Bifurcation Analysis of Endogenous Growth Models

By

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Abstract

It is important to recognize that bifurcation boundaries do not necessarily separate stable from unstable solution domains. Bifurcation boundaries can separate one kind of unstable dynamics domain from another kind of unstable dynamics domain, or one kind of stable dynamics domain from another kind, such as monotonic stability from damped periodic stability or from damped multiperiodic stability. There are not only an infinite number of kinds of unstable dynamics, some very close to stability in appearance, but also an infinite number of kinds of stable dynamics. Hence subjective prior views on whether the economy is or is not stable provide little guidance without mathematical analysis of model dynamics.

The thesis analyzes, within its feasible parameter space, the dynamics of the Uzawa-Lucas endogenous growth model. We examine the stability properties of both centralized and decentralized versions of the model and locate Hopf and transcritical bifurcation boundaries. In an extended analysis, we investigate the existence of Andronov-Hopf bifurcation, branch point bifurcation, limit point cycle bifurcation, and period doubling bifurcations. While these all are local bifurcations, the presence of global bifurcation is confirmed as well. We find evidence that the model could produce chaotic dynamics, but our analysis cannot confirm that conjecture.

Further this thesis analyses the dynamics of a variant of Jones semi-endogenous growth model "Sources of US Economic growth in a World of Ideas" The American Economic Review, March 2002, Vol 92 No. 1, pp 220-239. A detailed bifurcation analysis is done within the feasible parameter space of the models. We showed the existence of codimension-1 bifurcations (Hopf, Branch Point, Limit Point of Cycles, and Period Doubling). In addition some codimension-2 (Bogdanov-Takens and Generalized Hopf) bifurcations are detected in the modified Jones model.

While the aforementioned are all local bifurcations, the Uzawa-Lucas model also shows the presence of global bifurcation.

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First and foremost, I would like to thank my advisor, Prof. William A. Barnett, for accepting me as his PhD student and giving me the motivation to start and complete this research. I highly value his steadfast support and guidance without which this thesis would have been a daunting journey. My special thanks also goes to my committee members Prof. John Keating, Prof. Weishi Liu, Prof. Erik Van Vleck and Prof. Shu Wu for encouraging me to think through the problem in greater details. I am also grateful to our supporting staff – Teri Chambers, Michelle Huslig and Leanea Wales. Finally, I would like to thank my family and friends who have provided the support to my commitment to the research.

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Chapter 1. Introduction

The Uzawa-Lucas (Uzawa (1965) and Lucas (1988)) model is among the most important endogenous growth models. It has two sectors producing human capital and physical capital, respectively. Individuals spend certain amount of their time in producing goods and devote the remainder in training themselves. They have the same level of work qualification and expertise (H).

The social planner solution for the Uzawa-Lucas model is saddle path stable, however, the representative agent's equilibrium exhibits indeterminacy, as shown by Benhabib and Perli (1994). The presence of externalities in human capital results in the market solution being different from the social planner solution. The externality leads to differences between return on capital, as perceived by the representative agent, and return on capital from a social point of view.

We solve for the steady states and provide a detailed bifurcation analysis of the model. The task of this thesis is to examine whether the dynamics of the model change within the feasible parameter space of the model. A system undergoes a bifurcation, if a small, smooth change in a parameter produces a sudden qualitative or topological change in the nature of singular points and trajectories of the system. The presence of bifurcation damages the inference robustness of the dynamics, when inferences are based on point estimates of the model. Hence, knowing the stability boundaries within the feasible region of the parameter space, especially near the point estimates, can lead to more robust inferences and more reliable policy conclusions.

Using Mathematica, we locate transcritical and Hopf bifurcation boundaries in two-dimension and three-dimension diagrams. The numerical continuation package, Matcont, is used to analyze further the stability properties of the limit cycles generated by Hopf bifurcations and the presence of other kinds of bifurcations. We show the existence of Hopf, branch-point, limit-point-of-cycles, and period-doubling bifurcations within the feasible parameters set of the model's parameter space. While these are all local bifurcations, presence of global bifurcation is also confirmed. There is some evidence of the possibility of chaotic dynamics through the detected series of period-doubling bifurcations. Some of these results have not previously been demonstrated in the literature on endogenous growth models.

In Uzawa- Lucas model, it is the human capital formation itself that, by non-decreasing marginal returns, creates endogenous growth. On the other hand, Romer(1990) proposed the idea of growth being driven by technological change that results from research and development of profit maximizing agents and led the foundation for endogenous growth models. These models suggest that the long-run growth rate of per capita income should be rising with the increases in R&D intensity or time spent accumulating skills.

Knowledge can be used by many people simultaneously without any loss. This indicates the presence of increasing returns to scale in production associated with any new idea which in turn depends on population (number of researchers). This is the "strong" scale effect of the first generation idea based growth models (Romer(1990) and Grossman and Helpman(1991)) where the growth rate of the economy is an increasing function of scale (population). Contrary to these results, the US data shows that the economy is fluctuating around its balanced growth path although educational attainment and research intensity has been steadily rising. Jones(2002) model tries to explain these facts using a model that exhibits "weak" scale effects. Jones found that long-run growth arises from the worldwide discovery of ideas, which depends on *rate* of population growth of the countries contributing to world research rather than the population. Such models are often called semi-endogenous growth models.

We incorporate a human capital accumulation in Jones model where technological progress (the invention of new varieties of intermediate goods) can positively, negatively or not influence at all the investment in skill acquisition. Compared to Bucci (2008), we introduce the possibility of decreasing returns to scale associated with human capital itself and time spent accumulating it in the human capital production equation. Along the balanced growth path of this modified Jones model, the long run growth can be even positive with no population growth. Hence reinforcing Bucci's (2008) result that economic growth is no longer semi-endogenous and is ultimately driven by private incentives to invest in human capital.

We showed the existence of codimension-1 bifurcations (Andronov-Hopf, Branch Point, Limit Point of Cycles, and Period Doubling). In addition some codimension-2 (Bogdanov-Takens and Generalized Hopf) bifurcations are detected in the modified Jones model.

Stability analysis is critical in understanding the dynamics of the model. Benhabib and Perli (1994) analyzed the stability property of the long-run equilibrium in the Lucas (1988) model. Arnold (2000a,b) analyzed the stability of equilibrium in the Romer (1990) model. Arnold (2006) has done the same for the Jones (1995) model. Mondal (2008) examined the dynamics of the Grossman-Helpman (1991b) model of endogenous product cycles. The results derived in these papers provide important insights to researchers considering different policies. But a detailed bifurcation analysis has not been provided so far for many of these popular endogenous growth models. The current dissertation thesis aims to fill this gap.

As pointed out by Banerjee et al (2011): "Just as it is important to know for what parameter values a system is stable or unstable, it is equally important to know the nature of stability (e.g. monotonic convergence, damped single periodic convergence, or damped multi-periodic convergence) or instability (periodic, multi-periodic, or chaotic)." Barnett and his coauthors have made significant contribution in this area. Barnett and He (1999, 2001, 2002) examined the dynamics of the Bergstrom-Wymer continuous-time dynamic macroeconometric model of the UK economy. Both transcritical bifurcation boundaries and the Hopf bifurcation boundaries for the model were found. Barnett and He (2008) have found singularity bifurcation boundaries within the parameter space for Leeper and Sims (1994) model. Barnett and Duzhak (2010) found Hopf and period doubling bifurcations in a New Keynesian model. More recently, Banerjee et al (2011) examined the possibility of cyclical behavior in the Marshallian Macroeconomic Model.

Chapter 2 and 3 discuss the advantages of continuous time models and use of non-linear dynamics in economics, respectively. Chapter 4 provides survey of work done in bifurcation and chaos in economics. Chapter 5 describes the Uzawa-Lucas model and the derivations of the dynamic equations for the centralized and the decentralized economy. In Chapter 6 we discuss the possibility of the existence of various bifurcations in the model. Chapter 7 describes the modified Jones model and its balanced growth path with Chapter 8 presenting the bifurcation analysis for the model. Finally, Chapter 9 concludes the thesis.

Chapter 2. Advantages of Continuous Time Models

This section provides a very brief overview of the advantages of continuous time models used in macroeconomics (Bergstrom, 1996). The first advantage is that a continuous time model can account better the interaction between the variables during the unit observation period. Most macroeconomic variables are measured at discrete intervals like quarterly or annually. But in reality the variables adjust at much shorter random intervals due to economic agents making uncoordinated decisions at different points of time. This fact is completely ignored in discrete time models. Furthermore, economic theory provides information on the particular interactions of these variables. If the sample size is small it is important that one uses all this information for the purpose of estimation. This again can be accomplished using continuous time models.

The second advantage is the ability of continuous time models to represent a causal system. In a causal system variables responds directly to stimulus provided by a proper subset only of the other variables of the model even though all variables interact during the unit observation period. Causal chain models are able to take account of the a priori information regarding the causal orderings of variables. For example, consider the case of aggregate consumer expenditure on a particular day. In this case variables known to the consumer (such as the personal income, personal assets and prices for that particular day) will affect expenditure. Variables such as exports, imports or investments for that day will not affect expenditure. The use of this information can reduce variance of parameter estimates greatly but to do this efficiently, one would need to use continuous time models.

The third advantage of a continuous time models is that they allow for more accurate representation of the partial adjustment processes in dynamic disequilibrium model. A typical equation in a dynamic disequilibrium model have two parts. The first part can be derived from microeconomic equilibrium theory which relates the partial equilibrium level of the causally dependent variable to a proper subset of other variables in the model. The second part is a differential equation of first or higher order, representing the adjustment of the causally dependent variable in response to the deviation of its current level from its partial equilibrium level.

The fourth advantage is that a continuous time model provides the basis for a more accurate estimation of the distributed lags with which each variable depends on the variables on which it is directly causally dependent.

The standard estimation procedures for discrete time models treat stock and flow variables in the same manner thus leading to bias due to specification error. The stock variables (e.g. money supply or stock of capital) are measured in points of time whereas the flow variables (e.g. output and consumption) are measured as integrals. Hence the fifth advantage is that the procedures for estimating continuous time models can distinguish between stock and flow variables.

The sixth advantage is that the form of a continuous time model does not depend on the unit observation period. Discrete time models are not flexible enough since the form of any particular discrete time model will depend on the unit observation period. This is a serious drawback due to many different types of data available. However, continuous time models are not affected by this drawback as they do not depend on the observation period. This is an advantage for econometricians who generally work on available data rather than choose the observation period. Moreover even if variables are observable at discrete intervals of time, continuous time models can be used to generate continuous time paths for such variables.

A seventh advantage is that a continuous time model can be used to generate forecasts of the continuous time paths of the variables. Such forecasts are of considerable value even though the variables are actually observable only at discrete intervals of time. For example, forecast of the continuous time path of the gross domestic product could be used by businessmen for sales forecasting or by the government for policy formulation.

Chapter 3. Non-linear Dynamics in Economics

Most economic phenomena are non-linear which are represented in continuous time by systems of non-linear differential equations. For these non-linear differential equations it is extremely difficult or in many cases impossible to find a solution even though the existence and uniqueness theorem for a solution is satisfied. The analytical study of differential equations follows two main approaches: the qualitative and the quantitative (Gandalfo 2008). Under the qualitative (or topological) approach the properties of the solutions of a differential equation (system) studied without actually knowing the solution itself by examining phase diagrams, Liapunov's second method, etcetera. The quantitative approach however, requires to find the explicit analytical solution of the differential equation or to approximate it by using power series and other methods. The qualitative approach is hence very important in economic dynamics since in many cases we do not know the exact form of the functions involved in the model. Having emphasized this my thesis focuses on bifurcation and chaos theory as they are an integral part of non-linear qualitative dynamics

Grandmont (1985) found that the parameter space of even the most simple classical models can have stable solutions or more complex solution like cycle or chaos. This is because the parameter space are stratified into different bifurcation regions. Due to the presence of bifurcation the dynamics of the model change within the feasible parameter space of the model. A system undergoes a bifurcation, if a small, smooth change in a parameter produces a sudden qualitative or topological change in the nature of singular points and trajectories of the system. The presence of bifurcation damages the inference robustness of the dynamics, when inferences are based on point estimates of the model. Hence, knowing the stability boundaries within the feasible region of the parameter space, especially near the point estimates, can lead to more robust dynamical inferences and more reliable policy conclusions. Hence it is important that the modeler is aware of the existence of such boundaries inside the confidence intervals of parameters point estimates.

Local bifurcations can be analyzed entirely through the local stability properties of equilibria or periodic orbits or other invariant sets of a system when parameters cross a critical value. The mathematical technique for examining local bifurcations involves linearizing a non-linear system around its equilibrium or steady state. In doing so we make an implicit assumption that the qualitative feature of the linearized system represents the qualitative feature of the original system in a very small neighbourhood of the equilibrium. That is we calculate and evaluate the Jacobian matrix of the original system at the equilibrium and then study the eigenvalues of the Jacobian. It is important to recognize that bifurcation boundaries do not necessarily separate stable from unstable solution domains. Bifurcation boundaries can separate one kind of unstable dynamics domain from another kind of unstable dynamics domain, or one kind of stable dynamics domain from another kind, such as monotonic stability from damped periodic stability or from damped multiperiodic stability. There are not only an infinite number of kinds of unstable dynamics, some very close to stability in appearance, but also an infinite number of kinds of stable dynamics. Hence subjective prior views on whether the economy is or is not stable provide little guidance without mathematical analysis of model dynamics.

Global bifurcations often occur when larger invariant sets of the system 'collide' with each other, or with equilibria of the system. This causes changes in the topology of the trajectories in the phase space which cannot be confined to a small neighbourhood, as is the case with local bifurcations. The changes in topology extend out to an arbitrarily large distance (hence 'global').

Chaos occurs when the dynamical systems are aperiodic and exhibit sensitive dependence on initial conditions. Sensitive dependence on initial conditions means that for a very small change in the initial state will have progressively larger changes in later system states. As initial states are not known exactly in real-world systems, the time evolution of the dynamical system appears random.

Chapter 4: Bifurcation and Stability Analysis in Economics

This section provides a brief survey of the literature in bifurcation and stability analysis done for various economic models. Benhabib and Perli (1994) analyzed the stability property of the long-run equilibrium in the Lucas (1988) model. Arnold (2000a,b) analyzed the stability of equilibrium in the Romer (1990) model. Arnold demonstrated that the steady-state solution of the optimal growth problem in Romer's (1990) model is globally saddle-point stable. He showed that a unique and monotonic growth path converges to the steady state. Furthermore, there is no indeterminacy in the system as already shown by Benhabib, Perli and Xie. Hence instability and cyclical behavior are ruled out as the equilibrium growth path is well behaved. Arnold showed that the optimal growth path can be attained as a market equilibrium through a (unique) combination of production and R&D subsidies. Arnold (2006) has done the steady state analysis for the Jones (1995) model. He provided an analytical treatment of the model's transitional dynamics. He showed that with constant returns to labor in R&D, a unique trajectory converging to balance growth path exists. The equilibrium growth path can be monotonic or oscillatory. Mondal (2008) provided the local stability analysis of the Grossman-Helpman (1991b) model of endogenous product cycles. He showed that there exists a unique saddle path converging to the steady state growth equilibrium in two versions of the model.

The results derived in these papers provide important insights to researchers considering different policies. But a detailed bifurcation analysis has not been provided so far for many of these popular endogenous and semi-endogenous growth models. My work aims to fill this gap for the two endogenous growth models, the Uzawa-Lucas model and a modified version of Jones model.

As pointed out by Banerjee et al (2011): "Just as it is important to know for what parameter values a system is stable or unstable, it is equally important to know the nature of stability (e.g. monotonic convergence, damped single periodic convergence, or damped multi-periodic convergence) or instability (periodic, multi-periodic, or chaotic)." Barnett and his coauthors have made significant contribution in this area. Barnett and Chen (1988) and Barnett, Gallant, Hinich, Jungeilges, Kaplan and Jensen (1997) have tested for chaos and for other forms of non-linearity in univariate time series. Barnett and He (1999, 2001, 2002) examined the dynamics of the Bergstrom-Wymer continuous-time dynamic macroeconometric model of the UK economy to answer if stabilization policy would indeed result in stability. Both transcritical bifurcation boundaries and the Hopf bifurcation boundaries for the model were found. They have numerically constructed bifurcation boundaries that intersect with the statistical confidence regions for the model. The model shows that the policy conclusions drawn from the model could even be wrong if bifurcation boundaries are not accounted for. Barnett and He (2008) have found singularity bifurcation boundaries within the parameter space for Leeper and Sims (1994) model. Barnett and Duzhak (2010) found Hopf and period doubling bifurcations in a New Keynesian model. More recently, Banerjee et al (2011) examined the possibility of cyclical behavior in the Marshallian Macroeconomic Model.

Onozakia, Sieg and Yokood (2003) provide an example in which a slight behavioral heterogeneity can fundamentally change the dynamical properties of the model. It is a nonlinear cobweb market model with a quadratic cost function and an isoelastic demand function and two types of producers, cautious adapters and naïve optimizers. They showed that in a market of naive optimizers a single cautious adapter stabilizes the otherwise exploding market. In a market of cautious adapters a single naive optimizer may destabilize the market and there may appear many (possibly infinite) coexisting periodic attractors. Hornmesa, Nusse and Simonovits (1995), deals with the dynamics of a continuous piecewise linear model of a socialist economy. The model has Hicksian-type nonlinearities which means that it is a linear model with ceilings and floors. In addition to stable cycles the model demonstrates possibility of chaotic and quasi-periodic behavior. Border-crossing bifurcations exists inside the confidence region of the parameters point estimates.

Hommes, Huang, Wang (2005) investigate the dynamics in an adaptive evolutionary asset pricing model with fundamentalists, trend followers and a market maker. Agents can choose between a fundamentalist strategy or choose a trend following strategy. Agents asynchronously update their strategy according to realized net profits in the recent past. They showed that when agents become more sensitive to differences in strategy performance, the steady state becomes unstable and multiple steady states may arise. As the traders' sensitivity to differences in fitness increases, a bifurcation route to chaos sets in due to homoclinic bifurcations of stable and unstable manifolds of the fundamental steady state.

Deissenberg and Nyssen (1998) study global dynamics of a discrete-time model of endogenous growth with a market for the resource used for innovation-creating investment. They show that the level of investment may fluctuate chaotically for a compact range of model parameters, as a consequence of the explicit intermediation and market imperfections created by the temporary monopoly power that the firm achieves following an innovation.

Chiarella C. and Flaschel P. (1998) investigate an open monetary growth model with sluggish prices and quantities. The model combines the dynamics of Rose's employment cycle and Metzler's inventory cycle with internal nominal dynamics of Tobin and external nominal dynamics of Dornbusch type. These intrinsically nonlinear system demonstrate asymptotically stable dynamics for low adjustment speeds of prices and expectations. But Hopf-bifurcations emerges as adjustment parameters are increased and we may have explosive behavior with a further change in adjustment parameter. Nishimura and Yano (1995) demonstrated the possibility of ergodically chaotic optimal accumulation in the case in which future utilities are discounted arbitrarily weakly. In a two-sector model with Leontief production functions with a condition such that the optimal transition function is unimodal and expansive, they showed that the set of parameter values satisfying that condition is nonempty no matter how weakly the future utilities are discounted. Nishimura and Mitra (2001) paper studies the relationship between the discount rate and the nature of long-run behavior in dynamic optimization models under two conditions. The first is history independence, which rules out multiple limit sets. The second is a condition that avoids the reversion to a stable steady state, as the discount factor is lowered, once cycles have emerged.

Mitra (2000) provides a sufficient condition for topological chaos for unimodal maps which can be satisfied when the well-known Li-Yorke condition is not satisfied. He shows how this result can be applied to a model of endogenous growth with externalities to establish the existence of chaotic equilibrium growth paths in that framework. Mitra,(1996) explores the precise extent of discounting needed to generate period-three cycles in a standard aggregative dynamic optimization framework. He showed that there is a ``universal constant'', M, such that (i) if an optimal program of any dynamic optimization model exhibits a period-three cycle, then the discount factor is less than M and (ii) if the discount factor is smaller than M, then it is possible to construct a transition possibility set and a utility function such that the resulting dynamic optimization model exhibits a period-three cycle.

Benhabib and Nishimura (1979) study a multisector optimal growth model and use bifurcation analysis to show that the optimal growth path (steady state) becomes a closed orbit for some values of the discount rate within the theoretical feasible region. Nishimura and Takahashi (1992) consider a multisector neoclassical optimal growth model and show that for a given discount factor, Hopf bifurcations can happen based on the factor intensity within each sector.

Mathematical Definitions of Bifurcations

Some of the important mathematical concept used in my dissertation are presented in this section (Kuznetsov, 1998 and Matcont , 2006)

Hyperbolic Equilibrium in Continuous Time

Consider a continuous time dynamical system defined by

$$\dot{x} = f(x), x \in \mathbb{R}^n$$

where f is smooth. Let $x_0 = 0$ be an equilibrium of the system and let A denote the Jacobian matrix $\frac{df}{dx}$, evaluated at x_0. Letn_-,n_0, n_+, be the number of eigen values of A (counting multiplicities) with negative, zero and positive real parts respectively.

Definition 4.1 An equilibrium is called hyperbolic if $n_0 = 0$, that is if there are no eigen- values on the imaginary axis.

Consider two dynamical systems:

$$\dot{x} = f(x, \alpha), x \in \mathbb{R}^n, \alpha \in \mathbb{R}^m \tag{4.1}$$

$$\dot{y} = f(y, \beta), y \in \mathbb{R}^n, \beta \in \mathbb{R}^m \tag{4.2}$$

with smooth right hand sides and the same number of variables and parameters

Definition 4.2 Dynamical systems (4.1) is topologically equivalent to dynamical system (4.2) if

a. there exists a homeomorphism of the parameter space

$$p : \mathbb{R}^m \to \mathbb{R}^m, \beta = p(\alpha)$$

b. there is a parameter-dependent homeomorphism of the phase space

$$h_\alpha : \mathbb{R}^n \to \mathbb{R}^n, y = h_\alpha(x)$$

$h_\alpha(x)$, mapping orbits of the system (4.2) at parameter values $\beta = p(\alpha)$, preserving the direction of time.

Definition 4.3 The appearance of a topologically non-equivalent phase portrait under variation of parameters is called a bifurcation.

Sufficiently small perturbations of parameters do not lead to changes in structural stability of a hyperbolic equilibrium. Thus bifurcation of equilibrium takes place at only non-hyperbolic points.

Consider a continuous time dynamical system that depends on parameters represented

As

$$(4.3) \qquad \dot{x} = f(x, \alpha), x \in \mathbb{R}^n, \alpha \in \mathbb{R}^m$$

where x represents phase variables and α represents parameters respectively.

Definition 4.4 The codimension of a bifurcation in system (4.3) is the difference between the dimension of the parameter space and the dimension of the corresponding bifurcation boundary.

A more practical definition of codimension as in Kuznetsov [1998] is the number of independent conditions determining the bifurcation boundary.

Consider a continuous time system depending on a parameter α

$$(4.4) \qquad \dot{x} = f(x,\alpha), x \in \mathbb{R}^n, \alpha \in \mathbb{R}^m$$

where f is smooth with respect to both x and α.

Definition 4.5 The bifurcation associated with the appearance of $\lambda_1 = 0$ is called a saddle- node or fold bifurcation.

Here λ_1 is a simple real eigenvalue of the system.

Definition 4.6 The bifurcation corresponding to the presence of $\lambda_{1,2} = \mp i\omega_0, \omega_0 > 0$, is called a Hopf(or Andronov-Hopf) bifurcation.

Here $\lambda_{1,2}$ are the complex conjugate eigenvalues of the continuous time dynamical system.

Note that unlike for the three codimension one bifurcations, namely fold, transcritical and pitchfork, Hopf bifurcation requires at least a 2X2 system.

Now I will outline the mathematical concepts used by Matcont for detecting bifurcations. Some Mathematical definition

Consider a differential equation

$$\frac{du}{dt} = f\,(u,\alpha), u \in \mathbb{R}^n, \alpha \in \mathbb{R}, \quad f: \mathbb{R}^{n+1} \to \mathbb{R}^n$$

We are interested in an equilibrium curve, i.e. $f\,(u,\alpha) = 0$. The defining function is therefore:

$$F(x) = f\,(u,\alpha) = 0$$

with $x = (u,\alpha) \in \mathbb{R}^{n+1}$. Denote by $v \in \mathbb{R}^{n+1}$ the tangent vector to the equilibrium curve at x.

