The Numerical Approximation for the Integrability Problem and the Measure of Welfare Changes, and its Applications

by

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THE NUMERICAL APPROXIMATION FOR THE INTEGRABILITY PROBLEM AND THE MEASURE OF WELFARE CHANGES, AND ITS APPLICATIONS

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Abstract

This dissertation mainly studied on numerical approximation methods as a solution of the integrability problem and the measure of welfare changes, and demonstrated how numerical algorithms can be applied in empirical studies as a solution method.

In general, the integrability problem is described as a system of the partial differential equations (PDE) in terms of the expenditure function, and the measure of welfare changes is defined by the difference between the expenditure function at two different time periods. Both problems can be solved using the same method since solutions for these questions mainly relied on how to recover the compensated income (expenditure) from the ordinary demand function.

In order to investigate whether numerical approximation methods can be applied to the integrability problem and the measure of welfare changes, first, we studied the integrability problem mainly focusing on how to transform the system of the partial differential equations to the system of the ordinary differential equation since this transform possibility provides a way to solve the integrability problem using the numerical method. Second, several numerical methods were investigated as a possible solution of both problem including the Vartia, the RK-4th order algorithm, and the Adams Fourth-Order Predictor-Corrector algorithm. In addition, the Rotterdam and Almost Ideal demand system were investigated since the demand system played an important role on recovering the expenditure.

Two empirical studies are performed. In the first application, using both the U.S consumer expenditure (CE) data and the consumer price index (CPI), the AI and Rotterdam demand system were estimated, and the expenditure was recovered from the estimated demand system using numerical approximation methods. From this, we could demonstrate the power and the applicability of numerical algorithms. In the second application, we paid attention to analyze the welfare effect on the U.S elderly population when prices changed. The burden index and the compensating variation were calculated using the numerical algorithm. From the evaluation, we could confirm that the welfare changes and consumer welfare losses of the elderly population were larger than that of the general U.S population.

To my family and friends

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1. Introduction

The numerical approximation to the integrability problem and the measure of welfare changes is the main topic of this paper. At first glance, it seems that there exists no relationships between the integrability problem and the measure of the welfare changes. However, the common fact in both problems, in terms of the solution, can be found when studying both problems a little deeper. In fact, the solution for both problems is identified in terms of the expenditure function. For example, the integrability problem is defined as the system of the partial differential equations in terms of the expenditure function (compensated income[1]), and the measure of welfare changes is described by the expenditure function throughout the compensating and equivalent variations. From this point of view, we could easily notice that the expenditure function plays an important role in both problems, and if there is a way to recover the expenditure function from other well-known systems, such as the demand function or the labor supply function, both problems can be solved. Although both problems have the expenditure function in common, there exist differences on the usage of the expenditure function between the integrability problem and the measure of the welfare changes. For example, the expenditure function in the integrability problem is used as an intermediate tool to deal with more fundamental questions in economic theories. On the contrary, in case of the measure of welfare changes, the expenditure functions are used more directly to estimate the size of the welfare change when price changes.

The integrability problem introduced by Varian (1992) and Mas-Colell et al (1995) is related to the following old economic question. "Given a system of demand functions which has the symmetric, negative semi-definite substitution matrix, can we derive the utility function (or

[1] According to the situation on the expenditure function, the expenditure function has the different name. The expenditure function is called *the money metric indirect utility function* if the price vector p is fixed when the indirect utility function changed. Similarly, the expenditure function is called *the income compensation function* if the indirect utility function is fixed when price vector p varies.

preference) from that demand function?" In general, the integrability problem is described as a system of the partial differential equations (PDE) in terms of the expenditure function. The well-known result of the integrability problem is that the expenditure function can be integrated back from the ordinary demand function, if the ordinary demand system suffices integrability conditions. The Slutsky matrix (substitution matrix of demand function) plays an important role in finding the solution of the integrability problem. General approaches are explained by Samuelson (1950); however, the main results are shown by Hurwicz and Uzawa (1971).

Similarly, the measure of welfare changes is another old topic in economic fields. At the starting point of the welfare analysis, the estimated demand functions, which are sufficiently flexible to capture the variation in behavior observed in data, play an important role in analyzing the welfare effect when the price changed, since the welfare analysis is mainly based on the expenditure function[2]. In general, the measure of welfare effects are usually estimated by the compensating and equivalent variations. These concepts on the measure of welfare changes are introduced by Hicks (1931), and defined by the difference between the expenditure function at two different time periods under the assumption that consumer utilities are held in constant. In fact, the measure of welfare changes can be determined by the indirect utility function derived from the rational preference. However, the approach using the expenditure function is more generally used, since these methods provide more a convenient way to estimate welfare changes expressed in dollar units.

Many economists have studied solutions for the integrability problem after Hurwiz and Uzawa (1971) introduced the PDE system. Apart from the integrability problem, studies on the measure of welfare changes were performed after Hicks(1931) introduced the compensating and equivalent variation in economics. Though both problems started from different research interests, both problems can be solved using the same method, since solutions for these questions mainly relied on how to recover the compensated income (expenditure) from the ordinary demand function. Therefore, if there existed a way to recover the expenditure function from the ordinary demand function, both problems can be solved. As a solution

[2] The role of the demand function in welfare analysis is based on the fact that the expenditure function can be recovered from the demand function when it suffices the integrability conditions.

method for this problem, several numerical approximation methods are introduced and used in economic fields[3].

Hausman (1981) proposed algebraical way to recover the compensated income from the ordinary demand function as a solution of the integrability problem and the measure of welfare changes. This paper opened the possibility of applying the numerical approximation method to welfare analysis after showing that the system of partial differential equations (PDEs) can be converted to the system of ordinary differential equation (ODE). In addition, he showed that the compensated income can be calculated algebraically from the linear or log-linear ordinary demand function. However, except fer some simple special cases, the algebraic integration of differential equation is too difficult to solve, particularly if more than one good is involved. Therefore, for more general cases, numerical algorithms for the ODE system with initial value are required to calculate the compensated income. Later, Hausman introduced the extrapolation method in the work with Newey (1995), which is well known as the best way to obtain high accuracy solutions for the ODE system.

Vartia (1983) proposed the numerical method based on the implicit trapezoidal method in order to generate the algorithm instead of the Taylor method which is very familiar to the economist. This Vartia algorithm pointed out exactly how to recover the expenditure (compensated income) from the ordinary demand function under no closed form solutions of the underlying utility function. However, the demerit is exited on the algorithm itself[4]. In order to calculate the compensated income, this algorithm requires an additional procedure, which is the iteration method, to achieve the solution. This iteration procedure makes the Vartia algorithm slow and inefficient. The applicability of the Vartia algorithm is demonstrated by Porter-Hudak and Hayes (1986, 1991). They applied the Vartia algorithm for recovering the compensating income from the estimated demand function (more specifically, calculating the cost-of-living index).

The Taylor higher order method is used in many economic studies including Mackenzie

[3]In general, difficulties existed in identifying the functional form of the estimated demand system because, except for very special cases, no closed form utility or expenditure function existed so it is hard to identify the demand function from unknown utility or expenditure function. However, the numerical approximation methods can be applied to both bases 1) there existed the functional form demand function and 2) opposite case.

[4]It used "implicit" instead of "explicit", and this implies that there is no direct way to obtain the exact solution from the algorithm

and Pearce (1976), Breslaw and Smith (1995), and Irvine and Sims (1998). It partly depends on the fact that the Taylor method is familiar to the economist, and this method provides an convenient way of the approximation around the initial condition. The advantage of the Taylor second and higher order method, compared to the Vartia algorithm, is that this algorithm provides a simple way to calculate the variance of the estimated share which the Vartia algorithm is unsuitable for. However, in general, the Taylor method is seldom used in practice, since this algorithm has the disadvantage that it needs the evaluation of derivatives of the object function. Sometimes, computing derivatives of the objective function is more difficult than calculating the algorithm itself if we do not have the exact functional form of the objective function(in here, the expenditure function.)

In addition, a couple of new methods which are variations of the Taylor method are proposed First, Dumagan and Mount (1997) proposed a REversible Second-ORder Taylor (RESORT) as an approximation method for calculating the compensated income from the ordinary demand function. The Taylor backward and forward second order method is used to generate the RESORT algorithm. However, the algorithm itself is more related to the Vartia algorithm. In their paper, they insisted that the Vartia algorithm is the first order case of the RESORT algorithm, and the main advantage of using the RESORT is in the built-in procedure which makes it possible to check the validity of the compensated income using the Slutsky matrix. However, like the Vartia algorithm, this algorithm requires an additional procedure to calculate the compensated income, since this algorithm has the unknown value in both sides of the equation.

In fields of numerical analysis, Runge-Kutta methods are considered as important approximation methods for the system of ordinary differential equations. Surprisingly, no economist applied RK methods, that are regarded as the standard solution of ODE, until Choi's working paper (2010). Like the RESORT algorithm, RK methods are derived from the Taylor higher order method, more specifically, the Taylor 4th order, but at the first glance, it seems that there existed no relationship between the Taylor method and the RK-4th order method. Moreover, it seems to be awkward and complicated to use it. However, it provides a simpler way to obtain the solution comparing to the Vartia algorithm and the Taylor higher order method, since this algorithm does not require the additional iteration method in the Vartia algorithm, or derivatives of the object function in the Taylor method. In addition, the accuracy of the RK-4th order method is more precise than that of other methods including the Vartia and the Taylor method according to the precision of the estimation results.

In this paper, we mainly investigate the integrability problem and the related numerical approximation methods which make it possible to recover the expenditure from the ordinary demand. In addition, as a related topic, we also investigate the measure of welfare changes when price changes. The remainder of this paper is organized as follows: In Chapter 2, we introduce the general description of the integrability problem and the measure of welfare changes. More specifically, we identified how to transform the system of the partial differential equations to the system of the ordinary differential equation, and how the compensating and equivalent variation can be used as the measure of welfare changes in terms of the approximation. Chapter 3 discusses numerical approximation methods which is proposed as the solution of the integrability problem and methods of measuring welfare changes in economics. In Chapter 4, we introduce two famous demand systems, Almost Ideal and Rotterdam demand system, which are used to estimate the consumption patterns in the U.S. Later, using both estimated coefficients from demand systems and approximation methods, the cost-of-living and welfare changes were calculated for analyzing purposes. Chapter 5 and 6 is about empirical applications derived from the welfare measure and integrability problem. More specifically, in Chapter 5, we introduce the new numerical approximation for measuring cost-of-living indices, and show that the newly proposed numerical method can be used as the alternative of Vartia's famous approximation method. In Chapter 6, we investigate the welfare changes in U.S elderly using the compensating variation and the burden index. Finally, in Chapter 7, the summary and conclusion of this research is provided.

2. Integrability problem and Measure of Welfare Changes

2.1. Integrability problem

2.1.1. Integrability problem

The integrability problem introduced by Varian (1992) and Mas-Colell et al (1995) is related the following old economic question. "Given a system of demand functions which has the symmetric, negative semi-definite substitution matrix, can we derive the utility function (or preference) from that demand function?" In general, the integrability problem is described as a system of the partial differential equations (PDE) between a expenditure and a demand function, and it is possible to solve the PDE system if the expenditure can be integrated back from the demand function. The Slutsky matrix, substitution matrix of demand function, plays an important role in finding the solution.

Let's consider a consumer who maximizes the utility. Let $q \in R^N_+$ denotes a vector of quantities and $p \in R^N_{++}$ denotes a corresponding vector of prices. Let, further, $U(q)$, $v(p, y)$ and $e(p, u)$ denote the consumer utility, indirect utility and expenditure function respectively. In here, u denotes a utility level and y denotes the total expenditure. We assume that $U(\cdot)$, $v(\cdot)$ and $e(\cdot)$ satisfy usual regularity conditions. In addition, we assume that $e(\cdot)$ and $v(\cdot)$ is the continuously differentiable in the open neighborhood around (p^0, u^0) and $\partial e(p^0, u^0) > 0$.

Given a system of demand functions $q_i(p, y)$ which is the solution of the utility maximization problem and is continuously differentiable, then the demand function satisfied following five conditions;

Hurwicz-Uzawa Integrability Conditions

1. Homogeneity

   The demand function $q_i(p, y)$ is homogeneous of degree zero in prices p, and total expenditure y

   - $q(tp, ty) = q(p, y)$

2. Summability (Budget Balance)

   The weighted sum of demand functions where weights are given by prices p is equal to total expenditure

   - $p \times q(p, y) = y$

3. Non-negativity

   The quantity consumed is non negative for all p, and y.

   - $q(p, y) \geq 0$

4. symmetry

   The matrix of compensated price effects for demand functions must be symmetric

   - $\dfrac{\partial q_i(p, e(\cdot))}{\partial p_j} + \dfrac{\partial q_i(p, e(\cdot))}{\partial e} \cdot q_j(p, e(\cdot)) = \dfrac{\partial q_j(p, e(\cdot))}{\partial p_i} + \dfrac{\partial q_j(p, e(\cdot))}{\partial e} \cdot q_i(p, e(\cdot))$

5. Negative-Semi-definite

   The compensated own-price substitution effect for the demand function is non positive

   - $\sum_j \sum_i \lambda_j \lambda_i \left(\dfrac{\partial q_i(p, e(p, u^0))}{\partial p_j} + \dfrac{\partial q_i(p, e(p, u^0))}{\partial e} \cdot q_j(p, e(p, u^0)) \right) \leq 0$

Main results of the integrability problem are summarized that these five integrability conditions are in fact sufficient as well as necessary conditions for the integrability process. These results imply that the expenditure can be derived from the observable data of demand quantities. General approaches are performed by Samuelson (1950); however, main results are shown by Hurwicz and Uzawa (1971).

To describe the integrability problem, the utility, an indirect utility, and expenditure function might be used. However, in order to find a utility function from a given system of demand functions, we need an equation to be integrated. It can be achieved by dealing with the integrability problem in terms of the expenditure function. From the Shepard's lemma, the duality between the Hicksian and Marshallian demand[1], and the given specific boundary condition of the expenditure $e(p^0, u^0)$, the integrability problem in terms of the expenditure function can be defined by followings

$$\begin{aligned} \frac{\partial e(p, u^0)}{\partial p_i} &= q_i(p, e(p, u^0)) \qquad i = 1, 2, \cdots, n \\ e(p^0, u^0) &= y^0 \end{aligned} \tag{2.1.1}$$

where $e(p^0, u^0) = y^0$ is the initial condition.

These system of equations (2.1.1) are called "a system of partial differential equations with the initial value." This system (2.1.1) has a unique solution if this PDE system suffices following conditions;

Hurwicz-Uzawa Global Existence Conditions

1. Differentiable

   Each component function $q_i(p, y)$ is differentiable in p and y.

2. Symmetry

   The Slutsky matrix which consist of the compensated own- and cross-price effects for

[1] We begin by posting a system of demand equations. Marshallian demand function $q(p, y)$ can be derived from the indirect utility function using the Roy's identity:

$$q(p, y) = -\frac{\partial v(p, y)/\partial p}{\partial v(p, y)/\partial y}$$

and the Hicksian demand function $h(p, u)$ can be derived from the expenditure function by the Shepard lemma:

$$h(p, u^0) = \frac{\partial e(p^0, u^0)}{\partial p}$$

If we used the duality condition at the initial equilibrium, we have the following relationship between Marshalian demand and Hicksian demand function:

$$\begin{aligned} h(p, u^0) &= q(p, e(p, u^0)) \\ &= q(p, y) \end{aligned}$$

The expenditure function, $e(p, v(q, y))$, is called as the income compensation function when the price vector,p , changes but the indirect utility, $v(q, y)$, is fixed.

demand functions is symmetric,

- $\dfrac{\partial q_i(p,e(\cdot))}{\partial p_j} + \dfrac{\partial q_i(p,e(\cdot))}{\partial e} \cdot q_j(p,e(\cdot)) = \dfrac{\partial q_j(p,e(\cdot))}{\partial p_i} + \dfrac{\partial q_j(p,e(\cdot))}{\partial e} \cdot q_i(p,e(\cdot))$

3. No Demand

   For each $i = 1, \ldots, n$, and p, we have

   - $q_i(p,0) = 0$

   This implies that the demand equals zero for all goods whenever the income equals zero.

4. Boundary Condition

   The function q_i satisfies the following boundary condition on the partial derivative *w.r.t* the income. For every $0 < \underline{a} < \bar{a}$, there exists a (finite) real number M such that for all $y \geq 0$

   - $\underline{a} \leq p \leq \bar{a} \quad \Longrightarrow \left| \dfrac{\partial q_i(p,y)}{\partial y} \right| \leq M \quad i = 1, \ldots, n$

2.1.2. From a system of partial differential equations to a system of the ordinary differential equation

As we described in the section above, if the PDE system suffices integrability conditions, then the expenditure (or utility) function could be recovered from the demand function. However, generally, there is no easy way to solve the PDE system algebraically and numerically so tricks are demanded to transform this system of partial differential equations to the system of the ordinary differential equation (ODE). Under the assumption that the utility is remained at same utility level[2] though there exist price changes from p^0 to p^1 as a result of the policy change, the PDE system can be transformed to the ODE system. This possibility are introduced by Hausman (1981) and Vartia (1983) respectively.

[2] We assume that the demand function are moving on the same indifference surface. In fact, the necessary and sufficient condition for the demand function $q(p,y)$ moving on the same indifference surface is that the rate of changes in utility equal zero.

Hausman's approach

Hausman approach (1981) begins with the Roy's identity[3] which described the relationship between the ordinary demand and the indirect utility function. In addition, the implicit function theorem are employed for transforming procedures.

Assuming that $p(t)$ denotes a price path with $p(0) = p^0$ and $p(1) = p^1$, and $e(p(t), u^0)$ be a compensated income, satisfying

$$v(p(t), e(p(t), u^0)) = v(p^0, e^0) \tag{2.1.2}$$

where u^0 is the utility level at the reference (base) period, and $e^0 = e(p(0), u^0)$, the initial value of total expenditure, is constant.

Differentiating both sides of the equation (2.1.2) *w.r.t* t yields

$$\sum_i \frac{\partial v\left(p(t), e(p(t), u^0)\right)}{\partial p_i(t)} \frac{\partial p_i(t)}{\partial t} + \frac{\partial v\left(p(t), e(p(t), u^0)\right)}{\partial e(p(t), u^0)} \frac{\partial e(p(t), u^0)}{\partial t} = 0 \tag{2.1.3}$$

Applying the implicit function theorem and Roy's identity on the equation (2.1.3) yields the following transformed equation,

$$\begin{aligned} \frac{\partial e(p(t), u^0)}{\partial p_i(t)} &= -\frac{\sum_i \partial v\left(p(t), e(p(t), u^0)\right)/\partial p_i(t)}{\partial v\left(p(t), e(p(t), u^0)\right)/\partial e(p(t), u^0)} \\ &= \sum_i q_i\left(p(t), e(p(t), u^0)\right) \end{aligned} \tag{2.1.4}$$

Multiplying $\frac{dp_i(t)}{dt}$ on both sides of the equation (2.1.4) , then finally we have

$$\frac{de(p(t), u^0)}{dt} = \sum_i q_i\left(p(t), e(p(t), u^0)\right) \cdot \frac{dp_i(t)}{dt} \tag{2.1.5}$$

This equation (2.1.5) is the system of the ordinary differential equation since it contains the derivative function for one independent variable[4]. Now, we know that the PDE system in

[3] This identity shows that the demand function can be derived from the indirect utility function.

[4] The difference between the ordinary differential equation and the partial differential equation are in the number of independent variable in the system. In fact, the ordinary differential equation contains the derivative

the equation (2.1.1) is transformed to the ODE system in the equation (2.1.5).

Vartia's approach

Another method of transforming the PDE system to the ODE system is introduced by Vartia (1983). This procedure is very similar to Hausman (1981) described above, however, it provides a more compact way to transform the the PDE system to the ODE system. Similar to Hausman's approach, Vartia's approach also employed the Roy's identity to solve the problem, and assume that price changes happened on the same indifference curve surfaces.

Let's consider the price path with $p(0) = p^0$ and $p(1) = p^1$, where $0 \leq t \leq 1$. In addition, $v(p(t), e(p(t), u^0)$ is the corresponding indirect utility function satisfying

$$v(p(t), e(p(t), u^0)) = v(t) \tag{2.1.6}$$

Differentiating the indirect utility function in the equation (2.1.6) *w.r.t* t produced

$$\sum_i \frac{\partial v\left(p(t), e(p(t), u^0)\right)}{\partial p_i(t)} \frac{\partial p_i(t)}{\partial t} + \frac{\partial v\left(p(t), e(p(t), u^0)\right)}{\partial e(p(t), u^0)} \frac{\partial e(p(t), u^0)}{\partial t} = \frac{dv(t)}{dt} \tag{2.1.7}$$

The above equation (2.1.7) gives the rate of changes in the utility at every point t when price p and expenditure e change in an arbitrary way. Rearranging the equation (2.1.7) after applying Roy's identity yields

$$\frac{dv\left(p(t), e(\cdot)\right)}{dt} = \lambda\left(p(t), e(\cdot)\right) \left[\frac{de(p(t), u^0)}{dt} - \sum q_i\left(p(t), e(\cdot)\right) \cdot \frac{dp_i(t)}{dt}\right] \tag{2.1.8}$$

where $\lambda\left(p(t), e(\cdot)\right) = q_i\left(p(t), e(\cdot)\right) \times \dfrac{\partial v\left(p(t), e(\cdot)\right)}{\partial e(\cdot)}$

Since we assume that price changes happened on the same indifference curve surfaces, this assumption leads the equation (2.1.8) to the ODE system in terms of $e(p(t), u^0)$.

$$\frac{de(p(t), u^0)}{dt} = \sum_i q_i\left(p(t), e(p(t), u^0)\right) \cdot \frac{dp_i(t)}{dt} \tag{2.1.9}$$

functions for the only one independent variable. In contrary, the partial differential equation contains partial derivatives functions of several independent variables. In generally speaking, the ordinary differential problems is easier than the partial differential equation case.

This equation (2.1.9) is the same equation in Hausman approach (see the equation (2.1.5)).

2.2. Measure of Welfare Changes

The measure of welfare changes when the price has varied is the main topic in this section. For this, the following economic situation is considered that the price changes from p^0 to p^1 as a result of the policy, such as tax policy, that leads changes in market prices. The postscripts 0 and 1 represent the before and the after policy changes, respectively.

In order to determine whether the consumer welfare is better or worse off when the price changes, the indirect utility function derived from the rational preference is enough for making this comparison. However, the money metric indirect utility function[5] which is constructed by means of the expenditure function provides more convenient way to handle this problem.

Let's consider the indirect utility function $v(p, y)$, an arbitrary price vector $\bar{p} \gg 0$, and the expenditure function $e\left(\bar{p}, v(p, y)\right)$. The expenditure function[6] $e\left(\bar{p}, v(p, y)\right)$ provides the wealth required to reach the utility level $v(p, y)$ when prices are $\bar{p}$. As a function of (p, y), the expenditure function is an indirect utility function for preference ordering. In addition, the difference of expenditures between different price levels

$$e(p, v(p^1, y)) - e(p, v(p^0, y)) \tag{2.2.1}$$

provides a measure of the welfare change expressed in dollars terms. In fact, the money metric indirect utility function can be formulated in this manner for any arbitrary price vector $\bar{p} \gg 0$. Moreover, this measure is unaffected by the choice of the initial indirect utility function, it depends only on the consumer's preference. Two particularly natural choices for the price vector $\bar{p}$ are the initial price vector p^0 and the new price vector p^1. These choices lead to two well-known measures of welfare changes called the *compensating* and the *equivalent variation*.

[5]It leads to measurement of the welfare change expressed in dollar units.
[6]This expenditure is strictly increasing as a function of the level $v(p, y)$

2.2.1. Compensating and Equivalent Variation

The compensating variation (CV) and the equivalent variation (EV) are used to measure the utility changes in economic fields. These concepts are introduced by Hicks (1939). In order to investigate the compensating and the equivalent variations, some technical assumptions on the utility level and the expenditure are required. Formally, we assume that $u^0 = v(p^0, y)$, $u^1 = v(p^1, y)$, and $e(p^0, y^0) = e(p^1, u^1) = y$.

Compensating-Variation. The compensating variation (CV) measures the net revenue of a planner who must compensate the consumer for price changes after it occurs in order to bring her back to the original utility level.

In fact, the compensating variation, which is defined in terms of the expenditure function reflects changes of expenditures required to maintain the consumer at the original level of utility when prices have changed from price level p^0 to p^1. It is defined by the following;

$$\begin{aligned} CV(p^0 \to p^1) &= e(p^1, u^0) - e(p^0, u^0) \\ &= e(p^1, u^0) - y \end{aligned} \tag{2.2.2}$$

where y is the income at the initial period, $v(p, y)$ is the indirect function, and $u^0 = v(p^0, y)$ is the utility corresponding to the pre-change situation.

Therefore, if the CV has the positive sign, it implies that more spending is required to achieve the same utility level as before price changes. Thus, this means the decrease in consumer welfare. By contrast, if the CV has the negative sign, it implies a drop in spending. In other words, this means a gain in consumer welfare. From these points of views, compensating variation can be used to find the effect of a price change on the consumer's net welfare.

Equivalent-Variation. The equivalent variation (EV) is defined as the amount of money paid to an individual consumer with base prices and income that lead to the same satisfaction as that generated by price changes.

In fact, the equivalent variation is the amount of money one has to give to consumers so they could attain the same utility level possible with new prices. As the same manner of the compensating variation (CV), when prices have changed from price level p^0 to p^1, the equivalent variation (EV) is defined by the following;

$$\begin{aligned} EV(p^0 \to p^1) &= e(p^1, u^1) - e(p^0, u^1) \\ &= y - e(p^0, u^1) \end{aligned} \tag{2.2.3}$$

where y is the income at the final period , $v(p,y)$ is the indirect function, and $u^1 = v(p^1, y)$ is the utility corresponding to the post-change situation.

2.2.2. the Hicksian Demand Curve and Approximations of CV and EV

The compensating variation can be represented in terms of the Hicksian demand function after applying the Shepard lemma on the expenditure function. This leads the following result.

$$\begin{aligned} CV(p^0 \to p^1) &= e(p^1, u^0) - e(p^0, u^0) \\ &= \sum_i \int_{p^0}^{p^1} \frac{\partial e(p, u^0)}{\partial p_i} dp_i \\ &= \sum_i \int_{p^0}^{p^1} h_i\left(p, u^0\right) dp_i \end{aligned} \tag{2.2.4}$$

where $h_i(p, u^0)$ is the Hicksian demand function.

Above equation (2.2.4) implies that changes in consumer welfare measured in terms of the CV can be represented by the area lying between p^0 and p^1. In other words, the CV can be calculated by the area which is located in the left of the Hickisan demand curve associated with the utility level u^0.

The approximation of the equation (2.2.4) in terms of the Taylor higher order series around at (p^0, u^0) is

$$
\begin{aligned}
CV(p^0 \to p^1) &= e(p^0,u^0) + \frac{\partial e(p^0,u^0)}{\partial p}\Delta + \frac{1}{2!}\frac{\partial^2 e(p^0,u^0)}{\partial^2 p}\Delta^2 + \cdots + R - e(p^0,u^0) \\
&= \frac{\partial e(p^0,u^0)}{\partial p}\Delta + \frac{1}{2!}\frac{\partial^2 e(p^0,u^0)}{\partial^2 p}\Delta^2 + \cdots + R \\
&= h(p^0,u^0)\Delta + \frac{1}{2!}\frac{\partial h(p^0,u^0)}{\partial p}\Delta^2 + \cdots + R \qquad (2.2.5)
\end{aligned}
$$

where $\Delta = (p^1 - p^0)$ is the price change, and R is the remainder error term.

