Super-Stretched and Countable Cohen-Macaulay Type

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# Abstract

This dissertation defines what it means for a Cohen-Macaulay ring to to be super-stretched. In particular, a super-stretched Cohen-Macaulay ring of positive dimension has $h$-vector $(1)$, $(1,n)$, or $(1,n,1)$. It is shown that Cohen-Macaulay rings of graded countable Cohen-Macaulay type are super-stretched. Further, one dimensional standard graded Gorenstein rings of graded countable type are shown to be hypersurfaces; this result is not known in higher dimensions.

In Chapter 1, some background material is given along with some preliminary definitions. This chapter defines what it means to be stretched and super-stretched. The chapter ends by showing a couple of scenarios when these two notions coincide.

Chapter 2 deals with super-stretched rings that are standard graded. We begin the chapter by exploring the graded category and defining what it means to be graded countable Cohen-Macaulay type. Equivalent characterizations of super-stretched are discovered and it is shown that rings of graded countable Cohen-Macaulay type are super-stretched. The chapter ends by analyzing standard graded rings that are super-stretched with minimal multiplicity. In Chapter 3, we examine what it means for a local ring to be super-stretched.

Finally, Chapter 4 uses the previous results to give a partial answer to the following question: Let $R$ be a standard graded Cohen-Macaulay ring of graded countable Cohen-Macaulay representation type, and assume that $R$ has an isolated singularity. Is

$R$ then necessarily of graded finite Cohen-Macaulay representation type? In particular, it is shown there is a positive answer when the ring is not Gorenstein. Throughout the chapter, many different cases of graded countable Cohen-Macaulay type are classified. Further, the Gorenstein case is studied is shown to be helpful in giving support to the following folklore conjecture: a Gorenstein ring of countable Cohen-Macaulay representation type is a hypersurface. It is shown that the conjecture holds for one dimensional standard graded Cohen-Macaulay rings of graded countable Cohen-Macaulay type.

*to Jennifer, Eva, Amalie, and Cora*

*thanks for your patience*

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# Chapter 1

# Introduction

According to I. Burban and Y. Drozd [9], the study of maximal Cohen-Macaulay modules can be traced back to the theory of integral representations of finite groups in the nineteenth century. In particular, it originated from the classification of crystallographic groups which is generalized in Hilbert's $18^{th}$ problem [15]. There has been much progress on the subject over the years and the theory has taken many directions. The study of maximal Cohen-Macaulay modules in this dissertation is motivated by the following question of C. Huneke and G. Leuschke [17]:

**Question 1.0.1.** *Let R be a complete local Cohen-Macaulay ring of countable Cohen-Macaulay representation type, and assume that R has an isolated singularity. Is R then necessarily of finite Cohen-Macaulay representation type?*

Here, a local Cohen-Macaulay ring is said to have **finite Cohen-Macaulay type** (respectively, **countable Cohen-Macaulay type**) if it has only finitely (respectively, countably) many isomorphism classes of indecomposable maximal Cohen-Macaulay modules. In the case of countable Cohen-Macaulay type, it is sometimes nice to distinguish between rings of finite Cohen-Macaulay type and rings that are countable Cohen-Macaulay type but actually have infinitely many indecomposables up to isomorphism. We call such rings of **countably infinite Cohen-Macaulay type**.

## 1.1 Current Results and Background

Question 1.0.1 has a positive answer if the ring is a hypersurface. In particular, it was shown in [19, 8] that if $R$ is a complete hypersurface containing an algebraically closed field $k$ (of characteristic different from 2), then $R$ is of finite Cohen-Macaulay type if and only if $R$ is the local ring of a simple hypersurface singularity in the sense of [1]. For example, if we let $k = \mathbb{C}$, then $R$ is one of the complete ADE singularities over $\mathbb{C}$. That is, $R$ is isomorphic to $k[\![x, y, z_2, \ldots, z_r]\!]/(f)$, where $f$ is one of the following polynomials rings:

$$(A_n): x^{n+1} + y^2 + z_2^2 + \cdots + z_r^2,\ n \geqslant 1;$$

$$(D_n): x^{n-1} + xy^2 + z_2^2 + \cdots + z_r^2,\ n \geqslant 4;$$

$$(E_6): x^4 + y^3 + z_2^2 + \cdots + z_r^2;$$

$$(E_7): x^3 y + y^3 + z_2^2 + \cdots + z_r^2;$$

$$(E_8): x^5 + y^3 + z_2^2 + \cdots + z_r^2.$$

It was further shown in [8] that a complete hypersurface singularity over an algebraically closed uncountable field $k$ has (infinite) countable Cohen-Macaulay type if and only if it is isomorphic to one of the following

$$(A_\infty): k[\![x, y, z_2, \ldots, z_r]\!]/(y^2 + z_2^2 + \cdots + z_r^2); \tag{1.1}$$

$$(D_\infty): k[\![x, y, z_2, \ldots, z_r]\!]/(xy^2 + z_2^2 + \cdots + z_r^2). \tag{1.2}$$

As both of the above rings are non-isolated singularities, if a hypersurface has an isolated singularity and is countable type then it must have finite type. In 2011, R. Karr and R. Wiegand [18, Theorem 1.4] showed the one dimensional case as well; assuming that the integral closure $S$ of the ring $R$ is finitely generated as an $R$-module.

Other classes of rings that have been classified are scrolls of type $(a_1,\dots,a_r)$ and are defined below.

**Definition 1.1.1.** Let $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_r > 0$ be given integers and let $\{x_{ij} \mid 0 \leqslant j \leqslant a_i, 1 \leqslant i \leqslant r\}$ be a set of variables over a field $k$. Then, take the ideal $I$ of the polynomial ring $S_1 = k[x_{ij} \mid 0 \leqslant j \leqslant a_i, 1 \leqslant i \leqslant r]$ generated by all the $2\times 2$-minors of the matrix:

$$\begin{pmatrix} x_{10} & x_{11} & \cdots & x_{1a_1-1} & | & \cdots & | & x_{r0} & x_{r1} & \cdots & x_{ra_r-1} \\ x_{11} & x_{12} & \cdots & x_{1a_1} & | & \cdots & | & x_{r1} & x_{r2} & \cdots & x_{ra_r} \end{pmatrix}.$$

Define the graded ring $R_1$ to be quotient $S_1/I$ with $\deg(x_{ij}) = 1$ for all $i,j$, and call $R$ the **scroll** of type $(a_1,\dots,a_r)$. It is known that $R$ is a domain of dimension $r+1$

**Example 1.1.2.** Let us consider the scroll of type $(2,1)$. As shown by M. Auslander and I. Reiten [5, Theorem 2.1], this ring has graded finite Cohen-Macaulay type. Here $r = 2$ and

$$S_1 = k[x_{10}, x_{11}, x_{12}, x_{20}, x_{21}].$$

The ideal $I$ is defined by

$$I = \det_2 \begin{pmatrix} x_{10} & x_{11} & x_{20} \\ x_{11} & x_{12} & x_{21} \end{pmatrix} = (x_{10}x_{12} - x_{11}^2, x_{10}x_{21} - x_{11}x_{20}, x_{11}x_{21} - x_{12}x_{20})$$

and hence the scroll of type $(2,1)$ is

$$R_1 = \frac{k[x_{10}, x_{11}, x_{12}, x_{20}, x_{21}]}{(x_{10}x_{12} - x_{11}^2, x_{10}x_{21} - x_{11}x_{20}, x_{11}x_{21} - x_{12}x_{20})}.$$

Scrolls where studied extensively by M. Auslander and I. Reiten in both the local and graded case. In particular, they were able to determine when exactly when a scroll was of finite Cohen-Macaulay type.

**Theorem 1.1.3** ([5, Theorem 3.2])**.** *Let $R$ be a scroll of type $(a_1,\ldots,a_r)$ over an infinite field $k$. Then $R$ is of finite Cohen-Macaulay type if and only if $R$ is of type $(m)$, $(1,1)$, or $(2,1)$.*

It was also shown by M. Auslander that finite Cohen-Macaulay type implies the ring has an isolated singularity. A fact that has been used several times over by many different authors.

**Theorem 1.1.4** ([2])**.** *Let $(R,\mathfrak{m},k)$ be a complete local Cohen-Macaulay ring. If $R$ is of finite Cohen-Macaulay type, then $R$ has an isolated singularity.*

In [11], D. Eisenbud and J. Herzog completely classify the standard graded Cohen-Macaulay rings of graded finite representation type in the category of graded maximal Cohen-Macaulay modules over a ring $R$ and degree-preserving homomorphisms. In doing this, they show that such rings are stretched as introduced by J. Sally [23] (see Definition 1.2.4):

In researching Question 1.0.1, we have developed the stronger notion of super-stretched (see Definition 1.2.7) and were able to obtain the following property of rings with graded countable Cohen-Macaulay type.

**Theorem 2.4.4.** *Let $(R,\mathfrak{m},k)$ be a standard graded Cohen-Macaulay ring with uncountable residue field $k$. If $R$ is of graded countable Cohen-Macaulay type then it is super-stretched.*

As Proposition 1.2.9 points out, it is not hard to show that super-stretched implies stretched. It turns out, the ADE singularities, along with the two hypersurfaces

(1.1) and (1.2) of countable type, are super-stretched. On the other hand, the ring $R = \mathbb{C}[x,y]/(x^3y - xy^3)$, which is of uncountable Cohen-Macaulay type, is stretched but not super-stretched (see Example 1.2.10).

The main tool in the proof of Theorem 2.4.4 is the ability to recover an ideal from its $d^{th}$ syzygy. That is, given a free resolution of a carefully chosen $\mathfrak{m}$-primary ideal $J$, we are able to regain the ideal from the $d^{th}$ syzygy of the resolution. With this mechanism, we construct uncountably many non-isomorphic, indecomposable maximal Cohen-Macaulay modules; thus arriving at a contradiction.

It would be nice if the super-stretched condition implied finite type. However, as mentioned above, (1.1) and (1.2) are super-stretched but are not finite type.

In Section 1.2 we give some preliminary definitions and define what it means to be stretched and super-stretched. The section ends by showing a couple scenarios when these two notions coincide. Chapter 2 deals with super-stretched rings that are standard graded. We begin the chapter by exploring the graded category and defining what it means to be graded countable Cohen-Macaulay type. Equivalent characterizations of super-stretched are discovered and it is shown that rings of graded countable Cohen-Macaulay type are super-stretched. The chapter ends by analyzing standard graded rings that are super-stretched with minimal multiplicity. In Chapter 3, we examine what it means for a local ring to be super-stretched.

Chapter 4 uses the previous results to give a partial answer to the graded version of Question 1.0.1. In particular, it is shown there is a positive answer to the following question when the ring is not Gorenstein.

**Question 1.1.5.** *Let R be a standard graded Cohen-Macaulay ring of graded countable Cohen-Macaulay representation type, and assume that R has an isolated singularity. Is R then necessarily of graded finite Cohen-Macaulay representation type?*

In attempting to answer Question 1.1.5, many different cases of graded countable Cohen-Macaulay type are classified. Further, the Gorenstein case is studied and Theorem 2.4.4 is shown to be helpful in giving support to the following folklore conjecture.

**Conjecture 1.1.6.** *A Gorenstein ring of countable Cohen-Macaulay representation type is a hypersurface.*

Using Theorem 2.4.4 we show that the conjecture holds for one dimensional standard graded Cohen-Macaulay rings of graded countable Cohen-Macaulay type.

## 1.2 Preliminary Definitions

For simplicity, we will always assume residue fields are uncountable of characteristic 0. This is not really necessary, but will make the statements and proofs more readable.

Given a local Cohen-Macaulay ring $(R,\mathfrak{m},k)$ of dimension $d$ and an ideal $I$ in $R$, recall that the **associated graded ring**, $\mathrm{gr}_I R$, is defined by

$$\mathrm{gr}_I R := R/I \,\oplus\, I/I^2 \,\oplus\, I^2/I^3 \oplus \,\cdots.$$

If $f \in R$, we denote by $f^* \in \mathrm{gr}_I R$ the **leading form** of $f$ in $\mathrm{gr}_I R$. Namely, $f^*$ is the image of $f$ in $I^n/I^{n+1}$, where $f \in I^n$, and $f \notin I^{n+1}$. In the case where $R$ is a local Noetherian ring, $\cap I^n = 0$, so such an $n$ exists.

The **Hilbert function**, $H_R(n)$, is the vector space dimension of the $n^{th}$ summand of $\mathrm{gr}_{\mathfrak{m}} R$; that is,

$$H_R(n) := \dim_k(\mathfrak{m}^n/\mathfrak{m}^{n+1}).$$

If $k$ is an infinite field, then D. Northcott and D. Rees [21] showed there exists $x_1,\ldots,x_d$ in $\mathfrak{m}$ such that $\mathfrak{m}^{n+1} = (x_1,\ldots,x_d)\mathfrak{m}^n$. The sequence $x_1,\ldots,x_d$ is called a **minimal**

**reduction** of $\mathfrak{m}$. A fact that will be used constantly is that in a Cohen-Macaulay ring, minimal reductions are regular sequences. Factoring out such a reduction yields an Artinian ring $\overline{R}$, and thus there exists an $s$ such that for $n > s$, the Hilbert function $H_{\overline{R}}(n) = 0$ and $H_{\overline{R}}(s) \neq 0$. We call the sequence of integers

$$(H_{\overline{R}}(0), H_{\overline{R}}(1), \ldots, H_{\overline{R}}(s))$$

the $h$**-vector** of $R$ and denote it by $h(R)$. In general, the Hilbert function can be represented as a reduced rational function

$$H_R(t) = \frac{f(t)}{(1-t)^{d-1}},$$

where $f(t) \in \mathbb{Z}[t]$. When $R$ is not Cohen-Macaulay, we define the $h$-vector as the vector of coefficients of $f(t)$.

The notion of a stretched Cohen-Macaulay ring was first given in 1979 by J. Sally and is defined as follows.

**Definition 1.2.1** ([23])**.** Let $(R, \mathfrak{m}, k)$ be a $d$ dimensional local Cohen-Macaulay ring with embedding dimension $v$. Let $l := e + d - v$ where $e := e(R)$ is the multiplicity of the ring $R$. Then $R$ is **stretched in the sense of Sally** if there exists a minimal reduction $(x_1, \ldots, x_d)$ of $\mathfrak{m}$ such that $l$ is the minimal integer $i$ such that $\mathfrak{m}^{i+1} \subseteq (x_1, \ldots, x_d)R$.

It is worth noting that, according to this definition, regular local rings are not stretched. Indeed, if $(R, \mathfrak{m})$ is a regular local ring, then $\mathfrak{m} = (x_1, \ldots, x_d)$ is generated by a minimal reduction and the embedding dimension of $R$ is $d$. As the multiplicity of $R$ is $e = 1$, we must have that $l = 1$ as well. However, in order for $R$ to be stretched we need $l = 0$.

The concepts of the next proposition are contained in [23].

**Proposition 1.2.2.** *Let $(R,\mathfrak{m},k)$ be local Cohen-Macaulay ring of dimension $d$ and infinite residue field $k$. The following are equivalent:*

*(1) $R$ is either stretched in the sense of Sally or a regular local ring;*

*(2) there exists a minimal reduction $\mathbf{x}=(x_1,\dots,x_d)$ of $\mathfrak{m}$ such that*

$$\dim_k\left(\operatorname{gr}_{\frac{\mathfrak{m}}{\mathbf{x}}}\left(\frac{R}{\mathbf{x}}\right)\right)_i \leqslant 1$$

*for all $i\geqslant 2$.*

*Proof.* Assume that (1) holds. If $R$ is regular then $\mathbf{x}=\mathfrak{m}$ and we have that

$$\operatorname{gr}_{\frac{\mathfrak{m}}{\mathbf{x}}}\left(\frac{R}{\mathbf{x}}\right)=\frac{R}{\mathfrak{m}}.$$

Thus (2) holds. If $R$ is stretched in the sense of Sally with embedding dimension $e-l+d$, then there exists a minimal reduction $\mathbf{x}=(x_1,\dots,x_d)$ of $\mathfrak{m}$ such that $\mathfrak{m}^{l+1}\subseteq\mathbf{x}R$ and $\mathfrak{m}^l\nsubseteq\mathbf{x}R$. Consider the ring

$$G:=\operatorname{gr}_{\frac{\mathfrak{m}}{\mathbf{x}}}\left(\frac{R}{\mathbf{x}}\right)=\frac{R}{\mathfrak{m}}\oplus\frac{\mathfrak{m}}{\mathfrak{m}^2+\mathbf{x}}\oplus\cdots\oplus\frac{\mathfrak{m}^l+\mathbf{x}}{\mathfrak{m}^{l+1}+\mathbf{x}}\oplus\frac{\mathfrak{m}^{l+1}+\mathbf{x}}{\mathfrak{m}^{l+2}+\mathbf{x}}\oplus\cdots.$$

As $\mathbf{x}$ is a reduction of $\mathfrak{m}$, we have that

$$\dim_k\left(\frac{\mathfrak{m}}{\mathfrak{m}^2+\mathbf{x}}\right)=e-l.$$

*Claim.* For $G$ as above, $\dim_k G_l\neq 0$.

If the claim is true, then the fact that $\lambda(G)=e(R)$ implies that $\dim_k G_i=1$ for $2\leqslant i\leqslant l$. Here $\lambda(G)$ is the length of $G$. To see the claim, notice that if $\dim_k G_l=0$, then $\mathfrak{m}^{l+1} + \mathbf{x} = \mathfrak{m}^l + \mathbf{x}$. Hence

$$\mathfrak{m}^l \subseteq \mathbf{x} + \mathfrak{m}^{l+1}. \tag{1.3}$$

By Nakayama's lemma, this forces $\mathfrak{m}^l \subseteq \mathbf{x}R$, a contradiction.

For the converse assume that $R$ is not regular and that $e - l + d$ be the embedding dimension. Define

$$G := \mathrm{gr}_{\frac{\mathfrak{m}}{\mathbf{x}}}\left(\frac{R}{\mathbf{x}}\right) = \frac{R}{\mathfrak{m}} \oplus \frac{\mathfrak{m}}{\mathfrak{m}^2 + \mathbf{x}} \oplus \cdots \oplus \frac{\mathfrak{m}^l + \mathbf{x}}{\mathfrak{m}^{l+1} + \mathbf{x}} \oplus \frac{\mathfrak{m}^{l+1} + \mathbf{x}}{\mathfrak{m}^{l+2} + \mathbf{x}} \oplus \cdots .$$

As $\lambda(G) = e(R)$, condition (2) forces

$$\dim_k G_i = \dim_k \left(\frac{\mathfrak{m}^i + \mathbf{x}}{\mathfrak{m}^{i+1} + \mathbf{x}}\right) = \begin{cases} 1 & i = 0 \\ e - l & i = 1 \\ 1 & 2 \leqslant i \leqslant l \\ 0 & i > l \end{cases}$$

Since $\dim_k G_l = 1$, we have that $\mathfrak{m}^l + \mathbf{x} \neq \mathfrak{m}^{l+1} + \mathbf{x}$. This forces $\mathfrak{m}^l \nsubseteq \mathbf{x}R$. To see that $\mathfrak{m}^{l+1} \subseteq \mathbf{x}R$, notice that $\dim_k G_{l+1} = 0$ and hence $\mathfrak{m}^{l+2} + \mathbf{x} = \mathfrak{m}^{l+1} + \mathbf{x}$. As in (1.3) intersect with $\mathfrak{m}^{l+1}$ and use Nakayama's lemma to see that $\mathfrak{m}^{l+1} \subseteq \mathbf{x}R$. □

*Remark* 1.2.3. The notion of a Cohen-Macaulay ring $R$ being "stretched" can be viewed as a condition on the $h$-vector. As such, it is sometimes convenient to allow regular local rings to be considered as stretched. As J. Sally's notion of stretched Cohen-Macaulay ring omits regular local rings, for the results in this manuscript, we shall use the following definition.

**Definition 1.2.4.** Let $(R,\mathfrak{m},k)$ be a local Cohen-Macaulay ring of dimension $d$. We say that $R$ is **stretched** if any of the conditions in Proposition 1.2.2 are satisfied.

If $R$ is a standard graded ring (see page 16), we only need to consider condition (2) of Proposition 1.2.2 for the definition of stretched.

**Definition 1.2.5.** Let $(R,\mathfrak{m},k)$ be a standard graded Cohen-Macaulay ring of dimension $d$. We say $R$ is **stretched** if there exists a homogeneous minimal reduction $\mathbf{x} = (x_1,\ldots,x_d)$ of $\mathfrak{m}$ such that

$$\dim_k \left(\frac{R}{\mathbf{x}}\right)_i \leqslant 1$$

for all $i \geqslant 2$.

In particular, if a ring $R$ is stretched, then there exists a minimal reduction $(x_1,\ldots,x_d)$ of the maximal ideal $\mathfrak{m}$ such that the $h$-vector is $(1,a,1,1,\ldots,1)$. Here,

$$a = \dim_k \left(\frac{\mathfrak{m}+(x_1,\ldots,x_d)}{\mathfrak{m}^2+(x_1,\ldots,x_d)}\right) = \dim_k \left(\frac{\mathfrak{m}}{\mathfrak{m}^2}\right) - d.$$

In 1988, D. Eisenbud and J. Herzog were able to relate the stretched condition to finite Cohen-Macaulay type with the following theorem.

**Theorem 1.2.6** ([11, Theorem A])**.** *Standard graded Cohen-Macaulay rings of graded finite Cohen-Macaulay type are stretched.*

We discovered that a stronger condition holds which we call super-stretched. Even more, we are able to prove graded countable Cohen-Macaulay type implies super-stretched when the ring in question is standard graded. We define this stronger notion of stretched by considering the Hilbert series of the quotient of any system of parameters. In particular, we introduce the following definition.

**Definition 1.2.7.** A local Cohen-Macaulay ring $(R,\mathfrak{m},k)$ of dimension $d$ is said to be **super-stretched** if for all homogeneous system of parameters $x_1^*,\ldots,x_d^*$ in $\mathrm{gr}_{\mathfrak{m}}R$,

$$\dim_k\left(\frac{\mathrm{gr}_{\mathfrak{m}}R}{(x_1^*,\ldots,x_d^*)}\right)_i \leqslant 1 \tag{1.4}$$

for all $i \geqslant \sum_{j=1}^{d}\deg(x_j^*) - d + 2$.

Similarly, we define what is meant for a standard graded Cohen-Macaulay ring to be super-stretched.