Bifurcations

In continuous-time systems there are two generic codimension 1 bifurcations that can be detected along the equilibrium curve

- fold, also known as limit point (LP)
- Hopf-point, denoted by H

The equilibrium curve can also have branch points. These are denoted with BP. To detect these singularities, we first define 3 test functions:

$$\phi_1(u\ \alpha) = det\begin{pmatrix} F_x \\ v^T \end{pmatrix}$$

$$\phi_2(u\ \alpha) = \left(\begin{bmatrix} (2f_u(u,\alpha)\odot I_n) & w_1 \\ {v^T}_1 & d \end{bmatrix} \backslash \begin{pmatrix} 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix}\right)_{n+1} \qquad (4.5)$$

$$\phi_2(u\ \alpha) = v_{n+1}$$

where $\odot$ is the bialternate matrix product and v_1, w_1 are $\frac{n(n-1)}{2}$ vectors chosen so that the square matrix in (4.5) is non-singular. Using these test functions we can define the singularities:

- BP: $\phi_1 = 0$
- H: $\phi_2 = 0$
- LP: $\phi_3 = 0, \phi_1 \neq 0$

A saddle-node bifurcation is a collision and disappearance of two equilibria in dynamical systems. In systems generated by autonomous ODEs, this occurs when the critical equilibrium has one zero eigenvalue. This phenomenon is also called fold or limit point bifurcation.

Andronov-Hopf bifurcation is the birth of a limit cycle from an equilibrium in dynamical systems generated by ODEs, when the equilibrium changes stability via a pair of purely imaginary eigenvalues. The bifurcation can be supercritical or subcritical, resulting in stable or unstable (within an invariant two-dimensional manifold) limit cycle, respectively.

Continuation of limit cycles

Consider the following differential equation

$$\frac{du}{dt} = f(u, \alpha), u \in \mathbb{R}^n, \alpha \in \mathbb{R}$$

A periodic solution with period T satisfies the following system

$$\begin{cases} \dfrac{du}{dt} = f(u, \alpha) \\ u(0) = u(T) \end{cases}$$

For simplicity the period T is treated as a parameter resulting in the system

$$\begin{cases} \dfrac{du}{d\tau} = T\, f\,(u,\alpha) \\ u(0) = u(1) \end{cases} \qquad (4.6)$$

If u(τ) is its solution then the shifted solution u(τ +s) is also a solution to (4.6) for any value of s. To select one solution, a phase condition is added to the system. The complete BVP (boundary value problem) is

$$\begin{cases} \dfrac{du}{d\tau} - T\, f\,(u,\alpha) = 0 \\ u(0) - u(1) = 0 \\ \int_0^1 \langle u(t), \dot{u}_{old}(t)\rangle\, dt = 0 \end{cases} \qquad (4.7)$$

where $\dot{u}_{old}$ is the derivative of a previous solution. A limit cycle is a closed phase orbit corresponding to this periodic solution.

Bifurcations

On a limit cycle curve the following bifurcations can occur

- Branch Point of Cycles, this will be denoted as BPC
- Period Doubling, denoted as PD
- Fold, also known as Limit Point of Cycles, this will be denoted as LPC
- Neimark-Sacker, this will be denoted as NS

Continuation of limit cycles from the Hopf point, can give rise to limit point (Fold/ Saddle Node) bifurcation of cycles. From the family of limit cycles bifurcating from the Hopf point, limit point cycle (LPC) is a fold bifurcation, where two limit cycles with different periods are present near the LPC point. Continuing computation from a Hopf point may also give rise to period doubling (flip) bifurcations. Period doubling bifurcation is defined as a situation in which a new limit cycle emerges from an existing limit cycle, and the period of the new limit cycle is twice that of the old one.

The test function for the Period Doubling bifurcation is defined by the following system

$$\begin{cases} \dot{v}(\tau) - Tf_u(u,\alpha)v(\tau) + G\phi(\tau) = 0 \\ v(0) + v(1) = 0 \\ \int_0^1 \langle \psi(\tau), v(\tau) \rangle \, d\tau = 1 \end{cases}$$

here ϕ and ψ are so-called bordering vector-functions [Kuznetsov, 1998]. The system is discretized using orthogonal collocation and solved using the standard matlab sparse system solver. The solution component G ∈ R of this system is the test function and equals zero when there is a Period Doubling bifurcation.

The Fold bifurcation is detected in the same way as the Fold bifurcation of equilibria, the last component of the tangent vector (the α component) is used as the test function. The Neimark-Sacker bifurcation is detected by monitoring the eigenvalues of the monodromy matrix for the cycle. The monodromy matrix is computed by a block elimination in the discretized form of the Jacobian of (4.7).

BPC cycles are not generic in families of limit cycles, but they are common in the case of symmetries, if the branch parameter is also the continuation parameter. CL MatCont uses a strategy that requires only the solution of linear systems; it is based on the fact that in a symmetry-breaking BPC cycle M_D has rank defect two, where M_D is the square matrix M_D, obtained from the discretized form of the Jacobian of (4.7). To be precise, if $h \epsilon C^1([0,1], \mathbb{R}^n)$then

$$M_h = \begin{bmatrix} \dot{h} - Tf_x(x(t), \alpha)h \\ h(0) - h(1) \end{bmatrix}$$

And

$$M_D(h)_{d_m} = \begin{bmatrix} (\dot{h} - Tf_x(x(t), \alpha)h)_{d_c} \\ h(0) - h(1) \end{bmatrix}$$

where $()d_m$ and $()d_c$ denote discretization in mesh points and in collocation points, respectively.

Therefore we border M_D with two additional rows and columns to obtain

$$M_{Dbb} = \begin{pmatrix} M_D & w_1 & w_2 \\ v_1^* & 0 & 0 \\ v_2^* & 0 & 0 \end{pmatrix}$$

so that M_{Dbb} is nonsingular in the BPC cycle. Then we solve the systems

$$M_{Dbb} \begin{pmatrix} \psi_{11} & \psi_{12} \\ gBPC_{11} & gBPC_{12} \\ gBPC_{21} & gBPC_{22} \end{pmatrix} = \begin{pmatrix} 0_{(Nm+1)n} & 0_{(Nm+1)n} \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where ψ_{11}, ψ_{12} have *(Nm + 1)n* components, and $gBPC_{11}, gBPC_{12}, gBPC_{21}$ and $gBPC_{22}$ are scalar test functions for the BPC. In the BPC cycle they all vanish.

The singularity matrix is

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & - & - & - & - \\ - & - & - & - & - & 0 & - & - \\ - & - & - & - & - & - & 0 & - \\ - & - & - & - & 1 & - & 1 & 0 \end{pmatrix}$$

The first row corresponds to the BPC. It contains 4 zeros which indicates that $gBPC_{11}, gBPC_{12}, gBPC_{21}$ and $gBPC_{22}$should vanish. The last row corresponds to the NS. Because we have to exclude that all four testfunctions of the BPC are zeros, we introduce an extra testfunction which corresponds to the norm of these four testfunctions. A NS is detected if this norm is nonzero,the test function for the fold is nonzero and the testfunction for the NS is equal to zero.

Hopf Continuation to detect Codimension-2 Bifurcations

In the MatCont / CL MatCont toolbox Hopf curves are computed by minimally extended defining systems, The Hopf curve is defined by the following system

$$\begin{cases} f(u,\alpha) = 0 \\ g_{i_1 j_1}(u,\alpha,k) = 0 \\ g_{i_2 j_2}(u,\alpha,k) = 0 \end{cases}$$

with the unknowns $u, \alpha, k, (i_1, j_1, i_2, j_2) \in \{1, 2\}$ and where $g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ is obtained by

solving

$$\begin{pmatrix} f_u^2 + kI_n & W_{bor} \\ V_{bor}^T & 0 \end{pmatrix} \begin{bmatrix} V \\ G \end{bmatrix} = \begin{bmatrix} 0_{n,2} \\ I_2 \end{bmatrix} \quad (4.4)$$

Where f_u has eigenvalues $\pm i\omega, \omega > 0$, k = ω^2 and $V_{bor}, W_{bor} \in R^{nx2}$ are chosen such that the matrix in (4.4) is nonsingular. $i_1, j_1, i_2, j_2, V_{bor}$ and W_{bor} are auxiliary variables that can be adapted

Bifurcations

In continuous-time systems there are four generic codimension 2 bifurcations that can be detected along the Hopf curve:

- Bogdanov - Takens. We will denote this bifurcation by BT
- Zero - Hopf point, denoted by ZH
- Double - Hopf point, denoted by HH
- Generalized Hopf point, denoted by GH

The Bogdanov-Takens bifurcation is a bifurcation of an equilibrium point in a two-parameter family of autonomous ODEs at which the critical equilibrium has a zero eigenvalue of (algebraic) mulitplicity two. For nearby parameter values, the system has two equilibria (a saddle and a nonsaddle) which collide and disappear via a saddle-node bifurcation. The nonsaddle equilibrium undergoes an Andronov-Hopf bifurcation generating a limit cycle. This cycle degenerates into an orbit homoclinic to the saddle and disappears via a saddle homoclinic bifurcation.

The Generalized Hopf (Bautin) bifurcation is a bifurcation of an equilibrium at which the critical equilibrium has a pair of purely imaginary eigenvalues and the first Lyapunov coefficient for the Andronov-Hopf bifucation vanishes. The bifurcation point separates branches of sub- and supercritical Andronov-Hopf bifurcations in the parameter plain. For nearby parameter values, the system has two limit cycles which collide and disappear via a saddle-node bifurcation of periodic orbits.

The Zero -Hopf bifurcation is a bifurcation of an equilibrium point in a two-parameter family of autonomous ODEs at which the critical equilibrium has a zero eigenvalue and a pair of purely imaginary eigenvalues

To detect these singularities, we first define 4 test functions:

$$\phi_1 = \mathrm{k}$$

$$\phi_2 = det(f_u)$$

$$\phi_3 = \left(\begin{bmatrix} (2f_u(u,\alpha) \odot I_n) & w_1 \\ {v^T}_1 & d \end{bmatrix} \backslash \begin{pmatrix} 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix} \right)_{n+1}$$

$$\phi_4 = l_1 (\text{first Lyapunov coefficient})$$

where $v_1, w_1$are carefully constructed and updated $\frac{n(n-1)}{2} X2$ matrices.

In this case the singularity matrix is:

$$S = \begin{pmatrix} 0 & 0 & - & - \\ 1 & 0 & - & - \\ - & - & 0 & - \\ - & - & - & 0 \end{pmatrix}$$

Chapter 5. The Uzawa Lucas Model

The production function in the physical sector is defined as follows:

$$Y = AK^{\alpha}(\varepsilon hL)^{1-\alpha}h_a^{\zeta}, \qquad 0 < \alpha < 1,$$

where Y is output, A is constant technology level, K is physical capital, α is the share of physical capital, L is labor, and h is human capital per person . In addition, ε and $(1-\varepsilon)$ are the fraction of labor time devoted to produce output and human capital, respectively, where $0 < \varepsilon < 1$. εhL is the quantity of labor in efficiency units employed to produce output. h_a^{ζ} measures the externality associated with average human capital of the labor force, h_a, where ζ is the positive externality parameter in the production of human capital. In per capita terms, $y = Ak^{\alpha}(\varepsilon h)^{1-\alpha}h_a^{\zeta}$.

The physical capital accumulation equation is $\dot{K} = AK^{\alpha}(\varepsilon hL)^{1-\alpha}h_a^{\zeta} - C - \delta K$. In per capita terms, $\dot{k} = Ak^{\alpha}(\varepsilon h)^{1-\alpha}h_a^{\zeta} - c - (n+\delta)k$, and the human capital accumulation equation is $\dot{h} = \eta h(1-\varepsilon)$, where η is defined as schooling productivity.

The decision problem is

$$\max_{c_t,\varepsilon_t} \int_t^{\infty} e^{-(\rho-n)t} \; [c(\tau)^{1-\sigma} - 1] \,/\, (1-\sigma) \quad dt \qquad (1)$$

subject to

$$\dot{k} = Ak^{\alpha}(\varepsilon h)^{1-\alpha}h_a^{\zeta} - c - (n+\delta)k$$

and

$$\frac{\dot{h}}{h} = \eta(1-\varepsilon),$$

where $\rho\ (\rho > n > 0)$ is the subjective discount rate, and $\sigma\ (\geq 0)$ is the inverse of the intertemporal elasticity of substitution in consumption.

5.1. Social Planner Problem

The social planner takes into account the externality associated with human capital when solving the maximization problem (1) subject to physical capital accumulation equation and the human capital accumulation equation. From the first order conditions (see Appendix 1), we derive the equations describing the economy of the Uzawa-Lucas model from a social planner's perspective

$$\frac{\dot{k}}{k} = Ak^{\alpha-1}\varepsilon^{1-\alpha}h^{1-\alpha+\zeta} - \frac{c}{k} - (n+\delta)$$

$$\frac{\dot{h}}{h} = \eta(1-\varepsilon)$$

$$\frac{\dot{c}}{c} = \frac{\alpha Ak^{\alpha-1}\varepsilon^{1-\alpha}h^{1-\alpha+\zeta} - (\rho+\delta)}{\sigma}$$

$$\frac{\dot{\varepsilon}}{\varepsilon} = \eta\frac{(1-\alpha+\zeta)}{(1-\alpha)}\varepsilon + \eta\frac{(1-\alpha+\zeta)}{\alpha} - \frac{c}{k} + \frac{(1-\alpha)}{\alpha}(n+\delta)$$

$$\frac{\dot{L}}{L} = n$$

Let $m = \frac{Y}{K} = Ak^{\alpha-1}\varepsilon^{1-\alpha}h^{1-\alpha+\zeta}$ & $g = \frac{c}{k}$. Taking logarithms of m and g and differentiating with respect to time, the following 2 equations define the dynamics of Uzawa Lucas model

$$(I) \qquad \frac{\dot{m}}{m} = -(1-\alpha)m + \frac{(1-\alpha)}{\alpha}(n+\delta) + \eta\frac{(1-\alpha+\zeta)}{\alpha}$$

$$(II) \qquad \frac{\dot{g}}{g} = \left(\frac{\alpha}{\sigma} - 1\right) m - \frac{\rho}{\sigma} - \delta\left(\frac{1}{\sigma} - 1\right) + g + n$$

The steady state (m^*, g^*) is given by $\dot{m} = \dot{g} = 0$ and derived to be

$$m^* = \eta \frac{(1 - \alpha + \zeta)}{\alpha} + \frac{(n + \delta)}{\alpha}$$

$$g^* = \frac{\rho - n}{\sigma} + \frac{(1 - \alpha)}{\alpha}(n + \delta) + \eta \frac{(1 - \alpha + \zeta)}{\alpha(1 - \alpha)} \frac{(\sigma - \alpha)}{\sigma}$$

A unique steady state exists, if

$$\Lambda = \frac{(1-\alpha+\zeta)}{\alpha}(\sigma - 1)\eta(1 - \varepsilon) + \rho > 0.$$

as Λ is the necessary and sufficient for the transversality condition to hold for the consumer's utility maximization problem**.** Following the footsteps of Barro and Sala-i-Martín (2003) and Mattana (2004), it can be shown that social planner solution is saddle path stable. We linearize around the steady state, $s^* = (m^*, g^*)$, to analyze the local stability properties of the system (I) and (II). The result is

$$\begin{bmatrix} \dot{m} \\ \dot{g} \end{bmatrix} = \underbrace{\begin{bmatrix} \left.\frac{\partial \dot{m}}{\partial m}\right|_{s^*} & \left.\frac{\partial \dot{m}}{\partial g}\right|_{s^*} \\ \left.\frac{\partial \dot{g}}{\partial m}\right|_{s^*} & \left.\frac{\partial \dot{g}}{\partial g}\right|_{s^*} \end{bmatrix}}_{J_s} \begin{bmatrix} m_t - m^* \\ g_t - g^* \end{bmatrix}$$

where

$$J_s = \begin{bmatrix} -(1 - \alpha)m^* & 0 \\ \left(\frac{\alpha}{\sigma} - 1\right) g^* & g^* \end{bmatrix}$$

As $m^*, g^* > 0$, it follows that $\mathrm{Det}(J_s) = -(1-\alpha)m^*g^* < 0$, hence saddle path stable.

5.2. Representative Agent Problem

From the first order conditions (see Appendix 2) and setting $h = h_a$, we derive the equations describing the economy of the Uzawa-Lucas model from a decentralized economy's perspective.

$$\frac{\dot{k}}{k} = Ak^{\alpha-1}\varepsilon^{1-\alpha}h^{1-\alpha+\zeta} - \frac{c}{k} - (n+\delta)$$

$$\frac{\dot{h}}{h} = \eta(1-\varepsilon)$$

$$\frac{\dot{c}}{c} = \frac{\alpha Ak^{\alpha-1}\varepsilon^{1-\alpha}h^{1-\alpha+\zeta} - (\rho+\delta)}{\sigma}$$

$$\frac{\dot{\varepsilon}}{\varepsilon} = \eta\frac{(\alpha-\zeta)}{\alpha}\varepsilon + \eta\frac{(1-\alpha+\zeta)}{\alpha} - \frac{c}{k} + \frac{(1-\alpha)}{\alpha}(n+\delta)$$

$$\frac{\dot{L}}{L} = n$$

Let $m = \frac{Y}{K}$ and $g = \frac{c}{k}$. Taking logarithms of m and g and differentiating with respect to time, the following 3 equations define the dynamics of Uzawa Lucas model

(i) $\quad \frac{\dot{m}}{m} = -(1-\alpha)m + \frac{(1-\alpha)}{\alpha}(n+\delta) + \eta\frac{(1-\alpha+\zeta)}{\alpha} - \eta\frac{\zeta}{\alpha}\varepsilon$

(ii) $\quad \frac{\dot{g}}{g} = \left(\frac{\alpha}{\sigma} - 1\right)m - \frac{\rho}{\sigma} - \delta\left(\frac{1}{\sigma} - 1\right) + g + n$

(iii) $\quad \frac{\dot{\varepsilon}}{\varepsilon} = \eta\frac{(\alpha-\zeta)}{\alpha}\varepsilon + \eta\frac{(1-\alpha+\zeta)}{\alpha} - g + \frac{(1-\alpha)}{\alpha}(n+\delta)$

The steady state $(m^*, g^*, \varepsilon^*)$ is given by $\dot{m} = \dot{g} = \dot{\varepsilon} = 0$ and derived to be

$$\varepsilon^* = 1 - \frac{(1-\alpha)(\rho - n - \eta)}{\eta[\zeta - \sigma(1 - \alpha + \zeta)]}$$

$$m^* = \eta \frac{[1 - \alpha + \zeta(1 - \varepsilon^*)]}{\alpha(1-\alpha)} + \frac{n}{\alpha}$$

$$g^* = \eta \frac{[1 - \alpha + \zeta(1 - \varepsilon^*) + \alpha\varepsilon^*]}{\alpha} + \frac{n(1-\alpha)}{\alpha}$$

Note that $0 < \frac{(1-\alpha)(\rho-n-\eta)}{\eta[\zeta-\sigma(1-\alpha+\zeta)]} < 1 \; is \; necessary \; for \; 0 < \varepsilon^* < 1.$

A unique steady state exists if

$$\Lambda = \frac{(1-\alpha+\zeta)}{\alpha}(\sigma-1)\eta(1-\varepsilon) + \rho > 0$$

$$0 < \varepsilon < 1$$

as Λ is the necessary and sufficient for the transversality condition to hold for the consumer's utility maximization problem and $0 < \varepsilon^* < 1$ is necessary for $m^*, g^* > 0$**.** We linearize the system (i), (ii) and (iii) around the steady state, $s^* = (m^*, g^*, \varepsilon^*)$, to acquire

$$\begin{bmatrix} \dot{m} \\ \dot{g} \\ \dot{\varepsilon} \end{bmatrix} = \underbrace{\begin{bmatrix} \left.\frac{\partial \dot{m}}{\partial m}\right|_{s^*} & \left.\frac{\partial \dot{m}}{\partial g}\right|_{s^*} & \left.\frac{\partial \dot{m}}{\partial \varepsilon}\right|_{s^*} \\ \left.\frac{\partial \dot{g}}{\partial m}\right|_{s^*} & \left.\frac{\partial \dot{g}}{\partial g}\right|_{s^*} & \left.\frac{\partial \dot{g}}{\partial \varepsilon}\right|_{s^*} \\ \left.\frac{\partial \dot{\varepsilon}}{\partial m}\right|_{s^*} & \left.\frac{\partial \dot{\varepsilon}}{\partial g}\right|_{s^*} & \left.\frac{\partial \dot{\varepsilon}}{\partial \varepsilon}\right|_{s^*} \end{bmatrix}}_{J_m} \begin{bmatrix} m_t - m^* \\ g_t - g^* \\ \varepsilon_t - \varepsilon^* \end{bmatrix}$$

where, $J_m = \begin{bmatrix} -(1-\alpha)m^* & 0 & -\eta\frac{\zeta}{\alpha}m^* \\ \left(\frac{\alpha}{\sigma} - 1\right)g^* & g^* & 0 \\ 0 & -\varepsilon^* & \eta\frac{(\alpha-\zeta)}{\alpha}\varepsilon^* \end{bmatrix}.$

The characteristic equation associated with J_m is $q^3 + c_2 q^2 + c_1 q + c_0 = 0$, where

$$c_0 = \eta \frac{[\sigma(1-\alpha+\zeta)-\zeta]}{\sigma} m^* g^* \varepsilon^*,$$

$$c_1 = \eta^2 \frac{(\alpha-\zeta)}{\alpha} \varepsilon^{*2} - (1-\alpha) m^* g^*,$$

$$c_2 = -\eta \frac{(2\alpha-\zeta)}{\alpha} \varepsilon^*.$$

Chapter 6. Bifurcation Analysis of Uzawa-Lucas Model

In this chapter, we examine the existence of codimension 1 and 2, transcritical, and Hopf bifurcations in the system (i), (ii), and (iii). The codimension, as defined by Kuznetsov (2004), is the number of independent conditions determining the bifurcation boundary. Varying a single parameter help to identify codimension-1 bifurcation, and varying 2 parameters helps to identify codimension-2 bifurcation.

An equilibrium point, $\boldsymbol{s}*$, of the system is called hyperbolic, if the coefficient matrix, J_m, has no eigenvalues with zero real parts. For small perturbations of parameters, there are no structural changes in the stability of a hyperbolic equilibrium, provided that the perturbations are sufficiently small. Therefore, bifurcations occur at nonhyperbolic equilibria only.

A transcritical bifurcation occurs, when a system has a nonhyperbolic equilibrium at the bifurcation point with a geometrically simple zero eigenvalue, and also additional transversality conditions must be satisfied, as given by Sotomayor's Theorem [Barnett and He (1999)]. So the first condition we are going to use to find the bifurcation boundary is $c_0 = \det(J_m) = 0$. The result is the following.

Theorem 6.1: J_m has a zero eigen value, if

$$\eta \frac{[\sigma(1-\alpha+\zeta)-\zeta]}{\sigma} m^* g^* \varepsilon^* = 0. \qquad \text{(a)}$$

Hopf bifurcations occur at points at which the system has a nonhyperbolic equilibrium with a pair of purely imaginary eigenvalues, but without zero eigenvalues. Also additional transversality conditions must be satisfied. We use the following theorem, based upon the version of the Hopf Bifurcation Theorem in Guckenheimer and Holmes (1983): J_m has precisely one pair of pure imaginary eigen values, if $c_0 - c_1 c_2 = 0$ and $c_1 > 0$. If $c_0 - c_1 c_2 \neq 0$ and $c_1 > 0$, then J has no pure imaginary eigen values. The result is:

Theorem 6. 2: The matrix J_m has precisely one pair of pure imaginary eigen values, if

$$\left.\begin{array}{c} \alpha m^* g^*\big((\alpha-1)\alpha\sigma + \zeta(\sigma-\alpha)\big) + \eta^2 \sigma\, {\varepsilon^*}^2 (2\alpha-\zeta)(\alpha-\zeta) = 0 \\ \text{and} \\ \frac{\eta^2}{\alpha} {\varepsilon^*}^2 (\alpha-\zeta) - (1-\alpha) m^* g^* > 0. \end{array}\right\} \qquad (b)$$

6.1 Case Studies

To be able to display the boundaries, we consider two or three parameters. But the procedure is applicable to any number of parameters.

