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Electrical analogs of atomic radiative decay processes

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Simple electrical circuits are analyzed, and the results show that for high frequencies they have frequency and time responses identical to the spontaneous radiative decays of atoms. As an illustration of the analogy a two-circuit electrical system is compared with a two-level atom. The comparison leads to the identification of electrical analogs for quantum-mechanical quantities. It is also shown that the responses of an appropriate electrical circuit can be compared with the decay characteristics of coupled three-level atomic systems.

1. INTRODUCTION

The solution of the time-dependent Schrödinger equation of the radiative decay of an atom has a number of features in common with the solution of classical systems. The comparison can establish various points of correspondence and it can be useful in understanding quantum-mechanical behavior. The quantum-mechanical calculations contain a number of results that are conceptually difficult to interpret. For instance, the probability that a photon with a given frequency has been emitted can be decreasing during a certain interval of time.^{1,2} The frequency distribution of emitted photons from a coupled three-level atom can have holes where no photons are emitted.¹ The classical electrical circuits, on the other hand, can be readily analyzed and their frequency and time response easily interpreted.

In the atomic case, the interaction between the atom and the radiation field causes the excited state to decay with the emission of a wave packet containing a distribution of frequencies. For a simple two-level atom the excited state decays exponentially, and the frequency distribution is Lorentzian. For a three-level atom having two excited states coupled through an external perturbation the decay properties and frequency responses are more complex.

The two-level atom has a simple two-loop circuit as an electrical analog, in which the atom is represented by an LC circuit and the radiation field by an LR circuit. The coupling arises through a mutual inductance. The electrical analog for the three-level atom consists of two LC circuits with capacitive coupling. A decaying LR circuit represents the radiation field.

A number of initial conditions are considered, and it is shown that a Fourier analysis of the energy dissipated by the resistor is equivalent to the frequency distribution of radiation emitted by the decaying atom.

2. ELECTRICAL ANALOG OF A SIMPLE ATOMIC SYSTEM

A pure oscillatory circuit coupled to an inductor-resistance circuit (see Fig. 1) through a mutual inductance represents a classical analog of a simple atomic system. By a simple atom we mean an atom with two energy levels that are coupled by interaction with a radiation field.

The oscillatory circuit is excited by a delta function potential. In the atomic case this corresponds to having the atom initially in the excited state. The energy dissipated in the resistor of the second circuit is calculated here and

Fourier analyzed so that its frequency dependence can be studied. If one uses the rotating wave approximation at high frequencies, the energy emitted from the resistor has the same frequency and time dependence as that of the probability of the two-level atom.³

In Fig. 1, L_1 and C_1 are the inductance and capacitance in the oscillatory circuit (first circuit) and L and R are the inductance and resistance in the dissipating circuit, respectively. In the analogy with the atomic decay process the first circuit represents the atomic system, the decaying circuit represents the radiation field, and the mutual inductance M accounts for the interaction. The oscillatory circuit is excited by a pulse of the form

$$E(t) = E_0\delta(t),$$

where $\delta(t)$ is the Dirac delta function and E_0 is a constant.

From Kirchhoff's law, the following equations are obtained for the currents in the two circuits⁴:

$$L_1 \frac{dI_1}{dt} + \frac{1}{C_1} \int I_1 dt - M \frac{dI}{dt} = E(t)$$

and (2.1)

$$L \frac{dI}{dt} + RI - M \frac{dI_1}{dt} = 0,$$

where I_1 and I are the currents in circuits 1 and 2, respectively.

In terms of the charges Q_1 and Q , Eq. (2.1) reads

$$L_1 \frac{d^2Q_1}{dt^2} + \frac{1}{C_1} Q_1 - M \frac{d^2Q}{dt^2} = E_0\delta(t),$$

$$L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} - M \frac{d^2Q_1}{dt^2} = 0. \quad (2.2)$$

If one uses the Fourier transformations

$$Q(t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} q(\omega) \exp(-i\omega t) d\omega$$

and (2.3)

$$\delta(t) = (2\pi)^{-1} \int_{-\infty}^{\infty} \exp(-i\omega t) d\omega,$$

and rearranges the terms in Eq. (2.2), one gets

$$-\omega^2 q_1(\omega) + (1/L_1 C_1) q_1(\omega) + \omega^2 (M/L_1) q(\omega) = [E_0/(2\pi)^{1/2} L_1], \quad (2.4)$$

$$-\omega^2 q(\omega) - i\omega (R/L) q(\omega) + \omega^2 (M/L) q_1(\omega) = 0. \quad (2.5)$$

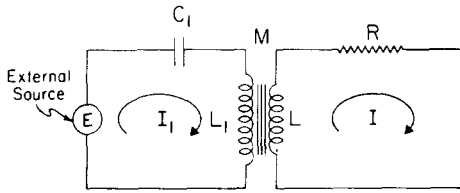


Fig. 1. A pure oscillatory circuit with capacitance C_1 and inductance L_1 is coupled to a dissipative circuit with inductance L and resistance R through a mutual inductance M . The currents I_1 and I represent the currents in the oscillatory and decaying circuits, respectively. An external delta function potential E excites the oscillatory circuit at time $t = 0$.