**Definition 1.2.8.** A standard graded Cohen-Macaulay ring $(R,\mathfrak{m},k)$ of dimension $d$ is said to be **super-stretched** if for all homogeneous systems of parameters $x_1,\ldots,x_d$,

$$\dim_k\left(\frac{R}{(x_1,\ldots,x_d)}\right)_i \leqslant 1 \tag{1.5}$$

for all $i \geq \sum\deg(x_j) - d + 2$.

If a Cohen-Macaulay ring $R$ is super-stretched, then for any homogenous system of parameters $(x_1,\ldots,x_d)$, the $h$-vector is $(1,a_1,a_2,\ldots,a_{D-1},1,\ldots,1)$. Here, $D = \sum\deg(x_i) - d + 2$ and

$$a_j = \dim_k\left(\frac{\mathfrak{m}^j + (x_1,\ldots,x_d)}{\mathfrak{m}^{j+1} + (x_1,\ldots,x_d)}\right).$$

**Proposition 1.2.9.** *If a ring $(R,\mathfrak{m},k)$ is super-stretched with an infinite residue field k, then it is also stretched.*

*Proof.* To see this in the standard graded case, choose a homogeneous minimal reduction $x_1,\ldots,x_d$ of degree one. Then Equation (1.5) holds for all $i \geqslant \sum\deg(x_j) - d + 2 = 2$; i.e. $R$ is stretched. For the local case, consider a homogeneous minimal reduction

$x_1^*, \ldots, x_d^*$ of $\oplus_{i>0} G_i$ in $G := \mathrm{gr}_{\mathfrak{m}} R$. Let $\mathbf{x} := (x_1, \ldots, x_d)R$. As $\mathbf{x} \cap \mathfrak{m}^i \supseteq \mathbf{x}\mathfrak{m}^{i-1}$, we always have that

$$\dim_k \left( \mathrm{gr}_{\frac{\mathfrak{m}}{\mathbf{x}}} \left( \frac{R}{\mathbf{x}} \right) \right)_i = \dim_k \frac{\mathfrak{m}^i}{\mathfrak{m}^{i+1} + \mathbf{x} \cap \mathfrak{m}^i} \leqslant \dim_k \frac{\mathfrak{m}^i}{\mathfrak{m}^{i+1} + \mathbf{x}\mathfrak{m}^{i-1}} = \dim_k \left( \frac{\mathrm{gr}_{\mathfrak{m}} R}{\mathbf{x}} \right)_i .$$

Therefore, if the local ring $R$ is super-stretched, we have that

$$\dim_k \left( \mathrm{gr}_{\frac{\mathfrak{m}}{\mathbf{x}}} \left( \frac{R}{\mathbf{x}} \right) \right)_i \leqslant \dim_k \left( \frac{\mathrm{gr}_{\mathfrak{m}} R}{\mathbf{x}} \right)_i \leqslant 1,$$

and hence $R$ is stretched as well. □

As we see in the next example, if a ring is stretched, it is not necessarily super-stretched.

**Example 1.2.10.** The standard graded Cohen-Macaulay ring $R = \mathbb{C}[x,y]/(x^3y - xy^3)$ is stretched but not super-stretched. To see this, notice that $x + 2y$ is a regular element. As a vector space over $\mathbb{C}$,

$$\dim_{\mathbb{C}} \left( \frac{R}{(x+2y)} \right)_2 = \dim_{\mathbb{C}} \left( \frac{R}{(x+2y)} \right)_3 = 1$$

and

$$\dim_{\mathbb{C}} \left( \frac{R}{(x+2y)} \right)_i = 0$$

for all $i \geqslant 4$. In order for $R$ to be super-stretched, $\dim_{\mathbb{C}} R_i \leqslant 1$ for all

$$i \geqslant \deg((x+2y)^2) - 1 + 2 = 3.$$

However, going modulo $(x+2y)^2$ yields

$$\dim_{\mathbb{C}}\left(\frac{R}{(x+2y)^2}\right)_3 = 2.$$

and thus $R$ is not super-stretched.

It turns out that there are a few instances when stretched and super-stretched coincide. In particular, this happens when the ring in question is zero dimensional or when the ring is a complete intersection that is not a hypersurface.

A zero dimensional ring does not have any system of parameters. Hence, we see that the expression $\sum \deg(x_i) - d + 2$ in the definition of super-stretched becomes just 2. From here it is easy to see that the two definitions are equivalent. A little more work is needed to see this for complete intersections.

**Proposition 1.2.11.** *Let $(R, \mathfrak{m}, k)$ be a standard graded complete intersection that is stretched with $k$ an infinite field. Then $R$ is a hypersurface or defined by two quadrics.*

*Proof.* Let $S = k[y_1, \ldots, y_n]$ and $R = S/(f_1, \ldots, f_m)$ with $\dim(R) = d$ and $\deg(f_i) = d_i$. Further, let $\mathbf{x} = (x_1, \ldots, x_d)$ be a minimal reduction of the maximal ideal $\mathfrak{m}$. Given that $R$ is a complete intersection, we know that the Hilbert function of $R/\mathbf{x}$ is

$$\begin{aligned} H_{\frac{S}{(f_1,\ldots,f_m,x_1,\ldots,x_d)}}(t) &= \frac{(1-t^{d_1})\cdots(1-t^{d_m})\cdot(1-t)^d}{(1-t)^n} \\ &= (1+t+\cdots+t^{d_1-1})\cdots(1+t+\cdots+t^{d_m-1}). \end{aligned} \tag{1.6}$$

As $R$ is Gorenstein and stretched, we know that the $h$-vector is of the form $(1,1)$ or $(1,N,1)$ for some $N > 0$. It is enough to only consider the $h$-vector $(1,N,1)$.

If the $h$-vector is $(1,N,1)$, then (1.6) is of the form $1+Nt+t^2$. In particular, the only case to consider is when $N=2$. In this case, (1.6) is $(1+t)(1+t)$ and thus the ideal $I$ is generated by two quadrics. □

Example 1.2.10 showed that a hypersurface can be stretched but not super-stretched. As it turns out, this is not the case when the ring is a complete intersection defined by 2 quadrics. Before we move on to the next result, we need the following definition.

**Definition 1.2.12.** Let $R$ be a Cohen-Macaulay ring with $h$-vector

$$(H_{\overline{R}}(0), H_{\overline{R}}(1), \ldots, H_{\overline{R}}(s)).$$

The **socle degree** of $R$ is defined to be $\operatorname{SocDeg}(R) = s$.

**Corollary 1.2.13.** *Let $(R,\mathfrak{m},k)$ be a standard graded complete intersection that is not a hypersurface. Then R is stretched if and only if R is super-stretched.*

*Proof.* It is enough to show that stretched implies super-stretched. To do this we show that the socle degree of $R$ modulo a homogeneous system of parameters is not too large. Since $R$ is not a hypersurface, Proposition 1.2.11 implies that $R = k[y_1,\ldots,y_n]/(f_1,f_g)$, $\dim(R) = d$, and $\deg(f_i) = 2$. Let $\mathbf{x} = (x_1,\ldots,x_d)$ be an ideal generated by a homogeneous system of parameters. As the $h$-vector of $R$ is $(1,2,1)$, we have that the socle degree of $R/\mathbf{x}$ is

$$\operatorname{SocDeg}(R/\mathbf{x}) = \deg(f_1) + \deg(f_2) + \sum \deg x_j - (2+d) = \sum \deg x_j - d + 2.$$

Thus for $i \geqslant \sum \deg x_j - d + 2$,

$$\dim_k \left(\frac{R}{\mathbf{x}}\right)_i = \begin{cases} 1 & \text{if } i = \sum \deg x_j - d + 2, \\ 0 & \text{if } i > \sum \deg x_j - d + 2 \end{cases}$$

for any homogeneous system of parameters $\mathbf{x}$ of $R$. Therefore $R$ is super-stretched as well. $\square$

# Chapter 2

# Super-Stretched Graded Rings

In 1988, D. Eisenbud and J.Herzog [11] were able to show that standard graded rings of finite Cohen-Macaulay type are stretched. In this chapter, standard graded rings of countable Cohen-Macaulay type are shown to be stretched. In particular, we are able to show that a standard graded ring of graded countable Cohen-Macaulay type is super-stretched. Before this concept is developed, the graded category is examined and explained. The chapter ends with some applications to standard graded rings with minimal multiplicity.

We say that a ring $R$ is **standard graded** if, as an abelian group, it has a decomposition $R = \bigoplus_{i \geqslant 0} R_i$ such that $R_i R_j \subseteq R_{i+j}$ for all $i, j \geqslant 0$, $R = R_0[R_1]$, and $R_0$ is a field. For the remainder of this section, we will denote $(R, \mathfrak{m}, k)$ by the standard graded ring with $\mathfrak{m}$ being the irrelevant maximal ideal, that is, $\mathfrak{m} = \sum_{i=1}^{\infty} R_i$, and $k := R_0 = R/\mathfrak{m}$ being an uncountable field of characteristic 0. Further, we will always assume that a standard graded ring is Noetherian.

As is, the definition of super-stretched (see Definition 1.2.8) is difficult to check; one needs to check every homogenous system of parameters against condition (1.5). Our goal is to find equivalent conditions (Theorem 2.2.9) that will allow us to find examples, and be useful in proving results about super-stretched rings.

## 2.1 The Graded Category

All of the results in Chapter 2 will be considered in the "graded category" of finitely generated graded $R$-modules. The objects of the category are the obvious choices, but there is a bit of ambiguity as to what the maps are. As such, we give a precise definition of what this category is. We also explore alternate ways of defining rings of graded finite (and countable) Cohen-Macaulay type. Understanding the nomenclature will help us avoid various snares along the way. An illustration of such a pitfall can be found in Example 2.4.1.

For a local Noetherian ring $R$, let $\mathfrak{Mod}(R)$ denote the category of finitely generated $R$-modules. Here the objects are defined as finitely generated $R$-modules and the morphisms are $R$-module homomorphisms. The full and faithful subcategory $\mathfrak{MCM}(R)$ is the category of maximal Cohen-Macaulay modules whose morphisms are defined by $R$-module homomorphisms between maximal Cohen-Macaulay modules. If $R$ is a standard graded ring, we define a subcategory of $\mathfrak{Mod}(R)$ that respects the grading. In particular, we let $\mathfrak{Mod}^{\mathrm{gr}}(R)$ be the category whose objects are finitely generated graded modules. The morphisms of $\mathfrak{Mod}^{\mathrm{gr}}(R)$ are graded $R$-module homomorphisms of degree zero. As with $\mathfrak{MCM}(R)$, we define the full and faithful subcategory of graded maximal Cohen-Macaulay modules by $\mathfrak{MCM}^{\mathrm{gr}}(R)$ where the morphisms are graded degree zero $R$-module homomorphisms.

We say that a result $\mathfrak{P}$ holds in the **graded category** of a ring $R$ if $\mathfrak{P}$ holds for all modules and morphisms in $\mathfrak{Mod}^{\mathrm{gr}}(R)$. If $M$ is an object of $\mathfrak{Mod}^{\mathrm{gr}}(R)$ we denote the **shift of** $M$ **by** $n$ by $M(n)_i = M_{i+n}$. Further, we say two modules $M$ and $N$ in $\mathfrak{Mod}^{\mathrm{gr}}(R)$ are **isomorphic up to shift in degree** if there exists an integer $n$ such that $M \simeq N(n)$ in the graded category.

We are now ready to define what is meant by a graded Cohen-Macaulay type of a standard graded ring.

**Definition 2.1.1.** A standard graded Cohen-Macaulay ring $(R, \mathfrak{m}, k)$ ring is said to have **graded finite Cohen-Macaulay type** (respectively, **graded countable Cohen-Macaulay type**) if it has only finitely (respectively, countably) many indecomposable modules in $\mathfrak{MCM}^{\mathrm{gr}}(R)$ up to a shift in degree.

It is worth pointing out that there are a few possible choices in the definition of graded finite Cohen-Macaulay type of a standard graded ring $R$. For example, one could use any of the following characterizations for graded finite Cohen-Macaulay type:

(A) there are finitely many graded indecomposable modules up to isomorphism in $\mathfrak{MCM}^{\mathrm{gr}}(R)$;

(B) there are finitely many graded indecomposable modules in $\mathfrak{MCM}^{\mathrm{gr}}(R)$ up to shifts in degrees;

(C) there are finitely many graded indecomposable modules up to isomorphism in $\mathfrak{MCM}(R)$;

(D) there are finitely many indecomposable modules up to isomorphism in $\mathfrak{MCM}(\widehat{R})$.

Here and throughout the rest of the thesis, we denote the completion with respect to the $\mathfrak{m}$-adic topology by $\widehat{*}$. As it turns out, using (A) as the definition would not be very helpful, since in general $\{R(n)\}$ is an infinite family of non-isomorphic graded indecomposable maximal Cohen-Macaulay modules. In short, only the zero ring would have graded finite Cohen-Macaulay type. Thus we can safely remove (A) from the list of possible definitions. Since we have adopted (B) as the definition, the question is, how do (C) and (D) fit into the picture? In Corollary 2.1.7, we see that (B), (C) and (D) are equivalent definitions, which follows as consequence of the work of M. Auslander and I. Reiten.

**Proposition 2.1.2** ([4, Proposition 8 and 9])**.** *Let $A,B$ be objects in $\mathfrak{MCM}^{\mathrm{gr}}(R)$ where $(R,\mathfrak{m},k)$ is a standard graded Cohen-Macaulay ring.*

*(1) The graded module $A$ is indecomposable in $\mathfrak{MCM}^{\mathrm{gr}}(R)$ if and only if $\widehat{A}$ is indecomposable in the $\mathfrak{MCM}(\widehat{R})$;*

*(2) If $A$ and $B$ are indecomposable, then $\widehat{A} \simeq \widehat{B}$ in $\mathfrak{MCM}(\widehat{R})$ if and only if there is some integer $n$ such that $A \simeq B(n)$ in $\mathfrak{MCM}^{\mathrm{gr}}(R)$.*

**Corollary 2.1.3.** *Let $(R,\mathfrak{m},k)$ be a standard graded Cohen-Macaulay ring and $M,N$ be indecomposable objects in $\mathfrak{MCM}^{\mathrm{gr}}(R)$. Then, $M \simeq N$ in $\mathfrak{MCM}(R)$ if an only if there is some integer $n$ such that $M \simeq N(n)$ in $\mathfrak{MCM}^{\mathrm{gr}}(R)$.*

*Proof.* This follows from the fact that completion is faithfully flat and Proposition 2.1.2 part (2). □

Another immediate corollary of Proposition 2.1.2 is the fact that the completion "bounds" the Cohen-Macaulay type.

**Corollary 2.1.4.** *Let $(R,\mathfrak{m},k)$ be a standard graded Cohen-Macaulay ring and $\widehat{R}$ the $\mathfrak{m}$-adic completion. If $\widehat{R}$ is of finite (respectively countable) Cohen-Macaulay type, then $R$ is of graded finite (respectively graded countable) Cohen-Macaulay type.*

The next proposition shows the equivalence of (B) and (C) for rings of finite Cohen-Macaulay type.

**Proposition 2.1.5.** *Let $(R,\mathfrak{m},k)$ be a standard graded Cohen-Macaulay ring. Then (B) and (C) are equivalent statements. In particular, either statement could be used as the definition of graded finite Cohen-Macaulay type.*

*Proof.* To see that (B) implies (C), notice that condition (C) has more isomorphisms in each class of indecomposable maximal Cohen-Macaulay modules than there are in each class satisfying condition (C). Thus, if (B) holds true, then (C) must be also be fulfilled.

It is left to show that (C) implies (B). By contradiction, assume there are infinitely many graded indecomposable modules in $\mathfrak{MCM}^{\mathrm{gr}}(R)$ up to shits in degree. We let $\{M_\alpha\}_{\alpha\in\Lambda}$ be a family of representatives, one from each isomorphism class. Let $\alpha,\beta \in \Lambda$ and assume that $M_\alpha \simeq M_\beta$ in $\mathfrak{MCM}(R)$. By Corollary 2.1.3, there exists an $n$ such that $M_\alpha \simeq M_\beta(n)$ in $\mathfrak{MCM}^{\mathrm{gr}}(R)$. In other words, $M_\alpha$ and $M_\beta$ are in the same isomorphism class up to shift. Therefore we must have that $\alpha = \beta$. This forces infinitely many graded indecomposable modules up to isomorphism in $\mathfrak{MCM}(R)$, a contradiction. □

When considering indecomposable maximal Cohen-Macaulay modules $M,N$ in the graded category, if there is an isomorphism between $M$ and $N$, then Corollary 2.1.3 says there exists a graded isomorphism between the two modules. Another nice result about the graded category is that the "finiteness" of $\mathfrak{MCM}^{\mathrm{gr}}(R)$ and $\mathfrak{MCM}(\widehat{R})$ are the same.

**Theorem 2.1.6** ([4, Theorem 5])**.** *Let $(R,\mathfrak{m},k)$ be a standard graded Cohen-Macaulay ring and $\widehat{R}$ the completion with respect to the maximal ideal $\mathfrak{m}$. Then $R$ is of finite graded Cohen-Macaulay type if and only if $\widehat{R}$ is of finite Cohen-Macaulay type.*

With Theorem 2.1.6 in hand, we are able to combine it with Proposition 2.1.5 to obtain the following immediate corollary.

**Corollary 2.1.7.** *Let $(R,\mathfrak{m},k)$ be a standard graded Cohen-Macaulay ring and $\widehat{R}$ the completion with respect to the maximal ideal $\mathfrak{m}$. Then conditions (B), (C), and (D) are equivalent.*

As with graded finite Cohen-Macaulay type, there are a few possible choices for the definition of graded countable Cohen-Macaulay type of a standard graded ring $R$. As described above, we could use any of the following for the definition:

(A') there are countably many graded indecomposable modules up to isomorphism in $\mathfrak{MCM}^{\mathrm{gr}}(R)$;

(B') there are countably many graded indecomposable modules in $\mathfrak{MCM}^{\mathrm{gr}}(R)$ up to shifts in degrees;

(C') there are countably many graded indecomposable modules up to isomorphism in $\mathfrak{MCM}(R)$;

(D') there are countably many indecomposable modules up to isomorphism in the category $\mathfrak{MCM}(\widehat{R})$.

Unlike the finite case, using (A') as the definition has potential. Notice that condition (A') is just removing the shifts and only allowing degree zero homomorphism between the modules. By removing the shifts, we are only adding up to countably many new isomorphism classes with condition (A'). Hence (A') does not really add anything new. In Proposition 2.1.8, we see that conditions (A'), (B'), and (C') are equivalent. Further, Corollary 2.1.10 describes the relation of (D') with the other statements.

**Proposition 2.1.8.** *Let $(R, \mathfrak{m}, k)$ be a standard graded Cohen-Macaulay ring. Then (A'), (B'), and (C') are equivalent statements. In particular, any of the statements could be used as the definition of graded countable Cohen-Macaulay type.*

*Proof.* To show that (A') implies (B'), notice that by removing the shifts we are adding more isomorphism classes. Therefore (B') follows. A similar argument as in Proposition 2.1.5 shows that (B') implies (C').

To see that (B') implies (A'), assume by contradiction that there exists uncountably many graded indecomposable maximal Cohen-Macaulay modules up to isomorphism in $\mathfrak{MCM}^{\text{gr}}(R)$. Let $\{M_\alpha\}_{\alpha\in\Lambda}$ be a family of representatives, one from each indecomposable class. Consider the isomorphism classes up to shifts in degrees. That is for each $\alpha \in \Lambda$, there exists a subset $I \subseteq \Lambda$, with the property that for each $\beta \in I$, there exists an integer $n$ such that $M_\alpha \simeq M_\beta(n)$. Let $\beta, \gamma \in I$ and assume that there exist an integer $n$ such that

$$M_\gamma(n) \simeq M_\alpha \simeq M_\beta(n).$$

As all of the isomorphisms above are degree zero, we have that $M_\beta = M_\gamma$ (i.e. $\beta = \gamma$). Hence, when we include the shifts to our assumption, for each $\alpha \in \Lambda$ we only associate countably many indecomposables up to shifts. Hence there are uncountably many graded indecomposable modules in $\mathfrak{MCM}^{\text{gr}}(R)$ that are isomorphic up to shifts in degrees, a contradiction.

It is left to show that (C') implies (B'). By contradiction, assume there are uncountably many graded indecomposable modules in $\mathfrak{MCM}^{\text{gr}}(R)$ up to shits in degree. Let $\{M_\alpha\}_{\alpha\in\Lambda}$ be an uncountable family of representatives from each isomorphism class. We wish to form the isomorphism classes described in (C'). Let $\alpha, \beta \in \Lambda$ and assume that $M_\alpha \simeq M_\beta$ in $\mathfrak{MCM}(R)$. Thus by Corollary 2.1.3, there exists an $n$ such that $M_\alpha \simeq M_\beta(n)$ in $\mathfrak{MCM}^{\text{gr}}(R)$. In other words, $M_\alpha$ and $M_\beta$ are in the same isomorphism class up to shift. Therefore we must have that $\alpha = \beta$. This forces uncountably many graded indecomposable modules up to isomorphism in $\mathfrak{MCM}(R)$, a contradiction. □

*Remark* 2.1.9. If a standard graded ring $R$ is of graded finite Cohen-Macaulay type then it is also of graded countable Cohen-Macaulay type. This does not mean that there are finitely many isomorphism classes as defined in conditions (A'), (B'), and (C'). By

Proposition 2.1.5, the ring being of finite type only says that there are finitely many isomorphism classes as defined in the conditions (B') and (C').

**Corollary 2.1.10.** *Let $(R, \mathfrak{m}, k)$ be a standard graded Cohen-Macaulay ring. If condition (D') holds, then so do the statements (A'), (B'), and (C').*

*Proof.* This is a direct application of Corollary 2.1.4 and Proposition 2.1.8. □

As a further extension to countable type, we would like to modify Theorem 2.1.6 to the case when $R$ is of graded countable Cohen-Macaulay type (see Question 2.1.16). But first, we need to clarify what it means for a graded ring to have isolated singularity in the graded category.