Let $\vartheta^* = \{\eta = 0.05,\ \zeta = 0.1,\ \alpha = 0.65,\ \rho = 0.0505,\ \sigma = 0.15,\ n = 0,\ \delta = 0\}$ and

$$\omega^* = \{\eta = 0.05, \zeta = 0.1, \alpha = 0.75, \rho = 0.0505, \sigma = 0.15, n = 0, \delta = 0\}.$$

Case I: Free parameters, α, η.

Assume that other parameters operate at ϑ^*. The result is illustrated in Figure 1. The red line gives a range of α and η satisfying the Hopf bifurcation conditions, while the blue line depicts the value of α and η satisfying conditions for a transcritical bifurcation boundary.

Similarly, the following cases gives the range of parameter values satisfying condition (a) and condition (b), in blue and red respectively, while the rest of the parameters are set at ϑ^*.

Case II: Free parameters, ζ, α (figure 2).

Case III: Free Parameters, σ , α (figure 3).

Case IV: Free Parameters, ζ, ρ (figure 4). Notice that for case IV we do not have a Hopf bifurcation boundary.

We now add another parameter as a free parameter and continue with the analysis. The following cases give the range of parameter values satisfying condition (a) and condition (b), in blue and red regions respectively, while the rest of the parameters are assumed to be at ω^*.

Case V: Free parameters, α , ζ , ρ (figure 5).

Case VI: Free parameters, η, ζ, σ (figure 6).

Case VII: Free parameters, α , η, ρ (figure 7). For case VII, we do not have a Hopf bifurcation boundary.

Case VIII: Free parameters, α , σ, ρ (figure 8).

Case IX: Free parameters, α , η, σ (figure 9).

The following is an approach to exploring cyclical behavior in the model. Suppose there is a change in policy, encouraging increase in savings rate. Consumption decreases initially, when intertemporal substitution for consumption is high (σ is low), as people start saving more. This will encourage a movement of labor from output production to human capital production. Human capital begins increasing. This implies faster accumulation of physical capital when sufficient externality to human capital is present in production of physical capital. If people care about the future more (subjective discount rate ρ is lower), consumption starts rising gradually with faster capital accumulation, leading to greater consumption-goods production in the future. This will eventually lead to a decline in savings rate. Hence two opposing effects exist, when the savings rate is different from the equilibrium rate. A lower savings rate now will cause a slower rate of physical capital accumulation. When ρ is lower, consumption starts falling over the time leading to an increase in savings rate. Interaction between different parameters can cause cyclical convergence to equilibrium (figure 10) or may cause instability; and for some parameter values we may have convergence to cycles (figure 11).

Using the numerical continuation package Matcont, we further investigate the stability properties of cycles generated by different combinations of parameters. While some of the limit cycles generated by Andronov-Hopf bifurcation are stable (supercritical bifurcation), there could be some unstable limit cycles (subcritical bifurcation) created as well. Table 1 reports the values of the share of capital (α), externality in production of human capital (ζ), and the inverse of intertemporal elasticity of substitution in consumption (σ). A positive value of the first Lyapunov coefficient indicates creation of subcritical Hopf bifurcation. Thus for each of the cases reported in Table 1, an unstable limit cycle (periodic orbit) bifurcates from the equilibrium. All of these cases also produce branch points (pitchfork/transcritical bifurcations).

Continuation of limit cycles from the Hopf point, when α is the free parameter, gives rise to limit point (Fold/ Saddle Node) bifurcation of cycles. From the family of limit cycles bifurcating from the Hopf point, limit point cycle (LPC) is a fold bifurcation, where two limit cycles with different periods are present near the LPC point at α = 0.738.

Continuing computation further from a Hopf point gives rise to a series of period doubling (flip) bifurcations. Period doubling bifurcation is defined as a situation in which a new limit cycle emerges from an existing limit cycle, and the period of the new limit cycle is twice that of the old one. The first period doubling bifurcation, at α = 0.7132369, has positive normal form coefficients, indicating existence of unstable double-period cycles. The rest of the period doubling bifurcations have negative normal-form coefficients, giving rise to stable double-period cycles.

The period of the cycle rapidly increases for very small perturbation in parameter α, as is evident in figure 12(C). The limit cycle approaches a global homoclinic orbit. A homoclinic orbit is a dynamical system trajectory, which joins a saddle equilibrium point to itself. In other words, a homoclinic orbit lies in the intersection of an equilibrium's stable manifold and unstable manifold. There exists the possibility of reaching chaotic dynamics through the series of period doubling bifurcations.

For the cases in which ζ and σ are free parameters, we carry out the continuation of limit cycle from the first Hopf point. Both cases give rise to limit point cycles with a nonzero normal-form coefficient, indicating the limit cycle manifold has a fold at the LPC point. Similar results are found, if we carry out the continuation of limit cycles from the second Hopf point for each of these cases, and hence we do not report those results.

Table 1

Parameters Varied	Equilibrium Bifurcation	Bifurcation of Limit Cycle
$\boldsymbol{\alpha}$	Figure 12 (A)	Figure 12 (B)
(Figure 12) Other parameters set at ϑ^*	**Hopf (H)** First Lyapunov coefficient = 0.00242, α =0.738207	**Limit point cycle (LPC)** period = 231.206, $\boldsymbol{\alpha}$ = 0.7382042 Normal form coefficient = 0.007
		Period Doubling (PD) period = 584.064, $\boldsymbol{\alpha}$ = 0.7132369 Normal form coefficient = 0.910
		Period Doubling (PD) period = 664.005, $\boldsymbol{\alpha}$ = 0.7132002 Normal form coefficient = -0.576
		Period Doubling (PD) period = 693.988, $\boldsymbol{\alpha}$ = 0.7131958 Normal form coefficient = -0.469
		Period Doubling (PD) period = 713.978, $\boldsymbol{\alpha}$ = 0.7131940 Normal form coefficient = -0.368
		Period Doubling (PD) period = 725.667, $\boldsymbol{\alpha}$ = 0.7131932 Normal form coefficient = -0.314
		Period Doubling (PD) period = 784.104, $\boldsymbol{\alpha}$ = 0.7131912 Normal form coefficient = -0.119
	Branch Point (BP)	
$\boldsymbol{\zeta}$	Figure 13 (A)	Figure 13 (B)
(Figure 13) Other parameters set at ω^*	**Hopf (H)** First Lyapunov coefficient = 0.00250, $\boldsymbol{\zeta}$ =0.107315	**Limit point cycle (LPC)** period = 215.751, $\boldsymbol{\zeta}$ = 0.1073147 Normal form coefficient = 0.009
	Hopf (H) First Lyapunov coefficient = 0.00246 $\boldsymbol{\zeta}$ =0.052623	
	Branch Point (BP) $\boldsymbol{\zeta} = 0.047059$	
$\boldsymbol{\sigma}$	Figure 14 (A)	Figure 14 (B)
(Figure 14) Other parameters set at ω^*	**Hopf (H)** First Lyapunov coefficient = 0.00264 $\boldsymbol{\sigma}$ =0.278571	**Limit point cycle (LPC)** period = 213.83, $\boldsymbol{\sigma}$ = 0.1394026 Normal form coefficient = 0.009
	Hopf (H) First Lyapunov coefficient = 0.00249 $\boldsymbol{\sigma}$ =0.139394	
	Branch Point (BP) $\boldsymbol{\sigma} = 0.278571$	