Applying the duality theorem to the equation (2.2.5), then we could represent the CV in terms of the ordinary demand function which is observable. That is

$$CV(p^0 \to p^1) = q(p^0,y^0)\Delta + \frac{1}{2!}S(p^0,y^0)\Delta^2 + \cdots + R \qquad (2.2.6)$$

where $S(p,y) = \frac{\partial h(p,u^0)}{\partial p} = \frac{\partial q(p,y)}{\partial p} + q(p,u^0)\frac{\partial q(p,y)}{\partial y}$ is the Slutsky matrix, and $y = e(p,u^0)$ is the income or the level of the expenditure.

From the equation (4.1.22), the fact that the CV can be calculated by using only the ordinary demand function and relative terms is checked.

Similarly, the EV could be defined by the same manner in the CV. From the equation (2.2.3), the equivalent variation could be presented by the following when the price changes from p^0 to p^1

$$
\begin{aligned}
EV(p^0 \to p^1) &= e(p^1,u^1) - e(p^0,u^1) \\
&= \sum_i \int_{p^0}^{p^1} \frac{\partial e(p,u^1)}{\partial p_i} dp_i \\
&= \sum_i \int_{p^0}^{p^1} h_i\left(p,u^1\right) dp_i \qquad (2.2.7)
\end{aligned}
$$

Since the Taylor higher order approximation of $e(p^0,u^1)$ around (p^0,u^0) is

$$e(p^0,u^1) = e(p^0,u^0) + \frac{\partial e(p^1,u^0)}{\partial p}\Delta + \frac{1}{2!}\frac{\partial^2 e(p^1,u^0)}{\partial^2 p}\Delta^2 + \cdots + R \qquad (2.2.8)$$

So the EV could be expressed in terms of the Taylor higher order series after plugging the equation (2.2.8) into equation (2.2.7)

$$\begin{aligned} EV(p^0 \to p^1) &= e(p^1, u^1) - \left[e(p^0, u^0) + \frac{\partial e(p^1, u^0)}{\partial p} \Delta + \frac{1}{2!} \frac{\partial^2 e(p^1, u^0)}{\partial^2 p} \Delta^2 + \cdots + R \right] \\ &= -\left[e(p^0, u^0) + \frac{\partial e(p^1, u^0)}{\partial p} \Delta + \frac{1}{2!} \frac{\partial^2 e(p^1, u^0)}{\partial^2 p} \Delta^2 + \cdots + R \right] \\ &= -\left[h(p^0, u^0)\Delta + \frac{1}{2!} \frac{\partial h(p^0, u^0)}{\partial p} \Delta^2 + \cdots + R \right] \end{aligned} \tag{2.2.9}$$

where $e(p^0, y^0) = e(p^1, u^1) = y,\ \Delta = (p^1 - p^0)$ is the price change, and R is the remainder error term.

Using equation (4.1.21), finally, we could represent the EV in terms of the ordinary demand function similarly in equation (2.2.6)

$$EV(p^0 \to p^1) = -\left[q(p^0, y^0)\Delta + \frac{1}{2!} S(p^0, y^0)\Delta^2 + \cdots + R \right] \tag{2.2.10}$$

where $S(p, y) = \dfrac{\partial h(p, u^0)}{\partial p} = \dfrac{\partial q(p, y)}{\partial p} + q(p, u^0)\dfrac{\partial q(p, y)}{\partial y}$ is the Slutsky matrix, and $y = e(p, u^0)$ is the income or the level of the expenditure.

From the equation (2.2.6) and equation (2.2.10), the relationship existed between the CV and EV is identified. That is

$$EV(p^0 \to p^1) = -CV(p^0 - p^1) \tag{2.2.11}$$

Above equation (2.2.11) represent that the equivalent variation can be calculated in terms of the compensating variation, either.

3. Numerical approximation for recovering the expenditure from the ordinary demand

Main results of the previous section are following. One is if the given demand system $q_i(p, y)$ which suffices integrability conditions, the utility in terms of the expenditure function can be derived from that demand function (Integrability problems). Another is the measure of welfare changes when price changes from p^0 to p^1 could be calculated in terms of the expenditure function from the definition of the compensating and equivalent variations.

Both problems are related to the expenditure function itself, and the solution of both problems can be boiled down to the question *how to recover the expenditure function from the ordinary demand function*. It can be simply described by the initial value problems of the ordinary differential equation. Several numerical methods are proposed by several economists including Hausman (1981 and 1995) and Vartia (1983) as a solution. In this section, the numerical approximation methods used in the economic fields are briefly introduced.

3.1. Numerical Approximation Methods

In order to apply numerical approximation methods on problems in the previous section, a couple of assumptions are considered. First assumption is that the price moves on the same indifference curve surface. This implies that no changes in utility level are existed though the price changes. In addition, we assume that the price change can be divided into smaller price steps. For this, following auxiliary terms are considered. For a given integer N which is a number of the interval, the price range is defined by the linear combination of prices p^0

and p^1

$$p(t) = p^0 + t(p^1 - p^0), \quad t \in [0,1] \tag{3.1.1}$$

where, $p^k = p(t_k)$, and $t_k = \dfrac{k}{N}$.

Using these auxiliary terms, technical terms in the previous section are simplified. Finally, the expenditure and the ordinary demand function are satisfied the following

$$e\left(p(t_k), u^0\right) = e\left(p^k, u^0\right) \tag{3.1.2}$$

and

$$q(p(t_k), e(p(t_k), u^0)) = q\left(p^k, e(p^k, u^0)\right) \tag{3.1.3}$$

Moreover, suppose that the compensated income $e\left(p^k, u^0\right)$ is the solution for the ODE system with the initial-value, then the equation (2.1.5) and (2.1.9) become

$$\begin{aligned} \frac{de\left(p^k, u^0\right)}{dt} &= \sum_i q_i\left(p^k, e(p^k, u^0)\right) \cdot \frac{dp_i(t)}{dt} \\ e(p^0, u^0) &= y^0 \end{aligned} \tag{3.1.4}$$

where $e\left(p^k, u^0\right)$ has $(n+1)$ continuous derivatives.

In general, the data obtained from the real world has not a continuous but discrete form, so the discrete form of the equation (3.1.4) is required in order to apply numerical algorithms to the given problem. The following is called the difference equation associated with the above equation (3.1.4).

$$\begin{aligned} e(p^k, u^0) &= e(p^{k-1}, u^0) + \sum_i q_i\left(p^k, e(p^k, u^0)\right) \cdot \Delta \\ e(p^0, u^0) &= y^0 \end{aligned} \tag{3.1.5}$$

Where $\Delta = (p(t_{k+1}) - p(t_k)) = \dfrac{(p^1 - p^0)}{N}$.

This equation (3.1.5) are generally used to solve the ODE system with the initial value. In general, the accuracy of the algorithm depends mainly on the size of intervals.

3.1.1. Taylor Higher-Order Type Methods

As a solution of the equation (3.1.5), the Taylor higher-order method is mainly recommended by various papers including Mckenzie and Pearce(1976), Breslaw and Smith(1995), and Irvine and Sims(1998). It is based on the fact that the Taylor method is familiar to the economist, and provides a convenient way of the approximation around the initial value. Though a couple of economist recommended the Taylor higher-order method, in many cases, the order of the Taylor method is limited to the order one or two. It is partly because of the difficulty in calculating higher order derivatives of the object function.

3.1.1.1. Taylor higher order Method

The Taylor higher order method as a solution for the ODE system with the initial value can be described by following. Let's consider the solution $e\left(p(t_{k+}), u^0\right)$ is evaluated at t_{k+1}. This solution can be expressed in terms of its nth Taylor polynomial around t_k,

$$e\left(p(t_{k+1}), u^0\right) = e\left(p(t_k), u^0\right) + e^{(1)}\left(p(t_k), u^0\right)\Delta + \frac{1}{2!}e^{(2)}\left(p(t_k), u^0\right)\Delta^2 + \cdots + R \qquad (3.1.6)$$

Where $\Delta = (p(t_{k+1}) - p(t_k)) = \dfrac{(p^1 - p^0)}{N}$, and R is the reminder term.

since $e^{(1)}\left(p(t_k)\right) = q\left(p(t_k), e(p(t_k), u^0)\right)$, $e^{(2)}\left(p(t_k)\right) = q^{(1)}(\cdot)$, and generally, $e^{(n)}\left(p(t_k)\right) = q^{(n-1)}(\cdot)$. Substituting these results into the equation (3.1.6) produced the following

$$\begin{aligned} e\left(p(t_{k+1}), u^0\right) &= e\left(p(t_k), u^0\right) + q\left(p(t_k), e(p(t_k), u^0)\right)\Delta \\ &\quad + \frac{1}{2!}q^{(1)}\left(p(t_k), e(p(t_k), u^0)\right)\Delta^2 + \cdots + R \end{aligned} \qquad (3.1.7)$$

Since $y^0 = e(p^0, u^0)$, and $e\left(p(t_k), u^0\right) = e\left(p^k, u^0\right)$, for simplicity, the Taylor higher order algorithm for the ODE system with the initial value are defined by the following,

For $i = 0, 1, \ldots, N$,

$$
\begin{aligned}
e\left(p^{k+1}, u^0\right) &= e\left(p^k, u^0\right) + T^{(k)}\left(p^k, e(p^k, u^0)\right) \cdot \Delta \\
e\left(p^0, u^0\right) &= y^0
\end{aligned}
\tag{3.1.8}
$$

where $T^{(k)}\left(p^k, e(p^k, u^0)\right) = e\left(p^k, u^0\right) + q\left(p^k, e(p^k, u^0)\right)\Delta + \frac{1}{2!}q^{(1)}\left(p^k, e(p^k, u^0)\right)\Delta^2 + \cdots + R$, k is the number of steps, and $\Delta = \frac{(p^1 - p^0)}{N}$.

The equation (3.1.8) is called the difference equation associated with Taylor method.

The advantage of using the Taylor method when calculating the solution of ODE system is that the economist and researcher[1] are familiar to the Taylor method so not much background or knowledge on the numerical analysis are required. However, in terms of the numerical algorithm, in real practice, this approximation does not used often. It mainly depends on that derivatives of the objective function are required in order to solve the ODE system. In fact, the computation of derivatives is a complicated and time-consuming procedure for most problems even that the closed form of objective function are known. However, if considering the situation in real world, problems become more complicated in order to achieve the goal since the exact functional form of the demand is generally unknown except very special cases. This disadvantage can be considered as the common problem of the numerical method which is used the Taylor expansion to generate the algorithm, either.

[1] In fact, the Taylor higher order approximation is used in various articles and papers to investigate the economic model and phenomena.

3.1.1.2. Euler Method

The Euler method is the special case of Taylor higher order method since this method is the Taylor method of the order one. The Euler method can be described by the following,

For $i = 0, 1, \ldots, N$

$$\begin{aligned} e\left(p^{k+1}, u^0\right) &= e\left(p^k, u^0\right) + q\left(p^k, e(p^k, u^0)\right) \cdot \Delta \\ e\left(p^0, u^0\right) &= y^0 \end{aligned} \tag{3.1.9}$$

where $\Delta = \dfrac{(p^1 - p^0)}{N}$, and k is the number of steps.

The equation (3.1.9) is called the difference equation associated with the Euler method. Though any economist do not mentioned the Euler method in their paper, the Euler method are appeared as the special case of the Taylor method. In fact, this method provides an easier and simpler way to recover the compensated income from the ordinary demand function so this algorithm can increase the understanding of other numerical approximation methods[2]. However, the precision of this methods is not enough accurate compared to the Taylor method and other algorithms.

3.1.1.3. RESORT Algorithm

The RESROT (REversible Second-ORder Taylor) method is proposed by Dumagan and Mount (1997) as an alternative for the Vartia algorithm to calculate the compensate income and the cost-of-living.

To generate the RESROT Algorithm, Dumagan and Mount (1997) employed the forward and backward Taylor second order methods. The RESROT algorithm is described by followings. At the first stage, for $i = 0, 1, \ldots, N$, in order to recover the compensated income from the ordinary demand, the forward second order Taylor approximation of $e\left(p^{k+1}, u^0\right)$ is

[2]In general, the Euler method is considered as the starting point of learning procedure since this algorithm is the very basic method for the initial value problems of the ordinary differential equation.

calculated,

$$e\left(p^{k+1},u^0\right) = e\left(p^k,u^0\right) + q\left(p^k, e(p^k,u^0)\right)\cdot\Delta + \frac{1}{2}S\left(p^k, e(p^k,u^0)\right)\cdot\Delta^2 \tag{3.1.10}$$

Similarly, the backward second order Taylor approximation to $e\left(p^k,u^0\right)$ is estimated at the second step,

$$e\left(p^k,u^0\right) = e\left(p^{k+1},u^0\right) - q\left(p^{k+1}, e(p^{k+1},u^0)\right)\cdot\Delta + \frac{1}{2}S\left(p^{k+1}, e(p^{k+1},u^0)\right)\cdot\Delta^2 \tag{3.1.11}$$

At the third stage, Dumagan and Mount combines above Taylor second order formulas to generate the RESORT algorithm, and solve it in terms of $e(p^{k+1},u^0)$. The following mutual unknown form is generated as a solution.

$$\begin{aligned} e\left(p^{k+1},u^0\right) &= e\left(p^k,u^0\right) + \frac{1}{2}q\left(p^k, e(p^k,u^0)\right)\cdot\Delta + \frac{1}{2}q\left(p^{k+1}, e(p^{k+1},u^0)\right)\cdot\Delta \\ &\quad + \frac{1}{4}S\left(p^k, e(p^k,u^0)\right)\cdot\Delta^2 - \frac{1}{4}S\left(p^{k+1}, e(p^{k+1},u^0)\right)\cdot\Delta^2 \qquad (3.1.12) \\ e\left(p^0,u^0\right) &= y^0 \end{aligned}$$

where $\Delta = \dfrac{(p^1 - p^0)}{N}$, and $S(p^k, e(p^k,u^0)) = \dfrac{q\left(p^k, e(p^k,u^0)\right)}{\partial p^k}\cdot\Delta$ is the Slutsky matrix.

This equation (3.1.12) is called the RESORT algorithm, and yields unique approximation to the compensated income. The value $e(p^{k+1},u^0)$ and $e\left(p^k,u^0\right)$ sufficed both the forward solution in equation (3.1.10) and the backward solution in equation (3.1.11).

At first glance, this algorithm can be considered as the special case of the Taylor method since the backward and forward second order Taylor method are used in the generating process. Moreover, according to Dumagan and Mount(1997), the RESORT algorithm has a couple of advantages. First of all, the RESORT algorithm is a general case of the Vartia algorithm[3]. This result can be confirmed from the first order of RESORT algorithm. That is

[3]For more details, see the Vartia(1983) or Vartia's algorithm in this paper

$$e\left(p^{k+1}, u^0\right) = e\left(p^k, u^0\right) + \frac{1}{2}\left[q\left(p^k, e(p^k, u^0)\right) + q\left(p^{k+1}, e(p^{k+1}, u^0)\right)\right] \cdot \Delta \qquad (3.1.13)$$

where $\Delta = \dfrac{(p^1 - p^0)}{N}$, and k is the number of steps.

The equation (3.1.13) has the exactly same expression of the equation (3.1.17) in the section 3.1.4. Another advantage of using the RESORT method when calculating compensated income is that this approximation provides the built-in procedure to check integrability conditions which the algorithm includes the Slutsky matrix in.

However, analogous other algorithm which used the Taylor method to generate the algorithm, this algorithm required derivatives of the objective function. This implies that if there exist no exact form of the ordinary demand function, the solution can not be easily generated from the above algorithm. Moreover, the RESORT algorithm contains the unknown value $e(p^{k+1}, u^0)$ in RHS of equation (3.1.12) so that the additional procedure[4] is required in order to complete the calculation.

[4]To estimated the the unknown value $e(p^{k+1}, u^0)$, iterating is required until the estimated $e(p^{k+1}, u^0)$ at step n is converged to the real value of $e(p^{k+1}, u^0)$.

3.1.2. Runge-Kutta 4th order Method

In numerical analysis, the RK-4th order method is considered as the standard solution for the ODE system with the initial value. This technique was developed by the German mathematicians C. Runge and M.W. Kutta around 1900s. Nevertheless, no papers in economic fields are applied this algorithm to calculate the compensated income or measure of welfare changes Until Choi(2010). In my opinion, it is partly because, at first glance, the expression in RK-4th order algorithm is not familiar to the economist. In addition, any closed relationship between the algorithm and the economic theory are not identified. However, in the strict sense of word, the relevance in those two topic can be identified if we studied these more deeper. In fact, the RK-4th order algorithm is another derivation of the Taylor higher order algorithm since this algorithm has developed from the Taylor method with order 4th[5].

[5] In here, the general description are used to derive the RK-4th order algorithm from the Taylor 4th order method. Assuming that the initial value problem for an ordinary differential equation satisfies

$$\frac{dy(t)}{dt} = f(t, y(t))$$

together with the initial condition

$$y(t_0) = y_0$$

The RK-4th order algorithm can be described by the following

$$y_{k+1} = y_k + (\alpha_1 k_1 + \alpha_2 k_2 + \alpha_3 k_3 + \alpha_4 k_4) \cdot h$$

where $h = (t_{k+1} - t_k)$, the knowing the value of $y = y_t$ at t_k, then we can find the value of $y = y_{t+1}$ at t_{k+1}. Equation described above is equated to first five terms of Taylor series

$$\begin{aligned} y_{k+1} &= y_k + \frac{dy}{dt}(t_{k+1} - t_k) + \frac{1}{2!}\frac{d^2y}{dt^2}(t_{k+1} - t_k)^2 + \frac{1}{3!}\frac{d^3y}{dt^3}(t_{k+1} - t_k)^3 + \frac{1}{4!}\frac{d^4y}{dt^4}(t_{k+1} - t_k)^4 \\ &= y_k + f(t_k, y_k) \cdot h + \frac{1}{2!}f^{'}(t_k, y_k) \cdot h^2 + \frac{1}{3!}f^{''}(t_k, y_k) \cdot h^3 + \frac{1}{4!}f^{'''}(t_k, y_k) \cdot h^4 \\ &= y_k + \left[f(t_k, y_k) + \frac{1}{2!}f^{'}(t_k, y_k) \cdot h + \frac{1}{3!}f^{''}(t_k, y_k) \cdot h^2 + \frac{1}{4!}f^{'''}(t_k, y_k) \cdot h^3 \right] \cdot h \end{aligned}$$

From this last equation, if we modified it properly, the RK-4th algorithm are derived.

$$y_{k+1} = y_k + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \cdot h$$

where

$$\begin{aligned} k_1 &= f(t_k, y_k) \\ k_2 &= f\left(t_k + \frac{1}{2}h, y_k + \frac{1}{2}k_1 h\right) \\ k_3 &= f\left(t_k + \frac{1}{2}h, y_k + \frac{1}{2}k_2 h\right) \\ k_4 &= f\left(t_k + h, y_k + k_3 h\right) \end{aligned}$$

The RK-4th order method is defined by the following;

$$
\begin{aligned}
e(p^{k+1}, u^0) &= e(p^k, u^0) + \frac{K_1 + 2K_2 + 2K_3 + K_4}{6} \\
e\left(p^0, u^0\right) &= y^0
\end{aligned}
\tag{3.1.14}
$$

where

$$
\begin{aligned}
K_1 &= q\left(p^k, e(p^k, u^0)\right) \cdot \Delta \\
K_2 &= q\left(\frac{p^{k+1} + p^k}{2}, e(p^k, u^0) + \frac{1}{2}K_1\right) \cdot \Delta \\
K_3 &= q\left(\frac{p^{k+1} + p^k}{2}, e(p^k, u^0) + \frac{1}{2}K_2\right) \cdot \Delta \\
K_4 &= q\left(p^k, e(p^k, u^0) + K_3\right) \cdot \Delta
\end{aligned}
$$

where $\Delta = \dfrac{(p^1 - p^0)}{N}$, and k is the number of steps.

As with other algorithms, the accuracy of the RK-4th order algorithm mainly depends on the size of intervals. In addition, this algorithm is easy to be programmed with any computational programing language, and allows to evaluate welfare changes easily from the demand function which is estimated.

Compared to the Euler or the Taylor approximation, the RK-4th order method seems to be very awkward and complicated at the starting point. However, this approximation provides a simpler way to solve the ODE system with the higher precisions, because this algorithm does not require an additional procedure, the iteration procedure, to calculate undetermined variables in equation. In fact, the $e(p^k, u^0)$ can be calculated only using values from the previous step such as $e(p^{k-1}, u^0)$, $q_i(p^{k-1}, e(p^{k-1}, u^0))$, and p. Moreover, on the contrary to the Taylor method, this RK-4th order algorithm contains no evaluations of the second or higher order derivatives though it has the high-order local truncation error of the Taylor method. This provides a huge advantages to the RK-4th order algorithm when solving the ODE system with the initial value.

3.1.3. Bulirsch-Stoer Method

The Bulirsh-Stoer method is generally called the Extrapolation method in numerical analysis. This method is used in Hausman and Newey (1995). However, not much details on this algorithm are provided throughout paper[6]. Because of the insufficient information on the algorithm, except for Hausman and Newey (1995) which is the original paper, no papers in economic fields used this method though it is well known as the best way to obtain high-accuracy solutions to the ODE system with minimal computational effort. It is partly because, by contrast to numerical methods introduced in previous section, the procedure of this algorithm is quite complicated to be programed. Moreover, this numerical approximation required much more background on the numerical analysis itself.

Bulirsh-Store algorithm

This numerical algorithm provides the numerical solution for the ODE system as with numerical methods introduced in previous section. To generate the algorithm, Bulirsh and Stoer (2002)[7] employed three powerful ideas such as : 1) *Richardson extrapolation*[8]: they used the Richardson-type rational function extrapolation in algorithm, 2) *the midpoint method*: was employed in order to obtain numerical solutions to the ODE system, and 3) *the end point correction*: was used to modify the solution with the high accuracy and comparatively little computational effort. It is sometimes called the Gragg–Bulirsch–Stoer (GBS) algorithm because of the important contribution of William B. Gragg in results about the error function of the modified midpoint method.

The following is the general description of the Bulirsh-Stoer algorithm for calculating the solution of the ODE system.

[6]In fact, Hausman and Newey shortly mentioned the algorithm name and the source where they hired the Bulirsh-Stoer Method in the introduction. In their paper, instead of focusing the numerical algorithm itself, Hausman and Newy focus on how to generate the ordinary demand function using non-parametric methods which used the kernel predictor for it.

[7]There existed the time difference between Hausman and Newey (1995) paper and Bulirsh and Store (2002). The book written by Bulirsh and Stoer are used in order to study the Bulirsh and Store algorithm instead of the original paper.

[8]The Richardson extrapolation in numerical analysis is a sequence acceleration method and used to improve the rate of convergence of a sequence. It is named after Lewis Fry Richardson, who introduced the technique in the early 20th century.

- Step.1 : The Midpoint Method
  - The solution $e\left(p^{k},u^{0}\right)$ are calculated by using the midpoint method at step k

$$e\left(p^{k+1},u^{0}\right)=e\left(p^{k},u^{0}\right)+2q\left(p^{k},e(p^{k},u^{0})\right)\cdot\Delta \tag{3.1.15}$$

  - Two starting values both $e\left(p^{1},u^{0}\right)$ and $e\left(p^{0},u^{0}\right)$ are required before the first midpoint approximation $e\left(p^{2},u^{0}\right)$. We assume that the initial condition for $e\left(p^{0},u^{0}\right)=y^{0}$ are known, and the Euler method are used to determine the second starting value $e\left(p^{1},u^{0}\right)$
  - A series of approximations obtained from equation (3.1.15) are generated at step k
- Step.2 : The end point correction using the extrapolation method
  - The extrapolation technique are used in order to execute the end point correction,

$$e\left(p^{k+1},u^{0}\right)=e\,\widehat{(p^{k},u^{0})}+\sum_{j=1}^{\infty}\delta_{j}\Delta^{2j} \tag{3.1.16}$$

  - where the δ_j is a constant term related to derivatives of the solution $e\left(p^{k},u^{0}\right)$, and is generated from the extrapolation method. The important point is that the δ_k do not depend on the step size Δ.

3.1.4. Vartia algorithm

The numerical approximation method proposed by Vartia(1983) is the numerical integration of equation (3.1.4). In economic literature, the Vartia algorithm is used to entail calculating compensated income at step k. The applicability of the Vartia algorithm is demonstrated by Porter-Hudak and Hayes (1986, 1991) when calculating cost-of-living indices and the compensated income. This Vartia algorithm is generally called *Implicit Trapezoidal method* in numerical analysis because the trapezoidal rule[9] was used to generate the algorithm. Since

[9] The *Trapezoidal Rule* is used for approximating the following integral

$$\int_{a}^{b}f(x)dx$$

this algorithm has the implicit form, iterations are required to estimate the solution. This algorithm usually applies for the stiff function case.

The Vartia algorithm is defined by the following

$$\begin{aligned} e(p^{k+1}, u^0) &= e(p^k, u^0) + \tfrac{1}{2}\left(q(p^k, e(p^k, u^0) + q(p^{k+1}, e(p^{k+1}, u^0)\right) \cdot \Delta \qquad (3.1.17) \\ e\left(p^0, u^0\right) &= y^0 \end{aligned}$$

where $\Delta = \dfrac{(p^1 - p^0)}{N}$, and k is the number of steps.

Weather the trapezoidal rule is employed or not can be checked after integrating both sides of the equation (3.1.4). The result is that

$$e(p^{k+1}, u^0) = e(p^k, u^0) + \int_{t_k}^{t_{k+1}} q\left(p^k, e(p^k, u^0)\right) \Delta \cdot dt$$

In order to calculate the area of $\int_{t_k}^{t_{k+1}} q\left(p^k, e(p^k, u^0)\right) \Delta \cdot dt$, the trapezoidal method is used. This leads to the equation (3.1.17).

As with algorithms described previous parts, the accuracy of the Vartia algorithm mainly depends on the size of intervals. In general, the precision of this method is not good enough compared to the RESORT and the RK-4th order algorithm. Moreover, this algorithm requires

---

let $x_0 = a$, $x_1 = b$, and $h = b - a$, and used the linear Lagrange polynomial $P_1(x)$ as an approximation of $f(x)$

$$P_1(x) = \frac{(x - x_1)}{(x_0 - x_1)} f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} f(x_1) \approx f(x)$$

If applying the Lagrange polynomial to the integral, then we have

$$\int_b^a f(x)dx = \int_{x_0}^{x_1} \left[\frac{(x - x_1)}{(x_0 - x_1)} f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} f(x_1)\right] dx$$

Above equation implies that

$$\begin{aligned} \int_b^a f(x)dx &= \left[\frac{(x - x_1)^2}{2(x_0 - x_1)} f(x_0) + \frac{(x - x_0)^2}{2(x_1 - x_0)} f(x_1)\right]_{x_0}^{x_1} \\ &= \frac{(x_1 - x_0)}{2}\left[f(x_0) + f(x_1)\right] \\ &= \frac{1}{2}\left[f(x_0) + f(x_1)\right] \cdot h \end{aligned}$$

This is called the Trapezoidal rule because when f is a function with the positive values, $\int_a^b f(x)dx$ is approximated by the area in a trapezoid.

the additional procedure, the iterated method, to calculate the compensated income since both sides of the equation (3.1.17) has the unknown value $e(p^k, u^0)$. To estimated the the unknown value $e(p^{k+1}, u^0)$ in equation (3.1.17), the iterated method is required until the estimated $e(p^{k+1}, u^0)$ at step k is converged to the real value of $e(p^{k+1}, u^0)$. This iteration makes the algorithm a little bit slow compared to other numerical methods though the convergence has achieved with the rapid speed.

However, this algorithm can be applied that the demand system was very inelastic (the stiff equation case). In this case, numerical algorithms introduced in the previous section can not be applied in order to recover the compensated income from the demand function since the approximation procedure could not converge to the solution but explode.