### 2.1.1 The Graded Singular Locus

Let $(R, \mathfrak{m}, k)$ be a standard graded ring and $\mathfrak{p}$ a homogeneous prime ideal in $R$. We define the **graded localization** of $R$ at $\mathfrak{p}$ to be

$$R_{(\mathfrak{p})} = \left\{ \frac{x}{a} \mid x \in R,\ a \text{ is homogeneous in } R \setminus \mathfrak{p} \right\}.$$

*Remark* 2.1.11. Notice that for a homogeneous prime ideal $\mathfrak{p}$ in the standard graded ring $R$, $R_{(\mathfrak{p})}$ is graded but not necessarily standard graded. In particular, if $x \in R$ is a homogeneous element, we define the $\deg(x/a) := \deg(x) - \deg(a)$ for $x/a \in R_{(\mathfrak{p})}$. Thus the grading on $R_{(\mathfrak{p})}$ is defined by

$$\left(R_{(\mathfrak{p})}\right)_i := \{x/a \in R_{(\mathfrak{p})} \mid x \text{ is homogeneous}, \ \deg(x/a) = i\}.$$

Hence $R_{(\mathfrak{p})} = \oplus_{i \in \mathbb{Z}} (R_{(\mathfrak{p})})_i$ where $(R_{(\mathfrak{p})})_i = 0$ if and only if $(R_{(\mathfrak{p})})_{-i} = 0$. Further notice that, unlike $R$, the units of $R_{(\mathfrak{p})}$ are not necessarily centralized in $(R_{(\mathfrak{p})})_0$.

As in Y. Yoshino's book [25, Chapter 15], we define the **graded singular locus** of $R$ to be

$$\operatorname{Sing}^{\mathrm{gr}}(R) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathfrak{p} \text{ is homogeneous and } R_{(\mathfrak{p})} \text{ is not regular } \}$$

and say that $R$ is a **graded isolated singularity** if $\operatorname{Sing}^{\mathrm{gr}}(R) = \mathfrak{m}$. In the definition of $\operatorname{Sing}^{\mathrm{gr}}(R)$, we say that $R_{(\mathfrak{p})}$ is not regular if there is a maximal ideal $\eta$ (not necessarily homogeneous) such that $(R_{(\mathfrak{p})})_\eta$ is not a regular local ring.

It turns out that a distinction between the graded singular locus in the graded category and the usual singular locus is not necessary. We see why in Propositions 2.1.12 and 2.1.13 below.

For an ideal $I$ in a graded ring $R$, let $I^*$ denote the ideal generated by all of the homogeneous elements of $I$. To help understand the situation, we have the following known results from [7].

**Proposition 2.1.12.** *Let $(R, \mathfrak{m}, k)$ be a standard graded ring and $\mathfrak{p}$ be a prime ideal in $R$, not necessarily homogeneous.*

*(1) $\mathfrak{p}^*$ is a homogeneous prime ideal;*

*(2) $R_\mathfrak{p}$ is a regular local ring if and only if $R_{\mathfrak{p}^*}$ is a regular local ring.*

*Proof.* The proof of (1) can be found in [7, Lemma 1.5.6 (a)]. Part (2) is an exercise in [7, Exercise 2.2.24 (a)]. □

**Proposition 2.1.13.** *Let $(R, \mathfrak{m}, k)$ be a standard graded ring and $\mathfrak{p}$ be a homogeneous prime ideal in $R$. Then $R_{(\mathfrak{p})}$ is regular if and only if $R_\mathfrak{p}$ is a regular local ring.*

*Proof.* First assume that $R_{(\mathfrak{p})}$ is regular. Thus the localization of any prime ideal in $R_{(\mathfrak{p})}$ is also regular. Notice that $\mathfrak{p}R_{(\mathfrak{p})}$ is a prime ideal and that $(R_{(\mathfrak{p})})_{\mathfrak{p}R_{(\mathfrak{p})}} = R_\mathfrak{p}$. Therefore $R_\mathfrak{p}$ is a regular local ring.

Conversely, assume that $R_{(\mathfrak{p})}$ is not regular. Notice that by definition of $R_{(\mathfrak{p})}$, if $\eta$ was maximal ideal in $R_{(\mathfrak{p})}$, we must have that $\eta \supseteq \mathfrak{p}R_{(\mathfrak{p})}$. As $\eta \setminus \mathfrak{p}R_{(\mathfrak{p})}$ does not contain any homogeneous elements, we know that $\eta^* = \mathfrak{p}R_{(\mathfrak{p})}$.

Since $R_{(\mathfrak{p})}$ is not regular, there exists a maximal ideal $\eta$ in $R_{(\mathfrak{p})}$ such that $(R_{(\mathfrak{p})})_\eta$ is not a regular local ring. However, by Proposition 2.1.12 (2), $R_\mathfrak{p} = (R_{(\mathfrak{p})})_{\eta^*}$ is not a regular local ring. □

**Corollary 2.1.14.** *Let $(R, \mathfrak{m}, k)$ be a standard graded ring and $\mathfrak{p}$ be a ideal in $R$. Then $\mathfrak{p} \in \operatorname{Sing}(R)$ if and only if $\mathfrak{p}^* \in \operatorname{Sing}^{\mathrm{gr}}(R)$.*

*Proof.* This is a straight forward application of Propositions 2.1.12 and 2.1.13. □

Although it is consistent to the graded category to speak of $\operatorname{Sing}^{\mathrm{gr}}(R)$, it is computationally easier to deal with $\operatorname{Sing}(R)$. Thus, when dealing with the singular locus of a standard graded ring, Corollary 2.1.14 allows us to switch between the two concepts.

With this language, we are also able to state the graded version of Theorem 1.1.4.

**Theorem 2.1.15** ([4, Proposition 4])**.** *Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay standard graded ring. If $R$ is of graded finite Cohen-Macaulay type then $R$ is an isolated singularity.*

Given Theorem 2.1.15, it is worth reconsidering the relation of condition (D') on page 21 with conditions (A'), (B') and (C'). In Corollary 2.1.10, we saw that (D') implies the other conditions. As a standard graded ring of graded finite Cohen-Macaulay type always has an isolated singularity, it might be possible to extend Theorem 2.1.6 to the countable case if we assume isolated singularity. In particular, we would like to know when (D') is actually equivalent to the other conditions; we leave this as a question.

**Question 2.1.16.** *Let* $(R,\mathfrak{m},k)$ *be a standard graded Cohen-Macaulay ring and* $\widehat{R}$ *the completion with respect to the maximal ideal* $\mathfrak{m}$*. If* $R$ *has an isolated singularity and is of countable graded Cohen-Macaulay type, then is* $\widehat{R}$ *of countable Cohen-Macaulay type?*

## 2.2 Graded super-stretched

Before we can show equivalent characterizations of super-stretched (Theorem 2.2.9), we need to build up the theory of super-stretched standard graded rings. For the next results, we will use the following notation. Let $y_1,\dots,y_n$ and $x_1,\dots,x_m$ be sequences in a ring $R$ such that $(y_1,\dots,y_n) \subseteq (x_1,\dots,x_m)$. We let $\underline{y} = A\underline{x}$ represent the relations $y_i = \sum_{j=1}^m a_{ij}x_j$ and denote the containment of $(y_1,\dots,y_n)$ in $(x_1,\dots,x_m)$ by $(\underline{y}) \overset{A}{\subseteq} (\underline{x})$. Further, we define $(y_1,\dots,y_n)^{[t]} := (y_1^t,\dots,y_n^t)$.

**Lemma 2.2.1.** *Let* $(R,\mathfrak{m},k)$ *be a standard graded ring and* $x_1,\dots,x_n$ *a homogeneous regular sequence in* $R$*. Then for all* $t \geqslant 1$*,*

$$(x_1,\dots,x_n)^{[t]} : (x_1\cdots x_n)^{t-1} = (x_1,\dots,x_n).$$

*Proof.* It is enough to show that $(x_1,\dots,x_n)^{[t]} : (x_1\cdots x_n)^{t-1} \subseteq (x_1,\dots,x_n)$. To prove this we will induct on $n$. For $n=1$, we need to show that $(x_1^t) : (x_1^{t-1}) \subseteq (x_1)$. Let $r \in R$ such that $rx_1^{t-1} = ax_1^t$ where $a$ is an element of $R$. As $x_1$ is a non-zero divisor on $R$, we have that $r = ax_1$.

Assume the result holds for all homogeneous regular sequences of length $n-1$ and let

$$r(x_1\cdots x_n)^{t-1} = a_1x_1^t + \cdots + a_nx_n^t$$

where the $a_i$'s are elements of $R$. Subtracting $a_n x_n^t$ from each side gives us that

$$x_n^{t-1}\left(r(x_1\cdots x_{n-1})^{t-1}-a_nx_n\right)=a_1x_1^t+\cdots+a_{n-1}x_{n-1}^t.$$

Note that $x_1^t,\ldots,x_{n-1}^t,x_n$ is a regular sequence as $x_1,\ldots,x_n$ is a regular sequence. This forces $r(x_1\cdots x_{n-1})^{t-1}\in(x_1^t,\ldots,x_{n-1}^t,x_n)$. Further, as $R$ is a standard graded ring, a graded version of Nakayama's lemma holds and we have that every permutation of the sequence $x_1,\ldots,x_n$ is also a regular sequence. Thus, the image of the regular sequence $x_1^t,\ldots,x_{n-1}^t,x_n$ in $R$ modulo $x_n$ is also a regular sequence. We can write

$$r(x_1\cdots x_{n-1})^{t-1}=b_1x_1^t+\cdots+b_{n-1}x_{n-1}^t+b_nx_n$$

where $b_i$ are elements of $R$. Let $\overline{*}$ denote the image of an element in $R/x_nR$. Modulo $x_n$, the above relation becomes

$$\overline{r}(\overline{x}_1\cdots\overline{x}_{n-1})^{t-1}=\overline{b}_1\overline{x}_1^t+\cdots+\overline{b}_{n-1}\overline{x}_{n-1}^t.$$

Since $\overline{x}_1^t,\overline{x}_2^t,\ldots,\overline{x}_{n-1}^t$ is a regular sequence, we may use the induction hypothesis to show that $\overline{r}\in(\overline{x}_1,\ldots,\overline{x}_{n-1})$. Lifting back to $R$ shows us that $r\in(x_1,\ldots,x_n)$. □

The next lemma is needed for Proposition 2.2.3 and is stated without proof.

**Lemma 2.2.2** ([12, Corollary 2.5])**.** *Let $R$ be a commutative ring with identity, $d=\dim(R)$ and let $(y_1,\ldots,y_d)\subset(x_1,\ldots,x_d)$. Suppose that there exist two matrices $A=(a_{ij})$ and $B=(b_{ij})$ such that $y_i=\sum_{j=1}^{d}a_{ij}x_j=\sum_{j=1}^{d}b_{ij}x_j$. Then $(y_1\cdots y_d)^d(\det A-\det B)\in(y_1^{d+1},\ldots,y_d^{d+1})$.*

**Proposition 2.2.3.** *Let $(R,\mathfrak{m},k)$ be a standard graded Cohen-Macaulay ring of dimension $d$ and assume that $(y_1,\ldots,y_d)\subseteq(x_1,\ldots,x_d)$ are ideals generated by a homoge-*

*neous system of parameters. Let* $A = (a_{ij})$ *such that* $y_i = \sum_{j=1}^{d} a_{ij} x_j$ *for* $i = 1, \ldots, d$. *If* $\Delta = \det A$, *then the map* $R/(x_1, \ldots, x_d) \xrightarrow{\cdot \Delta} R/(y_1, \ldots, y_d)$ *is injective.*

*Proof.* Notice that the above map is well-defined as $\Delta(x_1, \ldots, x_d) \subset (y_1, \ldots, y_d)$. Let $r \in R$ such that $r \cdot \Delta \in (y_1, \ldots, y_d)$. Since $y_1, \ldots, y_d$ is a homogeneous system of parameters, there exists a positive integer $t$ and a matrix $B$ such that $(x_1, \ldots, x_d)^{[t]} \overset{B}{\subseteq} (y_1, \ldots, y_d)$. Hence we have the following inclusions:

$$(x_1, \ldots, x_d)^{[t]} \overset{B}{\subseteq} (y_1, \ldots, y_d) \overset{A}{\subseteq} (x_1, \ldots, x_d)$$
$$(x_1, \ldots, x_d)^{[t]} \overset{D}{\subseteq} (x_1, \ldots, x_d),$$

where $D$ is the diagonal matrix with entries $x_i^{t-1}$. Let $E = AB$. By Lemma 2.2.2 we obtain

$$(x_1 \cdots x_d)^{td} (\det\, E - \det\, D) \in (x_1, \ldots, x_d)^{[td+t]}.$$

As $\det\, D = (x_1 \cdots x_d)^{t-1}$, we have

$$(x_1 \cdots x_d)^{td} (\det\, B)\Delta - (x_1 \cdots x_d)^{td} (x_1 \cdots x_d)^{t-1} \in (x_1, \ldots, x_d)^{[td+t]}$$

and thus multiplication by $r$ yields

$$r(x_1 \cdots x_d)^{td} (\det\, B)\Delta - r(x_1 \cdots x_d)^{td} (x_1 \cdots x_d)^{t-1} \in (x_1, \ldots, x_d)^{[td+t]}. \tag{2.1}$$

Since $r\Delta \in (y_1, \ldots, y_d)$, we have that

$$r \cdot (x_1 \cdots x_d)^{td} \cdot \det B \cdot \Delta \in (x_1 \cdots x_d)^{td} \cdot \det B \cdot (y_1, \ldots, y_d). \tag{2.2}$$

By definition of $B$, $(y_1,\dots,y_d)\cdot\det\,B \subseteq (x_1,\dots,x_d)^{[t]}$ and hence

$$(x_1\cdots x_d)^{td}\cdot\det B\cdot(y_1,\dots,y_d) \subseteq (x_1\cdots x_d)^{td}(x_1,\dots,x_d)^{[t]} \subset (x_1,\dots,x_d)^{[td+t]}. \quad (2.3)$$

Combining (2.2) and (2.3) we see that $r\cdot(x_1\cdots x_d)^{td}\cdot\det B\cdot\Delta \in (x_1,\dots,x_d)^{[td+t]}$. Therefore, (2.1) gives $r(x_1\cdots x_d)^{td}(x_1\cdots x_d)^{t-1} \in (\underline{x})^{[td+t]}$ and thus

$$r(x_1\cdots x_d)^{td+t-1} \in (\underline{x})^{[td+t]}.$$

Since $R$ is Cohen-Macaulay, we have that our sequence $(x_1,\dots,x_d)$ is actually a homogeneous regular sequence. Applying Lemma 2.2.1 shows that $r \in (x_1,\dots,x_d)$ and hence multiplication by $\Delta$ is injective. □

**Proposition 2.2.4.** *Let $(R,\mathfrak{m},k)$ be a graded Cohen-Macaulay ring of dimension $d$. If $y_1,\dots,y_d$ is a homogeneous system of parameters satisfying (1.5), and $x_1,\dots,x_d$ is a homogeneous system of parameters such that $(y_1,\dots,y_d)\subseteq(x_1,\dots,x_d)$, then $x_1,\dots,x_d$ satisfies (1.5) as well.*

*Proof.* Since $(y_1,\dots,y_d)\subseteq(x_1,\dots,x_d)$, we can write $y_i = \sum_{j=1}^{d} a_{ij}x_j$ and let $A=(a_{ij})$ be a $d\times d$ matrix of elements in $R$. Let the $\deg(y_i)=f_i$ and $\deg(x_i)=e_i$. This forces the $\deg(a_{ij}) = f_i - e_j$. Now let $\Delta=\det(A)$ and notice that $\deg(\Delta)=\sum_{i=1}^{d} f_i - \sum_{i=1}^{d} e_i$. Since $R$ is Cohen-Macaulay, $y_1,\dots,y_d$ and $x_1,\dots,x_d$ are regular sequences. Let $c = \sum_{i=1}^{d} e_i - d + 2$. By Lemma 2.2.3, multiplication by $\Delta$ is an injection, in particular,

$$\left(\frac{R}{(x_1,\dots,x_d)}\right)_c \xhookrightarrow{\Delta} \left(\frac{R}{(y_1,\dots,y_d)}\right)_{c+\deg(\Delta)}$$

is a graded homomorphism. Thus, if

$$\dim_k(R/(x_1,\ldots,x_d))_c > 2,$$

so is $\dim_k(R/(y_1,\ldots,y_d))_{c+\deg(\Delta)}$. But $y_1,\ldots,y_d$ satisfies (1.5), thus we must have that $x_1,\ldots,x_d$ satisfies the same property. $\square$

This next lemma is a basic fact that we will use multiple times and we state it here without proof.

**Lemma 2.2.5.** *Let $R$ be a commutative ring with identity. For ideals $I,J,K \subseteq R$ with $K \subseteq I$, we have the following:*

$$\frac{I+J}{K+J} \simeq \frac{I}{(I\cap J)+K}.$$

This next proposition distinguishes super-stretched rings from that of stretched rings as not all stretched rings have this property. Further, when $R$ is a zero dimensional super-stretched ring, this proposition fails as can be seen by the ring $k[x]/(x^4)$.

**Proposition 2.2.6.** *If $(R,\mathfrak{m},k)$ is a standard graded Cohen-Macaulay ring of dimension $d>0$ that is super-stretched, then for all homogeneous minimal reductions $(x_1,\ldots,x_d)$ of the maximal ideal $\mathfrak{m}$, we have $\mathfrak{m}^3 = (x_1,\ldots,x_d)\mathfrak{m}^2$.*

*Proof.* Induct on $d$. For the dimension one case let $x$ be a minimal reduction. (Note that $\deg(x) = 1$.) Because $R$ is super stretched, we have that $\dim_k(R/xR)_2 = 1$. By Lemma 2.2.5 we have

$$\dim_k \frac{\mathfrak{m}^2+(x)}{\mathfrak{m}^3+(x)} = \dim_k \frac{\mathfrak{m}^2}{\mathfrak{m}^3+((x)\cap\mathfrak{m}^2)} = \dim_k \frac{\mathfrak{m}^2}{\mathfrak{m}^3+x\mathfrak{m}} = 1.$$

Note that the second equality is always true as $(x) \cap \mathfrak{m}^2 = x\mathfrak{m}$ if $x \notin \mathfrak{m}^2$. The displayed equality says that there is a $y \in \mathfrak{m}^2 - (x\mathfrak{m} + \mathfrak{m}^3)$ such that $\mathfrak{m}^2 = x\mathfrak{m} + (y) + \mathfrak{m}^3$. By Nakayama's lemma we have $\mathfrak{m}^2 = x\mathfrak{m} + (y)$.

By the super-stretched hypothesis we can also consider $R$ modulo $x^2$. Due to the grading we have that $(x^2) \cap \mathfrak{m}^3 = x^2\mathfrak{m}$. This gives us

$$\dim_k \frac{\mathfrak{m}^3}{\mathfrak{m}^4 + x^2\mathfrak{m}} = 1.$$

Thus, as before, there exists $z \in \mathfrak{m}^3 - (x^2\mathfrak{m} + \mathfrak{m}^4)$ such that $\mathfrak{m}^3 = x^2\mathfrak{m} + (z) + \mathfrak{m}^4$. Nakayama's lemma shows that $\mathfrak{m}^3 = x^2\mathfrak{m} + (z)$.

Notice that we can choose $z$ to be anything in $\mathfrak{m}^3 - (x^2\mathfrak{m} + \mathfrak{m}^4)$. We would like to choose $z = xy$, but first we must show

*Claim.* The element $xy$ is not in $x^2\mathfrak{m} + \mathfrak{m}^4$.

If the claim holds, then we have

$$\begin{aligned} \mathfrak{m}^3 &= x^2\mathfrak{m} + (z) \\ &= x^2\mathfrak{m} + (xy) \\ &= x(x\mathfrak{m} + (y)) \\ &= x\mathfrak{m}^2. \end{aligned}$$

This is the desired result for dimension one.

To show the claim, let $n$ be minimally chosen such that $\mathfrak{m}^n = x\mathfrak{m}^{n-1}$ and suppose $xy \in x^2\mathfrak{m} + \mathfrak{m}^4$. Since $\mathfrak{m}^2 = x\mathfrak{m} + (y)$, we have that $x\mathfrak{m}^2 \subseteq x^2\mathfrak{m} + \mathfrak{m}^4$. Assume that $n > 3$ and multiply by $\mathfrak{m}^{n-3}$. As $R$ is Cohen-Macaulay, we have that $x$ is a non-zero divisor. Cancel the $x$'s to observe that $\mathfrak{m}^{n-1} \subseteq x\mathfrak{m}^{n-2} + \mathfrak{m}^n$. This forces $\mathfrak{m}^{n-1} = x\mathfrak{m}^{n-2}$, a contradiction since $n$ was chose to be minimal. Thus $xy \notin (x^2\mathfrak{m} + \mathfrak{m}^4)$.

For higher dimensions we may assume that

$$\frac{\mathfrak{m}^3+(x_d)}{(x_d)}=\frac{(x_1,\ldots,x_{d-1})\mathfrak{m}^2+(x_d)}{(x_d)}.$$

Lifting gives us the inclusion

$$\mathfrak{m}^3\subseteq(x_1,\ldots,x_{d-1})\mathfrak{m}^2+(x_d).$$

Notice by the grading, and the regularity of $x_d$, we have that $\mathfrak{m}^2=(\mathfrak{m}^3:x_d)$ and hence

$$\begin{aligned}\mathfrak{m}^3&=(x_1,\ldots,x_{d-1})\mathfrak{m}^2+x_d(\mathfrak{m}^3:x_d)\\&=(x_1,\ldots,x_d)\mathfrak{m}^2.\end{aligned}$$

□

The next lemma is well known and is useful in proving Proposition 2.2.8

**Lemma 2.2.7.** *Let $a_1,\ldots,a_k$ be elements of a ring $R$. Then for every positive integer $m$,*

$$(a_1^m,\ldots,a_k^m)(a_1,\ldots,a_k)^{(k-1)(m-1)}=(a_1,\ldots,a_k)^{(m-1)k+1}.$$

*Proof.* Page 152 of [24]. □

The next proposition is helpful in obtaining a converse to Proposition 2.2.6.