The solutions of Eqs. (2.4) and (2.5) are

$$q_1(\omega) = \frac{E_0}{(2\pi)^{1/2} L_1} \left(-\omega^2 + \frac{1}{L_1 C_1} - i\omega\gamma(\omega) \right)^{-1} \quad (2.6)$$

and

$$q(\omega) = \frac{ME_0}{(2\pi)^{1/2} L_1 L} \left(\frac{\omega}{\omega + iR/L} \right) \times \left(-\omega^2 + \frac{1}{L_1 C_1} - i\omega\gamma(\omega) \right)^{-1}, \quad (2.7)$$

where

$$\gamma(\omega) = i \frac{M^2}{L_1 L} \left(\frac{\omega^2}{\omega + iR/L} \right). \quad (2.8)$$

In order to compare these results with the decay characteristics of an atom, one has to consider the high-frequency behavior of the solutions. In this range ω is large compared to R/L and thus the real part of $\gamma(\omega)$ is given by

$$\begin{aligned} \text{Re}[\gamma(\omega)] &= \frac{R M^2}{L L_1 L} \left(\frac{\omega^2}{\omega^2 + R^2/L^2} \right) \\ &\cong \frac{R M^2}{L_1 L^2} \equiv \gamma_0. \end{aligned} \quad (2.9)$$

Similarly, the imaginary part is

$$\text{Im}[\gamma(\omega)] = \frac{M^2}{L_1 L} \left(\frac{\omega^3}{\omega^2 + R^2/L^2} \right) \simeq \omega \frac{M^2}{L_1 L}. \quad (2.10)$$

Hence, Eq. (2.7) becomes

$$q(\omega) = \frac{1}{(2\pi)^{1/2}} \frac{ME_0}{L_1 L} \left(\frac{\omega}{\omega + iR/L} \right) \times \left[\left(-1 + \frac{M^2}{L_1 L} \right) \omega^2 + \frac{1}{L_1 C_1} - i\omega\gamma_0 \right]^{-1}.$$

It is assumed here that circuit 1 is weakly coupled to the decaying circuit, and thus the term $M^2/L_1 L$ is much smaller than unity and therefore can be neglected. Thus,

$$I(t) = \frac{ME_0}{2L_1 L \omega_0} \left[\frac{R^2}{L^2} \frac{2\omega_0}{\omega^2 + (\gamma_0/2 - R/L)^2} \exp\left(-\frac{R}{L} t\right) + \frac{(\omega_0 - i\gamma_0/2)^2 \exp(-i\omega_0 t - \gamma_0 t/2)}{\omega_0 - i(\gamma_0/2 - R/L)} - \frac{(\omega_0 + i\gamma_0/2)^2 \exp(i\omega_0 t - \gamma_0 t/2)}{\omega_0 + i(\gamma_0/2 - R/L)} \right] \quad (2.19)$$

$$q(\omega) = \frac{ME_0}{(2\pi)^{1/2} L_1 L} \left(\frac{\omega}{\omega + iR/L} \right) \times \left(-\omega^2 + \frac{1}{L_1 C_1} - i\omega\gamma_0 \right)^{-1},$$

which can be written in the following form:

$$q(\omega) = -\frac{ME_0}{(2\pi)^{1/2} L_1 L} \left(\frac{\omega}{\omega + iR/L} \right) \frac{1}{2\omega_0} \times \left[\left(\omega + \frac{i}{2} \gamma_0 - \omega_0 \right)^{-1} - \left(\omega + \frac{i}{2} \gamma_0 + \omega_0 \right)^{-1} \right], \quad (2.11)$$

where

$$\omega_0 = \left(\frac{1}{L_1 C_1} - \frac{\gamma_0^2}{4} \right)^{1/2}. \quad (2.12)$$

The current in the decaying circuit can be Fourier transformed in the following way:

$$I(t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} I(\omega) \exp(-i\omega t) d\omega. \quad (2.13)$$

Comparison of this equation with Eq. (2.11) shows that

$$I(\omega) = -i\omega q(\omega) = i \frac{ME_0}{(2\pi)^{1/2} L_1 L} \left(\frac{\omega^2}{\omega + iR/L} \right) \frac{1}{2\omega_0} \times \left[\left(\omega + \frac{i}{2} \gamma_0 - \omega_0 \right)^{-1} - \left(\omega + \frac{i}{2} \gamma_0 + \omega_0 \right)^{-1} \right]. \quad (2.14)$$

The total energy emitted before time t is

$$W(t) = R \int_0^t |I(t')|^2 dt', \quad (2.15)$$

which can be written as

$$W(t) = \int_{-\infty}^{\infty} W(\omega, t) d\omega, \quad (2.16)$$

where $W(\omega, t)$ is the energy per unit frequency interval, emitted before time t with frequency ω . It is defined as

$$W(\omega, t) = R |I(\omega, t)|^2, \quad (2.17)$$

where $I(\omega, t)$ is the Fourier transform of the current $I(t)$ for a finite interval of time:

$$I(\omega, t) = (2\pi)^{-1/2} \int_0^t I(t') \exp(i\omega t') dt'. \quad (2.18)$$

By combining Eqs. (2.16)–(2.18) one can show that Eqs. (2.15) and (2.16) are equivalent.