3.1.5. Adams Fourth-Order Predictor-Corrector Method

Until now, the one-step method[10] is investigated. Although these one-step methods might be used for evaluating information at points between p^k and p^{k+1}, they do not retain information for the future approximation since all information used in one-step methods are obtained within sub intervals that the solution is approximated. For example, the Euler and Taylor method refer to only one previous point and its derivatives to estimate the value of the current step, and the RK-4th order method takes some intermediate steps, for example, a mid point, to calculate the current value, but finally discards all previous information before taking a second step.

However, the approximation solution is available at every mesh points $p^0, p^1, \ldots, p^k$ before the approximation at p^{k+1} is achieved. So it seems reasonable to employ methods that use more than one previous points when approximating the solution. In fact, the method using more than one points to determine the approximation at the current step is called the multistep method. The multistep method attempts to utilize the efficiency by keeping and using the information from previous steps rather than discarding it. Consequently, the multistep method refers to several previous points to calculate the next value so the approximation of the current value can be a linear combination of previous values.

[10]Because the approximation for the mesh point t_{i+1} involves information from only one for the previous mesh point t_i

The multistep method for solving the ODE system is generally defined by the following difference equation. In order to find the approximation $e(p^{k+1}, u^0)$ at the mesh point t_{i+1}, for $i = m-1, m, \ldots, N-1$

$$\begin{aligned} e(p^{k+1}, u^0) &= a_{m-1} \cdot e(p^k, u^0) + a_{m-2} \cdot e(p^{k-1}, u^0) + \cdots a_0 \cdot e(p^{k+1-m}, u^0) \\ &\quad + [b_m \cdot q\left(p^{k+1}, e(p^{k+1}, u^0)\right) + b_{m-1} \cdot q\left(p^k, e(p^k, u^0)\right) \\ &\quad + \cdots b_0 \cdot q\left(p^{k+1-m}, e(p^{k+1-m}, u^0)\right)] \cdot \Delta \end{aligned} \tag{3.1.18}$$

Where $\Delta = \dfrac{(p^1 - p^0)}{N}$, and k is the number of steps. Coefficients a_0, $a_1, \ldots, a_{m-1}$, and b_0, $b_1, \ldots, b_m$ presented in equation (3.1.18) are all constant, and starting values $e(p^0, u^0) = \alpha$, $e(p^1, u^0) = \alpha_1,\ e(p^2, u^0) = \alpha_2, \ldots, e(p^{m-1}, u^0) = \alpha_{m-1}$ are specified[11].

To obtain the numerical solution using the multistep method, the implicit and the explicit method are generally used. The method is called the *explicit (or forward) method* when $b_m = 0$. w_{i+1} is expressed as a linear combination of known past values and functions of w_i, as a consequence, w_{i+1} is computed easily in terms of previously determined values. On the contrary, when $b_m \neq 0$, the method is called *the implicit (or iterative) method*. It is based on the fact that w_{i+1} occurs on both sides of equation (3.1.18). In fact, the equation (3.1.18) is specified only implicitly. Therefore, the iterative procedure is generally required in order to compute the w_{i+1}.

Several multistep methods in numerical analysis are commonly used as the solution for the ODE system. One is called the Adams-Bashforth (A-B) method family which used the explicit method to estimate the value of the current step, the other is called Adams-Moulton (A-M) method family which used the implicit method to evaluate the present value.

The following is examples of the multistep explicit and implicit method.

[11]In general, we assume that the initial value of the ordinary differential equation is known. However, except the initial value, there are no assumptions on the next values of the mesh points. In order to initiate the values in algorithm, another algorithms including the RK method or some other one-step techniques are adapted.

Example. Explicit Method

The A-B 4th order method is described by the following,

$$
\begin{aligned}
e(p^{k+1}, u^0) &= e(p^k, u^0) + \frac{1}{24}[55q\left(p^k, e(p^k, u^0)\right) \\
&\quad -59q\left(p^{k-1}, e(p^{k-1}, u^0)\right) + 37q\left(p^{k-2}, e(p^{k-2}, u^0)\right) - 9q\left(p^{k-3}, e(p^{k-3}, u^0)\right)] \cdot \Delta
\end{aligned}
$$

and

$$e(p^0, u^0) = \alpha_0, e(p^1, u^0) = \alpha_1, e(p^2, u^0) = \alpha_2, \text{ and } e(p^3, u^0) = \alpha_3$$

Where $\Delta = \dfrac{(p^1 - p^0)}{N}$, and k is the number of steps.

Example. Implicit Method

The A-M 4th order method is described by the following,

$$
\begin{aligned}
e(p^{k+1}, u^0) &= e(p^k, u^0) + \frac{h}{24}[9q\left(p^{k+1}, e(p^{k+1}, u^0)\right) \\
&\quad +19q\left(p^k, e(p^k, u^0)\right) - 5q\left(p^{k-1}, e(p^{k-1}, u^0)\right) + q\left(p^{k-2}, e(p^{k-2}, u^0)\right)] \cdot \Delta
\end{aligned}
$$

$$\text{and } e(p^0, u^0) = \alpha_0, e(p^1, u^0) = \alpha_1, \text{ and } e(p^2, u^0) = \alpha_2$$

Where $\Delta = \dfrac{(p^1 - p^0)}{N}$, and k is the number of steps.

The Adams 4th Order Predictor-Corrector algorithm

It is well known that the implicit method provides better results than the explicit method of the same order but the implicit method is not always possible[12]. To derive the better estimation result, the combination of the implicit and the explicit method are often used to compute the solution for the ODE system. In common, the explicit method is used to "predict" the value of w_{i+1}, and then the predicted value is used inside the implicit formula to

[12] It is not surprising facts that the greater difficult of using the implicit formula with $b_m \neq 0$ are existed when calculating the w_{i+1}.

"correct" the estimated value later. The combination of the explicit and the implicit technique leads so called *a predictor-corrector method*.

The *Adams 4th Order Predictor-Corrector Method* is a multistep method since it used more than one mesh points to approximate the solution for the ODE system. Moreover, this algorithm is the predictor-corrector method since it employed both the explicit and implicit method in algorithm. In fact, Adams 4th Order Predictor-Corrector Method used total three methods, including both the fourth order A-B and A-M method, to generate the algorithm.

The general description of Adams Fourth-Order Predictor-Corrector Method is following. First, the RK-4th order algorithm is used for calculating four starting values. Second, the explicit method (A-B method) is employed to predict an approximation, and finally, the implicit multistep method (A-M method) is used to improve approximations obtained by the explicit method. For simplicity, following notations, $e\left(p^0, u^0\right) = w_0$, $e\left(p^1, u^0\right) = w_1$, and $e\left(p^k, u^0\right) = w_k$ are used.

- Step.1 : the RK method of the order four
  - To calculate the first four starting values, $e(p^0, u^0) = \alpha_0$, $e(p^1, u^0) = \alpha_1$, $e(p^2, u^0) = \alpha_2$, and $e(p^3, u^0) = \alpha_3$, the RK-4th order method are used.
  - Four staring values are used in the explicit Adamas-Bashforth method as staring values
- Step.2 :The explicit Adams-Bashfouth method as a predictor,
  - An approximation $w_4^{(0)}$ which is a prediction of $e\left(p^4, u^0\right)$ was calculated by the explicit Adams-Bashfouth method, where superscript (0) means the number of iterations

$$\begin{aligned} w_4^{(0)} &= e\left(p^3, u^0\right) + \frac{1}{24}[55q\left(p^3, e(p^3, u^0)\right) \\ &\quad -59q\left(p^2, e(p^2, u^0)\right) + 37q\left(p^1, e(p^1, u^0)\right) - 9q\left(p^0, e(p^0, u^0)\right)] \cdot \Delta \end{aligned}$$

  - where $\Delta = \dfrac{(p^1 - p^0)}{N}$

- Step.3 : The implicit Admas-Moulton method as a corrector.
  - The estimated values $w_4^{(1)}$, which is a approximation of $e\left(p^4, u^0\right)$, is improved by inserting $w_4^{(0)}$ inside of the three-step implicit Adams-Moulton method, where superscript (1) means the number of iterations

$$
\begin{aligned}
w_4^{(1)} &= e\left(p^3, u^0\right) + \frac{1}{24}[9q\left(p^4, w_4^{(0)}\right) \\
&\quad +19q\left(p^3, e(p^3, u^0)\right) - 5q\left(p^2, e(p^2, u^0)\right) + q\left(p^1, e(p^1, u^0)\right)] \cdot \Delta
\end{aligned}
$$

  - where $\Delta = \dfrac{(p^1 - p^0)}{N}$

- Step.4 : Iterating this procedure (from Step.1 to Step.3) until an approximation of $e\left(p^k, u^0\right) = w_k^{(j)}$ are calculated.
  - In fact, the improved approximations to $e\left(p^{k+1}, u^0\right)$ could be obtained by iterating the implicit Adams-Moulton formula

$$
\begin{aligned}
w_{k+1}^{(j+1)} &= e\left(p^k, u^0\right) + + \frac{1}{24}[9q\left(p^{k+1}, w_{k+1}^{(j)}\right) \\
&\quad +19q\left(p^k, e(p^k, u^0)\right) - 5q\left(p^{k-1}, e(p^{k-1}, u^0)\right) + q\left(p^{k-2}, e(p^{k-2}, u^0)\right)] \cdot \Delta
\end{aligned}
$$

  - where j is the iteration size.

The $\{w_{k+1}^{(j+1)}\}$ converges to the approximation given by the implicit formula rather than to the solution $e\left(p^{k+1}, u^0\right)$, and it is usually more efficient to use a reduction in the step size if improved accuracy is needed.

4. Demand systems

In this section, we introduce two demand systems such as the Rotterdam and the Almost Ideal demand system. Each demand system is estimated as a starting point to calculate the compensated income, the measure of welfare changes, and the cost-of-living index.

4.1. The Almost Ideal Demand system

4.1.1. Model description

The almost ideal demand system of Deaton and Muellbauer (1980) is fully consistent with the economic theory, and possesses properties of the exact aggregation. The AI demand system is developed from a particular cost function taken from the general class of "price-independent, generalized logarithmic" or PIGLOG cost function which permits the exact aggregation over consumers.

The PIGLOG cost function is defined by

$$\ln C\,(p, U) = (1 - U) \ln \{a(p)\} + U \ln \{b(p)\} \tag{4.1.1}$$

where p is a $n \times 1$ vector of unit price, and U denotes the utility index which can be scaled under cases of subsistence $(U = 0)$ and bliss$(U = 1)$. In addition, $\ln a(p)$ and $\ln b(p)$ regarded as costs of the subsistence and the bliss have the specific flexible functional form;

$$\ln a(p) = \alpha_0 + \sum_k \alpha_k \ln(p_k) + \frac{1}{2} \sum_k \sum_j \gamma^*_{kj} \ln p_k \ln p_j \tag{4.1.2}$$

and

$$\ln b(p) = \ln a(p) + \beta_0 \prod_k p_k^{\beta_k} \tag{4.1.3}$$

Applying equation (4.1.2) and (4.1.3) into equation (4.1.1) leads to the AIDS cost function.

$$\ln C(p,U) = \alpha_0 + \sum_k \alpha_k \ln(p_k) + \frac{1}{2}\sum_k \sum_j \gamma^*_{kj} \ln p_k \ln p_j + U\beta_0 \prod_k p_k^{\beta_k} \tag{4.1.4}$$

Share equations which are an objective function in AIDS can be derived easily after applying the Shepard's lemma to the AIDS cost function. This leads to a set of share equations:

$$w_i = \alpha_i + \sum_j \gamma_{ij} \ln p_j + \beta_i(\ln M - \ln P) \tag{4.1.5}$$

where w_i is the expenditure share on commodity i, $\log p_j$ and M is the log price of commodities and total expenditure respectively.

The price index $\ln P$ is given by

$$\ln P = \alpha_0 + \sum_i \alpha_i \ln p_i + \frac{1}{2}\sum_i \sum_j \gamma_{ij} \ln p_i \ln p_j \tag{4.1.6}$$

The constant term α_i denotes the value of the budget share when prices and total outlays remained constant. The effect of price changes is transmitted through the parameter γ_{ij}. And the parameter β_i measures the effect of changes in total real expenditure. The consistency with economic theory requires that parameters of the system satisfy the following adding-up, homogeneity and symmetry restrictions:

- Adding-up and Homogeneity Condition :

$$\sum_i \alpha_i = 1, \quad \sum_j \gamma_{ij} = 0, \quad \sum_i \beta_i = 0$$

- Symmetry Condition :

$$\gamma_{ij} = \gamma_{ji}$$

For the estimation purpose, the LA-AIDS model, the linear approximation of the AI demand system, is often used. The Stone index is used in order to approximate the translog price index which are use in the AI demand system;

$$\ln P^S = \sum_i w_i \ln p_i \tag{4.1.7}$$

In general, the estimation of the LA-AIDS model is simpler than that of the AI demand system. However, there is a well-known couple of demerits using the LA-AIDS instead of the AIDS. First, according to Moschini (1995) using the Stone index can cause the simultaneity bias problem when estimating the demand system, because the expenditure share w_i appears on both sides of share equations. Moreover, deriving and calculating elasticities of the LA-AIDS is more complicated than that of the AI demand system. In fact, the expenditure share w_i on the Stone index causes these difficulties. In general, the advantage in estimation of LA-AI demand system is offset by difficulties in deriving elasticities. More details are described in the next section.

4.1.2. Income and Price Elasticities of the AI demand System

Results in Barnett and Seck (2007) and Green and Alston(1990) are used to investigate elasticities of the AI demand system. Income elasticities of the AIDS model can be derived after applying the logarithm rule $dz = z \cdot d \log z$, on the definition of income elasticities,

$$\eta_{iM} = \frac{\partial \log q_i}{\partial \log M} = 1 + \frac{\partial \log w_i}{\partial \log M} = 1 + \frac{1}{w_i} \frac{\partial w_i}{\partial \log M} \tag{4.1.8}$$

where q_i is the ordinary demand, w_i is the expenditure share of good i, and M is the total expenditure.

Uncompensated price elasticities can be calculated with the same manner,

$$\eta_{ij} = \frac{\partial \log q_i}{\partial \log p_j} = -\delta_{ij} + \frac{\partial \log w_i}{\partial \log p_j} = -\delta_{ij} + \frac{1}{w_i} \frac{\partial w_j}{\partial \log p_j} \tag{4.1.9}$$

where p_i is the price of good i , and $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ is the Kronecker delta term.

In case of AI demand system, partial derivatives of the budget share (4.1.5) *w.r.t* the income and the price j are

$$\frac{\partial w_i}{\partial \log M} = \beta_i \tag{4.1.10}$$

and

$$\frac{\partial w_j}{\partial \log p_j} = \gamma_{ij} - \beta_i \frac{\partial \log P}{\partial \log p_j} = \gamma_{ij} - \beta_i \left(\alpha_j + \sum \gamma_{jk} \log p_k \right) \tag{4.1.11}$$

where $\frac{\partial \log P}{\partial \log p_j} = \alpha_j + \sum \gamma_{jk} \log p_k$.

Therefore, plugging equation (4.1.10) and (4.1.11) into the equation (4.1.8) and equation (4.1.9), then income and the uncompensated elasticities of AI demand system become

$$\eta_{iM} = 1 + \frac{\beta_i}{w_i} \tag{4.1.12}$$

and

$$\eta_{ij} = -\delta_{ij} + \frac{\gamma_{ij} - \beta_i \left(\alpha_j + \sum_k \gamma_{jk} \ln p_k \right)}{w_i} \tag{4.1.13}$$

where $\alpha_j + \sum_k \gamma_{jk} \ln p_k = w_j - \beta_j (\ln M - \ln P)$.

On the contrary, differently from the AIDS case, obtaining elasticities of LA-AI demand system is more complicated and sensitive. This mainly depends on that the LA-AI demand system used the Stone index as the price index instead of the translog price index used in AI demand model.

Partial derivatives of the budget share equation (4.1.5) in case of LA-AI demand system, *w.r.t* the income and the price j are

$$\frac{\partial w_i}{\partial \log M} = \beta_i - \beta_i \frac{\partial \log P^S}{\partial \log M} \tag{4.1.14}$$

and

$$\frac{\partial w_j}{\partial \log p_j} = \gamma_{ij} - \beta_i \frac{\partial \log P^S}{\partial \log p_j} \tag{4.1.15}$$

respectively.

In order to evaluate the equation (4.1.14) and (4.1.15), exact values of partial derivatives on the Stone index are required. In fact, partial derivatives of the Stone index *w.r.t* the income and the price j are following

$$\frac{\partial \log P^S}{\partial \log M} = \sum_j \log p_j \frac{\partial w_j}{\partial \log M} = \sum_j w_j \log p_j (\eta_{jM} - 1) \tag{4.1.16}$$

and

$$\frac{\partial \log P^S}{\partial \log p_j} = w_j + \sum_k w_k \log p_k \frac{\partial \log w_k}{\partial \log p_j} = w_j + \sum_k w_k \log p_k (\eta_{kj} + \delta_{kj}) \tag{4.1.17}$$

Finally, applying the equation (4.1.16) and (4.1.17) into the equation (4.1.8) and equation (4.1.9), then income and uncompensated elasticities of the LA-AI demand system are obtained by

$$\eta_{iM} = 1 + \frac{\beta_i}{w_i} \left[1 - \sum w_j \log p_j (\eta_{jM} - 1)\right] \tag{4.1.18}$$

and

$$\eta_{ij} = -\delta_{ij} + \frac{\gamma_{ij}}{w_i} - \frac{\beta_i \left(w_j + \sum_k w_k \log p_k (\eta_{kj} + \delta_{kj})\right)}{w_i} \tag{4.1.19}$$

respectively.

In fact, the form of elasticities in the LA-AI demand system are more complicated those of the AI demand system. However, practically, in order to calculate income and uncompensated elasticities of the LA-AI demand system, the matrix form of the equation (4.1.20) and (4.1.21) are required. After rearranging components in above equations, income and uncompensated price elasticities can be expressed in matrix form as

$$\mathbf{N} = \left[\mathbf{BC} + \mathbf{I}\right]^{-1} \mathbf{B} \tag{4.1.20}$$

and

Table 4.1.: Income and uncompensated elasticities of AI and LA-AI demand system

<table>
<tr><th colspan="3"></th><th colspan="2">Demand System</th></tr>
<tr><th colspan="3"></th><th>AIDS</th><th>LA-AIDS</th></tr>
<tr><td rowspan="3">Elasticities</td><td colspan="2">Income</td><td>$\eta_{iM} = \frac{\beta_i}{w_i} + 1$</td><td>$\eta_{iM} = 1 + \frac{\beta_i}{w_i}\left[1 - \sum w_j \log p_j (\eta_{jM} - 1)\right]$</td></tr>
<tr><td rowspan="2">Price</td><td>Uncompensated</td><td>$\eta_{ij} = -\delta_{ij} + \frac{\gamma_{ij} - \beta_i \left(\alpha_j + \sum_k \gamma_{jk} \ln p_k\right)}{w_i}$</td><td>$\eta_{ij} = -\delta_{ij} + \frac{\gamma_{ij}}{w_i} - \frac{\beta_i \left(w_j + \sum_k w_k \log p_k (\eta_{kj} + \delta_{kj})\right)}{w_i}$</td></tr>
<tr><td>Compensated</td><td>$\eta^*_{ij} = \eta_{ij} + w_j \left(1 + \frac{\beta_i}{w_i}\right)$</td><td>$\eta^*_{ij} = \eta_{ij} + w_j \left(1 + \frac{\beta_i}{w_i}\left[1 - \sum w_j \log p_j (\eta_{jM} - 1)\right]\right)$</td></tr>
</table>

$$\mathbf{E} = \left[\mathbf{BC} + \mathbf{I}\right]^{-\mathbf{1}} \left[\mathbf{A} + \mathbf{I}\right] - \mathbf{I} \tag{4.1.21}$$

where $\mathbf{A} = -\delta_{ij} + \dfrac{\gamma_{ij}}{w_i} - \beta_i \dfrac{w_j}{w_i}$, $\mathbf{B} = \dfrac{\beta_i}{w_i}$, and $\mathbf{C} = w_j \ln P_j$ which is a $(1 \times n)$ matrix.

In addition, the compensated elasticities (η^*_{ij}) can be derived from the Slutsky matrix easily. Since Slutsky matrix is defined by

$$\frac{\partial h_i(p,u)}{\partial p_j} = \frac{\partial q_i(p,M)}{\partial p_j} + \frac{\partial q_i(p,M))}{\partial M} \cdot q_j(p,M) \tag{4.1.22}$$

after applying the logarithm rule, $dz = z \cdot d \log z$.

Rearranging of the above equation (4.1.22), then compensated price elasticities are defined by

$$\eta^*_{ij} = \eta_{ij} + w_j \cdot \eta_{iM} \tag{4.1.23}$$

From this definition, compensated price elasticities of AIDS and LA-AIDS can be obtained. For AIDS case, compensated elasticities is

$$\eta^*_{ij} = \eta_{ij} + w_j \left(1 + \frac{\beta_i}{w_i}\right) \tag{4.1.24}$$

and compensated elasticities for the LA-AI demand system is

$$\eta^*_{ij} = \eta_{ij} + w_j \left(1 + \frac{\beta_i}{w_i}\left[1 - \sum w_j \log p_j (\eta_{jM} - 1)\right]\right) \tag{4.1.25}$$

4.1.3. Deriving the functional form of the ordinary demand from the AI demand model

The functional form of the AIDS ordinary demand can be derived from share equations. Let's consider the general form of expenditure share,

$$w_i = \frac{p_i q_i(p, M)}{M} \tag{4.1.26}$$

where $M = \sum_i p_i q_i$ is the total expenditure, and differently called the true compensated income.

Plugging equation (4.1.5) into the equation (4.1.26), then following functional relationships of the share of good i are obtained.

$$w_i = \frac{p_i q_i(p, M)}{C} = \alpha_i + \sum_j \gamma_{ij} \ln p_j + \beta_i \ln \left(\frac{M}{P} \right) \tag{4.1.27}$$

From the above expenditure share equation (4.1.27), we have

$$q_i(p, M) = \frac{M}{p_i} \left[\alpha_i + \sum_j \gamma_{ij} \ln p_j + \beta_i \ln \left(\frac{M}{P} \right) \right] \tag{4.1.28}$$

This equation (4.1.28) is the function form of the ordinary demand in the AIDS, and can be used for calculating the compensated income in numerical algorithms.

4.2. Rotterdam Model

4.2.1. Model description

The Rotterdam demand system was introduced by Theil (1967) and Barten (1966) as an estimation method for the differential approach. This demand system expressed the rate of change in the quantity demanded of each commodity, weighted by the corresponding budget share, as a function of the rate of change in prices and the rate of changes in total real spending (income).

The Rotterdam model is started from the utility maximization. Differently from the functional form of demand system, the Rotterdam demand system has no specific assumptions on the utility and the expenditure itself.

Let's consider the utility maximization problem

$$\begin{aligned} &\max_q u(q) \\ \text{s.t.} \quad &\sum p_i \cdot q_i = M \end{aligned} \tag{4.2.1}$$

where p_i and q_i are the price and the demand quantity for good for good i, and M is total expenditure.

Above utility maximization problem can be solved after applying the first order condition and the Lagrangian method which is a auxiliary function. It leads the following ordinary demand function which is the solution for the utility maximization problem,

$$q_i = q_i(p, M) \quad i = 1, 2, \ldots .n, \tag{4.2.2}$$

Applying total differentiation on the equation (4.2.1) leads to

$$dq_i = \frac{\partial q_i(\cdot)}{\partial M} dM + \sum_j \frac{\partial q_i(\cdot)}{\partial p_j} dp_j \quad i = 1, 2, \ldots .n \tag{4.2.3}$$

In fact, this equation (4.2.3) is the starting point of the differential approach and the Rotterdam demand system. Multiplying both sides of equation (4.2.3) by p_i/M and applying the logarithm rule, $dz = z \cdot d \log z$, then the following logarithmic differential equation is obtained

$$w_i d \log q_i = \theta_i d \log M + \sum_j \frac{p_i p_j}{M} \cdot \frac{\partial q_i}{\partial p_j} \cdot d \log p_j \tag{4.2.4}$$

where $w_i = \dfrac{p_i q_i}{M}$ is the budget share of the ith good, and $\theta_i = \dfrac{p_i \partial q_i}{\partial M}$ is the marginal budget shares of commodity i.

From the Slutsky Substitution Matrix, equation (4.1.22), we know that

$$\frac{\partial q_i}{\partial p_j} = \frac{\partial h_i}{\partial p_j} - \frac{\partial q_i}{\partial M} q_j \tag{4.2.5}$$

where $h(p, u)$ is the Hicksian compensated demand function.

Plugging equation (4.2.5) into equation (4.2.4), then this procedure yields

$$w_i d\log x_i = \theta_i \left(\underbrace{d\log M - \sum w_j d\log p_j}_{A} \right) + \sum_j \frac{p_i p_j}{M} \cdot \frac{\partial h_i}{\partial p_j} \cdot d\log p_j \tag{4.2.6}$$

Where $\sum_j w_j d\log p_j = d\log P$.

The $d\log P$, the budget share multiplied by weighted average of log price changed, is called the Divisia price index which is proposed by Divisia (1925), French economist. Similarly, the Divisia Quantity index are defined as the budget share multiplied by weighted average of log quantity change instead of the price, $\sum_j w_j d\log q_j = d\log Q$. The relationship between Divisia price and quantity index can be find in the budget constraint in the problems of the utility maximization.

Applying total differentiation and logarithm rule, $dz = z \cdot d\log z$, on the budget constraint, then we have

$$\sum_j w_j d\log p_j + \sum_j w_j d\log q_j \quad = \quad d\log M \tag{4.2.7}$$

From the definition of Divisia indices, above equation can be described in terms of the Divisia quantity index. That is

$$d\log Q = d\log M - d\log P \tag{4.2.8}$$

Moreover, $\frac{p_i p_j}{M}$ is the constant because we assume that prices and expenditures are known, as a result, the functional form for the i th equation in this demand system takes the following form

$$w_i d\log q_i = \theta_i d\log Q + \sum_{j=1} \pi_{ij} d\log p_j \tag{4.2.9}$$

where $\pi_{ij} = \frac{\partial h_i}{\partial p_j} = \left(\frac{\partial q_i}{\partial p_j} + -\frac{\partial q_i}{\partial M} q_j \right)$ is the Slutsky matrix.

Since the data obtained from the real world is not continuous but discrete, the discrete form of equation (4.2.9) are required in order to estimate the demand system. In fact, the finite changes version of equation (4.2.9) is

$$w_{it}^* Dq_{it} = \theta_i DQ_t + \sum_{j=1} \pi_{ij} Dp_{jt} \tag{4.2.10}$$

where $w_i^* = \dfrac{1}{2}(w_{it} + w_{i,t-1})$ denotes the average budget share of commodity i at time t and $t-1$, $DQ_t = \log Q_t - \log Q_{t-1}$, $Dq_{it} = \log q_{it} - \log q_{i,t-1}$, and $Dp_{it} = \log p_{it} - \log p_{i,t-1}$. This equation (4.2.10) is called *the Absolute Price Version of the Rotterdam Model* when the coefficient θ_i and π_{ij} are treated as constants.

Parameters in the Rotterdam model have the following interpretations. π_{ij} , the Slutsky matrix, measures total substitution effects on the demand for good i of a compensated change in the price of commodity j. And θ_i, marginal budget shares, measures marginal effects on the budget when the income changes. Differently from budget shares, marginal budget shares are not always positive (for example, $\theta_i < 0$ if good i is an inferior good) but, like budget shares, sum to unity, $\sum_i \theta_i = 1$.