**Proposition 2.2.8.** *Let $(R,\mathfrak{m},k)$ be a $d$-dimensional standard graded ring and let the ideal $(x_1,\ldots,x_d)$ be a homogeneous reduction of $\mathfrak{m}$ such that $(x_1,\ldots,x_d)\mathfrak{m}^2=\mathfrak{m}^3$. If $R$ is stretched, then $(x_1^t,\ldots,x_d^t)$ satisfies (1.5) for all $t>0$.*

*Proof.* We would like to show that for all $t > 0$,

$$\dim_k \left( \frac{R}{(x_1^t, \ldots, x_d^t)} \right)_i < 2$$

for each $i \geqslant dt - d + 2$. In particular, we need that

$$\dim_k \left( \frac{\mathfrak{m}^{dt-d+2}}{\mathfrak{m}^{dt-d+3} + (x_1^t, \ldots, x_d^t) \cap \mathfrak{m}^{dt-d+2}} \right) < 2.$$

To show this, we need to first show the equality

$$\dim_k \left( \frac{\mathfrak{m}^{dt-d+2}}{\mathfrak{m}^{dt-d+3} + (x_1^t, \ldots, x_d^t) \cap \mathfrak{m}^{dt-d+2}} \right) = \dim_k \left( \frac{\mathfrak{m}^{dt-d+2}}{(x_1^t, \ldots, x_d^t)\mathfrak{m}^{dt-d+2-t}} \right). \quad (2.4)$$

In order to see equation (2.4), notice that

$$\mathfrak{m}^{dt-d+3} + (x_1^t, \ldots, x_d^t) \cap \mathfrak{m}^{dt-d+2} = \mathfrak{m}^{dt-d+3} + (x_1^t, \ldots, x_d^t)\mathfrak{m}^{dt-d+2-t}. \quad (2.5)$$

As $(x_1, \ldots, x_d)\mathfrak{m}^2 = \mathfrak{m}^3$, we have that for any positive integer $N$, $(x_1, \ldots, x_d)^N \mathfrak{m}^2 = \mathfrak{m}^{N+2}$. Further, if we consider $\mathfrak{m}^{dt-d+3}$ and let $N = d(t-1)+1$, we have that

$$\mathfrak{m}^{dt-d+3} = \mathfrak{m}^{d(t-1)+3} = (x_1, \ldots, x_d)^{d(t-1)+1}\mathfrak{m}^2.$$

By Lemma 2.2.7 we have that $(x_1, \ldots, x_d)^{d(t-1)+1} = (x_1^t, \ldots, x_d^t)(x_1, \ldots, x_d)^{(d-1)(t-1)}$. Therefore we have that

$$\begin{aligned} \mathfrak{m}^{dt-d+3} &= (x_1^t, \ldots, x_d^t)(x_1, \ldots, x_d)^{(d-1)(t-1)}\mathfrak{m}^2 \\ &= (x_1^t, \ldots, x_d^t)\mathfrak{m}^{(d-1)(t-1)+2} \\ &\subseteq (x_1^t, \ldots, x_d^t)\mathfrak{m}^{(d-1)(t-1)+1} = (x_1^t, \ldots, x_d^t)\mathfrak{m}^{dt-d+2-t}. \end{aligned}$$

Applying this fact to Equation (2.5) allows us to write

$$\mathfrak{m}^{dt-d+3}+(x_1^t,\ldots,x_d^t)\cap\mathfrak{m}^{dt-d+2}=(x_1^t,\ldots,x_d^t)\mathfrak{m}^{dt-d+2-t},$$

and hence equality holds in Equation (2.4).

The next step is to show that

$$(x_1,\ldots,x_d)^{dt-d}=(x_1^t,\ldots,x_d^t)(x_1,\ldots,x_d)^{dt-d-t}+(x_1x_2\cdots x_d)^{t-1}. \tag{2.6}$$

Notice that the generators of $(x_1,\ldots,x_d)^{dt-d-t}$ are all the monomials in $x_1,\ldots,x_d$ of degree $dt-d-t$. Thus the generators of the product $(x_1^t,\ldots,x_d^t)(x_1,\ldots,x_d)^{dt-d-t}$ are monomials $m$ in $x_1,\ldots,x_d$ of degree $d(t-1)$ such that $x_j^t|m$ for some $j=1,2,\ldots d$. Call the set of these monomials $M$. The monomials in $M$ are also a part of a minimal generating set of the ideal $(x_1,\ldots,x_d)^{dt-d}$. In fact, the set $N=\{x_1^{n_1}x_2^{n_2}\cdots x_d^{n_d}\mid\sum n_i=dt-d\}$ is a generating set for the ideal $(x_1,\ldots,x_d)^{dt-d}$. Hence, the elements in $M$ that are not in $N$ are

$$N\setminus M=\{m=x_1^{n_1}x_2^{n_2}\cdots x_d^{n_d}\mid\sum n_i=d(t-1)\text{ and such that }x_j^t\not|m\text{ for all }j\}.$$

This implies that $m\in N\setminus M$ is an element of the ideal $(x_1x_2\cdots x_d)^{t-1}$ as $\deg(m)=d(t-1)$ and $x_j^t$ does not divide $m$. We therefore have the equality in Equation (2.6).

Since $x_1,\dots,x_d$ is a reduction, we can write $\mathfrak{m}^{dt-d+2} = \mathfrak{m}^2(x_1,\dots,x_d)^{dt-d}$. Combining this with Equation (2.6) yields

$$\begin{aligned}
\mathfrak{m}^{dt-d+2} &= \mathfrak{m}^2(x_1,\dots,x_d)^{dt-d} \\
&= \mathfrak{m}^2((x_1^t,\dots,x_d^t)(x_1,\dots,x_d)^{dt-d-t} + (x_1x_2\cdots x_d)^{t-1}) \\
&= (x_1^t,\dots,x_d^t)(x_1,\dots,x_d)^{dt-d-t}\mathfrak{m}^2 + (x_1x_2\cdots x_d)^{t-1}\mathfrak{m}^2 \\
&= (x_1^t,\dots,x_d^t)\mathfrak{m}^{dt-d-t+2} + (x_1x_2\cdots x_d)^{t-1}\mathfrak{m}^2.
\end{aligned}$$

Because $R$ is stretched, we may proceed as in Proposition 2.2.6 and choose a $y \in \mathfrak{m}^2 - ((x_1,\dots,x_d)\mathfrak{m} + \mathfrak{m}^3)$ such that $\mathfrak{m}^2 = (x_1,\dots,x_d)\mathfrak{m} + (y)$. Substituting into the above relation yields

$$\begin{aligned}
\mathfrak{m}^{dt-d+2} &= (x_1^t,\dots,x_d^t)\mathfrak{m}^{dt-d-t+2} + (x_1x_2\cdots x_d)^{t-1}((x_1,\dots,x_d)\mathfrak{m} + (y)) \\
&= (x_1^t,\dots,x_d^t)\mathfrak{m}^{dt-d-t+2} + (y(x_1x_2\cdots x_d)^{t-1}).
\end{aligned}$$

Thus, modulo $(x_1^t,\dots,x_d^t)\mathfrak{m}^{dt-d-t+2}$, we have that $\mathfrak{m}^{dt-d+2}$ is one dimensional. □

We are now ready to state and prove an equivalent definition of super-stretched; this is also the main result of this section and is used to better understand what rings of graded countable Cohen-Macaulay type.

**Theorem 2.2.9.** *Let $(R,\mathfrak{m},k)$ be a standard graded Cohen-Macaulay ring of dimension $d > 0$. The following are equivalent:*

*(1) $R$ is super-stretched;*

*(2) $R$ is stretched and $J\mathfrak{m}^2 = \mathfrak{m}^3$ for every homogeneous reduction $J$ of the maximal ideal;*

*(3) $R$ is stretched and $J\mathfrak{m}^2 = \mathfrak{m}^3$ for some homogeneous reduction $J$ of the maximal ideal.*

*Proof.* (1) $\Rightarrow$ (2): This is an application of Proposition 2.2.6.

(2) $\Rightarrow$ (3): This is straightforward.

(3) $\Rightarrow$ (1): Assume that $J$ is as in (3) and is generated by $x_1, \ldots, x_d$. Let $(y_1, \ldots, y_d)$ be an ideal of $R$ generated by a homogeneous system of parameters. We have that there exists a $t$ such that $\mathfrak{m}^t \subseteq (y_1, \ldots, y_d)$. In particular, $(x_1^t, \ldots, x_d^t) \subseteq (y_1, \ldots, y_d)$. By proposition 2.2.4, $(y_1, \ldots, y_d)$ satisfies (1.5) since $(x_1^t, \ldots, x_d^t)$ satisfies (1.5) by proposition 2.2.8. Therefore, $R$ is super-stretched. □

The next two propositions are immediate corollaries of Theorem 2.2.9 that describe the $h$-vector of a super-stretched ring.

**Proposition 2.2.10.** *Assume that $(R, \mathfrak{m}, k)$ is a standard graded super-stretched ring of dimension $d > 0$ with infinite residue field $k$. Then the $h$-vector of $R$ is of one of the following forms: $(1)$, $(1, n)$, or $(1, n, 1)$ for some non-zero positive integer $n$.*

*Proof.* Let $J$ be a minimal reduction of the maximal ideal $\mathfrak{m}$. Since $R$ is super-stretched, we have by Theorem 2.2.9 that $J\mathfrak{m}^2 = \mathfrak{m}^3$. Let $\overline{R} = R/J$ and notice that by Lemma 2.2.5 we have

$$H_{\overline{R}}(3) = \dim_k \left( \frac{\mathfrak{m}^3}{\mathfrak{m}^4 + J \cap \mathfrak{m}^3} \right) = \dim_k \left( \frac{\mathfrak{m}^3}{\mathfrak{m}^4 + J\mathfrak{m}^2} \right) = \dim_k \left( \frac{\mathfrak{m}^3}{\mathfrak{m}^4 + \mathfrak{m}^3} \right) = 0.$$

This forces $H_{\overline{R}}(n) = 0$ for all $n > 2$. The fact that $R$ is stretched forces $H_{\overline{R}}(2) \leqslant 1$. Therefore, the only possible $h$-vectors are $(1)$, $(1, n)$, or $(1, n, 1)$ where $n$ is a non-zero positive integer. □

**Proposition 2.2.11.** *A standard graded hypersurface with multiplicity at most 3 is super-stretched.*

*Proof.* Let $R$ be a hypersurface with multiplicity $e \leqslant 3$. As the sum of the $h$-vector is the multiplicity, the only possible $h$-vectors are $(1)$, $(1,1)$, and $(1,1,1)$. All of these satisfy condition 3 of Theorem 2.2.9. □

## 2.3 Examples of super-stretched

Given Theorem 2.2.9, we are able to give some examples of a standard graded ring that is super-stretched. We list several examples in Tables 2.3 and 2.3 without exposition. Many of these examples are straight forward computations and can be checked by hand or with the computer algebra software Macaulay2 [13]. References are giving when appropriate and the following key is used in interpreting the tables. Further, we let $k$ be an algebraically closed uncountable field of characteristic 0.

| Acronym | Meaning |
|---|---|
| GFT | Graded finite Cohen-Macaulay type |
| GCT | Graded Countable Cohen-Macaulay type |
| NGFT | Not GFT but maybe GCT |
| NGCT | Not GCT |
| FT | Finite Cohen-Macaulay type |
| CT | Countable Cohen-Macaulay type |
| NFT | Not FT but maybe CT |
| NCT | Not CT |
| DR | Drozd-Roĭter conditions, Proposition 4.2.3 |

| Ring | $\dim(R)$ | CM Type | $h$-vector | Red. Num. | Stretched | Sup. Stret. | Min'l Red. | Ref |
|---|---|---|---|---|---|---|---|---|
| $k[x,y]/(x^2+y^2)$ | 1 | GFT | $(1,1)$ | 1 | yes | yes | $x$ | [11] |
| $k[x,y]/(xy)$ | 1 | GFT | $(1,1)$ | 1 | yes | yes | $x+y$ | [11] |
| $k[x,y]/(xy(x+y))$ | 1 | GFT | $(1,1,1)$ | 2 | yes | yes | $x+2y$ | [11] |
| $k[x,y,z]/\det_2\begin{pmatrix} x & y & z \\ y & z & x \end{pmatrix}$ | 1 | GFT | $(1,2)$ | 1 | yes | yes | $x$ | [11] |
| $k[x,y]/(x^3y-xy^3)$ | 1 | NGCT | $(1,1,1,1)$ | 3 | yes | no | $x+2y$ | DR |
| $k[\![t^3,t^7]\!]$ | 1 | NCT | $(1,1,1)$ | 2 | yes | yes | $t^3$ | DR |
| $k[x,y,z]/(x^2+y^2,x^2+z^2)$ | 1 | NGCT | $(1,2,1)$ | 2 | yes | yes | $x$ | DR |
| $k[\![t^4,t^5,t^7]\!]$ | 1 | NCT | $(1,2,1)$ | 2 | yes | yes | $t^4$ | DR |
| $k[\![t^4,t^5,t^6,t^7]\!]$ | 1 | NCT | $(1,3)$ | 1 | yes | yes | $t^4$ | DR |
| $k[x,y,z]/(xy,xz,z^2)$ | 1 | NGCT | $(1,2)$ | 1 | yes | yes | $x+y$ | [20, Ex. 13.25] |
| $k[\![x,y]\!]/(xy^2)$ | 1 | CT | $(1,1,1)$ | 2 | yes | yes | x+y | $(D_\infty)$ |
| $k[\![x,y]\!]/(y^2)$ | 1 | CT | $(1,1)$ | 1 | yes | yes | $x$ | $(A_\infty)$ |
| $k[\![t^4,t^5]\!]$ | 1 | NCT | $(1,1,1,1)$ | 3 | yes | no | $t^4$ | DR |

Table 2.1: One dimensional examples

| Ring | $\dim(R)$ | CM Type | $h$-vector | Red. Num. | Stretched | Sup. Stret. | Min'l Red. | Ref |
|---|---|---|---|---|---|---|---|---|
| $k[x,y,z]/(x^2+y^2+z^2)$ | 2 | GFT | $(1,1)$ | 1 | yes | yes | $x,y$ | [11] |
| $k[x_1,\dots,x_{n+1}]/\det_2\begin{pmatrix} x_1 & \dots & x_n \\ x_2 & \dots & x_{n+1}\end{pmatrix}, n \geqslant 2$ | 2 | GFT | $(1,n-1)$ | 1 | yes | yes | $x_1,x_{n+1}$ | [11] |
| $k[\![x,y,a,b,z]\!]/(xa,xb,ya,yb,xz-y^2,az-b^2)$ | 2 | CT | $(1,3)$ | 1 | yes | yes | $x+a,z$ | [20, Pg 326] |
| $k[\![a,b,c,z]\!]/\det_2\begin{pmatrix} a & z^2 & b \\ z^2 & b & c\end{pmatrix}$ | 2 | NFT | $(1,2)$ | 1 | yes | yes | $a+c,z$ | [20, Ex. 13.28] |
| $k[\![x,y,z]\!]/(y^2+z^2)$ | 2 | CT | $(1,1)$ | 1 | yes | yes | $x,z$ | $(A_\infty)$ |
| $k[x,y,z,t]/(x^2+y^2+z^2+t^2)$ | 3 | GFT | $(1,1)$ | 1 | yes | yes | $x,y,z$ | [11] |
| $k[x_1,\dots,x_5]/\det_2\begin{pmatrix} x_1 & x_2 & x_4 \\ x_2 & x_3 & x_5\end{pmatrix}$ | 3 | GFT | $(1,2)$ | 1 | yes | yes | $x_1,x_3+x_4,x_5$ | [11] |
| $k[x_1,\dots,x_6]/\det_2(\text{sym}3\times 3)$ | 3 | GFT | $(1,3)$ | 1 | yes | yes | $x_1,x_4,x_6$ | [11] |
| $k[\![x,y,z,t]\!]/(y^2+z^2+t^2)$ | 3 | CT | $(1,1)$ | 1 | yes | yes | $x,y,z$ | $(A_\infty)$ |

Table 2.2: Two and three dimensional examples

## 2.4 Super-Stretched and Graded Countable Type

Given Theorem 2.2.9, we are now able to generalize D. Eisenbud and J. Herzog's result to standard graded rings of countable Cohen-Macaulay type. We restate it here for convenience.

**Theorem 1.2.6** ([11, Theorem A]). *Let $(R, \mathfrak{m}, k)$ be a standard graded Cohen-Macaulay ring with infinite residue field k. If R is of graded finite Cohen-Macaulay type, then R is stretched.*

It is worth noting that Theorem 1.2.6 is only known to be true in the graded category and the classification of rings of finite Cohen-Macaulay type (not graded finite Cohen-Macaulay type) is still open. However, by Proposition 2.1.5 we are able to drop the restriction of graded isomorphisms and look at any isomorphism between graded indecomposibles when considering graded finite type. The proof of D. Eisenbud and J. Herzog assumes the ring is not stretched and then constructs an infinite family $\{M_\alpha\}_{\alpha \in k}$ of non-isomorphic (in the graded category) graded maximal Cohen-Macaulay modules to form a contradiction. Unfortunately, as written the proof is not correct. The authors of [11] assume that since the family $\{M_\alpha\}$ has a uniform bound on the rank, there must be infinitely many, non-isomorphic up to shifts in degree, graded, indecomposable maximal Cohen-Macaulay modules. As shown by the following example, this is not the case.

**Example 2.4.1.** Let $(R, \mathfrak{m}, k)$ be a standard graded Cohen-Macaulay ring. Note that $R$ is a graded indecomposable maximal Cohen-Macaulay module and consider the family of graded maximal Cohen-Macaulay modules $\{R \oplus R(-i)\}_{i \in \mathbb{Z}_{>0}}$. There is not a graded isomorphism between any two modules in the family and each module is of rank 2. Thus we have an infinite family of non-isomorphic (in the graded category) maximal

Cohen-Macaulay modules of bounded rank. However, up to isomorphism in shifts of degree, there is only one indecomposable maximal Cohen-Macaulay module.

Even though the proof is not quite correct, a slight modification (as shown by Y. Yoshino [25, Theorem 17.7]) fixes the error and shows Theorem 1.2.6 to be true. Instead of using a rank argument, Y. Yoshino reduces the family of maximal Cohen-Macaulay modules to a family of non-isomorphic (in the graded category) indecomposable maximal Cohen-Macaulay modules. In essence, he directly shows that if the ring is not stretched, then there are infinitely many non-isomorphic (in the graded category) indecomposable maximal Cohen-Macaulay modules.

For the countable case (and hence the finite case), we are able to show the stronger notion of super-stretched. The proof of Theorem 2.4.4 is an extension of D. Eisenbud and J. Herzog's proof. However, we have chosen to be a little more explicit by directly showing the existence of an uncountable family of non-isomorphic indecomposable modules in $\mathfrak{MCM}^{\mathrm{gr}}(R)$ for a standard graded ring $R$ that is not super-stretched.

The following lemmata a helpful in proving Theorem 2.4.4. The proof of Lemma 2.4.2 is straight forward, however, we write out the statement here as it is used multiple times in the proof of Theorem 2.4.4.

**Lemma 2.4.2.** *Let $(R,\mathfrak{m},k)$ be a standard graded ring and $M$, $N$ be finitely generated graded $R$-modules such that $N = Rx_1 + \cdots + Rx_s$, $M = Ry_1 + \cdots + Ry_t$ with $\deg(x_i) = n_i$, $\deg(y_j) = m_j$ and $t = \max\{n_i\} < \min\{m_j\}$. Let $\phi$ and $\psi$ be a graded presentations of $N$ and $N+M$ (respectively) sending $R(-n_i)$ to $x_i$ and $R(-m_j)$ to $y_j$. Consider the following commutative diagram defined by the canonical injection:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & \bigoplus R(-n_i) & \xrightarrow{\phi} & N & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow h & & \big\downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & \bigoplus R(-n_i) \oplus \bigoplus R(-m_j) & \xrightarrow{\psi} & N+M & \longrightarrow & 0
\end{array}$$

*where $K$ and $L$ are the respective kernels. If $z \in L$ is such that $\deg(z) \leqslant t$, then $z \in \operatorname{im}(h)$. In other words, any syzygy of $N+M$ with degree less than $t+1$ comes from a syzygy of $N$.*

*Proof.* By choice of $z \in L$, we know that $z$ is also in $\bigoplus R(-n_i)$ since $\max\{n_i\} < \min\{m_j\}$. Since $\phi$ is $\psi$ restricted $\bigoplus R(-n_i)$, we see that $z$ is indeed in the image of $h$. □

**Lemma 2.4.3.** *If $(R, \mathfrak{m}, k)$ is a standard graded ring such that $\dim_k(R_i) > 1$ for some $i > 0$, then there exists $|k|$ many distinct homogeneous ideals in $R$.*

*Proof.* Let $x, y$ be distinct basis elements of $R_i$ and let $\alpha, \beta \in k$. Assume that

$$(x + \alpha y) = (x + \beta y).$$

Since $R$ is graded, there exist $\gamma \in k$ such that $x + \alpha y = \gamma(x + \beta y)$ in $R_i$. Hence

$$(1-\gamma)x + (\alpha - \beta\gamma)y = 0 \tag{2.7}$$

in $R$. In particular, this relation holds in the vector space $R_i$ as $\alpha, \beta, \gamma \in k$. Thus the coefficients of $x$ and $y$ are zero and we have that $\gamma = 1$ and $\alpha = \beta$. Therefore the conclusion follows. □

We are now ready to prove the generalization of D. Eisenbud and J. Herzog's result (Theorem 1.2.6).

**Theorem 2.4.4.** *Let $(R, \mathfrak{m}, k)$ be a standard graded Cohen-Macaulay ring of dimension $d > 0$ with uncountable residue field $k$. If $R$ is of graded countable Cohen-Macaulay type then it is super-stretched.*

*Proof.* Let $R$ be as above with $\dim(R) = d$. Assume $R$ is not super-stretched and let $x_1, \ldots, x_d$ be a homogeneous system of parameters such that

$$\dim_k(R/(x_1, \ldots, x_d))_c \geqslant 2$$

for some

$$c \geqslant \sum_{j=1}^{d} \deg(x_j) - d + 2. \tag{2.8}$$

If we have that $\dim_k(R/(x_1, \ldots, x_d))_c \geqslant 2$, then $\dim_k(R/(x_1, \ldots, x_d))_l \geqslant 2$ for all $2 \leqslant l \leqslant c$. Therefore, without losing any generality, we can assume equality in (2.8). Let $\overline{R} = R/(x_1, \ldots, x_d)$ and consider $\overline{y} \in (\overline{R})_c$. Define $I_{\overline{y}} \subseteq R$ to be a preimage of $(\overline{y})$. For each $\overline{y} \in \overline{R}_c$, we shall associate a graded maximal Cohen-Macaulay module $M_{\overline{y}}$ such that the family $\{M_{\overline{y}}\}_{\overline{y} \in \overline{R}_c}$ has the following properties:

(1) Let $(M_{\overline{y}})_i$ be the graded components of $M_{\overline{y}}$. We have that $\dim_k(M_{\overline{y}})_{<t} = 0$ and $\dim_k(M_{\overline{y}})_t = 1$ where $t = \sum \deg(x_j)$.

(2) There is a unique indecomposable summand $N_{\overline{y}}$ of $M_{\overline{y}}$ such that $(N_{\overline{y}})_t = (M_{\overline{y}})_t$.