The current $I(t)$ in the decaying circuit is calculated by using the Fourier transformation given in Eq. (2.13). The integral is evaluated by contour integration. The contour is closed by extending the integration along a semicircle of infinite radius in the lower half of the complex plane. The contribution to the integral from the semicircle is zero. The result is

One can now substitute $I(t)$ from Eq. (2.19) into Eq. (2.18) and integrate. This procedure yields

$$I(\omega, t) = -i \frac{ME_0}{2(2\pi)^{1/2} L_1 L \omega_0} \left[\frac{2R^2 \omega_0}{L^2 [\omega_0^2 + (\gamma_0/2 - R/L)^2]} \frac{\exp[i\omega t - (R/L)t] - 1}{\omega + iR/L} + \frac{(\omega_0 - i\gamma_0/2)^2}{\omega_0 - i(\gamma_0/2 - R/L)} \frac{\exp[i(\omega - \omega_0)t - \gamma_0 t/2] - 1}{\omega - \omega_0 + i\gamma_0/2} - \frac{(\omega_0 + i\gamma_0/2)^2}{\omega_0 + i(\gamma_0/2 - R/L)} \frac{\exp[i(\omega + \omega_0)t - \gamma_0 t/2] - 1}{\omega + \omega_0 + i\gamma_0/2} \right]. \quad (2.20)$$

For high frequencies, the contributions from the first and last terms in Eq. (2.20) are negligible compared to the contribution from the second term. To justify this approximation we note that the first term in Eq. (2.20) contains the factor $2R^2/L^2\omega_0\omega$ and the third term contains $\omega_0/(\omega + \omega_0)$. Both of these are very small compared to the factor $\omega_0/(\omega - \omega_0)$ appearing in the second term.

With these approximations Eq. (2.20) reduces to

$$I(\omega, t) = i \frac{ME_0}{2(2\pi)^{1/2} L_1 L \omega_0} \frac{(\omega_0 - i\gamma_0/2)^2}{\omega_0 - i(\gamma_0/2 - R/L)} \times \frac{\exp[i(\omega - \omega_0)t - \gamma_0 t/2] - 1}{\omega - \omega_0 + i\gamma_0/2}. \quad (2.21)$$

The spectral density of energy dissipated in the resistor before time t is given by

$$W(\omega, t) = R |I(\omega, t)|^2 = \frac{R}{2\pi} \left(\frac{ME_0}{2L_1 L} \right)^2 \times \frac{1 + \exp(-\gamma_0 t) - 2 \cos[(\omega - \omega_0)t] \exp(-\gamma_0 t/2)}{(\omega - \omega_0)^2 + \gamma_0^2/4}, \quad (2.22)$$

where the term

$$[(\omega_0^2 + \gamma_0^2/4)^2 \{\omega_0^2 [\omega_0^2 + (\gamma_0/2 - R/L)^2]\}^{-1}]$$

has been replaced by unity (high-frequency approximation). This result can easily be obtained if one makes the "rotating wave approximation" in Eq. (2.14) before the time integration. This approximation removes the second term which contains $\omega + \omega_0$ in the denominator. In addition, $\omega^2(\omega + iR/L)^{-1}$ can be replaced by ω in the high-frequency approximation. The remaining term can then be used to find $W(\omega, t)$, which is identical with the one in Eq. (2.22).

The energy $W(\omega, t)$ has the same frequency and time dependence as that of the quantum-mechanical probability of the decay of a simple atom.⁵

In the electrical system, if one removes the coupling between the two circuits by setting $M = 0$, the first circuit oscillates without loss in energy and γ_0 becomes zero. For the atomic system, this is equivalent to saying that, if one removes the radiation interaction, the atom remains in the excited state and the decay constant γ_0 vanishes. In the electrical system the interaction between the two loops of the circuit produces a frequency shift

$$\Delta\omega = (1/L_1 C_1 - \gamma_0^2/4)^{1/2} - (1/L_1 C_1)^{1/2}.$$

In Fig. 2 the frequency distribution of the energy dissipated in the resistor is plotted as a function of $(\omega - \omega_0)/\gamma_0$ for different times. For small times the distribution is quite broad, but as t increases, the central peak narrows and a number of secondary maxima can be observed. For $t \rightarrow \infty$, the frequency distribution becomes Lorentzian with a linewidth equal to γ_0 . In Fig. 3 the spectral energy density is plotted as a function of $\gamma_0 t$ for different values of $(\omega -$

$\omega_0)/\gamma_0$. For $\omega - \omega_0 \neq 0$ the energy density oscillates at small times and approaches a constant value as t becomes large. The frequency of oscillation is approximately equal to $\omega - \omega_0$.