Two set restrictions on parameters of the Rotterdam model are required to suffice the theory of the demand. The first set of week restrictions on the consumer demand is the adding-up and the homogeneity of the demand equations

- Adding up conditions

$$\sum_{i=1}^{n} \theta_i = 1 \quad \text{and} \quad \sum_{j=1}^{n} \pi_{ij} = 0 \;\; \text{for all } i = 1, 2, \ldots, n$$

- Demand homogeneity required

$$\sum_{j=1} \pi_{ij} = 0 \quad \text{for all } i = 1, \cdots, n$$

In addition, under the standard assumptions of economic theory, θ_i and π_{ij} must also satisfy the following strong restrictions

- Slutsky symmetry

$$\pi_{ij} = \pi_{ji}$$

- Concavity

$$[\pi_{ij}] \text{ is negative semi-definite}$$

Above restrictions are not independent. Typically, the adding-up, the homogeneity, and the symmetry are imposed in estimation while the negative semi-definiteness of $[\pi_{ij}]$ matrix is empirically confirmed. Parameters are assumed to be constant under the Rotterdam parametrization, and the average budget shares over the sample period are used.

4.2.2. Income and Price Elasticities of the AI demand System

Compared to the AI demand system, one advantage using the Rotterdam model is that income and price elasticities can be calculated with the convenient manner since the objective function contains the log differential in equation.

From the Rotterdam demand equation (4.2.9), income and compensated price elasticities are easily calculated

$$\eta_{iM} = \frac{d \log q_i}{d \log Q} = \frac{\theta_i}{w_i}, \quad , i = 1, \cdots, n$$

$$\eta^*_{ij} = \frac{d \log q_i}{d \log p_j} = \frac{\pi_{ij}}{w_i}, \quad , i, j = 1, \cdots, n$$

Moreover, from the equation (4.1.23), the uncompensated price elasticities are obtained

$$\eta_{ij} = \eta^*_{ij} - w_j \cdot \eta_{iM}$$

4.2.3. Calculating ordinary demand From the Rotterdam model

The Rotterdam model is employed to demonstrate the applicability of the numerical algorithm in real situation where there existed no exact form of the expenditure or demand function. Though no the functional form of ordinary demand is existed in the Rotterdam model, the quantity demand at time t can be calculated.

There are two way to calculated the estimated demand quantity of the Rotterdam model. The first method starts from the equation (4.2.10), which is the absolute version of the Rotterdam model. The equation (4.2.10) can be reorganized by the following.

$$
\begin{aligned}
w^*_{it} Dq_{it} \quad &= \quad \theta_i DQ_t + \sum_{j=1} \pi_{ij} Dp_{jt} \\
&\rightarrow \quad w^*_{it}(\log q_{it} - \log q_{i,t-1}) = \theta_i DQ_t - + \sum_j \pi_{ij}(\log p_{it} - \log p_{i,t-1}) \\
&\rightarrow \quad \log q_{it} - \log q_{i,t-1} = \left[\underbrace{\frac{\theta_i DQ_t + \sum_j \pi_{ij}(\log p_{it} - \log p_{i,t-1})}{w^*_{it}}}_{A} \right] \qquad (4.2.11) \\
&\rightarrow \quad \log q_{it} = \underbrace{A + \log q_{i,t-1}}_{B} \\
&\therefore \quad q_{it} = exp(B)
\end{aligned}
$$

Using above procedure and estimated parameters from the Rotterdam model, whole calculation can be done from the first period to the last period. In fact, we could calculate the estimated demand quantity of good i at each time period of the Rotterdam model. However, this estimated demand quantity is defined as a function of Divisia quantity index and differences of log prices instead of the expenditure and prices[1]. Therefore, we can not directly apply this estimated demand quantity to the numerical algorithm in order to recover the compensated income. Therefore, the new approach is required to do it.

We considered the way of calculating the demand quantity from estimated expenditure shares as with the AI demand system. According to the Theil(1981), in the Rotterdam model, the estimated share $\hat{w}_{it}$ is measured by

$$
\hat{w}_{it} = w_{i,t-1} + \text{prediction of } \Delta w_{it} \qquad (4.2.12)
$$

where $\hat{w}_{it}$ is the estimated shares of w_{it}, and the prediction of Δw_{it} implies the changes in the budget share from period $t-1$ to t.

Theil(1981) described how to calculate the prediction of Δw_{it} in his book. This can be accomplished after applying total differentiation and the following logarithm rule, $z \cdot d \log z = dz$, on the expenditure share. Then, we have

[1] In order to apply the ordinary demand to the numerical method, the ordinary demand should be the function of the expenditure and price.

$$\Delta w_{it} = w_{it} d \log p_{it} + w_{it} d \log q_{it} - w_{it} d \log M_t \tag{4.2.13}$$

Plug in the equation (4.2.13) into the equation (4.2.12), finally we have

$$\hat{w}_{it} = w_{i,t-1} + w_{it} d \log p_{it} + w_{it} d \log q_{it} - w_{it} d \log M_t \tag{4.2.14}$$

From this equation (4.2.14), the estimated share at time t is evaluated. To obtain more accurate estimated share, updated values instead of original expenditure shares are used. For example, first, we calculate the estimated value of second period using the original expenditure share values such that

$$\hat{w}_{i2} = w_{i1} + \text{prediction of } \Delta w_{i2}$$

Then, except for the first periods, updated values of expenditure shares are kept using until time t

$$\hat{w}_{it} = \hat{w}_{it-1} + \text{prediction of } \Delta w_{it}$$

However, values we actually needed in numerical algorithm is values of the demand quantity so demand quantities are calculated by applying equation (4.2.14) to the share equation (4.2.12). Finally, ordinary demand quantities of good i at time t are given by

$$\begin{aligned} q_{it}(p_t, M_t) &= \frac{w_{it} M_t}{p_{it}} \\ &= \frac{[w_{i,t-1} + w_{it} d \log p_{it} + w_{it} d \log q_{it} - w_{it} d \log M_t] \times M_t}{p_{it}} \end{aligned} \tag{4.2.15}$$

Unlike the AI demand system, obtaining the functional form of ordinary demand from the Rotterdam system is impossible, however, we could finally calculate the demand quantity of good i at each time period t. This implies that quantity demand at time t can be applied to numerical approximation method in order to calculate the compensated income.

5. Application 1: Numerical Approximation to Calculate the Cost-of-living and Its Empirical Solution

5.1. Introduction

The purpose of this paper is to propose an empirically feasible method to measure the cost-of-living index based on the RK-4th order algorithm which is the solution of the ordinary differential equation, and demonstrate that the newly proposed approximation method can be an alternative way to calculate the cost-of-living index instead of the Vartia algorithm. For this research purpose, the accuracy and the applicability of the RK-4th order algorithm are investigated. First, the simulation method is employed in order to check the accuracy of the algorithm. For this, one of the flexible functional form demand system : Almost Ideal Demand System (AIDS)[1] is employed. The cost-of-living index[2] and the conventional price indices[3] are calculated by using the numerical algorithm[4] and estimated parameters from the AI demand system. From the comparison between estimated cost-of-living indices and conventional price indices, the accuracy and the power of the newly proposed numerical method are confirmed. Second, the demand system which has no closed form solution is considered in order to identify the applicability of the newly proposed numerical method for calculating cost-of-living in the empirical world, since the exact functional form of the

[1]The demand quantity function are required to calculate the compensated income using the numerical algorithm. The AIDS demand system has the exact form of the expenditure function, and this implies that the exact form of the demand quantity function can be derived from it. See the demand model description in Chapter 4 in this paper for more details.

[2]Here, true cost-of-living means the Konüs price index. It measure the proportional change in the minimum cost of maintaining some fixed level of economic welfare when prices change. More description can be found in Section 2.

[3]In general, when we mentioned the conventional price index, it include the following the price indices: the Laspeyres, Paasche, Fisher Ideal, and Törnqvist Price Index. In this paper, these price indices are calculated by the each specific formulas described in Section 2.

[4]To calculate the cost-of-living index, we hire the Vartia's algorithm which is a standard method in calculating the cost-of-living index and the RK-4th algorithm as an alternative of Vartia's method. More detail can be find in Section 3, or Chapter 3.

demand does not exited in general. For this, the Rotterdam model is employed since this theoretical demand system does not have any assumption on the utility or the expenditure so no specific ordinary demand function can be derived from the demand system. We could demonstrate that the numerical algorithm can be applied in any kinds of situations, whether the exact form of demand system existed or not, to calculate the cost-of-living indices after the comparison of the cost-of-living indices between AI and Rotterdam model.

The cost-of-living index (COLI) measures the relative cost-of-living over times. The usefulness of the COLI is that this index provides a method for comparing costs of maintaining a certain level of living in different demographic groups, years, and geographic areas. In economic fields, generally, the Konüs cost-of-living index, which is defined as the ratio of the compensated incomes corresponding to the terminal and initial prices given the reference level of the utility, is used in order to calculate the cost-of-living index. Based on the definition, the Konüs cost-of-living index can be calculated if compensated incomes are known at each time period.

There are well known two methods for calculating the compensated income from the ordinary demand function. One is that the exact form of expenditure or indirect utility functions existed. For example, the Translog or the AI demand system[5]. In this case, the demand function with an explicit parametric form can be derived from the indirect utility or the expenditure functions. Therefore, the compensated income can be evaluated by using this ordinary demand function with the exact from. In the same manner, the Konüs cost-of-living index can be evaluated after the compensated income is obtained. In this paper, this case is used

[5]Both the Translog and the AI demand system have the exact functional form demand system. In case of the Translog demand system, it has the functional form of the indirect utility function such as

$$\ln V(p,y) = \alpha_0 + \sum_k \alpha_k \ln\left(\frac{p_k}{y}\right) + \frac{1}{2}\sum_k\sum_j \gamma_{kj} \ln\left(\frac{p_k}{y}\right)\ln\left(\frac{p_j}{y}\right)$$

From the above equation, the compensated income function under the restrictions on translog demand system can be derived.

$$e(p,u) = \frac{\sum_k \alpha_k \ln p_k + \frac{1}{2}\sum_k\sum_j \gamma_{kj} \ln p_k \ln p_k - \ln V}{-1 + \sum_k\sum_j \gamma_{kj} \ln p_k}$$

Similarly, the expenditure function of the AI demand system is defined by

$$\ln C(p,U) = \alpha_0 + \sum_k \alpha_k \ln(p_k) + \frac{1}{2}\sum_k\sum_j \gamma^*_{kj} \ln p_k \ln p_j + U\beta_0 \prod_k p_k^{\beta_k}$$

in order to check the accuracy of the numerical algorithm. However, in general, closed form solutions for parametric demand systems are not available. This implies that the compensated income can not be obtained from the above method. An alternative way to calculate the compensated income and the cost-of-living index is required.

Many economists including Breslaw and Smith (1995), Hausman and Newey(1995), Dumagan and Mount (1997), and Vartia (1983) proposed the different ways to calculate the compensated income using the numerical algorithm[6]. The solution they proposed are related the integrability problem[7] in economics. Simply, the integrability problem can be described by followings: "Given a system of demand functions, Can a utility function or expenditure function are derived?" In economics, the integrability problem is described as the system of the partial differential equations, more specifically, represents that relationship between the partial derivative of the expenditure function and ordinary demand functions with the given initial value. In fact, the integrability problem can be solved if demand functions suffices integrability conditions defined by the Slutsky matrix.

In this paper, as the solution of the ordinary differential approach, we propose the RK-4th order algorithm that is the general solution of the ODE problems in numerical analysis, and represents that this algorithm can be the alternative tools to calculate the cost-of-living indices after comparing its precision and accuracy to other conventional price indices, and the Vartia algorithm. The reminder of this paper is organized as follows: section 5.2 and 5.3, we will consider the numerical method related to the computation of the cost-of-living index, and briefly introduce the definition of conventional price indices. Section 5.4 discusses data, and reports the estimation results of both AIDS and Rotterdam demand system. In addition, we report the index numbers based on numerical approximation, and describes the general results. In section 5.5, the summary and conclusion of this research are provided.

[6]Numerical algorithms are related to the solution of the system of the ordinary differential equation with the initial values. Using numerical approximation methods, the compensated income(expenditure function) can be recovered from the ordinary demand system. Numerical methods can be applied both 1) there exists the closed form of the ordinary demand function and 2) there is no closed form solutions of demand systems. Especially, numerical approximation methods are very useful in the case under no closed form solution of demand system since, generally, if there existed the parametric form demand system such as the AIDS or more specifically defined demand system, one could directly calculated the compensated income using the ordinary demand function without numerical approximations. However, if there is no closed form solution of the demand function, it is impossible to calculate the compensated income. In this case, the numerical algorithm can be applied to calculate the compensated income.

[7]See the Varian(1992) and Mas-Colell and et el(1995) for more details

5.2. Price indices

A price index is a normalized average of prices during a given interval of time. It is a statistically designed to help to compare how these prices, taken as a whole, differ between time periods or geographical locations. In this reason, the price index is generally used for measuring the price level or a cost-of-living of the economy. In this section, we briefly introduced the concept of the Konüs type price index, and the conventional price indices including Laspeyres, Paasche, Fisher, and Törnqvist price indices.

5.2.1. Konüs cost-of-living Index

A cost-of-living index is a theoretical price index which measures the relative cost of living over time or regions, and also allows to measure differences in the price of goods and services. In economic fields, the Konüs price index, at least, is considered as the ideal for measuring the cost-of-living. A *Konüs cost-of-living index* proposed by the Russian economist Konüs(1924) is defined as the ratio of the minimum expenditure required to attain a particular level of satisfaction in two price situations, a comparison period and a base period.

$$I(p^0, p^1, u^0) = \frac{e(p^1, u^0)}{e(p^0, u^0)} \tag{5.2.1}$$

where u^0 is the reference level of utility, e is the minimum expenditure of the utility level at u when the consumer is facing a price vector p , and the index I represents the minimum expenditure(cost) of the reference level of the utility when a consumer is facing with two different price level, the price vector p^1 and p^0.

By the definition, the Konüs cost-of-living index $I(p^0, p^1, u^0)$ can be calculated at each data point if the $e(p, u)$ for a given time period and a given level of utility is known. In addition, this index can be used an alternative way to evaluate the Hicksian welfare change because it is defined as the difference between the compensated incomes in the different period.

5.2.2. Conventional Price indices

Four price indices in the economic field are used for a method of estimating the cost-of-living. Two most basic formulas used for calculating the cost-of-living are the Paasche and the Laspeyres price index. The Fisher and the Törnqvist price formula are also frequently used to calculate the cost-of-living.

In general, the Laspeyres index tends to overstate inflation while the Paasche index tends to understate it, because these indices do not explain that consumers typically react to price changes by changing quantities they purchase. As a solution of these problems, later, the Fisher and the Törnqvist price index formulas are appeared.

We will start from the definition of the expenditure share to explain price indices. The expenditure share are defined by the following way :

$$w_i^0 = \frac{q_i^0 p_i^0}{\sum_i q_i^0 p_i^0}, \quad w_i^T = \frac{q_i^T p_i^T}{\sum_i q_i^T p_i^T}, \quad \sum_i w_i^0 = 1 \quad \sum_i w_i^T = 1 \tag{5.2.2}$$

where 0 and T represent the initial and the terminal time period, respectively.

Combining the equation (5.2.1) and (5.2.2) leads to the Laspeyres and the Paasche price indices, respectively.

The Laspeyres price index is defined by

$$I_p^L = \frac{e(p^T, q^0, u^0)}{e(p^0, q^0, u^0)} = \frac{p^T \cdot q^0}{p^0 \cdot q^0} = \sum_i w_i^0 \left(\frac{p_i^T}{p_i^0} \right) \tag{5.2.3}$$

where p^0 and p^T is the vector of prices at the initial and terminal time period. As the same manner, Paasche index is defined by

$$I_p^p = \frac{e(p^T, q^T, u^T)}{e(p^0, q^T, u^T)} = \frac{p^T \cdot q^T}{p^0 \cdot q^T} = \frac{1}{\sum_i w_i^T \left(\frac{p_i^0}{p_i^T} \right)} \tag{5.2.4}$$

The Fisher and the Törnqvist price index formulas are used to be presented by the combination of the Laspeyres and Paasche price indices. First, the Fisher Ideal Index (I_p^F) is the geometric mean of those price indices, and is represented by

$$I_p^F = (I_p^L I_p^P)^{1/2} = \exp\left\{\frac{1}{2}\ln\left(\sum_i w_i^0\left(\frac{p_i^T}{p_i^0}\right)\right) - \frac{1}{2}\ln\left(\sum_i w_i^T\left(\frac{p_i^0}{p_i^T}\right)\right)\right\} \tag{5.2.5}$$

By contrast to the Fisher Ideal Index, the relationship between Törnqvist price Index and above two indices are not clear but the close relationship with the Laspeyres and the Paasche price indices can be identified from the following definition.

$$I_p^T = \exp\left\{\frac{1}{2}\sum_i\left(w_i^0 + w_i^T\right)\ln\left(\frac{p_i^T}{p_i^0}\right)\right\} \tag{5.2.6}$$

In fact, the Törnqvist price Index (I_p^T) can be considered as a discrete approximation to a continuous Divisia price index[8].

Here, above four price indices are used when checking the accuracy of the RK-4th order algorithm.

5.3. Numerical Solutions

The expenditure function can be recovered from the ordinary demand function if it suffices integrability conditions described in Chapter 2. As a solution method, numerical methods are discussed in this section. Numerical algorithms for the ODE system, in general, consist of

[8] A Divisia index is a theoretical index number series for continuous-time data on prices and quantities of goods exchanged. It is designed for incorporate quantity and price changes over time.

The Divisia indices are developed from the following line integral such as

$$\ln(p(t)\cdot x(t)) - \ln(p(t^{'})\cdot x(t^{'})) = \int_{t'}^{t}\frac{d\ln(p(\tau)\cdot x(\tau))}{d\tau}d\tau$$

where τ is time variable, $p(t)$ and $x(t)$ is a price vector and quantity vector respectively.
From above equation, the following equation are obtained

$$\frac{d\ln(p(\tau)\cdot x(\tau))}{d\tau} = \sum_{n=1} s_n(\tau)\frac{d\ln x_n(\tau)}{d\tau} + \sum_{n=1} s_n(\tau)\frac{d\ln p_n(\tau)}{d\tau}$$

This means that the growth rate of value are divide by
the Divisia price index

$$\frac{d\ln P^{Div}(t)}{dt} = \sum s_i(t)\left[\frac{d\ln p_i(t)}{dt}\right]$$

and
the Divisia Quantity index

$$\frac{d\ln Q^{Div}(t)}{dt} = \sum s_i(t)\left[\frac{d\ln x_i(t)}{dt}\right]$$

two elements. One is a discretization concept that steps are equally distributed where solutions are evaluated. Another is an associated difference equations which determine the solution using the previous value. Practically, except for some special cases where the differential equations is linear, the ODE system cannot be solved exactly so numerical approximations are required for identifying the solution of it.

5.3.1. Vartia Algorithm

This numerical approximation method proposed by Vartia(1983) is the numerical integration of equation (2.1.1) in Chapter 2. Vartia's algorithm entails calculating compensated income at step k of the algorithm. This algorithm is useful to calculate the compensated income under no closed form solutions for the underlying utility or expenditure function. It is defined by the following

$$e(p^k, u^0) = e(p^{k-1}, u^0) + \frac{1}{2}\sum_i \Big(q_i(p^k, e(p^k, u^0) + q_i(p^{k-1}, e(p^{k-1}, u^0) \Big) \cdot (p_k - p_{k-1}) \quad (5.3.1)$$

where $p_k = \frac{k}{N} \cdot (p^1 - p^0)$ is the price increment, and N is the number of steps in the algorithm.

The accuracy of the Vartia algorithm depends mainly on the size of intervals. This method provides an extremely rapid numerical procedure that can be easily be programmed, and allows to calculate welfare changes from the demand function. However, this algorithm requires the additional procedure, the iteration method, to calculate the compensated income, since the unknown value $e(p^k, u^0)$ is appeared in both sides of the equation (5.3.1) simultaneously. To estimated the unknown value $e(p^k, u^0)$, the iterated method is used until the estimated value of $e(p^k, u^0)$ is converged to the real value of $e(p^k, u^0)$. This iteration makes the algorithm slow though the convergence has achieved with the rapid speed.

5.3.2. Runge-Kutta 4th order Algorithm

In numerical analysis, the RK-4th order algorithm is regarded as the standard solution of the ODE system with initial value. This technique was developed around 1900 by the German mathematicians C. Runge and M.W. Kutta.

It is defined by the following

$$e(p^k, u^0) = e(p^{k-1}, u^0) + \frac{1}{N}\frac{K_1 + 2K_2 + 2K_3 + K_4}{6} \tag{5.3.2}$$

where

$$\begin{aligned}
K_1 &= \sum_i q_i\left(p^{k-1}, e(p^{k-1}, u^0)\right) \cdot (p_i^1 - p_i^0) \\
K_2 &= \sum_i q_i\left(\frac{p^{k-1} + p^k}{2}, e(p^{k-1}, u^0) + \frac{1}{2}K_1\right) \cdot (p_i^1 - p_i^0) \\
K_3 &= \sum_i q_i\left(\frac{p^{k-1} + p^k}{2}, e(p^{k-1}, u^0) + \frac{1}{2}K_2\right) \cdot (p_i^1 - p_i^0) \\
K_4 &= \sum_i q_i\left(p^k, e(p^{k-1}, u^0) + k3\right) \cdot (p_i^1 - p_i^0)
\end{aligned}$$

where k is the step number, $p^k = p(t_k)$, and $t_k = \frac{k}{N}$.

Similar to the Vartia algorithm, the accuracy of the RK-4th order algorithm depends mainly on the size of intervals, and this algorithm is easy to be programmed with any computational programing language. Compared to the Vartia method, this method seems to be very awkward and complicated at the starting point. However, this algorithm provides a simpler way to solve ODE system with higher precisions, because this algorithm do not need an iteration procedure to calculate the undetermined in the equation. In fact, the $e(p^k, u^0)$ can be calculated only using values from the previous step such as $e(p^{k-1}, u^0)$, $q_i(p^{k-1}, e(p^{k-1}, u^0))$, and p. In addition, on the contrary to the Taylor method[9], this RK-4th algorithm contains no evaluations of the second or higher order derivatives though it has the high-order local truncation error of the Taylor method.

[9]See the Chapter 3 for more details.

5.4. Estimation results and the cost-of-living index

The annual time series data of the *Consumer Expenditure Survey (CE)* from 1984 to 2008 are used for the estimation purpose. These time series variables were constructed by the Bureau of Labor Statistics (BLS) in the U.S., and were taken directly from the BLS web site (`http://www.bls.gov/cex/`). According to categories of the *Consumer Price Index (CPI)*, total consumption is divided into seven commodities: "Food and Beverage", "Housing", "Apparel", "Transportation", "Health Care", "Recreation" and "Other goods and services".

A measure of relative prices is required in order to estimate a system of demand. *The Consumer Price Index (CPI*[10]*)* reported by The BLS is employed as a measure of the average changes in prices over time. Similar to the CE Data, the CPI were directly collected from the BLS web site (`http://www.bls.gov/cpi/`), too. Among the CPIs in BLS Web pages, the *CPI for all urban consumers (CPI-U)* are used in the estimation since it covers 87% of the population of the U.S..

During the data period, total expenditure and the expenditure of each categories have gradually increased. And the CPI-U also has a increasing tendency at each data period. But, according to the Figure 5.4.1, the share of each commodities present different properties during the same data period. For example, shares of "Food and Beverage", "Apparel" and "Other goods and service" have generally decreased, but, in case of "House" and "the Health Care", shares of these commodities have increased during the same period. From variations of shares in each item category, changes of consumption patterns of the U.S. consumer are identified.

In order to demonstrate the power of the Runge-Kutta-4th algorithm, we start with estimating the AI demand system, which is one of the flexible functional form demand systems in economic theory. Since the AI demand system has the functional form of the cost, the compensated income and cost-of-living can be algebraically calculated. The equation (4.1.5) in Chapter 4 is estimated for the data period from 1984 to 2008 with the translog price index. The Iterated Linear Least Squares estimator (ILLE) procedure proposed by Blundell and Robin (1999) is used for estimating the non-linear simultaneous system with six share

[10]When CPI is calculated, it is based on a fixed 'market basket' of goods and services.

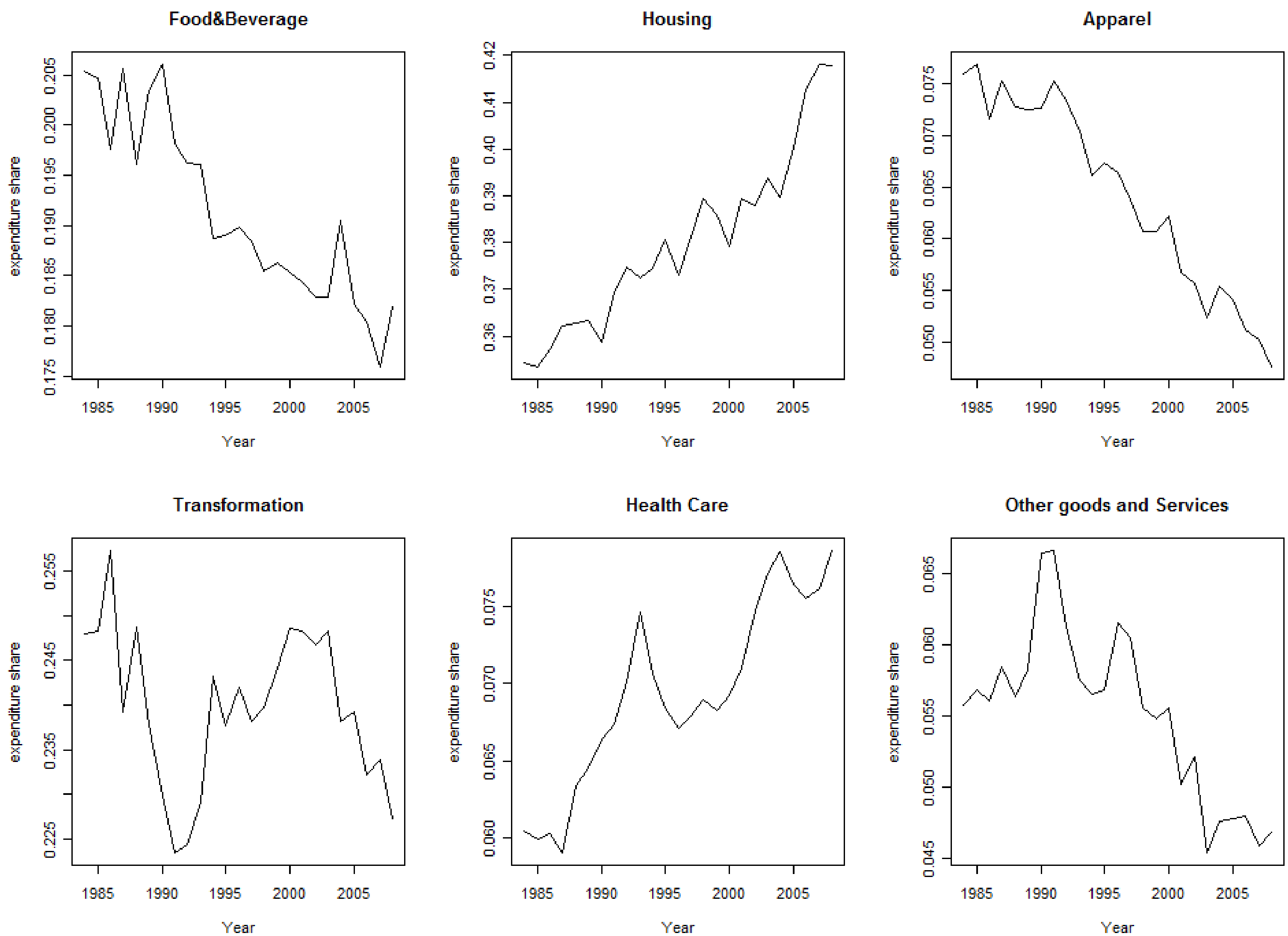

Figure 5.4.1.: The expenditure share according to Time change

equations (the share, "Other good and service" is dropped to avoid a covariance matrix singularity). The last share equation is recovered by using the restriction on the AI demand system. The `systemfit` and `micEcon` packages in R (`http://CRAN.R-project.org/`) are employed for the AI demand system estimation and the matrix calculation respectively.

