}^{\infty}\left((m+1)^{\frac{1}{3}} - 2m^{\frac{1}{3}} + (m-1)^{\frac{1}{3}}\right)^3.$$

That is, C_h corresponds to the constant C_K from Proposition 5.15 with $K = 1$.

Proof of (5.28). By Mean Value and the above computation for condition (T.5), $|\omega(j,k,1)| \leq C(j+k)^{-\gamma}$ for some $\gamma > 1$. Hence, for each $j \geq 2$,

$$\begin{aligned}\sum_{k=0}^{\lfloor nt\rfloor-1} |\omega(j,k,1)| &\leq C\sum_{k=0}^{\lfloor nt\rfloor-1}(j+k)^{-\gamma}\\ &\leq C\int_{j-1}^{\infty} u^{\gamma}du \leq C(j-1)^{1-\gamma}.\end{aligned}$$

It follows that we have

$$\begin{aligned}n^{-1}\sum_{j,k=0}^{\lfloor nt\rfloor-1}|\omega(j,k,1)| &= n^{-1}\sum_{k=0}^{\lfloor nt\rfloor-1}\left(|\omega(0,k,1)|+|\omega(1,k,1)|\right) + n^{-1}\sum_{j=2}^{\lfloor nt\rfloor-1}\sum_{k=0}^{\lfloor nt\rfloor-1}|\omega(j,k,1)|\\ &= n^{-1}\sum_{k=0}^{\lfloor nt\rfloor-1}\left(\left[(k+2)^h - 2(k+1)^h + k^h\right] + \left[(k+3)^h - 2(k+2)^h + (k+1)^h\right]\right)\\ &\quad + Cn^{-1}\sum_{j=2}^{\lfloor nt\rfloor-1}(j-1)^{1-\gamma}\\ &\leq Cn^{-1} + Cn^{-1}\lfloor nt\rfloor^{2-\gamma}\end{aligned}$$

which converges to 0 since $\gamma > 1$. □

5.4 Proof of Technical Lemmas

5.4.1 Proof of Lemma 5.3

We may assume $t_1 = 0$. For this proof we use Malliavin calculus to represent $\Delta X_{\frac{j}{n}}^5$ as a Skorohod integral. Consider the Hermite polynomial identity $x^5 = H_5(x) + 10H_3(x) + 15H_1(x)$. Using the isometry $H_p(X(h)) = \delta^p(h^{\otimes p})$ (when $\|h\|_{\mathfrak{H}} = 1$) we obtain for each $0 \leq j \leq \lfloor nt_2\rfloor - 1$,

$$\Delta X_{\frac{j}{n}}^5 = \delta^5(\partial_{\frac{j}{n}}^{\otimes 5}) + 10\|\Delta X_{\frac{j}{n}}\|_{L^2}^2\delta^3(\partial_{\frac{j}{n}}^{\otimes 3}) + 15\|\Delta X_{\frac{j}{n}}\|_{L^2}^4\delta(\partial_{\frac{j}{n}}). \tag{5.29}$$

With this representation, we can expand

$$\sum_{j,k=0}^{\lfloor nt_2\rfloor-1}\mathbb{E}\left[f^{(5)}(\widetilde{X}_{\frac{j}{n}})f^{(5)}(\widetilde{X}_{\frac{k}{n}})\Delta X_{\frac{j}{n}}^5\Delta X_{\frac{k}{n}}^5\right]$$

into 9 sums of the form

$$C\sum_{j,k=0}^{\lfloor nt_2\rfloor-1}\|\Delta X_{\frac{j}{n}}\|_{L^2}^{5-p}\|\Delta X_{\frac{k}{n}}\|_{L^2}^{5-q}\mathbb{E}\left[f^{(5)}(\widetilde{X}_{\frac{j}{n}})f^{(5)}(\widetilde{X}_{\frac{k}{n}})\delta^p(\partial_{\frac{j}{n}}^{\otimes p})\delta^q(\partial_{\frac{k}{n}}^{\otimes q})\right] \tag{5.30}$$

where p,q take values 1, 3, or 5. By Lemma 2.1.d and (2.6), each term of the form (5.30) can be further expanded into terms of the form

$$C\sum_{j,k=0}^{\lfloor nt_2\rfloor-1}\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{r}\|\Delta X_{\frac{j}{n}}\|_{L^2}^{5-p}\|\Delta X_{\frac{k}{n}}\|_{L^2}^{5-q}\mathbb{E}\left[f^{(5)}(\widetilde{X}_{\frac{j}{n}})f^{(5)}(\widetilde{X}_{\frac{k}{n}})\delta^{p+q-2r}(\partial_{\frac{j}{n}}^{\otimes p-r}\otimes\partial_{\frac{k}{n}}^{\otimes q-r})\right]$$

$$=C\sum_{j,k=0}^{\lfloor nt_2\rfloor-1}\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{r}\|\Delta X_{\frac{j}{n}}\|_{L^2}^{5-p}\|\Delta X_{\frac{k}{n}}\|_{L^2}^{5-q}$$
$$\times\mathbb{E}\left[\left\langle D^{p+q-2r}\left(f^{(5)}(\widetilde{X}_{\frac{j}{n}})f^{(5)}(\widetilde{X}_{\frac{k}{n}})\right),\partial_{\frac{j}{n}}^{\otimes p-r}\otimes\partial_{\frac{k}{n}}^{\otimes q-r}\right\rangle_{\mathfrak{H}^{\otimes p+q-2r}}\right]$$

where $0\le r\le p\wedge q$ and $p,q\in\{1,3,5\}$. For $0\le m=p+q-2r\le 10$, we have

$$D^m\left[f^{(5)}(\widetilde{X}_{\frac{j}{n}})f^{(5)}(\widetilde{X}_{\frac{k}{n}})\right]=\sum_{a+b=m}D^a\left(f^{(5)}(\widetilde{X}_{\frac{j}{n}})\right)D^b\left(f^{(5)}(\widetilde{X}_{\frac{k}{n}})\right)$$
$$=\sum_{a+b=m}f^{(5+a)}(\widetilde{X}_{\frac{j}{n}})f^{(5+b)}(\widetilde{X}_{\frac{j}{n}})\widetilde{\varepsilon}_{\frac{j}{n}}^{\otimes a}\otimes\widetilde{\varepsilon}_{\frac{k}{n}}^{\otimes b}.$$

Hence, we expand (5.30) again into terms of the form:

$$C\sum_{j,k=0}^{\lfloor nt_2\rfloor-1}\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{r}\|\Delta X_{\frac{j}{n}}\|_{L^2}^{5-p}\|\Delta X_{\frac{k}{n}}\|_{L^2}^{5-q}\mathbb{E}\left[f^{(5+a)}(\widetilde{X}_{\frac{j}{n}})f^{(5+b)}(\widetilde{X}_{\frac{k}{n}})\right]$$
$$\times\left\langle\widetilde{\varepsilon}_{\frac{j}{n}}^{\otimes a}\otimes\widetilde{\varepsilon}_{\frac{k}{n}}^{\otimes b},\partial_{\frac{j}{n}}^{\otimes p-r}\otimes\partial_{\frac{k}{n}}^{\otimes q-r}\right\rangle_{\mathfrak{H}^{\otimes a+b}},$$

where $a+b=p+q-2r$. With this representation, we are now ready to develop estimates for each term. By condition (T.0),

$$\left|\mathbb{E}\left[f^{(5+a)}(\widetilde{X}_{\frac{j}{n}})f^{(5+b)}(\widetilde{X}_{\frac{k}{n}})\right]\right|\le\left(\sup_{0\le j<\lfloor nt_2\rfloor}\mathbb{E}\left[f^{(5+a)}(\widetilde{X}_{\frac{j}{n}})\right]\right)^{\frac{1}{2}}\left(\sup_{0\le k<\lfloor nt_2\rfloor}\mathbb{E}\left[f^{(5+b)}(\widetilde{X}_{\frac{k}{n}})\right]\right)^{\frac{1}{2}}\le C;$$

and by condition (T.1),

$$\sup_{0\le j<\lfloor nt_2\rfloor}\|\Delta X_{\frac{j}{n}}\|_{L^2}^{5-p}\sup_{0\le k<\lfloor nt_2\rfloor}\|\Delta X_{\frac{k}{n}}\|_{L^2}^{5-q}\le Cn^{-\frac{10-(p+q)}{6}}.$$

If $a\ge 1$ with $a+b=p+q-2r$, by condition (T.4)

$$\left|\left\langle\widetilde{\varepsilon}_{\frac{j}{n}}^{\otimes a}\otimes\widetilde{\varepsilon}_{\frac{k}{n}}^{\otimes b},\partial_{\frac{j}{n}}^{\otimes p-r}\otimes\partial_{\frac{k}{n}}^{\otimes q-r}\right\rangle_{\mathfrak{H}^{\otimes a+b}}\right|\le Cn^{-(a+b-1)\lambda}\left|\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}\right|,$$

with a similar term in k if $a=0$ and $b\ge 1$. Hence, assuming $a\ge 1$, each term in the expansion of (5.30) has an upper bound of

$$C\sum_{j,k=0}^{\lfloor nt_2\rfloor-1}\left|\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|^{r}n^{-\frac{10-(p+q)}{6}-(a+b-1)\lambda}\left|\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}\right|.$$

To show each term has the desired upper bound, first assume $r \geq 1$. Then $\left|\left\langle \widetilde{\varepsilon}_{\frac{j}{n}}, \partial_{\frac{j}{n}} \right\rangle_{\mathfrak{H}}\right| \leq Cn^{-\lambda}$, and by Lemma 5.2.d we have an upper bound of

$$Cn^{-\frac{10-(p+q)}{6}-(a+b)\lambda} \sum_{j,k=0}^{\lfloor nt_2 \rfloor -1} \left|\left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}}\right|^r \leq C\lfloor nt_2 \rfloor n^{-\frac{5}{3}+\frac{p+q-2r}{6}-(a+b)\lambda} = C\lfloor nt_2 \rfloor n^{-\frac{5}{3}-(a+b)(\lambda-\frac{1}{6})}$$

which is less than or equal to $C\lfloor nt_2 \rfloor n^{-\frac{5}{3}}$ because $\lambda > 1/6$. For cases with $r = 0$, then either $a \geq 1$ or $b \geq 1$, so without loss of generality assume $a \geq 1$. For this case with Lemma 5.2.c we have an upper bound of

$$C \sum_{j,k=0}^{\lfloor nt_2 \rfloor -1} n^{-\frac{10-(p+q)}{6}-(a+b-1)\lambda} \left|\left\langle \widetilde{\varepsilon}_{\frac{j}{n}}, \partial_{\frac{j}{n}} \right\rangle_{\mathfrak{H}}\right| \leq C\lfloor nt_2 \rfloor n^{-\frac{5}{3}-(a+b)(\lambda-\frac{1}{6})+\lambda},$$

which is less than $C\lfloor nt_2 \rfloor n^{-\frac{4}{3}}$ since $\lambda \leq 1/3$.

5.4.2 Proof of Lemma 5.6

Without loss of generality, assume $a = 0$. First we want to show that for each integer $0 \leq k \leq b-1$,

$$\mathbb{E}\left|\sum_{j=0}^{k} f^{(3)}(\widetilde{X}_{\frac{j}{n}})\Delta X_{\frac{j}{n}}\right| \leq C. \tag{5.31}$$

Using the Taylor expansion similar to Section 5.2,

$$\begin{aligned} f''(X_{\frac{j+1}{n}}) - f''(X_{\frac{j}{n}}) &= \left(f''(X_{\frac{j+1}{n}}) - f''(\widetilde{X}_{\frac{j}{n}})\right) - \left(f''(X_{\frac{j}{n}}) - f''(\widetilde{X}_{\frac{j}{n}})\right) \\ &= f^{(3)}(\widetilde{X}_{\frac{j}{n}})\Delta X_{\frac{j}{n}} + \frac{1}{24} f^{(5)}(\widetilde{X}_{\frac{j}{n}})\Delta X_{\frac{j}{n}}^3 + \frac{1}{2^5 5!} f^{(7)}(\widetilde{X}_{\frac{j}{n}})\Delta X_{\frac{j}{n}}^5 + B_n^+(j) - B_n^-(j) \end{aligned}$$

where $B_n^+(j), B_n^-(j)$ have the form $Cf^{(9)}(\xi_j)\Delta X_{\frac{j}{n}}^7$. Hence we can write,

$$\begin{aligned} \mathbb{E}\left|\sum_{j=0}^{k} f^{(3)}(\widetilde{X}_{\frac{j}{n}})\Delta X_{\frac{j}{n}}\right| &\leq \mathbb{E}\left|\sum_{j=0}^{k} \left(f''(X_{\frac{j+1}{n}}) - f''(X_{\frac{j}{n}})\right)\right| + \frac{1}{24}\mathbb{E}\left|\sum_{j=0}^{k} f^{(5)}(\widetilde{X}_{\frac{j}{n}})\Delta X_{\frac{j}{n}}^3\right| \\ &\quad + \frac{1}{2^5 5!}\mathbb{E}\left|\sum_{j=0}^{k} f^{(7)}(\widetilde{X}_{\frac{j}{n}})\Delta X_{\frac{j}{n}}^5\right| + \mathbb{E}\sum_{j=0}^{k} \left|B_n^+(j)\right| + \left|B_n^-(j)\right|. \end{aligned}$$

We have the following estimates: By condition (T.0),

$$\mathbb{E}\left|\sum_{j=0}^{k} \left(f''(X_{\frac{j+1}{n}}) - f''(X_{\frac{j}{n}})\right)\right| \leq \mathbb{E}\left|f''(X_{\frac{k+1}{n}}) - f''(X_0)\right| \leq C;$$

by Lemma 5.3,

$$\mathbb{E}\left|\sum_{j=0}^{k} f^{(7)}(\widetilde{X}_{\frac{j}{n}})\Delta X_{\frac{j}{n}}^5\right| \leq Cn^{-\frac{2}{3}}(k+1)^{\frac{1}{2}};$$

and by Lemma 5.4,

$$\sum_{j=0}^{k} \mathbb{E}\left|B_n^+(j)\right| + \mathbb{E}\left|B_n^-(j)\right| \le Cn^{-\frac{7}{3}}(k+1)^2.$$

This leaves the ΔX^3 term. Using the Hermite polynomial identity $x^3 = H_3(x) + 3H_1(x)$, we can write

$$\mathbb{E}\left|\sum_{j=0}^{k} f^{(5)}(\widetilde{X}_{\frac{j}{n}})\Delta X_{\frac{j}{n}}^3\right| \le \mathbb{E}\left|\sum_{j=0}^{k} f^{(5)}(\widetilde{X}_{\frac{j}{n}})\delta^3(\partial_{\frac{j}{n}}^{\otimes 3})\right| + \mathbb{E}\left|\sum_{j=0}^{k} \left\|\Delta X_{\frac{j}{n}}\right\|_{L^2}^2 f^{(5)}(\widetilde{X}_{\frac{j}{n}})\delta(\partial_{\frac{j}{n}})\right|.$$

For the first term we have

$$\begin{aligned}
&\mathbb{E}\left|\sum_{j=0}^{k} f^{(5)}(\widetilde{X}_{\frac{j}{n}})\delta^3(\partial_{\frac{j}{n}}^{\otimes 3})\right|^2 \\
&\quad= \sum_{j,\ell=0}^{k} \mathbb{E}\left[f^{(5)}(\widetilde{X}_{\frac{j}{n}})f^{(5)}(\widetilde{X}_{\frac{\ell}{n}})\delta^3\left(\partial_{\frac{j}{n}}^{\otimes 3}\right)\delta^3\left(\partial_{\frac{\ell}{n}}^{\otimes 3}\right)\right] \\
&\quad= \sum_{j,\ell=0}^{k}\sum_{r=0}^{3} r!\binom{r}{3}^2 \mathbb{E}\left[f^{(5)}(\widetilde{X}_{\frac{j}{n}})f^{(5)}(\widetilde{X}_{\frac{\ell}{n}})\delta^{6-2r}\left(\partial_{\frac{j}{n}}^{\otimes 3-r}\otimes_r \partial_{\frac{\ell}{n}}^{\otimes 3-r}\right)\right] \\
&\quad= \sum_{j,\ell=0}^{k}\sum_{r=0}^{3} r!\binom{r}{3}^2 \mathbb{E}\left[\left\langle D^{6-2r}\left[f^{(5)}(\widetilde{X}_{\frac{j}{n}})f^{(5)}(\widetilde{X}_{\frac{\ell}{n}})\right], \partial_{\frac{j}{n}}^{\otimes 3-r}\otimes_r \partial_{\frac{\ell}{n}}^{\otimes 3-r}\right\rangle_{\mathfrak{H}^{\otimes 6-2r}}\right]\left\langle \partial_{\frac{j}{n}}, \partial_{\frac{\ell}{n}}\right\rangle_{\mathfrak{H}}^r \\
&\quad\le \sum_{j,\ell=0}^{k}\sum_{r=0}^{3}\sum_{\substack{a+b=\\6-2r}} \left|\left\langle \widetilde{\varepsilon}_{\frac{j}{n}}^{\otimes a}\otimes\widetilde{\varepsilon}_{\frac{\ell}{n}}^{\otimes b}, \partial_{\frac{j}{n}}^{\otimes 3-r}\otimes \partial_{\frac{\ell}{n}}^{\otimes 3-r}\right\rangle_{\mathfrak{H}^{\otimes 6-2r}}\right| \left|\left\langle \partial_{\frac{j}{n}}, \partial_{\frac{\ell}{n}}\right\rangle_{\mathfrak{H}}\right|^r.
\end{aligned}$$

For this sum, if $r = 0$ we use Lemma 5.2.a and 5.2.b for each pair (a,b) to obtain terms of the form

$$\begin{aligned}
\sum_{j,\ell=0}^{k}\left|\left\langle \widetilde{\varepsilon}_{\frac{j}{n}}^{\otimes a}\otimes\widetilde{\varepsilon}_{\frac{\ell}{n}}^{\otimes b}, \partial_{\frac{j}{n}}^{\otimes 3}\otimes \partial_{\frac{\ell}{n}}^{\otimes 3}\right\rangle_{\mathfrak{H}^{\otimes 6}}\right| &\le \sup_{j,\ell}\left|\left\langle \widetilde{\varepsilon}_{\frac{j}{n}}, \partial_{\frac{\ell}{n}}\right\rangle_{\mathfrak{H}}\right|^3 \sup_{j}\left|\left\langle \widetilde{\varepsilon}_{\frac{j}{n}}, \partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}\right| \sum_{j,\ell=0}^{k}\left\langle \widetilde{\varepsilon}_{\frac{j}{n}}, \partial_{\frac{\ell}{n}}\right\rangle_{\mathfrak{H}}^2 \\
&\le Cn^{-1-3\lambda}(k+1),
\end{aligned}$$

where we use the fact that $r = 0$ implies $a \ge 3$ or $b \ge 3$. If $r \ge 1$, we use Lemma 5.2.a and 5.2.d to obtain terms of the form

$$Cn^{-(6-2r)\lambda}\sum_{j,\ell=0}^{k}\left|\left\langle \partial_{\frac{j}{n}}, \partial_{\frac{\ell}{n}}\right\rangle_{\mathfrak{H}}\right|^r \le Cn^{-(6-2r)\lambda-\frac{r}{3}}(k+1),$$

noting that $(6-2r)\lambda + \frac{r}{3} > 1$.