3. ELECTRICAL ANALOG OF A THREE-LEVEL ATOMIC SYSTEM

Two pure oscillatory circuits numbered 1 and 2, each containing an inductor and a capacitor, are coupled by a capacitor C , as indicated in Fig. 4. A dissipative circuit having an inductance L and a resistance R is coupled with both of the oscillatory circuits through mutual inductances M_1 and M_2 . At $t = 0$ a delta function pulse excites the first circuit.

From Kirchoff's law, the currents in the different circuits satisfy the following equations:

$$\begin{aligned} L_1 \frac{dI_1}{dt} + \frac{1}{C_1'} \int I_1 dt - \frac{1}{C} \int I_2 dt - M_1 \frac{dI}{dt} &= E_0 \delta(t), \\ L_2 \frac{dI_2}{dt} + \frac{1}{C_2'} \int I_2 dt - \frac{1}{C} \int I_1 dt - M_2 \frac{dI}{dt} &= 0, \\ L \frac{dI}{dt} + RI - M_1 \frac{dI_1}{dt} - M_2 \frac{dI_2}{dt} &= 0, \end{aligned} \quad (3.1)$$

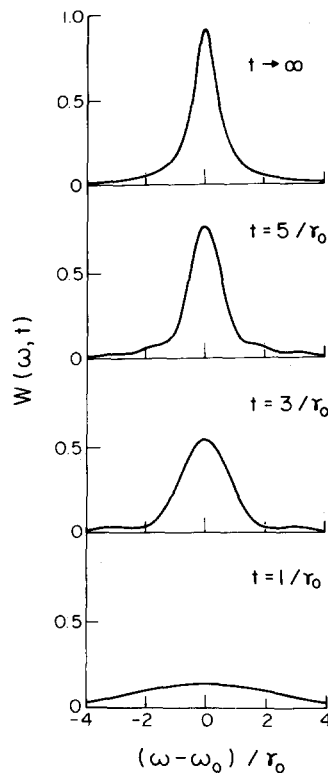


Fig. 2. Frequency distribution of the energy dissipated in the resistor at different times. The decay constant of the circuit is γ_0 , and its natural frequency is ω_0 .

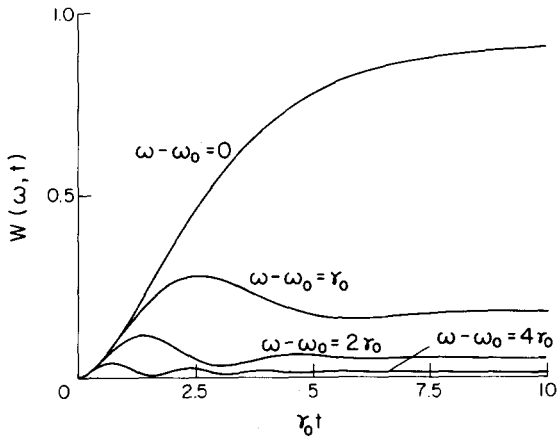


Fig. 3. Spectral energy density as a function of time for different values of $\omega - \omega_0$. The decay constant of the circuit is γ_0 , and its natural frequency is ω_0 .

where I_1 and I_2 are the currents and L_1, C_1' and L_2, C_2' are the inductances and capacitances in circuits 1 and 2, respectively. The capacitances C_1' and C_2' are series combination of C_1, C and C_2, C , respectively:

$$C_1' = C_1 C / (C_1 + C); \quad C_2' = C_2 C / (C_2 + C).$$

The mutual inductance M_1 couples the decaying circuit with the pure oscillatory circuit 1 and M_2 with the circuit 2. The current in the decaying circuit is represented by I .

In terms of the charges Q_1 , Q_2 , and Q , Eq. (3.1) becomes

$$\frac{d^2 Q_1}{dt^2} + \frac{1}{L_1 C_1'} Q_1 - \frac{1}{L_1 C} Q_2 - \frac{M_1}{L_1} \frac{d^2 Q}{dt^2} = \frac{E_0}{L_1} \delta(t),$$

$$\frac{d^2 Q_2}{dt^2} + \frac{1}{L_2 C_2'} Q_2 - \frac{1}{L_2 C} Q_1 - \frac{M_2}{L_2} \frac{d^2 Q}{dt^2} = 0,$$

$$q(\omega) = \left(\frac{E_0}{(2\pi)^{1/2} L_1} \right) \left(\frac{\omega}{\omega + iR/L} \right) \left[\frac{M_1}{L} \left(-\omega^2 + \frac{1}{L_2 C_2'} - i\omega\gamma_{22} \right) + \frac{M_2}{L} \left(\frac{1}{L_2 C} + i\omega\gamma_{21} \right) \right] \\ \times \left[\left(-\omega^2 + \frac{1}{L_2 C_2'} - i\omega\gamma_{22} \right) \left(-\omega^2 + \frac{1}{L_1 C_1'} - i\omega\gamma_{11} \right) - \left(\frac{1}{L_1 C} + i\omega\gamma_{12} \right) \left(\frac{1}{L_2 C} + i\omega\gamma_{21} \right) \right]^{-1}, \quad (3.4)$$

where

$$\gamma_{lm} = i \frac{M_l M_m}{L_l L} \frac{\omega^2}{\omega + iR/L}, \quad l, m = 1, 2.$$

