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DIVERSIFICATION IN A THREE-MOMENT WORLD

*Michael A. Simkowitz and William L. Beedles**

I. Introduction

Of the behavioral recommendations garnered from modern capital market theory, few, if any, generalizations have been documented as convincingly as the simple advice to hold several assets in one's portfolio. Sharpe made such a conclusion perfectly clear when he stated [27, p. 184]:

> If the market is efficient and if an investor is privy to no special information or predictive power, what should he do? First, and most important: diversify.

Such advice is supported by the theoretical capital asset pricing model (CAPM) of Sharpe [28], Lintner [20], Mossin [23], and others, and by the empirical results of, among others, Evans and Archer [11] and Fielitz [13]. These empirical works demonstrated that virtually all of the risk (as measured with standard deviation or mean absolute deviation of returns) of a portfolio that can be diversified away is eliminated in a portfolio with a small number of securities.

The intent here is to show that diversification is not necessarily desirable for investors who base their decisions on the first three moments of return distributions. This task is accomplished as follows: Part II includes a brief literature review noting some works concerned with skewness of returns. Next, the asset combination process is stated algebraically and the theoretical behavior of portfolio skew is examined. The empirical distribution of characteristic function errors is examined in Part IV, and a discussion of the behavior of portfolio distributional statistics follows. Finally, some conclusions are drawn and behavioral recommendations forwarded.

II. Three-Parameter Decision Making

Following the lead of Markowitz [21], modern capital market theory has been couched in terms of the first two moments of investment return distributions--the mean and the variance. As a formal matter, use of only the first two moments

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* *Indiana University, and University of Kansas, respectively.* *The authors wish to thank an anonymous referee and participants in Indiana's Finance Workshop for helpful comments on earlier drafts.*

is quite restrictive since only quadratic utility functions or normally distributed asset returns are appropriate. Although Tsiang [29, 30] has provided persuasive arguments favoring the use of quadratic utility as a useful approximation in practical problems, many remain skeptical of its adequacy [7, 9, 19]. Furthermore, Fama [12] and others have provided evidence which indicates that normality is not an empirically justifiable assumption for return distributions of common stocks.

Relatively recently, considerable attention has been directed to the topic of skewness of asset and portfolio returns. Arditti [2, 3] has shown the empirical importance of skewness as an explainer of ex-post returns and Jean [16, 17] has made several important normative contributions to the subject. Francis [14] has presented evidence from which one may infer that skewness is not as important as some may think, although Arditti [4] has taken him to task. Arditti and Levy [5] have developed a multi-period three-parameter partial equilibrium model, while more recently, Kraus and Litzenberger [18] have provided a three-moment variation of the single-period asset pricing model based upon separable utility functions. In a work relevant to the present one, Reback [27] has used hypothetical return distributions in a Monte Carlo simulation to study the distributional characteristics of portfolios of options. Finally, McEnally [22] has used individual asset returns and presented evidence which supports the notion of skewness preference. However, since investors often own multi-asset portfolios, the behavior of distribution measures of diversified holdings is of interest. The theoretical behavior of the measures is investigated in the next section.

III. The Analytics of Portfolios Skew

The excess return of any asset i can be generally stated as:

$$\tilde{R}_i - R_f = a_i + \tilde{S}_i + \tilde{\varepsilon}_i \qquad (1)$$

where circumflexes denote random variables, $\tilde{R}_i$ is the asset's total return, R_f is the return on riskfree assets, a_i is a unique return portion, $\tilde{S}_i$ represents a systematic return portion, and $\tilde{\varepsilon}_i$ is an error term with the following properties:

$$E(\tilde{\varepsilon}_i) = 0 \ , \qquad (2a)$$

$$E(\tilde{\varepsilon}_i \tilde{\varepsilon}_j) = 0 \ , \text{ and} \qquad (2b)$$

(2c) $$E(\tilde{S}_i \tilde{\varepsilon}_i) = 0 ,$$

where E is the expectation operator.

Subtracting the expectation of (1) from (1), squaring, taking expectations, and using (2) yields

(3) $$\sigma^2_{\tilde{R}_i} = \sigma^2_{\tilde{S}_i} + \sigma^2_{\tilde{\varepsilon}_i}$$

where σ^2 denotes the variance. Equation (3) is simply a representation of the well-known generalization that asset return variance can be divided into systematic risk (variance) and unsystematic risk (variance).