For the other term, we have by Lemma 5.2.b,

$$\begin{aligned}
&\mathbb{E}\left(\sum_{j=0}^{k}\left\|\Delta X_{\frac{j}{n}}\right\|_{L^2}^2\ f^{(5)}(\widetilde{X}_{\frac{j}{n}})\delta(\partial_{\frac{j}{n}})\right)^2\\
&\le \sup_{0\le j\le k}\left\|\Delta X_{\frac{j}{n}}\right\|_{L^2}^4\sum_{j,\ell=0}^{k}\left|\mathbb{E}\left[f^{(5)}(\widetilde{X}_{\frac{j}{n}})f^{(5)}(\widetilde{X}_{\frac{\ell}{n}})\left(\delta^2\left(\partial_{\frac{j}{n}}\otimes\partial_{\frac{\ell}{n}}\right)+\left\langle\partial_{\frac{j}{n}},\partial_{\frac{\ell}{n}}\right\rangle_{\mathfrak{H}}\right)\right]\right|\\
&\le Cn^{-\frac{2}{3}}\sum_{j,\ell=0}^{k}\left|\mathbb{E}\left[\left\langle D^2\left[f^{(5)}(\widetilde{X}_{\frac{j}{n}})f^{(5)}(\widetilde{X}_{\frac{\ell}{n}})\right],\partial_{\frac{j}{n}}\otimes\partial_{\frac{\ell}{n}}\right\rangle_{\mathfrak{H}^{\otimes 2}}\right]\right|+\left|\mathbb{E}\left[f^{(5)}(\widetilde{X}_{\frac{j}{n}})f^{(5)}(\widetilde{X}_{\frac{\ell}{n}})\right]\left\langle\partial_{\frac{j}{n}},\partial_{\frac{\ell}{n}}\right\rangle_{\mathfrak{H}}\right|\\
&\le Cn^{-\frac{2}{3}}\sum_{j,\ell=0}^{k}\sum_{a+b=2}\left|\left\langle\widetilde{\varepsilon}_{\frac{j}{n}}^{\otimes a}\otimes\widetilde{\varepsilon}_{\frac{\ell}{n}}^{\otimes b},\partial_{\frac{j}{n}}\otimes\partial_{\frac{\ell}{n}}\right\rangle_{\mathfrak{H}^{\otimes 2}}\right|+Cn^{-\frac{2}{3}}\sum_{j,\ell=0}^{k}\left|\left\langle\partial_{\frac{j}{n}},\partial_{\frac{\ell}{n}}\right\rangle_{\mathfrak{H}}\right|\\
&\le Cn^{-\frac{2}{3}}\left(\sum_{j=0}^{k}\left|\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}\right|\right)\left(\sum_{\ell=0}^{k}\left|\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{\ell}{n}}\right\rangle_{\mathfrak{H}}\right|\right)+\left(\sum_{j=0}^{k}\left|\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}\right|\right)^2+Cn^{-1}(k+1)\\
&\le Cn^{-1}(k+1)^{1-\theta}+Cn^{-\frac{4}{3}}(k+1)^{2-2\theta}+Cn^{-1}(k+1)\le C,
\end{aligned}$$

where the estimates follow from Lemma 5.2.c and 5.2.d. Hence, by Cauchy-Schwarz

$$\mathbb{E}\left|\sum_{j=0}^{k}\left\|\Delta X_{\frac{j}{n}}\right\|_{L^2}^2\ f^{(5)}(\widetilde{X}_{\frac{j}{n}})\delta(\partial_{\frac{j}{n}})\right|\le C,$$

which proves (5.31). Now we define

$$G_n(j)=\sum_{k=0}^{j}f^{(3)}(\widetilde{X}_{\frac{k}{n}})\Delta X_{\frac{k}{n}},$$

and by Abel's formula and condition (T.3) we have

$$\begin{aligned}
\mathbb{E}\left|\sum_{j=0}^{b-1}\left\|\Delta X_{\frac{j}{n}}\right\|_{L^2}^2 f^{(3)}(\widetilde{X}_{\frac{j}{n}})\Delta X_{\frac{j}{n}}\right| &\le \left\|\Delta X_{\frac{b}{n}}\right\|_{L^2}^2\mathbb{E}\left|G_n(b-1)\right|+\sum_{j=0}^{b-1}\mathbb{E}\left|G_n(j)\right|\left(\left\|\Delta X_{\frac{j+1}{n}}\right\|_{L^2}^2-\left\|\Delta X_{\frac{j}{n}}\right\|_{L^2}^2\right)\\
&\le Cn^{-\frac{1}{3}}+Cn^{-\frac{1}{3}}\sum_{j=4}^{b-1}(j-1)^{-\nu}\\
&\le Cn^{-\frac{1}{3}}.
\end{aligned}$$

5.4.3 Proof of Lemma 5.10

Proof of (5.18). Let $a_j=\lfloor nt_{j-1}\rfloor$ and $b_j=\lfloor nt_j\rfloor$. By Lemma 2.1.b,

$$\begin{aligned}
D^2F_n^j &= \sum_{k=a_j}^{b_j-1}D^2\delta^3\left(f^{(3)}(\widetilde{X}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 3}\right)\\
&= \sum_{k=a_j}^{b_j-1}\left\{\delta^3\left(f^{(5)}(\widetilde{X}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 3}\right)\widetilde{\varepsilon}_{\frac{k}{n}}^{\otimes 2}+6\ \delta^2\left(f^{(4)}(\widetilde{X}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 2}\right)\partial_{\frac{k}{n}}\otimes\widetilde{\varepsilon}_{\frac{k}{n}}+6\ \delta\left(f^{(3)}(\widetilde{X}_{\frac{k}{n}})\partial_{\frac{k}{n}}\right)\partial_{\frac{k}{n}}^{\otimes 2}\right\}
\end{aligned}$$

and

$$DF_n^k = \sum_{m=a_k}^{b_k-1} \left\{ \delta^3 \left(f^{(4)}(\widetilde{X}_{\frac{m}{n}}) \partial_{\frac{m}{n}}^{\otimes 3} \right) \widetilde{\varepsilon}_{\frac{m}{n}} + 3\, \delta^2 \left(f^{(3)}(\widetilde{X}_{\frac{m}{n}}) \partial_{\frac{m}{n}}^{\otimes 2} \right) \partial_{\frac{m}{n}} \right\}. \tag{5.32}$$

With these two expansions, it follows that the expectation

$$\mathbb{E}\left[\left\langle u_n^i, D^2 F_n^j \otimes DF_n^k \right\rangle_{\mathfrak{H}^{\otimes 3}}^2 \right]$$

consists of terms of the form

$$\sum_{p,p'=a_i}^{b_i-1} \sum_{q,q'=a_j}^{b_j-1} \sum_{m,m'=a_k}^{b_k-1} \mathbb{E}\left[G(p,p') \prod_{\ell=1}^{4} \delta^{r_\ell} \left(g_\ell(\widetilde{X}_{\frac{j_\ell}{n}}) \partial_{\frac{j_\ell}{n}}^{\otimes r_\ell} \right) \right] \left\langle \widetilde{\varepsilon}_{\frac{q}{n}}, \partial_{\frac{p}{n}} \right\rangle_{\mathfrak{H}}^{r_1-1} \left\langle \partial_{\frac{q}{n}}, \partial_{\frac{p}{n}} \right\rangle_{\mathfrak{H}}^{3-r_1} \left\langle \widetilde{\varepsilon}_{\frac{q'}{n}}, \partial_{\frac{p'}{n}} \right\rangle_{\mathfrak{H}}^{r_2-1}$$

$$\times \left\langle \partial_{\frac{q'}{n}}, \partial_{\frac{p'}{n}} \right\rangle_{\mathfrak{H}}^{3-r_2} \left\langle \widetilde{\varepsilon}_{\frac{m}{n}}, \partial_{\frac{p}{n}} \right\rangle_{\mathfrak{H}}^{r_3-2} \left\langle \partial_{\frac{m}{n}}, \partial_{\frac{p}{n}} \right\rangle_{\mathfrak{H}}^{3-r_3} \left\langle \widetilde{\varepsilon}_{\frac{m'}{n}}, \partial_{\frac{p'}{n}} \right\rangle_{\mathfrak{H}}^{r_4-2} \left\langle \partial_{\frac{m'}{n}}, \partial_{\frac{p'}{n}} \right\rangle_{\mathfrak{H}}^{3-r_4} \tag{5.33}$$

where $G(p,p') := f^{(3)}(\widetilde{X}_{\frac{p}{n}}) f^{(3)}(\widetilde{X}_{\frac{p'}{n}})$, r_1, r_2 take values 1,2 or 3; r_3, r_4 take values 2 or 3; each g_i represents the appropriate derivative of f, and $(j_1, j_2, j_3, j_4) = (q, q', m, m')$. Without loss of generality, we will assume that $u_n^i, D^2 F_n^j$, and DF_n^k are all defined over the interval $[0,t]$, and that all sums are over the set $\{0, \ldots, \lfloor nt \rfloor - 1\}$. Let $R = r_1 + r_2 + r_3 + r_4$, and note that $6 \le R \le 12$. It follows from Lemma 5.2.a, 5.2.c, and/or 5.2.d that

$$\begin{aligned}
&\sum_{p,p'=0}^{\lfloor nt \rfloor -1} \Big| \left\langle \widetilde{\varepsilon}_{\frac{q}{n}}, \partial_{\frac{p}{n}} \right\rangle_{\mathfrak{H}}^{r_1-1} \left\langle \partial_{\frac{q}{n}}, \partial_{\frac{p}{n}} \right\rangle_{\mathfrak{H}}^{3-r_1} \left\langle \widetilde{\varepsilon}_{\frac{q'}{n}}, \partial_{\frac{p'}{n}} \right\rangle_{\mathfrak{H}}^{r_2-1} \left\langle \partial_{\frac{q'}{n}}, \partial_{\frac{p'}{n}} \right\rangle_{\mathfrak{H}}^{3-r_2} \left\langle \widetilde{\varepsilon}_{\frac{m}{n}}, \partial_{\frac{p}{n}} \right\rangle_{\mathfrak{H}}^{r_3-2} \\
&\quad \times \left\langle \partial_{\frac{m}{n}}, \partial_{\frac{p}{n}} \right\rangle_{\mathfrak{H}}^{3-r_3} \left\langle \widetilde{\varepsilon}_{\frac{m'}{n}}, \partial_{\frac{p'}{n}} \right\rangle_{\mathfrak{H}}^{r_4-2} \left\langle \partial_{\frac{m'}{n}}, \partial_{\frac{p'}{n}} \right\rangle_{\mathfrak{H}}^{3-r_4} \Big| \\
&\le \sum_{p=0}^{\lfloor nt \rfloor -1} \left| \left\langle \widetilde{\varepsilon}_{\frac{q}{n}}, \partial_{\frac{p}{n}} \right\rangle_{\mathfrak{H}}^{r_1-1} \left\langle \partial_{\frac{q}{n}}, \partial_{\frac{p}{n}} \right\rangle_{\mathfrak{H}}^{3-r_1} \left\langle \widetilde{\varepsilon}_{\frac{m}{n}}, \partial_{\frac{p}{n}} \right\rangle_{\mathfrak{H}}^{r_3-2} \left\langle \partial_{\frac{m}{n}}, \partial_{\frac{p}{n}} \right\rangle_{\mathfrak{H}}^{3-r_3} \right| \\
&\quad \times \sum_{p'=0}^{\lfloor nt \rfloor -1} \left| \left\langle \widetilde{\varepsilon}_{\frac{q'}{n}}, \partial_{\frac{p'}{n}} \right\rangle_{\mathfrak{H}}^{r_2-1} \left\langle \partial_{\frac{q'}{n}}, \partial_{\frac{p'}{n}} \right\rangle_{\mathfrak{H}}^{3-r_2} \left\langle \widetilde{\varepsilon}_{\frac{m'}{n}}, \partial_{\frac{p'}{n}} \right\rangle_{\mathfrak{H}}^{r_4-2} \left\langle \partial_{\frac{m'}{n}}, \partial_{\frac{p'}{n}} \right\rangle_{\mathfrak{H}}^{3-r_4} \right| \\
&\le C n^{-\Lambda},
\end{aligned}$$

where the exponent Λ is determined by $\{r_1, \ldots, r_4\}$ as follows: First, suppose $r_1 = 3$. Then by Lemma 5.2.a and 5.2.c,

$$\begin{aligned}
\sum_{p=0}^{\lfloor nt \rfloor -1} \left| \left\langle \widetilde{\varepsilon}_{\frac{q}{n}}, \partial_{\frac{p}{n}} \right\rangle_{\mathfrak{H}}^{2} \left\langle \widetilde{\varepsilon}_{\frac{m}{n}}, \partial_{\frac{p}{n}} \right\rangle_{\mathfrak{H}}^{r_3-2} \left\langle \partial_{\frac{m}{n}}, \partial_{\frac{p}{n}} \right\rangle_{\mathfrak{H}}^{3-r_3} \right| &\le \sup_{m,p} \left| \left\langle \widetilde{\varepsilon}_{\frac{m}{n}}, \partial_{\frac{p}{n}} \right\rangle_{\mathfrak{H}}^{r_3-2} \left\langle \partial_{\frac{m}{n}}, \partial_{\frac{p}{n}} \right\rangle_{\mathfrak{H}}^{3-r_3} \right| \sum_{p=0}^{\lfloor nt \rfloor -1} \left| \left\langle \widetilde{\varepsilon}_{\frac{q}{n}}, \partial_{\frac{p}{n}} \right\rangle_{\mathfrak{H}}^{2} \right| \\
&\le C n^{-2\lambda - (r_3-2)\lambda - \frac{1}{3}(3-r_3)} \\
&= C n^{-(r_1+r_3-3)\lambda - \frac{1}{3}(6-r_1-r_3)}.
\end{aligned}$$

On the other hand, if $r_1 = 1$ or 2 then by Lemma 5.2.a and 5.2.d,

$$\sum_{p=0}^{\lfloor nt\rfloor-1}\left|\left\langle\widetilde{\varepsilon}_{\frac{q}{n}},\partial_{\frac{p}{n}}\right\rangle_{\mathfrak{H}}^{r_1-1}\left\langle\partial_{\frac{q}{n}},\partial_{\frac{p}{n}}\right\rangle_{\mathfrak{H}}^{3-r_1}\left\langle\widetilde{\varepsilon}_{\frac{m}{n}},\partial_{\frac{p}{n}}\right\rangle_{\mathfrak{H}}^{r_3-2}\left\langle\partial_{\frac{m}{n}},\partial_{\frac{p}{n}}\right\rangle_{\mathfrak{H}}^{3-r_3}\right|$$

$$\leq \sup_{q,p}\left|\left\langle\widetilde{\varepsilon}_{\frac{q}{n}},\partial_{\frac{p}{n}}\right\rangle_{\mathfrak{H}}^{r_1-1}\right|\sup_{m,p}\left|\left\langle\widetilde{\varepsilon}_{\frac{m}{n}},\partial_{\frac{p}{n}}\right\rangle_{\mathfrak{H}}^{r_3-2}\left\langle\partial_{\frac{m}{n}},\partial_{\frac{p}{n}}\right\rangle_{\mathfrak{H}}^{3-r_3}\right|\sum_{p=0}^{\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{q}{n}},\partial_{\frac{p}{n}}\right\rangle_{\mathfrak{H}}^{3-r_1}\right|$$

$$\leq Cn^{-(r_1+r_3-3)\lambda-\frac{1}{3}(6-r_1-r_3)}.$$

Combining this with a similar computation for the sum over p', we obtain

$$\Lambda = \lambda(R-6)+\frac{1}{3}(12-R) = 2-\left(\frac{1}{3}-\lambda\right)(R-6).$$

In particular, $\Lambda = 2$ if $R = 6$ and $\Lambda = \frac{5}{3}+\lambda$ for $R = 7$. It follows that we want to find bounds for terms of the form

$$Cn^{-\Lambda}\sup_{p,p'}\sum_{j_1,j_2,j_3,j_4}\left|\mathbb{E}\left[G(p,p')\prod_{i=1}^{4}\delta^{r_i}\left(g_i(\widetilde{X}_{\frac{j_i}{n}})\partial_{\frac{j_i}{n}}^{\otimes r_i}\right)\right]\right|. \tag{5.34}$$

By repeated use of Lemma 2.1.d, we can expand each product of the form

$$\prod_{i=1}^{4}\delta^{r_i}\left(g_i(\widetilde{X}_{\frac{j_i}{n}})\partial_{\frac{j_i}{n}}^{\otimes r_i}\right)$$

into a sum of terms of the form

$$C_M\delta^M\left(\Psi_n\partial_{\frac{j_1}{n}}^{\otimes b_1}\otimes\partial_{\frac{j_2}{n}}^{\otimes b_2}\otimes\partial_{\frac{j_3}{n}}^{\otimes b_3}\otimes\partial_{\frac{j_4}{n}}^{\otimes b_4}\right)\left\langle\partial_{\frac{j_1}{n}},\partial_{\frac{j_2}{n}}\right\rangle_{\mathfrak{H}}^{\alpha_1}\left\langle\partial_{\frac{j_1}{n}},\partial_{\frac{j_3}{n}}\right\rangle_{\mathfrak{H}}^{\alpha_2}\left\langle\partial_{\frac{j_1}{n}},\partial_{\frac{j_4}{n}}\right\rangle_{\mathfrak{H}}^{\alpha_3}$$
$$\times\left\langle\partial_{\frac{j_2}{n}},\partial_{\frac{j_3}{n}}\right\rangle_{\mathfrak{H}}^{\alpha_4}\left\langle\partial_{\frac{j_2}{n}},\partial_{\frac{j_4}{n}}\right\rangle_{\mathfrak{H}}^{\alpha_5}\left\langle\partial_{\frac{j_3}{n}},\partial_{\frac{j_4}{n}}\right\rangle_{\mathfrak{H}}^{\alpha_6},$$

where C_M is a combinatorial constant from Lemma 2.1.d, $\Psi_n = \prod_{i=1}^{4} g_i(\widetilde{X}_{\frac{j_i}{n}})$; each $\alpha_i \in \{0,1,2\}$, such that $A := \sum_{i=1}^{6}\alpha_i \leq R/2$; each nonnegative integer b_i satisfies $b_i \leq r_i$; and the exponent M satisfies:

$$M = b_1+b_2+b_3+b_4 = R-2A.$$

With this representation, and using the Malliavin duality (2.6), we want to bound terms of the form

$$Cn^{-\Lambda}\sup_{p,p'}\sum_{j_1,j_2,j_3,j_4}\left|\mathbb{E}\left[\left\langle D^M G(p,p'),\Psi_n\partial_{\frac{j_1}{n}}^{\otimes b_1}\otimes\cdots\otimes\partial_{\frac{j_4}{n}}^{\otimes b_4}\right\rangle_{\mathfrak{H}^{\otimes M}}\right]\left\langle\partial_{\frac{j_1}{n}},\partial_{\frac{j_2}{n}}\right\rangle_{\mathfrak{H}}^{\alpha_1}\cdots\left\langle\partial_{\frac{j_3}{n}},\partial_{\frac{j_4}{n}}\right\rangle_{\mathfrak{H}}^{\alpha_6}\right|. \tag{5.35}$$

Consider first the case $A = 0$. Then $M = R \geq 6$, and each $b_i = r_i \geq 1$. Hence

$$Cn^{-\Lambda}\sup_{p,p'}\sum_{j_1,j_2,j_3,j_4}\left|\mathbb{E}\left[\left\langle D^M G(p,p'),\Psi_n\partial_{\frac{j_1}{n}}^{\otimes b_1}\otimes\cdots\otimes\partial_{\frac{j_4}{n}}^{\otimes b_4}\right\rangle_{\mathfrak{H}^{\otimes M}}\right]\right|$$

$$\leq Cn^{-\Lambda}\sup_{p,j}\left|\left\langle\widetilde{\varepsilon}_{\frac{p}{n}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}\right|^{R-4}\left(\sup_{p}\sum_{j=0}^{\lfloor nt\rfloor-1}\left|\left\langle\widetilde{\varepsilon}_{\frac{p}{n}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}\right|\right)^4$$

By Lemma 5.2.a and 5.2.c, this is bounded by $Cn^{-1-2\lambda}$, since $\Lambda \geq 1$ for all R and $R \geq 6$.

If $A \geq 1$, by permutation of indices we may assume that $\alpha_1 \geq 1$, so (5.35) may be bounded using Lemma 5.2.c and 5.2.d:

$$\begin{aligned}
&Cn^{-\Lambda} \sup_{p,p'} \sup_{j_1,j_2} \sum_{j_3,j_4=0}^{\lfloor nt\rfloor -1} \left| \mathbb{E}\left[\left\langle D^M G(p,p'), \Psi_n \partial_{\frac{j_1}{n}}^{\otimes b_1} \otimes \cdots \otimes \partial_{\frac{j_4}{n}}^{\otimes b_4} \right\rangle_{\mathfrak{H}^{\otimes M}} \right] \right| \\
&\qquad \times \sup_{j,k} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}} \right|^{A-1} \sum_{j_1,j_2=0}^{\lfloor nt\rfloor -1} \left| \left\langle \partial_{\frac{j_1}{n}}, \partial_{\frac{j_2}{n}} \right\rangle_{\mathfrak{H}} \right| \\
&\quad \leq C\lfloor nt\rfloor^3 n^{-\Lambda-\frac{1}{3}} \sup_{p,j} \left| \left\langle \widetilde{\varepsilon}_{\frac{p}{n}}, \partial_{\frac{j}{n}} \right\rangle_{\mathfrak{H}} \right|^M \sup_{j,k} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}} \right|^{A-1} \\
&\quad \leq \lfloor nt\rfloor^3 n^{-\Theta},
\end{aligned}$$

where

$$\Theta = 4 + (R-6+M)\lambda - \frac{R-A}{3} = 4 + (R-A)(2\lambda - \frac{1}{3}) - 6\lambda.$$

Since $A \leq R/2$, $R \geq 6$, and $\lambda > 1/6$, we have $\Theta > 3$ for all cases except when $R = 6$, $A = 3$. This case has the form,

$$\begin{aligned}
&Cn^{-2} \sup_{p,p'} \sum_{j_1,j_2,j_3,j_4} \left| \mathbb{E}\left[G(p,p')\Psi_n \right] \left\langle \partial_{\frac{j_1}{n}}, \partial_{\frac{j_2}{n}} \right\rangle_{\mathfrak{H}}^{\alpha_1} \cdots \left\langle \partial_{\frac{j_3}{n}}, \partial_{\frac{j_4}{n}} \right\rangle_{\mathfrak{H}}^{\alpha_6} \right| \\
&\leq Cn^{-2} \sup_{j_1,j_2} \sum_{j_3,j_4} \left| \left\langle \partial_{\frac{j_1}{n}}, \partial_{\frac{j_2}{n}} \right\rangle_{\mathfrak{H}}^{\alpha_1-1} \left\langle \partial_{\frac{j_1}{n}}, \partial_{\frac{j_3}{n}} \right\rangle_{\mathfrak{H}}^{\alpha_2} \cdots \left\langle \partial_{\frac{j_3}{n}}, \partial_{\frac{j_4}{n}} \right\rangle_{\mathfrak{H}}^{\alpha_6} \right| \sum_{j_1,j_2=0}^{\lfloor nt\rfloor -1} \left| \left\langle \partial_{\frac{j_1}{n}}, \partial_{\frac{j_2}{n}} \right\rangle_{\mathfrak{H}} \right| \\
&\leq C\lfloor nt\rfloor n^{-\frac{7}{3}} \left[\left(\sup_k \sum_{j=0}^{\lfloor nt\rfloor -1} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}} \right| \right)^2 + \sup_{j,k} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}} \right| \sum_{j_3,j_4=0}^{\lfloor nt\rfloor -1} \left| \left\langle \partial_{\frac{j_3}{n}}, \partial_{\frac{j_4}{n}} \right\rangle_{\mathfrak{H}} \right| \right] \\
&\leq C\lfloor nt\rfloor^2 n^{-3},
\end{aligned}$$

by Lemma 5.2.d.