Figure 1: Free Parameters α, η

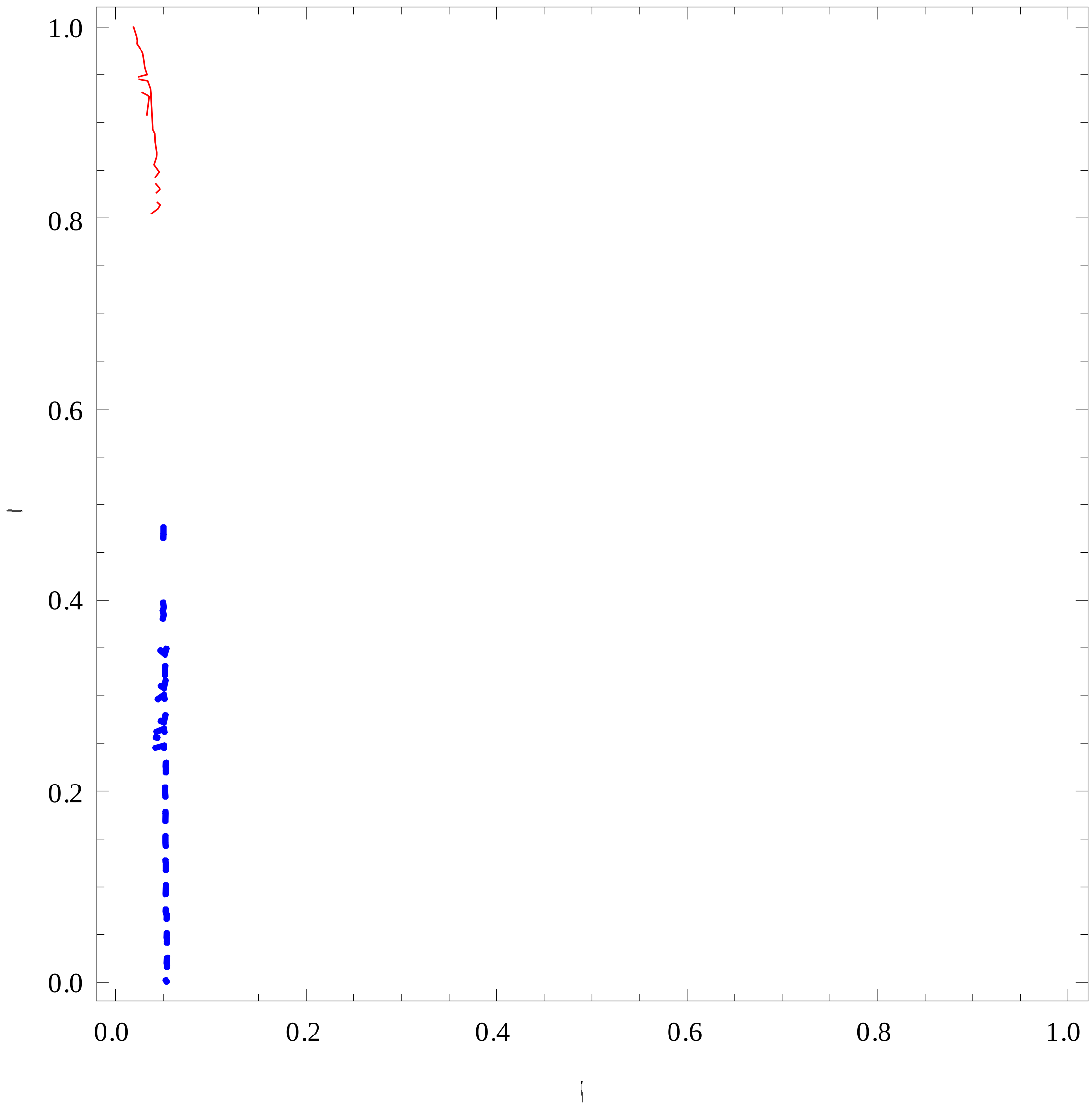


Figure 2: Free parameters, ζ, α

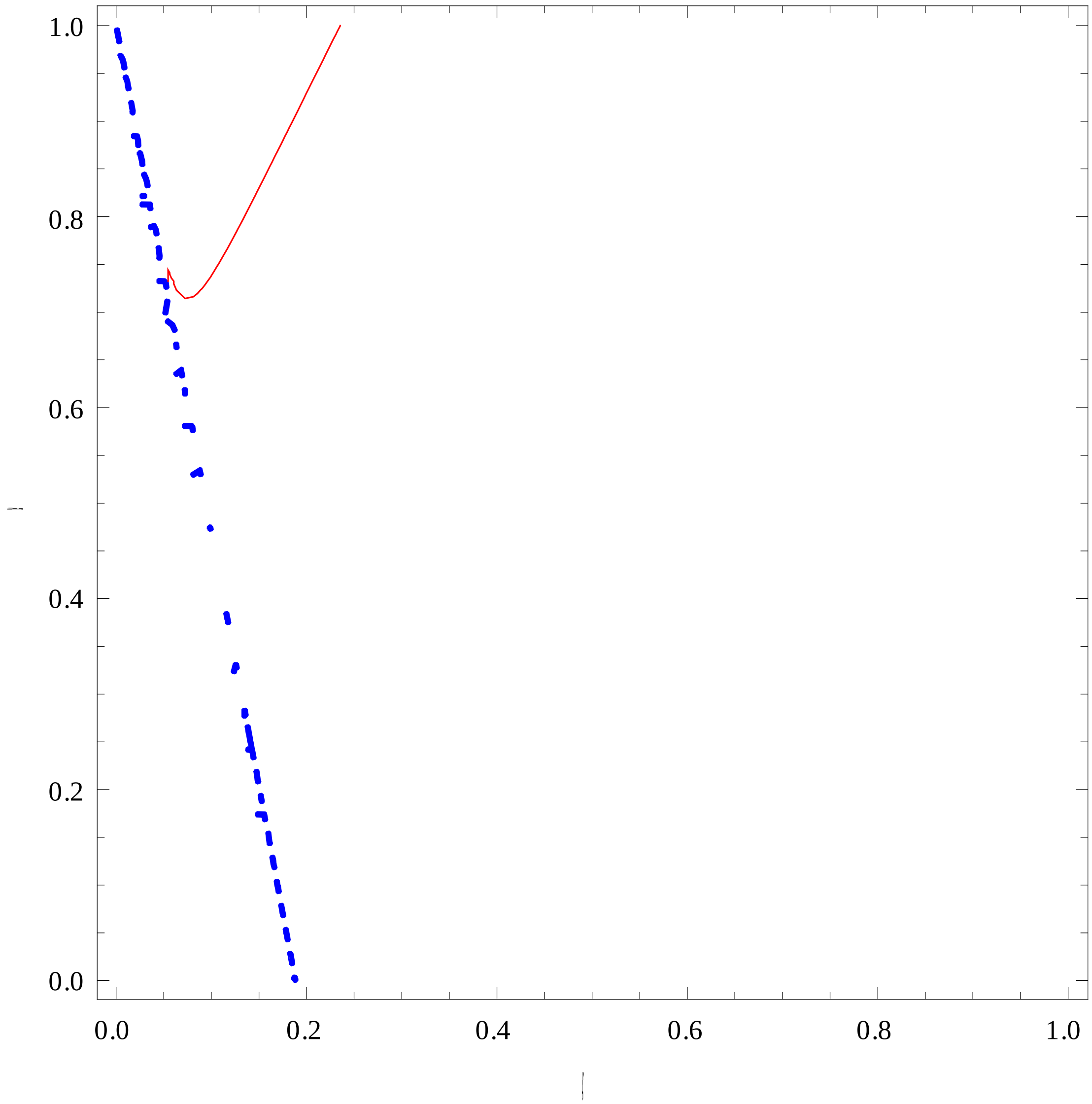


Figure 3: Free Parameters, σ , α

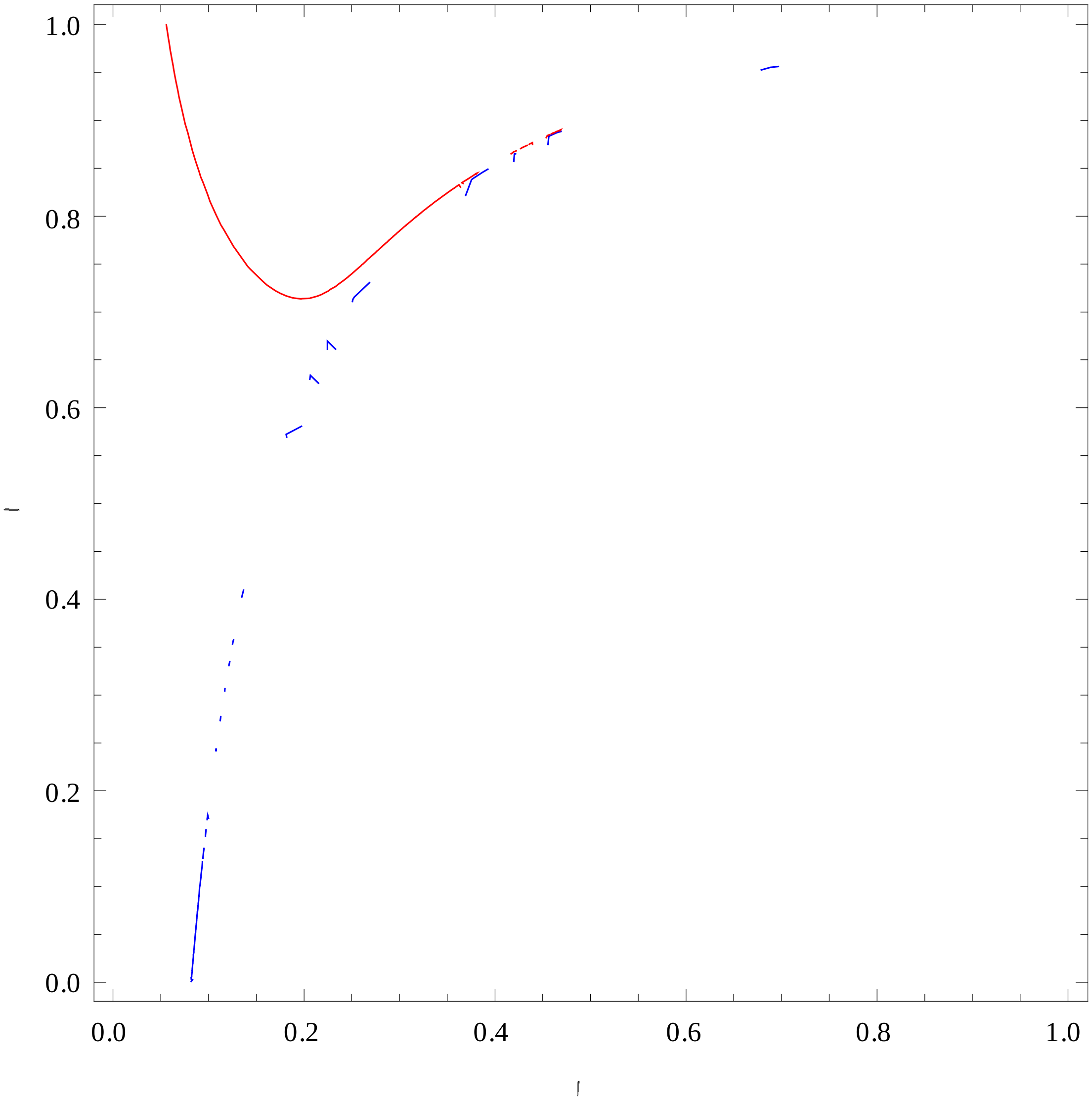


Figure 4: Free Parameters ζ , ρ

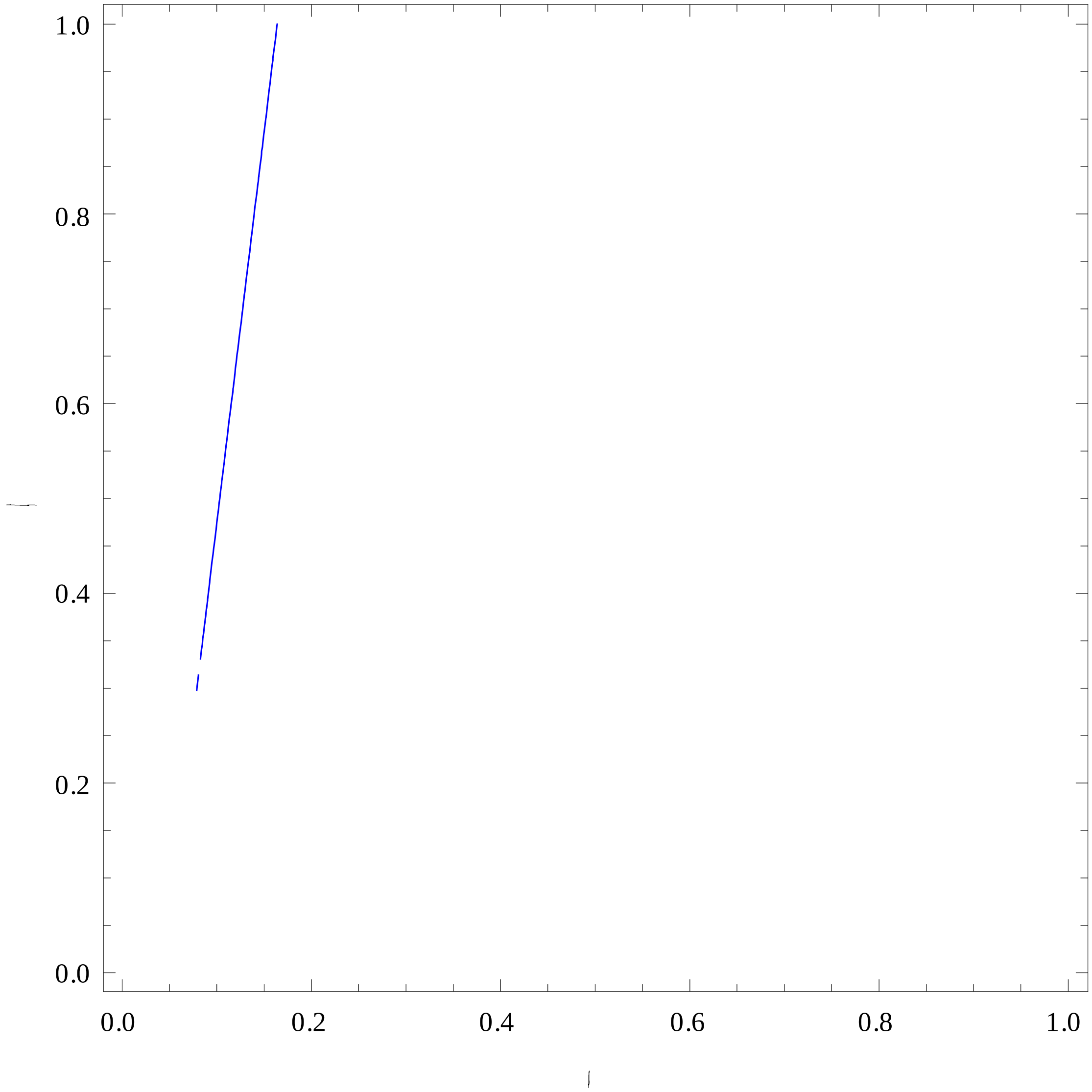


Figure 5: Free parameters α , ζ, ρ

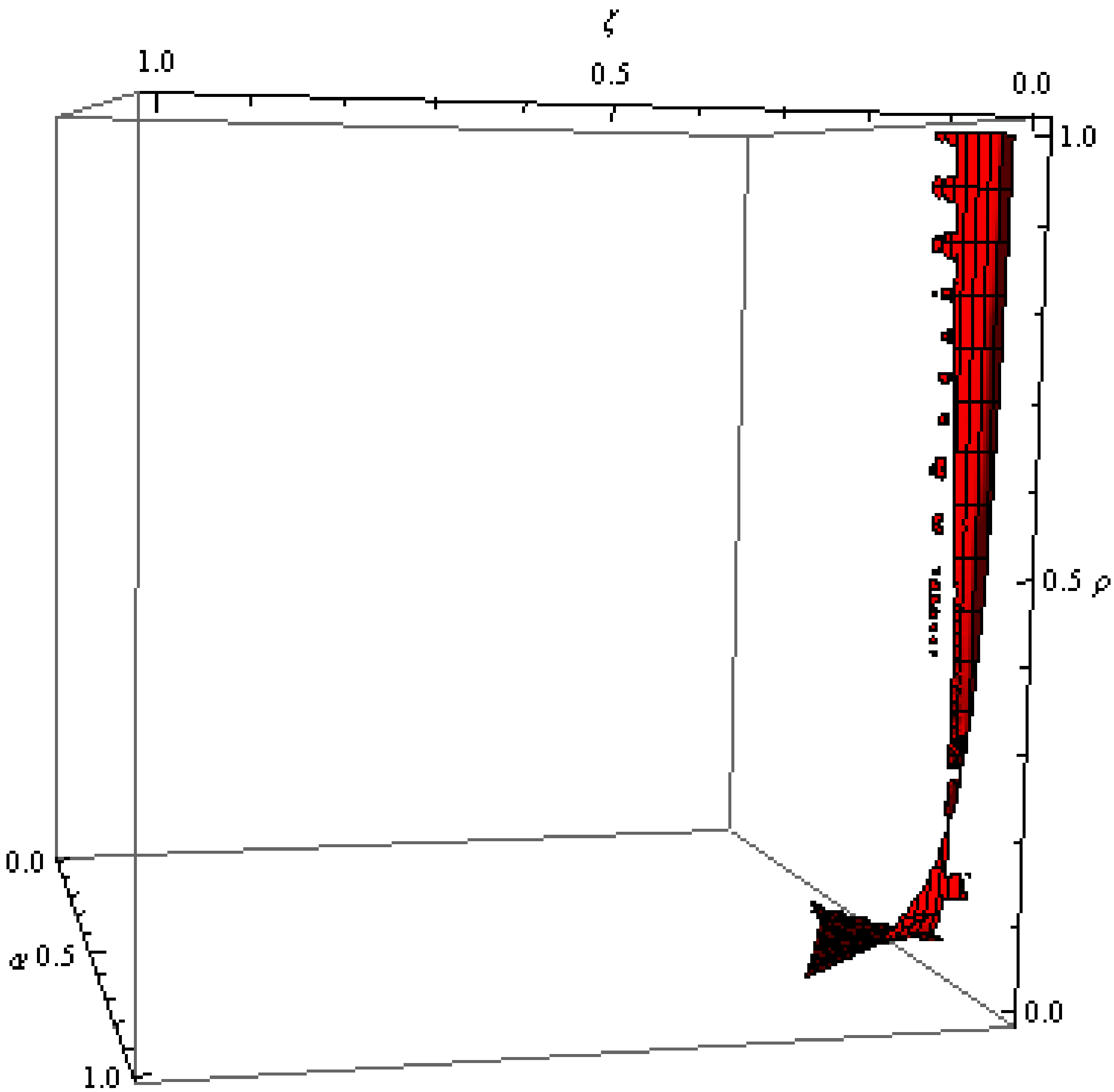


Figure 6: Free Parameters η, ζ, σ

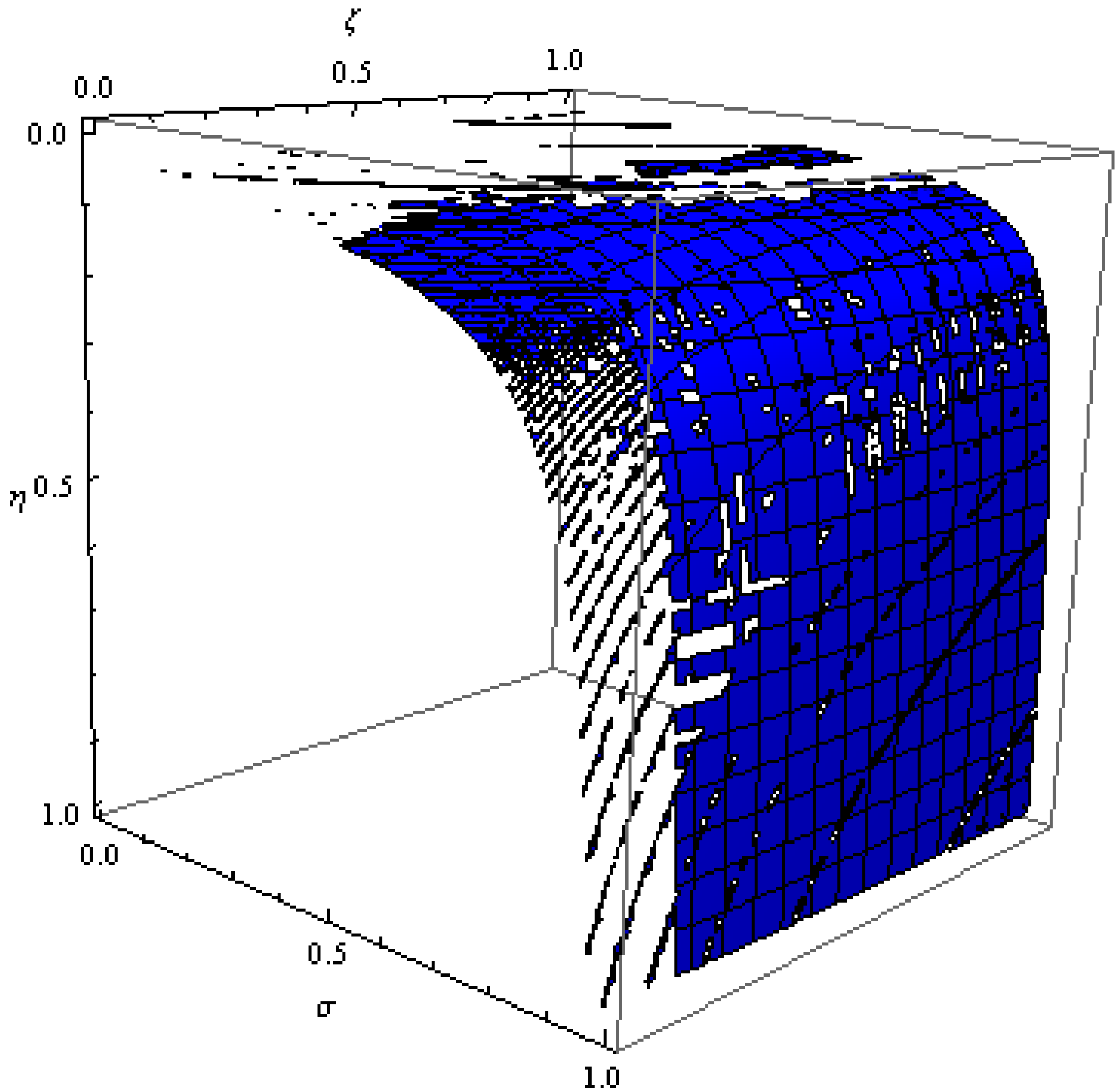


Figure 7: Free parameters α, η, ρ

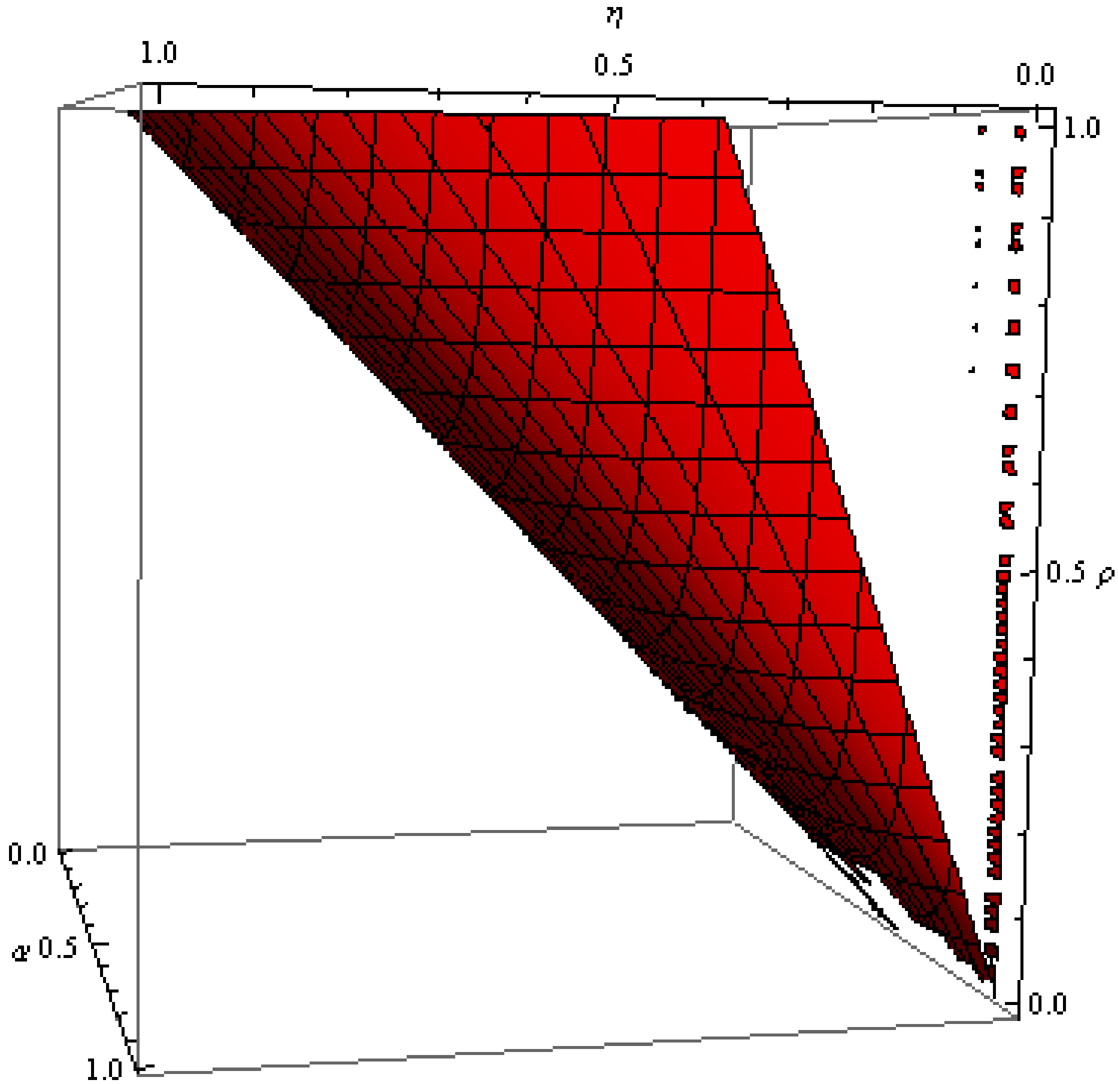


Figure 8: Free parameters, α , σ, ρ 8: α

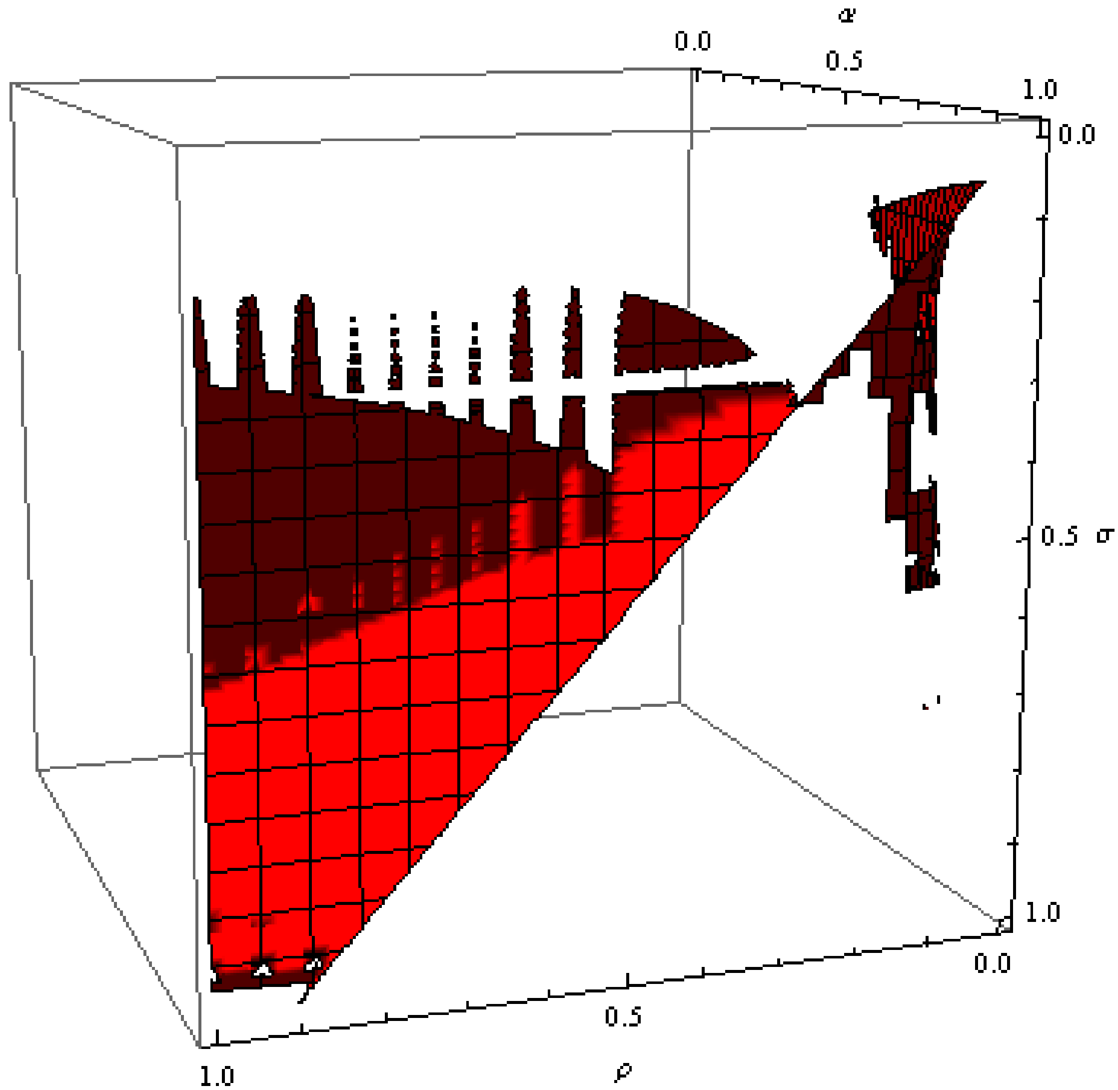


Figure 9: Free parameters, α , η, σ

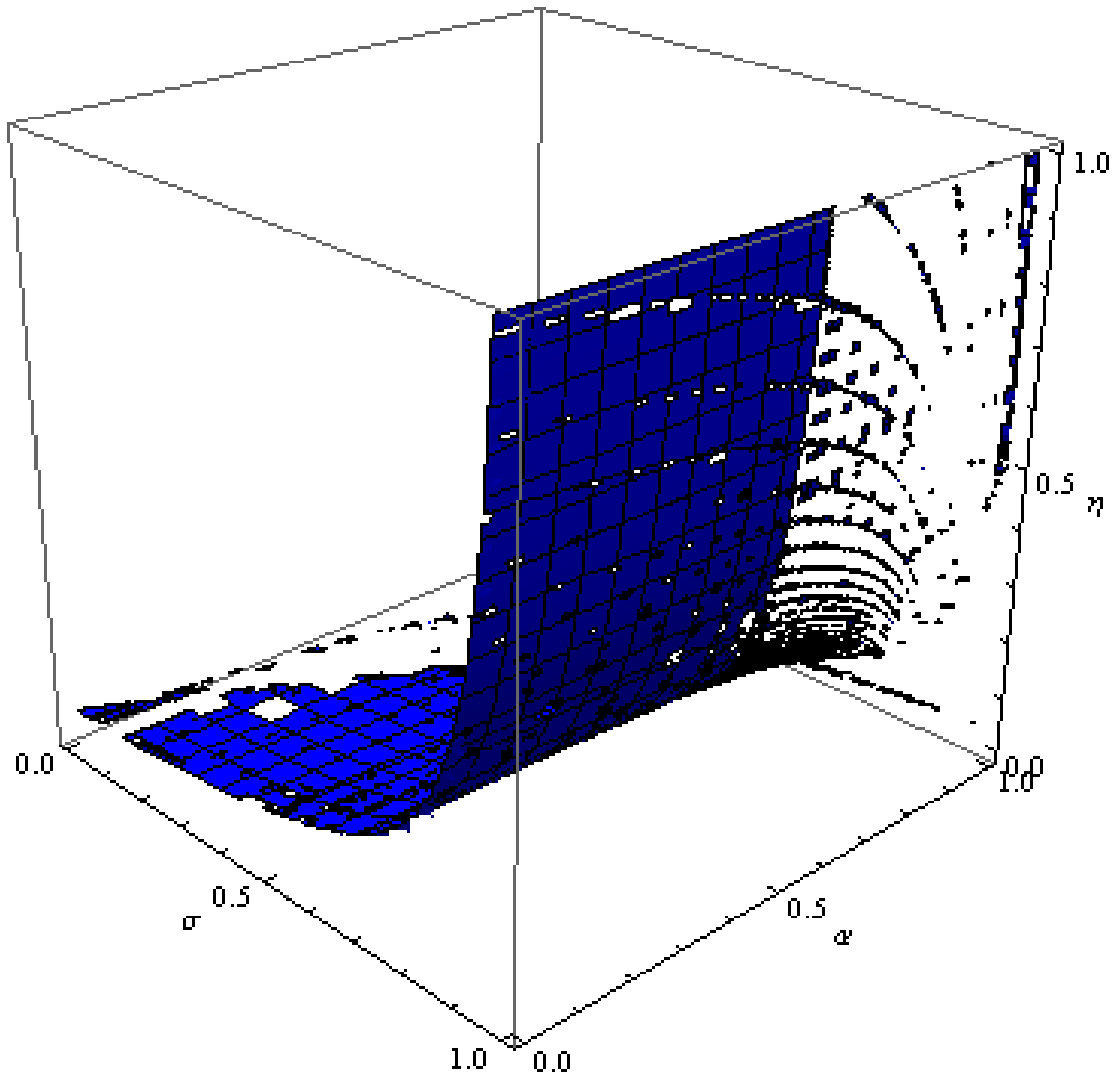


Figure 10

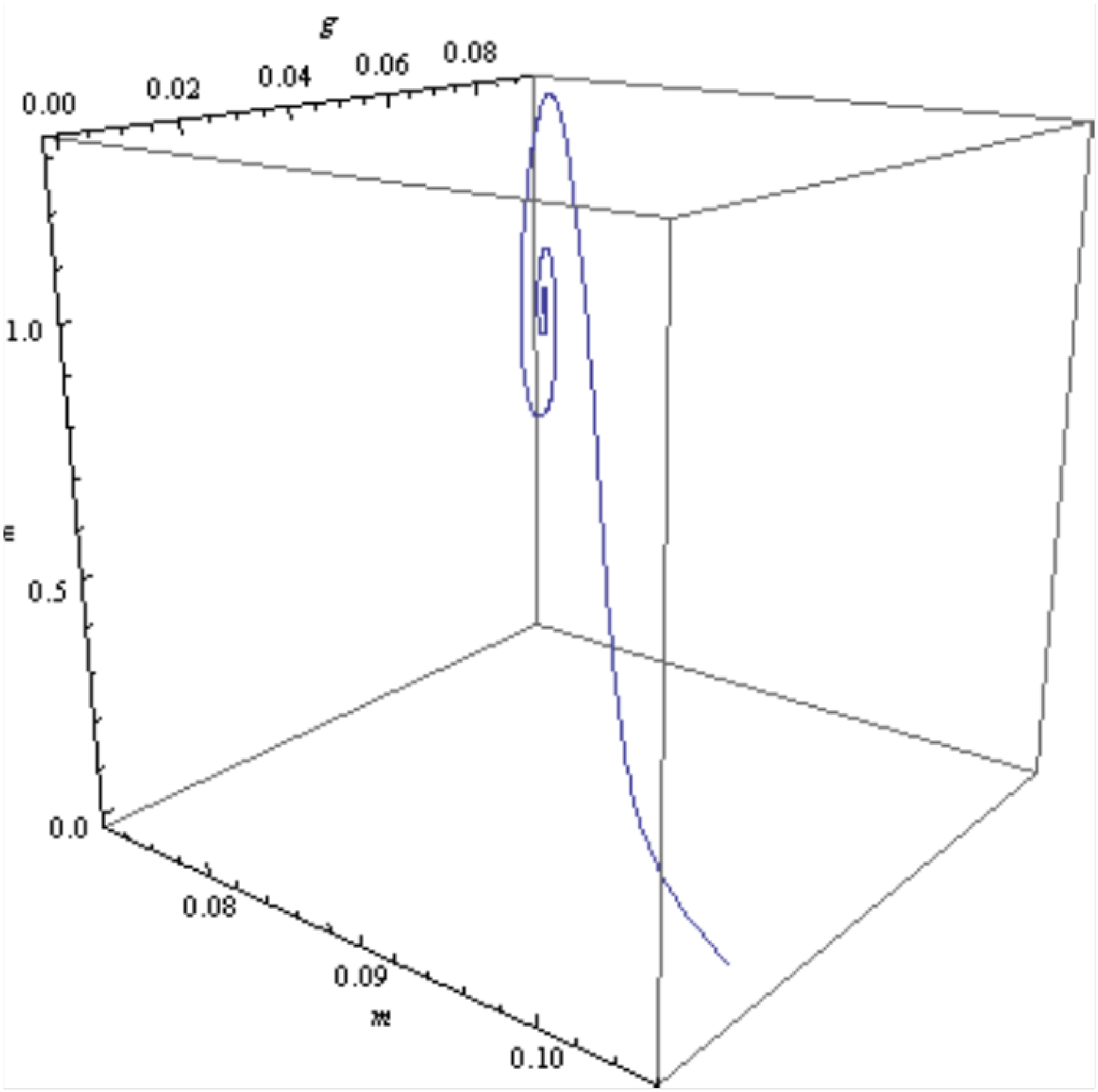