Analogously, subtracting the expectation of (1) from (1), cubing, taking expectations, and noting (2) yields:

(4) $$\begin{aligned} {}_{\tilde{R}_i}\mu^3 = E(\tilde{R}_i - \bar{R}_i)^3 &= {}_{\tilde{S}_i}\mu^3 + {}_{\tilde{\varepsilon}_i}\mu^3 \\ &+ 3\{E[(\tilde{S}_i - \bar{S}_i)^2 \tilde{\varepsilon}_i] + E[(\tilde{S}_i - \bar{S}_i)\tilde{\varepsilon}_i^{\,2}]\} \end{aligned}$$

where $\bar{X}_i$ and ${}_{\tilde{X}_i}\mu^3$ represent, respectively, the expectation and skew of any random variable. Hence, the third moment of $\tilde{R}_i$ $(= {}_{\tilde{R}_i}\mu^3)$ can be separated into four terms: the systematic skew $(= {}_{\tilde{S}_i}\mu^3)$, the unsystematic skew $(= {}_{\tilde{\varepsilon}_i}\mu^3)$, and two cross-product terms involving the cross-expectations between the error term and the systematic deviations. Note that when the model (1) is properly specified, the cross-product terms will equal zero. In particular, $E[(\tilde{S}_i - \bar{S}_i)^2 \tilde{\varepsilon}_i] = 0$ when $(\tilde{S}_i - \bar{S}_i)^2$ is not a determinant of $\tilde{R}_i$ (that is, when the model does not suffer from excluded variable bias). The last term in (4)--$E[(\tilde{S}_i - \bar{S}_i)\tilde{\varepsilon}_i^2]$--equals zero when the error terms are homoscedastic with respect to $(\tilde{S}_i - \bar{S}_i)$.

The foregoing analysis can be applied to any risky portfolio p. Rewriting (1) gives:

(1') $$(\tilde{R}_{p_{(N)}} - R_f) = a_{p_{(N)}} + \tilde{S}_{p_{(N)}} + \varepsilon_{p_{(N)}}$$

where the subscript $p_{(N)}$ represents a portfolio composed of N securities. Since $\tilde{\varepsilon}_{p_{(N)}}$ approaches zero as N approaches the number of assets in the market

(M), (3) can be rewritten:

$$(3') \qquad \sigma^2_{\tilde{R}_{p(N)}} = \alpha^2_{\tilde{S}_{p(N)}} \quad \text{as } N \to M.$$

Analogously, (4) may be restated as:

$$(4') \qquad \tilde{R}_{p(N)}\, \mu^3 = \tilde{S}_{p(N)}\, \mu^3 \quad \text{as } N \to 0$$

if the cross expectations are indeed zero.[1]

Thus, since systematic expected return ($\bar{S}_{p(N)}$), variance ($\sigma^2_{\tilde{S}_{p(N)}}$), and skewness (${}_{S_{p(N)}}M\ \mu^3$) remain constant as diversification occurs ($N \to M$), portfolio variance decreases (since $\sigma^2_{\varepsilon_{p(N)}} \geq 0$), and portfolio skewness either decreases, remains unchanged, or increases depending upon the second term of equation (4) taking on a value that is positive, zero, or negative. Thus, the behavior of $\tilde{\varepsilon}_{p(N)}\ \mu^3$ will be the major determinant of the nature of the change in portfolio skew as diversification occurs.[2] Consequently, if the skew of the error is

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[1]As an empirical matter, the likelihood of the cross products precisely equalling zero is small. Thus, the "speed" with which (4) approaches (4') will depend upon the relative magnitudes of the terms in (4). The estimates of these values, which are provided in footnote 4, Section IV below, indicate that the unsystematic skew is indeed the dominant factor.

[2]This observation is the analogue to the reduction of variance with diversification in the two-moment model, which requires less than perfect correlation between the errors $\tilde{\varepsilon}_i$ and $\tilde{\varepsilon}_j$. When considering skewness, though, the behavior of the triplet $\tilde{\varepsilon}_i$, $\tilde{\varepsilon}_j$, $\tilde{\varepsilon}_k$ is interesting. To see this, let $\tilde{Z}$ represent a linear combination of $\tilde{\varepsilon}_i$ with (for simplicity) equal weighting. That is:

$$\tilde{Z} = \sum_i^N \frac{1}{N} \tilde{\varepsilon}_i .$$

Since the errors have zero expectations, the skew of $\tilde{Z}$ is simply:

$$E(\tilde{Z}^3) = \sum_i \sum_j \sum_k \frac{1}{N^3} E(\tilde{\varepsilon}_i \tilde{\varepsilon}_j \tilde{\varepsilon}_k) .$$

If $\underset{\neq}{\Sigma\Sigma\Sigma}$ designates the triple sum excluding the terms where i=j=k, then

$$E(\tilde{Z}^3) = \sum_i \frac{1}{N^3} E(\tilde{\varepsilon}_i^3) + \underset{\neq}{\Sigma\Sigma\Sigma} \frac{1}{N^3} E(\tilde{\varepsilon}_i \tilde{\varepsilon}_j \tilde{\varepsilon}_k) .$$

For notational ease, designate

positive/zero/negative, diversification will destroy/leave unchanged/increase portfolio skew. This prospect is empirically examined in the next two sections.