Proof of (5.19). For this term, we see that

$$\mathbb{E}\left[\left\langle u_n^i, DF_n^j \otimes DF_n^k \otimes DF_n^\ell \right\rangle_{\mathfrak{H}^{\otimes 3}}^2 \right]$$

consists of terms with the form

$$\begin{aligned}
&\sum_{p,p'=a_i}^{b_i-1} \sum_{j_1,j_2=a_j}^{b_j-1} \sum_{j_3,j_4=a_k}^{b_k-1} \sum_{j_5,j_6=a_\ell}^{b_\ell-1} \mathbb{E}\left[G(p,p')\delta^{r_1}\left(g_1(\widetilde{X}_{\frac{j_1}{n}}) \partial_{\frac{j_1}{n}}^{\otimes r_1} \right) \cdots \delta^{r_6}\left(g_6(\widetilde{X}_{\frac{j_6}{n}}) \partial_{\frac{m'}{n}}^{\otimes r_6} \right) \right] \\
&\qquad \times \left\langle \widetilde{\varepsilon}_{\frac{j_1}{n}}, \partial_{\frac{p}{n}} \right\rangle_{\mathfrak{H}}^{r_1-2} \left\langle \partial_{\frac{j_1}{n}}, \partial_{\frac{p}{n}} \right\rangle_{\mathfrak{H}}^{3-r_1} \cdots \left\langle \widetilde{\varepsilon}_{\frac{j_6}{n}}, \partial_{\frac{p}{n}} \right\rangle_{\mathfrak{H}}^{r_6-2} \left\langle \partial_{\frac{j_6}{n}}, \partial_{\frac{p}{n}} \right\rangle_{\mathfrak{H}}^{3-r_6}.
\end{aligned}$$

where each $r_i \in \{2,3\}$ and $G(p,p'), g_i(x)$ are as defined above. As with (5.18) above, we assume that all components are defined over the time interval $[0,t]$ for some $t \le T$. As above, let $R = \sum_{i=1}^{6} r_i$, and note that for this case $12 \le R \le 18$. Similar to the above case, we obtain

$$\sum_{p,p'} \left| \left\langle \widetilde{\varepsilon}_{\frac{j_1}{n}}, \partial_{\frac{p}{n}} \right\rangle_{\mathfrak{H}}^{r_1-2} \left\langle \partial_{\frac{j_1}{n}}, \partial_{\frac{p}{n}} \right\rangle_{\mathfrak{H}}^{3-r_1} \cdots \left\langle \widetilde{\varepsilon}_{\frac{j_6}{n}}, \partial_{\frac{p}{n}} \right\rangle_{\mathfrak{H}}^{r_6-2} \left\langle \partial_{\frac{j_6}{n}}, \partial_{\frac{p}{n}} \right\rangle_{\mathfrak{H}}^{3-r_6} \right| \le Cn^{-\Lambda'},$$

where $\Lambda' = 2 - (\frac{1}{3} - \lambda)(R-12)$. It follows that, similar to (5.34), we want to obtain bounds for terms of the form

$$Cn^{-\Lambda'} \sup_{p,p'} \sum_{j_1,\ldots,j_6=0}^{\lfloor nt \rfloor -1} \left| \mathbb{E}\left[G(p,p') \prod_{i=1}^{6} \delta^{r_i} \left(g_i(\widetilde{X}_{\frac{j_i}{n}}) \partial_{\frac{j_i}{n}}^{\otimes r_i} \right) \right] \right|.$$

Using Lemma 2.1.d and the Malliavin duality as before, we obtain terms of the form

$$Cn^{-\Lambda} \sup_{p,p'} \sum_{j_1,\ldots,j_6=0}^{\lfloor nt \rfloor -1} \left| \mathbb{E}\left[\left\langle D^M G(p,p), \delta^M \left(\widetilde{\Psi}_n \partial_{\frac{j_1}{n}}^{\otimes b_1} \otimes \cdots \otimes \partial_{\frac{j_6}{n}}^{\otimes b_6} \right) \right\rangle_{\mathfrak{H}^{\otimes M}} \right] \right| \prod_{\{j_1,\ldots,j_6\}} \left| \left\langle \partial_{\frac{j_\ell}{n}}, \partial_{\frac{j_m}{n}} \right\rangle_{\mathfrak{H}}^{\alpha_i} \right|, \tag{5.36}$$

where $\widetilde{\Psi}_n = \prod_{i=1}^{6} g_i(\widetilde{X}_{\frac{j_i}{n}})$, each α_i and each b_i take values from $\{0,1,2,3\}$; and the product includes all 15 possible pairs from the set $\{j_1,\ldots,j_6\}$ such that $A := \sum_{i=1}^{15} \alpha_i \le R/2$. As in the above case, for each R we have M and A satisfying $M = \sum_{i=1}^{6} b_i$ and $M = R - 2A$.

In the product

$$\left\langle D^M G(p,p), \delta^M \left(\widetilde{\Psi}_n \partial_{\frac{j_1}{n}}^{\otimes b_1} \otimes \cdots \otimes \partial_{\frac{j_6}{n}}^{\otimes b_6} \right) \right\rangle_{\mathfrak{H}^{\otimes M}} \prod_{\{i_1,\ldots,i_6\}} \left\langle \partial_{\frac{j_\ell}{n}}, \partial_{\frac{j_m}{n}} \right\rangle_{\mathfrak{H}}^{\alpha_i} \tag{5.37}$$

each of the indices $\{j_1,\ldots,j_6\}$ must appear at least once. Note that by Lemma 5.2.a we have (possibly up to a fixed constant)

$$\sup_{0 \le j,k \le \lfloor nt \rfloor} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}} \right| \le \sup_{0 \le j,p \le \lfloor nt \rfloor} \left| \left\langle \widetilde{\varepsilon}_{\frac{p}{n}}, \partial_{\frac{j}{n}} \right\rangle_{\mathfrak{H}} \right|,$$

and by Lemmas 5.2.c and 5.2.d we have

$$\sup_{0 \le k \le \lfloor nt \rfloor} \sum_{j=0}^{\lfloor nt \rfloor -1} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}} \right| \le \sup_{0 \le p \le \lfloor nt \rfloor} \sum_{j=0}^{\lfloor nt \rfloor -1} \left| \left\langle \widetilde{\varepsilon}_{\frac{p}{n}}, \partial_{\frac{j}{n}} \right\rangle_{\mathfrak{H}} \right|.$$

Hence, we may conclude that (5.37) contains terms less than or equal to

$$\left\langle \widetilde{\varepsilon}_{\frac{p}{n}}, \partial_{\frac{j_1}{n}} \right\rangle_{\mathfrak{H}} \left\langle \widetilde{\varepsilon}_{\frac{p}{n}}, \partial_{\frac{j_3}{n}} \right\rangle_{\mathfrak{H}} \left\langle \widetilde{\varepsilon}_{\frac{p}{n}}, \partial_{\frac{j_3}{n}} \right\rangle_{\mathfrak{H}},$$

and, by Lemma 5.2.c, (5.36) is bounded in absolute value by

$$\begin{aligned} Cn^{-\Lambda'} \sum_{j_4,j_5,j_6=0}^{\lfloor nt \rfloor -1} \sup_{p,j} \left| \left\langle \widetilde{\varepsilon}_{\frac{p}{n}}, \partial_{\frac{j}{n}} \right\rangle_{\mathfrak{H}} \right|^{M-3} \sup_{j,k} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}} \right|^{A} \left(\sup_{p} \sum_{j=0}^{\lfloor nt \rfloor -1} \left| \left\langle \widetilde{\varepsilon}_{\frac{p}{n}}, \partial_{\frac{j}{n}} \right\rangle_{\mathfrak{H}} \right| \right)^3 \\ \le C \lfloor nt \rfloor^3 n^{-\Theta'}, \end{aligned}$$

where, using the fact that $R = M + 2A$,

$$\Theta' = 2 - (\frac{1}{3} - \lambda)(R - 12) + (M - 3)\lambda + \frac{A}{3} = 6 + (R - A)(2\lambda - \frac{1}{3}) - 15\lambda.$$

Observe that $\Theta' > 3$ whenever $R - A > 6$. The case $R - A = 6$ occurs only when $R = 12$, $A = 6$, and $M = 0$; so in this case we have an upper bound of

$$Cn^{-2} \sum_{j_4, j_5, j_6 = 0}^{\lfloor nt \rfloor - 1} \sup_{j,k} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}} \right|^3 \left(\sup_k \sum_{j=0}^{\lfloor nt \rfloor - 1} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}} \right| \right)^3$$

$$\leq C \lfloor nt \rfloor^3 n^{-2-\frac{6}{3}} \leq Cn^{-1}.$$

Chapter 6

Simpson's rule and fBm with $H = 1/10$

6.1 Introduction

In this chapter we consider another critical case from Proposition 3.3, which was also mentioned in [16]. This is the Simpson's rule Riemann sum, defined as

$$S_n^S(t) = \sum_{j=0}^{\lfloor nt \rfloor - 1} \frac{1}{6} \left(f'(B_{\frac{j}{n}}) + 4f'(\widetilde{B}_{\frac{j}{n}}) + f'(B_{\frac{j+1}{n}}) \right) \Delta B_{\frac{j}{n}},$$

where $B = \{B_t^H, t \geq 0\}$ is fBm, and we recall the notation from Section 2.4,

$$\widetilde{B}_{\frac{j}{n}} = \frac{1}{2}\left(B_{\frac{j}{n}} + B_{\frac{j+1}{n}} \right) \text{ and } \Delta B_{\frac{j}{n}} = B_{\frac{j+1}{n}} - B_{\frac{j}{n}}.$$

As shown in Proposition 3.3.c, given $t \geq 0$, $S_n^S(t)$ converges in probability to $f(B_t) - f(0)$ for fBm with $H > 1/10$, but generally does not converge in probability for $H \leq 1/10$. Similar to the main theorems of Chapters 4 and 5, here we consider the critical case of $H = 1/10$, and we employ Theorem 2.3 to show that, conditioned on the path $\{B_s, s \leq t\}$, we have

$$(B_t, S_n^S(t)) \xrightarrow{\mathscr{L}} \left(f(B_t) - f(0) - \frac{\beta}{2880} \int_0^t f^{(5)}(B_s)\, dW_s \right),$$

where β is a known constant, and W is a standard Brownian motion, independent of B.

This result is similar in form to the preceding results for $S_n^{M1}(t)$, $S_n^{M2}(t)$ and $S_n^T(t)$. Indeed, the result was not suprising, though the explicit value of the constant β was previously unknown. Moreover, this case was different in that the integral correction term arises from a sum of two, independent Gaussian random variables instead of only one in the previous cases.

Unlike Chapters 4 and 5, this case was done for fBm only, though it could be extended to a generalized Gaussian process using an approach similar to Chapters 4 and 5. This generalization would likely include some types of bifractional Brownian motion with some range of values for parameters H, K such that $HK = 1/10$. In addition, weak convergence was only estabilised for the pointwise case (*i.e.* fixed t). A finite-dimensional distributions argument should be possible using a version of Corollary 2.5, though this was not pursued in the present writing.

It is also expected that the techniques of this chapter could be applied to the 'Milne's rule' sum for the case $H = 1/14$, see Proposition 3.3.d. In that case, we would expect an integral correction term involving $f^{(7)}$.

The organization of this chapter is as follows: in Section 2 we state and prove the main result, which is Theorem 6.1. Finally, Section 6.3 contains proofs of three of the longer lemmas from Section 6.2.

6.2 Results

Throughout the rest of this paper, we will assume that $f : \mathbb{R} \to \mathbb{R}$ is a $\mathscr{C}^\infty$ function, such that f and all derivatives satisfy moderate growth conditions. Note that this implies $\mathbb{E}\left[\sup_{t\in[0,T]}\left|f^{(n)}(B_t)\right|^p\right] < \infty$ for all $n = 0,1,2,\dots$ and $1 \le p < \infty$.

Theorem 6.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a $\mathscr{C}^\infty$ function such that f and its derivatives have moderate growth conditions, and let $\{B_t, t \ge 0\}$ be a fractional Brownian motion with $H = 1/10$. For $t \ge 0$ and integers $n \ge 2$, Define*

$$S_n^S(t) = \sum_{j=0}^{\lfloor nt\rfloor-1} \frac{1}{6}\left(f'(B_{\frac{j}{n}}) + 4f'\left((B_{\frac{j}{n}} + B_{\frac{j+1}{n}})/2\right) + f'(B_{\frac{j+1}{n}})\right)\left(B_{\frac{j+1}{n}} - B_{\frac{j}{n}}\right).$$

Then as $n \to \infty$

$$\left(B_t, S_n^S(t)\right) \xrightarrow{\mathscr{L}} \left(B_t, f(B_t) - f(0) + \frac{\beta}{2^5\cdot 90}\int_0^t f^{(5)}(B_s)\,dW_s\right),$$

where $W = \{W_t, t \ge 0\}$ is a Brownian motion, independent of B, and

$$\beta = \sqrt{(5!)2^{-5}\kappa_5 + 75\kappa_3},\ \text{for}\ \ \kappa_5 = \sum_{p\in\mathbb{Z}}\left((p+1)^{\frac{1}{5}} - 2p^{\frac{1}{5}} + (p-1)^{\frac{1}{5}}\right)^5,\ \text{and}$$

$$\kappa_3 = \sum_{p\in\mathbb{Z}}\left((p+1)^{\frac{1}{5}} - 2p^{\frac{1}{5}} + (p-1)^{\frac{1}{5}}\right)^3.$$

Consequently,

$$f(B_t) \stackrel{\mathscr{L}}{=} f(0) + \int_0^t f'(B_s)\,d^S B_s - \frac{\beta}{2880}\int_0^t f^{(5)}(B_s)\,dW_s,$$

where $\int_0^t f'(B_s)\,d^S B_s$ denotes the weak limit of the 'Simpson's rule' sum $S_n^S(t)$.

The rest of this section is given to proof of Theorem 6.1, and follows in Sections 6.2.1 - 6.2.3. Following the telescoping series argument given in the proof of Proposition 3.3.c (see (3.6)), we can write

$$f(B_t) - f(0) = S_n^S(t) - \frac{1}{2^5\,90}\sum_{j=0}^{\lfloor nt\rfloor-1} f^{(5)}(\widetilde{B}_{\frac{j}{n}})\Delta B_{\frac{j}{n}}^5 - A_7\sum_{j=0}^{\lfloor nt\rfloor-1} f^{(7)}(\widetilde{B}_{\frac{j}{n}})\Delta B_{\frac{j}{n}}^7 - A_9\sum_{j=0}^{\lfloor nt\rfloor-1} f^{(9)}(\widetilde{B}_{\frac{j}{n}})\Delta B_{\frac{j}{n}}^9$$
$$- \frac{1}{6(7!)}\sum_{j=0}^{\lfloor nt\rfloor-1}\int_0^{\Delta B_{j/n}}\left(f^{(11)}(\xi) + f^{(11)}(\eta)\right)u^8(\Delta B_{\frac{j}{n}} - u)^2 du + \left(f(B_t) - f(B_{\frac{\lfloor nt\rfloor}{n}})\right).$$

As in the proof of Proposition 3.3.c, for $H = 1/10$ it follows from Lemma 3.4 that the terms including A_7, A_9 and the integral term all tend to zero in $L^2(\Omega)$ as $n \to \infty$, and the term $\big(f(B_t) - f(B_{\lfloor nt \rfloor/n})\big)$ also tends to zero ucp as $n \to \infty$. The main task to prove Theorem 6.1, then, is to show convergence in law of the error term

$$\sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(5)}(\widetilde{B}_{\frac{j}{n}}) \Delta B_{\frac{j}{n}}^5. \tag{6.1}$$

6.2.1 Malliavin calculus representation.

In order to apply our convergence theorem (Theorem 2.3), we wish to find a Malliavin calculus representation for the term (6.1). Consider the Hermite polynomial identity $H_5(x) = x^5 - 10H_3(x) - 15x$. Taking $x = \Delta B_{j/n}/\|\Delta B_{j/n}\|_{L^2(\Omega)} = n^H \Delta B_{j/n}$, we have

$$n^{5H} \Delta B_{\frac{j}{n}}^5 = H_5(n^H \Delta B_{\frac{j}{n}}) + 10H_3(n^H \Delta B_{\frac{j}{n}}) + 15n^H \Delta B_{\frac{j}{n}}.$$

Using (2.4), this gives

$$\begin{aligned}\sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(5)}(\widetilde{B}_{\frac{j}{n}}) \Delta B_{\frac{j}{n}}^5 = &\sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(5)}(\widetilde{B}_{\frac{j}{n}}) \delta^5(\partial_{\frac{j}{n}}^{\otimes 5}) \\ &+ 10n^{-2H} \sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(5)}(\widetilde{B}_{\frac{j}{n}}) \delta^3(\partial_{\frac{j}{n}}^{\otimes 3}) + 15n^{-4H} \sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(5)}(\widetilde{B}_{\frac{j}{n}}) \Delta B_{\frac{j}{n}}.\end{aligned}$$

We first show that the last term tends to zero in $L^1(\Omega)$.