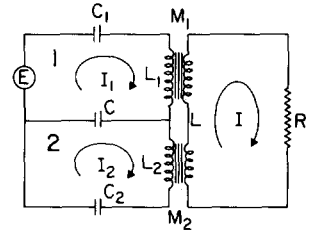
If one makes the high-frequency approximation as was done in the previous case, one can write

$$\text{Re}(\gamma_{lm}) = \frac{RM_l M_m}{L_l L^2} \left(\frac{\omega^2}{\omega^2 + R^2/L^2} \right) \approx \frac{RM_l M_m}{L_l L^2} \equiv \gamma_{lm}^0, \quad \text{Im}(\gamma_{lm}) = \omega \frac{M_l M_m}{L_l L} \left(\frac{\omega^2}{\omega^2 + R^2/L^2} \right) \approx \omega \frac{M_l M_m}{L_l L}, \\ l, m = 1, 2$$

and

$$\left(-\omega^2 + \frac{1}{L_1 C_1'} - i\omega\gamma_{11}^0 \right)^{-1} = -\frac{1}{2\omega_1} \left[\left(\omega - \omega_1 + \frac{i\gamma_{11}^0}{2} \right)^{-1} - \left(\omega + \omega_1 + \frac{i\gamma_{11}^0}{2} \right)^{-1} \right] \\ \approx -\left[2\omega_1 \left(\omega - \omega_1 + \frac{i\gamma_{11}^0}{2} \right) \right]^{-1}, \quad (3.5) \\ \left(-\omega^2 + \frac{1}{L_2 C_2'} - i\omega\gamma_{22}^0 \right)^{-1} \approx -\left[2\omega_2 \left(\omega - \omega_2 + \frac{i\gamma_{22}^0}{2} \right) \right]^{-1},$$

Fig. 4. Two pure oscillatory circuits numbered 1 and 2 are coupled through mutual inductances M_1 and M_2 , respectively, to a decaying circuit with inductance L and resistance R . The two oscillatory circuits are coupled by a capacitor C . The inductances and capacitances in circuits 1 and 2 are L_1, C_1 and L_2, C_2 , respectively. The currents in the circuits 1 and 2 and in the decaying circuit are represented by I_1 , I_2 , and I , respectively. Initially, circuit 1 is excited by a delta function potential E .



where

$$\omega_1 = [1/L_1 C_1' - (\gamma_{11}^0)^2/4]^{1/2} \quad \text{and} \quad \omega_2 = [1/L_2 C_2' - (\gamma_{22}^0)^2/4]^{1/2}.$$

Thus, assuming that $M_l M_m / L_l L \ll 1$ ($l, m = 1, 2$) (weak coupling of circuits) and using the above high-frequency approximations, one gets

$$q(\omega) = \left\{ \frac{-E_0}{(2\pi)^{1/2} L_1} \left(\frac{\omega}{\omega + iR/L} \right) \left[\frac{M_1}{2\omega_1 L} \left(\omega - \omega_2 + \frac{i\gamma_{22}^0}{2} \right) - \frac{M_2}{2\omega_2 L} \left(\frac{1}{L_2 C} + i\omega\gamma_{21} \right) \right] \right\} \\ \times \left[\left(\omega - \omega_1 + \frac{i\gamma_{11}^0}{2} \right) \left(\omega - \omega_2 + \frac{i\gamma_{22}^0}{2} \right) - \left(\frac{1}{L_1 C} + i\omega\gamma_{12} \right) \left(\frac{1}{L_2 C} + i\omega\gamma_{21} \right) (4\omega_1\omega_2)^{-1} \right]^{-1}.$$

The Fourier component $I(\omega)$ of the current $I(t)$ in the decaying circuit is now given by

$$I(\omega) = -i\omega q(\omega) = \left\{ i \frac{E_0}{(2\pi)^{1/2} L_1} \left(\frac{\omega^2}{\omega + iR/L} \right) \left[\frac{M_1}{2\omega_1 L} \left(\omega - \omega_2 + \frac{i\gamma_{22}^0}{2} \right) + \frac{M_2}{2\omega_2 L} V_{21} \right] \right\} \\ \times \left[\left(\omega - \omega_1 + \frac{i\gamma_{11}^0}{2} \right) \left(\omega - \omega_2 + \frac{i\gamma_{22}^0}{2} \right) - V_{21} V_{12} \right]^{-1}, \quad (3.6)$$

where

$$V_{lm} = -(1/2\omega_m)[1/L_l C + i\omega\gamma_{lm}], \quad l \neq m = 1, 2. \quad (3.7)$$