(3) If $\overline{M_{\overline{y}}} = M_{\overline{y}}/(x_1, \ldots, x_d)M_{\overline{y}}$, then $\operatorname{ann}_R(\overline{M_{\overline{y}}})_t = I_{\overline{y}}$ where $t$ is as in (1).

Assuming these three claims, we show there exists uncountably many graded indecomposable maximal Cohen-Macaulay modules up to isomorphism in $\mathfrak{MCM}^{\mathrm{gr}}(R)$. Hence by Proposition 2.1.8, $R$ cannot be of graded countable Cohen-Macaulay type.

As in (2), let $N_{\overline{y}}$ and $N_{\overline{y}'}$ be the unique indecomposable summands of $M_{\overline{y}}$ and $M_{\overline{y}'}$ for $\overline{y}, \overline{y}' \in \overline{R}_c$. Suppose that there is an isomorphism $N_{\overline{y}} \simeq N_{\overline{y}'}$ in $\mathfrak{MCM}^{\mathrm{gr}}(R)$. Thus we have that $Re \mapsto Re'$ where $e$ and $e'$ are generators of $(N_{\overline{y}})_t$ and $(N_{\overline{y}'})_t$ (respectively). This implies

$$N_{\overline{y}}/(x_1, \ldots, x_d)N_{\overline{y}} \simeq N_{\overline{y}'}/(x_1, \ldots, x_d)N_{\overline{y}'}.$$

Once again, under this isomorphism, $R\overline{e} \mapsto R\overline{e}'$. From (3), we have that $\operatorname{ann}_R(\overline{e}) = I_{\overline{y}}$ and $\operatorname{ann}_R(\overline{e}') = I_{\overline{y}'}$, which forces $I_{\overline{y}} = I_{\overline{y}'}$. Thus $(\overline{y}) = (\overline{y}')$. Note that $\dim_k(\overline{R}_c) \geqslant 2$, so by Lemma 2.4.3 there exists uncountably many ideals $(\overline{y})$, where $\overline{y} \in \overline{R}_c$. As such there must be uncountably many graded indecomposable maximal Cohen-Macaulay modules up to isomorphism in $\mathfrak{MCM}^{\mathrm{gr}}(R)$ and we are finished by Proposition 2.1.8.

To show property (1), consider the Koszul complex $K_.$ of the homogeneous system of parameters $x_1, \ldots, x_d$,

$$K_. : 0 \longrightarrow K_d \longrightarrow \cdots \longrightarrow K_1 \longrightarrow R \longrightarrow R/(x_1, \ldots, x_d)R \longrightarrow 0.$$

Note that for $J \subseteq \{1, 2, \ldots, d\}$, $K_i \simeq \bigoplus_{|J|=i} R(-\sum_{j \in J} \deg x_j)$. Let $\Omega_i$ be the $i^{th}$ syzygy of $K_.$ and fix $\overline{y} \in (\overline{R})_c$. Further, let $I_{\overline{y}} = (x_1, \ldots, x_d, y)$ be a preimage of $(\overline{y})$ and consider the minimal resolution $F_.$ of $R/I_{\overline{y}}$. From $F_.$, we have the short exact sequence

$$0 \longrightarrow M_2 \longrightarrow \bigoplus R(-\deg(x_i)) \oplus R(-c) \longrightarrow (x_1, \ldots, x_d, y) \longrightarrow 0$$

where $M_2$ is the second syzygy of $R/I_{\overline{y}}$. Notice that

$$c = \sum_{j=1}^{d} \deg(x_j) - d + 2 > \max\{\deg x_i\}.$$

Therefore by Lemma 2.4.2, we have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_2 & \longrightarrow & K_1 & \longrightarrow & (x_1, \ldots, x_d) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_2 & \longrightarrow & \bigoplus R(-\deg(x_j)) \oplus R(-c) & \longrightarrow & (x_1, \ldots, x_d, y) & \longrightarrow & 0
\end{array}$$

with the minimal graded components of $M_2$ coming from the image of $\Omega_2$.

In the resolution $F.$, any shift coming from $y$ will be bounded below by $c+i$. Since

$$c+i=\sum_{j=1}^{d}\deg(x_j)-(d-(i+1))+2>\max\{\sum_{\substack{|J|=i+1\\ j\in J}}\deg x_j\},$$

we have that

$$\text{shift from } y \geqslant c+i>\max\{\sum_{\substack{|J|=i+1\\ j\in J}}\deg x_j\}.$$

Thus, we for each $i$ we have the commutative diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & \Omega_i & \longrightarrow & K_{i-1} & \longrightarrow & \Omega_{i-1} & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & M_i & \longrightarrow & \bigoplus_{|J|=i-1} R(-\sum_{j\in J}\deg(x_j))\oplus\bigoplus R(-d_{i-1,j}) & \longrightarrow & M_{i-1} & \longrightarrow 0
\end{array}$$

and each minimal graded component of $M_i$ is an image of a minimal generator of $\Omega_i$. When $i=d$, we have that $K_d\hookrightarrow M_d$ and any minimal graded generator of $M_d$ is of degree $t=\sum_{j=1}^{d}\deg x_j$. The injection shows this is unique. Thus these modules satisfy property (1). We denote $M_d$ by $M_{\overline{y}}$.

Property (2) is a straight forward consequence of (1). As there is a unique minimal element in $M_{\overline{y}}$, say $Re=(M_{\overline{y}})_t$, it must be contained in a unique indecomposable summand $N_{\overline{y}}$ of $M_{\overline{y}}$. Therefore $(N_{\overline{y}})_t=(M_{\overline{y}})_t$.

To prove (3), let $\overline{F}_.=F_.\otimes R/(x_1,\ldots,x_d)R$ and consider $\operatorname{Tor}_d^R(R/(x_1,\ldots,x_d),R/I_{\overline{y}})$. Since $R$ is Cohen-Macaulay, any system of parameters is a regular sequence. Therefore,

$$\operatorname{Tor}_d^R(R/(x_1,\ldots,x_d),R/I_{\overline{y}})\simeq H_d(x_1,\ldots,x_d;R/I_{\overline{y}}),$$

where $H_d(x_1,\ldots,x_d;R/I_{\overline{y}})$ is the $d^{th}$ homology of the Koszul complex of the homogeneous system of parameters $x_1,\ldots,x_d$ with values in $R/I_{\overline{y}}$. Since the $x_i$ annihilate $R/I_{\overline{y}}$, we have

$$H_d(x_1,\ldots,x_d;R/I_{\overline{y}}) \simeq R/I_{\overline{y}}(-t).$$

Apply $\cdot \otimes_R R/(x_1,\ldots,x_d)$ to the short exact sequence

$$0 \longrightarrow M_{\overline{y}} \longrightarrow F_{d-1} \longrightarrow M_{d-1} \longrightarrow 0$$

to get

$$0 \longrightarrow \mathrm{Tor}_1(R/(x_1,\ldots,x_d),M_{d-1}) \longrightarrow M_{\overline{y}}/(x_1,\ldots,x_d)M_{\overline{y}} \longrightarrow \overline{F}_{d-1}.$$

Since $\mathrm{Tor}_1(R/(x_1,\ldots,x_d),M_{d-1}) \simeq \mathrm{Tor}_d(R/x_1,\ldots,x_d,R/I_{\overline{y}})$, we have

$$0 \longrightarrow R/I_{\overline{y}}(-t) \longrightarrow M_{\overline{y}}/(x_1,\ldots,x_d)M_{\overline{y}} \xrightarrow{\alpha} \overline{F}_{d-1}.$$

Since $e \in M_{\overline{y}}$ corresponds to the generator of $K_d$, it is clear that $\overline{e} \mapsto 0$, and it follows from the exact sequence that $R\overline{e} \simeq R/I_{\overline{y}}(-t)$. □

*Remark* 2.4.5. It is worth noting that as for Theorem 1.2.6, we are able to lift the restriction of the isomorphism classes in the definition of graded countable Cohen-Macaulay type by way of Proposition 2.1.8. Thus Theorem 2.4.4 holds if we have countably many graded indecomposable modules up to isomorphism in $\mathfrak{MCM}(R)$. This is a much stronger statement as we do not require graded isomorphisms.

It would be nice if super-stretched implied finite Cohen-Macaulay type. Unfortunately this is not the case. Consider the complete hypersurface singularity $k[\![x,y]\!]/(y^2)$ found on page 2. This ring is not finite Cohen-Macaulay type but is super-stretched.

However, this ring is not an isolated singularity, so there is still hope to show that given an isolated singularity, super-stretched implies finite Cohen-Macaulay type.

Next we have a couple of immediate corollaries. The first one is interpreting Theorem 2.4.4 in terms of graded finite Cohen-Macaulay type while the second is attributing properties of super-stretched to standard graded rings of graded countable Cohen-Macaulay type.

**Corollary 2.4.6.** *Let $(R, \mathfrak{m}, k)$ be a standard graded Cohen-Macaulay ring with uncountable residue field k. If R is of graded finite Cohen-Macaulay type then it is super-stretched.*

**Corollary 2.4.7.** *Let $(R, \mathfrak{m}, k)$ be a standard graded Cohen-Macaulay ring with graded countable Cohen-Macaulay type. Then the possible h-vectors are $(1)$, $(1,n)$, or $(1,n,1)$ for some integer n.*

*Proof.* Combine Theorem 2.4.4 and proposition 2.2.10. □

## 2.5 Minimal Multiplicity and Graded Countable Type

Throughout this sub-section we will continue the assumption we are working in the graded category $\mathfrak{Mod}^{\mathrm{gr}}(R)$ where $(R, \mathfrak{m}, k)$ is a standard graded ring. In particular, we will be considering graded maximal Cohen-Macaulay modules in $\mathfrak{MCM}^{\mathrm{gr}}(R)$.

In [11], D. Eisenbud and J. Herzog showed that standard graded rings of graded finite Cohen-Macaulay type and $\dim(R) > 1$ have minimal multiplicity. Using Theorem 2.4.4, we are able to extend this result to graded countable Cohen-Macaulay type with $\dim(R) > 2$.

**Definition 2.5.1.** Let $(R,\mathfrak{m},k)$ be a $d$-dimensional standard graded Cohen-Macaulay ring and $e(R)$ be the multiplicity of $R$. If $e(R) = \dim_k(\mathfrak{m}/\mathfrak{m}^2) - \dim R + 1$ then $R$ is said to have **minimal multiplicity**.

An immediate result is the following proposition.

**Proposition 2.5.2.** *Let $(R,\mathfrak{m},k)$ be a $d$-dimensional standard graded Cohen-Macaulay ring. If $k$ is infinite, the following are equivalent:*

*(1) $R$ has minimal multiplicity;*

*(2) there exists a regular sequence $x_1,\ldots,x_d$ such that $\mathfrak{m}^2 = (x_1,\ldots,x_d)\mathfrak{m}$;*

*(3) the h-vector of $R$ is of the form $(1,n)$.*

The proofs of the results below use the well known Serre conditions and Normality. We state the definitions here for convenience.

**Definition 2.5.3.** A Noetherian ring $R$ satisfies Serre's condition $(S_n)$ (respectively $(R_n)$) of the following holds:

$(S_n)$: If $\mathfrak{p} \in \operatorname{Spec}(R)$, then $\operatorname{depth}(R_\mathfrak{p}) \geqslant \min\{n, \dim(R_\mathfrak{p})\}$;

$(R_n)$: If $\mathfrak{p} \in \operatorname{Spec}(R)$ and $\dim(R_\mathfrak{p}) \leqslant n$, then $R_\mathfrak{p}$ is a regular local ring.

**Definition 2.5.4.** A ring is **normal** if all localizations at prime ideals are integrally closed domains.

Further, we state a popular criterion for determining if a Noetherian ring is normal.

**Theorem 2.5.5** (Serre's Criterion, [24, Theorem 4.5.3])**.** *A Noetherian ring is normal if and only if the ring satisfies $(R_1)$ and $(S_2)$.*

The next theorem is used in [11] to classify the rings of graded finite Cohen-Macaulay type and is essential in proving Proposition 2.5.7. We will also use this theorem in showing the countable analog so we state it here without proof.

**Theorem 2.5.6** ([11, Theorem B]). *Let $B$ be a standard graded Cohen-Macaulay domain, and let $\overline{R}$ be the graded Artinian ring obtained by reducing modulo a maximal regular sequence of elements of degree 1. If there exists a degree 1 element of $\overline{R}$ which is in the socle of $\overline{R}$, then the square of the maximal ideal of $\overline{R}$ is 0 (that is, $R$ has minimal multiplicity).*

Proposition 2.5.7 is a result of D. Eisenbud and J. Herzog that is embedded in the proof the classification of standard graded rings of graded finite Cohen-Macaulay type.

**Proposition 2.5.7** ([11]). *Let $(R, \mathfrak{m}, k)$ be a standard graded Cohen-Macaulay ring of graded finite Cohen-Macaulay type and* $\dim R \geqslant 2$*. Then $R$ must have minimal multiplicity.*

*Proof.* If $R$ is Gorenstein, then $R = k[x_1, \ldots, x_n]/(x_1^2 + \cdots + x_n^2)$ as shown in the proof of the main theorem in [11]. Since $x_1, \ldots, x_{n-1}$ is a minimal reduction of $(x_1, \ldots, x_n)R$, we see that the $h$-vector of $R$ is $(1,1)$. Thus $R$ has minimal multiplicity.

If $R$ is not Gorenstein, then Auslander's results (Theorem 2.1.15) says that $(R_1)$ holds. As $R$ is Cohen-Macaulay, we have that $R$ is also $(S_2)$. Thus by Serre's Criterion (Theorem 2.5.5) we have that $R$ is a normal domain . As $R$ is stretched, we know the $h$-vector is of the form $(1, n, 1, \ldots, 1)$. Since $R$ is not Gorenstein, we know that there is an element of the socle in degree 1. Theorem $B$ above forces $R$ to be of minimal multiplicity. □

Before we can reproduce Proposition 2.5.7 for the graded countable Cohen-Macaulay type, we need a standard result of chains of homogeneous prime ideals in a standard graded ring. It is stated here without proof.

**Lemma 2.5.8.** *Let $(R,\mathfrak{m},k)$ be a standard graded ring and $\mathfrak{p}$ a homogeneous prime ideal of height n. Then there exists a chain of distinct homogeneous prime ideals*

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}.$$

The next theorem is a graded version of Theorem 1.3 in [17] and is used as a replacement of Auslander's result in Proposition 2.5.7. The proof below is a straight forward translation form the local case to the graded case and we place it here for completeness. Even though Theorem 2.5.9 is stated in the graded category, by Corollary 2.1.14 we are able to consider all prime ideals in the singular locus, not just the homogeneous prime ideals.

**Theorem 2.5.9** (Graded version of [17, Theorem 1.3]). *Let $(R,\mathfrak{m},k)$ be a Cohen-Macaulay standard graded ring of dimension d such that k is uncountable. If R has graded countable Cohen-Macaulay type, then the singular locus of R has dimension at most one.*

*Proof.* Assume that the $\dim(\operatorname{Sing}(R)) \geqslant 2$. Since $R$ is standard graded, we have that the singular locus is defined by a homogeneous ideal $J$ such that $\operatorname{ht}(J) < d-1$. Let $\{M_i\}_{i=1}^{\infty}$ be a complete list of representatives for the isomorphism classes of graded indecomposable MCM $R$-modules. Consider the set $\Lambda$ of homogeneous prime ideals defined as follows:

$$\Lambda = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathfrak{p} = \operatorname{ann}(\operatorname{Ext}^1_R(M_i,M_j)), \text{ for some } i,j, \text{ and } \dim(R/\mathfrak{p}) = 1\}.$$

We know that $\operatorname{Ext}^1_R(M_i,M_j)$ is a graded object and the annihilator of a graded module is homogeneous. Thus the set $\Lambda$ is well defined. Notice that $\Lambda$ is at most countable. It is also true that $J$ is contained in each $\mathfrak{p} \in \Lambda$. To see this, assume $\mathfrak{p} \in \Lambda$ and that $R_\mathfrak{p}$

is regular. As $\mathfrak{p} \in \Lambda$, there exists $M_i, M_j$ such that $\mathfrak{p} = \mathrm{ann}(\mathrm{Ext}^1_R(M_i, M_j))$ and hence $\mathfrak{p} \supseteq \mathrm{ann}(M_i)$. This forces $(M_i)_\mathfrak{p}$ to be a maximal Cohen-Macaulay module over $R_\mathfrak{p}$, hence free. Therefore $\mathrm{Ext}^1_R(M_i, M_j)_\mathfrak{p} = 0$, a contradiction as $\mathfrak{p} = \mathrm{ann}(\mathrm{Ext}^1_R(M_i, M_j))$.

Notice that the $k$-vector space $R_1$ is not contained in the countable union of all the subspaces $R_1 \cap \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal in $\Lambda$. So, there exists an element $f \in R_1 \setminus \bigcup_{\mathfrak{p}\in\Lambda}(R_1 \cap \mathfrak{p})$. By Lemma 2.5.8, we can choose a homogeneous prime $\mathfrak{q}$ containing $f$ and $J$ such that $\dim(R/\mathfrak{q}) = 1$; then of course $\mathfrak{q} \notin \Lambda$.

Let $X$ (resp. $Y$) be a $(d-1)^{\text{th}}$ (resp. $d^{\text{th}}$) syzygy coming from a graded free resolution of $R/\mathfrak{q}$. Then $X$ and $Y$ are both graded maximal Cohen-Macaulay $R$-modules and we have a non-split graded short exact sequence

$$0 \longrightarrow Y \longrightarrow \oplus R(-di) \longrightarrow X \longrightarrow 0, \tag{2.9}$$

where $\oplus R(-di)$ is a direct sum of $R$ with shifts. We claim that $\mathrm{ann}(\mathrm{Ext}^1_R(X,Y)) = \mathfrak{q}$. It is clear that $\mathfrak{q}$ kills $\mathrm{Ext}^1_R(X,Y) \simeq \mathrm{Ext}^{d+1}_r(R/\mathfrak{q}, Y)$. To see the opposite containment, note that since $\mathfrak{q}$ contains $J$, $R_{(\mathfrak{q})}$ is not regular and hence $R_\mathfrak{q}$ is not regular. The resolution of the residue field of $R/\mathfrak{q}$ is thus infinite, and neither $X_\mathfrak{q}$ nor $Y_\mathfrak{q}$ is free, so (2.9) is non-split when localized at $\mathfrak{q}$. Thus $\mathrm{ann}(\mathrm{Ext}^1_R(X,Y)) = \mathfrak{q}$.

We can write both $X$ and $Y$ as direct sums of copies of the indecomposables $M_i$, and further write

$$\mathrm{Ext}^1_R(X,Y) \cong \bigoplus_{i,j} \mathrm{Ext}^1_R(M_i, M_j)^{a_{ij}}$$

with all but finitely many of the $a_{ij}$ equal to zero. Then $\mathfrak{q}$ is the intersection of the annihilators of the nonzero Ext modules appearing in the above decomposition. Since $\mathfrak{q}$ is prime, it must equal one of these annihilators, and then $\mathfrak{q} \in \Lambda$, a contradiction. $\square$

Using the above results, we are now able to expand D. Eisenbud and J. Herzog's result (Proposition 2.5.7) to the graded countable case. In the case that $R$ is of graded finite Cohen-Macaulay type, Theorem 2.1.15 shows that $R$ has an isolated singularity. Unfortunately this is not known to be true for graded countable Cohen-Macaulay type. However, we do have (Theorem 2.5.9) that the singular locus is of dimension at most one. Therefore, for rings with dimension larger than two we have the following result.

**Proposition 2.5.10.** *Let $(R, \mathfrak{m}, k)$ be a standard graded Cohen-Macaulay ring of countable Cohen-Macaulay type that is not Gorenstein and $\dim R \geqslant 3$. Then $R$ must be a domain and have minimal multiplicity.*

*Proof.* By Theorem 2.5.9, we know that the dimension of the singular locus is at most one. Since $\dim(R) \geqslant 3$, we have that $R$ satisfies Serre's condition $(R_1)$. Further, as $R$ is Cohen-Macaulay, we know that $R$ also satisfies Serre's condition $(S_2)$. Thus by Serre's criterion (Theorem 2.5.5), $R$ must be a normal. Because $R$ normal, we can write it as a finite direct product of integrally closed domains [24, Lemma 2.1.15]. As $R$ is standard graded, we have that $R_0 = k$ and thus there is only one term in the direct product. Hence $R$ is also a domain.

By Corollary 2.4.7, we know that $R$ is either of minimal multiplicity or has $h$-vector $(1, n, 1)$ for some positive integer $n$. Since $R$ is not Gorenstein, there must be a socle element in degree one. However, by Theorem 2.5.6 this forces $R$ to have minimal multiplicity. □

*Remark* 2.5.11. It is worth noting that standard graded Cohen-Macaulay rings of countable Cohen-Macaulay type and dimension at least 3 are normal domains. Even though Proposition 2.5.10 assumed the ring was not Gorenstein, the argument to show that the ring was a normal domain still holds.

# Chapter 3

# Super-Stretched Local Rings

Throughout this chapter, we will explore the concept of super-stretched in the local case. Changing notation, let $(R,\mathfrak{m},k)$ be a local ring of dimension $d$ with maximal ideal $\mathfrak{m}$, and uncountable residue field $k$ of characteristic 0. Further, we will let $G := \mathrm{gr}_{\mathfrak{m}}R$ and $\mathfrak{M} = \oplus_{i>0}\mathfrak{m}^i/\mathfrak{m}^{i+1}$. Note that $\dim(G) = \dim(R)$ [24, Proposition 5.1.6]. Recall the following definition.

**Definition 1.2.7.** *A local Cohen-Macaulay ring $(R,\mathfrak{m},k)$ of dimension $d$ is said to be **super-stretched** if for all homogeneous system of parameters $x_1^*,\ldots,x_d^*$ in $\mathrm{gr}_{\mathfrak{m}}R$,*

$$\dim_k\left(\frac{\mathrm{gr}_{\mathfrak{m}}R}{(x_1^*,\ldots,x_d^*)}\right)_i \leqslant 1$$

*for all* $i \geqslant \sum_{j=1}^{d} \deg(x_j^*) - d + 2.$

As in the graded case, we will be considering minimal reductions of the maximal ideal. It is well known that minimal reductions of length $d = \dim(R)$ exist if $R/\mathfrak{m}$ is infinite [24]. Further, a minimal reduction of the maximal ideal is the preimage in $R$ of a degree one homogeneous system of parameters in $\mathrm{gr}_{\mathfrak{m}}R$. Thus, if $x_1,\ldots,x_d$ is a minimal reduction of $\mathfrak{m}$, we have that $\deg(x_i^*) = 1$.