Lemma 6.2. *Under the assumptions of Theorem 6.1, there is a constant $C > 0$ such that*

$$\mathbb{E}\left[\left(n^{-4H} \sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(5)}(\widetilde{B}_{\frac{j}{n}}) \Delta B_{\frac{j}{n}}\right)^2\right] \le Cn^{-2H}.$$

Proof. We start with a 2-sided Taylor expansion of $f^{(4)}$ of order 7. That is,

$$f^{(4)}(B_{\frac{j+1}{n}}) - f^{(4)}(\widetilde{B}_{\frac{j}{n}}) = \sum_{\ell=1}^{6} \frac{f^{(4+\ell)}(\widetilde{B}_{\frac{j}{n}})}{2^\ell \ell!} \Delta B_{\frac{j}{n}}^\ell + \frac{f^{(11)}(\xi_j)}{2^7 7!} \Delta B_{\frac{j}{n}}^7$$

and

$$f^{(4)}(\widetilde{B}_{\frac{j}{n}}) - f^{(4)}(B_{\frac{j}{n}}) = \sum_{\ell=1}^{6} \frac{(-1)^{\ell+1} f^{(4+\ell)}(\widetilde{B}_{\frac{j}{n}})}{2^\ell \ell!} \Delta B_{\frac{j}{n}}^\ell + \frac{f^{(11)}(\eta_j)}{2^7 7!} \Delta B_{\frac{j}{n}}^7,$$

for some intermediate values ξ_j, η_j between $B_{j/n}$ and $B_{(j+1)/n}$. Adding the above equations, we obtain

$$\begin{aligned}f^{(4)}(B_{\frac{j+1}{n}}) - f^{(4)}(B_{\frac{j}{n}}) = f^{(5)}(\widetilde{B}_{\frac{j}{n}}) \Delta B_{\frac{j}{n}} + \frac{f^{(7)}(\widetilde{B}_{\frac{j}{n}})}{24} \Delta B_{\frac{j}{n}}^3 + \frac{f^{(9)}(\widetilde{B}_{\frac{j}{n}})}{2^4 5!} \Delta B_{\frac{j}{n}}^5 \\ + \frac{f^{(11)}(\xi_j) + f^{(11)}(\eta_j)}{2^7 7!} \Delta B_{\frac{j}{n}}^7.\end{aligned} \tag{6.2}$$

It follows that we can write

$$\mathbb{E}\left[\left(n^{-4H}\sum_{j=0}^{\lfloor nt\rfloor-1} f^{(5)}(\widetilde{B}_{\frac{j}{n}})\Delta B_{\frac{j}{n}}\right)^2\right] \le 4\mathbb{E}\left[\left(n^{-4H}\sum_{j=0}^{\lfloor nt\rfloor-1}\left(f^{(4)}(B_{\frac{j+1}{n}})-f^{(4)}(B_{\frac{j}{n}})\right)\right)^2\right]$$
$$+4\mathbb{E}\left[\left(n^{-4H}\sum_{j=0}^{\lfloor nt\rfloor-1}\frac{f^{(7)}(\widetilde{B}_{\frac{j}{n}})}{24}\Delta B^3_{\frac{j}{n}}\right)^2\right]+4\mathbb{E}\left[\left(n^{-4H}\sum_{j=0}^{\lfloor nt\rfloor-1}\frac{f^{(9)}(\widetilde{B}_{\frac{j}{n}})}{2^4 5!}\Delta B^5_{\frac{j}{n}}\right)^2\right]$$
$$+4\mathbb{E}\left[\left(n^{-4H}\sum_{j=0}^{\lfloor nt\rfloor-1}\frac{f^{(11)}(\xi_j)+f^{(11)}(\eta_j)}{2^7 7!}\Delta B^7_{\frac{j}{n}}\right)^2\right].$$

By growth assumptions on $f^{(4)}$,

$$\mathbb{E}\left[\left(n^{-4H}\sum_{j=0}^{\lfloor nt\rfloor-1}\left(f^{(4)}(B_{\frac{j+1}{n}})-f^{(4)}(B_{\frac{j}{n}})\right)\right)^2\right]=n^{-8H}\mathbb{E}\left[\left(f^{(4)}(B_{\frac{\lfloor nt\rfloor}{n}})-f^{(4)}(0)\right)^2\right]\le Cn^{-8H}.$$

By Lemma 3.4,

$$\mathbb{E}\left[\left(n^{-4H}\sum_{j=0}^{\lfloor nt\rfloor-1}\frac{f^{(7)}(\widetilde{B}_{\frac{j}{n}})}{24}\Delta B^3_{\frac{j}{n}}\right)^2\right]\le C\sup_{0\le j\le\lfloor nt\rfloor}\|f^{(7)}(\widetilde{B}_{\frac{j}{n}})\|^2_{\mathbb{D}^{6,2}}\lfloor nt\rfloor n^{-14H},$$

and

$$\mathbb{E}\left[\left(n^{-4H}\sum_{j=0}^{\lfloor nt\rfloor-1}\frac{f^{(9)}(\widetilde{B}_{\frac{j}{n}})}{2^4 5!}\Delta B^5_{\frac{j}{n}}\right)^2\right]\le C\sup_{0\le j\le\lfloor nt\rfloor}\|f^{(9)}(\widetilde{B}_{\frac{j}{n}})\|^2_{\mathbb{D}^{10,2}}\lfloor nt\rfloor n^{-18H}.$$

Then by (B.1),

$$\mathbb{E}\left[\left(n^{-4H}\sum_{j=0}^{\lfloor nt\rfloor-1}\frac{f^{(11)}(\xi_j)+f^{(11)}(\eta_j)}{2^7 7!}\Delta B^7_{\frac{j}{n}}\right)^2\right]$$
$$\le C\left(\mathbb{E}\left[\sup_{s\in[0,t]}\left|f^{(11)}(B_s)^4\right|\right]\right)^{\frac{1}{2}}n^{-8H}\left(\sum_{j=0}^{\lfloor nt\rfloor-1}\|\Delta B^7_{\frac{j}{n}}\|_{L^4(\Omega)}\right)^2\le C\lfloor nt\rfloor^2 n^{-22H}\le Cn^{-2H}.$$

This proves the lemma. □

Lemma 6.2 shows that only the terms

$$\sum_{j=0}^{\lfloor nt\rfloor-1} f^{(5)}(\widetilde{B}_{\frac{j}{n}})\delta^5\left(\partial^{\otimes 5}_{\frac{j}{n}}\right)+10n^{-2H}\sum_{j=0}^{\lfloor nt\rfloor-1} f^{(5)}(\widetilde{B}_{\frac{j}{n}})\delta^3(\partial^{\otimes 3}_{\frac{j}{n}})$$

are significant. Using Lemma 2.1.a, we can write the first term as

$$\sum_{j=0}^{\lfloor nt\rfloor-1}\delta^5\left(f^{(5)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 5}\right)+\sum_{r=1}^{5}\binom{5}{r}\sum_{j=0}^{\lfloor nt\rfloor-1}\delta^{5-r}\left(f^{(5+r)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes(5-r)}\right)\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}^{r}.$$

By Lemma 2.1.c and (B.1), we have the estimate

$$\left\|\delta^{(5-r)}\left(f^{(5+r)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes(5-r)}\right)\right\|_{L^2(\Omega)}\le C\left\|\partial_{\frac{j}{n}}^{\otimes(5-r)}\right\|_{\mathfrak{H}^{\otimes 5-r}}\le Cn^{(r-5)H}.$$

It follows that for $r=1,\dots,5$, we can use Lemma 3.1.b,

$$\mathbb{E}\left|\binom{5}{r}\sum_{j=0}^{\lfloor nt\rfloor-1}\delta^{(5-r)}\left(f^{(5+r)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes(5-r)}\right)\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}^{r}\right|$$
$$\le Cn^{(r-5)H}\sum_{j=0}^{\lfloor nt\rfloor-1}\left|\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}^{r}\right|\le Cn^{-(3+r)H}.$$

By a similar computation,

$$10n^{-2H}\sum_{j=0}^{\lfloor nt\rfloor-1}f^{(5)}(\widetilde{B}_{\frac{j}{n}})\delta^3(\partial_{\frac{j}{n}}^{\otimes 3})=10n^{-2H}\sum_{j=0}^{\lfloor nt\rfloor-1}\delta^3\left(f^{(5)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 3}\right)$$
$$+10n^{-2H}\sum_{r=1}^{3}\binom{3}{r}\sum_{j=0}^{\lfloor nt\rfloor-1}\delta^{(3-r)}\left(f^{(5+r)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes(3-r)}\right)\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}^{r},$$

where

$$n^{-2H}\mathbb{E}\left|\sum_{r=1}^{3}\binom{3}{r}\sum_{j=0}^{\lfloor nt\rfloor-1}\delta^{(3-r)}\left(f^{(5+r)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes(3-r)}\right)\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}^{r}\right|\le Cn^{-4H}.$$

Therefore, we define

$$F_n:=\sum_{j=0}^{\lfloor nt\rfloor-1}\delta^5\left(f^{(5)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 5}\right)=\delta^5(u_n),\quad\text{where }\ u_n=\sum_{j=0}^{\lfloor nt\rfloor-1}f^{(5)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 5};\text{ and}$$

$$G_n:=10n^{-2H}\sum_{j=0}^{\lfloor nt\rfloor-1}\delta^3\left(f^{(5)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 5}\right)=\delta^3(v_n),\quad\text{where }\ v_n=10n^{-2H}\sum_{j=0}^{\lfloor nt\rfloor-1}f^{(5)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 3}.$$

It follows that for large n, the term (6.1) may be represented as $F_n+G_n+\varepsilon_n$, where $\varepsilon_n\to 0$ in $L^1(\Omega)$. Hence, we will apply Theorem 2.3 to the vector sequence (F_n,G_n).

6.2.2 Conditions of Theorem 2.3.

Our main task in this step is to show that the sequence of random vectors (F_n,G_n) satisfies the conditions of Theorem 2.3. The first condition is that (F_n,G_n) is bounded in $L^1(\Omega)$. In fact, we have a stronger result that will also be helpful with later conditions.

Lemma 6.3. *Fix real numbers $0 < t \leq T$ and $p \geq 2$, and integer $n \geq 2$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a $\mathscr{C}^\infty$ function such that ϕ and all its derivatives have moderate growth. For integer $1 \leq q \leq 5$, define*

$$w_n = \sum_{j=0}^{\lfloor nt \rfloor - 1} \phi(\widetilde{B}_{\frac{j}{n}}) \partial_{\frac{j}{n}}^{\otimes q}.$$

Then for integers $0 \leq a \leq 5$, there exists a constant $c_{q,a}$ such that

$$\|D^a \delta^q(w_n)\|^2_{L^p(\Omega;\mathfrak{H}^{\otimes a})} \leq c_{q,a} \sup_{0 \leq j \leq \lfloor nt \rfloor} \left\| \phi(\widetilde{B}_{\frac{j}{n}}) \right\|^2_{\mathbb{D}^{q+a,p}} \lfloor nt \rfloor n^{-2qH} \leq C n^{1-2qH}.$$

In particular,

$$\|D^a F_n\|_{L^p(\Omega;\mathfrak{H}^{\otimes a})} + \|D^a G_n\|_{L^p(\Omega;\mathfrak{H}^{\otimes a})} \leq C. \tag{6.3}$$

Proof. This proof follows a similar result in [23], see Theorem 5.2. First, note that by Lemma 3.1.c and growth conditions on ϕ, for each integer $b \geq 0$,

$$\begin{aligned}
\left\| D^b w_n \right\|^2_{\mathfrak{H}^{\otimes q+b}} &= \left\| \sum_{j=0}^{\lfloor nt \rfloor - 1} \phi^{(b)}(\widetilde{B}_{\frac{j}{n}}) \partial_{\frac{j}{n}}^{\otimes q} \otimes \widetilde{\varepsilon}_{\frac{j}{n}}^{\otimes b} \right\|^2_{\mathfrak{H}^{\otimes q+b}} \\
&\leq \sup_{0 \leq j \leq \lfloor nt \rfloor} \left| \phi^{(b)}(\widetilde{B}_{\frac{j}{n}}) \right|^2 \sup_{0 \leq j,k \leq \lfloor nt \rfloor} \left| \left\langle \widetilde{\varepsilon}_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{k}{n}} \right\rangle^b \right| \sum_{j,k=0}^{\lfloor nt \rfloor - 1} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle^q_{\mathfrak{H}} \right| \\
&\leq C \lfloor nt \rfloor n^{-2qH} \sup_{0 \leq j \leq \lfloor nt \rfloor} \left| \phi^{(b)}(\widetilde{B}_{\frac{j}{n}}) \right|^2.
\end{aligned}$$

It follows that for $p \geq 2$,

$$\left\| D^b w_n \right\|^2_{L^p(\Omega;\mathfrak{H}^{\otimes q+b})} \leq C \lfloor nt \rfloor n^{-2qH} \mathbb{E}\left[\sup_{0 \leq j \leq \lfloor nt \rfloor} \left| \phi^{(b)}(\widetilde{B}_{\frac{j}{n}}) \right|^p \right]^{\frac{2}{p}}.$$

Then, using the Meyer inequality (see [23], Proposition 1.5.7),

$$\|D^a \delta^q(w_n)\|^2_{L^p(\Omega;\mathfrak{H}^{\otimes a})} \leq \|\delta^q(w_n)\|^2_{\mathbb{D}^{a,p}} \leq C \lfloor nt \rfloor n^{-2qH} \sup_{0 \leq j \leq \lfloor nt \rfloor} \left\| \phi(\widetilde{B}_{\frac{j}{n}}) \right\|^2_{\mathbb{D}^{q+a,p}(\mathfrak{H}^q)} \leq C \lfloor nt \rfloor n^{-2qH}. \tag{6.4}$$

For (6.3), we have

$$\|D^a F_n\|^2_{L^p(\Omega;\mathfrak{H}^{\otimes a})} = \left\| D^a \delta^5(u_n) \right\|^2_{L^p(\Omega;\mathfrak{H}^{\otimes a})} \leq C \lfloor nt \rfloor n^{-10H} \sup_{0 \leq j \leq \lfloor nt \rfloor} \left\| f^{(5)}(\widetilde{B}_{\frac{j}{n}}) \right\|^2_{\mathbb{D}^{5+a,p}(\mathfrak{H}^{\otimes 5})} \leq C,$$

and

$$\|D^a G_n\|^2_{L^p(\Omega;\mathfrak{H}^{\otimes a})} = \left\| n^{-2H} D^a \delta^3(u_n) \right\|^2_{L^p(\Omega;\mathfrak{H}^{\otimes a})} \leq C \lfloor nt \rfloor n^{-10H} \sup_{0 \leq j \leq \lfloor nt \rfloor} \left\| f^{(5)}(\widetilde{B}_{\frac{j}{n}}) \right\|^2_{\mathbb{D}^{3+a,p}(\mathfrak{H}^{\otimes 3})} \leq C.$$

□

The fact that (F_n, G_n) is bounded in $L^1(\Omega)$ follows by taking $a=0$. Next, we consider condition (a) of Theorem 2.3.

Lemma 6.4. *Under the assumptions of Theorem 6.1, (F_n, G_n) satisfies condition (a) of Theorem 2.3. That is, we have*

(a) For arbitrary $h \in \mathfrak{H}^{\otimes 5}$ and $g \in \mathfrak{H}^{\otimes 3}$,

$$\lim_{n\to\infty} \mathbb{E}\left|\langle u_n, h\rangle_{\mathfrak{H}^{\otimes 5}}\right| = \lim_{n\to\infty} \mathbb{E}\left|\langle v_n, g\rangle_{\mathfrak{H}^{\otimes 3}}\right| = 0.$$

(b) $\lim_{n\to\infty} \mathbb{E}\left|\left\langle u_n, \bigotimes_{i=1}^{s} D^{a_i} F_n \bigotimes_{i=s+1}^{r} D^{a_i} G_n \otimes h\right\rangle_{\mathfrak{H}^{\otimes 5}}\right| = 0$*, where* $0 \le a_i < 5$, $1 \le a_1 + \cdots + a_r < 5$*, and* $h \in \mathfrak{H}^{\otimes 5-(a_1+\cdots+a_r)}$*; and* $\lim_{n\to\infty} \mathbb{E}\left|\left\langle v_n, \bigotimes_{i=1}^{s} D^{b_i} F_n \bigotimes_{i=s+1}^{r} D^{b_i} G_n \otimes g\right\rangle_{\mathfrak{H}^{\otimes 3}}\right| = 0$*, where* $0 \le b_i < 3$, $1 \le b_1 + \cdots + b_r < 3$*, and* $g \in \mathfrak{H}^{\otimes 3-(b_1+\cdots+b_r)}$.

(c) $\lim_{n\to\infty} \mathbb{E}\left|\left\langle u_n, \bigotimes_{i=1}^{s} D^{a_i} F_n \bigotimes_{i=s+1}^{r} D^{a_i} G_n\right\rangle_{\mathfrak{H}^{\otimes 5}}\right| = 0$*, where* $r \ge 2$, $0 \le a_i < 5$ *and* $a_1 + \cdots + a_r = 5$*; and*

$\lim_{n\to\infty} \mathbb{E}\left|\left\langle v_n, \bigotimes_{i=1}^{s} D^{b_i} F_n \bigotimes_{i=s+1}^{r} D^{b_i} G_n\right\rangle_{\mathfrak{H}^{\otimes 3}}\right| = 0$*, where* $r \ge 2$, $0 \le b_i < 3$ *and* $b_1 + \cdots + b_r = 3$.

The proof of this lemma is deferred to Section 6.3 due to its length. To verify condition (b) of Theorem 2.3, we have four terms to consider:

- $\left\langle u_n, D^5 G_n\right\rangle_{\mathfrak{H}^{\otimes 5}}$
- $\left\langle v_n, D^3 F_n\right\rangle_{\mathfrak{H}^{\otimes 3}}$
- $\left\langle u_n, D^5 F_n\right\rangle_{\mathfrak{H}^{\otimes 5}}$
- $\left\langle v_n, D^3 G_n\right\rangle_{\mathfrak{H}^{\otimes 3}}$

We deal with the first two terms in the following lemma. The proof is given in Section 6.3.

Lemma 6.5. *Under the assumptions of Theorem 6.1, we have*

(a) $\lim_{n\to\infty} \mathbb{E}\left|\left\langle u_n, D^5 G_n\right\rangle_{\mathfrak{H}^{\otimes 5}}\right| = 0$

(b) $\lim_{n\to\infty} \mathbb{E}\left|\left\langle v_n, D^3 F_n\right\rangle_{\mathfrak{H}^{\otimes 3}}\right| = 0.$

This leaves the variance terms. Lemma 2.1.b allows us to write

$$\begin{aligned}
\left\langle u_n, D^5 F_n\right\rangle_{\mathfrak{H}^{\otimes 5}} &= \sum_{j,k=0}^{\lfloor nt\rfloor -1} \left\langle f^{(5)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 5}, D^5 \delta^5\left(f^{(5)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 5}\right)\right\rangle_{\mathfrak{H}^{\otimes 5}} \\
&= \sum_{z=0}^{4} \binom{5}{z}^2 z! \sum_{j,k=0}^{\lfloor nt\rfloor -1} \left\langle f^{(5)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 5}, \delta^{5-z}\left(f^{(10-z)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 5-z}\right)\partial_{\frac{k}{n}}^{\otimes z} \otimes \widetilde{\varepsilon}_{\frac{k}{n}}^{\otimes 5-z}\right\rangle_{\mathfrak{H}^{\otimes 5}} \\
&\quad + 5! \sum_{j,k=0}^{\lfloor nt\rfloor -1} \left\langle f^{(5)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 5}, f^{(5)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 5}\right\rangle_{\mathfrak{H}^{\otimes 5}}.
\end{aligned}$$

We first deal with the case $0 \le z \le 4$. We have

$$\mathbb{E}\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left|\left\langle f^{(5)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 5}, \delta^{5-z}\left(f^{(10-z)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 5-z}\right)\partial_{\frac{k}{n}}^{\otimes z}\otimes\widetilde{\varepsilon}_{\frac{k}{n}}^{\otimes 5-z}\right\rangle_{\mathfrak{H}^{\otimes 5}}\right|$$

$$\le C\sup_{0\le j\le\lfloor nt\rfloor}\left\|f^{(5)}(\widetilde{B}_{\frac{j}{n}})\right\|_{L^2(\Omega)}\sup_{0\le k\le\lfloor nt\rfloor}\left\|\delta^{5-z}\left(f^{(10-z)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 5-z}\right)\right\|_{L^2(\Omega)}$$

$$\times\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{z}\left\langle\partial_{\frac{j}{n}},\widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{5-z}\right|.$$

By (B.1) and Lemma 2.1.c, we have

$$\sup_{0\le k\le\lfloor nt\rfloor}\left\|\delta^{5-z}\left(f^{(10-z)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 5-z}\right)\right\|_{L^2(\Omega)}\le C\|\partial_{\frac{1}{n}}\|_{\mathfrak{H}}^{5-z}\le Cn^{-(5-z)H},$$

so for the case $z=0$, we have

$$\sup_{0\le j\le\lfloor nt\rfloor}\left\|f^{(5)}(\widetilde{B}_{\frac{j}{n}})\right\|_{L^2(\Omega)}\sup_{0\le k\le\lfloor nt\rfloor}\left\|\delta^{5-z}\left(f^{(10-z)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 5-z}\right)\right\|_{L^2(\Omega)}$$

$$\times\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{z}\left\langle\partial_{\frac{j}{n}},\widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{5-z}\right|$$

$$\le Cn^{-5H}\sup_{0\le j\le\lfloor nt\rfloor}\left\{\sup_{s\in[0,t]}\left|\left\langle\partial_{\frac{j}{n}},\varepsilon_s\right\rangle_{\mathfrak{H}}^{4}\right|\right\}\sum_{k=0}^{\lfloor nt\rfloor-1}\sup_{0\le k\le\lfloor nt\rfloor}\sum_{j=0}^{\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{j}{n}},\widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|.$$

By (B.4) and Lemma 3.1.a, respectively,

$$\sup_{0\le j\le\lfloor nt\rfloor}\left\{\sup_{s\in[0,t]}\left|\left\langle\partial_{\frac{j}{n}},\varepsilon_s\right\rangle_{\mathfrak{H}}^{4}\right|\right\}\le Cn^{-8H}\ \text{ and }\ \sup_{0\le k\le\lfloor nt\rfloor}\sum_{j=0}^{\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{j}{n}},\widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|\le C,$$

so this gives

$$Cn^{-5H}\sup_{0\le j\le\lfloor nt\rfloor}\left\{\sup_{s\in[0,t]}\left|\left\langle\partial_{\frac{j}{n}},\varepsilon_s\right\rangle_{\mathfrak{H}}^{4}\right|\right\}\sum_{k=0}^{\lfloor nt\rfloor-1}\sup_{0\le k\le\lfloor nt\rfloor}\sum_{j=0}^{\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{j}{n}},\widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|\le C\lfloor nt\rfloor n^{-13H}\le Cn^{-3H}.$$