The total energy per unit frequency interval, dissipated in the resistor with frequency ω , is

$$W(\omega) = |I(\omega)|^2 R = \left\{ \left[\frac{R^{1/2} E_0}{(2\pi)^{1/2} L_1} \left(\frac{\omega^2}{\omega + iR/L} \right) \left[\frac{M_1}{2\omega_1 L} \left(\omega - \omega_2 + \frac{i\gamma_{22}^0}{2} \right) + \frac{M_2}{2\omega_2 L} V_{21} \right] \right] \right\} \\ \times \left[\left(\omega - \omega_1 + \frac{i\gamma_{11}^0}{2} \right) \left(\omega - \omega_2 + \frac{i\gamma_{22}^0}{2} \right) - V_{12} V_{21} \right]^{-1}. \quad (3.8)$$

This result can now be compared with the decay characteristics of a three-level atom.⁶ Circuits 1 and 2 correspond to the two excited states, and the decaying circuit represents the radiation field. The capacitor C which connects circuits 1 and 2 is the equivalent of the static perturbation coupling the two excited states, and the mutual inductances M_1 and M_2 represent the interactions of the atom with the radiation field.

To indicate some of the special features of this circuit a few special cases are considered. As was done before, the equations are approximated for the high-frequency range. In the equations below the following approximations have been used:

$$\frac{\omega^4}{(\omega^2 + R^2/L^2)\omega_1^2} \approx \frac{\omega^4}{(\omega^2 + R^2/L^2)\omega_2^2} \approx 1, \\ \frac{\omega^2}{\omega^2 + R^2/L^2} \approx 1, \\ \frac{1}{L_l C} + i\omega\gamma_{lm} \approx \frac{1}{L_l C}, \quad l \neq m = 1, 2.$$

In the third approximation it was also assumed that circuits 1 and 2 are weakly coupled to the decaying circuit.⁷

Case (i)

The decaying circuit is coupled with the first circuit but not with the second one. This implies that the quantities M_2 , γ_{22}^0 , γ_{21} , and γ_{12} are zero. Thus the total energy $W(\omega)$ emitted from the resistor reduces to

$$W(\omega) = \frac{R}{2\pi} \left(\frac{M_1 E_0}{2L_1 L} \right)^2 \\ \times \left| \frac{\omega - \omega_2}{(\omega - \omega_2)(\omega - \omega_1 + i\gamma_1/2) - V_{12} V_{21}} \right|^2, \quad (3.9)$$

where γ_1 stands for γ_{11}^0 . This result has the same frequency dependence as that of the probability that a photon has been

emitted from an atom initially in a decaying state which is coupled to a nondecaying state through a static perturbation. In Fig. 5 this frequency distribution is plotted as a function of $[\omega - (\omega_1 + \omega_2)/2] \gamma_1^{-1}$ for different values of $V = V_{12} = V_{21}$.⁸ There are two maxima: one at $\omega = (\omega_1 + \omega_2)/2 + [(\omega_1 - \omega_2)^2 + 4V^2]^{1/2}/2$ and the other at $\omega = (\omega_1 + \omega_2)/2 - [(\omega_1 - \omega_2)^2 + 4V^2]^{1/2}/2$. These maxima have equal heights but unequal linewidths. The linewidth of the first maximum decreases and the second one increases with increasing V . A "hole" is observed in the frequency spec-

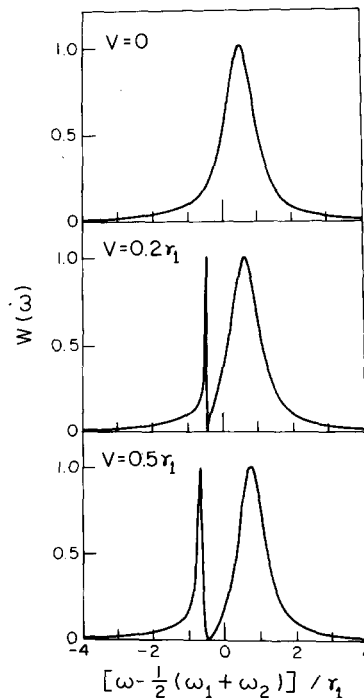


Fig. 5. Frequency distribution of the energy dissipated in the resistor for the case when circuit 2 is not coupled to the decaying circuit. The decay constant of the circuit 1-decaying circuit combination is γ_1 . The natural frequencies of circuits 1 and 2 are ω_1 and ω_2 , respectively. The frequency difference between circuits 1 and 2 is γ_1 .