Figure 11

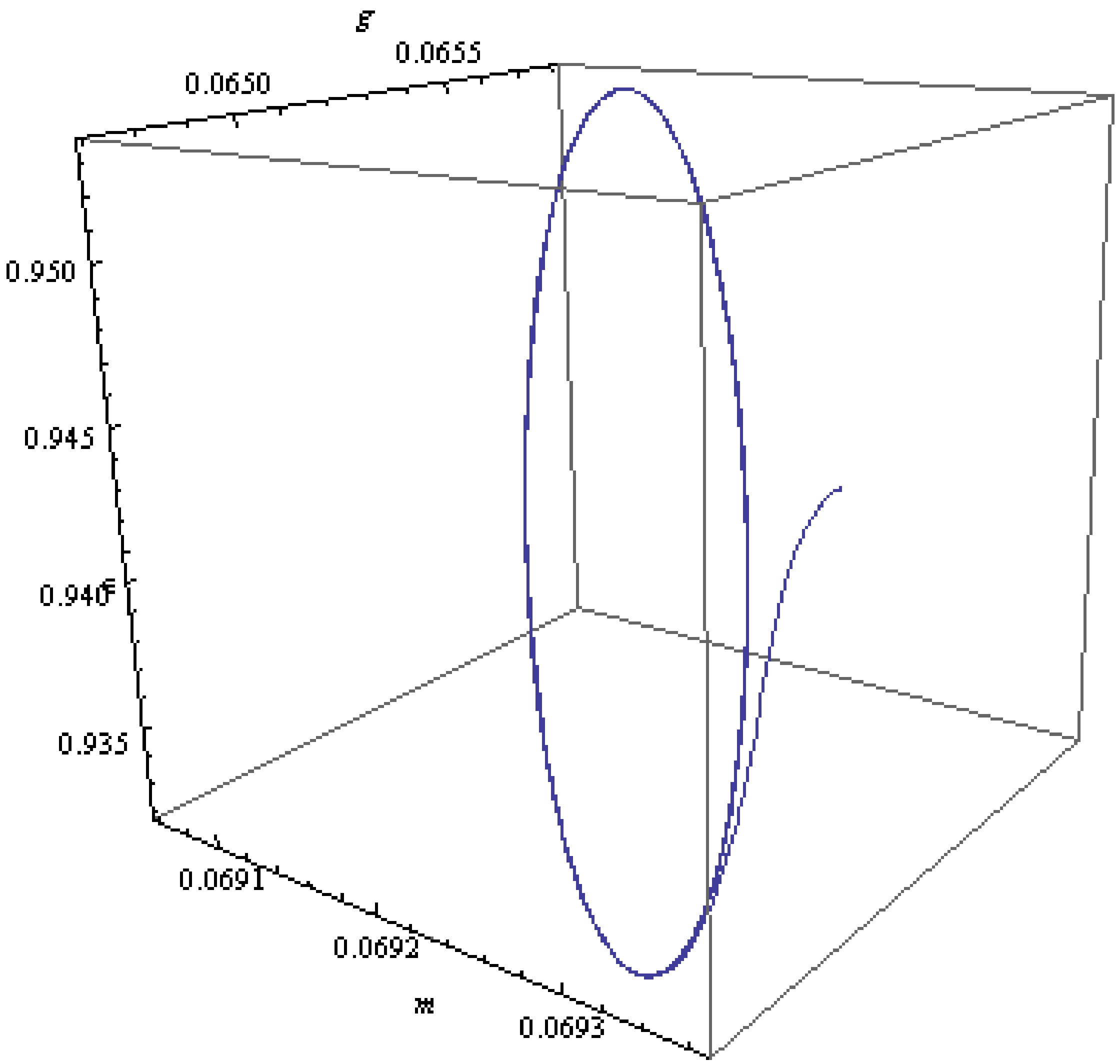


Figure 11a: Parameters on the Hopf Bifurcation

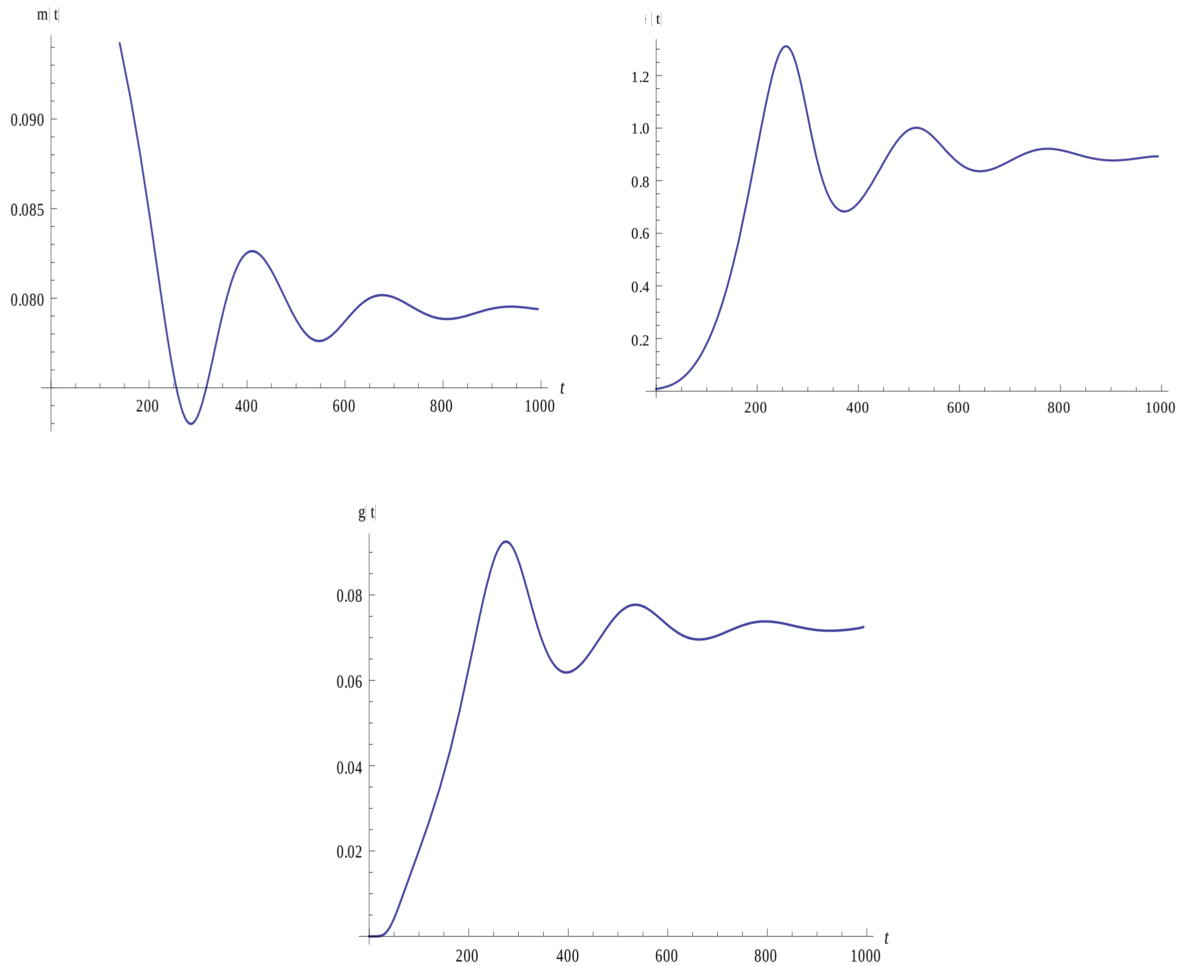


Figure 11b: Parameters on the Hopf Bifurcation

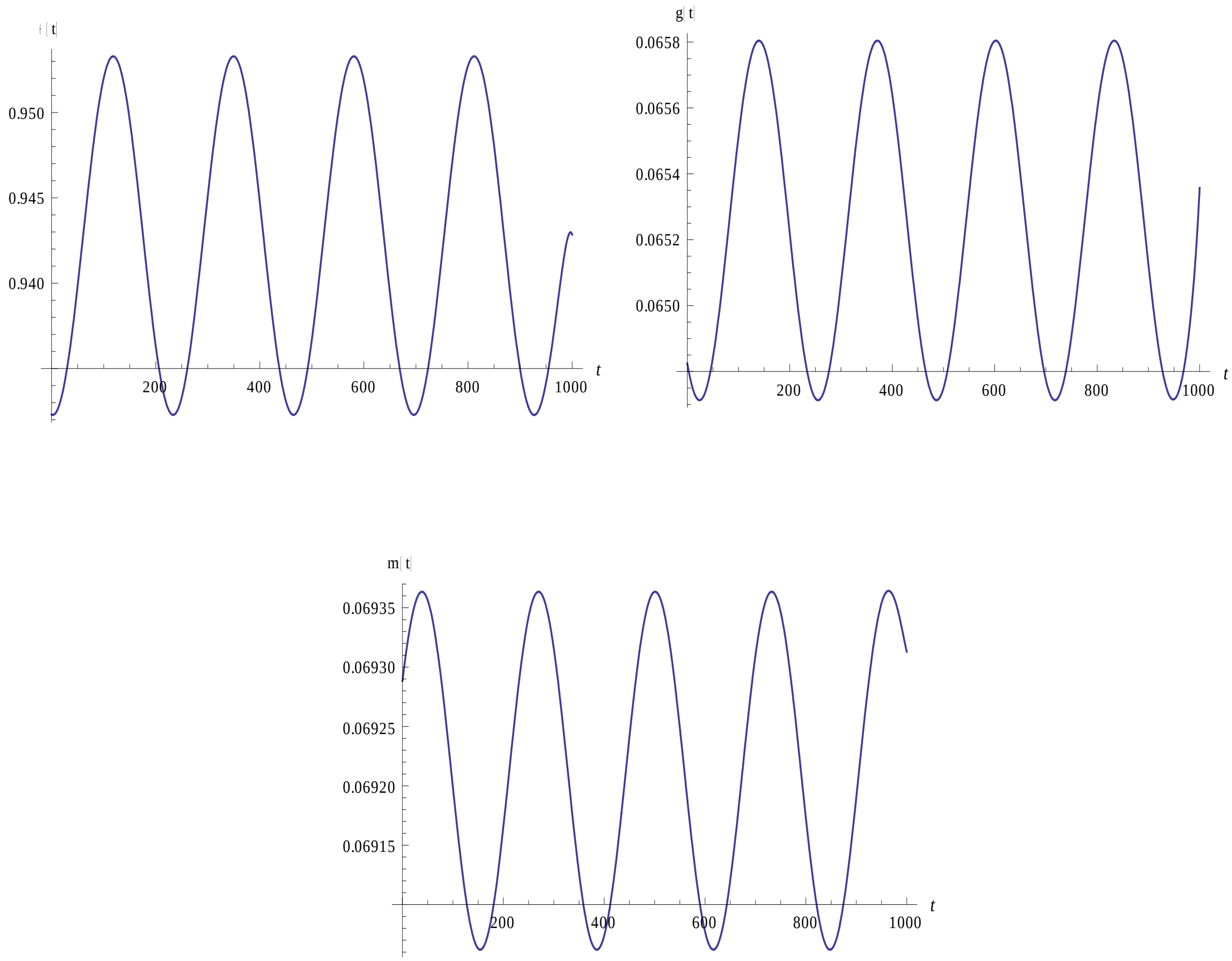


Figure 12(A)

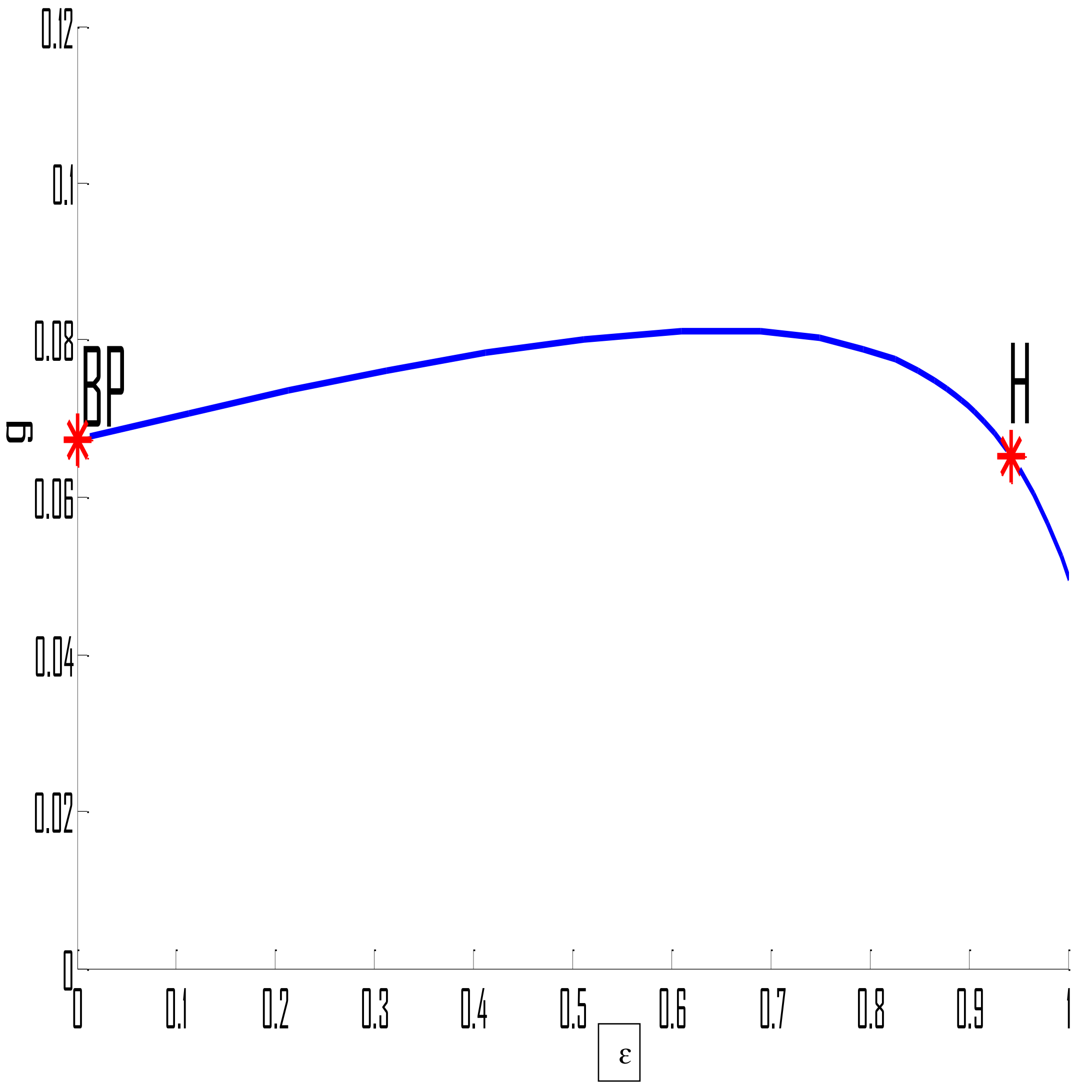
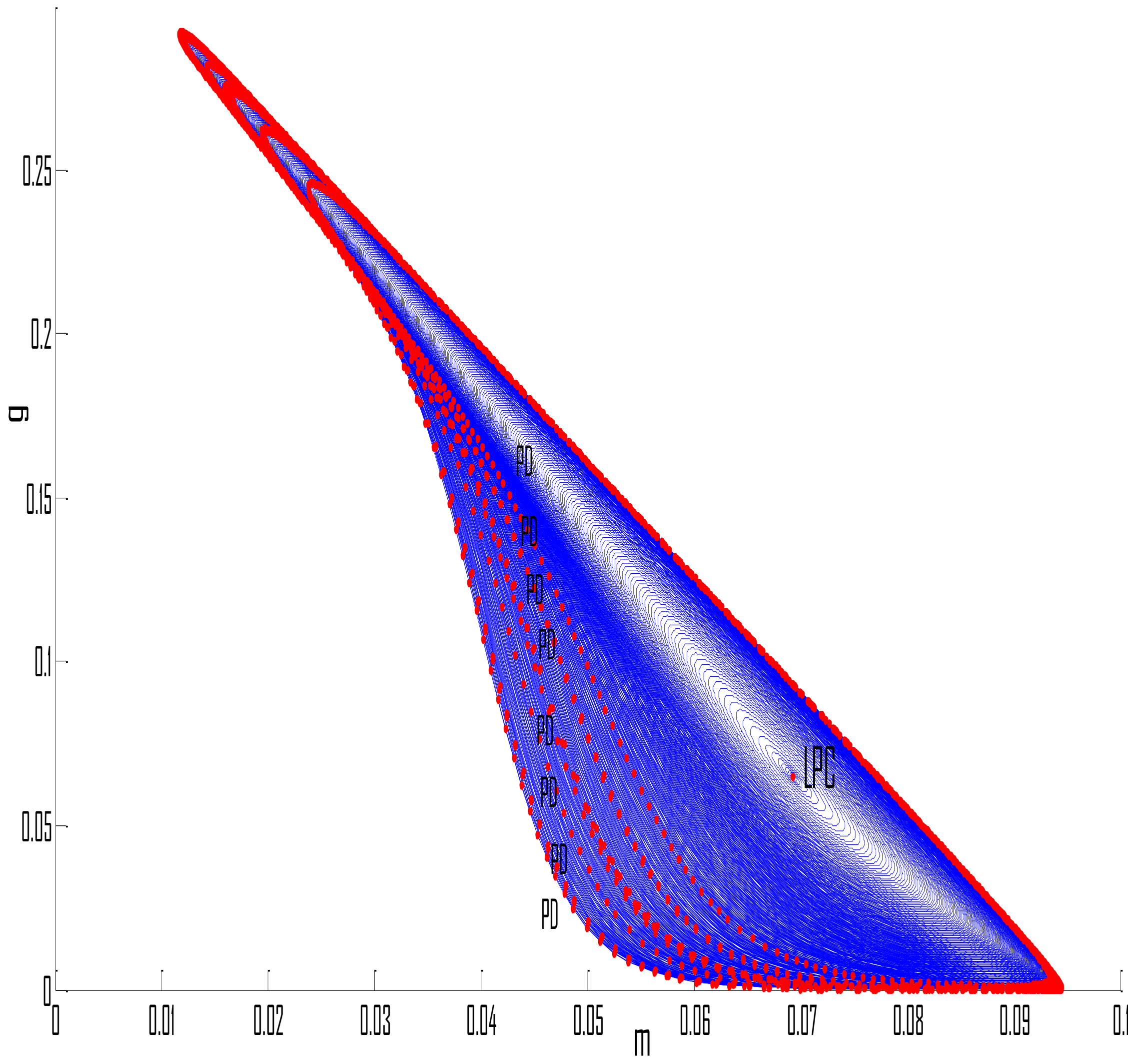
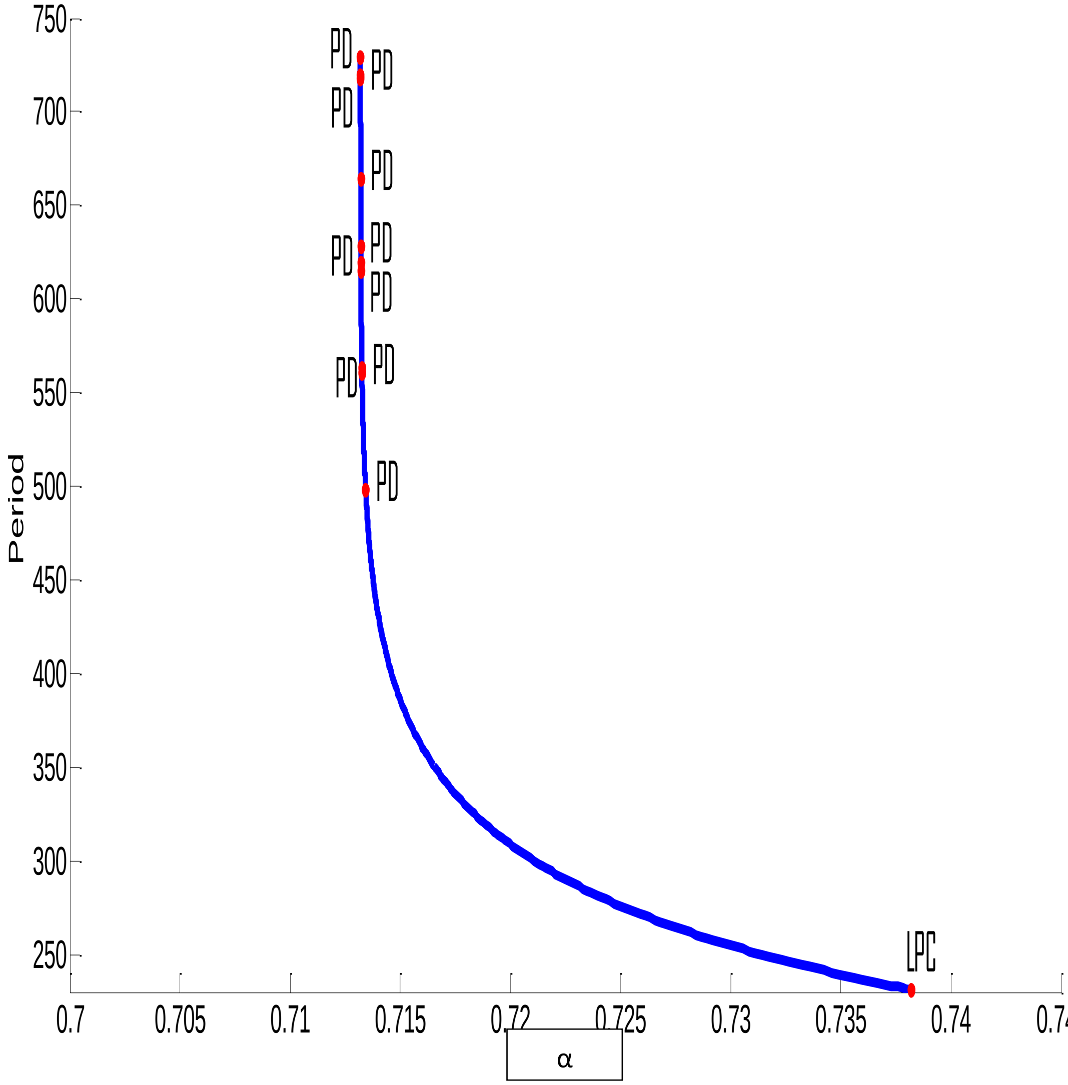


Figure 12(B)



0.25
0.2
0.15
0.1
0.05
0
g
PD
PD
PD
PD
PD
PD
PD
PD
LPC
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1
m

Figure 13(A)



750
700
650
600
550
500
450
400
350
300
250
Period
PD
PD
PD
PD
PD
PD
PD
PD
PD
PD
LPC
0.7
0.705
0.71
0.715
0.72
0.725
0.73
0.735
0.74
α

Figure 13(B)

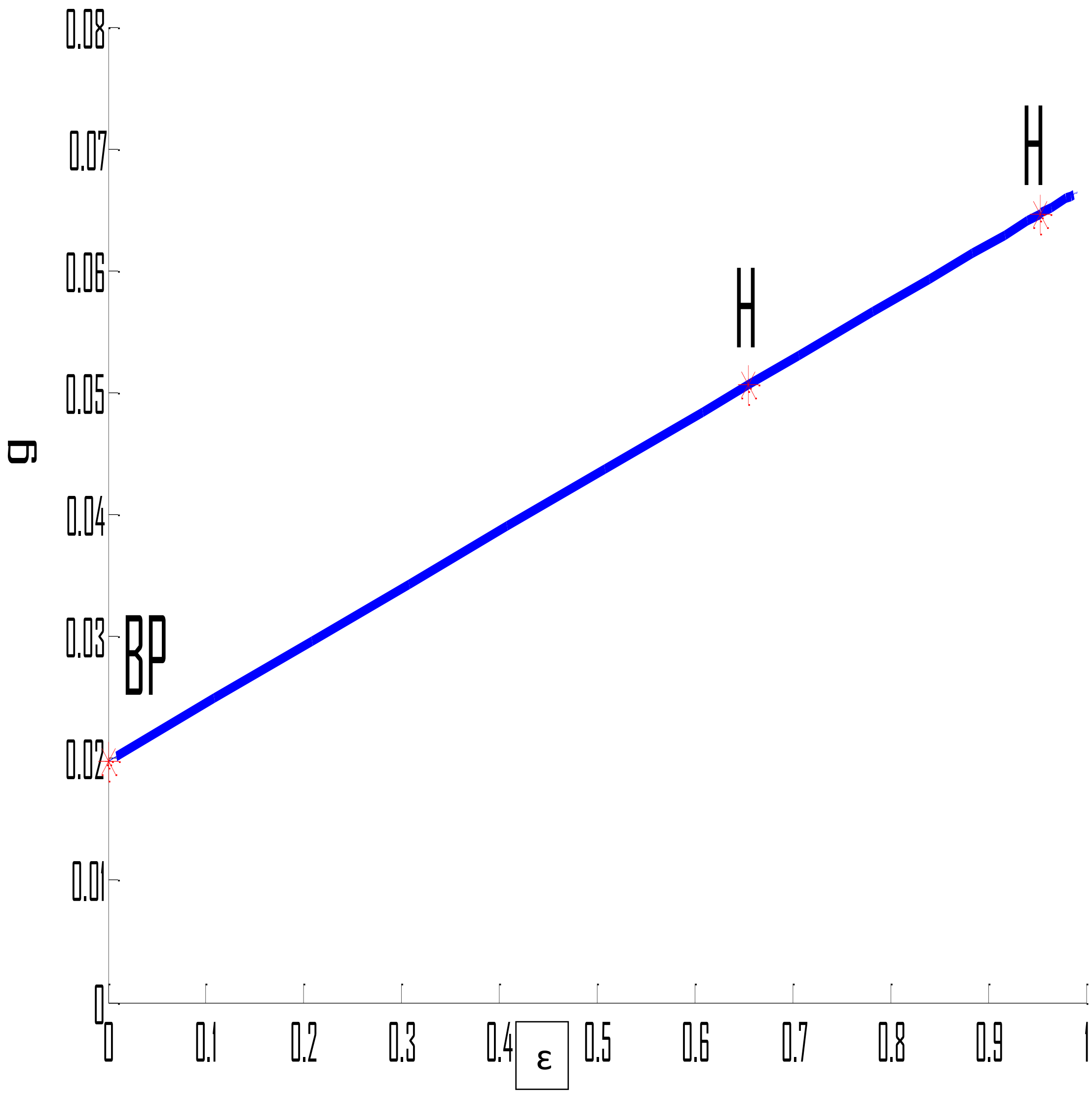
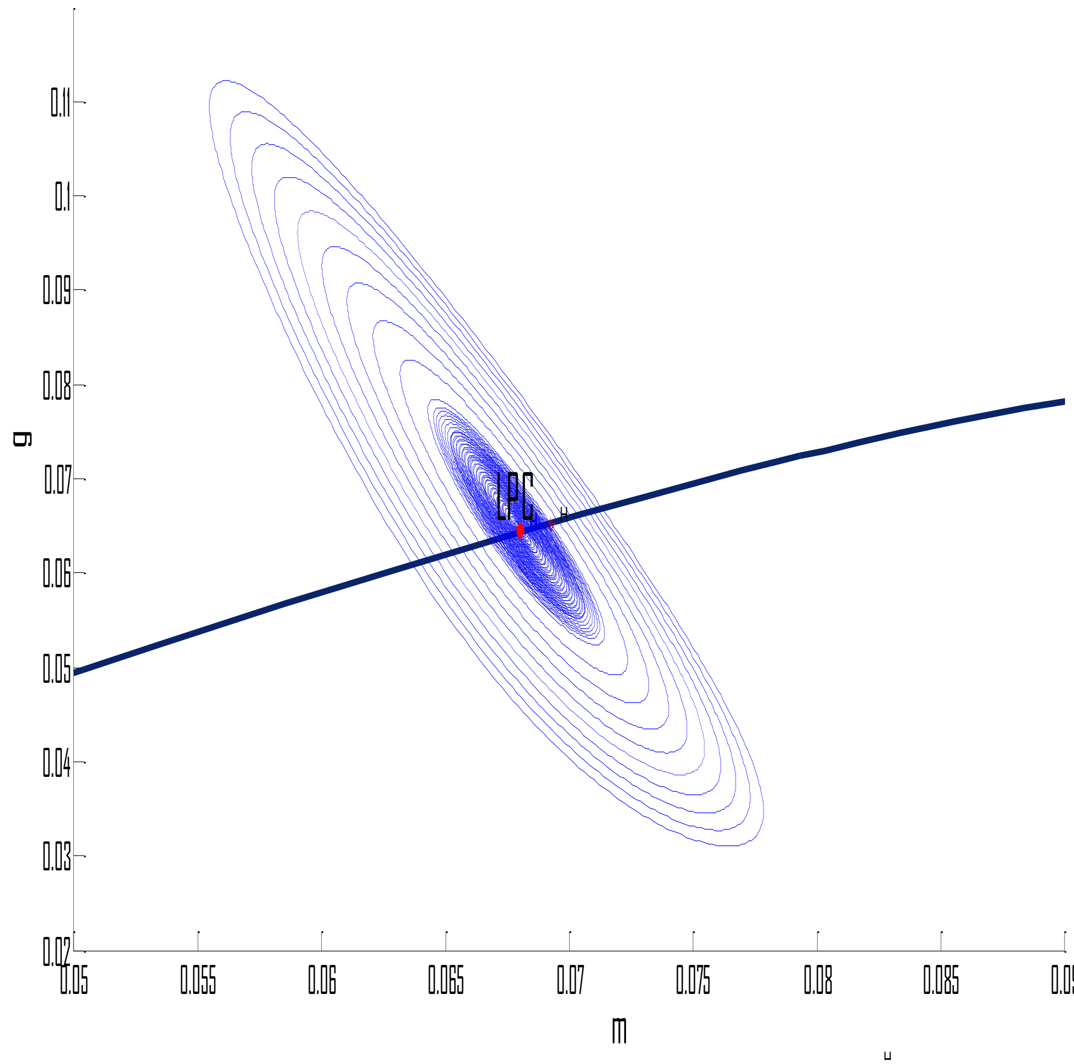