If $1\le z\le 4$, then by (B.1), (B.4) and Lemma 3.1.c we have an upper bound of

$$\sup_{0\le j\le\lfloor nt\rfloor}\left\|f^{(5)}(\widetilde{B}_{\frac{j}{n}})\right\|_{L^2(\Omega)}\sup_{0\le k\le\lfloor nt\rfloor}\left\|\delta^{5-z}\left(f^{(10-z)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 5-z}\right)\right\|_{L^2(\Omega)}\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{z}\left\langle\partial_{\frac{j}{n}},\widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{5-z}\right|$$

$$\le C\|\partial_{\frac{1}{n}}\|_{\mathfrak{H}}^{5-z}\sup_{0\le j\le\lfloor nt\rfloor}\left\{\sup_{s\in[0,t]}\left|\left\langle\partial_{\frac{j}{n}},\varepsilon_s\right\rangle^{5-z}\right|\right\}\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{z}\right|\le C\lfloor nt\rfloor n^{-(15-z)H}\le Cn^{-H},$$

because $z<5$. It follows that the term corresponding to each $z=0,\dots,4$ vanishes in $L^1(\Omega)$, and we have that only the term with $z=5$ is significant. For the case $z=5$, we use a result from [23], see proof of Theorem 5.2.

$$\begin{aligned}
&5!\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left\langle f^{(5)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 5}, f^{(5)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 5}\right\rangle_{\mathfrak{H}^{\otimes 5}}\\
&\qquad = 5!\sum_{j,k=0}^{\lfloor nt\rfloor-1} f^{(5)}(\widetilde{B}_{\frac{j}{n}})f^{(5)}(\widetilde{B}_{\frac{k}{n}})\left(\mathbb{E}\left[\Delta B_{\frac{j}{n}},\Delta B_{\frac{k}{n}}\right]\right)^5\\
&\qquad = \frac{5!}{2^5 n^{10H}}\sum_{p=-\infty}^{\infty}\sum_{j=(0\vee -p)}^{(\lfloor nt\rfloor-1)\wedge(\lfloor nt\rfloor-1-p)} f^{(5)}(\widetilde{B}_{\frac{j}{n}})f^{(5)}(\widetilde{B}_{\frac{j+p}{n}})\left(|p+1|^{2H}-2|p|^{2H}+|p-1|^{2H}\right)^5,
\end{aligned}$$

which (for $H=1/10$) converges in $L^1(\Omega)$ to

$$\frac{5!}{2^5}\kappa_5\int_0^t f^{(5)}(B_s)^2\,ds,\ \text{where } \kappa_5=\sum_{p\in\mathbb{Z}}\left(|p+1|^{\frac15}-2|p|^{\frac15}+|p-1|^{\frac15}\right)^5. \tag{6.5}$$

Hence, we have that

$$\lim_{n\to\infty}\left\langle u_n, D^5F_n\right\rangle_{\mathfrak{H}^{\otimes 5}}=\frac{5!}{2^5}\kappa_5\int_0^t f^{(5)}(B_s)^2 ds. \tag{6.6}$$

Similarly, we have

$$\left\langle v_n, D^3G_n\right\rangle_{\mathfrak{H}^{\otimes 3}}=10^2n^{-4H}\sum_{z=0}^{3}\binom{3}{z}^2 z!\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left\langle f^{(5)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 3},\delta^{3-z}\left(f^{(8-z)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 3-z}\right)\partial_{\frac{k}{n}}^{\otimes z}\otimes\widetilde{\varepsilon}_{\frac{k}{n}}^{\otimes 3-z}\right\rangle_{\mathfrak{H}^{\otimes 3}}.$$

For $z=0$,

$$\begin{aligned}
&100n^{-4H}\mathbb{E}\left|\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left\langle f^{(5)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 3},\delta^{3}\left(f^{(8)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 3}\right)\widetilde{\varepsilon}_{\frac{k}{n}}^{\otimes 3}\right\rangle_{\mathfrak{H}^{\otimes 3}}\right|\\
&\quad\le 100n^{-4H}\sup_{0\le j\le\lfloor nt\rfloor}\left\|f^{(5)}(\widetilde{B}_{\frac{j}{n}})\right\|_{L^2(\Omega)}\sup_{0\le k\le\lfloor nt\rfloor}\left\|\delta^3\left(f^{(8)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 3}\right)\right\|_{L^2(\Omega)}\sup_{j,k}\left|\left\langle\partial_{\frac{j}{n}},\widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^2\right|\\
&\qquad\times\sum_{k=0}^{\lfloor nt\rfloor-1}\sup_{s\in[0,t]}\sum_{j=0}^{\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{j}{n}},\varepsilon_s\right\rangle_{\mathfrak{H}}\right|\\
&\quad\le C\lfloor nt\rfloor n^{-11H}\le Cn^{-H}.
\end{aligned}$$

For $z=1$ or $z=2$, by (B.4) and Lemma 3.1.c,

$$\begin{aligned}
&100\binom{3}{z}^2 z!n^{-4H}\mathbb{E}\left|\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left\langle f^{(5)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 3},\delta^{3-z}\left(f^{(8-z)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 3-z}\right)\partial_{\frac{k}{n}}^{\otimes z}\otimes\widetilde{\varepsilon}_{\frac{k}{n}}^{\otimes 3-z}\right\rangle_{\mathfrak{H}^{\otimes 3}}\right|\\
&\quad\le Cn^{-4H}\sup_{0\le j\le\lfloor nt\rfloor}\left\|f^{(5)}(\widetilde{B}_{\frac{j}{n}})\right\|_{L^2(\Omega)}\sup_{0\le k\le\lfloor nt\rfloor}\left\|\delta^{3-z}\left(f^{(8-z)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 3-z}\right)\right\|_{L^2(\Omega)}\sup_{j,k}\left|\left\langle\partial_{\frac{j}{n}},\widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{3-z}\right|\\
&\qquad\times\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{z}\right|\\
&\quad\le C\lfloor nt\rfloor n^{-(13-z)H}\le Cn^{-H},
\end{aligned}$$

because $z \leq 2$. Then for $z=3$, we have

$$600n^{-4H} \sum_{j,k=0}^{\lfloor nt \rfloor -1} \left\langle f^{(5)}(\widetilde{B}_{\frac{j}{n}}) \partial_{\frac{j}{n}}^{\otimes 3}, f^{(5)}(\widetilde{B}_{\frac{k}{n}}) \partial_{\frac{k}{n}}^{\otimes 3} \right\rangle_{\mathfrak{H}^{\otimes 3}}$$
$$= \frac{600}{2^3 n^{10H}} \sum_{j,k=0}^{\lfloor nt \rfloor -1} f^{(5)}(\widetilde{B}_{\frac{j}{n}}) f^{(5)}(\widetilde{B}_{\frac{k}{n}}) \left(|j-k+1|^{2H} - 2|j-k|^{2H} + |j-k-1|^{2H}\right)^3.$$

Similar to (6.5), this converges in $L^1(\Omega)$ to

$$75\kappa_3 \int_0^t f^{(5)}(B_s)^2 \, ds, \text{ where } \kappa_3 = \sum_{p \in \mathbb{Z}} \left(|p+1|^{\frac{1}{5}} - 2|p|^{\frac{1}{5}} + |p-1|^{\frac{1}{5}}\right)^3. \tag{6.7}$$

Hence, we have that

$$\lim_{n \to \infty} \langle v_n, D^3 G_n \rangle_{\mathfrak{H}^{\otimes 3}} = 75\kappa_3 \int_0^t f^{(5)}(B_s)^2 ds. \tag{6.8}$$

6.2.3 Proof of Theorem 6.1.

By Sections 6.2.1, the term (6.1) is dominated in probability by $\frac{1}{2880}(F_n + G_n)$. By the results of Section 6.2.2, the vector (F_n, G_n) satisfies Theorem 2.3, that is, (F_n, G_n) converges stably as $n \to \infty$ to a mean-zero Gaussian random vector (F_∞, G_∞) with independent components, whose variances are given by (6.6) and (6.8), respectively. It follows that $F_n + G_n$ converges in distribution to a centered Gaussian random variable with variance

$$s^2 = \frac{5!}{2^5}\kappa_5 \int_0^t f^{(5)}(B_s)^2 \, ds + 75\kappa_3 \int_0^t f^{(5)}(B_s)^2 \, ds = \beta^2 \int_0^t f^{(5)}(B_s)^2 \, ds,$$

where $\beta^2 = (5!)2^{-5}\kappa_5 + 75\kappa_3$. The result of Theorem 6.1 then follows from the Itô isometry. This concludes the proof.

6.3 Proof of Technical Lemmas

6.3.1 Proof of Lemma 6.4.

For $\theta \in \{0, 2\}$ define

$$w_n(\theta) = n^{-\theta H} \sum_{j=0}^{\lfloor nt \rfloor -1} f^{(5)}(\widetilde{B}_{\frac{j}{n}}) \partial_{\frac{j}{n}}^{\otimes 5-\theta}; \text{ and } \Phi_n(\theta) = \delta^{5-\theta}(w_n(\theta)).$$

This allows us to write $u_n = w_n(0)$, $F_n = \Phi_n(0)$, $v_n = 10w_n(2)$, and $G_n = 10\Phi_n(2)$. Following Remark 2.4, we may assume that $h \in \mathfrak{H}^{\otimes 5-\theta}$ has the form $\varepsilon_{t_1} \otimes \cdots \otimes \varepsilon_{t_{5-\theta}}$, for some set of times

$\{t_1,\dots,t_{5-\theta}\}$ in $[0,T]^{5-\theta}$. Then for (a), using (B.4) and Lemma 3.1.a,

$$\begin{aligned}\mathbb{E}\left|\langle w_n(\theta),h\rangle_{\mathfrak{H}^{\otimes 5-\theta}}\right| &= n^{-\theta H}\mathbb{E}\left|\sum_{j=0}^{\lfloor nt\rfloor-1}\left\langle f^{(5)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 5-\theta},\varepsilon_{t_1}\otimes\cdots\otimes\varepsilon_{t_{5-\theta}}\right\rangle_{\mathfrak{H}^{\otimes 5-\theta}}\right|\\ &\le n^{-\theta H}\mathbb{E}\left[\sup_{s\in[0,t]}\left|f^{(5)}(B_s)\right|\right]\sum_{j=0}^{\lfloor nt\rfloor-1}\prod_{k=1}^{5-\theta}\left|\left\langle\partial_{\frac{j}{n}},\varepsilon_{t_k}\right\rangle_{\mathfrak{H}}\right|\\ &\le Cn^{-(8-\theta)H}\le Cn^{-6H},\end{aligned}$$

where the last inequality follows because $\theta\le 2$.

Next, for (b), consider integers $0\le a_i<5-\theta$, $0\le s\le r<5-\theta$, $r\ge 1$ and q, such that $s\le r$, $1\le a_1+\cdots+a_r<5-\theta$ and $q=5-\theta-(a_1+\cdots+a_r)\ge 1$. We have

$$\begin{aligned}&\mathbb{E}\left|\left\langle w_n(\theta),\bigotimes_{i=1}^{s}D^{a_i}F_n\bigotimes_{i=s+1}^{r}D^{a_i}G_n\otimes h\right\rangle_{\mathfrak{H}^{\otimes 5-\theta}}\right|\\ &\le n^{-\theta H}\mathbb{E}\sum_{j=0}^{\lfloor nt\rfloor-1}\left|f^{(5)}(\widetilde{B}_{\frac{j}{n}})\prod_{i=1}^{s}\left\langle\partial_{\frac{j}{n}}^{\otimes a_i},D^{a_i}F_n\right\rangle_{\mathfrak{H}^{\otimes a_i}}\left(\prod_{i=s+1}^{r}\left\langle\partial_{\frac{j}{n}}^{\otimes a_i},D^{a_i}G_n\right\rangle_{\mathfrak{H}^{\otimes a_i}}\right)\left\langle\partial_{\frac{j}{n}}^{\otimes q},h\right\rangle_{\mathfrak{H}^{\otimes q}}\right|.\end{aligned}$$

Using (B.1), Lemma 6.3, and Lemma 3.1.a, this is bounded by

$$\begin{aligned}&n^{-\theta H}\sup_{0\le j\le\lfloor nt\rfloor}\left\|f^{(5)}(\widetilde{B}_{\frac{j}{n}})\right\|_{L^p(\Omega)}\prod_{i=1}^{r}\sup_j\left\|\partial_{\frac{j}{n}}^{\otimes a_i}\right\|_{\mathfrak{H}^{\otimes a_i}}\prod_{i=1}^{s}\|D^{a_i}F_n\|_{L^p(\Omega;\mathfrak{H}^{\otimes a_i})}\\ &\qquad\times\prod_{i=s+1}^{r}\|D^{a_i}G_n\|_{L^p(\Omega;\mathfrak{H}^{\otimes a_i})}\sum_{j=0}^{\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{j}{n}}^{\otimes q},h\right\rangle_{\mathfrak{H}^{\otimes q}}\right|\le Cn^{-(3+q)H},\end{aligned}$$

where $p=r+1$.

For (c), we want to consider terms of the form

$$\mathbb{E}\left|\left\langle w_n(\theta_0),\bigotimes_{i=1}^{r}D^{a_i}\Phi_n(\theta_i)\right\rangle_{\mathfrak{H}^{\otimes 5-\theta_0}}\right|,$$

where $\theta_i\in\{0,2\}$, $2\le r\le 5-\theta_0$, $0\le a_i\le 4-\theta_0$, and $a_1+\cdots+a_r=5-\theta_0$. For example, the term

$$\left\langle u_n,D^3F_n\otimes D^2G_n\right\rangle_{\mathfrak{H}^{\otimes 3}}$$

corresponds to the case $(\theta_0,\theta_1,\theta_2)=(0,0,2)$, $a_1=3$, $a_2=2$. We will show that terms of this type tend to zero in $L^2(\Omega)$ as $n\to\infty$. Using the above definitions for $w_n(\theta_i)$, $\Phi_n(\theta_i)$, we have

$$\begin{aligned}&\mathbb{E}\left[\left\langle w_n(\theta_0),\bigotimes_{i=1}^{r}D^{a_i}\Phi_n(\theta_i)\right\rangle_{\mathfrak{H}^{\otimes 5-\theta_0}}^2\right]\\ &=n^{-2H(\theta_0+\cdots+\theta_r)}\mathbb{E}\sum_{p,p'=0}^{\lfloor nt\rfloor-1}\sum_{j_1,\dots,j_r=0}^{\lfloor nt\rfloor-1}\sum_{k_1,\dots,k_r=0}^{\lfloor nt\rfloor-1}\left\langle f^{(5)}(\widetilde{B}_{\frac{p}{n}})\partial_{\frac{p}{n}}^{\otimes 5-\theta_0},\bigotimes_{i=1}^{r}D^{a_i}\delta^{5-\theta_i}\left(f^{(5)}(\widetilde{B}_{\frac{j_i}{n}})\partial_{\frac{j_i}{n}}^{\otimes 5-\theta_i}\right)\right\rangle_{\mathfrak{H}^{\otimes 5-\theta_0}}\\ &\qquad\times\left\langle f^{(5)}(\widetilde{B}_{\frac{p'}{n}})\partial_{\frac{p'}{n}}^{\otimes 5-\theta_0},\bigotimes_{i=i}^{r}D^{a_1}\delta^{5-\theta_i}\left(f^{(5)}(\widetilde{B}_{\frac{k_i}{n}})\partial_{\frac{k_i}{n}}^{\otimes 5-\theta_i}\right)\right\rangle_{\mathfrak{H}^{\otimes 5-\theta_0}}.\qquad(6.9)\end{aligned}$$

By Lemma 2.1.b,

$$D^{a_i}\delta^{5-\theta_i}\left(f^{(5)}(\widetilde{B}_{\frac{j_i}{n}})\partial_{\frac{j_i}{n}}^{\otimes 5-\theta_i}\right) = \sum_{\ell_i=0}^{(5-\theta_i)\wedge a_i} \ell_i! \binom{5-\theta_i}{\ell_i}\binom{a_i}{\ell_i}\delta^{5-\theta_i-\ell_i}\left(f^{(5+a_i-\ell_i)}(\widetilde{B}_{\frac{j_i}{n}})\partial_{\frac{j_i}{n}}^{\otimes 5-\theta_i-\ell_i}\right)\partial_{\frac{j_i}{n}}^{\otimes \ell_i}\otimes \widetilde{\varepsilon}_{\frac{j_i}{n}}^{\otimes a_i-\ell_i}.$$

Applying this to each term, we can expand the inner product

$$\left\langle f^{(5)}(\widetilde{B}_{\frac{p}{n}})\partial_{\frac{p}{n}}^{\otimes 5-\theta_0}, D^{a_1}\delta^{5-\theta_1}\left(f^{(5)}(\widetilde{B}_{\frac{j_1}{n}})\partial_{\frac{j_1}{n}}^{\otimes 5-\theta_1}\right)\otimes\cdots\otimes D^{a_r}\delta^{5-\theta_r}\left(f^{(5)}(\widetilde{B}_{\frac{j_r}{n}})\partial_{\frac{j_r}{n}}^{\otimes 5-\theta_r}\right)\right\rangle_{\mathfrak{H}^{\otimes 5-\theta_0}}$$

into terms of the form

$$C_\ell f^{(5)}(\widetilde{B}_{\frac{p}{n}})\delta^{b_1}\left(f^{(\lambda_1)}(\widetilde{B}_{\frac{j_1}{n}})\partial_{\frac{j_1}{n}}^{\otimes b_1}\right)\cdots\delta^{b_r}\left(f^{(\lambda_r)}(\widetilde{B}_{\frac{j_r}{n}})\partial_{\frac{j_r}{n}}^{\otimes b_r}\right) \times \left\langle \partial_{\frac{p}{n}},\partial_{\frac{j_1}{n}}\right\rangle_{\mathfrak{H}}^{\ell_1}\left\langle \partial_{\frac{p}{n}},\widetilde{\varepsilon}_{\frac{j_1}{n}}\right\rangle_{\mathfrak{H}}^{a_1-\ell_1}\cdots\left\langle \partial_{\frac{p}{n}},\partial_{\frac{j_r}{n}}\right\rangle_{\mathfrak{H}}^{\ell_r}\left\langle \partial_{\frac{p}{n}},\widetilde{\varepsilon}_{\frac{j_r}{n}}\right\rangle_{\mathfrak{H}}^{a_r-\ell_r},$$

where $C_\ell = C_\ell(\ell_1,\ldots,\ell_r)$ is an integer constant, each $b_i = 5-\theta_i-\ell_i$, and each $\lambda_i = 5+a_i-\ell_i$. It follows that (6.9) is a sum of terms of the form

$$C_\ell C_{\ell'} n^{-2H(\theta_1+\cdots+\theta_r)}\mathbb{E}\sum_{p,p'=0}^{\lfloor nt\rfloor-1} f^{(5)}(\widetilde{B}_{\frac{p}{n}})f^{(5)}(\widetilde{B}_{\frac{p'}{n}}) \times\left(\sum_{j_1=0}^{\lfloor nt\rfloor-1}\delta^{b_1}\left(f^{(\lambda_1)}(\widetilde{B}_{\frac{j_1}{n}})\partial_{\frac{j_1}{n}}^{\otimes b_1}\right)\left\langle \partial_{\frac{p}{n}},\partial_{\frac{j_1}{n}}\right\rangle_{\mathfrak{H}}^{\ell_1}\left\langle \partial_{\frac{p}{n}},\widetilde{\varepsilon}_{\frac{j_1}{n}}\right\rangle_{\mathfrak{H}}^{a_1-\ell_1}\right) \times\cdots\times\left(\sum_{k_r=0}^{\lfloor nt\rfloor-1}\delta^{b'_r}\left(f^{(\lambda'_r)}(\widetilde{B}_{\frac{k_r}{n}})\partial_{\frac{k_r}{n}}^{\otimes b'_r}\right)\left\langle \partial_{\frac{p'}{n}},\partial_{\frac{k_r}{n}}\right\rangle_{\mathfrak{H}}^{\ell'_r}\left\langle \partial_{\frac{p'}{n}},\widetilde{\varepsilon}_{\frac{k_r}{n}}\right\rangle_{\mathfrak{H}}^{a_r-\ell'_r}\right). \quad (6.10)$$

For $0\le j_1,\ldots,j_r\le\lfloor nt\rfloor$ we have the estimate

$$\sum_{p=0}^{\lfloor nt\rfloor-1}\left|\left\langle \partial_{\frac{p}{n}},\partial_{\frac{j_1}{n}}\right\rangle_{\mathfrak{H}}^{\ell_1}\left\langle \partial_{\frac{p}{n}},\widetilde{\varepsilon}_{\frac{j_1}{n}}\right\rangle_{\mathfrak{H}}^{a_1-\ell_1}\cdots\left\langle \partial_{\frac{p}{n}},\partial_{\frac{j_r}{n}}\right\rangle_{\mathfrak{H}}^{\ell_r}\left\langle \partial_{\frac{p}{n}},\widetilde{\varepsilon}_{\frac{j_r}{n}}\right\rangle_{\mathfrak{H}}^{a_r-\ell_r}\right| \le \sup_{\mathscr{I}}\sum_{p=0}^{\lfloor nt\rfloor-1}\prod_{i=1}^{r}\left|\left\langle \partial_{\frac{p}{n}},\partial_{\frac{j_i}{n}}\right\rangle_{\mathfrak{H}}^{\ell_i}\left\langle \partial_{\frac{p}{n}},\widetilde{\varepsilon}_{\frac{j_i}{n}}\right\rangle_{\mathfrak{H}}^{a_i-\ell_i}\right|,$$

where $\mathscr{I}=\{0\le j_1,\ldots,j_r\le\lfloor nt\rfloor\}$. By Lemma 3.1.a and/or 3.1.c, this is bounded by $Cn^{-2H(5-\theta_0)}$ if $\ell_1+\cdots+\ell_r\ge 1$, and bounded by $Cn^{-2H(5-\theta_0-1)} = Cn^{-2H(4-\theta_0)}$ if and only if $\ell_1=\cdots=\ell_r=0$. Hence, we can write

$$\sup_{\mathscr{I},\mathscr{I}'}\sum_{p,p'=0}^{\lfloor nt\rfloor-1}\left|\left\langle \partial_{\frac{p}{n}},\partial_{\frac{j_1}{n}}\right\rangle_{\mathfrak{H}}^{\ell_1}\cdots\left\langle \partial_{\frac{p'}{n}},\widetilde{\varepsilon}_{\frac{k_r}{n}}\right\rangle_{\mathfrak{H}}^{a_r-\ell'_r}\right|\le Cn^{-\Lambda H}, \quad (6.11)$$

where $4H(4-\theta_0) \le \Lambda \le 4H(5-\theta_0)$.