Estimation results are reported in Table 5.1. Income parameters (β_i) of the estimated demand system measure the effect of changes in total expenditure. Commodities are necessities when the value of income parameter is negative, and are luxuries in the opposite case. According to the Table 5.1. "Food and Beverage", "Health Care", "Recreation" and "Other goods and services" turned out as necessities. In addition, income parameters (β_i) provide the information whether the preference is homothetic or not. In fact, the AI demand system is non-homothetic if and only if β_i is not zero. Based on this fact, it turned out that the estimated demand system has the non-homothetic preference.

Table 5.1.: Estimated Coefficient for the AI demand system from 1984 to 2008

	Estimated Coefficients									R^2
	α_i	γ_{i1}	γ_{i2}	γ_{i3}	γ_{i4}	γ_{i5}	γ_{i6}	γ_{i7}	β_i	
w_1	0.6254	0.0467	-0.0044	-0.0250	0.0681	-0.1225	0.0123	0.0249	-0.0801	0.865
	(0.423)	(0.118)	(0.090)	(0.051)	(0.118)	(0.075)	(0.063)	(0.069)	(0.079)	
w_2	0.2294	-0.0044	0.0658	-0.0395	-0.0410	0.1317	-0.0294	-0.0833	0.0158	0.938
	(0.597)	(0.090)	(0.098)	(0.030)	(0.112)	(0.069)	(0.041)	(0.044)	(0.112)	
w_3	-0.0055	-0.0250	-0.0395	0.0492	0.0288	-0.0221	-0.0243	0.0328	0.0125	0.963
	(0.534)	(0.051)	(0.030)	(0.013)	(0.103)	(0.052)	(0.015)	(0.012)	(0.100)	
w_4	-0.6610	0.0681	-0.0410	0.0288	-0.1730	0.0397	0.0415	0.0360	0.1721	0.215
	(0.608)	(0.118)	(0.112)	(0.103)	(0.219)	(0.129)	(0.117)	(0.132)	(0.116)	
w_5	0.4652	-0.1225	0.1317	-0.0221	0.0397	-0.0160	-0.0012	-0.0097	-0.0779	0.824
	(0.538)	(0.075)	(0.069)	(0.052)	(0.129)	(0.110)	(0.059)	(0.067)	(0.102)	
w_6	0.2243	0.0123	-0.0294	-0.0243	0.0415	-0.0012	0.0239	-0.0228	-0.0275	0.336
	(0.517)	(0.063)	(0.041)	(0.015)	(0.117)	(0.059)	(0.024)	(0.024)	(0.097)	
w_7	0.1221	0.0249	-0.0833	0.0328	0.0360	-0.0097	-0.0228	0.0220	-0.0150	0.798
	(0.571)	(0.069)	(0.044)	(0.012)	(0.132)	(0.067)	(0.024)	(0.028)	(0.109)	

The income and uncompensated price elasticities in the AI demand system are defined[11] by

- Income elasticities

$$\eta_{iM} = \frac{\beta_i}{w_i} + 1 \tag{5.4.1}$$

[11]For more details, see the Chapter 4.

- Uncompensated price elasticities

$$\eta_{ij} = -\delta_{ij} + \frac{\gamma_{ij} - \beta_i \left(\alpha_j + \sum_k \gamma_{jk} \ln p_k\right)}{w_i} \tag{5.4.2}$$

where $\alpha_j + \sum_k \gamma_{jk} \ln p_k = w_j - \beta_j (\ln M - \ln P)$,

$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ is the Kronecker delta term .

The full set of elasticities, income and uncompensated price elasticities, calculated with estimated parameters is reported in Table 5.2. Here, FB, H, A, TR, HC, R, and O represent "Food and Beverage", "Housing", "Apparel", "Transportation", "Health Care", "Recreation" and "Other goods", respectively. Estimated own-price elasticities has all negative values, as required by demand theory. This assumption is satisfied in all of the commodities considered here.

Table 5.2.: Income and Uncompensated price elasticities based on AI demand system

	Elasticities							
	Price							Income
	η_{i1}	η_{i2}	η_{i3}	η_{i4}	η_{i5}	η_{i6}	η_{i7}	η_{iM}
FB	-0.4688	0.1018	-0.1441	0.0941	-0.4895	0.1635	0.2000	0.5431
H	-0.0380	-0.8284	-0.1101	-0.0861	0.3477	-0.0911	-0.2382	1.0442
A	-0.5424	-0.7235	-0.1698	0.6199	-0.4680	-0.4524	0.5260	1.2101
TR	-0.1408	-0.3938	0.1303	-1.2757	-0.1737	0.0277	0.0629	1.7631
HC	-1.1945	2.3688	-0.3452	-0.1615	-0.6953	0.2277	0.0034	-0.2033
R	0.4285	-0.3302	-0.3708	0.3612	0.1731	-0.5506	-0.2935	0.5823
O	0.6545	-1.5422	0.6385	0.5144	-0.0548	-0.3849	-0.5340	0.7085

After estimating the AI demand system, according to the procedure demonstrated in the section 4.2.3, using estimated parameters which is represented in Table 5.2, The Expenditure, Shares, and Quantities from the estimated parameters of AI demand system are recovered. Finally, the true cost-of-living index are calculated based on the AI demand system. Next, for the comparison purpose, the conventional price indices including Laspeyres, Paasche, Fisher, and Törnqvist index are calculated using formula (5.2.3), (5.2.4), (5.2.5), and (5.2.6). All indices are calculated after assuming that the utility level U^0 is held fixed at 100 though prices change. In addition, cost-of-living indices using the Vartia and the RK-4th order algorithm are also calculated with the same objective and assumption. All evaluated indices were reported in Table 5.3 for the comparison. In addition, substitution biases of each index are reported in Table 5.4.

Not surprisingly, all price indices calculated is very similar to the true cost of living index based on the AI demand system. According to the Table 5.4, the substitution bias of each index is not larger than 4.0 point. Moreover, as expected from the general results from economic theory about price index[12], it turned out that the Laspeyres index has positive substitution biases and substitution biases of the Paasche index are negative. From this we could confirm the fact that the Laspeyres and Paasche indices is an upper and lower bound of the true cost-of-living index. However, we know that the estimated AI demand system represents the non-homothetic preference instead of the homothetic preference according to estimated income parameters[13]. In fact, the Laspeyres and Paasche indices do not necessarily bound of the true cost-of-living index under non-homothetic preference.

In addition, according to the Table 5.4, Törnqvist indices generally have a negative substitution bias but Fisher indices have a positive substitution bias as usual, and these two indices are, however, very much smaller than those of the Laspeyres and Paasche indices. Törnqvist indices have the smallest bias among them except for the indices calculated using numerical algorithms. For Fisher indices, we can confirm that biases are some exact combinations, the geometric mean, of the Laspeyres and Paasche biases. For Törnqvist indices, However, it is not clear how it's substitution biases are related algebraically to biases of the Laspeyres and Paasche indices because there is no explicit relationship between these three indices.

[12]In general, if the consumer's preference are homothetic, the Laspeyres index is the upper bound, and the Paasche index is the lower bound of the true cost-of-living, respectively.

[13]In general, preferences are not homothetic in most actual situations since household budget studies and most time-series evidence of systematic change in expenditure patterns show that Engel curve is not straight but also increasing. From this point of view, the homothetic preference is very unrealistic and unattractive assumption on the consumer demand.

Table 5.3.: True COLI and Price Indices based on the AI Demand System

	Price							Cost-of-Living Index						
year	p_1	p_2	p_3	p_4	p_5	p_6	p_7	True COLI for $U^0 =$ 100	Rk-4th	Vartia	Laspeyres	Paasche	Fisher	Törnqvist
1984	103.20	103.60	102.10	103.70	106.80	60.10	107.90	100	100	100	100	100	100	100
1985	105.60	107.70	105.00	106.40	113.50	66.40	114.50	104.2214	104.2214	104.222	104.2402	104.2029	104.2216	104.2214
1986	109.10	110.90	105.90	102.30	122.00	65.20	121.40	106.6329	106.6329	106.6335	106.7012	106.5624	106.6318	106.6326
1987	113.50	114.20	110.60	105.40	130.10	67.60	128.50	110.8386	110.8386	110.8394	110.9569	110.7184	110.8376	110.838
1988	118.20	118.50	115.40	108.70	138.60	74.90	137.00	116.2018	116.2018	116.2032	116.4174	115.9834	116.2002	116.2007
1989	124.90	123.00	118.60	114.10	149.30	80.00	147.70	122.2808	122.2808	122.2826	122.5935	121.9709	122.2818	122.2792
1990	132.10	128.50	124.10	120.50	162.80	79.70	159.00	128.6191	128.6191	128.6216	129.0255	128.2276	128.6259	128.6171
1991	136.80	133.60	128.70	123.80	177.00	82.80	171.60	134.3283	134.3283	134.3314	134.9386	133.7498	134.3429	134.3249
1992	138.70	137.50	131.90	126.50	190.10	84.10	183.30	138.4341	138.4341	138.4376	139.2545	137.6752	138.4626	138.4297
1993	141.60	141.20	133.70	130.40	201.40	90.70	192.90	143.2041	143.2041	143.208	144.2213	142.274	143.2443	143.1984
1994	144.90	144.80	133.40	134.30	211.00	92.70	198.50	146.9654	146.9654	146.9695	148.0802	145.9639	147.0182	146.9592
1995	148.90	148.50	132.00	139.10	220.50	94.50	206.90	151.0586	151.0586	151.0629	152.2999	149.9655	151.1282	151.052
1996	153.70	152.80	131.70	143.00	228.20	97.40	215.40	155.5008	155.5008	155.5053	156.8435	154.3281	155.5807	155.4934
1997	157.70	156.80	132.90	144.30	234.60	99.60	224.80	159.2559	159.2559	159.2606	160.7332	157.9672	159.3442	159.2467
1998	161.10	160.40	133.00	141.60	242.10	101.10	237.70	162.2472	162.2472	162.252	164.0413	160.6776	162.3507	162.2329
1999	164.60	163.90	131.30	144.40	250.60	102.00	258.30	166.1465	166.1465	166.1519	168.3435	164.2679	166.2932	166.1301
2000	168.40	169.60	129.60	153.30	260.80	103.30	271.10	171.551	171.551	171.5572	173.8562	169.6322	171.7312	171.5375
2001	173.60	176.40	127.30	154.30	272.80	104.90	282.60	176.6748	176.6748	176.6813	179.2153	174.5991	176.8921	176.6567
2002	176.80	180.30	124.00	152.90	285.60	106.20	293.20	179.9124	179.9124	179.9191	182.8446	177.5699	180.1879	179.8869
2003	180.50	184.80	120.90	157.60	297.10	107.50	298.70	184.035	184.035	184.042	187.0687	181.6861	184.3578	184.0088
2004	186.60	189.50	120.40	163.10	310.10	108.60	304.70	189.1383	189.1383	189.1457	192.3274	186.7211	189.5035	189.1111
2005	191.20	195.70	119.50	173.90	323.20	109.40	313.40	195.1114	195.1114	195.1198	198.4194	192.6397	195.5082	195.0893
2006	195.70	203.20	119.50	180.90	336.20	110.90	321.70	201.1875	201.1875	201.1962	204.5677	198.705	201.6151	201.1664
2007	203.30	209.59	119.00	184.68	351.05	111.44	333.33	207.2424	207.2424	207.2516	210.9407	204.5894	207.7408	207.2175
2008	214.23	216.26	118.91	195.55	364.07	113.25	345.38	215.0524	215.0524	215.0624	219.0608	212.1807	215.5933	215.029

Table 5.4.: Substitution biases based on AI demand system

years	True COLI	Substitution Biases					
		Rk-4th	Vartia	Laspeyres	Paasche	Fisher	Törnqvist
1984	100.0000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
1985	104.2214	0.000000	0.000600	0.018830	-0.018491	0.000168	-0.000005
1986	106.6329	0.000000	0.000600	0.068344	-0.070463	-0.001082	-0.000310
1987	110.8386	0.000001	0.000800	0.118315	-0.120192	-0.001003	-0.000552
1988	116.2018	0.000001	0.001400	0.215592	-0.218417	-0.001615	-0.001089
1989	122.2808	0.000001	0.001800	0.312631	-0.309927	0.000956	-0.001665
1990	128.6191	0.000002	0.002500	0.406446	-0.391468	0.006870	-0.002014
1991	134.3283	0.000003	0.003100	0.610310	-0.578455	0.014613	-0.003339
1992	138.4341	0.000003	0.003500	0.820391	-0.758855	0.028516	-0.004443
1993	143.2041	0.000003	0.003900	1.017247	-0.930111	0.040259	-0.005661
1994	146.9654	0.000003	0.004100	1.114792	-1.001482	0.052847	-0.006207
1995	151.0586	0.000003	0.004300	1.241279	-1.093139	0.069563	-0.006558
1996	155.5008	0.000003	0.004500	1.342675	-1.172683	0.079913	-0.007375
1997	159.2559	0.000003	0.004700	1.477292	-1.288765	0.088262	-0.009177
1998	162.2472	0.000004	0.004800	1.794052	-1.569602	0.103514	-0.014353
1999	166.1465	0.000004	0.005400	2.196963	-1.878638	0.146677	-0.016418
2000	171.551	0.000004	0.006200	2.305153	-1.918888	0.180146	-0.013541
2001	176.6748	0.000004	0.006500	2.540481	-2.075758	0.217304	-0.018113
2002	179.9124	0.000004	0.006700	2.932205	-2.342436	0.275585	-0.025493
2003	184.035	0.000005	0.007000	3.033756	-2.348865	0.322803	-0.026142
2004	189.1383	0.000005	0.007400	3.189021	-2.417281	0.365139	-0.027225
2005	195.1114	0.000005	0.008400	3.307988	-2.471674	0.396800	-0.022088
2006	201.1875	0.000006	0.008700	3.380237	-2.482428	0.427596	-0.021116
2007	207.2424	0.000006	0.009200	3.698289	-2.653057	0.498345	-0.024942
2008	215.0524	0.000006	0.010000	4.008374	-2.871737	0.540875	-0.023358

As expected, the RK-4th algorithm shows outstanding approximation results. The Table 5.3 and 5.4 are represented well these results. Compared to conventional price indices such as Fisher and Törnqvist, substitution biases of numerical algorithms are much smaller than those of conventional price indices. From this, the precision and power of numerical algorithms are confirmed. Similarly, substitution biases of the RK-4th order algorithm is the smallest among all of indices. In addition, the level of accuracies of this algorithm has improved by increasing the number of price steps. In fact, the RK-4th order algorithm represents the better performance than any other indices. From these results, we could conclude that the RK-4th algorithm has enough power to calculate the cost-of-living index.

Until this point, the power of the RK-4th algorithm are presented. The thing remained is to identify the applicability of the RK-4th order algorithm in empirical fields. To achieve this goal, the ability of numerical algorithm has to be demonstrated under no exact functional form of the expenditure or utility functions. For this, the Rotterdam model is selected. The main reason the Rotterdam model hired in this paper is that the Rotterdam model is based on the neoclassical consumer theory, and allows to impose and test cross-equation restrictions such as symmetry although this model avoids the necessity of using a particular functional form for the utility function.

Based on the equation (4.2.10), parameters of the demand system are estimated. The Iterated Seemingly Unrelated Regression(ISUR) are used for estimating the system of six share equations with the same data using in estimating the AI demand system. The share named 'Other good and service' is dropped to avoid a covariance matrix singularity. Parameters of the last equation is recovered by using the restriction on the Rotterdam demand system.

Estimation results are reported in Table 5.5. Marginal budget shares of commodities (θ_i) have positive signs. This means that the estimated commodities are all normal goods except for Health Care. To estimate the Rotterdam model, generally, adding-up, linear homogeneity, and symmetry are imposed in the estimation. But the negative semi-definite restriction on the $[\pi_{ij}]$ matrix has to be empirically confirmed. The negative semi-definite restriction on $[\pi_{ij}]$ matrix are simply checked from the Table 5.5. From the fact that All six π_{ii} are negative for each commodity and the matrix $[\pi_{ij}]$ is singular, we roughly confirmed that the negative semi-definite restriction on $[\pi_{ij}]$ matrix are satisfied.

Table 5.5.: Estimated coefficient for Absolute Rotterdam demand system

	Estimated coefficient							
	θ_i	π_{i1}	π_{i2}	π_{i3}	π_{i4}	π_{i5}	π_{i6}	π_{i7}
$w_1 d \log q_1$	0.0690	-0.0585	-0.0025	-0.0287	0.0653	-0.0529	0.0454	0.0320
	(0.060)	(0.075)	(0.082)	(0.023)	(0.041)	(0.029)	(0.026)	(0.034)
$w_2 d \log q_2$	0.3417	-0.0025	-0.0271	-0.0242	0.0399	0.0833	-0.0231	-0.0464
	(0.084)	(0.082)	(0.143)	(0.033)	(0.060)	(0.038)	(0.041)	(0.050)
$w_3 d \log q_3$	0.0711	-0.0287	-0.0242	-0.0100	0.0542	-0.0152	-0.0073	0.0312
	(0.029)	(0.023)	(0.033)	(0.014)	(0.019)	(0.012)	(0.013)	(0.015)
$w_4 d \log q_4$	0.3425	0.0653	0.0399	0.0542	-0.2019	-0.0107	0.0088	0.0445
	(0.098)	(0.041)	(0.060)	(0.019)	(0.069)	(0.018)	(0.024)	(0.026)
$w_5 d \log q_5$	-0.0005	-0.0529	0.0833	-0.0152	-0.0107	-0.0434	0.0321	0.0067
	(0.027)	(0.029)	(0.038)	(0.012)	(0.018)	(0.023)	(0.014)	(0.021)
$w_6 d \log q_6$	0.0772	0.0454	-0.0231	-0.0073	0.0088	0.0321	-0.0290	-0.0270
	(0.039)	(0.026)	(0.041)	(0.013)	(0.024)	(0.014)	(0.022)	(0.018)
$w_7 d \log q_7$	0.0992	0.0320	-0.0464	0.0312	0.0445	0.0067	-0.0270	-0.0409
		(0.034)	(0.050)	(0.015)	(0.026)	(0.021)	(0.018)	

Income and uncompensated price elasticities of the Rotterdam model can be easily calculated as follows

$$\eta_{iy} = \frac{d \log x_i}{d \log Q} = \frac{\theta_i}{w_i}, \quad , i = 1, \cdots, n \tag{5.4.3}$$

$$\eta_{ij} = \frac{d \log x_i}{d \log p_j} = \frac{\pi_{ij}}{w_i}, \quad , i = 1, \cdots, n \tag{5.4.4}$$

Elasticities of the Rotterdam model are reported in Table 5.6. Like the AI demand system, the Rotterdam model is well estimated since estimated own-price elasticities in the Rotterdam model are all negative as required by the demand theory.

Table 5.6.: Income and Uncompensated price Elasticities of Absolute Rotterdam demand system

	Elasticities							
	Income	Price						
	η_{iM}	η_{i1}	η_{i2}	η_{i3}	η_{i4}	η_{i5}	η_{i6}	η_{i7}
FB	0.3860	-0.3278	-0.0142	-0.1608	0.3653	-0.2959	0.2543	0.1790
H	0.9629	-0.0071	-0.0764	-0.0681	0.1125	0.2349	-0.0650	-0.1308
A	1.1845	-0.4787	-0.4025	-0.1669	0.9036	-0.2532	-0.1217	0.5192
TR	1.5301	0.2915	0.1783	0.2423	-0.9022	-0.0478	0.0393	0.1986
HC	-0.0077	-0.8161	1.2865	-0.2346	-0.1652	-0.6698	0.4955	0.1037
R	1.1611	0.6837	-0.3469	-0.1099	0.1323	0.4830	-0.4360	-0.4063
O	1.9259	0.6212	-0.9013	0.6052	0.8635	0.1304	-0.5244	-0.7947

Similar to the case of the AI demand system, the cost-of-living index using the Rotterdam demand estimation are calculated. Since the Rotterdam model has no exact form of the expenditure or demand function, estimated shares based on the equation (4.2.12) are employed to calculate the demand quantity at each time period. First, estimated shares of each good are calculated at each time period. Second, the definition of expenditure shares ($w_i = \frac{p_i q_i}{M}$) is applied to calculate the ordinary demand quantity at each time period[14]. Finally, using this ordinary demand quantity, compensated income are evaluated.

Cost-of-living indices of the Rotterdam model calculated by the Vartia and the RK-4th order algorithm are reported in Table 5.7. Not surprisingly, all price indices from each demand system using numerical algorithms are very similar, though price indices from the Rotterdam demand system which has no specific functional form are not much closer the true cost-of-living compared to those from the AI demand system. It seems to be partly because we calculated the true cost-of-living based on the AI demand system. Table 5.7 shows that, in Rotterdam case, RK-4th algorithm has better performance than the Vartia algorithm. Actually, the precision of the cost-of-living index calculating from RK-4th algorithm is more accurate than that of the Vartia algorithm. From this, we could represent the applicability of the RK-4th order algorithms in the empirical fields.

[14]See section 4.2.3 for more details.

Table 5.7.: True cost-of-living index using RK-4th algorithm : AIDS vs Rotterdam base year=1984

	True COLI	Rotterdam demand system		AI demand system	
		RK-4th	Vartia	RK-4th	Vartia
1984	1.00000	1.00000	1.00000	1.00000	1.00000
1985	1.04221	1.03891	1.03892	1.04221	1.04222
1986	1.06633	1.05287	1.05288	1.06633	1.06634
1987	1.10839	1.09164	1.09165	1.10839	1.10839
1988	1.16202	1.14066	1.14067	1.16202	1.16203
1989	1.22281	1.19757	1.19759	1.22281	1.22283
1990	1.28619	1.25861	1.25864	1.28619	1.28622
1991	1.34328	1.31079	1.31082	1.34328	1.34331
1992	1.38434	1.34860	1.34863	1.38434	1.38438
1993	1.43204	1.39277	1.39281	1.43204	1.43208
1994	1.46965	1.42867	1.42872	1.46965	1.46970
1995	1.51059	1.46849	1.46853	1.51059	1.51063
1996	1.55501	1.51053	1.51058	1.55501	1.55505
1997	1.59256	1.54408	1.54413	1.59256	1.59261
1998	1.62247	1.56569	1.56574	1.62247	1.62252
1999	1.66147	1.60144	1.60149	1.66147	1.66152
2000	1.71551	1.65895	1.65901	1.71551	1.71557
2001	1.76675	1.70250	1.70257	1.76675	1.76681
2002	1.79912	1.72523	1.72530	1.79912	1.79919
2003	1.84035	1.76491	1.76498	1.84035	1.84042
2004	1.89138	1.81409	1.81416	1.89138	1.89146
2005	1.95111	1.87863	1.87871	1.95111	1.95120
2006	2.01188	1.93962	1.93970	2.01188	2.01196
2007	2.07242	1.99448	1.99457	2.07242	2.07252
2008	2.15052	2.07327	2.07336	2.15052	2.15062

5.5. Concluding Remarks

In this paper, we introduced the RK-4th algorithm as an alternative of the famous Vartia algorithm which is considered as one of the standard solutions to recover the expenditure from the ordinary demand function in economic fields. For this, we estimated the demand system using both the U.S consumer expenditure (CEX) data and the consumer price index (CPI), and recovered the expenditure at each time period using estimated demand from the AIDS model. In addition, the cost-of-living index were evaluated at each time period in order to demonstrate the power of the RK-4th algorithm. From the comparison between price indices from this numerical algorithm and conventional price indices including Paasche and

Fisher ideal price indices, we confirmed the accuracy of RK-4th algorithm since substitution biases of this algorithm is much smaller than conventional price indices.

After then, we have tried to identify whether this numerical algorithm could be used in the case that no exact form of object function were existed. For this research purpose, the Rotterdam model was employed because the Rotterdam model had no exact form of the expenditure or utility function though this demand system suffices the neoclassical economic theory. For the comparison purpose, as with the AI demand system, cost-of-living indices based on the estimated results of the Rotterdam model were calculated. Results showed that all price indices from each demand system using numerical algorithms were very similar to each other even price indices which were calculated from the Rotterdam model. From this, we could confirm that the applicability of RK-4th algorithm in empirical worlds, and that the RK-4th algorithm could be used as an alternative of the Vartia method when calculating the cost-of-living indices.

6. Application 2 : The Measure of Welfare Changes in the U. S Elderly

6.1. Introduction

The purpose of this paper is to investigate the expenditure pattern of the elderly in U.S, which should be different from the general U.S population, and to evaluate consumer welfare effects on both population groups due to the increase in commodity prices. To accomplish the goal of the paper, first, consumption patterns of each population group are estimated by using the *Almost Ideal Demand System (AIDS)* since estimated parameters and elasticities of this demand system provide the information about expenditure patterns of consumers. Second, based on estimation results from the AI demand system, consumer welfare effects when commodity prices change (our case; Health Care and Housing) are analyzed by two different methods. One is the *compensating variation* which is one of the Hicksian welfare measure, another is the *burden index* which is defined as the ratio of the consumer welfare loss to income per person. Both methods provide specific information on welfare changes. For example, the compensating variation directly presents how much money are required to attain a certain level of utility, similarly, the burden index provides the information on welfare loss for individual in terms of dollars when the price has changed.

In 2011, the oldest baby boomers- Americans born between 1946 and 1964- will start to turn 65. According to *A profile of Older Americans* (2009) by U.S. Department of Health and Human Services, and *American Aging Population* (2011) by Population Bulletin, the older population (65+) numbered 38.9 million in 2008, in other words, over one in every eight, or 12.8% of the population is an older American. In addition, Reports expected that the pace of population aging would be accelerated in next 40 years. Finally, this number would be projected to be more than double, 89 million, by 2050[1]. This phenomenon, the increase of the old population, can be explained by the development in the medical system. It reduced mortality rates of the U.S elderly, and has improved health conditions of the elderly over past 30 years. As a consequence, the life expectancy of the U.S elderly has increased in 68.9 years 1950 to 79.2 years in 2009.

Separately from the aging population in the U.S, the old population has faced with slightly higher inflation rates than the general U.S population during the period from 1984 to 2008. According to Stewart (2008), Cashell (2008), the consumer price index of the elderly (CPI-E) for all items rose at an annual average rate of 3.3 percent, compared to increases of 3.1 percent for the consumer price index for urban population (CPI-U). In short, the average annual difference is 0.2 percent. This difference is very tiny, however, this small difference can be a fundamental source of different demand patterns in both the over 65 and general population groups. According to their paper, the higher inflation rate the elderly faced with mainly depends on the fact that the older Americans devotes a substantially larger share of their total budgets to "Health Care" and "Housing". During the same periods, Health care inflation increased more rapidly than most other goods and services, and the cost for the shelter have modestly outpaced overall inflation.

Though the elderly population has experienced slightly higher inflation rate compared to the general U.S population, annual average incomes of the elderly are much lower than those of the general U.S population. According to *A profile of Older Americans* (2009), 1) the median income of older persons in 2008 was $18,337[2], 2) 39.9% of the elderly population groups has the average income lower than $15,000, and 3) about 3.7 million elderly persons (9.7%) were below the poverty level in 2008. Moreover, the major source of the elderly population's income is not stable compared to the general U.S population group. Most cases, the income of the elderly American relies on the combination of Social Security benefits, pensions, retirement saving, and earnings from full-time or part-time work. More specifically,

[1] The old population has increased of 4.5 million or 13% since 1998. Moreover, the pace of population aging is projected to be accelerated in next 40 years

[2] According to "Age of reference person" of the *Consumption expenditure Survey* (2008), the average income of the U.S general population which is $25,425 per person.