0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
g
BP
H
H
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
ε

Figure 14(A)



0.11
0.1
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
g
LPC
0.05
0.055
0.06
0.065
0.07
0.075
0.08
0.085
m

Figure 14 (B)

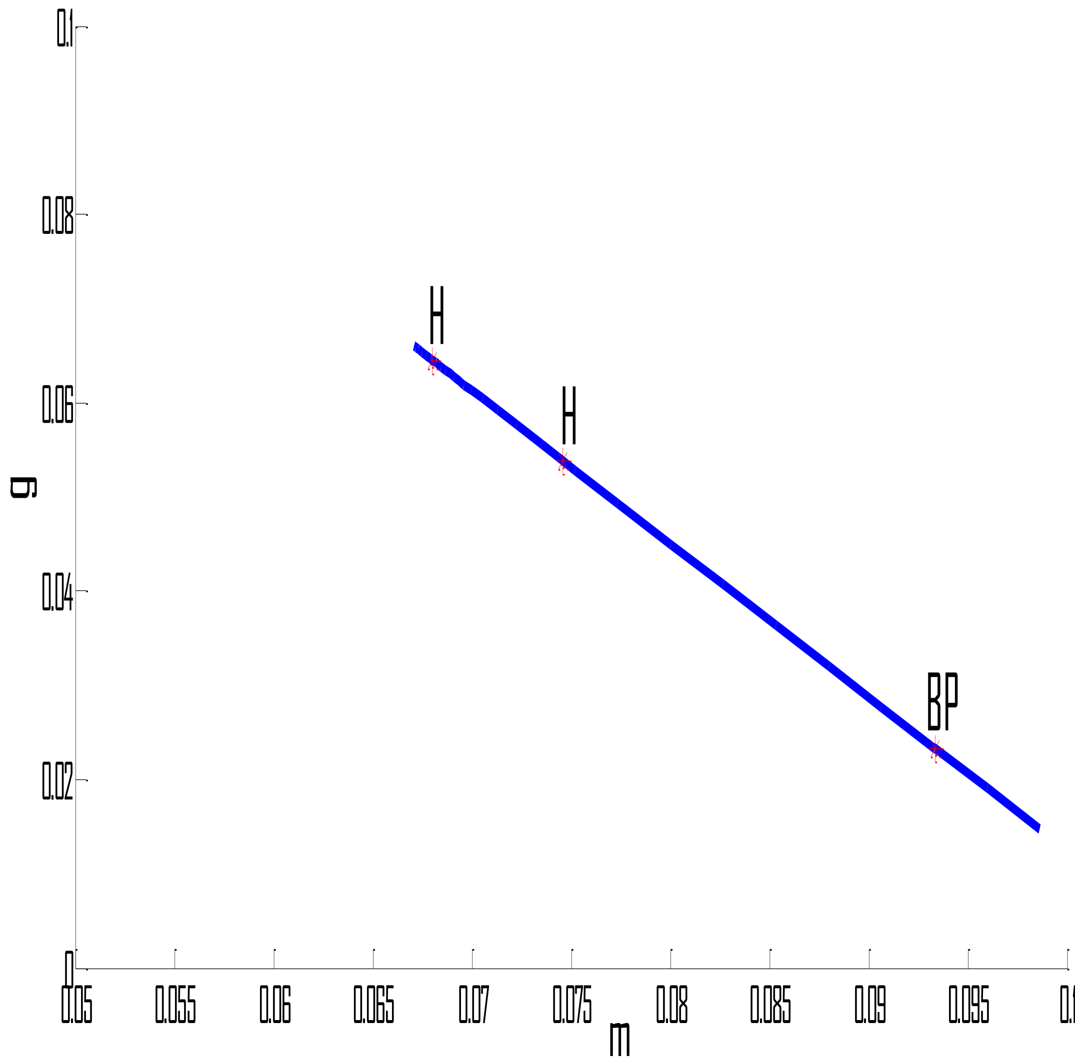
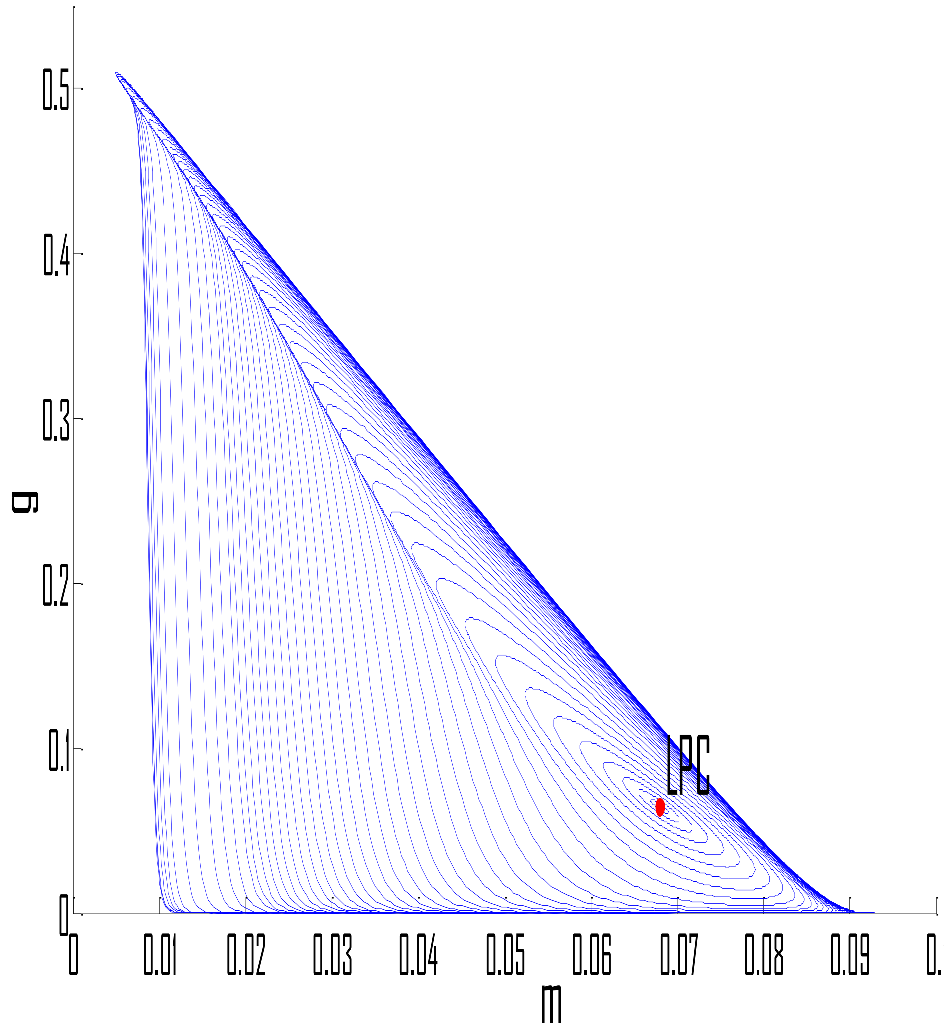

0.1
0.08
0.06
0.04
0.02
0
g
0.05
0.055
0.06
0.065
0.07
0.075
0.08
0.085
0.09
0.095
0.1
m
H
H
BP

Figure 14(B)



Chapter 7. The Modified Jones Model

There are N_t identical, infinitely lived individuals present with a population growth rate of $n\,(> 0)$. Each individual is endowed with one unit of time and divides this unit among producing goods, producing ideas and producing human capital.

$$(7.1)\quad L_{At} + L_{Yt} = L_t = \epsilon_t N_t$$

Where, at time t, L_t is employment, L_{Yt} is the total amount of raw labor employed in producing output, L_{At} is the total number of researchers. ϵ_t denotes the amount of time individuals spend working and $(1 - \epsilon_t)$ represents the amount of time the individual spends accumulating human capital.

Physical capital is accumulated by foregoing consumption.

$$(7.2)\quad \dot{K} = s_{kt}\,Y_t - dK_t\,, \qquad K_0 > 0$$

where s_{kt} is the fraction of output that is invested ((1-s_{kt}) is the fraction consumed). d is the exogenous, constant rate of depreciation. Y_t is the aggregate production of homogenous final good and K_t is capital stock. Hence, we can also write

$$(7.3)\quad \dot{K} = Y_t - C_t - dK_t$$

Output is produced using total quantity of human capital, H_{Yt} and a set of intermediaries j, which are obtained one-from-one from capital.

H_{Yt} is given by

$$(7.4)\quad H_{Yt} = h_t\,L_{Yt}$$

where h_t is human capital per person and L_{Yt} is total amount of raw labor employed in producing output. An individual's human capital (h_t) is produced by foregoing time in the labor force. Since individual spend $(1 - \epsilon_t)$ amount of time accumulating human capital,

$$(7.5) \quad \dot{h}_t = \eta {h_t}^{\beta_1}(1 - \epsilon_t)^{\beta_2} - \theta g_A h_t \,, \qquad 0 < \beta_1 \,,\, \beta_2 \,,\, \epsilon_t < 1,\ \eta > 0, (1 + \theta) > 0$$

Where η is the productivity of human capital in the production of new human capital, θ reflects the effect of technological progress on human capital investment and g_A is the growth rate of technology.

Equation (7.5) builds on the human capital accumulation equation from the Uzawa-Lucas (Uzawa, 1965 and Lucas, 1988) model. Firstly, it is modified to show that higher the level of human capital or time spent accumulating human capital, the more difficult it is to generate additional human capital (Gong, Greiner and Semmler, 2004). This is reflected in the equation by $0 < \beta_1$, $\beta_2 < 1$. Values of β_1 or $\beta_2 = 1$ imply that an increase in the time spent for education or higher level of human capital itself, raises the growth rate of human capital accumulation monotonically which in turn, raises the balanced growth rate. This can be interpreted as "strong" scale effect. US data clearly reject that as shown by Jones(2002). The US economy is fluctuating around its balanced growth path even though educational attainment and research intensity is steadily rising for last 50 years. Secondly, we incorporate the fact that faster technological progress (g_A) may influence the rate of human capital accumulation. This depends on the technological parameter θ ($\theta > -1$) as in Bucci (2008). Hence faster technological progress may increase, decrease or have no effect on human capital investment.

The production function is given by

$$(7.6)\quad Y_t = {H_{Yt}}^{1-\alpha} \int_0^A x(j)^\alpha \, dj$$

$x(j)$ is the input of intermediate j and A is the number of available intermediates, $\alpha \in (0,1)$ and $\frac{1}{1-\alpha}$ is the elasticity of substitution for any pair of intermediates. Research and development (R&D) enables firms to produce new intermediates. The R&D technology is

$$(7.7)\quad \dot{A} = \gamma\, {H_{At}}^{\lambda} {A_t}^{1-\phi}$$

According to this equation, new ideas produced at any point in time depends on the number of researchers (H_{At}) and existing stock of ideas(A_t). ϕ captures the externalities associated from R&D, $\phi > 0$. Past discoveries can either increase or decrease current research productivity. $0 < \lambda \leq 1$ captures the possibility of duplication in research. H_{At} is effective research effort given by

$$(7.8)\quad H_{At} = h_t\, L_{At}$$

7.1 Final Goods Sector

Faced with a price list $\{p(i): i \in R_+\}$ for all the producer durables, the representative final output firm choose a profit–maximizing quantity $x(i)$ for each durable

$$\max_{x,H_y} \int_0^\infty [H_y^{1-\alpha}\ x(i)^{\alpha} - p(i)x(i)]\, di - wH_y$$

Where ‘w’ is the rental rate per unit of human capital. Solving the maximization problem gives

$$(7.9)\quad p(i) = \alpha H_y^{1-\alpha}\, x(i)^{\alpha-1}$$

$$(7.10)\quad w = (1-\alpha)\frac{Y}{H_y}$$

7.2 Intermediate Goods Sector

Given the demand curve in equation (7.9), the producer of each specialized durable chooses the profit maximizing output to set. Faced with a given value of H_y and r, a firm that has already incurred the fixed cost of investment in a design will choose a level of output x to maximize its revenue minus variable cost at every date.

$$\pi = \max_x p(x)x - rx$$

where 'r' is the interest rate on loans denominated in goods. Solving the Monopoly profit maximization problem gives

$$(7.11)\quad p(i) = \bar{p} = \frac{r}{\alpha}$$

The flow of monopoly profit is

$$(7.12)\quad \pi(i) = \bar{\pi} = \bar{p}\bar{x} - r\bar{x} = (1-\alpha)\bar{p}\bar{x}$$

7.3 R&D Sector

The market for designs is competitive. Hence the price for designs P_A will bid up until it is equal to present value of the net revenue that a monopoly can extract.

$$(7.13)\quad \int_t^{\infty} e^{-\int_t^{\tau} r(s)ds}\pi(\tau)d\tau = P_A(t)$$

Because of the assumption that anyone engaged in research can freely take advantage of the entire existing stock of designs in doing research to produce new designs, its follow from R&D technology equation (7.7),

$$(7.14)\quad wH_A = P_A\, \gamma H_A^{\lambda}\, A^{1-\phi}$$

If $\upsilon(t)$ denote the value of the innovation

$$(7.15)\quad \upsilon(t) = \int_t^{\infty} e^{-\int_t^{\tau} r(s)ds}\pi(\tau)d\tau$$

Therefore, equation (7.14) can be equivalently written as

$$(7.16)\quad wH_A = \upsilon\gamma H_A^{\lambda} A^{1-\phi}$$

Also because of symmetry with respect to different intermediate, $K = Ax$, equation (7.6) is written as

$$(7.17)\quad Y = (AH_Y)^{1-\alpha}(K)^{\alpha}$$

Hence, from equation (7.10) and (7.17),

$$(7.18)\quad w = (1-\alpha)A\left(\frac{K}{AH_Y}\right)^{\alpha}$$

From zero profits in the final goods sector $\pi = {H_Y}^{1-\alpha}Ax^{\alpha} - pAx - wH_Y = 0$ and equation (7.10)

$$(7.19)\quad Y - wH_Y = Apx = \alpha Y$$

Notice that wages equalize across sectors due to free entry and exit.

7.4 Consumers

Each individual supply labor and receive some amount of consumption, $c(t)$. Individual maximize the intertemporal utility function choosing consumption and the fraction of time to devote in human capital production (or the fraction of time to devote in market work). Hence, the agent's problem is

$$\max_{c_t,\epsilon_t} \int_t^{\infty} e^{-(\rho-n)t} \; [c(\tau)^{1-\sigma} - 1] \,/\, (1-\sigma) \quad dt$$

Subject to

$$\dot{K} = r_t[K_t + v_t A_t] + w_t H_t - c_t N_t - v_t \dot{A}_t - \dot{v}_t A_t$$

$$\dot{h}_t = \eta {h_t}^{\beta_1}(1 - \epsilon_t)^{\beta_2} - \theta g_A h_t \text{ and,}$$

$$\epsilon_t \in [0,1]$$

where ρ $(\rho > n > 0)$ is the subjective discount rate, and σ (≥ 0) is the inverse of intertemporal elasticity of substitution in consumption.

Balanced Growth Path (BGP) Analysis

Definition 7.1. (Balanced Growth Path (BGP)) I define a BGP as a state where variables *A*, *K*, *H* and *Y* grow at a constant (possibly positive) rates, (ii) technological progress (*A*) and the available stock of human capital (*H*) grow at the same rate, $g_A = g_H$ and, (iii) r , $\frac{A}{K}$, $\frac{H}{K}$ and $\frac{H_A}{K}$ are constants (iv) the amount of time the individual spends on accumulating human capital is constant $\dot{\varepsilon} = 0$.

It can be shown that a BGP equilibrium exists (Bucci 2008) when the R&D technology has constant returns to scale in H_A and A together. Using equation (7.7) and (7.8) and using the fact that $\frac{\dot{L_A}}{L_A} = n$, the growth rate of A, $g_A = \gamma \frac{H_A^{\lambda}}{A^{\phi}}$, is constant along a BGP when, $g_A = \frac{\lambda}{\phi}[g_h + n]$ is constant, that is when $\lambda = \phi$.

Using Definition 7.1 and the fact that in a BGP R&D technology exhibits constant returns to scale, we get the following result along the BGP (see Appendix 4),

$$g_C = g_K = g_H = g_A = \frac{\rho - n(\sigma - \xi)}{(1+\theta)\xi + 1 - \sigma - \theta(1 - \beta_1)} \text{ and}$$

$$g_Y = (2 - \alpha) g_A, \qquad (1_{BGP})$$

$$\text{where } \xi = \frac{\beta_2 L_Y}{(1 - \varepsilon)N} = \text{constant}$$

Hence along a BGP, consumption (*C*), physical capital (*K*), human capital (*H*) and technology (*A*) grow at a same constant rate.

Proposition 7.1: The long run growth rate depends on the preference and technological parameters. A positive long run growth rate exists under certain combination of these parameters. While the effect of population growth on growth rate of output is ambiguous, positive long run output growth rate is still achievable with zero population growth.

Proof: A simple inspection of equation (1_{BGP}) proves the results of proposition 1.

Chapter 8. Bifurcation Analysis of the Modified Jones Model

In this chapter, we examine the existence of codimension 1 and codimension 2 bifurcations in the Modified Jones Model.

Let $m = \frac{Y}{K}$ and $g = \frac{cN}{K}$. Using equations (7.11), (7.19) and $K = Ax$ implies

$$(8.0) \quad r = \alpha^2 m$$

And the physical capital equation can be written as,

$$(8.1) \quad \frac{\dot{K}}{K} = m - g - d$$

The consumers intertemporal optimization conditions are (for proof see appendix 3)

$$(8.2) \quad \frac{\dot{c}}{c} = \frac{r - \rho}{\sigma} = \frac{\alpha^2 m - \rho}{\sigma}$$

$$(8.2') \quad -r + \frac{\dot{h}}{h}\left(\frac{\beta_2 L_Y}{(1-\varepsilon)N} + 1\right) + \theta g_A\left(\frac{\beta_2 L_Y}{(1-\varepsilon)N} - (1 - \beta_1)\right) = (\beta_2 - 1)\frac{\dot{(-\varepsilon)}}{(1-\varepsilon)} - \frac{\dot{w}}{w} - n$$

Substituting equations (8.0), (8.1), (8.2) and using $g = \frac{cN}{K}$, we can derive $\frac{\dot{g}}{g} = \frac{\dot{c}}{c} - \frac{\dot{N}}{N} - \frac{\dot{K}}{K}$.

$$(8.3) \quad \frac{\dot{g}}{g} = \left(\frac{\alpha^2}{\sigma} - 1\right)m - \frac{\rho}{\sigma} + n + g + d$$

Now multiplying both sides of equation (7.1) by h_t, and using the definitions of equations (7.4) and (7.8),

$$(8.4) \quad \underbrace{h_t L_{At}}_{H_{At}} + \underbrace{h_t L_{Yt}}_{H_{Yt}} = \underbrace{h_t L_t}_{H_t} = h_t \epsilon_t N_t$$

Let $u = \frac{\gamma}{A^{\phi}} h\epsilon N$ and $\nu = \frac{(1-\alpha)Y}{\upsilon A}$

Using equations (7.10), (7.16) and (8.24) in equation (7.7), $\frac{\dot{A}}{A} = \frac{\gamma H_A}{A^{\phi}}$, and setting λ=1 for the rest of the analysis,

$$(8.5) \quad \frac{\dot{A}}{A} = \underbrace{\frac{\gamma}{A^{\phi}} h\epsilon N}_{u} - \underbrace{\frac{(1-\alpha)Y}{\upsilon A}}_{\nu}$$

The following can be shown from equation (7.15) and using $\pi = \frac{\alpha(1-\alpha)Y}{A}$ (from equations (7.12) and (7.19)),

$$(8.6) \quad \frac{\dot{\upsilon}}{\upsilon} = r - \frac{\pi}{\upsilon} = \alpha^2 m - \alpha\nu$$

Let $f = \frac{\epsilon_t}{(1-\epsilon_t)}$. Using equation (7.10) and (7.16), it can be shown,

$$(8.7) \quad \frac{L_{Yt}}{(1-\epsilon_t)N_t} = \frac{1}{(1-\epsilon_t)h_t N_t}\frac{(1-\alpha)Y_t}{w_t} = \frac{\nu f}{u}$$

Let $z = \frac{\eta(1-\epsilon_t)^{\beta_2}}{{h_t}^{1-\beta_1}}$, equation (7.5) can be written as,

$$(8.8) \quad \frac{\dot{h}}{h} = z - \theta g_A$$

We can derive $\frac{\dot{w}}{w} = \frac{\dot{\upsilon}}{\upsilon} + (1-\phi)\frac{\dot{A}}{A}$ from equation (7.16) and substitute equation (8.5) and (8.6) in it, to get

$$(8.9) \quad \frac{\dot{w}}{w} = \alpha^2 m - \alpha\nu + (1-\phi)(u-\nu)$$

Equation (8.2') is simplified in the following way by using (8.7), (8.8) and (8.9)

$$(8.10)\quad \frac{\dot{\epsilon}}{\epsilon} = \frac{1}{f(\beta_2 - 1)}\left[-z - \theta g_A(\beta_1 - 2) + \alpha\nu - \beta_2 \frac{z\nu f}{u} - (1-\phi)(u-\nu) - n\right]$$

$$(8.11)\quad m = \frac{Y}{K} = \left(\frac{AH_Y}{K}\right)^{1-\alpha} \quad \text{from equation (7.17)}$$

Using equation (7.16), (7.18) and (8.11), $\upsilon\gamma A^{1-\phi} = w = (1-\alpha)A\left(\frac{K}{AH_Y}\right)^{\alpha} \Rightarrow m^{\frac{\alpha}{1-\alpha}} = \frac{(1-\alpha)A^{\phi}}{\upsilon\gamma}$

Substituting equations (8.5) and (8.6) in, $\frac{\dot{m}}{m} = \frac{(1-\alpha)}{\alpha}\left[-\frac{\dot{\upsilon}}{\upsilon} + \phi\frac{\dot{A}}{A}\right]$, derived from the above relation

$$(8.12)\quad \frac{\dot{m}}{m} = \frac{(1-\alpha)}{\alpha}\left[-\alpha^2 m + \alpha\nu + \phi(u-\nu)\right]$$

From equations (8.1) and (8.12) and using $\frac{\dot{Y}}{Y} = \frac{\dot{m}}{m} + \frac{\dot{K}}{K}$,

$$(8.13)\quad \frac{\dot{Y}}{Y} = \frac{(1-\alpha)}{\alpha}\left[-\alpha^2 m + \alpha\nu + \phi(u-\nu)\right] + (m - g - d)$$

Plugging in results from (8.5), (8.6) and (8.13) in $\frac{\dot{\nu}}{\nu} = \frac{\dot{Y}}{Y} - \frac{\dot{\upsilon}}{\upsilon} - \frac{\dot{A}}{A}$

$$(8.14)\quad \frac{\dot{\nu}}{\nu} = \left[(1-\alpha)m + \nu - g + \left\{\frac{(1-\alpha)\phi}{\alpha} - 1\right\}(u-\nu) - d\right]$$

Using equation (8.10) in $\frac{\dot{z}}{z} = -\beta_2 f\frac{\dot{\epsilon}}{\epsilon} - (1-\beta_1)\frac{\dot{h}}{h}$ and $\frac{\dot{f}}{f} = \frac{\dot{\epsilon}}{\epsilon}(1+f)$, we derive,

$$(8.15)\quad \frac{\dot{z}}{z} = \frac{1}{f(\beta_2-1)}\left[-z - \theta g_A(\beta_1 - 2) + \alpha\nu - \beta_2\frac{z\nu f}{u} - (1-\phi)(u-\nu) - n\right] -$$
$$(1-\beta_1)(z - \theta g_A)$$

$$(8.16)\quad \frac{\dot{f}}{f} = \frac{(1+f)}{f(\beta_2-1)}\left[-z - \theta g_A(\beta_1 - 2) + \alpha\nu - \beta_2\frac{z\nu f}{u} - (1-\phi)(u-\nu) - n\right]$$

Using equations (8.5), (8.8), (8.10) in $\frac{\dot{u}}{u} = \frac{\dot{h}}{h} + \frac{\dot{N}}{N} - \phi\frac{\dot{A}}{A} + \frac{\dot{\epsilon}}{\epsilon}$,

$$(8.17)\quad \frac{\dot{u}}{u} = z - \theta g_A + n - \phi(u - \nu) + \frac{\left[-z-\theta g_A(\beta_1-2)+\alpha\nu-\beta_2\frac{z\nu f}{u}-(1-\phi)(u-\nu)-n\right]}{f(\beta_2-1)}$$

Equations (8.3), (8.12), (8.14), (8.15), (8.16) and (8.17) represent the dynamic equations for the model.