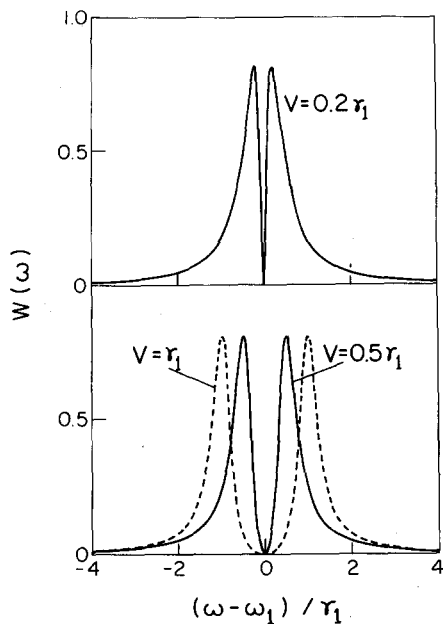


Fig. 6. Frequency distribution of the energy dissipated in the resistor for the case when circuit 2 is not coupled to the decaying circuit. The decay constant of the circuit 1-decaying circuit combination is γ_1 . The natural frequencies of circuits 1 and 2 are the same and equal to ω_1 .

trum at a frequency ω equal to the frequency of oscillation ω_2 of the second circuit. No energy is radiated through the resistor at this frequency. In Fig. 6 the special case $\omega_1 = \omega_2$ is plotted. Here the two maxima have the same linewidth and their separation increases as V increases.

$$W(\omega) = \left| \left[\left(\frac{R}{2\pi} \right)^{1/2} \left(\frac{M_1 E_0}{2L_1 L} \right) \left(\omega - \omega_1 + \frac{i\gamma}{2} \right) + \left(\frac{R}{2\pi} \right)^{1/2} \left(\frac{M_2 E_0}{2L_1 L} \right) V_{21} \right] \times \left[\left(\omega - \omega_1 + \frac{i\gamma}{2} \right) \left(\omega - \omega_2 + \frac{i\gamma}{2} \right) - V_{12} V_{21} \right]^{-1} \right|^2, \quad (3.11)$$

where $\gamma = \gamma_{11}^0 = \gamma_{22}^0$.

In Fig. 8 the spectral density of energy emitted through the resistor is plotted as a function of $[\omega - (\omega_1 + \omega_2)/2] \gamma^{-1}$

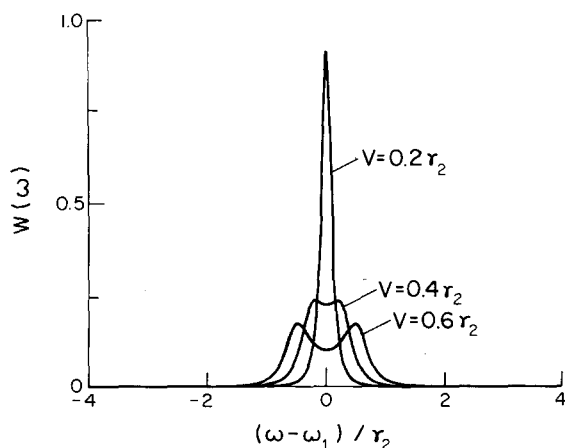


Fig. 7. Frequency distribution of the energy dissipated in the resistor for the case when circuit 1 is not coupled to the decaying circuit. The decay constant of the circuit 2-decaying circuit combination is γ_2 . The natural frequencies of circuits 1 and 2 are the same and equal to ω_1 .

Case (ii)

In this case the decaying circuit is coupled with the second circuit but not with the first one. This gives

$$M_1 = \gamma_{11}^0 = \gamma_{12} = \gamma_{21} = 0.$$

Thus the spectral distribution of energy $W(\omega)$ dissipated in the resistor with frequency ω becomes

$$W(\omega) = \frac{R}{2\pi} \left(\frac{M_2 E_0}{2L_1 L} \right)^2 \left| V_{21} \left[(\omega - \omega_1) \times \left(\omega - \omega_2 + \frac{i\gamma_2}{2} \right) - V_{12} V_{21} \right]^{-1} \right|^2, \quad (3.10)$$

where γ_2 stands for γ_{22}^0 .

The frequency dependence is the same as that of a three-level atom initially in a nondecaying state which is coupled to a decaying state through a static perturbation.⁹

The frequency distribution of Eq. (3.10) is plotted in Fig. 7 for $\omega_1 = \omega_2$ and for different values of $V = V_{12} = V_{21}$.¹⁰ For small values of V the spectrum has only one very narrow peak. As V increases, the width of the peak increases, and for higher values of V two peaks appear with a dip at the center.²

Case (iii)

Here the decaying circuit is coupled with both oscillatory circuits. The mutual inductances M_1 and M_2 , as well as the self-inductances L_1 and L_2 , are considered to be equal. Then γ_{11}^0 and γ_{22}^0 are identical.

Thus Eq. (3.8) reduces to

for different values of $V = V_{12} = V_{21}$ and for $\omega_1 - \omega_2 = 2\gamma$.¹¹ In general the two peaks have unequal heights. The separation between the two peaks increases with increasing V . The increasing V also enhances one peak and suppresses the other. This frequency dependence is the same as that of the probability of having a photon emitted from an atom with two decaying states which are coupled through a static perturbation.