The next theorem is a nice result of T. Puthenpurakal and shows that the length of $\mathfrak{m}^3/J\mathfrak{m}$ is invariant of the minimal reduction $J$ of $\mathfrak{m}$.

**Theorem 3.0.12** ([22, Theorem 1])**.** *Let $(R,\mathfrak{m},k)$ be a Cohen-Macaulay local ring of dimension $d \geqslant 1$ with infinite residue field $k$. If $J$ is a minimal reduction of $\mathfrak{m}$ then we have an equality*

$$\lambda(\mathfrak{m}^3/J\mathfrak{m}^2) = e(R) + \mu(\mathfrak{m})(d-1) - \mu(\mathfrak{m}^2) - \binom{d-1}{2}$$

*where $e(R)$ is the multiplicity of $R$ and $\mu(I)$ is the minimal number of generators of an ideal $I$ in $R$.*

Using Theorem 3.0.12, we establish the one dimensional local analog to Proposition 2.2.6.

**Lemma 3.0.13.** *If $(R,\mathfrak{m},k)$ is a local Cohen-Macaulay ring of dimension 1 that is super-stretched, then for all minimal reductions $(x)$ of the maximal ideal $\mathfrak{m}$, we have $\mathfrak{m}^3 = x\mathfrak{m}^2$.*

*Proof.* By Theorem 3.0.12, it is enough to consider a fixed minimal reduction $(x)$ of $\mathfrak{m}$ such that $\deg(x^*) = 1$. Notice that $x^*\mathfrak{M}^{n-1} = \mathfrak{M}^n$ for some integer $n$. Hence $x^*$ is a homogeneous parameter. Because $R$ is super-stretched, we have that $\dim_k(G/x^*G)_2 = 1$. By Lemma 2.2.5 we have

$$\dim_k \frac{\mathfrak{m}^2+(x)}{\mathfrak{m}^3+(x)} = \dim_k \frac{\mathfrak{m}^2}{\mathfrak{m}^3+((x)\cap\mathfrak{m}^2)} = \dim_k \frac{\mathfrak{m}^2}{\mathfrak{m}^3+x\mathfrak{m}} = 1.$$

Note that the second equality is always true as $(x)\cap\mathfrak{m}^2 = x\mathfrak{m}$ if $x \notin \mathfrak{m}^2$. The displayed equality says that there is a $y \in \mathfrak{m}^2 - (x\mathfrak{m}+\mathfrak{m}^3)$ such that $\mathfrak{m}^2 = x\mathfrak{m}+(y)+\mathfrak{m}^3$. By Nakayama's lemma we have $\mathfrak{m}^2 = x\mathfrak{m}+(y)$.

By the super-stretched hypothesis we can also consider $G$ modulo $(x^*)^2$. Due to the choice of $x$, we have that $(x^2) \cap \mathfrak{m}^3 = x^2\mathfrak{m}$. This gives us

$$\dim_k \frac{\mathfrak{m}^3}{\mathfrak{m}^4 + x^2\mathfrak{m}} = 1.$$

Thus, as before, there exists $z \in \mathfrak{m}^3 - (x^2\mathfrak{m} + \mathfrak{m}^4)$ such that $\mathfrak{m}^3 = x^2\mathfrak{m} + (z) + \mathfrak{m}^4$. Nakayama's lemma shows that $\mathfrak{m}^3 = x^2\mathfrak{m} + (z)$.

Notice that we can choose $z$ to be anything in $\mathfrak{m}^3 - (x^2\mathfrak{m} + \mathfrak{m}^4)$. We would like to choose $z = xy$, but first we must show

*Claim.* The element $xy$ is not in $x^2\mathfrak{m} + \mathfrak{m}^4$.

If the claim holds, then we have

$$\begin{aligned}\mathfrak{m}^3 &= x^2\mathfrak{m} + (z)\\ &= x^2\mathfrak{m} + (xy)\\ &= x(x\mathfrak{m} + (y))\\ &= x\mathfrak{m}^2.\end{aligned}$$

This is the desired result for dimension one.

To show the claim, let $n$ be minimally chosen such that $\mathfrak{m}^n = x\mathfrak{m}^{n-1}$ and suppose $xy \in x^2\mathfrak{m} + \mathfrak{m}^4$. Since $\mathfrak{m}^2 = x\mathfrak{m} + (y)$, we have that $x\mathfrak{m}^2 \subseteq x^2\mathfrak{m} + \mathfrak{m}^4$. Assume that $n > 3$ and multiply by $\mathfrak{m}^{n-3}$. As $R$ is Cohen-Macaulay, we have that $x$ is a non-zero divisor. Cancel the $x$'s to observe that $\mathfrak{m}^{n-1} \subseteq x\mathfrak{m}^{n-2} + \mathfrak{m}^n$. This forces $\mathfrak{m}^{n-1} = x\mathfrak{m}^{n-2}$, a contradiction since $n$ was chose to be minimal. Thus $xy \notin (x^2\mathfrak{m} + \mathfrak{m}^4)$. □

Before we move on, we need a couple of known results to help us. Theorem 3.0.16 is known in the folklore as **Sally's machine** and is named after Judy Sally who used it extensively in many of her papers. Although the theorem was not formally written down until S. Huckaba and T. Marley [16] proved the general case in 1997. For completeness, we offer a proof of Sally's machine.

**Definition 3.0.14.** Let $R$ be a ring and $I$ an ideal in $R$. We say that $x \in I$ is a **superficial element** of $I$ if there exists $c \in \mathbb{N}$ such that for all $n \geqslant c$, $(I^{n+1} : x) \cap I^c = I^n$. Further, a sequence $x_1, \ldots, x_s \in I$ is said to be a **superficial sequence** for I if for all $i = 1, \ldots, s$, the image of $x_i$ in $I/(x_1, \ldots, x_{i-1})$ is a superficial element of $I/(x_1, \ldots, x_{i-1})$.

**Lemma 3.0.15.** *Suppose $(R, \mathfrak{m}, k)$ be a local Noetherian ring. Let $x$ be a superficial element for an ideal I contained in R, and define $\overline{*}$ as the image of the quotient with xR. If* $\operatorname{depth}(\operatorname{gr}_I R) > 0$*, then $x^*$ is a non-zero divisor in* $\operatorname{gr}_I R$ *and*

$$\operatorname{gr}_{\overline{I}} \overline{R} \simeq \frac{\operatorname{gr}_I R}{x^* \operatorname{gr}_I R}.$$

*Proof.* Let $G := \operatorname{gr}_I R$ and $\mathfrak{M} := \oplus_{n>0} I^n / I^{n+1}$. Since $x$ is a superficial element of $I$, we have that for $N >> 0$, $(0 :_G x^*)_N = 0$. Therefore, for $N$ large enough, $\mathfrak{M}^N (0 :_G x^*) = 0$. However, $\mathfrak{M}^N$ contains a non-zero divisor as $\operatorname{depth}(\operatorname{gr}_I R) > 0$. Hence we have that $(0 :_G x^*) = 0$ and $x^*$ is a non-zero divisor on $G$. From here we have that $I^n \cap x = x I^{n-1}$ for all $n \geqslant 0$. Thus by Corollary 8.6.2 [24], we have that

$$\operatorname{gr}_{\overline{I}} \overline{R} \simeq \frac{\operatorname{gr}_I R}{x^* \operatorname{gr}_I R}.$$

□

**Theorem 3.0.16** ([24, Theorem 6.5, "Sally's Machine"])**.** *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring. Let $(x_1, \ldots, x_n) \subset I$ be a minimal reduction of I generated by a superficial sequence of length n. Fix $r \leqslant n$, and set $\mathbf{x} = (x_1, \ldots, x_r)$. Then* $\operatorname{depth}(\operatorname{gr}_I R) \geqslant r+1$ *if and only if* $\operatorname{depth}(\operatorname{gr}_{I/\mathbf{x}} R/\mathbf{x}) \geqslant 1$.

*Proof.* First assume that $r = 1$. If $\operatorname{depth}(\operatorname{gr}_I R) \geqslant 2 \geqslant 1$, then by Lemma 3.0.15, $x_1^*$ is a non-zero divisor and

$$\operatorname{depth}(\operatorname{gr}_{\frac{I}{x_1}}(\frac{R}{x_1})) = \operatorname{depth}(\frac{\operatorname{gr}_I R}{x_1^* \operatorname{gr}_I R}) \geqslant 1.$$

Conversely, we need to show that $x_1^*$ is a non-zero divisor. As the $\operatorname{depth}(\operatorname{gr}_{\overline{I}}\overline{R}) \geqslant 1$, we know there exists $y \in I^t \setminus I^{t+1}$ such that $\overline{y}$ is a non-zero divisor in $\operatorname{gr}_{\overline{I}}\overline{R}$ and hence $\overline{y}^p$ is a non-zero divisor for all $p \geqslant 1$. This means that

$$I^{pt+n} : y \subseteq (I^n, x_1)$$

for all $n, p \geqslant 1$. For convenience, define $x := x_1$. As $x$ is superficial, we know that for all $j \geqslant 1$ that $x^j$ is superficial as well. Hence there exists an integer $c$ such that

$$(I^{n+1} : x^j) \cap I^c = I^{n+1-j}$$

for all $n + 1 - j \geqslant c$. Fix $p \geqslant c/t$ and let $n \geqslant 1$ such that $j \leqslant n + 1$. These conditions force

$$y^p(I^{n+1} : x^j) \subseteq (I^{n+pt+1} : x^j) \cap I^c = I^{n+pt+1-j}.$$

Therefore

$$I^{n+1} : x^j \subseteq I^{pt+n+1-j} : y^p \subseteq (I^{n+1-j}, x). \tag{3.1}$$

When $j = 1$ in (3.1), we have that

$$I^{n+1} : x \subseteq (I^n, x) \cap (I^{n+1} : x) \subseteq I^n + (x) \cap (I^{n+1} : x) = I^n + x(I^{n+1} : x^2). \tag{3.2}$$

Likewise, when $j = 2$ notice that

$$I^{n+1} : x^2 \subseteq (I^{n-1}, x) \cap (I^{n+1} : x^2) \subseteq I^{n-1} + (x) \cap (I^{n+1} : x^2) = I^{n-1} + x(I^{n+1} : x^3). \quad (3.3)$$

This process stops when $j = n$ in (3.1), that is, when

$$I^{n+1} : x^n \subseteq (I, x^{n-1}) \cap (I^{n+1} : x^n) \subseteq I + (x) \cap (I^{n+1} : x^n) = I + x(I^{n+1} : x^{n+1}) = I. \quad (3.4)$$

Combining the above conditions for $j = 1, 2, \ldots, n$, we see that

$$I^n \subseteq I^{n+1} : x \subseteq I^n + xI^{n-1} + x^2 I^{n-2} + \cdots + x^{n-1} I + x^n \subseteq I^n.$$

Therefore $x^*$ is a non-zero divisor and

$$\operatorname{depth}(\operatorname{gr}_I R) = \operatorname{depth}(\frac{\operatorname{gr}_I R}{x^* \operatorname{gr}_I R}) + 1 = \operatorname{depth}(\operatorname{gr}_{\bar{I}} \overline{R}) + 1 \geqslant 1 + 1 = 2.$$

Now assume that $r > 1$. If we assume that $\operatorname{depth}(\operatorname{gr}_I R) \geqslant r + 1$, then iterating Lemma 3.0.15 shows that

$$\operatorname{depth}(\operatorname{gr}_{\bar{I}} \overline{R}) = \operatorname{depth}(\frac{\operatorname{gr}_I R}{(x_1^*, \ldots, x_r^*) \operatorname{gr}_I R}) \geqslant 1.$$

Conversely, proceed by induction on $r$ and assume that $\operatorname{depth}(\operatorname{gr}_{\bar{I}} \overline{R}) \geqslant 1$. By the induction hypothesis we know that

$$\operatorname{depth}\left(\operatorname{gr}_{\frac{I}{(x_1)}} \frac{R}{(x_1)}\right) \geqslant r - 1 + 1 = r \geqslant 1.$$

Once again, by induction, we have that $a := \operatorname{depth}(\operatorname{gr}_I R) \geqslant 2 > 0$. By Lemma 3.0.15 we know that $x_1^*$ is a non-zero divisor on $\operatorname{gr}_I R$ and that

$$r \leqslant \operatorname{depth}\left(\operatorname{gr}_{\frac{I}{(x_1)}} \frac{R}{(x_1)}\right) = \operatorname{depth}\left(\frac{\operatorname{gr}_I R}{x_1^* \operatorname{gr}_I R}\right) = a - 1.$$

Hence $r + 1 \leqslant a = \operatorname{depth}(\operatorname{gr}_I R)$. □

Here is another lemma that is useful proving Theorem 2.2.9.

**Lemma 3.0.17.** *Let $(R, \mathfrak{m}, k)$ be a local super-stretched ring of dimension $d$ with infinite residue field $k$. Let $x_1, \ldots, x_j$ be a superficial sequence in $R$ such that the initial forms $x_1^*, \ldots, x_j^*$, for some $1 \leqslant j \leqslant d-1$, is a part of a homogeneous system of parameters in $\operatorname{gr}_{\mathfrak{m}} R$ with $\deg(x_i^*) = 1$, then $R/(x_1, \ldots, x_j)$ is super-stretched.*

*Proof.* To prove this induct on $j$. For $j = 1$, consider the natural graded surjection

$$\frac{\operatorname{gr}_{\mathfrak{m}} R}{(x_1^*)} \longrightarrow \operatorname{gr}_{\frac{\mathfrak{m}}{(x_1)}} \frac{R}{(x_1)}. \tag{3.5}$$

From here choose a sequence $x_2^*, x_3^*, \ldots, x_d^*$ that is a homogeneous system of parameters for both rings defined in (3.5). We now have the following surjection

$$\frac{\operatorname{gr}_{\mathfrak{m}} R}{(x_1^*, \ldots, x_d^*)} \longrightarrow \frac{\operatorname{gr}_{\frac{\mathfrak{m}}{(x_1)}} \frac{R}{(x_1)}}{(x_2^*, x_3^*, \ldots, x_d^*)}.$$

In particular, we have that

$$\dim_k \left(\frac{\operatorname{gr}_{\frac{\mathfrak{m}}{(x_1)}} \frac{R}{(x_1)}}{(x_2^*, x_3^*, \ldots, x_d^*)}\right)_i \leqslant \dim_k \left(\frac{\operatorname{gr}_{\mathfrak{m}} R}{(x_1^*, \ldots, x_d^*)}\right)_i \leqslant 1$$

for $i \geqslant \sum_{j=1}^{d} \deg(x_j) - d + 2 = \sum_{j=2}^{d} \deg(x_j) - (d-1) + 2$. This shows that $R/x_1 R$ is indeed super-stretched.

Now assume the results holds for $1 \leqslant j \leqslant d-2$ and let $x_1^*, \ldots, x_{d-1}^*$ be a homogeneous system of parameters of $\mathrm{gr}_{\mathfrak{m}} R$ with $\deg(x_i^*) = 1$. By induction we know that $\overline{R} := R/(x_1, \ldots, x_{d-2})$ is super-stretched. Further, notice that $x_{d-1}^*$ is part of a homogeneous system of parameters in $\mathrm{gr}_{\overline{\mathfrak{m}}} \overline{R}$ of degree one. Applying the induction hypothesis once again shows that $\overline{R}/x_{d-1}\overline{R} \simeq R/(x_1, \ldots, x_{d-1})$ is super-stretched. □

We are now ready to show the local analog of Theorem 2.2.9.

**Theorem 3.0.18.** *Let $(R, \mathfrak{m}, k)$ be a local Cohen-Macaulay ring of dimension $d > 0$ with uncountable residue field k. The following are equivalent:*

*(1) R is super-stretched;*

*(2) $\mathrm{gr}_{\mathfrak{m}} R$ is super-stretched as a graded ring;*

*(3) R is stretched and $J\mathfrak{m}^2 = \mathfrak{m}^3$ for every minimal reduction J of the maximal ideal;*

*(4) R is stretched and $J\mathfrak{m}^2 = \mathfrak{m}^3$ for some minimal reduction J of the maximal ideal.*

*Proof.* First assume that $\dim(R) = 1$ and notice that Theorem 3.0.12 shows the equivalence of (3) and (4). Further, by Lemma 3.0.13 and Proposition 1.2.9, we know that (1) implies (3). Also, if we assume (2), then a straight forward application of the definition of the graded version of super-stretched shows that the local ring $R$ is super-stretched as well. To show the dimension one case, we only need to prove that (3) implies (2).

*Claim.* If $\dim(R) = 1$, then condition (3) implies that $\mathrm{gr}_m R$ is Cohen-Macaulay.

To prove the claim, assume (3) and let $x$ be a minimal reduction of $\mathfrak{m}$ coming from a homogeneous degree one element $x^* \in \mathrm{gr}_{\mathfrak{m}} R$. As $1 = \dim(R) = \dim(\mathrm{gr}_m R)$, we will be finished if we can show that $x^*$ is a non-zero divisor on $\mathrm{gr}_{\mathfrak{m}} R$. We know that $x\mathfrak{m}^2 = \mathfrak{m}^3$ and that $x$ is a non-zero divisor. Consider the image $x^*$ in $\mathrm{gr}_{\mathfrak{m}} R$ and assume there is a $y^*$ in $\mathrm{gr}_{\mathfrak{m}} R$ such that $x^* y^* = 0$. With out loss of generality we may assume that $y^*$ is an element of $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ for some $n$. Thus there is a $y$ in $\mathfrak{m}$ such that $xy \in \mathfrak{m}^{n+2}$. Since $n \geqslant 1$, $xy \in \mathfrak{m}^{n+2} = x\mathfrak{m}^{n+1}$. So, $xy = xr$ for $r \in \mathfrak{m}^{n+1}$. But $x$ is a non-zero divisor, so $y$ is an element of $\mathfrak{m}^{n+1}$; forcing $y^* = 0$ and $\mathrm{gr}_m R$ is Cohen-Macaulay.

Since $\mathrm{gr}_\mathfrak{m} R$ is Cohen-Macaulay and let $x$ be a superficial minimal reduction of $\mathfrak{m}$ coming from a homogeneous degree one element $x^* \in \mathrm{gr}_\mathfrak{m} R$. In particular, we have that $x\mathfrak{m}^2 = \mathfrak{m}^3$. This implies that $x^*\mathfrak{M}^2 = \mathfrak{M}^3$ where $\mathfrak{M} = \oplus_{i>0}(\mathrm{gr}_\mathfrak{m} R)_i$ is the irrelevant maximal ideal in $\mathrm{gr}_\mathfrak{m} R$. Further, as $R$ is stretched, Lemma 3.0.15 shows that $\mathrm{gr}_\mathfrak{m} R$ is stretched as well. Applying Theorem 2.2.9 shows us that $\mathrm{gr}_\mathfrak{m} R$ is super-stretched as well. Hence (3) implies (2) and the theorem is true for $\dim(R) = 1$.

Assume that $\dim(R) > 1$. As $R$ is Cohen-Macaulay, it is easy to check that (2) implies that $R$ is super-stretched. Conversely, assume that $R$ is super-stretched. It is enough to show that $\mathrm{gr}_\mathfrak{m} R$ is Cohen-Macaulay. Let $x_1, \ldots, x_{d-1}$ be a superficial sequence in $R$ such that the initial forms $x_1^*, \ldots, x_{d-1}^*$ are part of a minimal reduction of $\mathfrak{M}$ with $\deg(x_i^*) = 1$. By Lemma 3.0.17 we know that $\overline{R} := R/(x_1, \ldots, x_{d-1})$ is super-stretched of dimension one. As Theorem 3.0.18 holds for dimension one, we know that $\mathrm{gr}_{\overline{\mathfrak{m}}} \overline{R}$ is Cohen-Macaulay and hence $\mathrm{depth}(\mathrm{gr}_{\overline{m}} \overline{R}) = 1$. By Sally's machine (Theorem 3.0.16) we know that $\mathrm{depth}(\mathrm{gr}_\mathfrak{m} R) \geqslant d - 1 + 1 = d$. This forces $\mathrm{gr}_\mathfrak{m} R$ to be Cohen-Macaulay and thus super-stretched as a a graded ring. We now have that (1) is equivalent to (2).

To see the equivalence of (2), (3), and (4), we can apply Theorem 2.2.9 to $\mathrm{gr}_\mathfrak{m} R$ along with the fact that minimal reduction of $\mathfrak{m}$ in $R$ come from homogeneous system of parameters of degree one in the associated graded ring. □

An immediate result is the fact that $\mathrm{gr}_\mathfrak{m} R$ is Cohen-Macaulay for local super-stretched rings.

*Remark* 3.0.19. Let $(R, \mathfrak{m}, k)$ be a local Cohen-Macaulay ring of dimension $d > 0$ with uncountable residue field $k$. If $R$ is super-stretched, then $\mathrm{gr}_{\mathfrak{m}} R$ is Cohen-Macaulay.

*Proof.* As $R$ is super-stretched, Theorem 3.0.18 implies that $\mathrm{gr}_{\mathfrak{m}} R$ is super-stretched as well. In order for a standard graded ring to be super-stretched, in must also be Cohen-Macaulay. □

As Theorem 2.2.9 was used to show that graded countable Cohen-Macaulay type implies super-stretched, one has the following natural question.

**Question 3.0.20.** *Let $(R, \mathfrak{m}, k)$ be a local Cohen-Macaulay ring of dimension $d > 0$ with uncountable residue field $k$. If $R$ is of countable Cohen-Macaulay type, does this necessarily imply $R$ is super-stretched?*

# Chapter 4

# Partial Classification of Graded Countable Cohen-Macaulay Type

In this chapter we use the results in the previous chapters to classify various instances of graded countable Cohen-Macaulay type. The sections are broken into cases and the summary of partial classification can be found in Section 4.5. Throughout this chapter, $(R, \mathfrak{m}, k)$ is considered as a standard graded (Noetherian) ring with $k$ an algebraically closed, uncountable field of characteristic 0.

## 4.1 Zero Dimensional Rings

It is well known that a zero dimensional local ring $R$ is a hypersurface if and only if $R$ is of finite Cohen-Macaulay type. To be more precise,

**Theorem 4.1.1** ([14, Satz 1.5])**.** *Let $(R, \mathfrak{m}, k)$ be a zero dimensional equicharacteristic local ring. The following are equivalent:*

*(1) $R$ is an abstract hypersurface;*

*(2) $R$ is of finite Cohen-Macaulay type.*

We show the graded countable analog to this statement in Proposition 4.1.2, and prove the graded version of Theorem 4.1.1 along the way.