8.1 Steady State

Definition 8.1. (Steady State) We define a steady state as a state where variables g, m, v, z, f and u grow at a constant (possibly zero) rates. A steady state is a BGP with zero growth rate.

Therefore, the steady state $s^* = (g^*, m^*, v^*, z^*, f^*, u^*,)$ is such that, $\dot{g} = \dot{m} = \dot{v} = \dot{z} = \dot{f} = \dot{u} = 0$. It is derived by solving the following equations (I)-(VI).

$$(I)\quad \left(\frac{\alpha^2}{\sigma} - 1\right)m - \frac{\rho}{\sigma} + n + g + d = 0$$

$$(II)\quad -\alpha^2 m + \alpha\nu + \phi(u - \nu) = 0$$

$$(III)\quad (1 - \alpha)m + \nu - g - d + \left\{\frac{(1-\alpha)\phi}{\alpha} - 1\right\}(u - \nu) = 0$$

$$(IV)\quad z - \theta g_A + n - \phi(u - \nu) + \frac{\left[-z-\theta g_A(\beta_1-2)+\alpha\nu-\beta_2\frac{z\nu f}{u}-(1-\phi)(u-\nu)-n\right]}{f(\beta_2-1)} = 0$$

$$(V)\quad \frac{1}{f(\beta_2 - 1)}\left[-z - \theta g_A(\beta_1 - 2) + \alpha\nu - \beta_2\frac{z\nu f}{u} - (1 - \phi)(u - \nu) - n\right] - (1 - \beta_1)(z - \theta g_A) = 0$$

$$\text{(VI)} \quad -z - \theta g_A(\beta_1 - 2) + \alpha\nu - \beta_2 \frac{z\nu f}{u} - (1-\phi)(u-\nu) - n = 0$$

The steady state, $s^* = \{z^* = \frac{n*\theta}{\phi},\ v^* = \frac{\rho - n}{\alpha} + \frac{n*\sigma}{\phi*\alpha},\ u^* = v^* + \frac{n}{\phi},\ m^* = \frac{v^*}{\alpha} + \frac{n}{\alpha^2},$

$$g^* = \left(1 - \frac{\alpha^2}{\sigma}\right) * m^* + \frac{\rho}{\sigma} - n - d,\ f^* = \frac{u^*}{v^* * \beta_2} * \left(\frac{\phi*\rho}{\theta*n} - \frac{(\phi+1-\sigma)}{\theta} - (\beta_1 - 1)\right)\}$$

Theorem 8.1. A unique steady state exists if

$$\Lambda = (1+\phi)(\sigma - 1)g_A + \rho - n > 0$$

Proof: Λ is the necessary and sufficient for the transversality condition for the consumer's utility maximization problem to hold (appendix 3)

8.2 Local Bifurcation Analysis

We examine the existence of codimension 1 and 2 bifurcations in the dynamical system defined by equations (8.3), (8.12), (8.14), (8.15), (8.16), and (8.17). The codimension, as defined by Kuznetsov (2004), is the number of independent conditions determining the bifurcation boundary. This procedure of varying a single parameter helps us to identify codimension-1 bifurcation and varying 2 parameters helps us to identify codimension-2 bifurcation.

Andronov-Hopf bifurcation is the birth of a limit cycle from an equilibrium in the dynamical system, when the equilibrium changes stability via a pair of purely imaginary eigenvalues. We use the numerical continuation package Matcont to detect such bifurcations. While some of the limit cycles generated by Andronov-Hopf bifurcation are stable (supercritical bifurcation), there could be some unstable limit cycles (subcritical bifurcation) created as well. Table 2 reports the values of subjective discount rate (ρ), share of human capital and share of time devoted for the human capital production (β_1& β_2, respectively), effect of technological progress on human capital accumulation (θ) and the depreciation rate of capital,(d) at which Hopf bifurcation occurs when they are treated as free parameters.

A positive value of the first Lyapunov coefficient indicates creation of subcritical Hopf bifurcation. Thus for each of the cases reported in Table 2, an unstable limit cycle (periodic orbit) bifurcates from the equilibrium. When $\rho, \beta_1, \theta \& \ d$ are treated as free parameters, a slight perturbation of them give rise to Branch Points (Pitchfork/Transcritical bifurcations). Notice that some of the Hopf points detected are neutral saddles and are not bifurcations.

The cyclical behavior could occur for various reasons. For instance, suppose profit for monopolist increases. As the market for designs is competitive, the price for designs P_A, bids up until it is equal to present value of the net revenue that a monopoly can extract. From equation (7.14), wages in the R&D sector rises. As a result of higher wages in the research sector, labor move out of output production to research sector. When sufficient amount of externalities to R&D ($(1-\phi > 0$ in equation (7.7)) is present, the growth rate of technology g_A starts rising. If there is a negative effect of technical progress on human capital investment $(\theta > 0)$, human capital accumulation start declining. The price of final good durables is a positive function of the average quality of labor given by equation (7.4) and (7.9). This implies that prices start falling in the final goods sector due to decline in average quality of labor which in turn, implies that monopoly profits start falling.

We further investigate the stability properties of cycles generated by different combination of such parameters. Continuation of limit cycle from the Hopf point for the case when ρ is the free parameter gives rise to two Period Doubling (flip) bifurcations. Period doubling bifurcation is defined as a situation when a new limit cycle emerges from an existing limit cycle, and the period of the new limit cycle is twice that of the old one. The initial period doubling bifurcations occur at ρ = 0.0257 and ρ = 0.0258 with a negative normal form coefficients indicating stable double-period cycles are involved.

Continuing computation further from the Hopf point, gives rise to Limit Point (Fold/ Saddle Node) bifurcation of Cycles. From the family of limit cycles bifurcating from the Hopf point, Limit Point Cycle (LPC) is a fold bifurcation of the cycle where two limit cycles with different periods are present near LPC point at ρ = 0.0258. We get another Period Doubling (flip) bifurcations upon further computation.

We carry out the continuation of limit cycle from the second Hopf point for the case when θ is treated as the free parameter. We investigate the existence of codimension-2 bifurcations by allowing two free parameters θ and ρ for the first case and θ & β_1 for the second. Two points were detected corresponding to codim 2 bifurcations: Bogdanov-Takens and Generalized Hopf (Bautin) for each of the cases. At each Bogdanov-Takens point the system has an equilibrium with a double zero eigenvalue and the normal form coefficients (a; b) are reported in Table 2 which are all nonzero. The Generalized Hopf points are nondegenerate since the second Lyapunov coefficient l2 are nonzero. The Generalized Hopf (Bautin) bifurcation is a bifurcation of an equilibrium at which the critical equilibrium has a pair of purely imaginary eigenvalues and the first Lyapunov coefficient for the Andronov-Hopf bifucation vanishes. The bifurcation point separates branches of sub- and supercritical Andronov-Hopf bifurcations in the parameter plain. For nearby parameter values, the system has two limit cycles which collide and disappear via a saddle-node bifurcation of periodic orbits.

<table>
<tr><th colspan="3">Table 2</th></tr>
<tr><th>Parameters Varied</th><th>Equilibrium Bifurcation</th><th>Continuation</th></tr>
<tr><td>$\boldsymbol{\beta_1}$
(Figure i)
$\{\alpha = 0.4, \rho = 0.055,\ \beta_2 = 0.04, n = 0.01, d = 0, \theta = 0.4, \phi = 1, \sigma = 8\}$</td><td>Branch Point (BP)
$\boldsymbol{\beta_1} = 1$</td><td></td></tr>
<tr><td>$\boldsymbol{\beta_1}$
(Figure ii)
$\alpha = 0.4, \rho = 0.025772,\ \beta_2 = 0.04, n = 0.01, d = 0, \theta = 0.4, \phi = 0.8, \sigma = 0.08\ \}$</td><td>Hopf (H)
First Lyapunov coefficient = 0.0000230, $\boldsymbol{\beta_1} = 0.19$</td><td></td></tr>
<tr><td>$\boldsymbol{\beta_2}$
$\{\alpha = 0.4,\ \beta_1 = 0.19, \rho = 0.025772, n = 0.01, d = 0, \theta = 0.4, \phi = 0.8, \sigma = 0.08\}$</td><td>Hopf (H)
First Lyapunov coefficient =0.00002302, $\beta_2 = 0.040000$</td><td></td></tr>
<tr><td>$\boldsymbol{d}$
$\{\alpha = 0.4,\ \beta_1 = 0.19, \rho = 0.055,\ \beta_2 = 0.04, n = 0.01, \theta = 0.4, \phi = 1, \sigma = 8\}$</td><td>Branch Point (BP)
$\boldsymbol{d} = 0.826546$</td><td></td></tr>
<tr><td rowspan="6">$\boldsymbol{\rho}$
(Figure iii)
$\{\alpha = 0.4,\ \beta_1 = 0.19, \rho = 0.055,\ \beta_2 = 0.04, n = 0.01, d = 0, \theta = 0.4, \phi = 1, \sigma = 0.08\}$</td><td>Figure iii (A)</td><td>Figure iii (B)</td></tr>
<tr><td rowspan="5">Hopf (H)
First Lyapunov coefficient = 0.0000149
$\boldsymbol{\rho}$ =0.025772</td><td>Bifurcation of Limit Cycle</td></tr>
<tr><td>Period Doubling
(period = 1,569.64; $\boldsymbol{\rho}$ = 0.0257)
Normal form coefficient =
-4.056657e-013</td></tr>
<tr><td>Period Doubling
(period = 1,741.46; ρ = 0.0258)
Normal form coefficient =
-7.235942e-015</td></tr>
<tr><td>Limit point cycle
(period = 2,119.53; ρ = 0.0258)
Normal form coefficient=
7.894415e-004</td></tr>
<tr><td>Period Doubling
(period = 2,132.13; ρ = 0.0258)
Normal form coefficient =
-1.763883e-013</td></tr>
</table>

<table>
<tr><td rowspan="2"></td><td>Branch Point (BP)
$\boldsymbol{\rho}$=0.026726</td><td></td></tr>
<tr><td>Hopf (H), Neutral Saddle, $\boldsymbol{\rho}$=0.026698</td><td></td></tr>
<tr><td rowspan="9">$\boldsymbol{\theta}$
(Figure iv)
$\{\alpha = 0.4,\ \beta_1 = 0.19, \rho = 0.029710729,\ \beta_2 = 0.04, n = 0.01, d = 0, \phi = .69716983, \sigma = 0.08\}$</td><td>Figure iv (A)</td><td>Figure iv (B)</td></tr>
<tr><td>Hopf (H)
First Lyapunov coefficient =0.0000230, $\boldsymbol{\theta}$=0.400000</td><td></td></tr>
<tr><td rowspan="5">Hopf (H)
First Lyapunov coefficient = 0.00001973
θ=0.355216</td><td>Codimension-2 bifurcation</td></tr>
<tr><td>Generalized Hopf (GH)
$\boldsymbol{\theta} = 0.000044$, $\boldsymbol{\rho}$ = 0.580853
l2= (0.000001254)</td></tr>
<tr><td>Bogdanov-Takens (BT)
$\boldsymbol{\theta}$ =0, $\boldsymbol{\rho}$=0.644247
(a,b)= (0.000001642, -0.003441)</td></tr>
<tr><td>Generalized Hopf (GH)
$\boldsymbol{\theta} = 0.000055$, $\boldsymbol{\beta_1} = 0.584660$
l2=0.0000008949</td></tr>
<tr><td>Bogdanov-Takens (BT)
$\boldsymbol{\theta}$ =0, $\boldsymbol{\beta_1}$=0.903003
(a,b)=(0.000006407790,0.03291344)</td></tr>
<tr><td>Hopf (H)
Neutral saddle, $\boldsymbol{\theta}$=0.612624</td><td></td></tr>
<tr><td>Branch Point (BP),
$\boldsymbol{\theta} = 0.613596$</td><td></td></tr>
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Figure (i)

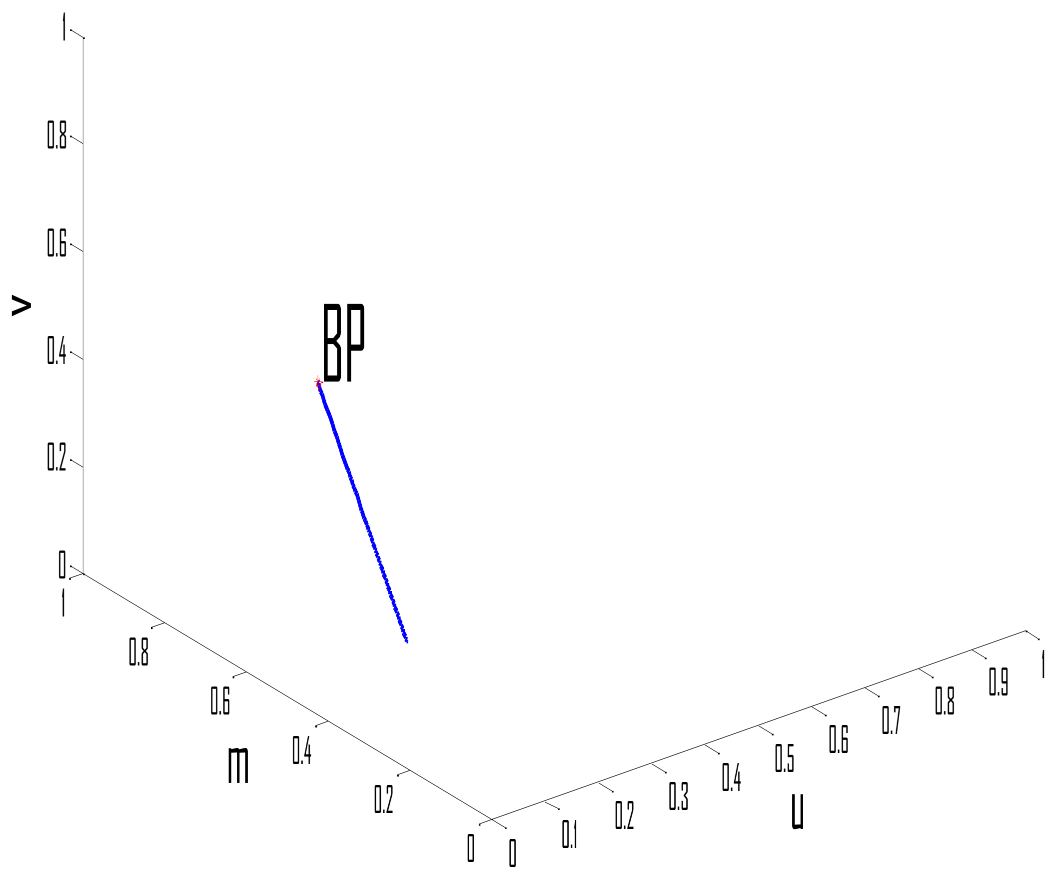


Figure (ii)

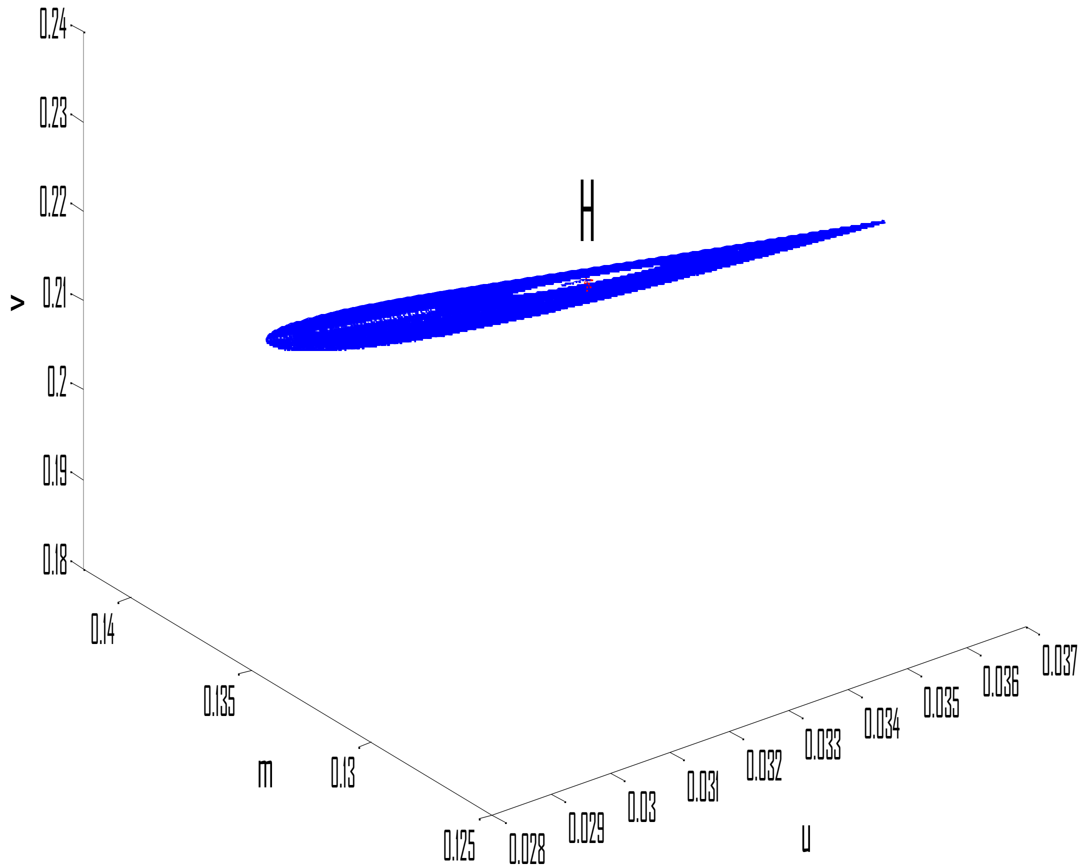

0.24
0.23
0.22
0.21
0.2
0.19
0.18
v
H
0.14
0.135
0.13
0.125
m
0.028
0.029
0.03
0.031
0.032
0.033
0.034
0.035
0.036
0.037
u