4. CONCLUSIONS

We have shown that some simple electrical circuits have similar decay characteristics as corresponding atomic systems. The energy is fed into the circuits by a delta function pulse at $t = 0$. In the atomic case this corresponds to having the atom initially in some excited state. In the three-circuit system one could have added another energy source in the second circuit that would have excited that circuit at the same time as the first one. The time and frequency response then would be identical in the high-frequency limit to the one of a coupled three-level atomic system initially excited in a superposition of states.

The analogy between electrical circuits and atomic sys-

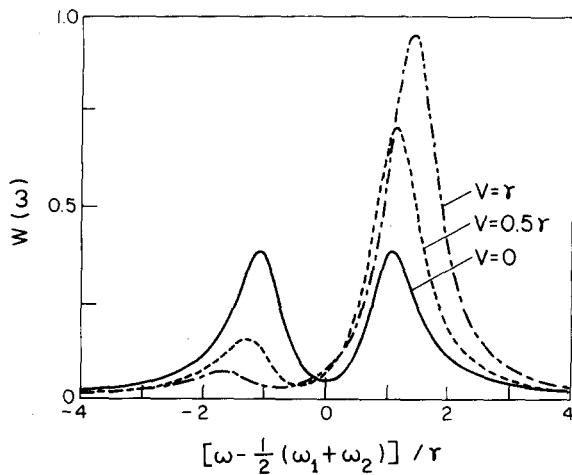


Fig. 8. Frequency distribution of the energy dissipated in the resistor for the case when both circuits 1 and 2 are coupled to the decaying circuit with common decay constant γ . The natural frequencies of circuits 1 and 2 are ω_1 and ω_2 , respectively. The frequency difference between circuits 1 and 2 is 2γ .

tems can be carried further. The decay characteristics of a many-level atom perturbed by static perturbations can be calculated by designing an appropriate circuit and Fourier transforming the Kirchhoff equations as was done in the examples in Secs. 2 and 3.

Frequently one is not interested in the full time dependence of the decay but wants to know the frequency spectrum at large times. In this case no integrations need to be done. In Sec. 2 the energy per unit frequency interval is given by Eq. (2.17). The Fourier transform $I(\omega, t)$ is obtained from Eq. (2.18), and $I(t')$ is calculated by using Eq. (2.13). For $t \rightarrow \infty$ the Fourier transform $I(\omega, \infty)$ becomes identical with $I(\omega)$, which is given by Eq. (2.14). To obtain

$I(\omega)$ one only needs $q(\omega)$, which can be easily obtained by solving the set of linear equations which are the Fourier transforms of Kirchhoff's equations.

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¹P. R. Fontana and R. P. Srivastava, Phys. Rev. A **7**, 1866 (1973).

²R. P. Srivastava and P. R. Fontana, J. Phys. B **7**, 911 (1974).

³J. R. Klauder and E. G. G. Sudarshan, *Fundamentals of Quantum Optics* (Benjamin, New York, 1968), p. 100.

⁴R. E. Scott, *Linear Circuits* (Addison-Wesley, Reading, MA, 1960), pp. 19-23.

⁵For a discussion of the spontaneous decay of a two-level atom, see W. Heitler, *The Quantum Theory of Radiation* (Oxford U. P., London, 1960), Chap. 5. The spectral density [Eq. (2.22)] is equivalent to the square of Eq. (6) (Heitler, p. 182), which gives the probability of photon emission.

⁶The decay characteristics of a three-level atom are discussed in detail in Ref. 1. The spectral density is equivalent to Eq. (10) in Ref. 1. Similar results were also obtained by M. P. Silverman and F. M. Pipkin [J. Phys. B **5**, 2236 (1972)], who used a time-dependent perturbation.

⁷In the atomic case, γ_{lm} ($l \neq m$) vanishes provided the levels are states of good angular momentum. For more details on this aspect, see G. Breit, Rev. Mod. Phys. **5**, 91 (1933), in particular the discussion on p. 118. There are, however, atomic systems where the γ_{lm} ($l \neq m$) do not vanish; see, for example, J. W. Czarnik and P. R. Fontana, J. Chem. Phys. **50**, 4071 (1969).

⁸This implies that $1/\omega_2 L_1 = 1/\omega_1 L_2$, and since $\omega_1 \simeq \omega_2$, one gets $L_1 \simeq L_2$.

⁹P. R. Fontana and D. J. Lynch, Phys. Rev. A **2**, 347 (1970).

¹⁰This implies that $1/\omega_2 L_1 = 1/\omega_1 L_2$, and thus in this case $L_1 = L_2$.

¹¹For $L_1 = L_2$, $V_{21} = (\omega_2/\omega_1)V_{12}$, and since $\omega_1 \simeq \omega_2$, this special case can be considered.