It follows that terms of the form (6.10) can be bounded in absolute value by

$$Cn^{-2H(\theta_0+\cdots+\theta_r)} \sup_{0\le p\le \lfloor nt\rfloor} \|f^{(5)}(\widetilde{B}_{\frac{p}{n}})\|^2_{L^{4r+2}(\Omega)} \sup_{\mathscr{I},\mathscr{I}'} \sum_{p,p'=0}^{\lfloor nt\rfloor -1} \left| \left\langle \partial_{\frac{p}{n}}, \partial_{\frac{j_1}{n}} \right\rangle_{\mathfrak{H}}^{\ell_1} \cdots \left\langle \partial_{\frac{p'}{n}}, \widetilde{\varepsilon}_{\frac{k_r}{n}} \right\rangle_{\mathfrak{H}}^{a_r-\ell'_r} \right|$$
$$\times \prod_{i=1}^{r} \left\| \sum_{j_i=0}^{\lfloor nt\rfloor -1} \delta^{b_i}\left(f^{(\lambda_i)}(\widetilde{B}_{\frac{j_i}{n}}) \partial_{\frac{j_i}{n}}^{\otimes b_i} \right) \right\|_{L^{2r+1}(\Omega)} \left\| \sum_{k_i=0}^{\lfloor nt\rfloor -1} \delta^{b'_i}\left(f^{(\lambda'_i)}(\widetilde{B}_{\frac{k_i}{n}}) \partial_{\frac{k_i}{n}}^{\otimes b'_i} \right) \right\|_{L^{2r+1}(\Omega)}.$$

By (6.11) and Lemma 6.3, this is bounded by

$$C\lfloor nt\rfloor^r n^{-2H(\theta_0+\cdots+\theta_r)-\Lambda H-H(b_1+\cdots+b_r+b'_1+\cdots+b'_r)}.$$

We have $\Lambda \ge 4H(4-\theta_0)$, and

$$b_1+\cdots+b_r = 5r-(\theta_1+\cdots+\theta_r)-(\ell_1+\cdots+\ell_r).$$

Since $\ell_i \le a_i$ for each i, then $\ell_1+\cdots+\ell_r \le a_1+\cdots+a_r = 5-\theta_0$, it follows that the exponent

$$\begin{aligned}&2H(\theta_0+\cdots+\theta_r)+\Lambda H+H(b_1+\cdots+b_r+b'_1+\cdots+b'_r)\\ &\quad\ge 2H(\theta_0+\cdots+\theta_r)+4H(4-\theta_0)+H(10r-2(\theta_1+\cdots+\theta_r)-2(5-\theta_0))\\ &\quad\ge 16H+10(r-1)H \ge 10rH+6H.\end{aligned}$$

Hence, we have an upper bound of

$$C\lfloor nt\rfloor^r n^{-10rH-6H} \le Cn^{-6H}$$

for each term of the form (6.10), so this term tends to zero in $L^2(\Omega)$, and we have (c). This concludes the proof of Lemma 6.4. □

6.3.2 Proof of Lemma 6.5.

Starting with (a), Lemma 2.1.b gives

$$\begin{aligned}\mathbb{E}\left|\left\langle u_n, D^5G_n\right\rangle_{\mathfrak{H}^{\otimes 5}}\right| &= n^{-2H}\mathbb{E}\left| \sum_{i=0}^{3}\binom{5}{i}\binom{3}{i} i! \sum_{j,k=0}^{\lfloor nt\rfloor -1} \left\langle f^{(5)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 5}, \delta^{3-i}\left(f^{(10-i)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 3-i}\right)\partial_{\frac{k}{n}}^{\otimes i}\otimes \widetilde{\varepsilon}_{\frac{k}{n}}^{\otimes 5-i}\right\rangle_{\mathfrak{H}^{\otimes 5}}\right|\\ &\le Cn^{-2H}\sum_{i=0}^{3} \sup_{0\le j\le\lfloor nt\rfloor}\left\|f^{(5)}(\widetilde{B}_{\frac{j}{n}})\right\|_{L^2(\Omega)} \sup_{0\le k\le\lfloor nt\rfloor}\left\|\delta^{3-i}\left(f^{(10-i)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 3-i}\right)\right\|_{L^2(\Omega)}\\ &\qquad\times \sum_{j,k=0}^{\lfloor nt\rfloor -1}\left|\left\langle \partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{i}\left\langle \partial_{\frac{j}{n}},\widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{5-i}\right|.\end{aligned}$$

By moderate growth conditions and (6.4), we have $\left\|f^{(5)}(\widetilde{B}_{\frac{j}{n}})\right\|_{L^2(\Omega)} \le C$ and $\left\|\delta^{3-i}\left(f^{(10-i)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 3-i}\right)\right\|_{L^2(\Omega)} \le C\|\partial_{\frac{1}{n}}\|_{\mathfrak{H}}^{3-i} = Cn^{-(3-i)H}$; so we have terms of the form

$$Cn^{-(5-i)H}\sum_{j,k=0}^{\lfloor nt\rfloor -1}\left|\left\langle \partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{i}\left\langle \partial_{\frac{j}{n}},\widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{5-i}\right|.$$

If $i>0$, then (B.4) and Lemma 3.1.c give an estimate of

$$Cn^{-(5-i)H}\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{i}\left\langle\partial_{\frac{j}{n}},\widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{5-i}\right|$$
$$\leq Cn^{-(15-3i)H}\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{i}\right|\leq C\lfloor nt\rfloor n^{-(15-3i)H}\leq Cn^{-2H},$$

because $i\leq 3$. On the other hand, if $i=0$, then by (B.4) and Lemma 3.1.a,

$$Cn^{-5H}\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{j}{n}},\widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{5}\right|\leq Cn^{-5H}\sum_{k=0}^{\lfloor nt\rfloor-1}\left\{\sup_{0\leq k\leq\lfloor nt\rfloor}\sum_{j=0}^{\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{j}{n}},\widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{5}\right|\right\}$$
$$\leq C\lfloor nt\rfloor n^{-13H}\leq Cn^{-3H},$$

hence (a) is proved.

For (b), again using Lemma 2.1.b we can write

$$\mathbb{E}\left|\left\langle v_n,D^3F_n\right\rangle_{\mathfrak{H}^{\otimes 3}}\right|=n^{-2H}\mathbb{E}\left|\sum_{i=0}^{3}\binom{5}{i}\binom{3}{i}i!\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left\langle f^{(5)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 3},\delta^{5-i}\left(f^{(8-i)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 5-i}\right)\partial_{\frac{k}{n}}^{\otimes i}\otimes\widetilde{\varepsilon}_{\frac{k}{n}}^{\otimes 3-i}\right\rangle_{\mathfrak{H}^{\otimes 3}}\right|$$
$$\leq Cn^{-2H}\sum_{i=0}^{3}\mathbb{E}\left|\sum_{j,k=0}^{\lfloor nt\rfloor-1}f^{(5)}(\widetilde{B}_{\frac{j}{n}})\delta^{5-i}\left(f^{(8-i)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 5-i}\right)\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{i}\left\langle\partial_{\frac{j}{n}},\widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{3-i}\right|.$$

We deal with three cases. First, assume $i=0$. Then we have a bound of

$$Cn^{-2H}\sum_{j,k=0}^{\lfloor nt\rfloor-1}\mathbb{E}\left|f^{(5)}(\widetilde{B}_{\frac{j}{n}})\delta^{5}\left(f^{(8)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 5}\right)\right|\ \left|\left\langle\partial_{\frac{j}{n}},\widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{3}\right|$$
$$\leq Cn^{-2H}\sup_{0\leq j\leq\lfloor nt\rfloor}\left\|f^{(5)}(\widetilde{B}_{\frac{j}{n}})\right\|_{L^2(\Omega)}\sup_{0\leq k\leq\lfloor nt\rfloor}\left\|\delta^{5}\left(f^{(8)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 5}\right)\right\|_{L^2(\Omega)}\sup_{j,k}\left|\left\langle\partial_{\frac{j}{n}},\widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{2}\right|$$
$$\times\sum_{k=0}^{\lfloor nt\rfloor-1}\left\{\sup_{0\leq k\leq\lfloor nt\rfloor}\sum_{j=0}^{\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{j}{n}},\widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|\right\}\leq C\lfloor nt\rfloor n^{-11H}\leq Cn^{-H},$$

where, as above, we use the estimates $\left\|f^{(5)}(\widetilde{B}_{\frac{j}{n}})\right\|_{L^2(\Omega)}\leq C$ and $\left\|\delta^{5}\left(f^{(8)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 5}\right)\right\|_{L^2(\Omega)}\leq Cn^{-5H}$; and

$$\sup_{j,k}\left|\left\langle\partial_{\frac{j}{n}},\widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{2}\right|\sum_{k=0}^{\lfloor nt\rfloor-1}\sum_{j=0}^{\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{j}{n}},\widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|\leq C\lfloor nt\rfloor n^{-4H}$$

follows from (B.4) and Lemma 3.1.a.

The next case is for $i=1$ or $i=2$. Using similar estimates we have

$$Cn^{-2H}\sum_{j,k=0}^{\lfloor nt\rfloor-1}\mathbb{E}\left|f^{(5)}(\widetilde{B}_{\frac{j}{n}})\delta^{5-i}\left(f^{(8-i)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 5-i}\right)\right|\left|\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{i}\left\langle\partial_{\frac{j}{n}},\widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{3-i}\right|$$
$$\leq Cn^{-2H}\sup_{0\leq j\leq\lfloor nt\rfloor}\left\|f^{(5)}(\widetilde{B}_{\frac{j}{n}})\right\|_{L^2(\Omega)}\sup_{0\leq k\leq\lfloor nt\rfloor}\left\|\delta^{5-i}\left(f^{(8-i)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 5-i}\right)\right\|_{L^2(\Omega)}\sup_{j,k}\left|\left\langle\partial_{\frac{j}{n}},\widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{3-i}\right|$$
$$\times\sum_{k=0}^{\lfloor nt\rfloor-1}\sum_{j=0}^{\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{i}\right|\leq C\lfloor nt\rfloor n^{-(7-i+6)H}\leq Cn^{-H},$$

because $7-i+6\geq 11$ for $i\leq 2$.

For the case $i=3$, we will use a different estimate, and show that the term with $i=3$ vanishes in $L^2(\Omega)$. Using Lemma 2.1.d we have,

$$\mathbb{E}\left[\left(n^{-2H}\sum_{j,k=0}^{\lfloor nt\rfloor-1}f^{(5)}(\widetilde{B}_{\frac{j}{n}})\delta^2\left(f^{(5)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 2}\right)\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{3}\right)^2\right]$$
$$=n^{-4H}\sum_{j,j',k,k'=0}^{\lfloor nt\rfloor-1}\mathbb{E}\left[f^{(5)}(\widetilde{B}_{\frac{j}{n}})f^{(5)}(\widetilde{B}_{\frac{j'}{n}})\delta^2\left(f^{(5)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 2}\right)\delta^2\left(f^{(5)}(\widetilde{B}_{\frac{k'}{n}})\partial_{\frac{k'}{n}}^{\otimes 2}\right)\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{3}\left\langle\partial_{\frac{j'}{n}},\partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}}^{3}\right]$$
$$=n^{-4H}\sum_{p=0}^{2}\binom{2}{p}^2p!\sum_{j,j',k,k'}\mathbb{E}\left[g(j,j')\delta^{4-2p}\left(g(k,k')\partial_{\frac{k}{n}}^{\otimes 2-p}\otimes\partial_{\frac{k'}{n}}^{\otimes 2-p}\right)\right]\left\langle\partial_{\frac{k}{n}},\partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}}^{p}\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{3}\left\langle\partial_{\frac{j'}{n}},\partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}}^{3},$$

where $g(j,j')=f^{(5)}(\widetilde{B}_{\frac{j}{n}})f^{(5)}(\widetilde{B}_{\frac{j'}{n}})$. Then by the Malliavin duality (2.6), this results in a sum of three terms of the form

$$Cn^{-4H}\sum_{j,j',k,k'}\mathbb{E}\left[\left\langle D^{4-2p}g(j,j'),g(k,k')\partial_{\frac{k}{n}}^{\otimes 2-p}\otimes\partial_{\frac{k'}{n}}^{\otimes 2-p}\right\rangle_{\mathfrak{H}^{\otimes 4-2p}}\right]\left\langle\partial_{\frac{k}{n}},\partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}}^{p}\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{3}\left\langle\partial_{\frac{j'}{n}},\partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}}^{3},\tag{6.12}$$

for $p=0,1,2$. When the index $p=0$, then $\mathbb{E}\left|\left\langle D^{4-2p}g(j,j'),g(k,k')\partial_{\frac{k}{n}}^{\otimes 2-p}\otimes\partial_{\frac{k'}{n}}^{\otimes 2-p}\right\rangle_{\mathfrak{H}^{\otimes 4-2p}}\right|$ consists of terms of the form

$$\mathbb{E}\left|\left(\frac{\partial^4}{\partial x_1^a\partial x_2^b}\Psi(\widetilde{B}_{\frac{j}{n}},\widetilde{B}_{\frac{j'}{n}})\right)g(k,k')\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{a}\left\langle\widetilde{\varepsilon}_{\frac{j'}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{2-a}\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}}^{b}\left\langle\widetilde{\varepsilon}_{\frac{j'}{n}},\partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}}^{2-b}\right|,\tag{6.13}$$

where $\Psi(x_1,x_2)=f^{(5)}(x_1)f^{(5)}(x_2)$ and $a+b=4$. By moderate growth and (B.4), we see that (6.13) is bounded by Cn^{-8H}, and so for the case $p=0$, (6.12) is bounded in absolute value by

$$Cn^{-12H}\sum_{j,j',k,k'}\left|\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{3}\left\langle\partial_{\frac{j'}{n}},\partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}}^{3}\right|=Cn^{-12H}\left(\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{3}\right|\right)^2$$
$$\leq C\lfloor nt\rfloor^2n^{-24H}\leq Cn^{-4H}.$$

By a similar estimate, when $p = 1$, then

$$\mathbb{E}\left|\left\langle D^2 g(j,j'), g(k,k')\partial_{\frac{k}{n}} \otimes \partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}^{\otimes 2}}\right| \le Cn^{-4H},$$

so that for $p = 1$, then (6.12) is bounded in absolute value by

$$Cn^{-8H} \sum_{j,j',k,k'} \left|\left\langle \partial_{\frac{k}{n}}, \partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}} \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^3 \left\langle \partial_{\frac{j'}{n}}, \partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}}^3\right|$$

$$\le Cn^{-8H} \sup_{k,k'}\left|\left\langle \partial_{\frac{k}{n}}, \partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}}\right| \left(\sum_{j,k=0}^{\lfloor nt\rfloor -1}\left|\left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^3\right|\right)^2 \le C\lfloor nt\rfloor^2 n^{-22H} \le Cn^{-2H}.$$

Last, the term in (6.12) with $p = 2$ has the form

$$Cn^{-4H} \sum_{j,j',k,k'} \mathbb{E}\left[g(j,j')g(k,k')\right] \left\langle \partial_{\frac{k}{n}}, \partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}}^2 \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^3 \left\langle \partial_{\frac{j'}{n}}, \partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}}^3.$$

This is bounded in absolute value by

$$Cn^{-4H} \sup_{0\le j\le \lfloor nt\rfloor}\left\|f^{(5)}(\widetilde{B}_{\frac{j}{n}})\right\|_{L^4(\Omega)}^4 \sum_{k,k'=0}^{\lfloor nt\rfloor -1}\left\langle \partial_{\frac{k}{n}}, \partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}}^2 \sum_{j=0}^{\lfloor nt\rfloor -1}\left|\left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^3\right| \sum_{j'=0}^{\lfloor nt\rfloor -1}\left|\left\langle \partial_{\frac{j'}{n}}, \partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}}^3\right|. \quad (6.14)$$

By Lemma 3.1.c, for every $0 \le k \le \lfloor nt\rfloor$ we have

$$\sum_{j=0}^{\lfloor nt\rfloor -1}\left|\left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^3\right| \le Cn^{-6H},$$

hence (6.14) is bounded by

$$Cn^{-16H}\sum_{k,k'=0}^{\lfloor nt\rfloor -1}\left\langle \partial_{\frac{k}{n}}, \partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}}^2 \le C\lfloor nt\rfloor n^{-20H} \le Cn^{-10H}.$$

Lemma 6.5 is proved. □

Chapter 7

CLT for an iterated integral with respect to fBm with $H > 1/2$

7.1 Introduction

Let $B = \{(B_t^1, \ldots, B_t^q), t \geq 0\}$ be a multidimensional fractional Brownian motion (fBm) with Hurst parameter $H > 1/2$. In this chapter, we study the asymptotic behavior as $k \to \infty$ of multiple stochastic integrals of the particular form:

$$Y_{k^t} := \int_1^{k^t} \int_1^{s_q} \cdots \int_1^{s_2} s_q^{-qH} dB_{s_1}^1 \ldots dB_{s_{q-1}}^{q-1} \, dB_{s_q}^q$$

where $t > 0$ and each iterated integral is a pathwise symmetric integral in the sense of Russo and Vallois [34]. We show that the pathwise symmetric integral is identical to the Malliavin divergence integral in this case. Our main result is a central limit theorem for the process $\{Y_{k^t}, t \geq 0\}$, namely that $\frac{Y_{k^t}}{\sqrt{\log k}}$ converges in distribution as $k \to \infty$ to a scaled Brownian motion. Our approach uses the techniques of Malliavin calculus, where we express Y_{k^t} in terms of the divergence integral δ, which coincides with the multiple Wiener-Itô stochastic integral in this case. In our proof, convergence of finite-dimensional distributions follows from a multi-dimensional version of the Fourth Moment Theorem [29, 31], which gives conditions for weak convergence to a Gaussian random variable (see section 2.4). Functional convergence to a Brownian motion is proved by investigating tightness. In addition to the proof, we are able to comment on the rate of convergence (which is fairly slow: $\sim (\log k)^{-\frac{1}{2}}$), using a result from Nourdin and Peccati [24] in their recent book on the Stein method.

The original motivation for this paper was [3], where Baudoin and Nualart studied a complex-valued fBm with $H > 1/2$. For $B_t = B_t^1 + iB_t^2$, $B_0 = 1$, they studied the integral

$$\int_0^t \frac{dB_s}{B_s} = \int_0^t \frac{B_s^1 dB_s^1 + B_s^2 dB_s^2}{|B_s|^2} + i \int_0^t \frac{B_s^2 dB_s^1 - B_s^1 dB_s^2}{|B_s|^2}. \tag{7.1}$$

When B is written in the form $\rho_t e^{i\theta_t}$, the angle θ_t is given by the imaginary part of (7.1). For standard Brownian motion, a well-known theorem by Spitzer [36] holds that as $t \to \infty$, the random variable $2\theta_t/(\log t)$ converges in distribution to a Cauchy random variable with parameter 1. In the case of Brownian motion, this integral in the complex plane has been studied in several papers [4, 14, 32]. We are not aware of a corresponding fBm version of Spitzer's theorem. In [3], the functional

$$Z_t := \int_1^t \frac{B_s^2 dB_s^1 - B_s^1 dB_s^2}{s^{2H}} \tag{7.2}$$

was proposed as an asymptotic approximation for θ_t as $t \to \infty$. It was shown (see Proposition 22 of [3]) that $\frac{Z_t}{\sqrt{\log t}}$ converges in distribution to a Gaussian random variable, with an expression for variance similar to our own result. Their proof also used Malliavin calculus, but did not use the Fourth Moment Theorem. For $q = 2$, since $B_t = \int_0^t dB_s$, Z_t is asymptotically equal in law to

$$Z_t' = \int_1^t \int_1^s \frac{dB_r^2 dB_s^1}{s^{2H}} - \int_1^t \int_1^s \frac{dB_r^1 dB_s^2}{s^{2H}},$$

and we have a new (and shorter) proof of the result in [3].