Figure 6.1.1.: The percentage changes on CPI-E vs CPI-U from 1984 to 2008

the major source of income for older people in 2007 were 1) Social Security (reported by 87% of older persons), income from assets (reported by 52%), private pensions (reported by 28%), government employee pensions (reported by 13%), and earnings from the full time or part-time work (reported by 25%). These facts imply that the elderly population can be effected severely by the increases of the commodity prices more than the general U.S population.

A couple of studies are studied to investigate welfare effects on the poor when price have changed dramatically including Wood, Nelson and Nogueira (2009) and Huang and Huang (2009). Similarly, a couple of studies are accomplished to identify the relationship between the economic growth and the aging population including Bloom, Canning and Finlay(2008) and Rogers, Toder and Johns (2000). However, there existed not much studies on the welfare analysis on the relationship between the elderly population and welfare changes though aging population are getting larger, and become the common phenomenon in U.S.

In this paper, after estimating the demand system of the U.S general and over 65 population groups, using the compensating variation, welfare effects on the U. S elderly and general population were investigated when prices have changes using the comparison method. The remainder of this paper is organized as follows: In section 6.2, we discuss the data that we hired to analyzes welfare effects on the U.S elderly population and the general U.S population. More specifically, the Consumer Expenditure data and the Consumer Price Index are used for the estimation purpose. Section 6.3 represents estimation results from the Linearly approximate AI demand system (LA-AIDS) of both U.S general population and the elderly. All estimations are based on the AIDS demand system described in Chapter 4. In section 6.4, general results of the welfare analysis are discussed. All measures of welfare changes when price changes are calculated by the burden index and the compensating variation which is estimated by the Vartia algorithm. In section 6.5, the summary and conclusion of this research is provide.

6.2. DATA

The annual times series data of the *Consumer Expenditure Survey (CE)* from 1984 to 2008 are used for the estimation purpose. These time series variables were constructed by the Bureau of Labor Statistics (BLS) in the U.S., and were taken directly from the BLS web site (`http://www.bls.gov/cex/`). According to categories of the *Consumer Price Index (CPI)*, total consumption is divided into seven commodities: *Food and Beverage* (FB), *Housing* (H), *Apparel* (A), *Transportation* (TR), *Health Care* (HC), *Recreation* (R) and *Other goods and services* (O). Two different consumer expenditure data sets are collected during the same period. One is for All American units, another is for Over 65 year population unit.

At the first glance, the data structure of consumer expenditure is investigated in order to identify differences in consumption patterns between over 65 and the general U.S population. According to Table 6.1, the largest difference in spending patterns between the elderly and the general population is in the share of expenditures accounted for “Health Care”. The over 65 year population group spent more than twice as large a share of their total outlays on health care as did the overall population. Moreover, in terms of averages, the elderly in the U.S spent less money on “Apparel”, “Recreation” , and “Transportation” than that of the general population, and spent as almost same as a share of their income on the Food&Beverage

Table 6.1.: Expenditure by Ages, from 1984 to 2008

		All Population	Over 65 Population
Expenditure Share	Food& Beverage	17.86	17.43
	Housing	35.48	34.98
	Apparel	6.00	4.49
	Transportation	22.38	18.51
	Health Care	6.48	13.75
	Recreation	6.65	5.59
	Others	5.15	5.27
	All	1.0	1.0

and Housing.

In order to estimate a system of demand, a measure of relative prices is required. The *Consumer Price Index (CPI)* reported by The Bureau of Labor Statistics (`http://www.bls.gov/cpi/`) is employed as a measure of the average changes in prices over time. Especially, we used the *CPI for all urban consumer*s (CPI-U) for estimating the general population's demand pattern, since it covers about 87 percent spending of the population of the United States. Moreover, the *CPI for the elderly* (CPI-E), the experimental price index, were used to estimate the demand system for over 62 or old population. The only difference between the CPI-E and the CPI-U are in the percentage weights price indices are used. The CPI-E used the 211 categories of goods and services in the CPI market basket of goods to reflect purchasing patterns among more elderly Americans instead of the general Americans.

During the period from 1984 to 2008, consumer price indices, both the CPI-U and the CPI-E, have an increasing tendency similar to total expenditure. In addition, the growth rate of the both the CPI-U and the CPI-E are reviewed for the comparison purpose. Over this 25-year period, the CPI-E for all items rose at an annual average rate of 3.3 percent, compared with increases of 3.1 percent for the CPI-U. This fact implies that the CPI-E faced slightly higher inflation rates than the CPI-U during sample data periods[3]. According to Horbijn and Lagakos (2003), Stewart (2008), Cashell (2008), a couple of underlying reasons for these differences are existed. The first reason of the differences is in the weights of the major goods categories that make up each index. For example, the older Americans devote a substantially larger share of their total budgets to "Health Care". The share of expenditures on Health Care

[3]See the Figure 6.1.1

costs by the CPI-E population is almost double that of the CPI-U population. The second reason of the differences is in the higher inflation rate of "Health care" and "Housing". Over the 1984-2008 period, Health care inflation increased more rapidly than most other goods and services, and the cost for the shelter have modestly outpaced overall inflation.

Though we assume that the CPI-E could explain the expenditure pattern of the elderly better than the CPI-U, we recognize that there existed couple of limitation in using the CPI-E as the price index for estimating the demand system of the elderly. Stewart (2008) and Horbijn and Lagakos (2003) indicated that the experimental price index for the elderly has the couple of methodological limitation when composing the price indices. The first methodological limitation is that expenditure weights used in the CPI-E are subject to higher sampling error than the CPI-U because the number of consumer units used for determining weights when the CPI-E constructed is much smaller than that of the CPI-U. In fact, approximately 18 percent of all consumer units met above definition for older Americans in 2008. The second limitation is that CPI-E used the same geographic areas and the same retail outlets as those used for the CPI-U. The outlets selected thus might not be representative of the location and types of stores used by the elderly population but the general population group. The third limitation is that items priced for the CPI-E are the same as those priced in the CPI-U because the items sampled within selected outlets are determined with the probabilities proportionate to total urban expenditures, not the elderly expenditure. Therefore, the specific items selected for pricing in each outlet many not be representative of the CPI-E population. In addition, with the above methodological limitations, there existed the gap between the population group of CPI-E and the population group over 65 since the CPI-E is composed of all urban consumer units who are at least 62 years of age[4]. Therefore, the demand estimation results of the elderly using the CPI-E could not provide accurate parameters of the demand system. However, we might assume that the CPI-E could explained the demand pattern better than the CPI-U though there existed the some weak points on the CPI-E, and

[4] The population of older Americans used in the CPI-E is composed of all urban non-institutionalized consumer units that meet one of the following three condition: 1) Unattached individuals who are at least 62 years of age; 2) Members of families whose reference person (as defined in the Consumer Expenditure Survey) or spouse is at least 62 years of ages, or 3) Members of groups of unrelated individuals living together who pool their resources to meet their living expenses and whose reference person is at least 62 years of age.

According to the 2007-2008 Consumer Expenditure Survey, Approximately 18 percent of all consumer units met this definition for older Americans the population group does not exactly matched[5].

6.3. Estimation Results

Parameters of the AI demand system are estimated by applying the Iterated Seemingly Unrelated Regression (ISUR) to the system of six share equations (the share, "Other good and service", is dropped to avoid covariance matrix singularity). The `systemfit` and `micEcon` package in R (`http://CRAN.R-project.org/`) are used to estimate the AI demand system and matrix calculation respectively. The AI demand system is selected to estimate the demand pattern of the U.S population since this demand system of Deaton and Muellbauer (1980) is fully consistent with the economic theory, possesses properties of the exact aggregation, and has the functional form of the expenditure[6] which made the estimation results more interpretable.

Three different demand systems are considered : 1) Over 65 population with CPI-E, 2) Over 65 population with CPI-U, and 3) All population with CPI-U. Equation (4.1.5) with Stone's index, LA-AI demand system, is used to estimate the demand system for the data period from 1984 to 2008. Estimation results from three different data sets are reported in Table 6.1. In fact, estimated parameters do represent the partial information about the demand system. The income parameters (β_i) of estimated demand system measure the effects of changes in total real expenditure. The commodities are necessities when the value of income parameter is negative, and is luxury when this value is positive. For example, according to the Table 6.2, in case of over 65 populations with the CPI-E, the commodities group of "Food and Beverage", "Health Care", and "Other goods and services" are necessities. In case of All populations with the CPI-U, in the contrary, the commodities group of "Food and Beverage", "Apparel", "Health Care", "Recreation" and "Other goods and services" are necessities, and the "Housing", "Transportation" are luxuries.

[5]From The own price elasticities from the estimated demand system in the following section, we find that the expenditure pattern (or demand system) of over 65 population might be explained well by the CPI-E instead of the CPI-U.

[6]From this functional form of expenditure function, the compensated income and the cost-of-living can be algebraically calculated.

Table 6.2.: The estimated parameter of AI demand system, from 1984 to 2008

	α_i	γ_{i1}	γ_{i2}	γ_{i3}	γ_{i4}	γ_{i5}	γ_{i6}	γ_{i7}	β_i	R^2
AIDS estimation of Over 65 population with CPI-E										
w_1	0.7040 (0.1907)	0.1056 (0.0549)	0.0295 (0.0527)	0.0055 (0.0141)	-0.0642 (0.0368)	0.0100 (0.0231)	0.0006 (0.0181)	-0.0870 (0.0358)	-0.1069 (0.0386)	0.887
w_2	0.2268 (0.2175)	0.0295 (0.0527)	-0.0624 (0.0820)	-0.0495 (0.0157)	0.0786 (0.0498)	0.0486 (0.0283)	0.0129 (0.0206)	-0.0576 (0.0487)	0.0224 (0.0441)	0.806
w_3	-0.2100 (0.1369)	0.0055 (0.0141)	-0.0495 (0.0157)	0.0429 (0.0073)	0.0070 (0.0156)	-0.0271 (0.0090)	-0.0213 (0.0075)	0.0426 (0.0133)	0.0517 (0.0281)	0.866
w_4	-0.2207 (0.2631)	-0.0642 (0.0368)	0.0786 (0.0498)	0.0070 (0.0156)	-0.0135 (0.0535)	-0.0548 (0.0250)	0.0002 (0.0191)	0.0468 (0.0395)	0.0833 (0.0539)	0.372
w_5	0.4453 (0.1988)	0.0100 (0.0231)	0.0486 (0.0283)	-0.0271 (0.0090)	-0.0548 (0.0250)	0.0113 (0.0231)	0.0046 (0.0126)	0.0073 (0.0311)	-0.0673 (0.0409)	0.716
w_6	-0.0824 (0.1250)	0.0006 (0.0181)	0.0129 (0.0206)	-0.0213 (0.0075)	0.0002 (0.0191)	0.0046 (0.0126)	0.0281 (0.0134)	-0.0251 (0.0187)	0.0310 (0.0255)	0.689
w_7	0.1369 (0.2065)	-0.0870 (0.0358)	-0.0576 (0.0487)	0.0426 (0.0133)	0.0468 (0.0395)	0.0073 (0.0311)	-0.0251 (0.0187)	0.0730 (0.0532)	-0.0142 (0.0420)	0.257
AIDS estimation of All populations with CPI-U										
w_1	0.8834 (0.2434)	0.0620 (0.0429)	0.0191 (0.0428)	-0.0346 (0.0122)	-0.0087 (0.0247)	-0.0927 (0.0153)	0.0380 (0.0196)	0.0168 (0.0184)	-0.1279 (0.0457)	0.849
w_2	-0.3055 (0.3572)	0.0191 (0.0428)	-0.0220 (0.0748)	-0.0087 (0.0153)	0.0012 (0.0439)	0.1235 (0.0219)	-0.0514 (0.0243)	-0.0617 (0.0291)	0.1123 (0.0674)	0.919
w_3	0.0866 (0.1231)	-0.0346 (0.0122)	-0.0087 (0.0153)	0.0385 (0.0059)	0.0266 (0.0108)	-0.0257 (0.0059)	-0.0166 (0.0084)	0.0206 (0.0076)	-0.0048 (0.0231)	0.963
w_4	-0.5751 (0.4500)	-0.0087 (0.0247)	0.0012 (0.0439)	0.0266 (0.0108)	-0.0329 (0.0427)	-0.0233 (0.0159)	0.0170 (0.0156)	0.0200 (0.0205)	0.1517 (0.0853)	0.158
w_5	0.6235 (0.1667)	-0.0927 (0.0153)	0.1235 (0.0219)	-0.0257 (0.0059)	-0.0233 (0.0159)	0.0075 (0.0126)	0.0165 (0.0098)	-0.0057 (0.0143)	-0.1065 (0.0315)	0.747
w_6	0.0952 (0.1404)	0.0380 (0.0196)	-0.0514 (0.0243)	-0.0166 (0.0084)	0.0170 (0.0156)	0.0165 (0.0098)	0.0211 (0.0159)	-0.0247 (0.0126)	-0.0039 (0.0263)	0.375
w_7	0.1921 (0.1978)	0.0168 (0.0184)	-0.0617 (0.0291)	0.0206 (0.0076)	0.0200 (0.0205)	-0.0057 (0.0143)	-0.0247 (0.0126)	0.0347 (0.0192)	-0.0210 (0.0374)	0.368
AIDS estimation of Over 65 population with CPI-U										
w_1	0.6914 (0.2069)	0.0450 (0.0635)	0.0397 (0.0528)	0.0077 (0.0200)	-0.0558 (0.0383)	0.0114 (0.0267)	0.0122 (0.0343)	-0.0602 (0.0309)	-0.1029 (0.0412)	0.888
w_2	0.1234 (0.2199)	0.0397 (0.0528)	-0.0968 (0.0644)	-0.0255 (0.0197)	0.0750 (0.0410)	0.0645 (0.0288)	-0.0238 (0.0319)	-0.0330 (0.0334)	0.0390 (0.0442)	0.813
w_3	-0.2370 (0.1526)	0.0077 (0.0200)	-0.0255 (0.0197)	0.0466 (0.0118)	-0.0003 (0.0180)	-0.0462 (0.0124)	-0.0191 (0.0168)	0.0367 (0.0149)	0.0579 (0.0309)	0.859
w_4	-0.2271 (0.2679)	-0.0558 (0.0383)	0.0750 (0.0410)	-0.0003 (0.0180)	-0.0001 (0.0494)	-0.0684 (0.0271)	0.0090 (0.0261)	0.0405 (0.0303)	0.0855 (0.0546)	0.384
w_5	0.5646 (0.2059)	0.0114 (0.0267)	0.0645 (0.0288)	-0.0462 (0.0124)	-0.0684 (0.0271)	0.0345 (0.0293)	0.0308 (0.0197)	-0.0266 (0.0297)	-0.0888 (0.0421)	0.735
w_6	-0.1057 (0.1545)	0.0122 (0.0343)	-0.0238 (0.0319)	-0.0191 (0.0168)	0.0090 (0.0261)	0.0308 (0.0197)	0.0215 (0.0319)	-0.0305 (0.0242)	0.0339 (0.0303)	0.663
w_7	0.1903 (0.2046)	-0.0602 (0.0309)	-0.0330 (0.0334)	0.0367 (0.0149)	0.0405 (0.0303)	-0.0266 (0.0297)	-0.0305 (0.0242)	0.0731 (0.0350)	-0.0247 (0.0415)	0.265

To investigate demand patterns of the All U.S populations and the elderly, income and uncompensated price elasticities are required since parameter estimates provide a clear understanding of the demand pattern of the U.S population and the elderly, summarized through income and price elasticities. Income and uncompensated price elasticities in the LA-AI demand system can be calculated by[7]:

- Income elasticities

$$\eta_{iM} = 1 + \frac{\beta_i}{w_i}\left[1 - \sum w_j \log p_j(\eta_{jM} - 1)\right] \tag{6.3.1}$$

- Price uncompensated price elasticities

$$\eta_{ij} = -\delta_{ij} + \frac{\gamma_{ij}}{w_i} - \frac{\beta_i\left(w_j + \sum_k w_k \log p_k(\eta_{kj} + \delta_{kj})\right)}{w_i} \tag{6.3.2}$$

where $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ is the Kronecker delta term.

The full set of elasticities, income and uncompensated price elasticities, based on estimates of parameters are reported in Table 6.3. In here, FB, H, A, TR, HC, R, and O represent "Food and Beverage", "Housing", "Apparel", "Transportation", "Health Care", "Recreation", and "Other goods and services" respectively. The first thing we find from the estimated income and price elasticities is that the estimated demand system on Over 65 populations with the CPI-U does not satisfied the required demand theory assumption since own price elasticities of "Apparel" has the positive sign instead of the negative sign. In contrary, the estimated own-price elasticities are in all cases negative in case of Over 65 populations with the CPI-E and All population with the CPI-U, as required by the demand theory. This implies that the estimated demand system are theoretically consistent demand model of all population and over 65 population, so we conclude that these estimated systems can be used to evaluate the welfare implications of the price changes. Second, not surprisingly, not much differences in expenditure pattens are identified between the U.S general and the over 65 population group in terms of income and uncompensated price elasticities. In fact, numbers of those in

[7]See Chapter 4 in this paper for more detail

Table 6.3.: Income and Price elasticities based on the estimated AIDS

	Marshallian (uncompensated) Price Elasticities							Income Elasticities
	η_{i1}	η_{i2}	η_{i3}	η_{i4}	η_{i5}	η_{i6}	η_{i7}	η_{iM}
AIDS estimation of Over 65 population with CPI-E								
FB	-0.2663	0.4057	0.0627	-0.2664	0.1601	0.0302	-0.4447	0.3628
H	0.0767	-1.2093	-0.1503	0.2212	0.1338	0.0355	-0.1788	1.0666
A	-0.0696	-1.5760	-0.0658	-0.0565	-0.8146	-0.5419	0.8470	2.1944
TR	-0.4369	0.2722	0.0170	-1.1611	-0.3812	-0.0184	0.2082	1.4677
HC	0.1588	0.5506	-0.1806	-0.3212	-0.8343	0.0560	0.1142	0.4917
R	-0.0825	0.0317	-0.4235	-0.1018	-0.0050	-0.5018	-0.5333	1.5762
O	-0.9625	-0.5964	0.4923	0.5609	0.1087	-0.2785	-0.1515	0.8383
AIDS estimation of All populations with CPI-U								
FB	-0.5307	0.4208	-0.1615	0.1083	-0.4953	0.2649	0.1883	0.2483
H	0.0102	-1.2016	-0.0445	-0.0669	0.3437	-0.1704	-0.2220	1.3325
A	-0.5942	-0.1187	-0.3216	0.4837	-0.4443	-0.2861	0.3694	0.9165
TR	-0.1402	-0.2862	0.0852	-1.3050	-0.1563	0.0408	0.0093	1.7116
HC	-1.2614	2.7110	-0.3204	-0.0127	-0.7653	0.3623	0.1120	-0.7268
R	0.6099	-0.7874	-0.2592	0.2820	0.2648	-0.6625	-0.3827	0.9386
O	0.2040	-0.5476	0.2243	0.2523	-0.0451	-0.2429	-0.6161	0.7835
AIDS estimation of Over 65 population with CPI-U								
FB	-0.6246	0.4515	0.0733	-0.2271	0.1518	0.1075	-0.3093	0.3769
H	0.0983	-1.3251	-0.0802	0.2010	0.1752	-0.0765	-0.1078	1.1153
A	-0.0430	-1.0607	0.0380	-0.2482	-1.2669	-0.5240	0.7430	2.3618
TR	-0.3930	0.2589	-0.0219	-1.0858	-0.4487	0.0245	0.1852	1.4808
HC	0.1958	0.7108	-0.3187	-0.3956	-0.6517	0.2675	-0.1404	0.3324
R	0.1208	-0.6493	-0.3791	0.0547	0.4834	-0.6375	-0.6179	1.6249
O	-0.6318	-0.2778	0.4254	0.5055	-0.2621	-0.3286	-0.1528	0.7221

Table 6.3 are similar. From this, we could assume that the expenditure pattern of the general population and the over 65 group is basically similar but has some differences.

6.4. Welfare Analysis

We started the welfare analysis from estimating the cost-of-living index for each population group, and move to the compensating variation and burden index since the cost-of-living index could provide basic information about cost-of-living over time or groups. Based on the estimated demand system, we calculate the cost-of-living index during the periods. The Vartia algorithm are applied to estimate it. Specifically, we calculate total three of cost-of-living index such as, for over 65 populations with the CPI-E, over 65 populations with the CPI-U, and all population with the CPI-U for the comparison. To calculate the cost-of-living index, we employed the Konüs cost-of-living index defined by following

$$I(p^0, p^1, u^0) = \frac{e(p^1, u^0)}{e(p^0, u^0)} \tag{6.4.1}$$

where u^0 is the reference level of utility, and e is the minimum expenditure of the utility level at u when the consumer is facing a price vector p . The index, I, represent the minimum expenditure(cost) of the reference level of the utility when a consumer is facing the price vector p^1 relative to the minimum expenditure of remaining at u^0 under a different set of prices p^0.

A *cost-of-living index* is a theoretical price index that measures relative cost of living over time or regions. This implies that if the cost of living of one group is higher than other group, the group with the high cost-of-living index required more money to maintain their life and lifestyle. In contrary, the *Consumer Price Index (CPI)* is a measure of the average change over time in the prices. According to the BLS, the CPI frequently is called a cost-of-living index, both the CPI and a cost-of-living index would reflect changes in the prices of goods and services. So if the cost-of-living index are well estimated, it will reflect the trend of the CPI as a similar manner. All calculated cost-of-living indices from 1984 to 2008 are reported in Table 6.4. All indices are calculated after assuming that the utility level U^0 is held fixed at 100 though prices change, and the base period is the year 1984. Not surprisingly, the cost-of-living indices of the Over 65 population group is larger than that of the general U.S population group. This result are well matched the fact that the CPI-E has faced with slightly higher inflation rates than the CPI-U during the period from 1984 to 2008. In addition, this result means that the Over 65 population group usually spent more money on maintaining the life than the general U.S population group in terms of the average. However, the average income of the over 65 population group is much lower than that of the U.S general population group, so it could cause the serious problem in welfare of the old generation. Moreover, we found that the cost-of-living for the over 65 population group with the CPI-U are smaller than that of using the CPI-E, either. Assuming that estimation results of the elderly with the CPI-E are correct and accurate, it implies that the cost-of-living indices for the over 65 population calculated using the CPI-U might be underestimated than the real indices.

Next, we want to investigate "How much money the elderly are required if the over 65 population is fully compensated in terms of income average of the general U.S population?" For this, the comparison method between the different two cost-of-living are applied. The following formula are used to calculate the required money for the full compensation to the elderly population group.

$$\text{Amout of money (\%)} = \frac{(COLI_{\text{Over 65 Population}} - COLI_{\text{All Population}})}{COLI_{\text{All Population}}} \times 100 \tag{6.4.2}$$

All calculated results based on the average income of over 65 population group in 2008 are reported at the last four columns in Table 6.4 for comparison. According to the Table 6.4, generally speaking, the required income for the elderly in order to maintain their lifestyle like an All population groups calculated by the CPI-E are two times larger than that of calculating by the CPI-U. More specifically, for example, if the elderly population want to be fully compensated, they required 4.1% of the increased income, in other words, need \$939.6 in terms of the CPI-E at 2004. In contrary, with the same situation, the elderly required 2.0% of the increased income, in other words, needs about \$468 in terms of the CPI-U. If we assume that estimated result are correct, these numbers implies that the elderly needed more money if they have fully compensated, and cost-of-living index based on the CPI-U can generate the wrong information on the spending of the elderly population. Generally speaking, it could be a problem when determining the direction of policy for elderly since every statistics about the over 65 population is calculated based on the CPI-U or the CPI-W instead of the CPI-E.

In order to define efficient policy responses to the welfare loss of both the elderly and the general population in the U.S, it is necessary to quantify the losses with accurate measure from the estimated demand system. To do this, based on estimated parameters on the demand system, the Vartia algorithm is applied to calculate the Hicksian compensating variation under various scenarios of price changes. According to Stewart(2008), "Health care" and "Housing" components account for a significant portion of the higher inflation rates in

Table 6.4.: Cost-of-living index and Need income for Elderly

	Cost-of-living indices: Over 65 Population CPI-E	Cost-of-living indices: Over 65 Population CPI-U	Cost-of-living indices: All Population CPI-U	Over 65 Population CPI-E %	Over 65 Population CPI-E $	Over 65 Population CPI-U %	Over 65 Population CPI-U $
1984	100.00	100.00	100.00	0.0	0.00	0.0	0.00
1985	105.59	105.40	105.28	0.3	68.56	0.1	27.19
1986	111.17	110.79	110.53	0.6	133.20	0.2	54.29
1987	116.75	116.18	115.78	0.8	194.49	0.4	81.17
1988	122.32	121.57	121.00	1.1	252.80	0.5	107.69
1989	127.90	126.95	126.22	1.3	308.47	0.6	133.83
1990	133.48	132.33	131.42	1.6	361.76	0.7	159.52
1991	139.05	137.71	136.61	1.8	412.90	0.8	184.74
1992	144.63	143.08	141.80	2.0	462.06	0.9	209.47
1993	150.21	148.46	146.98	2.2	509.41	1.0	233.69
1994	155.80	153.84	152.15	2.4	555.08	1.1	257.43
1995	161.39	159.22	157.31	2.6	599.16	1.2	280.65
1996	166.98	164.60	162.47	2.8	641.79	1.3	303.39
1997	172.57	169.98	167.63	3.0	683.04	1.4	325.64
1998	178.17	175.37	172.78	3.1	723.01	1.5	347.39
1999	183.78	180.76	177.92	3.3	761.77	1.6	368.69
2000	189.39	186.14	183.06	3.5	799.39	1.7	389.54
2001	195.00	191.54	188.20	3.6	835.93	1.8	409.94
2002	200.62	196.93	193.34	3.8	871.45	1.9	429.89
2003	206.24	202.32	198.47	3.9	906.00	1.9	449.44
2004	211.87	207.72	203.60	4.1	939.61	2.0	468.57
2005	217.50	213.12	208.73	4.2	972.35	2.1	487.30
2006	223.13	218.53	213.85	4.3	1004.28	2.2	505.65
2007	228.77	223.93	218.98	4.5	1035.40	2.3	523.63
2008	234.42	229.34	224.10	4.6	1065.74	2.3	541.23

the CPI-E over the past 25, and play important roles on the living cost of the elderly. Based on this fact, losses of the consumer welfare are estimated both Over 65 and general population groups to compare the welfare loss caused by the simultaneous increase of both "Housing" and "Health care" prices.

In Table 6.5, we present a total of 49 scenarios for combined changes in prices of "Health Care" and "Housing", ranging from 0 to 30 percent at every 5 percent intervals. According to Table 6.5, in case of Over 65 population group, a 10 percent increase in prices of both "Health Care" and "Housing" would increase the per capita total compensated expenditure or incur the consumer welfare loss of $1,242. Similarly, in case of All population groups, a 10 percent increase in prices of both "Health Care" and "Housing" would increase the per capita total compensated expenditure or incur the consumer welfare loss of $955. Similarly, in case of increases in "Health care" price by 15% and "Housing" price by 5%, the Table 6.5 shows that per capita total compensated expenditure would increase by about $1,072 in Over 65 population group and approximately $870 in All population group respectively. According to the Table 6.5, we could easily notice that the increase of Housing price have more serious effects on the welfare loss of both population groups than that of Health Care. It is because the share of "Housing" component is more twice larger than that of "Health care". In addition, generally speaking, the welfare loss of Over 65 populations from price changes are larger than that of All population groups in terms of the compensating variation.