**Proposition 4.1.2.** *Let $(R, \mathfrak{m}, k)$ be a 0-dimensional standard graded Cohen-Macaulay ring. Further assume that k is an uncountable field. Then the following are equivalent:*

*(1) R is of graded finite Cohen-Macaulay type;*

*(2) R is of graded countable Cohen-Macaulay type;*

*(3) R is a hypersurface ring.*

*Proof.* The implication (1) implies (2) is straight forward. To show (2) implies (3), assume that $R$ is not a hypersurface. Thus there must be two linear forms $a, b \in \mathfrak{m} \setminus \mathfrak{m}^2$ that are basis elements of $\mathfrak{m}/\mathfrak{m}^2$. By Lemma 2.4.3, there are uncountably many distinct homogeneous ideals $\{I_\alpha\}_{\alpha \in k}$ in $R$. In this context, we have that $I_\alpha = (a + \alpha b)R$. Consider the graded indecomposable maximal Cohen-Macaulay modules $\{R/I_\alpha\}_{\alpha \in k}$. As each of these modules have different annihilators, we know that they are not isomorphic. A contradiction as we assumed that $R$ was of graded countable type.

To prove that (3) implies (1), we consider the $\mathfrak{m}$-adic completion of $R$ and then apply Theorem 4.1.1 to see that the completion is of finite Cohen-Macaulay type. By Theorem 2.1.6, we know that $R$ also has graded finite Cohen-Macaulay type. □

We thus have a complete classification of graded countable Cohen-Macaulay type for zero dimensional standard graded rings. Further, Proposition 4.1.2 gives a positive answer to Question 1.1.5 for zero dimensional rings with uncountable residue field.

## 4.2 One Dimensional Rings

In the one-dimensional case, Question 1.1.5 has a positive answer as shown by R. Karr and R. Wiegand [18, Theorem 1.4]. In this section, we examine the Drozd-Roĭter conditions and give a partial classification of one dimensional graded countable Cohen-Macaulay type.

### 4.2.1 Finite Type and the Drozd-Roĭter conditions

As detailed by N. Cimen, R. Wiegand, and S. Wiegand [10], if $(R, \mathfrak{m}, k)$ is a one dimensional, reduced, local, Noetherian ring such that the integral closure of $R$, say $S$, is finitely generated as an $R$-module, then we know precisely when $R$ has finite Cohen-Macaulay type. This happens when the following conditions occur:

**DR1** $S$ is generated by 3 elements as an $R$-module;

**DR2** the intersection of the maximal $R$-submodules of $S/R$ is cyclic as an $R$-module.

These are called the **Drozd-Roĭter conditions**.

**Proposition 4.2.1** ([10, Prop 1.12]). *The Drozd-Roĭter conditions are equivalent to the following:*

$$\dim_k(S/\mathfrak{m}S) \leqslant 3; \tag{dr1}$$

$$\dim_k\left(\frac{R+\mathfrak{m}S}{R+\mathfrak{m}^2S}\right) \leqslant 1. \tag{dr2}$$

Using the above equivalent conditions, along with the theory of Artinian pairs, R. Karr and R. Wiegand were able to show the following theorem.

**Theorem 4.2.2** ([18, Theorem 1.4])**.** *Let $(R,\mathfrak{m},k)$ be a one-dimensional, reduced, local Noetherian ring with finite integral closure. Assume that either* ***DR1*** *or* ***DR2*** *fails. Let $n$ be an arbitrary positive integer.*

*(1) There exists an indecomposable maximal Cohen-Macaulay $R$-module of constant rank $n$.*

*(2) If the residue field $k$ is infinite, there exist $|k|$ pairwise non-isomorphic indecomposable maximal Cohen-Macaulay modules of constant rank $n$.*

In particular, assume that $(R,\mathfrak{m},k)$ is a one-dimensional local ring which satisfies the hypothesis of Theorem 4.2.2. If $k$ is an uncountable field, then under the conditions of Theorem 4.2.2, $R$ is either finite Cohen-Macaulay type or of uncountable Cohen-Macaulay type. That is, there does not exist a one-dimensional Cohen-Macaulay ring with an uncountable residue field that is infinite countable Cohen-Macaulay type. As reduced is equivalent to isolated singularity for one dimensional rings, this gives a positive answer to Question 1.0.1. Further, by Theorem 2.1.6, we have a positive answer to Question 1.1.5 as well.

In order to have a better grasp of what it means to satisfy the Drozd-Roĭter conditions, we have found another set of equivalent conditions. This result is stated in the next proposition where $e(R)$ is the multiplicity of the maximal ideal $\mathfrak{m}$, $\lambda$ represents length as an $R$-module, and $\overline{*}$ is the integral closure of ideals.

**Proposition 4.2.3.** *Let $(R,\mathfrak{m},k)$ be a one-dimensional, reduced, local Noetherian ring with finite integral closure and uncountable residue field $k$. Let $x$ be a minimal reduction of the maximal ideal $\mathfrak{m}$. The Drozd-Roĭter conditions are equivalent to the following:*

$$e(R) \leqslant 3; \tag{4.1}$$

$$\lambda(\overline{\mathfrak{m}^2}/x\mathfrak{m}) \leqslant 1. \tag{4.2}$$

*Proof.* To show that (4.1) holds, we will show that $e(R) = \dim_k(S/\mathfrak{m}S)$ where $S$ is the integral closure of $R$. As $x$ is a reduction of $\mathfrak{m}$, we know that $xS$ is also a reduction of $\mathfrak{m}S$. But this holds if and only if $\mathfrak{m}S \subseteq \overline{xS}$. As principal ideals are integrally closed in $S$, we have that

$$xS \subseteq \mathfrak{m}S \subseteq \overline{xS} = xS,$$

and hence $xS = \mathfrak{m}S$. By assumption, $S$ is finitely generated as an $R$-module. Therefore we have that the map $S \to S$ defined by multiplication by $x$ is an injection. Hence, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & 0 & & K & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R & \xrightarrow{\cdot x} & R & \longrightarrow & R/xR & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S & \xrightarrow{\cdot x} & S & \longrightarrow & S/xS & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & C' & & C' & & C & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $K$, $C'$, and $C$ are the respective kernel and cokernels. Consider the exact sequence

$$0 \longrightarrow K \longrightarrow R/xR \longrightarrow S/xS \longrightarrow C \longrightarrow 0. \tag{4.3}$$

Notice by the Snake Lemma applied to (4.3) that $\lambda(K) = \lambda(C)$; here $\lambda$ represents length as $R$-modules. Further we have that

$$\lambda(R/xR) + \lambda(C) = \lambda(S/xS) + \lambda(K).$$

As $C$ and $K$ have the same length, we see that $R/xR$ and $S/xS$ also have the same length. Since $x$ is a minimal reduction of the maximal ideal, we know that $e(R) = \lambda(R/xR)$. Therefore,

$$e(R) = \lambda(R/xR) = \lambda(S/xS) = \lambda(S/\mathfrak{m}S) = \dim_k(S/\mathfrak{m}S).$$

In order to show (4.2), first notice that

$$\frac{R+\mathfrak{m}S}{R+\mathfrak{m}^2S} \simeq \frac{\mathfrak{m}S}{\mathfrak{m}^2S+(R\cap\mathfrak{m}S)} \simeq \frac{\mathfrak{m}S}{\mathfrak{m}^2S+\mathfrak{m}} \simeq \frac{xS}{x^2S+\mathfrak{m}}. \tag{4.4}$$

For simplicity, we define $B$ as follows,

$$B := \dim_k\left(\frac{R+\mathfrak{m}S}{R+\mathfrak{m}^2S}\right) = \lambda\left(\frac{xS}{x^2S+\mathfrak{m}}\right).$$

We now consider the short exact sequence

$$0 \longrightarrow \frac{\mathfrak{m}^2S+\mathfrak{m}}{\mathfrak{m}^2S} \longrightarrow \frac{\mathfrak{m}S}{\mathfrak{m}^2S} \longrightarrow \frac{\mathfrak{m}S}{\mathfrak{m}^2S+\mathfrak{m}} \longrightarrow 0.$$

Rewriting the two terms on the left gives us

$$\frac{\mathfrak{m}^2S+\mathfrak{m}}{\mathfrak{m}^2S} \simeq \frac{\mathfrak{m}}{\mathfrak{m}^2S\cap\mathfrak{m}} \simeq \mathfrak{m}/\overline{\mathfrak{m}^2}; \tag{4.5}$$

$$\frac{\mathfrak{m}S}{\mathfrak{m}^2S} \simeq \frac{S}{\mathfrak{m}S} \simeq \frac{S}{xS}. \tag{4.6}$$

Combining (4.5) and (4.6) with the above short exact sequence yields

$$\lambda\left(\frac{S}{xS}\right) = \lambda\left(\mathfrak{m}/\overline{\mathfrak{m}^2}\right) + B. \tag{4.7}$$

On the other hand, consider the following short exact sequence

$$0 \longrightarrow \frac{\overline{\mathfrak{m}^2}+xR}{xR} \longrightarrow \frac{R}{xR} \longrightarrow \frac{R}{\overline{\mathfrak{m}^2}+xR} \longrightarrow 0, \tag{4.8}$$

along with the isomorphisms

$$\frac{\overline{\mathfrak{m}^2}+xR}{xR} \simeq \frac{\overline{\mathfrak{m}^2}}{xR\cap\overline{\mathfrak{m}^2}} = \frac{\overline{\mathfrak{m}^2}}{x\mathfrak{m}}. \tag{4.9}$$

Note that the equality in (4.9) can be justified as follows. Since $xS = \mathfrak{m}S$, we know that $\overline{\mathfrak{m}^2} = \mathfrak{m}^2S\cap R = x^2S\cap R$. If $y \in xR\cap\overline{\mathfrak{m}^2}$, then $y = xr \in \overline{\mathfrak{m}^2} = x^2S\cap R$. This forces $r \in xS\cap R = \mathfrak{m}$. Hence $y \in x\mathfrak{m}$. Equality follows as $x\mathfrak{m} \subseteq xR\cap\overline{\mathfrak{m}}$.

Computing length in (4.8) gives us

$$\lambda\left(\frac{R}{xR}\right) = \lambda\left(\frac{\overline{\mathfrak{m}^2}}{x\mathfrak{m}}\right) + \lambda\left(\frac{R}{\overline{\mathfrak{m}^2}+xR}\right).$$

We can repeat the above steps with the following short exact sequence and isomorphisms:

$$0 \longrightarrow \frac{\overline{\mathfrak{m}^2}+xR}{\overline{\mathfrak{m}^2}} \longrightarrow \frac{R}{\overline{\mathfrak{m}^2}} \longrightarrow \frac{R}{\overline{\mathfrak{m}^2}+xR} \longrightarrow 0;$$

$$\frac{\overline{\mathfrak{m}^2}+xR}{\overline{\mathfrak{m}^2}} \simeq \frac{xR}{\overline{\mathfrak{m}^2}\cap xR} \simeq \frac{xR}{\mathfrak{m}xR} \simeq \frac{R}{\mathfrak{m}}.$$

Once again, if we compute the length, we have that

$$\lambda\left(\frac{R}{\overline{\mathfrak{m}^2}}\right) = 1 + \lambda\left(\frac{R}{\overline{\mathfrak{m}^2}+xR}\right) \tag{4.10}$$

Combining (4.7) and (4.10) with the fact that $\lambda\left(\frac{R}{xR}\right) = \lambda\left(\frac{S}{xS}\right)$ and $\lambda\left(\frac{\mathfrak{m}}{\overline{\mathfrak{m}^2}}\right) = \lambda\left(\frac{R}{\overline{\mathfrak{m}^2}}\right) - 1$, we have

$$\begin{aligned}
\lambda\left(\frac{\overline{\mathfrak{m}^2}}{x\mathfrak{m}}\right) + \lambda\left(\frac{R}{\overline{\mathfrak{m}^2}+xR}\right) &= \lambda\left(\frac{\mathfrak{m}}{\overline{\mathfrak{m}^2}}\right) + B \\
&= \lambda\left(\frac{R}{\overline{\mathfrak{m}^2}}\right) - 1 + B \\
&= 1 + \lambda\left(\frac{R}{\overline{\mathfrak{m}^2}+xR}\right) - 1 + B.
\end{aligned}$$

Simplifying we see that $\lambda\left(\frac{\overline{\mathfrak{m}^2}}{x\mathfrak{m}}\right) = B$. We now have that

$$\begin{aligned}
e(R) &= \dim_k(S/\mathfrak{m}S) \\
\lambda\left(\frac{\overline{\mathfrak{m}^2}}{x\mathfrak{m}}\right) &= \dim_k\left(\frac{R+\mathfrak{m}S}{R+\mathfrak{m}^2S}\right).
\end{aligned}$$

Hence by Proposition 4.2.1, we have the desired result. □

Given Proposition 4.2.3, we can construct a couple of examples.

**Example 4.2.4.** Consider the ring $R = k[\![t^3, t^7]\!]$. This is a one-dimensional domain with $e(R) = 3$. The element $t^3$ is a minimal reduction of the maximal ideal $(t^3, t^7)R$. If we compute the length, we see that $\lambda(\overline{\mathfrak{m}^2}/t^3\mathfrak{m}) = 2$. Hence, by Proposition 4.2.3, we have that $R$ is not of finite type.

**Example 4.2.5.** Let $R = k[\![x,y]\!]/(x^3y - xy^3)$. This ring is one-dimensional and reduced. If we compute the multiplicity, we find that $e(R) = 4$. Thus, we immediately have from Proposition 4.2.3 that $R$ is not of finite type. Computing the length none-the-less, we find that $\lambda(\overline{\mathfrak{m}^2}/(x+2y)\mathfrak{m}) = 1$.

**Example 4.2.6.** With Proposition 4.2.3 in mind, it would be nice if we could generalize to higher dimensions. However, as the next example will show, any generalization of the Drozd-Roĭter conditions must not have a bound on multiplicity. The ring

$$R = k[x_1, \ldots, x_{n+1}]/\det_2 \begin{pmatrix} x_1 & \ldots & x_n \\ x_2 & \ldots & x_{n+1} \end{pmatrix}, \; n \geqslant 2$$

is a two-dimensional ring of finite type (see [11]) and $e(R) = n$.

### 4.2.2 Graded Countable Type

We wish to obtain the results of R. Karr and R. Weigand for rings of graded countable Cohen-Macaulay type. Note that by considering the completion, Theorem 2.1.6 and Corollary 2.1.4 allow us to safely apply the results discussed above to standard graded rings.

**Theorem 4.2.7.** *Let $(R, \mathfrak{m}, k)$ be a standard graded one dimensional Cohen-Macaulay ring with uncountable residue field k. If R is not of minimal multiplicity and not a hypersurface, then R is not of graded countable Cohen-Macaulay type.*

*Proof.* Let $x \in \mathfrak{m}$ be a homogeneous minimal reduction of the maximal ideal $\mathfrak{m}$. As $R$ is not a hypersurface, we know that $H_{R/xR}(1) \geqslant 2$. Further, since $R$ does not have minimal multiplicity, we know by Proposition 2.5.2 that $x\mathfrak{m} \neq \mathfrak{m}^2$. So let $a, b \in \mathfrak{m}$ be distinct elements of a minimal generating set of $\mathfrak{m}$ such that $a^2 \notin x\mathfrak{m}$ or $ab \notin x\mathfrak{m}$. Notice that any ideal of the form $(x, a + \alpha b)$, where $\alpha \in k$, is a graded indecomposable maximal Cohen-Macaulay module. By Proposition 2.1.8, it is enough to show there are uncountably many such ideals up to isomorphism in $\mathfrak{MCM}^{\mathrm{gr}}(R)$.

Consider the ideals $I_\alpha := (x, a + \alpha b)$ and $I_\beta := (x, a + \beta b)$ where $\alpha, \beta \in k$ and view them as objects in $\mathfrak{MCM}^{\mathrm{gr}}(R)$. Let $\varphi$ be an isomorphism between $I_\alpha$ and $I_\beta$ in

$\mathfrak{MCM}^{\text{gr}}(R)$. As such, $\varphi$ is a graded degree zero map

$$I_\alpha = (x, a+\alpha b) \overset{\varphi}{\simeq} (x, a+\beta b) = I_\beta$$

given by

$$x \longmapsto d_1 x + d_2(a+\beta b)$$

$$a+\alpha b \longmapsto d_3 x + d_4(a+\beta b).$$

Hence we see that the $d_i$'s are elements of $k$ for $i = 1,2,3,4$. Consider the relation

$$x\varphi(a+\alpha b) - (a+\alpha b)\varphi(x) = 0.$$

Hence we have

$$d_3x^2 + d_4x(a+\beta b) - d_1x(a+\alpha b) - d_2(a^2 + (\alpha+\beta)ab + \alpha\beta b^2) = 0. \tag{4.11}$$

From here we can focus on $d_2$. If $d_2 = 0$, then we have the relation

$$d_3x^2 + d_4x(a+\beta b) - d_1x(a+\alpha b) = 0. \tag{4.12}$$

Since $x$ is a non-zero divisor, we can cancel $x$ and rearrange (4.12) as a $k$-linear combination of $x, a, b$

$$d_3x + (d_4 - d_1)a + (\beta d_4 - \alpha d_1)b = 0.$$

As $x, a, b$ are independent over $k$, we have that the coefficients are zero. In particular $d_4 - d_1 = 0$. Since $d_1d_4 - d_2d_3 \neq 0$, we know that $d_1 = d_4 \neq 0$. Thus the fact that $\beta d_4 - \alpha d_1 = 0$ implies that $\alpha = \beta$. Hence there are uncountably many ideals $I_\alpha$ up to isomorphism in $\mathfrak{MCM}^{\text{gr}}(R)$.

If we assume that $d_2 \neq 0$, then (4.11) modulo $x\mathfrak{m}$, shows that

$$a^2 + (\alpha+\beta)ab + \alpha\beta b^2 \equiv 0.$$

If $a^2 \notin x\mathfrak{m}$, notice that $R_2 = (a^2, xR_1)$. Thus, there exists a fixed $\gamma, \sigma \in k$ such that modulo $x\mathfrak{m}$ we have

$$ab \equiv \gamma a^2;$$
$$b^2 \equiv \sigma a^2.$$

Therefore

$$a^2 \cdot (1 + \gamma(\alpha+\beta) + \sigma\alpha\beta) \equiv 0 \pmod{x\mathfrak{m}}. \tag{4.13}$$

As $a^2$ is non-zero modulo $x\mathfrak{m}$ and $1 + \gamma(\alpha+\beta) + \sigma\alpha\beta$ is a degree zero element, the grading forces

$$1 + \gamma(\alpha+\beta) + \sigma\alpha\beta = 0$$

in the field $k$. In particular, every $\alpha, \beta$ such that $I_\alpha \simeq I_\beta$ is a solution to

$$f(X,Y) = 1 + \gamma(X+Y) + \sigma XY \in k[X,Y].$$

This forces $f(X,Y)$ to be identically zero, a contradiction.

Similarly, if $ab \notin x\mathfrak{m}$ then there exists a fixed $\gamma', \sigma' \in k$ such that modulo $x\mathfrak{m}$ we have

$$a^2 \equiv \gamma' a^2;$$
$$b^2 \equiv \sigma' a^2.$$

Therefore

$$ab\cdot(\gamma' + (\alpha+\beta) + \sigma'\alpha\beta) \equiv 0 \ (\mathrm{mod}\ x\mathfrak{m}) \tag{4.14}$$

and we recover a similar contradiction as we did from Equation (4.13). □

Applying this Theorem 4.2.7 to rings of graded countable Cohen-Macaulay type brings to light some very useful structure. In particular, if the ring is Gorenstein, then we have a hypersurface.

**Corollary 4.2.8.** *Let $(R,\mathfrak{m},k)$ be a standard graded one dimensional Gorenstein ring with uncountable residue field $k$. If $R$ is of graded countable Cohen-Macaulay type, then $R$ is a hypersurface ring.*

*Proof.* By Corollary 2.4.7 and the fact that $R$ is Gorenstein, we know that the possible $h$-vectors are $(1)$, $(1,1)$, or $(1,n,1)$. Thus if $R$ has minimal multiplicity, then $R$ is a hypersurface. If $R$ is not of minimal multiplicity, Theorem 4.2.7 forces that $e(R) \leqslant 3$. Thus $n = 1$ and $R$ is a hypersurface. □

Turning to the case of minimal multiplicity, we find some more structure to the ring.