7.2 Main result

Fix $q \geq 2$. For $t > 0$ and integer $k \geq 2$, define

$$Y_{k^t} = \int_1^{k^t} \int_1^{s_q} \cdots \int_1^{s_2} s_q^{-qH} dB_{s_1}^1 \dots dB_{s_{q-1}}^{q-1} \, dB_{s_q}^q,$$

where the stochastic integrals are iterated symmetric integrals in the sense of Definition 3.6. Theorem 3.7 and the diagonal structure of Y_{k^t} allow us to identify the pathwise and Skorohod integrals.

Lemma 7.1. *For each $q \geq 2$, we have*

$$Y_{k^t} = \int_1^{k^t} \int_1^{s_q} \cdots \int_1^{s_2} s_q^{-qH} \delta B_{s_1}^1 \dots \delta B_{s_{q-1}}^{q-1} \, \delta B_{s_q}^q. \tag{7.3}$$

Proof. This follows from iterated application of Theorem 3.7, where the correction term is zero due to independence. Indeed, in the notation of (3.8), this is

$$Y_{k^t} = \delta^{(q)} \cdots \delta^{(1)} \left(s_q^{-qH} \mathbf{1}_{\{1 \leq s_1 < \cdots < s_q \leq k^t\}} \right).$$

□

Following is the main result of this section.

Theorem 7.2. *For $t \geq 0$, define*

$$X_k(0) = 0; \quad X_k(t) = \frac{Y_{k^t}}{\sqrt{\log k}}, \; t > 0.$$

Then as $k \to \infty$, the family $\{X_k(t), t \geq 0\}$ converges in distribution to the process $X = \{X(t), t \geq 0\}$, where X is a scaled Brownian motion with variance $\sigma_q^2 t$, and

$$\sigma_2^2 = \alpha_H \int_0^1 x^{-2H} R(1,x)(1-x)^{2H-2} dx; \;\; \textit{and for } q > 2, \tag{7.4}$$

$$\sigma_q^2 = \alpha_H^{q-1} \int_0^1 x_q^{-qH}(1-x_q)^{2H-2} \int_{\mathscr{M}} R(x_2,y_2) \prod_{i=2}^{q-1} |x_i - y_i|^{2H-2}\, dx_2\, dy_2 \dots dy_{q-1}\, dx_q, \tag{7.5}$$
$$\textit{where } \mathscr{M} = \{0 \le x_2 < \cdots < x_q; 0 \le y_2 < \cdots < y_{q-1} \le 1\}.$$

The proof of Theorem 7.2 follows the lemmas in Sections 7.2.1 and 7.2.2. Our first task is to investigate the covariance (Section 7.2.1), then verify two other conditions for weak convergence (Section 7.2.2).

7.2.1 Convergence of the covariance function

Let $A = \{1 \le s_1 < \cdots < s_q \le k^t\}$, and $\Lambda = \{(i_1,\dots,i_q) = (1,\dots,q)\}$. Lemma 7.1 allows us to write $Y_{k^t} = \delta^q(f_{k^t})$, where $f_{k^t} : ([0,\infty) \times \{1,\dots,q\})^q \to \mathbb{R}$ is given by

$$f_{k^t}\left((s_1,i_1),\dots,(s_q,i_q)\right) = s_q^{-qH} \mathbf{1}_A(s_1,\dots,s_q) \mathbf{1}_\Lambda(i_1,\dots,i_q). \tag{7.6}$$

Here, $f_{k^t} \in \mathfrak{H}^{\otimes q}$, where $\mathfrak{H} := \mathfrak{H}_q$ is the Hilbert space associated with a $q-$dimensional fBm (see Section 3.2). Clearly, f_{k^t} is not symmetric. Instead, we will work with the symmetrization defined in (2.1):

$$\widetilde{f}_{k^t}\left((s_1,i_1),\dots,(s_q,i_q)\right) = \frac{1}{q!} \sum_\sigma s_{\sigma(q)}^{-qH} \mathbf{1}_A(s_{\sigma(1)},\dots,s_{\sigma(q)}) \mathbf{1}_\Lambda(i_{\sigma(1)},\dots,i_{\sigma(q)}), \tag{7.7}$$

where σ covers all permutations of $\{1,\dots,q\}$. This gives equivalent results, by the relation $I_q(\widetilde{f}) = I_q(f)$ (see [27], Sec. 1.1.2).

By definition $\widetilde{f}_{k^t}$ is nonzero only if $1 \le s_{\sigma(1)} < \cdots < s_{\sigma(q)} \le k^t$ and $(i_{\sigma(1)},\dots,i_{\sigma(q)}) = (1,\dots,q)$, hence it is possible to express $\widetilde{f}_{k^t}$ without a sum. Let σ be an arbitrary permutation of $\{1,\dots,q\}$, and let $A_\sigma = \{1 \le s_{\sigma(1)}, < \cdots < s_{\sigma(q)} \le k^t\}$. Since the sets $\{A_\sigma\}$ form an almost-everywhere partition of $[1,k^t]^q$, we can write (7.7) as

$$\widetilde{f}_{k^t}\left((s_1,i_1),\dots,(s_q,i_q)\right) = \frac{1}{q!} s_{(q)}^{-qH} \mathbf{1}_{A_1}\left((s_1,i_1),\dots,(s_q,i_q)\right), \tag{7.8}$$

where $s_{(q)} = \max\{s_1,\dots,s_q\}$, and the set A_1 is defined by the following condition: when $s_1,\dots,s_q$ are arranged in $[1,k^t]$ such that $s_{(1)} < \cdots < s_{(q)}$, then $(i_{(1)},\dots,i_{(q)}) = (1,\dots,q)$.

In the next three results, we check the conditions of Theorem 2.6 for $\delta^q(\widetilde{f}_{k^t})$.

Lemma 7.3. *For each $q \ge 2$ and $t > 0$,*

$$t\sigma_q^2 = \lim_{k\to\infty} \mathbb{E}\left[X_k(t)^2\right]$$

exists, where σ_q^2 is given by (7.4) *and* (7.5) *for $q = 2$ and $q > 2$, respectively.*

Proof. Since f_{k^t} is deterministic, we use (2.5) and (3.10):

$$\mathbb{E}\left[X_k(t)^2\right] = \frac{1}{\log k} \mathbb{E}\left[\delta^q(f_{k^t})^2\right] = \frac{q!}{\log k} \left\langle \widetilde{f}_{k^t}, \widetilde{f}_{k^t} \right\rangle_{\mathfrak{H}^{\otimes q}}$$

$$= \frac{\alpha_H^q}{q!\log k} \sum_{i_1,\dots,i_q=1}^{q} \int_{[1,k^t]^{2q}} (r_{(q)}s_{(q)})^{-qH} \mathbf{1}_{A_1}(\mathbf{r},\mathbf{i}) \mathbf{1}_{A_1}(\mathbf{s},\mathbf{i}) \prod_{\ell=1}^{q} |r_\ell - s_\ell|^{2H-2}\, d\mathbf{r}\, d\mathbf{s}, \tag{7.9}$$

where $(\mathbf{r},\mathbf{i}) = \big((r_1,i_i),\dots,(r_q,i_q)\big)$, and similar for $(\mathbf{s},\mathbf{i})$. To evaluate (7.9), we decompose $[1,k^t]^{2q}$ into the union of the sets $\{A_\sigma \times A_{\sigma'}\}$, which form a partition almost everywhere. Since $\mathbf{1}_{A_1}(\mathbf{r},\mathbf{i})$ is nonzero only if $r_{\sigma(1)} < \cdots < r_{\sigma(q)}$ and $(i_{\sigma(1)},\dots,i_{\sigma(q)}) = (1,\dots,q)$, and similar for $\mathbf{1}_{A_1}(\mathbf{s},\mathbf{i})$, it follows that we integrate only over the diagonal sets, that is, when $\sigma = \sigma'$. Hence, (7.9) can be integrated as a sum of $q!$ equal terms, and we have

$$\mathbb{E}\left[X_k(t)^2\right] = \frac{\alpha_H^q}{\log k} \int_{\mathscr{A}} (r_q s_q)^{-qH} \prod_{i=1}^{q} |r_i - s_i|^{2H-2}\, dr_1\, ds_1 \dots dr_q\, ds_q, \tag{7.10}$$

where the integral is over the set

$$\mathscr{A} = \left\{1 \le r_1 < \cdots < r_q \le k^t,\ 1 \le s_1 < \cdots < s_q \le k^t\right\}.$$

Integrating over r_1, s_1, we have by L'Hôpital,

$$\lim_{k\to\infty} \frac{\alpha_H^{q-1}}{\log k} \int_{[1,k^t]^2} (r_q s_q)^{-qH} |r_q - s_q|^{2H-2} \int_{\mathscr{A}} R(r_2,s_2) \prod_{i=2}^{q-1} |r_i - s_i|^{2H-2}\, dr_2\, ds_2 \dots dr_q\, ds_q$$

$$= \lim_{k\to\infty} t k^t \alpha_H^{q-1} \int_1^{k^t} (r_q k^t)^{-qH} (k^t - r_q)^{2H-2} \int_{\mathscr{A}'} R(r_2,s_2) \prod_{i=2}^{q-1} |r_i - s_i|^{2H-2}\, dr_2\, ds_2 \dots ds_{q-1}\, dr_q,$$

where the set $\mathscr{A}' = \{1 \le r_2 < \cdots < r_q,\ 1 \le s_2 < \cdots < s_{q-1} \le k^t\}$ ($\mathscr{A}'$ is empty if $q = 2$). Using the change of variable $r_i = k^t x_i,\ s_i = k^t y_i$, this may be written

$$\lim_{k\to\infty} t\alpha_H^{q-1} \int_{\frac{1}{k^t}}^{1} x_q^{-qH} (1-x_q)^{2H-2} \int_{\mathscr{M}'} R(x_2,y_2) \prod_{i=2}^{q-1} |x_i - y_i|^{2H-2}\, dx_2\, dy_2 \dots dy_{q-1}\, dx_q \tag{7.11}$$

$$= t\alpha_H^{q-1} \int_0^1 x_q^{-qH} (1-x_q)^{2H-2} \int_{\mathscr{M}} R(x_2,y_2) \prod_{i=2}^{q-1} |x_i - y_i|^{2H-2}\, dx_2\, dy_2 \dots dy_{q-1}\, dx_q,$$

where $\mathscr{M}' = \{\frac{1}{k^t} < x_2 < \cdots < x_q; \frac{1}{k^t} < y_2 < \cdots < y_{q-1} \le 1\}$, $\mathscr{M}$ is as in (7.5) for $q > 2$, and we have (7.4) if $q = 2$. To show (7.4) and (7.5) are convergent, we use properties (R.1) and (R.2), so that

$$\sigma_2^2 = \alpha_H \int_0^1 x^{-2H} (1-x)^{2H-2} R(1,x)\, dx \le c_1 \alpha_H \int_0^1 x^{-H} (1-x)^{2H-2}\, dx < \infty$$

and for $q > 2$

$$\sigma_q^2 \le \alpha_H \int_0^1 x_q^{-qH} (1-x_q)^{2H-2} R(1,x_q)^{q-1} dx_q \le c_1^{q-1} \alpha_H \int_0^1 x_q^{-H} (1-x_q)^{2H-2}\, dx_q < \infty.$$

This concludes the proof. □

Lemma 7.4. *Let* $0 \le \tau \le t$. *For each* $q \ge 2$,

$$\lim_{k\to\infty} \mathbb{E}\left[X_k(t)X_k(\tau)\right] = \sigma_q^2 \tau;$$

and consequently $\lim_{k\to\infty} \mathbb{E}\left[X_k(s)X_k(t)\right] = \sigma_q^2 (s \wedge t)$ *for all* $0 \le s,t < \infty$.

Proof.

$$\begin{aligned}\mathbb{E}\left[X_k(t)X_k(\tau)\right] &= \mathbb{E}\left[(X_k(t)-X_k(\tau)+X_k(\tau))X_k(\tau)\right]\\ &= \frac{1}{\log k}\mathbb{E}\left[(Y_{k^t}-Y_{k^\tau})Y_{k^\tau}\right]+\mathbb{E}\left[X_k(\tau)^2\right],\end{aligned}$$

where $\mathbb{E}\left[X_k(\tau)^2\right] \to \sigma_q^2\tau$ by Lemma 7.3. Note that $Y_{k^t}-Y_{k^\tau} = \delta^q(\widetilde{f}_{k^t}) - \delta^q(\widetilde{f}_{k^\tau})$, where, recalling the notation of (7.8),

$$\begin{aligned}\delta^q(\widetilde{f}_{k^t}) - \delta^q(\widetilde{f}_{k^\tau}) &= \int_{[1,k^t]^{2q}} \frac{1}{q!s_{(q)}^{qH}}\mathbf{1}_{A_1}(\mathbf{s},\mathbf{i})\ \delta B_s - \int_{[1,k^\tau]^{2q}} \frac{1}{q!s_{(q)}^{qH}}\mathbf{1}_{A_1}(\mathbf{s},\mathbf{i})\ \delta B_s\\ &= \int_1^{k^t}\int_1^{s_{(q)}}\cdots\int_1^{s_{(2)}} \frac{1}{q!s_{(q)}^{qH}}\delta B^{(1)}_{s_{(1)}}\cdots\delta B^{(q-1)}_{s_{(q-1)}}\ \delta B^{(q)}_{s_{(q)}} - \int_1^{k^\tau}\int_1^{s_{(q)}}\cdots\int_1^{s_{(2)}} \frac{1}{q!s_{(q)}^{qH}}\delta B^{(1)}_{s_{(1)}}\cdots\delta B^{(q-1)}_{s_{(q-1)}}\ \delta B^{(q)}_{s_{(q)}}\\ &= \int_{k^\tau}^{k^t}\int_1^{s_{(q)}}\cdots\int_1^{s_{(2)}} \frac{1}{q!s_{(q)}^{qH}}\delta B^{(1)}_{s_{(1)}}\cdots\delta B^{(q-1)}_{s_{(q-1)}}\ \delta B^{(q)}_{s_{(q)}}.\end{aligned}$$

Hence, we can write $Y_{k^t}-Y_{k^\tau} = \delta^q(\widetilde{f}_{\Delta k})$, where

$$\widetilde{f}_{\Delta k} = \frac{1}{q!s_{(q)}^{qH}}\mathbf{1}_{A_1}\mathbf{1}_{\{k^\tau\le s_{(q)}\le k^t\}} = \widetilde{f}_{k^t}\mathbf{1}_{\{k^\tau\le s_{(q)}\le k^t\}}. \tag{7.12}$$

With this notation, it follows that

$$\begin{aligned}\frac{1}{\log k}\mathbb{E}\left[(Y_{k^t}-Y_{k^\tau})Y_{k^\tau}\right] &= \frac{q!}{\log k}\left\langle \widetilde{f}_{\Delta k}, \widetilde{f}_{k^\tau}\right\rangle_{\mathfrak{H}^{\otimes q}}\\ &= \frac{\alpha_H^q}{q!\log k}\sum_{i_1,\dots,i_q=1}^{q}\int_{[1,k^t]^{2q}}\left(r_{(q)}s_{(q)}\right)^{-qH}\mathbf{1}_{A_1}(\mathbf{r},\mathbf{i})\mathbf{1}_{A_1}(\mathbf{s},\mathbf{i})\mathbf{1}_{\{1\le s_{(q)}\le k^\tau\le r_{(q)}\le k^t\}}\prod_{\ell=1}^{q}|r_\ell-s_\ell|^{2H-2}d\mathbf{s}\,d\mathbf{r}.\end{aligned}$$

As in Lemma 7.3, we decompose $[1,k^t]^{2q}$ into the union of the sets $\{A_\sigma\times A_{\sigma'}\}$. Since $\mathbf{1}_{A_1}(\mathbf{r},\mathbf{i})$ is nonzero only if $r_{\sigma(1)}<\cdots<r_{\sigma(q)}$ and $(i_{\sigma(1)},\dots,i_{\sigma(q)}) = (1,\dots,q)$, and similar for $\mathbf{1}_{A_1}(\mathbf{s},\mathbf{i})$, it follows that we integrate only over the diagonal sets, that is, when $\sigma=\sigma'$. Hence, we have $q!$ equal terms of the form

$$\frac{\alpha_H^q}{q!\log k}\int_{k^\tau}^{k^t}\int_1^{k^\tau}\int_{[1,k^t]^{2q-2}}\left(r_qs_q\right)^{-qH}\mathbf{1}_{\{r_1<\cdots<r_q\}}\mathbf{1}_{\{s_1<\cdots<s_q\}}\prod_{\ell=1}^{q}|r_\ell-s_\ell|^{2H-2}d\mathbf{s}\,d\mathbf{r}. \tag{7.13}$$

By (R.1) and (R.2), for each $r_\ell\le r_q,\ s_\ell\le s_q$, we have the estimate

$$\begin{aligned}&\alpha_H\int_1^{r_{(\ell)}}\int_1^{s_{(\ell)}}|r_{(\ell-1)}-s_{(\ell-1)}|^{2H-2}dr_{(\ell-1)}\ ds_{(\ell-1)}\\ &\le \alpha_H\int_0^{r_{(q)}}\int_0^{s_{(q)}}|r-s|^{2H-2}dr\ ds = R(r_{(q)},s_{(q)}) \le c_1(r_qs_q)^H.\end{aligned} \tag{7.14}$$

It follows that

$$\frac{1}{\log k}\mathbb{E}\left[(Y_{k^t}-Y_{k^\tau})Y_{k^\tau}\right] \le \frac{C}{\log k}\int_{k^\tau}^{k^t}\int_1^{k^\tau}(r_q s_q)^{-qH}R(r_q,s_q)^{q-1}|r_q-s_q|^{2H-2}dr_q\,ds_q$$
$$\le \frac{C}{\log k}\int_{k^\tau}^{k^t}\int_1^{k^\tau}(r_q s_q)^{-2H}R(r_q,s_q)|r_q-s_q|^{2H-2}dr_q\,ds_q.$$

Using the change-of-variable $s_q = k^\tau x,\ r_q = k^\tau y$, this is bounded by

$$\frac{C}{\log k}\int_1^{k^{t-\tau}}\int_0^1 (xy)^{-2H}R(x,y)(y-x)^{2H-2}dx\,dy.$$

Using (R.3), we obtain the estimate,

$$\frac{C}{\log k}\int_1^{k^{t-\tau}}\int_0^1 \left(y^{-2H}(y-x)^{2H-2}+x^{1-2H}y^{-1}(y-x)^{2H-2}\right)\,dx\,dy,$$

where

$$\int_1^{k^{t-\tau}}\int_0^1 y^{-2H}(y-x)^{2H-2}dx\,dy \le \int_1^2 y^{-2H}\int_0^y (y-x)^{2H-2}dx\,dy+\int_2^{k^{t-\tau}}(y-1)^{-2}dy$$
$$\le C\int_1^2 y^{-1}dy + C\int_1^\infty y^{-2}dy<\infty,$$

and

$$\int_1^{k^{t-\tau}}\int_0^1 y^{-1}x^{1-2H}(y-x)^{2H-2}dx\,dy \le \int_1^2 y^{-1}\int_0^y x^{1-2H}(y-x)^{2H-2}dx\,dy+\int_2^{k^{t-\tau}}(y-1)^{2H-3}dy$$
$$\le C\int_1^2 y^{-1}dy + \int_1^\infty y^{2H-3}dy<\infty.$$

Hence, this term vanishes and Lemma 7.4 is proved. □

7.2.2 Conditions for weak convergence of $\{X_k(t)\}$

In the next two lemmas we verify additional properties of $\{X_k(t)\}$. In Lemma 7.5 we check condition (iv) of Theorem 2.6, and Lemma 7.6 is a tightness result.