Figure (iii) A

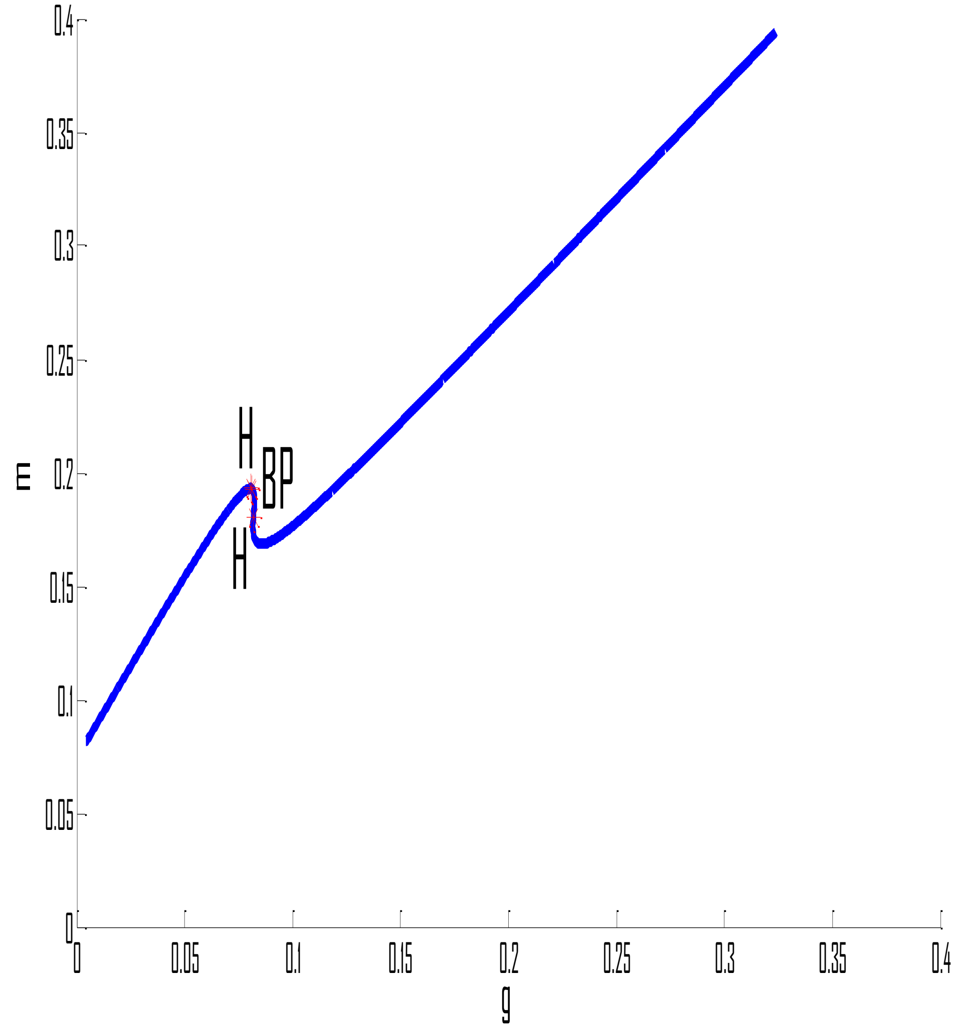
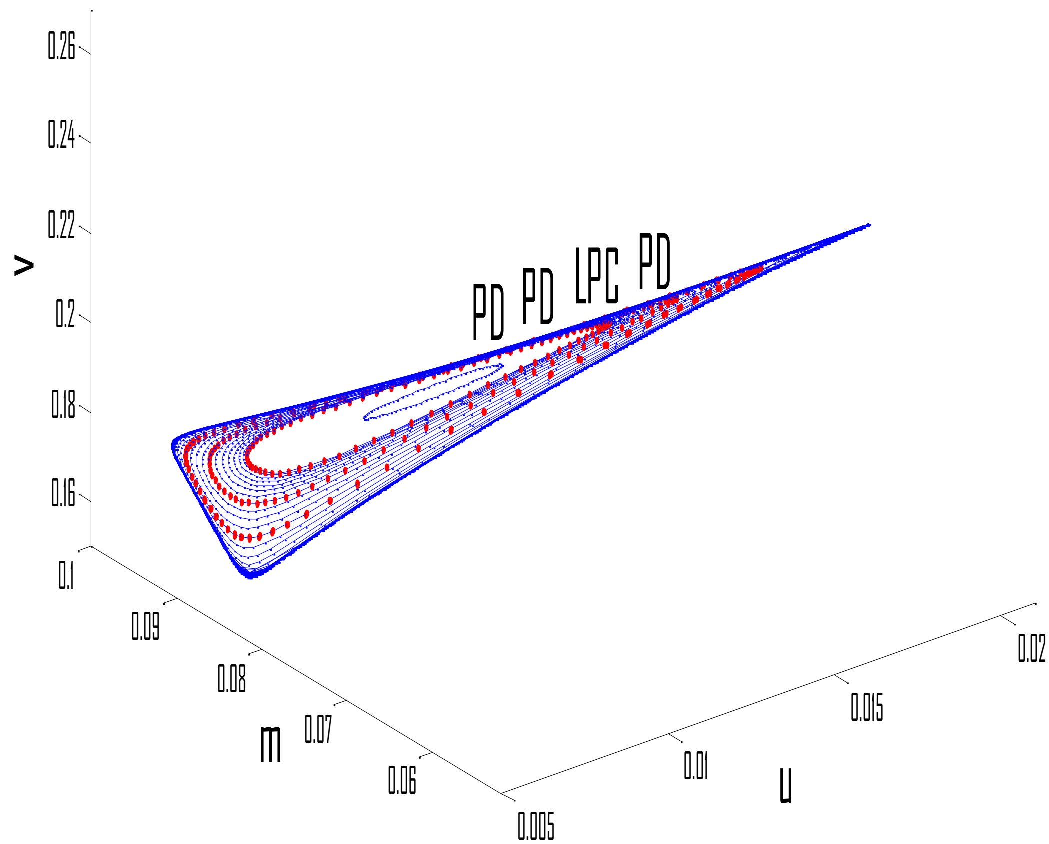
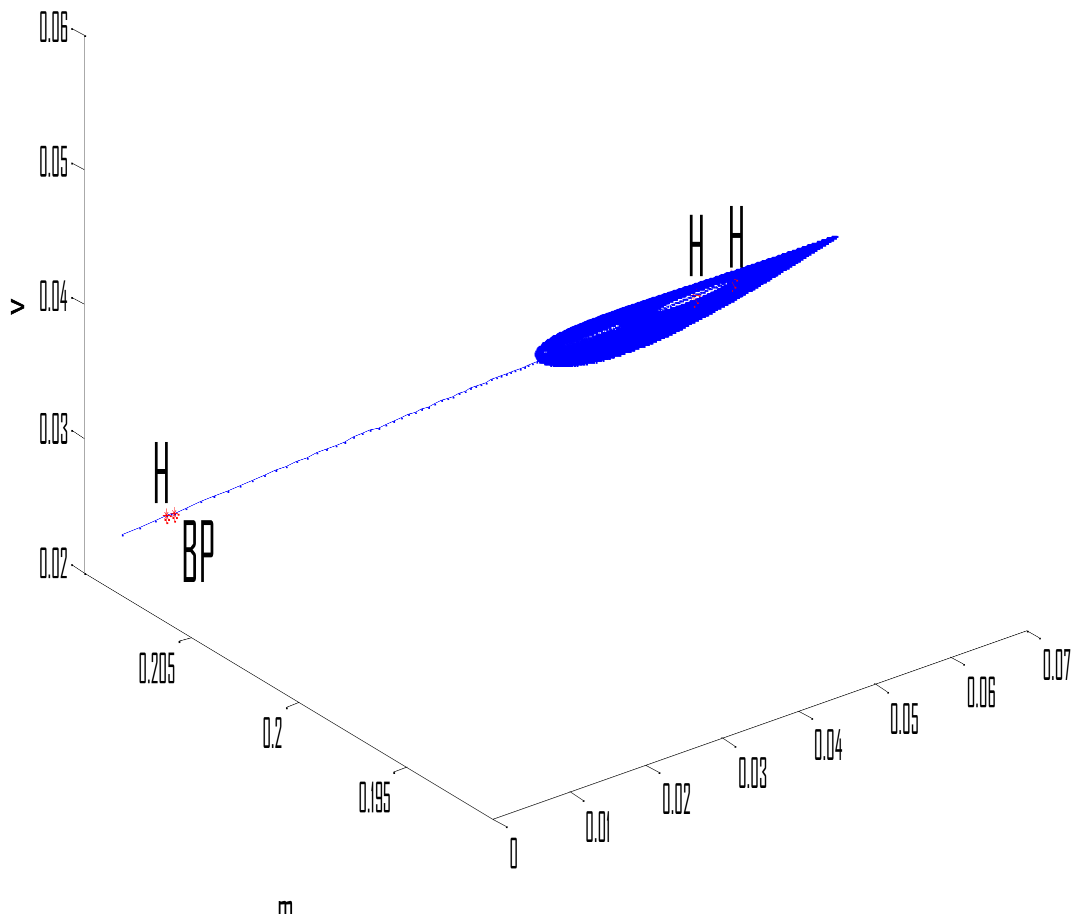


Figure (iii) B



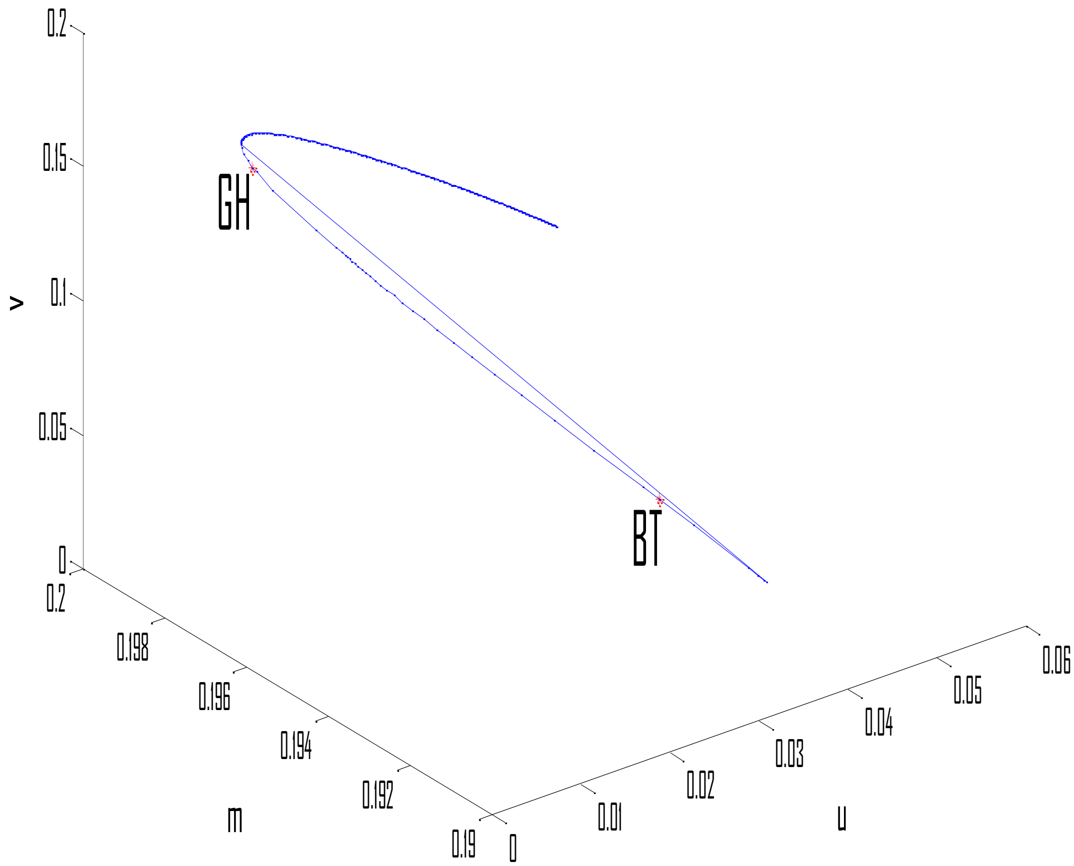
PD
PD
LPC
PD
0.26
0.24
0.22
0.2
0.18
0.16
0.1
0.09
0.08
0.07
0.06
0.005
0.01
0.015
0.02
v
m
u

Figure (iv) A



0.06
0.05
0.04
0.03
0.02
V
H
H
H
BP
0.205
0.2
0.195
m
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07

Figure (iv) B



0.2
0.15
0.1
0.05
0
v
GH
BT
0.2
0.198
0.196
0.194
0.192
0.19
m
0
0.01
0.02
0.03
0.04
0.05
0.06
u

Chapter 9. Conclusion

This thesis provides a detailed stability and bifurcation analysis of the Uzawa-Lucas model and the Modified Jones model. Transcritical bifurcation and Hopf bifurcation boundaries, corresponding to different combination of parameters, are located for the decentralized version of the Uzawa-Lucas model. Examination of the stability properties of the limit cycles from various Hopf bifurcations in the model depicts occurrence of limit point-of-cycles bifurcations and period-doubling bifurcations within the model's feasible parameter set. The series of Period Doubling bifurcations confirms the presence of global bifurcation. This also highlights the possibility of having chaotic dynamics in the model. On the contrary, the social planner solution for the Uzawa-Lucas model is always saddle path stable with no possibility of occurrence of bifurcation in the feasible parameter range of the model. Thus the externality of human capital parameter plays an important role in determining the dynamics of the decentralized Uzawa-Lucas model.

Our result emphasizes the need for simulations of decentralized macroeconometric models at settings throughout the parameter-estimates' confidence regions, rather than at the point estimates alone, since dynamical inferences otherwise can produce oversimplified conclusions subject to robustness problems.

Along the balanced growth path in the modified Jones model, I have shown that the long run growth rate of the model does depend on the rate of population growth. But the long run growth rate can even be positive with no population growth. Several Andronov-Hopf bifurcations and Branch Points are located. The stability properties of the limit cycles created from these Hopf bifurcations are examined. We showed the existence several codimension-1 bifurcations (Limit Point of Cycles and Period Doubling bifurcations) and codimension-2 bifurcations (Bogdanov-

Takens and Generalized Hopf) are located. The choice of certain parameters in locating various bifurcations emphasizes the role played by human capital in such a model where growth is driven by technological progress, which in turn, is ultimately driven by human capital investment. The parameters in the human capital accumulation among others equation play a key role in determining the dynamics of the model.

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Appendix

Appendix 1:

Social Planner Problem:

$$\mathcal{H} = \frac{[c(\tau)^{1-\sigma}-1]}{(1-\sigma)} + \lambda[Ak^{\alpha}\varepsilon^{1-\alpha}h^{1-\alpha+\zeta} - c - (n+\delta)K] + \mu[\eta h(1-\varepsilon)].$$

The first order conditions are

(1) c: $c^{-\sigma}e^{-(\rho-n)} = \lambda$

(2) ε: $\lambda(1-\alpha)Ak^{\alpha}\varepsilon^{-\alpha}h^{1-\alpha+\zeta} = \mu\eta h$

(3) k: $\lambda\left[\alpha Ak^{\alpha-1}\varepsilon^{1-\alpha}h^{1-\alpha+\zeta} - (n+\delta)\right] = -\dot{\lambda}$

(4) h: $\lambda(1-\alpha+\zeta)Ak^{\alpha}\varepsilon^{1-\alpha}h^{-\alpha+\zeta} + \mu\eta(1-\varepsilon) = -\dot{\mu}$

Appendix 2:

Decentralized or Market Solution:

$$\mathcal{H} = \frac{[c(\tau)^{1-\sigma}-1]}{(1-\sigma)} + \lambda[Ak^{\alpha}\varepsilon^{1-\alpha}h^{1-\alpha}h_a^{\zeta} - c - (n+\delta)K] + \mu[\eta h(1-\varepsilon)].$$

The first order conditions are

(1) c: $c^{-\sigma}e^{-(\rho-n)} = \lambda$

(2) ε: $\lambda(1-\alpha)Ak^{\alpha}\varepsilon^{-\alpha}h^{1-\alpha}h_a^{\zeta} = \mu\eta h$

(3) k: $\lambda\left[\alpha Ak^{\alpha-1}\varepsilon^{1-\alpha}h^{1-\alpha}h_a^{\zeta} - (n+\delta)\right] = -\dot{\lambda}$

(4) h: $\lambda(1-\alpha)Ak^{\alpha}\varepsilon^{1-\alpha}h^{-\alpha}h_a^{\zeta} + \mu\eta(1-\varepsilon) = -\dot{\mu}$

Appendix 3:

I use the zero profit condition $w_t H_{At} = \ v_t \dot{A}_t$ and equation (8.6), $A_t \dot{v}_t = A_t r_t v_t - A_t \pi_t$ in the wealth accumulation equation of the households $\dot{K} = r_t[K_t + v_t A_t] + w_t H_t - c_t N_t - \dot{A}_t v_t - A_t \dot{v}_t$, to get

$$\dot{K} = r_t K_t + w_t h_t (1 - l_{ht}) N_t - c_t N_t - w_t h_t L_{At} + A_t \pi_t$$

The relevant Hamiltonian for the consumer's problem is

$$\mathcal{H} = \ e^{-(\rho-n)t}[c(\tau)^{1-\sigma} - 1] \,/\, (1-\sigma) + \lambda[r_t K_t + w_t h_t (1 - l_{ht}) N_t - c_t N_t - w_t h_t L_{At} + A_t \pi_t] + \mu[\eta {h_t}^{\beta_1}(1 - \ \epsilon_t)^{\beta_2} - \ \theta g_A h_t \ ]$$

The first order conditions are

i. $\quad c^{-\sigma} e^{-(\rho-n)t} = \ \lambda N \qquad \Rightarrow \frac{\dot{c}}{c} = \frac{r-\rho}{\sigma}$

ii. $\quad \varepsilon\text{:} \quad -\lambda whN - \ \mu\eta h^{\beta_1}\beta_2 \,(1 - \ \varepsilon)^{\beta_2 - 1} = 0 \qquad \Rightarrow \frac{\lambda}{\mu} = \frac{\eta h^{\beta_1 - 1}(1-\varepsilon)^{\beta_2 - 1}\beta_2}{wN}$

iii. $\quad K\text{:} \quad \lambda r = \ -\dot{\lambda} \qquad \Rightarrow \frac{\dot{\lambda}}{\lambda} = -\mathrm{r}$

iv. $\quad h\text{:} \quad \lambda w\varepsilon N - \lambda w L_A + \ \mu\eta\beta_1 h^{\beta_1 - 1}(1 - \ \varepsilon)^{\beta_2} - \ \mu\theta g_A = \ -\dot{\mu}$

Dividing (iv) by μ and substituting (ii) in it,

$$\eta h^{\beta_1 - 1}(1 - \ \varepsilon)^{\beta_2}\left[\frac{\beta_2 L_Y}{(1-\varepsilon)N} + \ \beta_1\right] - \ \theta g_A = -\frac{\dot{\mu}}{\mu} \quad \text{(v)}$$

Now, from (ii),

$\frac{\dot{\lambda}}{\lambda} - \frac{\dot{\mu}}{\mu} = (\beta_2 - 1)\left(\frac{(-\dot{\varepsilon})}{(1-\varepsilon)}\right) + \ (\beta_1 - 1)\frac{\dot{h}}{h} - \frac{\dot{w}}{w} - \ n$ and substituting (iii)and (v) in it,

$$-r+\frac{\dot{h}}{h}\left(\frac{\beta_2 L_Y}{(1-\varepsilon)N}+1\right)+\theta g_A\left(\frac{\beta_2 L_Y}{(1-\varepsilon)N}-(1-\beta_1)\right)=(\beta_2-1)\frac{(\dot{-\varepsilon})}{(1-\varepsilon)}-\frac{\dot{w}}{w}-n$$

Transversality Conditions:

$$\lim_{t\to\infty}\lambda_t[K_t+v_t A_t]=0$$

$$\lim_{t\to\infty}\mu_t h_t=0$$

In a steady state, $\dot{g}=\dot{m}=\dot{v}=\dot{u}=\dot{f}=\dot{z}=0$, and using the fact that in the steady state, $\frac{H_Y}{H}$ is constant, we can derive the following relations

$$\dot{g}=0 \text{ and } \dot{m}=0 \text{ implies } \frac{\dot{K}}{K}=\frac{\dot{c}}{c}+n=\frac{\dot{Y}}{Y}$$

We use $\dot{z}=-\beta_2\frac{\dot{\epsilon}}{\epsilon}f-(1-\beta_1)\frac{\dot{h}}{h}=0$ and $\dot{f}=\frac{\dot{\epsilon}}{\epsilon}(1+f)=0$ to derive the following

$$\frac{\dot{\epsilon}}{\epsilon}=\frac{(-1+\beta_1)}{\beta_2}\frac{\dot{h}}{h}$$

Hence $\dot{u}=\frac{\dot{h}}{h}+\frac{\dot{N}}{N}-\phi\frac{\dot{A}}{A}+\frac{\dot{\epsilon}}{\epsilon}=0$ implies $\frac{\dot{h}}{h}=\frac{\beta_2}{(1-\beta_1+\beta_2)}[\phi g_A-n]$

$$\frac{\dot{H_Y}}{H_Y}=\frac{\dot{H}}{H}=\frac{\dot{h}}{h}+\frac{\dot{N}}{N}+\frac{\dot{\epsilon}}{\epsilon}=\phi g_A$$

From equation (8.12),

$$m=\frac{Y}{K}=\left(\frac{AH_Y}{K}\right)^{1-\alpha}$$

$$\frac{\dot{m}}{m} = (1-\alpha)\left[\frac{\dot{A}}{A} + \frac{\dot{H}_Y}{H_Y} - \frac{\dot{K}}{K}\right] = 0 \text{ implies } \frac{\dot{K}}{K} = \frac{\dot{A}}{A} + \frac{\dot{H}_Y}{H_Y} = (1+\phi)g_A$$

$$\text{Hence,} \quad \frac{\dot{K}}{K} = \frac{\dot{C}}{C} = \frac{\dot{Y}}{Y} = (1+\phi)g_A$$

Hence, the transversality condition implies that,

$$(1+\phi)(\sigma-1)g_A + \rho - n > 0$$

Appendix 4:

$\frac{A}{K}$ is constant $\Rightarrow$ $\bar{x}$ is constant since $K = Ax.$ The flow of monopoly profit is $\bar{\pi} = (1-\alpha)\bar{p}\bar{x}$. From equation (7.13), we know the PDV of this stream of profit must equal to price P_A of the design and using equation (7.9),

$$\upsilon = P_A = \frac{1}{r}\pi = \frac{1}{r}(1-\alpha)\bar{p}\bar{x} = \frac{(1-\alpha)}{r}\alpha H_Y^{1-\alpha}x^{\alpha-1}x = \frac{\alpha(1-\alpha)}{r}H_Y^{1-\alpha}x^{\alpha}$$

The condition determining the allocation of H_Y and H_A says that wages paid to human capital in each sector must be the same, from equation (7.16) and (7.18),

$$w = \upsilon\gamma H_A^{\lambda-1}A^{1-\phi} = (1-\alpha)\frac{H_Y^{1-\alpha}Ax^{\alpha}}{H_Y}$$

$$\Rightarrow H_Y = \frac{r}{\alpha}\frac{A^{\phi}}{\gamma H_A^{\lambda}}H_A = \frac{r}{\alpha g_A}H_A$$

From equation (7.11) and (7.12), we get, $\bar{\pi} = \frac{(1-\alpha)}{\alpha}r\bar{x}$ and from, equation (7.15),

$$\upsilon(t) = \frac{\pi}{r}, \frac{\dot{\upsilon}}{\upsilon} = 0$$

$$\frac{\dot{w}}{w} = \frac{\dot{\upsilon}}{\upsilon} + (\lambda - 1)\frac{\dot{H}_A}{H_A} + (1-\phi)\frac{\dot{A}}{A} = 0 \text{ under CRS}$$

Dividing the household's wealth accumulation equation $\dot{K} = rK + wH_Y - cN + A\pi$ by K and incorporating, $\frac{-\dot{\lambda}}{\lambda} = r$, we get

$$\frac{\dot{\lambda}}{\lambda} = -g_K + w\frac{H_Y}{K} - \frac{cN}{K} + \frac{A}{K}\pi \qquad \text{(a)}$$

For the first order conditions from the consumer’s optimization problem (see Appendix 3 for derivation),

$$\eta h^{\beta_1 - 1}(1-\varepsilon)^{\beta_2}\left[\frac{\beta_2}{(1-\varepsilon)}\frac{L_Y}{N} + \beta_1\right] - \theta g_A = \frac{-\dot{\mu}}{\mu}$$

$$\frac{\dot{\lambda}}{\lambda} - \frac{\dot{\mu}}{\mu} = (\beta_2 - 1)\,(-g_\varepsilon)\frac{\varepsilon}{(1-\varepsilon)} + (\beta_1 - 1)g_h - g_w - \eta$$

the human capital equation, $g_h = \eta\frac{(1-\varepsilon)^{\beta_2}}{h^{(1-\beta_1)}} - \theta g_A$, implies,

$$\frac{-\dot{\mu}}{\mu} = (g_h + \theta g_A)\left[\frac{\beta_2}{(1-\varepsilon)}\frac{L_Y}{N} + \beta_1\right] - \theta g_A \qquad \text{(b)}$$

and, as $H = hN\varepsilon$ and $g_\varepsilon = 0$ along a BGP, $g_h = g_H - n$ implies

$$\frac{\dot{\lambda}}{\lambda} - \frac{\dot{\mu}}{\mu} = (\beta_1 - 1)g_h - n \qquad (c)$$

Substituting (a) and (b) in (c),

$$-g_K + w\frac{H_Y}{K} - \frac{C}{K} + \frac{A}{K}\pi + (g_h + \theta g_A)\left[\frac{\beta_2}{(1-\varepsilon)}\frac{L_Y}{N} + \beta_1\right] - \theta g_A = (\beta_1 - 1)g_h - \eta$$

As $H_Y = \frac{r}{\alpha g_A} H_A$ and in a BGP, r and g_A are constants, $\frac{H_A}{H_Y}$ constant, that is $\frac{L_A}{L_Y}$ constant

As ε is constant in tha labor endowment equation $L_Y/_N$ is also a constant.

I assume $\frac{H_Y}{K}$, $\frac{\mathrm{C}}{\mathrm{K}}$ and $\frac{\mathrm{A}}{\mathrm{K}}$ is constant along a BGP implying $g_C = g_K = g_A = g_h + n$, $g_c = \frac{r-\rho}{\sigma} = g_C - n$. Substituting $r = \sigma(g_A - n) + \rho$ and using $g_\varepsilon = g_w = 0, g_A = g_h + n$ in the equation below (see consumer's optimization problem in Appendix I)

$$-r + \frac{\dot{h}}{h}\left(\frac{\beta_2 L_Y}{(1-\ \varepsilon)N} + 1\right) +\ \theta g_A\left(\frac{\beta_2 L_Y}{(1-\ \varepsilon)N} - (1-\ \beta_1)\right) =\ (\beta_2 - 1)\frac{\dot{(-\varepsilon)}}{(1-\varepsilon)} -\ \frac{\dot{w}}{w} -\ n$$

$$-\sigma(g_A - n) - \rho +\ (g_A - n)\left(\frac{\beta_2 L_Y}{(1-\ \varepsilon)N} + 1\right) +\ \theta g_A\left(\frac{\beta_2 L_Y}{(1-\ \varepsilon)N} - (1-\ \beta_1)\right) =\ -n$$

$$g_C = g_K = g_H = g_A = \frac{\rho - n(\sigma - \xi)}{(1+\theta)\xi + 1 - \sigma - \theta(1-\beta_1)} \text{ where } \xi = \frac{\beta_2 L_Y}{(1-\ \varepsilon)N} = \text{constant}$$

$\frac{Y}{K} =\ \left(\frac{AH_Y}{K}\right)^{1-\alpha}$ where $\frac{H_Y}{K}$ is constant along a BGP.

$$g_Y = (2-\alpha)g_A$$

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