**Theorem 4.2.9.** *Let $(R,\mathfrak{m},k)$ be a standard graded one dimensional Cohen-Macaulay ring with uncountable residue field $k$. If the $h$-vector of $R$ is $(1,n)$ with $n \geqslant 3$, then $R$ is not of graded countable Cohen-Macaulay type.*

*Proof.* Let $x$ be a minimal homogeneous reduction of the maximal ideal $\mathfrak{m}$, and let $x,u,v,w,c_4,\cdots,c_n \in \mathfrak{m}$ be elements of a minimal $k$-basis of $\mathfrak{m}/\mathfrak{m}^2$. By assumption $n \geqslant 3$, so we are guaranteed at least four elements in the basis of $\mathfrak{m}/\mathfrak{m}^2$. Without losing any generality, we assume that $w$ is the fourth basis element. Assume that there is a graded isomorphism

$$I_\alpha = (x, u+\alpha v) \simeq (x, u+\beta v) = I_\beta$$

where $\alpha, \beta$ are elements of $k$. As $\dim(R) = 1$, these ideals are graded indecomposable maximal Cohen-Macaulay modules. Since this isomorphism is graded of degree 0, we have that

$$\begin{aligned} x &\mapsto \delta_1 x + \delta_2(u + \beta v) \\ u + \alpha v &\mapsto \delta_3 x + \delta_4(u + \beta v) \end{aligned}$$

where $\det(\delta_i)$ is a unit and $\delta_i$ are elements of $k$. We have that

$$0 = \delta_3 x^2 + \delta_4 x(u + \beta v) - \delta_1 x(u + \alpha v) - \delta_2(u + \alpha v)(u + \beta v). \tag{4.15}$$

Notice that $(u + \alpha v)(u + \beta v)$ is an element of $\mathfrak{m}^2$. As $R$ is of minimal multiplicity, by Proposition 2.5.2 we have that $x\mathfrak{m} = \mathfrak{m}^2$. Hence we can view elements of $\mathfrak{m}^2$ as elements of $x\mathfrak{m}$. In particular we view $u^2, uv, v^2$ in the following way

$$\begin{pmatrix} u^2 \\ uv \\ v^2 \end{pmatrix} = \begin{pmatrix} x(a_{10}x + a_{11}u + a_{12}v + a_{13}w + a_{15}c_4 + \cdots + a_{1n}c_n) \\ x(a_{20}x + a_{21}u + a_{22}v + a_{23}w + a_{25}c_4 + \cdots + a_{2n}c_n) \\ x(a_{30}x + a_{31}u + a_{32}v + a_{33}w + a_{35}c_4 + \cdots + a_{3n}c_n) \end{pmatrix} = x \cdot A \cdot \begin{pmatrix} x \\ u \\ v \\ w \\ c_4 \\ \vdots \\ c_n \end{pmatrix}$$

where the matrix $A = (a_{ij})$, $1 \leqslant i \leqslant 3$, $0 \leqslant j \leqslant n$. Since $u^2, uv, v^2$ are homogeneous elements, the grading forces the entries of $A$ to be elements of $k$. Further, if we let

$$\Phi = \begin{pmatrix} 1 & \alpha + \beta & \alpha\beta \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} x & u & v & w & c_4 & \cdots & c_n \end{pmatrix}^t,$$

then we can use matrix notation to write

$$(u+\alpha v)(u+\beta v) = u^2 + (\alpha+\beta)uv + \alpha\beta v^2 = x \cdot \Phi \cdot A \cdot \mathbf{b}.$$

We can now cancel the $x$ in equation (4.15) and rewrite it as

$$0 = \begin{pmatrix} \delta_3 - \Phi A_0 \delta_2 \\ \delta_4 - \delta_1 - \Phi A_1 \delta_2 \\ \beta\delta_4 - \alpha\delta_1 - \Phi A_2 \delta_2 \\ -\Phi A_3 \delta_2 \\ -\Phi A_4 \delta_2 \\ \vdots \\ -\Phi A_n \delta_2 \end{pmatrix}^t \cdot \mathbf{b} \tag{4.16}$$

where $A_i$ are the columns of the matrix $A$. All of the elements in the coefficient matrix of (4.16) are elements of $k$ and hence equal zero as $x, u, v, w, c_4, \ldots, c_n$ form a $k$-basis.

At this point we focus on $\delta_2$. If $\delta_2 \neq 0$, then the fact that $\Phi A_3 \delta_2 = 0$ implies that

$$a_{13} + (\alpha+\beta)a_{23} + \alpha\beta a_{33} = 0 \tag{4.17}$$

in the field $k$. As the $a_{ij}$ are independent of our choice of $\alpha$ and $\beta$, Equation (4.17) shows that every $\alpha, \beta$ such that $I_\alpha \simeq I_\beta$ is a solution to

$$f(X,Y) = a_{13} + (X+Y)a_{23} + XYa_{33} \in k[X,Y].$$

This forces $f(X,Y)$ to be identically zero, a contradiction. Hence there are uncountably many $I_\alpha$ that are not isomorphic.

If we let $\delta_2 = 0$, then Equation (4.17) becomes the relation

$$\delta_3 x + (\delta_4 - \delta_1)u + (\beta\delta_4 - \alpha\delta_1)v = 0. \tag{4.18}$$

As $x, u, v$ are independent over $k$, we have that the coefficients are zero. In particular $\delta_4 - \delta_1 = 0$. Since $\delta_1\delta_4 - \delta_2\delta_3 \neq 0$, we know that $\delta_1 = \delta_4 \neq 0$. Thus the fact that $\beta\delta_4 - \alpha\delta_1 = 0$ implies that $\alpha = \beta$. Hence there are uncountably many non-isomrophic ideals $I_\alpha$. □

Given the above results, we are now ready to characterize one dimension standard graded rings of graded countable Cohen-Macaulay type.

**Corollary 4.2.10.** *Let $(R, \mathfrak{m}, k)$ be a one-dimensional standard graded Cohen-Macaulay ring with uncountable residue field k. If R is of graded countable Cohen-Macaulay type, then R is either of minimal multiplicity with h-vector $(1,2)$, or is isomorphic to one of the following hypersurfaces:*

*(1)* $k[x]$;

*(2)* $k[x,y]/(xy)$;

*(3)* $k[x,y]/(xy(x+y))$;

*(4)* $k[x,y]/(xy^2)$;

*(5)* $k[x,y]/(y^2)$.

*Proof.* A direct application of Theorem 4.2.7 and Theorem 4.2.9 show that $R$ is either a hypersurface ring, or has minimal multiplicity with $h$-vector $(1,2)$.

Concerning the hypersurfaces, items (1)-(3) have graded finite Cohen-Macaulay type as can be seen from [8] or [11]. The hypersurfaces (4) and (5) are not graded finite

Cohen-Macaulay type, but their completions are the one dimensional $(A_\infty)$ and $(D_\infty)$ hypersurface singularities shown in (1.1) and (1.2). It was shown by R. Buchweitz, G. Greuel, and F. Schreyer in [8] that these are the only hypersurfaces that are countable but not finite Cohen-Macaulay type. Hence by Corollary 2.1.4, the rings (4) and (5) are of graded countable Cohen-Macaulay type. □

**Corollary 4.2.11.** *Let $(R, \mathfrak{m}, k)$ be a one-dimensional standard graded Cohen-Macaulay ring with uncountable residue field k. If R is of graded countable Cohen-Macaulay type then $e(R) \leqslant 3$.*

*Proof.* Combining Corollaries 2.4.7 and 4.2.10, we know that the possible $h$-vectors of $R$ are the following:

$$(1),\ (1,1),\ (1,2),\ (1,1,1).$$

Hence we have that $e(R) \leqslant 3$. □

An obvious improvement to Corollary 4.2.10 would be to classify the rings of minimal multiplicity. As it is, we leave it as a question.

**Question 4.2.12.** *Given a one-dimensional standard graded Cohen-Macaulay $(R, \mathfrak{m}, k)$ with uncountable residue field k, if R is of graded countable Cohen-Macaulay type, and minimal multiplicity, then is R isomorphic to*

$$k[x,y,z]/\det_2 \begin{pmatrix} x & y & z \\ y & z & x \end{pmatrix}?$$

It is worth noting that a positive answer to Question 1.1.5, independent of Theorem 4.2.2, can be given using the above results.

**Corollary 4.2.13.** *Let $(R,\mathfrak{m},k)$ be a one-dimensional, reduced, standard graded Cohen-Macaulay ring with uncountable residue field $k$. If $R$ is of graded countable Cohen-Macaulay type, then $R$ is of graded finite type and isomorphic to one of the following:*

*(1)* $k[x]$;

*(2)* $k[x,y]/(x^2+y^2)$;

*(3)* $k[x,y]/(xy)$;

*(4)* $k[x,y]/(xy(x+y))$;

*(5)* $k[x,y,z]/\det_2\begin{pmatrix} x & y & z \\ y & z & x \end{pmatrix}$.

*Proof.* Since standard graded rings of minimal multiplicity with $h$-vector $(1,2)$ have $\lambda(\overline{\mathfrak{m}^2}/x\mathfrak{m}) \leqslant 1$, we can apply Proposition 4.2.3 and Corollary 4.2.10 to obtain the desired result. □

## 4.3 Non-Gorenstein Rings of Dimension at least 3

Let $(R,\mathfrak{m},k)$ be a standard graded Cohen-Macaulay ring of graded countable Cohen-Macaulay type, that is not Gorenstein and $\dim R \geqslant 3$. By Proposition 2.5.10, we know that $R$ must be a domain and have minimal multiplicity. As can be seen in Y. Yoshino's book [25, Page 166], standard graded Cohen-Macaulay domains of minimal multiplicity have the following classification:

(i) hypersurfaces $k[x_1,\ldots,x_n]/(f)$ for some quadratic polynomial $f$.

(ii) the ring $k[x_1,\ldots,x_6]/\det_2(\text{sym } 3\times 3)$

(iii) the scrolls defined in Definition 1.1.1.

The ring defined in (i) is Gorenstein and is discussed in section 4.4. The ring in (ii) is of graded finite (hence countable) Cohen-Macaulay type as can be seen in [11] or [25, Example 17.6.1]. So, we only need to consider the rings given in (iii).

As mentioned earlier, Theorem 1.1.3 classifies the scrolls of finite type. In the same paper, M. Auslander and I. Reiten show the following result for graded Cohen-Macaulay type.

**Theorem 4.3.1** ([5, Theorem 3.1])**.** *Let $(R,\mathfrak{m},k)$ be a standard graded scroll of type $(a_1,\ldots,a_r)$. If $r \geqslant 2$ and $R$ is not of type $(1,1)$ or $(2,1)$, then $R$ has $|k|$ many indecomposable graded Cohen-Macaulay modules, up to shifts.*

As it is, the graded scrolls of type $(1,1)$ and $(2,1)$ are the only graded scrolls of dimension at least 3 that have graded countable Cohen-Macaulay type. Hence, given Theorem 4.3.1, we have a nice corollary to Proposition 2.5.10.

**Corollary 4.3.2.** *Let $(R,\mathfrak{m},k)$ be a standard graded Cohen-Macaulay ring with uncountable residue field $k$. Further assume that $R$ is not Gorenstein and $\dim R \geqslant 3$. If $R$ is of graded countable Cohen-Macaulay type, then $R$ is of graded finite type and is isomorphic one of the following rings:*

*(1)* $k[x_1,\ldots,x_5]/\det_2\begin{pmatrix} x_1 & x_2 & x_4 \\ x_2 & x_3 & x_5 \end{pmatrix}$;

*(2)* $k[x_1,\ldots,x_6]/\det_2(\mathit{sym}\ 3\times 3)$.

*Proof.* Notice that a scroll of type $(1,1)$ is Gorenstein and is omitted. Thus combining Proposition 2.5.10 and the above discussion yields the desired result. □

Corollary 4.3.2 shows that for standard graded Cohen-Macaulay rings of dimension at least three, graded countable Cohen-Macaulay type is the same as graded finite type when the ring is not Gorenstein. Hence we have another case where Question 1.1.5 has a positive answer.

## 4.4 Dimension Two and Gorenstein Rings

It seems the two main difficulties in classifying rings of graded countable Cohen-Macaulay type lie in rings of dimension two and Gorenstein rings. These cases are still open, but some partial results are given below.

### 4.4.1 Non-Gorenstein Rings of Dimension Two

In [11], D. Eisenbud and J. Herzog exploit the fact that graded finite Cohen-Macaulay type implies the ring is an isolated singularity (see Proposition 2.5.7). As Theorem 2.5.9 articulates, rings of graded countable Cohen-Macaulay type do not have the luxury of an isolated singularity. However, if we assume an isolated singularity, we have a positive answer to Question 1.1.5 in the two dimensional non-Gorenstein case.

**Proposition 4.4.1.** *Let* $(R,\mathfrak{m},k)$ *be a standard graded Cohen-Macaulay ring that is not Gorenstein and* $\dim(R)=2$*. Further assume that* $R$ *has an isolated singularity and that* $k$ *is an uncountable field. If* $R$ *is of graded countable Cohen-Macaulay type, then* $R$ *is of graded finite type and is isomorphic to*

$$k[x_1,\ldots,x_{n+1}]/\det{}_2\begin{pmatrix} x_1 & \ldots & x_n \\ x_2 & \ldots & x_{n+1}\end{pmatrix},$$

*where* $n \geqslant 2$*.*

*Proof.* First notice that two dimensional ring with isolated singularity satisfies $(R_1)$ and $(S_2)$. Hence by Serre's criterion (Theorem 2.5.5), we have that $R$ is normal and is therefore a domain. As discussed in Section 4.3, we know that $R$ must be isomorphic to one of the following rings:

(i) hypersurfaces $k[x_1,\dots,x_n]/(f)$ for some quadratic polynomial $f$.

(ii) the ring $k[x_1,\dots,x_6]/\det_2(\text{sym } 3\times 3)$

(iii) the scrolls defined in Definition 1.1.1.

As (i) is Gorenstein and (ii) is three dimensional, we only need to concern ourselves with two dimensional scrolls. The only non-Gorenstein scrolls of dimension two are of type $(m)$ where $m \geqslant 2$ and are the ones listed in the statement. It is known that these rings are of graded finite Cohen-Macaulay type (see [11] or [3, Theorem 2.3]). □

Proposition 4.4.1 gives a partial (positive) answer to Question 1.1.5. Ideally though, we would like to remove the isolated singularity condition from the hypothesis of Proposition 4.4.1. Doing so would show that graded countable Cohen-Macaulay type implies graded finite Cohen-Macaulay type for non-Gorenstein two dimensional rings.

### 4.4.2 Gorenstein Rings

It is well known that standard graded Gorenstein rings of graded finite Cohen-Macaulay type are hypersurfaces [14, Satz 1.2]. This fact is heavily exploited in the classification of standard graded Cohen-Macaulay rings of graded finite Cohen-Macaulay type [11]. The countable analog of this fact is still unknown.

**Conjecture 1.1.6.** *A Gorenstein ring of countable Cohen-Macaulay type is a hypersurface.*

Using the concept of super-stretched, Theorem 4.4.2 shows this conjecture to be true for standard graded rings of dimension at most one, but the Conjecture 1.1.6 remains open for higher dimensions.

**Theorem 4.4.2.** *Let $(R, \mathfrak{m}, k)$ be a standard graded Gorenstein ring with* $\dim(R) \leqslant 1$. *If $R$ is of graded countable Cohen-Macaulay type, then $R$ is a hypersurface.*

*Proof.* This is a combination of Proposition 4.1.2 and Corollary 4.2.8. □

Not much is known about two dimensional Gorenstein rings of graded countable Cohen-Macaulay type. However, if we restrict to a domain of minimal multiplicity, then we have the following proposition.

**Proposition 4.4.3.** *Let $(R, \mathfrak{m}, k)$ is a standard graded Gorenstein domain of minimal multiplicity and* $\dim(R) \geqslant 2$*. If $R$ is of graded countable Cohen-Macaulay type, then $R$ is a hypersurface $k[x_1, \ldots, x_n]/(f)$ for some quadratic polynomial $f$.*

*Proof.* As discussed in Section 4.3, Cohen-Macaulay domains of minimal multiplicity are classified. The only Gorenstein domains are hypersurfaces $k[x_1, \ldots, x_n]/(f)$ for some quadratic polynomial $f$. □

According to Serre's criterion (Theorem 2.5.5), if we assume isolated singularity, then we have that a two dimensional standard graded ring is a normal domain. Hence we have an immediate corollary.

**Corollary 4.4.4.** *Let $(R, \mathfrak{m}, k)$ is a standard graded Gorenstein ring of minimal multiplicity and* $\dim(R) = 2$*. Further assume that $R$ has an isolated singularity. If $R$ is of graded countable Cohen-Macaulay type, then $R$ is a hypersurface $k[x_1, x_2, x_3]/(f)$ for some quadratic polynomial $f$.*

It was also shown in Remark 2.5.11 that standard graded rings of graded countable Cohen-Macaulay type and dimension at least three are normal domains. Hence we have the following result for Gorenstein domains of minimal multiplicity.

**Corollary 4.4.5.** *Let* $(R, \mathfrak{m}, k)$ *is a standard graded Gorenstein ring of minimal multiplicity and* $\dim(R) \geqslant 2$*. Further assume that R has an isolated singularity. If R is of graded countable Cohen-Macaulay type, then R is of graded finite Cohen-Macaulay type and is isomorphic to a hypersurface* $k[x_1, \ldots, x_n]/(f)$ *for some quadratic polynomial* $f$.

*Proof.* By Corollary 4.4.4 and Remark 2.5.11, we know that $R$ is isomorphic to a hypersurface $k[x_1, \ldots, x_n]/(f)$ for some quadratic polynomial $f$. Since $R$ has an isolated singularity, we can let $f = x_1^2 + \cdots + x_n^2$. By [11], this ring is of graded finite Cohen-Macaulay type. □

These results beg the following questions.

**Question 4.4.6.** *If* $(R, \mathfrak{m}, k)$ *is a standard graded Gorenstein ring of graded countable Cohen-Macaulay type and* $\dim(R) \geqslant 2$*, is R necessarily of minimal multiplicity?*

A positive answer to Question 4.4.6 would ultimately force the ring to be a hypersurface. Further it would show that Question 1.1.5 has an affirmative answer for Gorenstein rings of graded countable Cohen-Macaulay type. A slightly weaker, but interesting, question is the following.

**Question 4.4.7.** *If* $(R, \mathfrak{m}, k)$ *is a standard graded Gorenstein hypersurface ring of graded countable Cohen-Macaulay type and* $\dim(R) \geqslant 2$*, is R necessarily of minimal multiplicity?*

Since Conjecture 1.1.6 is still open, the natural place to look are rings of complete intersection. According to Proposition 1.2.11, we know that standard graded complete intersection that are stretched are either a hypersurface or defined by two quadrics. According to Theorem 2.4.4, graded countable Cohen-Macaulay type implies superstretched; in particular stretched. Thus we have the following corollary.

**Corollary 4.4.8.** *Let $(R, \mathfrak{m}, k)$ be a standard graded complete intersection of graded countable Cohen-Macaulay type with k an uncountable field. Then R is a hypersurface or defined by two quadrics.*

*Remark* 4.4.9. The natural desire is to somehow show that $R$ can only be a hypersurface. Using the results of L. Avramov an S. Iyengar [6, Theorem 7.4], one is able to show that standard graded rings of graded countable type are indeed hypersurfaces. These results are in the local case, thus in order to apply them we must pass to the completion.

## 4.5 Isolated Singularity and Graded Countable Type

Although a complete classification of the standard graded rings of graded countable Cohen-Macaulay type was not obtained in this thesis, we have found some interesting results. In this section we summarize the results of Chapter 4 as they relate to Question 1.1.5.

In the non-Gorenstein case, Question 1.1.5 is found to have a positive answer. However there is still work to be done in the Gorenstein case. Below is a table summarizing the results in the graded case. Let $(R, \mathfrak{m}, k)$ be a standard graded Cohen-Macaulay ring of graded countable Cohen-Macaulay type with an isolated singularity. The following table represents when $R$ is of graded finite Cohen-Macaulay type (GFT).

| Dimension of $R$ | $R$ non-Gorenstein | $R$ Gorenstein |
|---|---|---|
| 0 | GFT; Prop 4.1.2 | GFT; Prop 4.1.2 |
| 1 | GFT; Thm 4.2.2 or Cor 4.2.13 | GFT; Thm 4.2.2 or Cor 4.2.13 |
| 2 | GFT; Prop 4.4.1 | If Min. Mult. then GFT; Cor 4.4.5 |
| $d \geqslant 3$ | GFT; Cor 4.3.2 | If Min. Mult. then GFT; Cor 4.4.5 |

As discussed above, a positive answer to Question 4.4.6 shows Question 1.1.5 to be true for standard graded rings.

# Chapter 5

# Conclusion and Further Directions

This dissertation was centered around the following question of C. Huneke and G. Leuschke [17]:

**Question 1.0.1.** *Let R be a complete local Cohen-Macaulay ring of countable Cohen-Macaulay representation type, and assume that R has an isolated singularity. Is R then necessarily of finite Cohen-Macaulay representation type?*

In order to better understand Question 1.0.1, the standard graded case was analyzed. In doing so, a new class of rings was discovered and defined as being super-stretched. To avoid confusion, Chapter 2 began with an explanation of what it means for a standard graded ring to be of graded Cohen-Macaulay type and the graded version of Question 1.0.1 was stated. What it means for a Cohen-Macaulay ring to be super-stretched was then explored as detailed in the latter part of Chapter 2. One of the main results is that standard graded Cohen-Macaulay rings of graded countable Cohen-Macaulay type are super-stretched (see Theorem 2.4.4). With this fact, we are able to give a partial classification of standard graded rings of graded countable Cohen-Macaulay type (see Section 4.5), and thus a partial answer to the graded version of Question 1.0.1.

Further, we have defined what it means for a local ring to be super-stretched, and have given different characterizations of this definition (see Theorem 3.0.18). It was also noticed that the associated graded ring of a super-stretched local ring is always Cohen-Macaulay. As this phenomenon is not the norm, it is interesting to find new classes of rings with this property. Now that the definition of a super-stretched local ring has been established, it is natural to extend the graded results to the local case. This direction of study is not contained in this thesis and was left as a question.

**Question 3.0.20.** *Let $(R, \mathfrak{m}, k)$ be a local Cohen-Macaulay ring of dimension $d > 0$ with uncountable residue field $k$. If $R$ is of countable Cohen-Macaulay type, does this necessarily imply $R$ is super-stretched?*

As described in Section 4.4.2, standard graded Gorenstein rings of graded finite Cohen-Macaulay type are hypersurfaces. This result is not known for the Gorenstein rings of graded countable Cohen-Macaulay type and was stated as a conjecture.

**Conjecture 1.1.6.** *A Gorenstein ring of countable Cohen-Macaulay type is a hypersurface.*

We have shown that the conjecture holds for one dimensional standard graded Gorenstein rings (see Theorem 4.4.2). Is it possible to extend this result to higher dimensions?

In [11], D. Eisenbud and J. Herzog completely classify the standard graded Cohen-Macaulay rings of graded finite Cohen-Macaulay type. In doing so, they show that such rings have minimal multiplicity if the dimension is two or larger. For the non-Gorenstein case, we are able to show that standard graded rings of dimension three or larger are of minimal multiplicity, if the ring is of graded countable Cohen-Macaulay type. Further, if we assume the ring has an isolated singularity (and non-Gorenstein), we are able to reduce the dimension from three to two (see Chapter 4). With these results we are able to classify the non-Gorenstein rings with isolated singularity and of graded countable Cohen-Macaulay type.

As described in Chapter 4, the Gorenstein case is all that is missing from a classification of rings of graded countable Cohen-Macaulay type and of isolated singularity. We leave these questions here for future consideration.

**Question 4.4.6.** *If* $(R, \mathfrak{m}, k)$ *is a standard graded Gorenstein ring of graded countable Cohen-Macaulay type and* $\dim(R) \geqslant 2$*, is* $R$ *necessarily of minimal multiplicity?*

**Question 4.4.7.** *If* $(R, \mathfrak{m}, k)$ *is a standard graded Gorenstein hypersurface ring of graded countable Cohen-Macaulay type and* $\dim(R) \geqslant 2$*, is* $R$ *necessarily of minimal multiplicity?*

A positive answer to Question 4.4.6 would ultimately force the ring to be a hypersurface and thus show Conjecture 1.1.6 to be true. Further it would show the graded version of Question 1.0.1 has an affirmative answer. Question 4.4.7 is slightly weaker, but interesting none-the-less.

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