Table 6.5.: Compensating variation of increased price of Health care and Housing

Increased price of Housing by	Increased the price of Health Care by						
	OVER65 population (Dollars)						
	0%	5%	10%	15%	20%	25%	30%
0%	0.00	226.71	446.50	659.89	867.33	1069.22	1265.94
5%	393.80	626.69	852.49	1071.71	1284.82	1492.25	1694.35
10%	771.90	1010.80	1242.41	1467.29	1685.91	1898.70	2106.03
15%	1135.54	1380.26	1617.53	1847.90	2071.86	2289.85	2502.27
20%	1485.81	1736.19	1978.95	2214.65	2443.81	2666.86	2884.21
25%	1823.68	2079.56	2327.66	2568.56	2802.77	3030.74	3252.89
30%	2150.03	2411.26	2664.56	2910.51	3149.64	3382.40	3609.23
	All population (Dollars)						
	0%	5%	10%	15%	20%	25%	30%
0%	0.00	196.80	388.87	576.50	759.97	939.51	1115.35
5%	274.81	477.97	676.23	869.91	1059.29	1244.61	1426.10
10%	541.61	750.95	955.24	1154.80	1349.91	1540.85	1727.84
15%	801.00	1016.35	1226.51	1431.79	1632.50	1828.91	2021.26
20%	1053.52	1274.73	1490.61	1701.47	1907.64	2109.38	2306.96
25%	1299.65	1526.58	1748.03	1964.35	2175.84	2382.79	2585.47
30%	1539.82	1772.34	1999.24	2220.87	2437.56	2649.61	2857.27

The increased price of "Health care" and "Housing" would take away the purchasing power of consumers in both Over 65 and All population groups, however, it should hit harder on the elderly who can afford it least compared to the average American since, in many case, the average income of the old American group is lower than that of the U.S general population, and the elderly does not have the proper source of income in comparison with the general population group[8]. Therefore, it could be important to estimate the size

[8] According to the "*A profile of older Americans*" published by Department of Health & Human Services, the

of welfare losses of the elderly population when the cost of "Health Care" and "Housing" increased in terms of *income per person*. For this, first, the average income of both over 65 populations and all American population are required for the calculation. According to "Age of reference person" of the *Consumption expenditure Survey* (2008), in case of all consumer units, the average number in consumer units are 2.5 persons and the average income before taxes is $63,563 per unit. In other words, the average income of the general U.S population group is $25,425.2 per person. Similarly, in case of the over 65 population units, the average number in consumer unit is 1.7 persons, and the average income before taxes is $39,341 per unit. In other words, the average income of the over 65 population group is $23,141.8 per person.[9]

Based on the loss of consumer welfare under the situation with "Health Care" and "Housing" price changes, we calculated the "Burden indices", which defined by the ratio of the consumer welfare loss to income per person, of the general U.S and the over 65 population groups. All calculated burden indices are reported in Table 6.6 by the population group.

According to the Table 6.6, in case of Over 65 population, for example, the diagonal entries show that the burden indices would increase from 2.7 % to 15.6% because of increase in both "Health Care" and "Housing" prices from 5% to 30%. In the contrary, in case of All American population, the burden indices would increase from 1.8% to 11.2% at the same increase of both "Health Care" and "Housing" prices, substantially smaller than those of the over 65 population groups. In general, burden indices of Over 65 population group are generally larger than those of the all population group. This result implies that the Over 65 population group suffer more than all American group when prices go up.

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major source of income for older people in 2007 were 1) Social Security (reported by 87% of older persons), 2) income from assets(reported by 52%), 3) private pensions (reported by 28%), 4) government employee pensions (reported by 13%), and 5) earnings (reported by 25%). In addition, Social Security constituted 90% or more of income received by 35% of all Social Security beneficiaries, and about 3.7 million elderly persons (9.7%) were below the poverty level in 2008.

[9] According to the " A profile of older Americans", The median income of older persons in 2008 was $25.503 for males and $14,559 for females. It is different from the average income of Consumer Expenditure Survey. We used the average income of Consumer Expenditure Survey instead of the income in A profile of older American since the all estimated result are based on the data from BLS.

Table 6.6.: Burden indices of increased price of Health care and Housing

Increased price of Housing by	Increased the price of Health Care by						
	OVER65 population(%)						
	0%	5%	10%	15%	20%	25%	30%
0%	0.0000	0.9796	1.9294	2.8515	3.7479	4.6203	5.4704
5%	1.7017	2.7081	3.6838	4.6311	5.5520	6.4483	7.3216
10%	3.3355	4.3678	5.3687	6.3404	7.2851	8.2046	9.1006
15%	4.9069	5.9644	6.9896	7.9851	8.9529	9.8949	10.8128
20%	6.4205	7.5024	8.5514	9.5699	10.5602	11.5240	12.4632
25%	7.8805	8.9862	10.0583	11.0992	12.1113	13.0964	14.0563
30%	9.2907	10.4195	11.5141	12.5769	13.6102	14.6160	15.5962
	All population (%)						
	0%	5%	10%	15%	20%	25%	30%
0%	0.0000	0.7740	1.5295	2.2674	2.9890	3.6952	4.3868
5%	1.0809	1.8799	2.6597	3.4215	4.1663	4.8952	5.6090
10%	2.1302	2.9535	3.7570	4.5419	5.3094	6.0603	6.7958
15%	3.1504	3.9974	4.8240	5.6314	6.4208	7.1933	7.9498
20%	4.1436	5.0137	5.8627	6.6921	7.5029	8.2964	9.0735
25%	5.1116	6.0042	6.8752	7.7260	8.5578	9.3718	10.1689
30%	6.0563	6.9708	7.8632	8.7349	9.5872	10.4212	11.2379

6.5. Concluding Remarks

Nowadays, the over one in every eight of the population in U.S is and older American. This fact partly depends on the fact that the reduction in mortality at older ages and the improvement in the health care system. Moreover, the pace of aging population is projected to accelerate in the U.S in the next 40 years since the baby boomer generation will start to turn 65. So studies on the demand pattern and welfare effect of the elderly population compared to the general U.S population group are required since, differently from the general U.S population, the elderly in U.S have faced slightly higher inflation rate since the older Americans spend substantially more money of their total income to Health Care and Housing whose inflation rates are higher than any other commodities.

In this paper, first, we investigate the expenditure pattern of both the elderly in U.S and the general population using the different price index such as CPI-E and CPI-U. We assume that the expenditure pattern of the elderly in U.S should be different from the general U.S population. From the demand estimation based on the AI demand system, we could check that not much difference in expenditure patterns between two groups are existed. However, we identified that the expenditure pattern of Over 65 population group can not be properly identified by using the CPI-U. Second, based on the estimated parameters, we calculate the burden index and the compensating variation which is one of Hicksian welfare measure in order to identify welfare changes and effects when price changes. From the evaluation, we could confirm that welfare changes and consumer welfare losses of the elderly population are larger than that of the general U.S population. This result implies that the over 65 population group suffers more than the general U.S population with the situation prices go up.

7. Conclusion

In this paper, we mainly investigated numerical approximation methods to the integrability problem and the measure of welfare changes. The close relationship between the integrability problem and the measure of welfare changes could be found in the expenditure function which was the common fact in both problems. Both problems can be solved using same methods since solutions for these questions mainly relied on how to recover the compensated income (expenditure) from the ordinary demand function, and this procedure are mainly demonstrated by numerical approximation algorithms.

As a solution of both the integrability problem and the measure of welfare changes, numerical approximation methods were considered and discussed in this paper. In addition, a couple of applications were accomplished in order to demonstrate how the numerical method could be applied in the empirical studies as a solution method.

To achieve the goal of the research, in this dissertation, a certain numbers of objectives were pursued:

1. Studies on the integrability problem
2. Studies on the measure of welfare changes
3. Investigated numerical approximation methods as a solution of both the integrability problem and the measure of welfare changes
4. Investigated the demand system since it plays an important role when solving the integrability problems which is defined as a system of the ordinary differential equation.
5. Demonstrated the accuracy and applicability of the numerical algorithm, and how the numerical algorithm can be applied in empirical studies as a solution method.

At first, we studied the integrability problem mainly focusing on how to transform the system of the partial differential equations to the system of the ordinary differential equation. Generally, there is no easy way to solve the system of partial differential equations algebraically and numerically. This transform possibility provides a way to solve the integrability problem using the numerical method. In addition, the measure of welfare changes is studied in terms of the compensating and equivalent variation. Main results of this part are that this measure of welfare changes can be estimated by the numerical approximation methods since welfare changes are defined in the relationship between the expenditure function and the ordinary demand function.

As solutions of both the integrability problem and the measure of welfare changes, a couple of numerical methods are discussed. At the first stage, we investigate existing numerical methods including the Taylor higher order method and the Vartia algorithm. In addition, possible new approximations have also been investigated in this paper such as the RK-4th order algorithm and the Adams Fourth-Order Predictor-Corrector algorithm. Finally, using these numerical algorithms, a couple of economic indicators including the cost-of-living and welfare measures are estimated for research purposes. In fact, the cost-of-living index is used to check the accuracy and the applicability of the algorithm. In addition, the welfare effects on the price change are calculated in terms of the compensating variation using the proposed numerical algorithm.

In order to demonstrate how numerical algorithms can be applied in empirical studies as a solution method, two empirical studies are performed. In the first application, as an alternative of the Vartia algorithm, the RK-4th algorithm is proposed. Using both the U.S consumer expenditure (CEX) data and the consumer price index (CPI), the AI and Rotterdam demand system are estimated, and expenditures are recovered from the estimated demand system using the RK-4th algorithm. Moreover, based on recovered expenditures, the cost-of-living index and conventional price index are evaluated at each time period. From this, we could demonstrate the power and the applicability of the RK-4th algorithm as an alternative of the Vartia algorithm based on the fact that the substitution bias of this algorithm is much smaller than other price indices.

In the second application, we pay attention to the phenomenon called the aging population in U.S, and demonstrate the welfare effects on the U.S elderly when price changes. Using the U.S consumer expenditure (CEX) data and the different consumer price index (CPI), the demand pattern in both U.S general population and the elderly are estimated based on the AI demand system. From this estimation results, the different demand patterns on two different population groups are identified. In addition, in order to analyze the welfare effect when prices have changes, the burden index and the compensating variation are calculated using the numerical algorithm. From the evaluation, we could confirm that the welfare changes and consumer welfare losses of the elderly population are larger than that of the general U.S population. This result implies that the over 65 population group suffers more than the general U.S population when prices go up.

References

Alston, J., & Chalfant, J. (1993). The silence of the lambdas - a test of the almost ideal and rotterdam models. *American Journal Of Agricultural Economics*, *75*(2), 304-313.

Amble, N., & Stewart, K. (1994). Experimental price index for elderly consumers. *Monthly Lab. Rev.*, *117*, 11.

Attfield, C. L. F. (1997). Estimating a cointegrating demand system. *European Economic Review*, *41*(1), 61-73.

Balk, B. (1995). Approximating a cost-of-living index from demand functions: A retrospect. *Economics Letters*, *49*(2), 147–155.

Balk, B. M. (1995). Axiomatic price index theory: a survey. *International Statistical Review/Revue Internationale de Statistique*, *63*(1), 69-93.

Balk, B. M. (2005). Divisia price and quantity indices: 80 years after. *Statistica Neerlandica*, *59*(2), 119-158.

Balk, B. M., & Netherlands, S. (2010). Direct and chained indices: a review of two paradigms. *PRICE AND PRODUCTIVITY MEASUREMENT: Volume 6–Index Number Theory*, 217.

Banks, J., Blundell, R., & Lewbel, A. (1996). Tax reform and welfare measurement: do we need demand system estimation? *The Economic Journal*, *106*(438), 1227-1241.

Banks, J., Blundell, R., & Lewbel, A. (1997). Quadratic engel curves and consumer demand. *Review Of Economics And Statistics*, *79*(4), 527-539.

Barnett, W., & Serletis, A. (2009, Jan). *The differential approach to demand analysis and the rotterdam model* (Tech. Rep. No. 200902). Available from `http://ideas.repec.org/p/kan/wpaper/200902.html`

Barnett, W. A. (1979a). The joint allocation of leisure and goods expenditure. *Econometrica: Journal of the Econometric Society*, *47*(3), 539-563.

Barnett, W. A. (1979b). Theoretical foundations for the rotterdam model. *The Review of Economic Studies*, *46*(1), 109-130.

Barnett, W. A. (1981). *Consumer demand and labor supply*. North-Holland Amsterdam.

Barnett, W. A., & Binner, J. M. (2004). *Functional structure and approximation in econometrics*. Emerald Group Pub Ltd.

Barnett, W. A., & Pasupathy, M. (2003). Regularity of the generalized quadratic production model: a counterexample. *Econometric Reviews*, *22*(2), 135-154.

Barnett, W. A., & Seck, O. (2006). Rotterdam vs almost ideal models: Will the best demand specification please stand up?

Barnett, W. A., & Serletis, A. (2008). Consumer preferences and demand systems. *Journal of Econometrics*, *147*(2), 210-224.

Barnett, W. A., & Serletis, A. (2009). Measuring consumer preferences and estimating demand systems. *Quantifying consumer preferences: Estimating demand systems. Contributions to Economic Analysis. Elsevier, Amsterdam*.

Barten, A. P. (1993). Consumer allocation models: choice of functional form. *Empirical Economics*, *18*(1), 129-158.

Blackorby, C., & Russell, R. R. (1989). Will the real elasticity of substitution please stand up?(a comparison of the allen/uzawa and morishima elasticities). *The American Economic Review*, 882-888.

Bloom, D., Canning, D., & Finlay, J. (2008). Population aging and economic growth in asia. *PGDA Working Papers*.

Blundell, R., & Robin, J. M. (1999). Estimation in large and disaggregated demand systems: an estimator for conditionally linear systems. *Journal of Applied Econometrics*, *14*(3), 209-

232.

Boskin, M. J., Dulberger, E. R., Gordon, R. J., Griliches, Z., & Jorgenson, D. W. (1998). Consumer prices, the consumer price index, and the cost of living. *The Journal of Economic Perspectives*, *12*(1), 3-26.

Braithwait, S. D. (1980). The substitution bias of the laspeyres price index: An analysis using estimated cost-of-living indexes. *The American Economic Review*, *70*(1), 64-77.

Breslaw, J. A., & Smith, J. B. (1995). A simple and efficient method for estimating the magnitude and precision of welfare changes. *Journal of Applied Econometrics*, *10*(3), 313-327.

Browning, M., & Meghir, C. (1991). The effects of male and female labor supply on commodity demands. *Econometrica: Journal of the Econometric Society*, *59*(4), 925-951.

Burden, R., & Faires, J. (n.d.). Numerical analysis. 2001. *Brooks/Cole, USA*.

Cashell, B., & Congress. Congressional Research Service, L. of. (2010). A separate consumer price index for the elderly?

Caves, D. W., & Christensen, L. R. (1980). Global properties of flexible functional forms. *The American Economic Review*, *70*(3), 422-432.

Choi, K. (2010). The differential equations of income compensation function and its numerical solution.

Christensen, L. R., & Manser, M. E. (1976). Cost-of-living indexes and price indexes for us meat and produce, 1947-1971. *NBER Chapters*, 399-450.

Creedy, J. (2000). Measuring welfare changes and the excess burden of taxation*. *Bulletin of Economic Research*, *52*(1), 1-48.

Creedy, J. (2006). How to calculate welfare measures using only marshallian demand functions. *Australian Economic Review*, *39*(3), 340-346.

Davis, G. C., & Gauger, J. (1996). Measuring substitution in monetary-asset demand systems. *Journal of Business & Economic Statistics*, *14*(2), 203-208.

Deaton, A., & Muellbauer, J. (1980a). An almost ideal demand system. *The American Economic Review*, 312-326.

Deaton, A., & Muellbauer, J. (1980b). *Economics and consumer behavior*. Cambridge Univ Pr.

Diewert, W. E. (1971). An application of the shephard duality theorem: a generalized leontief production function. *The Journal of Political Economy*, *79*(3), 481-507.

Dumagan, J. C., & Mount, T. D. (n.d.). Re-examining the cost-of-living index and the biases of price indices.

Dumagan, J. C., & Mount, T. D. (1997). Approximating compensated income from ordinary demand functions. *Economics Letters*, *55*(2), 191-201.

Faroque, A. (2008). An investigation into the demand for alcoholic beverages in canada: a choice between the almost ideal and the rotterdam models. *Applied Economics*, *40*(16), 2045-2054. Available from http://ideas.repec.org/a/taf/applec/v40y2008i16p2045-2054.html

Green, R., & Alston, J. M. (1990). Elasticities in aids models. *American Journal of Agricultural Economics*, *72*(2), 442-445.

Greenberg, S. (2009). A profile of older americans: 2009. *Washington DC*.

Hammond, P. J. (1990). *Money metric measures of individual and social welfare allowing for environmental externalities*. Citeseer.

Hausman, J. (1981). Exact consumer's surplus and deadweight loss. *The American Economic Review*, *71*(4), 662–676.

Hausman, J. A., & Newey, W. K. (1995). Nonparametric estimation of exact consumers surplus and deadweight loss. *Econometrica: Journal of the Econometric Society*, *63*(6), 1445-1476.

Health, N. I. of, et al. (2007). Why population aging matters: A global perspective. *National Institute on Aging (Publication No. 07–6134) Washington, DC*.

Henningsen, A. (2010). miceconaids: Demand analysis with the almost ideal demand system (aids) [Computer software manual]. Available from http://CRAN.R-project.org/package=micEconAids (R package version 0.6-4)

Henningsen, A., & Hamann, J. D. (2007a). systemfit: A package for estimating systems of simultaneous equations in r. *Journal of Statistical Software*, *23*(4), 1-40. Available from http://www.jstatsoft.org/v23/i04/

Henningsen, A., & Hamann, J. D. (2007b). systemfit: A package for estimating systems of simultaneous equations in r. *Journal of Statistical Software*, *23*(4), 1–40. Available from http://www.jstatsoft.org/v23/i04/

Hicks, J. (1946). Value and capital, 1939. *Mathematical Appendix*, 311–2.

Hobijn, B., & Lagakos, D. (2003). Social security and the consumer price index for the elderly. *Current Issues in Economics and Finance, Vol. 9, No. 5, May 2003*.

Holt, M. T., & Goodwin, B. K. (2009). The almost ideal and translog demand systems. *MPRA Paper*.

Huang, S., & Huang, K. (2009). How increased food and energy prices affect consumer welfare.

Hulten, C. R. (1973). Divisia index numbers. *Econometrica: Journal of the Econometric Society*, *41*(6), 1017-1025.

Hurwicz, L., & Uzawa, H. (1971). On the integrability of demand functions. *Preferences, utility and demand*, 114–148.

Irvine, I., & Sims, W. (1998). Measuring consumer surplus with unknown hicksian demands. *The American Economic Review*, *88*(1), 314–322.

Jacobsen, L. (2011). *America's aging population*. Population Reference Bureau.

Jorgenson, D. W. (1990). Aggregate consumer behavior and the measurement of social welfare. *Econometrica: Journal of the Econometric Society*, *58*(5), 1007-1040.

Jorgenson, D. W. (1997). *Welfare: Aggregate consumer behavior*. The MIT Press.

Kim, H. Y. (1997). Inverse demand systems and welfare measurement in quantity space. *Southern Economic Journal*, *63*(3).

Kramer, J. F. w. c. f. A., & Friendly, M. (2010). sem: Structural equation models [Computer software manual]. Available from `http://CRAN.R-project.org/package=sem` (R package version 0.9-21)

LaFrance, J. T. (1998). The silence bleating! of the lambdas: comment. *American Journal of Agricultural Economics*, *80*(1), 221.

LaFrance, J. T. (2004). Integrability of the linear approximate almost ideal demand system. *Economics Letters*, *84*(3), 297-303.

Lebow, D. E., & Rudd, J. B. (2003). Measurement error in the consumer price index: where do we stand? *Journal of economic literature*, *41*(1), 159-201.

Lewbel, A., & Ng, S. (2005). Demand systems with nonstationary prices. *Review of Economics and Statistics*, *87*(3), 479-494.

Mas-Colell, A., Whinston, M., Green, J., & Empresarials, U. P. F. F. de Cičncies

Econňmiques i. (1995). *Microeconomic theory* (Vol. 1). Oxford university press New York.

Matsuda, T. (2006). Linear approximations to the quadratic almost ideal demand system. *Empirical Economics*, *31*(3), 663-675.

McKenzie, G. (1983). *Measuring economic welfare: new methods*. Cambridge Univ Pr.

McKenzie, G., & Pearce, I. (1976). Exact measures of welfare and the cost of living. *The Review of Economic Studies*, *43*(3), 465-468.

Moschini, G. (1995). Units of measurement and the stone index in demand system estimation. *American Journal of Agricultural Economics*, *77*(1), 63.

Moschini, G. (1998). The semiflexible almost ideal demand system. *European Economic Review*, *42*(2), 349-364.

Ogura, M. (n.d.). Estimating a demand system in the aids model: The case of japan. *Kobe University*.

Oulton, N. (2008). Chain indices of the cost-of-living and the path-dependence problem: An empirical solution. *Journal of Econometrics*, *144*(1), 306-324.

Oulton, N. (2009). How to measure living standards and productivity.

Parks, R. W. (1969). Systems of demand equations: an empirical comparison of alternative functional forms. *Econometrica: Journal of the Econometric Society*, *37*(4), 629-650.

Pollak, R. A. (1989). *The theory of the cost-of-living index*. Oxford University Press, USA.

Porter-Hudak, S., & Hayes, K. (1986). The statistical precision of a numerical methods estimator as applied to welfare loss. *Economics Letters*, *20*(3), 255–257.

Porter-Hudak, S., & Hayes, K. (1991). A numerical methods approach to calculating cost-of-living indices. *Journal of Econometrics*, *50*(1), 91-105.

Rao, C. R. (2002). Linear statistical inference and its applications, paperback.

Rogers, D., et al. (1997). Economic consequences of an aging population. *Ratio*, 100.

Ryan, L., & Wales, T. (1998). A simple method for imposing local curvature in some flexible consumer-demand systems. *Journal Of Business & Economic Statistics*, *16*(3), 331-338.

Samuelson, P. (1950). The problem of integrability in utility theory. *Economica*, *17*(68), 355–385.

Slesnick, D. T. (1998). Empirical approaches to the measurement of welfare. *Journal of*

Economic Literature, *36*(4), 2108-2165.

Stewart, K. (2008). Experimental consumer price index for elderly americans (cpi-e): 1982-2007, the. *Monthly Lab. Rev.*, *131*, 19.

Stewart, K., & Reed, S. (1999). Consumer price index research series using current methods, 1978-98. *Monthly Lab. Rev.*, *122*, 29.

Stoer, J., & Bulirsch, R. (2002). *Introduction to numerical analysis* (Vol. 12). Springer Verlag.

Theil, H. (1967). *Economics and information theory*. North-Holland Amsterdam.

Theil, H. (1980). *The system-wide approach to microeconomics*. University of Chicago Press Chicago.

Varian, H. R. (1992). Microeconomic analysis.

Vartia, Y. O. (1983). Efficient methods of measuring welfare change and compensated income in terms of ordinary demand functions. *Econometrica*, *51*(1), 79-98.

Woo, C., Zarnikau, J., & Kollman, E. (2010). Exact welfare measurement for double-log demand with partial adjustment. *Empirical Economics*, 1–10.

Wood, B., Nelson, C., & Nogueira, L. (2009). Food price crisis: welfare impact on mexican households. In *International agricultural trade research consortium analytic symposium, seattle, washington* (pp. 22–23).

Wood, F., Esbensen, K., & Geladi, P. (1987). Principal component analysis. *Chemometr. Intel. Lab. Syst*, *2*, 37-52.