Lemma 7.5. *Fix $q\ge 2$ and $t>0$. For each integer $1\le p\le q-1$,*

$$\lim_{k\to\infty}(\log k)^{-2}\|\widetilde{f}_{k^t}\otimes_p \widetilde{f}_{k^t}\|^2_{\mathfrak{H}^{\otimes 2(q-p)}}=0.$$

Proof. Let $1\le p\le q-1$. To compute the p^{th} contraction of $\widetilde{f}_{k^t}$, we use (3.11).

$$\widetilde{f}_{k^t}\otimes_p \widetilde{f}_{k^t} = \frac{\alpha_H^p}{(q!)^2}\sum_{i_1,\dots,i_q=1}^{q}\int_{[1,k^t]^{2p}}(r_{(q)}s_{(q)})^{-qH}\mathbf{1}_{A_1}(\mathbf{r},\mathbf{i})\mathbf{1}_{A_1}(\mathbf{s},\mathbf{i})\prod_{\ell=1}^{p}|r_\ell-s_\ell|^{2H-2}dr_1\,ds_1\dots dr_p\,ds_p. \tag{7.15}$$

Using (7.15), we want to compute

$$\|\widetilde{f}_{k^t} \otimes_p \widetilde{f}_{k^t}\|^2_{\mathfrak{H}^{\otimes 2(q-p)}} = \left\langle \widetilde{f}_{k^t} \otimes_p \widetilde{f}_{k^t}, \widetilde{f}_{k^t} \otimes_p \widetilde{f}_{k^t} \right\rangle_{\mathfrak{H}^{\otimes 2(p-q)}}$$

$$= \frac{\alpha_H^{2q}}{(q!)^4} \sum_{i_1,\dots,i_q=1}^{q} \int_{[1,k^t]^{4q}} \left(r_{(q)} s_{(q)} r'_{(q)} s'_{(q)}\right)^{-qH} \left(\mathbf{1}_{A_1}(\mathbf{r},\mathbf{i}) \cdots \mathbf{1}_{A_1}(\mathbf{s}',\mathbf{i})\right) \prod_{j=1}^{p} \left(|r_j - s_j||r'_j - s'_j|\right)^{2H-2}$$
$$\times \prod_{j=p+1}^{q} \left(|r_j - r'_j||s_j - s'_j|\right)^{2H-2} \, d\mathbf{r}\, d\mathbf{s}\, d\mathbf{r}' d\mathbf{s}'. \quad (7.16)$$

As in the proof of Lemma 7.3, we view integration over the set $[1,k^t]^{4q}$ as a sum of integrals over various cases corresponding to the orderings of the real variables $r_1,\dots,r_q$ (as in Lemma 7.3, the variables $\mathbf{s},\mathbf{r}',\mathbf{s}'$ must follow the same ordering). Up to permutation of indices, each integral term has the form

$$\frac{\alpha_H^{2q}}{(q!)^4} \int_{\mathscr{G}} (r_{(q)} s_{(q)} r'_{(q)} s'_{(q)})^{-qH} \prod_{j=1}^{p} \left(|r_j - s_j|\, |r'_j - s'_j|\right)^{2H-2} \prod_{j=p+1}^{q} \left(|r_j - r'_j|\, |s_j - s'_j|\right)^{2H-2} d\mathbf{r}\, d\mathbf{s}\, d\mathbf{r}'\, d\mathbf{s}', \quad (7.17)$$

where $\mathscr{G} = \left\{1 \le r_{(1)} < \cdots < r_{(q)} \le k^t; \dots; 1 \le s'_{(1)} < \cdots < s'_{(q)} \le k^t\right\}$. To evaluate (7.17), there are two cases to consider. The first case is if $r_{(q)} \in \{r_1,\dots,r_p\}$, that is, (7.17) contains the terms $|r_{(q)} - s_{(q)}|, |r'_{(q)} - s'_{(q)}|$. In this case, using (7.14) we can bound (7.17) by

$$\frac{\alpha_H^2}{(q!)^4} \int_{[1,k^t]^4} (r_{(q)} s_{(q)} r'_{(q)} s'_{(q)})^{-qH} \left(R(r_{(q)}, s_{(q)}) R(r'_{(q)}, s'_{(q)})\right)^{p-1} \left(R(r_{(q)}, r'_{(q)}) R(s_{(q)}, s'_{(q)})\right)^{q-p}$$
$$\times \left(|r_{(q)} - s_{(q)}|\, |r'_{(q)} - s'_{(q)}|\right)^{2H-2} dr_{(q)}\, ds_{(q)}\, dr'_{(q)}\, ds'_{(q)}$$
$$\le C \int_{[1,k^t]^4} (r s r' s')^{-2H} R(r,r')\, R(s,s') \left(|r-s|\, |r'-s'|\right)^{2H-2} dr\, ds\, dr'\, ds', \quad (7.18)$$

where we used (R.2) in the last estimate. The second case is the complement, that is, $r_{(q)} \in \{r_{p+1},\dots,r_q\}$, so that (7.17) contains the terms $|r_{(q)} - r'_{(q)}|, |s_{(q)} - s'_{(q)}|$. If this is the case, then (7.17) is bounded by

$$\frac{\alpha_H^2}{(q!)^4} \int_{[1,k^t]^4} (r_{(q)} s_{(q)} r'_{(q)} s'_{(q)})^{-qH} \left(R(r_{(q)}, s_{(q)}) R(r'_{(q)}, s'_{(q)})\right)^{p} \left(R(r_{(q)}, r'_{(q)}) R(s_{(q)}, s'_{(q)})\right)^{q-p-1}$$
$$\times \left(|r_{(q)} - s_{(q)}|\, |r'_{(q)} - s'_{(q)}|\right)^{2H-2} dr_{(q)}\, ds_{(q)}\, dr'_{(q)}\, ds'_{(q)}$$
$$\le C \int_{[1,k^t]^4} (r s r' s')^{-2H} R(r,s)\, R(r',s') \left(|r-r'|\, |s-s'|\right)^{2H-2} dr\, ds\, dr'\, ds'. \quad (7.19)$$

The result then follows by a change of variable and applying Lemma 7.8 to (7.18) and (7.19). □

Lemma 7.6. *There is a constant $0 < C < \infty$ such that for each $k \geq 2$ and any $0 \leq \tau < t < \infty$ we have*

$$\mathbb{E}\left[|X_k(t) - X_k(\tau)|^4\right] \leq C(t-\tau)^2.$$

Proof. Based on the hypercontractivity property (2.8), it is enough to show

$$\mathbb{E}\left[|X_k(t) - X_k(\tau)|^2\right] \leq C(t-\tau).$$

Using the notation of (7.12), we can write

$$\mathbb{E}\left[|X_k(t) - X_k(\tau)|^2\right] = \frac{1}{\log k}\mathbb{E}\left[|Y_{k^t} - Y_{k^\tau}|^2\right] = \frac{q!}{\log k}\left\langle \widetilde{f}_{\Delta k}, \widetilde{f}_{\Delta k}\right\rangle_{\mathfrak{H}^{\otimes q}}$$
$$= \frac{\alpha_H^q}{q!\log k}\sum_{i_1,\dots,i_q=1}^{q}\int_{[1,k^t]^{2q}}\left(r_{(q)}s_{(q)}\right)^{-qH}\mathbf{1}_{A_1}(\mathbf{r},\mathbf{i})\mathbf{1}_{A_1}(\mathbf{s},\mathbf{i})\mathbf{1}_{\{k^\tau\leq r_{(q)},s_{(q)}\leq k^t\}}\prod_{\ell=1}^{q}|r_\ell - s_\ell|^{2H-2}d\mathbf{s}\,d\mathbf{r}.$$

In the same manner as (7.13), this can be decomposed into a sum of $q!$ equal terms of the form

$$\frac{\alpha_H^q}{q!\log k}\int_{k^\tau}^{k^t}\int_{k^\tau}^{k^t}\int_{[1,k^t]^{2q-2}}\left(r_q s_q\right)^{-qH}\mathbf{1}_{\{r_1\leq\cdots\leq r_q\}}\mathbf{1}_{\{s_1\leq\cdots\leq s_q\}}\prod_{\ell=1}^{q}|r_\ell - s_\ell|^{2H-2}d\mathbf{s}\,d\mathbf{r}.$$

Similar to Lemma 7.5, we use (7.14) and a change-of-variable to obtain

$$\frac{1}{\log k}\mathbb{E}\left[|Y_{k^t} - Y_{k^\tau}|^2\right] \leq \frac{C}{\log k}\int_{k^\tau}^{k^t}\int_{k^\tau}^{k^t}(r_q s_q)^{-qH}R(r_q,s_q)^{q-1}|r_q - s_q|^{2H-2}dr_q\,ds_q$$
$$\leq \frac{C}{\log k}\int_{k^{\tau-t}}^{1}\int_{k^{\tau-t}}^{1}(xy)^{-2H}R(x,y)|x-y|^{2H-2}dx\,dy.$$

Without loss of generality, assume $x < y$. By (R.3), we have the estimate

$$\frac{C}{\log k}\int_{k^{\tau-t}}^{1}\int_{k^{\tau-t}}^{1}(xy)^{-2H}R(x,y)|x-y|^{2H-2}dx\,dy = \frac{C}{\log k}\int_{k^{\tau-t}}^{1}\int_{k^{\tau-t}}^{y}(xy)^{-2H}R(x,y)|x-y|^{2H-2}dx\,dy$$
$$\leq \frac{C}{\log k}\int_{k^{\tau-t}}^{1}\int_{0}^{y}\left(y^{-2H}(y-x)^{2H-2} + x^{1-2H}y^{-1}(y-x)^{2H-2}\right)dx\,dy$$
$$\leq \frac{C}{\log k}\int_{k^{\tau-t}}^{1}\left(y^{-2H}y^{2H-1} + y^{-1}\right)\,dy \leq C(t-\tau).$$

This concludes the proof. □

7.2.3 Proof of Theorem 7.2

Fix integers $q \geq 2$ and $d \geq 1$, and choose a set of times $0 \leq t_1 < \cdots < t_d$. Lemmas 7.3 and 7.4 show that the random vector sequence $\{(X_k(t_1),\dots,X_k(t_d)), k \geq 1\}$ meets the covariance conditions of Theorem 2.6. Moreover, Lemma 7.5 verifies condition (iv) of Theorem 2.6. Therefore, we conclude that as $k \to \infty$,

$$(X_k(t_1),\dots,X_k(t_d)) \xrightarrow{\mathscr{L}} (X(t_1),\dots,X(t_d)), \tag{7.20}$$

where each $X(t_i)$ has distribution $\mathcal{N}(0, \sigma_q^2 t_i)$, and $\mathbb{E}[X(t_i)X(t_k)] = \sigma_q^2(t_i \wedge t_k)$ for all $1 \le i,k \le d$. By Lemma 7.6, the sequence $\{X_k(t)\}$ is tight, hence it follows from (7.20) that the sequence converges in the sense of finite-dimensional distributions (see, for example, Theorem 13.5 of [6]). Thus, we conclude that the family $\{X_k(t), t \ge 0\}$ converges in distribution to the process $\{X(t), t \ge 0\} \stackrel{\mathcal{L}}{=} \{\sigma_q W_t, t \ge 0\}$, where W_t is a standard Brownian motion. This concludes the proof of Theorem 7.2.

7.2.4 Rate of convergence

Let $t > 0$ be fixed. By Theorem 7.2, it follows that the sequence $\{X_k(t), k \ge 1\}$ converges in distribution to a random variable $N(t)$, where $N(t) \sim \mathcal{N}(0, \sigma_q^2 t)$. Recent work by Nourdin and Peccati [24] has produced a stronger form of the Fourth Moment Theorem for the 1-dimensional case, that is, that the conditions of the Fourth Moment Theorem also imply convergence in the sense of total variation (as well as other metrics - see Theorem 5.2.6). The result below follows from Corollary 5.2.10 of [24].

Proposition 7.7. *Let $t \ge 0$. Then for sufficiently large k, there is a constant $0 < C < \infty$ such that*

$$d_{TV}(X_k(t), N(t)) \le \frac{C}{\sqrt{\log k}},$$

where $d_{TV}(\cdot,\cdot)$ is total variation distance. Hence $X_k(t)$ converges as $k \to \infty$ to Gaussian in the sense of total variation.

Proof. The result follows from an estimate in [24] (Cor. 5.2.10):

$$d_{TV}(X_k(t), N(t)) \le 2\sqrt{\frac{\mathbb{E}[X_k(t)^4] - 3\mathbb{E}[X_k(t)^2]^2}{3\mathbb{E}[X_k(t)^2]^2}} + \frac{2\left|\mathbb{E}\left[X_k(t)^2\right] - \sigma_q^2 t\right|}{\mathbb{E}[X_k(t)^2] \vee \sigma_q^2 t}. \tag{7.21}$$

To simplify notation, we will assume $t = 1$. To help interpret this estimate, the following identity is computed in [24] (see Lemma 5.2.4):

$$\mathbb{E}\left[X_k(1)^4\right] - 3\mathbb{E}\left[X_k(1)^2\right]^2 = \frac{3}{q(\log k)^2} \sum_{p=1}^{q-1} p(p!)^2 \binom{q}{p}^4 (2q-2p)! \|\widetilde{f}_k \,\widetilde{\otimes}_p\, \widetilde{f}_k\|^2_{\mathfrak{H}^{\otimes 2(q-p)}}. \tag{7.22}$$

From Lemma 7.5, we know $(\log k)^{-2}\|\widetilde{f}_k \,\widetilde{\otimes}_p\, \widetilde{f}_k\|^2_{\mathfrak{H}^{\otimes 2(q-p)}} \to 0$ at a rate $C/\log k$, hence it follows the first term of (7.21) is of order $C(\log k)^{-\frac{1}{2}}$. The second term depends on the covergence rate of (7.10). In the proof of Lemma 7.3, convergence follows from a limit of the form $\mathbb{E}\left[Y_k^2\right]/\log k$. By L'Hôpital's rule, it follows the rate of convergence has the form $C/\log k$, hence the first term controls. $\square$

7.3 A technical lemma

Lemma 7.8. *Fix* $T>0$. *Let* $1/2<H<1$, *and for nonnegative* x,y, *let* $R(x,y)=\frac{1}{2}\left(x^{2H}+y^{2H}-|x-y|^{2H}\right)$. *Then there is a constant* $0<K<\infty$ *such that*

$$\int_{[\frac{1}{T},1]^4}(xyuv)^{-2H}R(x,y)\,R(u,v)|x-u|^{2H-2}|y-v|^{2H-2}dx\,dy\,du\,dv\leq K\log T.$$

Proof. In the following computations, we will obtain estimates based on the order of integration. Due to the symmetries of the integral, it is enough to consider four distinct cases. We will make frequent use of (R.3), and for a second estimate, note that for $x<y<u$ we can write $(u-x)^{2H-2}\leq(u-y)^{-\alpha}(y-x)^{-\beta}$, where $\alpha,\beta>0$ satisfy $\alpha+\beta=2-2H$.

Case 1: $x\leq y\leq u\leq v$ We can write

$$\begin{aligned}
&\int_{\frac{1}{T}}^{1}\int_{\frac{1}{T}}^{v}(uv)^{-2H}R(u,v)\int_{\frac{1}{T}}^{u}y^{-2H}(v-y)^{2H-2}\int_{\frac{1}{T}}^{y}x^{-2H}R(x,y)(u-x)^{2H-2}\,dx\,dy\,du\,dv\\
&\leq C\int_{\frac{1}{T}}^{1}\int_{\frac{1}{T}}^{v}(uv)^{-2H}R(u,v)\int_{\frac{1}{T}}^{u}y^{-2H}(v-y)^{2H-2}(u-y)^{-\alpha}\int_{\frac{1}{T}}^{y}(y-x)^{-\beta}+x^{1-2H}y^{2H-1}(y-x)^{-\beta}\,dx\ldots dv\\
&\leq C\int_{\frac{1}{T}}^{1}\int_{\frac{1}{T}}^{v}(uv)^{-2H}R(u,v)(v-u)^{-\alpha}\int_{\frac{1}{T}}^{u}y^{1-2H-\beta}(u-y)^{-\beta-\alpha}\,dy\,du\,dv\\
&\leq C\int_{\frac{1}{T}}^{1}\int_{\frac{1}{T}}^{v}(uv)^{-2H}R(u,v)(v-u)^{-\alpha}u^{-\beta}\,du\,dv\\
&\leq C\int_{\frac{1}{T}}^{1}v^{-2H}\int_{\frac{1}{T}}^{v}u^{-2H}(v-u)^{-\alpha}\left(u^{2H}+uv^{2H-1}\right)\,du\,dv\\
&\leq C\int_{\frac{1}{T}}^{1}v^{-1}dv\leq K\log T.
\end{aligned}$$

Case 2: $x<y<v<u$ For this case, we use constants $\alpha,\beta>0$ such that $\alpha+\beta=2H-2$, and $\gamma,\delta>0$ such that $\gamma+\delta=\alpha$.

$$\begin{aligned}
&\int_{\frac{1}{T}}^{1}\int_{\frac{1}{T}}^{u}(uv)^{-2H}R(u,v)\int_{\frac{1}{T}}^{u}y^{-2H}(v-y)^{2H-2}\int_{\frac{1}{T}}^{y}x^{-2H}R(x,y)(u-x)^{2H-2}\,dx\,dy\,dv\,du\\
&\qquad\leq C\int_{\frac{1}{T}}^{1}\int_{\frac{1}{T}}^{u}(uv)^{-2H}R(u,v)\int_{\frac{1}{T}}^{u}y^{1-2H-\beta}(v-y)^{2H-2}(u-y)^{-\alpha}\,dy\,dv\,du\\
&\qquad\leq C\int_{\frac{1}{T}}^{1}\int_{\frac{1}{T}}^{u}(uv)^{-2H}R(u,v)(u-v)^{-\gamma}\int_{\frac{1}{T}}^{u}y^{1-2H-\beta}(v-y)^{2H-2-\delta}\,dy\,dv\,du\\
&\qquad\leq C\int_{\frac{1}{T}}^{1}u^{-2H}\int_{\frac{1}{T}}^{u}v^{-2H-\beta-\delta}\left(v^{2H}+vu^{2H-1}\right)(u-v)^{-\gamma}\,dv\,du\\
&\qquad\leq C\int_{\frac{1}{T}}^{1}u^{-1}\leq K\log T.
\end{aligned}$$

Case 3: $x < u < y < v$

$$\int_{\frac{1}{T}}^{1}\int_{\frac{1}{T}}^{v}(yv)^{-2H}(v-y)^{2H-2}\int_{\frac{1}{T}}^{y}u^{-2H}R(u,v)\int_{\frac{1}{T}}^{u}x^{-2H}R(x,y)(u-x)^{2H-2}\,dx\,du\,dy\,dv$$
$$\le C\int_{\frac{1}{T}}^{1}\int_{\frac{1}{T}}^{v}(yv)^{-2H}(v-y)^{2H-2}\int_{\frac{1}{T}}^{y}u^{-2H}\left(u^{2H}+uv^{2H-1}\right)\left(u^{2H-1}+y^{2H-1}\right)\,du\,dy\,dv$$
$$\le C\int_{\frac{1}{T}}^{1}\int_{\frac{1}{T}}^{v}(yv)^{-2H}(v-y)^{2H-2}\int_{\frac{1}{T}}^{y}\left(u^{2H-1}+y^{2H-1}+v^{2H-1}+u^{1-2H}(vy)^{2H-1}\right)\,du\,dy\,dv$$
$$\le C\int_{\frac{1}{T}}^{1}\int_{\frac{1}{T}}^{v}(yv)^{-2H}(v-y)^{2H-2}\left(y^{2H}+yv^{2H-1}\right)\,dy\,dv$$
$$\le C\int_{\frac{1}{T}}^{1}v^{-1}\,dv\le K\log T.$$

Case 4: $x < v < u < y$

$$\int_{\frac{1}{T}}^{1}\int_{\frac{1}{T}}^{y}(uy)^{-2H}\int_{\frac{1}{T}}^{u}v^{-2H}R(u,v)(y-v)^{2H-2}\int_{\frac{1}{T}}^{v}x^{-2H}R(x,y)(u-x)^{2H-2}\,dx\,dv\,du\,dy$$
$$\le C\int_{\frac{1}{T}}^{1}\int_{\frac{1}{T}}^{y}(uy)^{-2H}\int_{\frac{1}{T}}^{u}v^{-2H}R(u,v)(y-v)^{2H-2}(u-v)^{-\alpha}\int_{\frac{1}{T}}^{v}x^{-2H}\left(x^{2H}+xy^{2H-1}\right)(v-x)^{-\beta}\,dx\,dv\,du\,dy$$
$$\le C\int_{\frac{1}{T}}^{1}\int_{\frac{1}{T}}^{y}(uy)^{-2H}(y-u)^{-\alpha}\int_{\frac{1}{T}}^{u}v^{-2H}\left(v^{2H}+vu^{2H-1}\right)(u-v)^{-\alpha-\beta}\left(v^{1-\beta}+v^{2-2H-\beta}y^{2H-1}\right)\,dv\,du\,dy$$
$$\le C\int_{\frac{1}{T}}^{1}\int_{\frac{1}{T}}^{y}(uy)^{-2H}(y-u)^{-\alpha}\left(u^{2H-\beta}+y^{2H-1}u^{1-\beta}\right)\,du\,dy$$
$$\le C\int_{\frac{1}{T}}^{1}y^{-2H}\left(y^{1-\alpha-\beta}+y^{2H-1}\right)\,dt$$
$$\le C\int_{\frac{1}{T}}^{1}y^{-1}\,dy\le K\log T.$$

□

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