Wright, B. D., & Williams, J. C. (1988). Measurement of consumer gains from market stabilization. *American Journal of Agricultural Economics*, *70*(3), pp. 616-627. Available from `http://www.jstor.org/stable/1241500`

(Wright & Williams, 1988; Amble & Stewart, 1994; Attfield, 1997; B. Balk, 1995; B. M. Balk, 1995; B. M. Balk & Netherlands, 2010; Banks, Blundell, & Lewbel, 1996, 1997; W. A. Barnett, 1979a, 1979b, 1981; W. A. Barnett & Pasupathy, 2003; W. A. Barnett & Binner, 2004; W. A. Barnett & Seck, 2006; W. A. Barnett & Serletis, 2008, 2009; Barten, 1993; Blackorby & Russell, 1989; Blundell & Robin, 1999; Boskin, Dulberger, Gordon, Griliches, & Jorgenson, 1998; Braithwait, 1980; Breslaw & Smith, 1995; Burden & Faires, n.d.; Cashell & Congress. Congressional Research Service, 2010; Caves & Christensen, 1980; Choi, 2010; Christensen &

Manser, 1976; Creedy, 2000, 2006; Deaton & Muellbauer, 1980a, 1980b; Dumagan & Mount, 1997, n.d.; Green & Alston, 1990; Greenberg, 2009; J. Hausman, 1981; J. A. Hausman & Newey, 1995; Henningsen & Hamann, 2007a; Henningsen, 2010; Hobijn & Lagakos, 2003; Holt & Goodwin, 2009; Huang & Huang, 2009; Hurwicz & Uzawa, 1971; Irvine & Sims, 1998; Jacobsen, 2011; Jorgenson, 1990, 1997; Kramer & Friendly, 2010; LaFrance, 1998, 2004; Lebow & Rudd, 2003; Mas-Colell, Whinston, Green, & Empresarials, 1995; Matsuda, 2006; McKenzie & Pearce, 1976; McKenzie, 1983; Moschini, 1995, 1998; Health et al., 2007; Pollak, 1989; Porter-Hudak & Hayes, 1986, 1991; W. Barnett & Serletis, 2009; Faroque, 2008; Slesnick, 1998; Stewart, 2008; Stoer & Bulirsch, 2002; Theil, 1967, 1980; Varian, 1992; Vartia, 1983; B. Wood, Nelson, & Nogueira, 2009; ?, ?; Samuelson, 1950; Hicks, 1946)

A. Appendix 1 : R code

A.1. Demand estimation

A.1.1. LA-AIDS Estimation

```
###################################################################################
## this code is an examaple R code for LA-AI demand system                       ##
## In this code, 6 commodity system are considered                               ##
###################################################################################


###################################################################################
###              Read the data                                                 ####
###################################################################################

USCes1<-read.table("/Source of Data", header=TRUE)

###################################################################################
###               Call the Library                                             ####
###################################################################################
library(micEcon)         # for matrix calculation
library(systemfit)       # systemfit for the SUR

###################################################################################
###               Variable in Demand estimation                                ####
###################################################################################

W                       # Expenditure share
LW                      # the log of Expenditure share
P                       # the log of Relative Price
PP                      # Relative Price
LTE                     # log of total Expenditure
lnPP                    # Stone index


###################################################################################
###         LA-AIDS Estimation                                                 ####
###################################################################################
```

```
# Stone index for the LA-AIDS
lnPP<-matrix(0,25,1)
for(i in 1:nrow(W)){
lnPP[i]=t(W[i,])%*%P[i,]
}

#calculate the LTEP = LTE - lnPP
LTEP<-LTE-lnPP


## LA-AIDS estimation
# make each equation - 6 quations
LAeq1<- W[,1]~P[,1]+P[,2]+P[,3]+P[,4]+P[,5]+P[,6]+LTEP
LAeq2<- W[,2]~P[,1]+P[,2]+P[,3]+P[,4]+P[,5]+P[,6]+LTEP
LAeq3<- W[,3]~P[,1]+P[,2]+P[,3]+P[,4]+P[,5]+P[,6]+LTEP
LAeq4<- W[,4]~P[,1]+P[,2]+P[,3]+P[,4]+P[,5]+P[,6]+LTEP
LAeq5<- W[,5]~P[,1]+P[,2]+P[,3]+P[,4]+P[,5]+P[,6]+LTEP
LAeq6<- W[,6]~P[,1]+P[,2]+P[,3]+P[,4]+P[,5]+P[,6]+LTEP

## system without restriction
# make the system : the equations go into the single system
LAsystem<-list(LAeq1, LAeq2, LAeq3, LAeq4, LAeq5, LAeq6)
labels<-list("LS1","LS2","LS3","LS4")
#SUR with all equations without restriction
LAresult<-systemfit(LAsystem, method="SUR", maxiter=500)


## system with restriction
LAsystemR<-list(LAeq1, LAeq2, LAeq3, LAeq4, LAeq5)

# Make the Restriction R matrix
RRmat<-matrix(0,nrow=15, ncol=40)

# Assgin the the value to make the restiction

# Homogeniety for gamma
RRmat[1,2:7]<-1
RRmat[2,10:15]<-1
RRmat[3,18:23]<-1
RRmat[4,26:31]<-1
RRmat[5,34:39]<-1

# symetric restiction
RRmat[6,3]<-1
RRmat[7,4]<-1
RRmat[8,5]<-1
RRmat[9,6]<-1
RRmat[10,12]<-1
RRmat[11,13]<-1
RRmat[12,14]<-1
RRmat[13,21]<-1
RRmat[14,22]<-1
RRmat[15,30]<-1
```

```
RRmat[6,10]<- -1
RRmat[7,18]<- -1
RRmat[8,26]<- -1
RRmat[9,34]<- -1
RRmat[10,19]<- -1
RRmat[11,27]<- -1
RRmat[12,35]<- -1
RRmat[13,28]<- -1
RRmat[14,36]<- -1
RRmat[15,37]<- -1


# Make the qvec of RHS
qvec<-matrix(0, 15, 1)

TLAresultRI<-systemfit(LAsystemR, method="SUR", restrict.matrix=RRmat,
                                       restrict.rhs= qvec, maxiter=500)
summary(TLAresultRI)              # show the estimation result



##################################################################################
###          Coefficient Matrix                                              ####
##################################################################################
# extract the coefficients from the coefficient matrix

# setup the coefficient matrix at first
coeff.TLAresultRI<-matrix(0,8,5)
for(i in 1:40){
coef.TLAresultRI<-TLAresultRI$coefficient
coeff.TLAresultRI[i]<-coef.TLAresultRI[i]
}

# The coefficients of LA-AIDS estimation result
Tcoef.TLAresultRI<-t(coeff.TLAresultRI)

#Add row in the Tcoef.TLAresultRI to calculate the wq6 coefficients
TTcoef.TLAresultRI<-insertRow( Tcoef.TLAresultRI, 6, v = 0 )

#Call the TTcoef.TLAresultRI
TTcoef.TLAresultRI

#this calculation is on the Theoretical restriction
#adding-up, homogeneous, andn sysmetry condition

sumCoef1<-sum(TTcoef.TLAresultRI[,1])
sumCoef7<-sum(TTcoef.TLAresultRI[,7])
sumCoef8<-sum(TTcoef.TLAresultRI[,8])


for(i in 1:5){
TTcoef.TLAresultRI[6,i+1]<- TTcoef.TLAresultRI[i,7]
}
TTcoef.TLAresultRI[6,1]<-1-sumCoef1
```

```
TTcoef.TLAresultRI[6,7]<- -sumCoef7
TTcoef.TLAresultRI[6,8]<- -sumCoef8

TTcoef.TLAresultRI
#Complete of Coefficient Matrix

######################################################################################
##       calculate the elasticity of system                                     #####
######################################################################################

# the average of Expenditure Share and Price
Aver.W<-apply(W,2,mean)
Aver.P<-apply(P,2,mean)

## Uncompensated Price Elasticity of LA-AIDS
######################################################################################
# set up the Alpha and the Beta
Alpha<-TTcoef.TLAresultRI[,1]
Beta<-TTcoef.TLAresultRI[,6]

# set up the Kronecker Delta delta_ij
Delta<-matrix(0,6,6)
Delta[row(Delta)==col(Delta)]<- -1
KronDelta<-Delta

# set up coefficient r_ij
TTcoef.withoutA<-TTcoef.TLAresultRI[,-1]
TTcoef.withoutAB<-TTcoef.withoutA[,-7]

# set up r_ij/w_i
RW<-TTcoef.withoutAB/Aver.W

#set up w_j/w_i
WW<-matrix(0,6,6)
for(i in 1:6){
for(j in 1:6){
WW[i,j]<-Aver.W[j]/Aver.W[i]
}
}
Alpha<-TTcoef.TLAresultRI[,1]
Beta<-TTcoef.TLAresultRI[,8]

#calculate the b_i (w_j/w_i)
BetaWW<-Beta * WW

# calculate the b_i/ w_i
BetaW<-Beta/Aver.W

#calculate the a_i/w_i
AlphaW<-Alpha/Aver.W

#calculate the b_i*a_*/W_i
BetaAlphaW<-Beta%*%t(AlphaW)
```

```
# calcalate sumRP
SumRP<-TTcoef.withoutAB%*%Aver.P

#calculate the BetaW*SumRPP
BetaWSumRP<-BetaW%*%t(SumRP)


#calculate the SumWP
SumWP<-Aver.W*Aver.P

#formula 1 -Green and Alston -AIDS
####################################################################################
UPE.LAAIDS.formula1<-KronDelta+RW-BetaAlphaW-BetaWSumRP
UPE.LAAIDS.formula1


#formula 2 -Green and Alston
####################################################################################
UPE.LAAIDS.formula2<-KronDelta+RW
UPE.LAAIDS.formula2


#formula 3 -Green and Alston
####################################################################################
UPE.LAAIDS.formula3<-KronDelta+RW-BetaWW
UPE.LAAIDS.formula3


#formula 4 -Green and Alston
####################################################################################
#set A,B,C, and D
A<-UPE.LAAIDS.formula3
B<-BetaW
C<-SumWP
#set 8x8 Identity matrix
ID<-matrix(0,6,6)
ID[col(ID)==row(ID)]<- 1

# calculate the Uncompensated Price elasticity of LA-AIDS
UPE.LAAIDS.formula4<-solve(B%*%t(C)+ID)%*%(A+ID)-ID
UPE.LAAIDS.formula4



## Income elasticity of LA-AIDS
####################################################################################
ICE.LAAIDS<-matrix(0,6,1)
for(i in 1:6){
ICE.LAAIDS[i]<-1+TTcoef.TLAresultRI[i,8]/Aver.W[i]
}
ICE.LAAIDS
```

```
#Income elasticity of LA-AIDS based on Green and Alston formula 4
ICE.LAAID.formula4<-solve(ID+B%*%t(C))%*%B
ICE.LAAID.formula4
```

A.1.2. Rotterdam Estimation

```
##################################################################################
## An examaple R code for Absolute version of Rotterdam demand system          ##
## In this code, 6 commodity system are considered                             ##
##################################################################################


##################################################################################
###               Call the Library                                            ####
##################################################################################
library(micEcon)         # for matrix calculation
library(systemfit)       # systemfit for the SUR

##################################################################################
###               Variable in Demand estimation                               ####
##################################################################################

W                       # Expenditure share
Expenditure             # Expenditure of each commodity
Price                   # the log of Relative Price
TE                      # Total exepnditure
LTE                     # log of total Expenditure


## Log difference of Price and Quantity
DLP<-diff(log(Price))
Quantity<-Expenditure/Price
DlogQuantity<-diff(log(Quantity))

## Weighted sum of exepnditure Share
WeightedW<-matrix(0,nrow(W),ncol(W))
for(i in 1:nrow(W)){
for(j in 1:ncol(W)){
WeightedW[i,j]<-.5*(W[i,j]+W[i+1,j])
}
}

## the Divisa Quantity Index
DivisiaQ<-matrix(0,nrow(W),1)
for(i in 1:nrow(W)){
DivisiaQ[i]<-t(WeightedW[i,])%*%DlogQuantity[i,]
}

ARs<-WeightedW*DlogQuantity

##################################################################################
## Absolute Price Version of Rotterdam Model
##################################################################################


# make each equation - 6 equations
##################################################################################
```

```
APVReq1<- ARs[,1]~ -1+DivisiaQ+DLP[,1]+DLP[,2]+DLP[,3]+DLP[,4]+DLP[,5]+DLP[,6]
APVReq2<- ARs[,2]~ -1+DivisiaQ+DLP[,1]+DLP[,2]+DLP[,3]+DLP[,4]+DLP[,5]+DLP[,6]
APVReq3<- ARs[,3]~ -1+DivisiaQ+DLP[,1]+DLP[,2]+DLP[,3]+DLP[,4]+DLP[,5]+DLP[,6]
APVReq4<- ARs[,4]~ -1+DivisiaQ+DLP[,1]+DLP[,2]+DLP[,3]+DLP[,4]+DLP[,5]+DLP[,6]
APVReq5<- ARs[,5]~ -1+DivisiaQ+DLP[,1]+DLP[,2]+DLP[,3]+DLP[,4]+DLP[,5]+DLP[,6]
APVReq6<- ARs[,6]~ -1+DivisiaQ+DLP[,1]+DLP[,2]+DLP[,3]+DLP[,4]+DLP[,5]+DLP[,6]
########################################################################################


########################################################################################
## APVR system without restriction
########################################################################################
# make the system : the equations go into the single system

APVRsystem<-list(APVReq1, APVReq2, APVReq3, APVReq4, APVReq5, APVReq6)

# Estimation using ISUR system method with all equations without restriction
APVRresultI<-systemfit(APVRsystem, method="SUR", maxiter=500)
summary(APVRresult)


########################################################################################
## APVR system with restriction
########################################################################################
APVRsystemR<-list(APVReq1, APVReq2, APVReq3, APVReq4, APVReq5)


########################################################################################
###  Imposing the Restriction to the Model
########################################################################################

# Make the Restriction R matrix
RRmat<-matrix(0,nrow=15, ncol=35)

########################################################################################
# Assgin the the value to make the restiction
RRmat[1,2:7]<-1
RRmat[2,9:14]<-1
RRmat[3,16:21]<-1
RRmat[4,23:28]<-1
RRmat[5,30:35]<-1


RRmat[6,3]<-1
RRmat[7,4]<-1
RRmat[8,5]<-1
RRmat[9,6]<-1
RRmat[10,11]<-1
RRmat[11,12]<-1
RRmat[12,13]<-1
RRmat[13,19]<-1
RRmat[14,20]<-1
RRmat[15,27]<-1
```

```
RRmat[6,9]<- -1
RRmat[7,16]<- -1
RRmat[8,23]<- -1
RRmat[9,30]<- -1
RRmat[10,17]<- -1
RRmat[11,24]<- -1
RRmat[12,31]<- -1
RRmat[13,25]<- -1
RRmat[14,32]<- -1
RRmat[15,33]<- -1

# Make the qvec of RHS
qvec<-matrix(0, 15, 1)

####################################################################################
## Estimation Result of APVR,
## Estimation using ISUR system method with 5 equations with restriction, 500times
####################################################################################

# Estmation of APVR using ISUR=500, with restriction
APVRresultRI<-systemfit(APVRsystemR, method="SUR", restrict.matrix=RRmat,
                                              restrict.rhs= qvec, maxiter=500)
APVRresultRI
summary(APVRresultRI)


####################################################################################
## calculate the elasticity of system
####################################################################################
# extract the coefficients from the coefficient matrix
# 1. setup the coefficient matrix at first
coeff.APVRresultRI<-matrix(0,7,5)
for(i in 1:35){
coef.APVRresultRI<-APVRresultRI$coefficient
coeff.APVRresultRI[i]<-coef.APVRresultRI[i]
}


# The coefficients of APVRresultRI estimation result
Acoef.APVRresultRI<-t(coeff.APVRresultRI)

#Add row in the Tcoef.TLAresultRI to calculate the wq6 coefficients
APVRcoef.APVRresultRI<-insertRow(Acoef.APVRresultRI, 6, v = 0 )

#Call the TTcoef.TLAresultRI
APVRcoef.APVRresultRI
summary(APVRcoef.APVRresultRI)


# this calculation is on the Theoretical restriction
# adding-up, homogeneous, andn sysmetry condition

sumCoef1<-sum(APVRcoef.APVRresultRI[,1])
```

```
sumCoef7<-sum(APVRcoef.APVRresultRI[,7])


for(i in 1:5){
APVRcoef.APVRresultRI[6,i+1]<- APVRcoef.APVRresultRI[i,7]
}
APVRcoef.APVRresultRI[6,1]<-1-sumCoef1
APVRcoef.APVRresultRI[6,7]<- -sumCoef7



APVRcoef.APVRresultRI
#Complete of Coefficient Matrix
```

A.2. Numerical Algorithm

A.2.1. Vartia Algorithm

```
####################################################################################
###             Vartia's algorithm                                             ####
####################################################################################
init_c                                     # initial value of Expenditure
num_step                                   # number of Stepms
qv                                         # demand function

## Assign the matrix for storing the calculation result
resultVTA1<-matrix(0,25,1)
resultVTA1[1]<-y0

## Algorithm
####################################################################################

for(i in 1:num_step){
  oldp<-PP[i,]
  newp<-PP[i,]+(PP[i+1,]-PP[i,])
  oldx<-qv(oldp,init_c)
  oldc<-init_c

    newx<-qv(newp,oldc)

    tolerance<-1000
    iterN<-0

    while(tolerance>0.00001){                     # iteration precedure
       newx<-qv(newp,oldc)
newc<-init_c +0.5*(newx+oldx)%*%(PP[i+1,]-PP[i,])
temp<-oldc
oldc<-newc
tolerance<- abs(newc-temp)

iterN<-iterN+1

if(iterN>100){
print("not converged")
break
}
    }                                             # iteration end

  resultVTA1[i+1]<-newc
  resultVTB1<-cbind(i,t(iterN), t(newp),newx,newc)
  print(resultVTB1)
  init_c<-newc                                    # replace the value
}
}
```

A.2.2. RK-4th Algorithm

```
#####################################################################################
###              VRK-4th Algorithm                                              ####
#####################################################################################

oldc                                       # initial value of Expenditure
num_step
n<-num_step+1                                   # number of Stepms
qv                                         # demand function


## Assign the matrix for storing the calculation result
  resultRKA1<-matrix(0,num_step,1)
  resultRKA1[1]<-y0


## Algorithm
#####################################################################################

for(k in 1:num_step){

  oldp<-PP[k,]
  newp<-PP[k,]+(PP[k+1,]-PP[k,])

  k1=qv(oldp, oldc)%*%(PP[k+1,]-PP[k,])
  k2=qv(0.5*(newp+oldp),oldc+0.5*k1)%*%(PP[k+1,]-PP[k,])
  k3=qv(0.5*(newp+oldp),oldc+0.5*k2)%*%(PP[k+1,]-PP[k,])
  k4=qv(newp,oldc+k3)%*%(PP[k+1,]-PP[k,])

  newc_4<-oldc+1/6*(k1+2*k2+2*k3+k4)



  newx = qv(newp, newc_4)

  resultRKA[k+1]<-newc_4
  resultRKB<-cbind(k,t(newp),newx,newc_4)
  print(resultRKB)

  oldc<-newc_4

}
resultRKA

coliRK<-matrix(0,n,1)
for(i in 1:n){
coliRK[i]<-resultRKA1[i]/resultRKA1[1]
}
coliRKB<-coliRK*100
coliRKB
```

A.3. Index Calculation

```
#####################################################################################
###               Index Calculation                                             ####
#####################################################################################
## Example : LA-AIDS demand system

## Recall estimated parameters from the demand system
alpha                                     # alpha (nx1)
beta                                      # beta  (nx1)
gamm                                      # gamma (nxm)
n <- 25                                   # sample period
k <- ncol(alpha)

#########################################################################################
P                                         # log price
PP                                        # Price

#########################################################################################
## Calculate the basic component
Pgamm<-P%*%gamm
lnA<-matrix(0,n,1)
lnB<-matrix(0,n,1)
#########################################################################################


#########################################################################################
## calculate the LnA and LnB
for(i in 1:nrow(P)){
lnA[i]<-t(alpha)%*%P[i,]+0.5*(t(Pgamm[i,])%*%P[i,])
}

for(i in 1:nrow(P)){
lnB[i]<-t(beta)%*%P[i,]
}

#########################################################################################
## calculation of expenditure
Yexpend<-exp(lnA)*100^exp(lnB)
Yexpend
#initial value of expenditure
y0<-exp(lnA[1,])*100^exp(lnB[1,])

#########################################################################################
## Calculate the Cost of living index

Coli<-matrix(0,n,1)

for (j in 1:nrow(Yexpend)){
Coli[j]<-Yexpend[j]/Yexpend[1]
}
COLI<-100*Coli
```

```
#####################################################################################
LTE<-log(Exp.All[,1])

#####################################################################################
## Calculate the Expenditure share of Model
wShare<-matrix(0,n,k)
for(j in 1:nrow(P)){
for(i in 1:k){
wShare[j,i]<- alpha[i]+gamm[i,]%*%P[j,]+beta[i]*(LTE[j]-lnA[j])
}
}

wShare<-matrix(0,n,k)
for(j in 1:nrow(P)){
for(i in 1:k){
wShare[j,i]<- alpha[i]+gamm[i,]%*%P[j,]+beta[i]*(log(Yexpend[j])-lnA[j])
}
}

## Laspeyres price ratio
LasPR<-matrix(0, n,k)
for(i in 1:nrow(PP)){
LasPR[i,]<-PP[i,]/PP[1,]
}
##Parssche price ratio
ParPR<-matrix(0, n,k)
for(i in 1:nrow(PP)){
ParPR[i,]<-PP[1,]/PP[i,]
}
## Log Price ratio of Laspeyres price rational
LogPR<-log(LasPR)

#####################################################################################
## Calculate the Layspeyres index
LasIndex<-matrix(0,n,1)
for(i in 1:n){
LasIndex[i]<-wShare[1,]%*%LasPR[i,]
}
LasprIndex<-LasIndex*100
#####################################################################################
## Calculate the Parssche index
ParIndex<-matrix(0,n,1)
for(i in 1:n){
ParIndex[i]<-wShare[i,]%*%ParPR[i,]
}
ParsscIndex<-1/ParIndex*100
#####################################################################################
## Calculate the Fisher Ideal index
FisIndex<-matrix(0,n,1)
for(i in 1:n){
FisIndex[i]<-(.5*log(LasIndex[i])-.5*log(ParIndex[i]))
}
FisherIndex<-exp(FisIndex)*100
#####################################################################################
```

```
## Calculate  Torqvist
TorIndex<-matrix(0,n,1)
TorIndex1<-matrix(0,n,k)
for(i in 1:k){
for(j in 1:n){
TorIndex1[j,i]<-.5*(wShare[1,i]+wShare[j,i])*LogPR[j,i]
}
}

for(i in 1:n){
TorIndex[i]<-sum(TorIndex1[i,])
}
TornqvIndex<-exp(TorIndex)*100
#########################################################################################
## Summary of Price index
## 1. Summary
SummaryOfIndex<-cbind(COLI,LasprIndex,ParsscIndex,FisherIndex, TornqvIndex)
SummaryOfIndex
## 2. Bias
SummaryOfSubBias<-cbind(LasprIndex-COLI,ParsscIndex-COLI,FisherIndex-COLI, TornqvIndex-COLI)
SummaryOfSubBias
#########################################################################################
```

B. Appendix 2 : RATS code for Demand estimation

B.1. LA-AIDS

```
****************************************************************************
* Almost Ideal Demand System Estimation Using RATS
****************************************************************************

****************************************************************************
* Read the data
****************************************************************************

OPEN DATA "Z:\RATS\Book1.xls"
DATA(FORMAT=XLS,ORG=COLUMNS)


****check the data which the RATS will use later
tables
****************************************************************************
** generate data
****************************************************************************

** 1. log of Expenditure
set lTE = log(TE);

** 2. log of Prices
set lp1 = log(P1); set lp2 = log(P2); set lp3 = log(P3);
set lp4 = log(P4); set lp5 = log(P5); set lp6 = log(P6)

** 3. Stone's Index
set lnP = w1*lp1+w2*lp2+w3*lp3+w4*lp4+w5*lp5+w6*lp6

set lTP = (lTE-lnP)


****************************************************************************
***** imposing the restriction on LA-AIDS
****************************************************************************
```

```
************ Note *******************
* in this analysis, we used 6 item categories
* After applying homogeneity assumption,
* we estimate only 5 of the 6 equation.
* we will remove the last one for estimation purpose.

**** Create the parameter and Demand system
* 1. Define the parameters which are used in the Rotterdam demand system
nonlin(parms=base) a1 a2 a3 a4 a5 a6 p11 p12 p13 p14 p15 p16  p21 p22 p23 p24 p25 p26  p31 p32
p33 p34 p35 p36  p41 p42 p43 p44 p45 p46  p51 p52 p53 p54 p55 p56 b1 b2 b3 b4 b5

* 2. Symmetry condition
nonlin(parms=symmetry) p12=p21 p13=p31 p14=p41 p15=p51 p23=p32 p24=p42
p25=p52 p34=p43 p35=p53 p45=p54

* 3. Homogeneity condition
nonlin(parms=homogeneity) p11+p12+p13+p14+p15+p16==0   p21+p22+p23+p24+p25+p26==0
p31+p32+p33+p34+p35+p36==0   p41+p42+p43+p44+p45+p46==0   p51+p52+p53+p54+p55+p56==0


****************************************************************************
*** Set up the LA-AIDS demand system
****************************************************************************

frml eq1 w1= a1 + p11*(lp1) + p12*(lp2) + p13*(lp3) + p14*(lp4) + p15*(lp5) + p16*lp6 + b1*lTP
frml eq2 w2= a2 + p21*(lp1) + p22*(lp2) + p23*(lp3) + p24*(lp4) + p25*(lp5) + p26*lp6 + b2*lTP
frml eq3 w3= a3 + p31*(lp1) + p32*(lp2) + p33*(lp3) + p34*(lp4) + p35*(lp5) + p36*lp6 + b3*lTP
frml eq4 w4= a4 + p41*(lp1) + p42*(lp2) + p43*(lp3) + p44*(lp4) + p45*(lp5) + p46*lp6 + b4*lTP
frml eq5 w5= a5 + p51*(lp1) + p52*(lp2) + p53*(lp3) + p54*(lp4) + p55*(lp5) + p56*lp6 + b5*lTP


****************************************************************************
**** set up the initial values of parameters
****************************************************************************
compute a1=a2=a3=a4=a5=0.0
compute p11=p12=p13=p14=p15=p16=p21=p22=p23=p24=p25=p26=p31=p32=p33=p34=p35=p36=p41=p42=p43
=p44=p45=p46=p51=p52=p53=p54=p55=p56=0.0
compute b1=b2=b3=b4=b5=0.0



****************************************************************************
*** estimate the demand system
****************************************************************************
* 1. without Restriction
nlsystem(parmset=base) / eq1 eq2 eq3 eq4 eq5

* 2. only with Symmetry
nlsystem(parmset=base+symmetry)  / eq1 eq2 eq3 eq4 eq5

* 3. only with Homogeneity
nlsystem(parmset=base+homogeneity)  / eq1 eq2 eq3 eq4 eq5
```

```
compute a1=a2=a3=a4=a5=0.0
compute p11=p12=p13=p14=p15=p16=p21=p22=p23=p24=p25=p26=p31=p32=p33=p34=p35=p36=p41=p42=p43=p44
=p45=p46=p51=p52=p53=p54=p55=p56=0.0
compute b1=b2=b3=b4=b5=0.0

* 4. symmetry+homogeneity
nlsystem(parmset=base+symmetry+homogeneity)  / eq1 eq2 eq3 eq4 eq5
```

B.2. Rotterdam

```
*****************************************************************************
* Rotterdam Demand System Estimation Using RATS
*****************************************************************************

*****************************************************************************
* Read the data
*****************************************************************************

OPEN DATA "Z:\RATS\Book1.xls"
DATA(FORMAT=XLS,ORG=COLUMNS)

*check the data which the RATS will use later
tables

*****************************************************************************
** generate data
*****************************************************************************

*********Divisia Quantity Index ******************
set DivisiaQ = sw1*dq1 +  sw2*dq2 +  sw3*dq3 +  sw4*dq4 +  sw5*dq5 +  sw6*dq6


********* dependent variable in Rotterdam demand system
set wdq1 = sw1*dq1 ; set wdq2 =sw2*dq2
set wdq3 = sw3*dq3 ; set wdq4 =sw4*dq4
set wdq5 = sw5*dq5 ; set wdq6 =sw6*dq6


*check the variable after generating process
tables


*****************************************************************************
**** Rotterdam demand Analysis
*****************************************************************************

********* define the Rotterdam demand system
set Rdq = wdq1 +wdq2 +wdq3 + wdq4 +wdq5 +wdq6

***** imposing the restriction on The Rotterdam demand system

* Homogeneity condition
************ Note  ********************
* in this analysis, we used 6 item categories
* After applying homogeneity assumption,
* we estimate only 5 of the 6 equation.
* we will remove the last one for estimation purpose.

**** Create the parameter and Demand system
* 1. Define the parameters which are used in the Rotterdam demand system
nonlin(parms=base) a1 a2 a3 a4 a5  p11 p12 p13 p14 p15 p21 p22 p23 p24 p25 p31
```

```
p32 p33 p34 p35 p41 p42 p43 p44 p45 p51 p52 p53 p54 p55

* 2. This is for the last equation
nonlin(parms=relax) ep16 ep26 ep36 ep46 ep56

* Symmetry condition
nonlin(parms=symmetry) p12=p21 p13=p31 p14=p41 p15=p51 p23=p32 p24=p42
p25=p52 p34=p43 p35=p53 p45=p54

****************************************************************************
*** Set up the Rotterdam demand system with homogeneity
****************************************************************************

frml eq1 wdq1= a1*DivisiaQ + p11*(dp1-dp6) + p12*(dp2-dp6) + p13*(dp3-dp6)
                + p14*(dp4-dp6) + p15*(dp5-dp6) + ep16*dp6
frml eq2 wdq2= a2*DivisiaQ + p21*(dp1-dp6) + p22*(dp2-dp6) + p23*(dp3-dp6)
                + p24*(dp4-dp6) + p25*(dp5-dp6) + ep26*dp6
frml eq3 wdq3= a3*DivisiaQ + p31*(dp1-dp6) + p32*(dp2-dp6) + p33*(dp3-dp6)
                + p34*(dp4-dp6) + p35*(dp5-dp6) + ep36*dp6
frml eq4 wdq4= a4*DivisiaQ + p41*(dp1-dp6) + p42*(dp2-dp6) + p43*(dp3-dp6)
                + p44*(dp4-dp6) + p45*(dp5-dp6) + ep46*dp6
frml eq5 wdq5= a5*DivisiaQ + p51*(dp1-dp6) + p52*(dp2-dp6) + p53*(dp3-dp6)
                + p54*(dp4-dp6) + p55*(dp5-dp6) + ep56*dp6

****************************************************************************
**** set up the initial values of parameters
****************************************************************************
compute a1=a2=a3=a4=a5=0.0
compute p11=p12=p13=p14=p15=p16=p21=p22=p23=p24=p25=p26=p31=p32=p33=p34=p35
=p36=p41=p42=p43=p44=p45=p46=p51=p52=p53=p54=p55=p56=0.0
compute ep16=ep26=ep36=ep46=ep56=0.0


****************************************************************************
*** estimate the demand system
****************************************************************************

* 1. with Homogeniety only
nlsystem(parmset=base) / eq1 eq2 eq3 eq4 eq5
nlsystem(parmset=base+relax) / eq1 eq2 eq3 eq4 eq5

* 2. with Symmetric and Homogeneity
compute ep16=ep26=ep36=ep46=ep56=0.0
nlsystem(parmset=base+symmetry)  / eq1 eq2 eq3 eq4 eq5
```