IV. The Distribution of the Errors

The data used for this study are the same as those used for [22] as corrected in [6]--the monthly holding period returns of the 549 common stocks that were continuously listed on the New York Stock Exchange during the period January 1945-December 1965. Fisher's Arithmetic Index and the yield on 30-day Treasury Bills have been used to measure the returns of the market portfolio and riskfree assets respectively.

For estimation, the following regression equation was used:

(5)
$$\tilde{R}_i = \hat{a}_i + \hat{b}_i \tilde{R}_{m_t} + \tilde{e}_{i_t}$$

$$\bar{\mu}^3_{\tilde{\varepsilon}_i} = \frac{\Sigma E(\tilde{\varepsilon}_i^3}{N} \quad \text{and}$$

$$\bar{\mu}_{\tilde{\varepsilon}_i\tilde{\varepsilon}_j\tilde{\varepsilon}_k} = \frac{\Sigma \underset{\neq}{\Sigma} \Sigma E(\tilde{E}_i \tilde{E}_j \tilde{E}_k)}{N^3 - N}$$

Now,

$$E(\tilde{Z}^3) = \frac{1}{N^2} \bar{\mu}^3_{\tilde{\varepsilon}_i} + \frac{N^3 - N}{N^3} (\bar{\mu}_{\tilde{\varepsilon}_i\tilde{\varepsilon}_j\tilde{\varepsilon}_k}) .$$

Differentiating this expression with respect to the number of assets, N, yields:

$$\frac{\partial E(\tilde{Z}^3)}{\partial N} = -2N^{-3} \bar{\mu}^3_{\tilde{\varepsilon}_i} + 2N^{-3} \bar{\mu}_{\tilde{\varepsilon}_i\tilde{\varepsilon}_j\tilde{\varepsilon}_i}$$

which is negative if

$$\bar{\mu}^3_{\tilde{\varepsilon}_i} > \bar{\mu}_{\tilde{\varepsilon}_i\tilde{\varepsilon}_j\tilde{\varepsilon}_k} , \text{ and}$$

$$\bar{\mu}^3_{\tilde{\varepsilon}_i} > 0.$$

Note that the two extremes of the right-hand side of the first inequality are

$$\bar{\mu}_{\tilde{\varepsilon}_i\tilde{\varepsilon}_j\tilde{\varepsilon}_k} = 0 ,$$

which would be a result of statistical independence [see 1, pp. 8-9], and

$$\bar{\mu}_{\tilde{\varepsilon}_i\tilde{\varepsilon}_j\tilde{\varepsilon}_k} = \bar{\mu}^3_{\tilde{\varepsilon}_i} ,$$

which occurs if the errors are perfectly correlated.

The conclusion from all of this is that diversification will decrease skew as long as the errors are less than perfectly correlated (that is, statistical independence of the errors is not necessary) and on average are positively skewed. The former has been established elsewhere (for example, [25]), and the latter is examined in the next section.

where $\tilde{R}_{m_t}$ represents market returns, $\hat{a}_i$ and $\hat{b}_i$ are estimated parameters, and the $\tilde{e}_{i_t}$ are estimated residuals. This equation, which is simply the conventional market model,[3] has been estimated with ordinary least squares over the 252 months for all 549 securities. The resulting 138,348 error terms have been stratified into a frequency distribution, which is presented in Table 1. In the columns of Table 1 headed "Range," the error magnitudes are specified, while in the other columns the frequency of errors occurring within the associated ranges is given. Note that four residuals were greater than 1.00 and are so reported.

A careful inspection of the data presented in Table 1 reveals that the regression errors are markedly right skewed around the mean of zero.[4] Moreover, the essence of the Table 1 data is summarized in Table 2 below, where the number of errors falling within particular "standard deviation (SD) bounds" is given.[5]

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[3]Since a specific equilibrium pricing model has not been assumed here, any number of regression equations could have been used. The market model was selected due to its simplicity and popularity in the extant literature.

In any event, all the empirics of this section have also been conducted using residuals from the following:

$$(\tilde{R}_{i_t} - R_{f_t}) = \hat{a}'_i + \hat{b}'_{1_i}(\tilde{R}_{m_t} - R_{f_t}) + \hat{b}'_{2_i}(\tilde{R}_{m_t} - \bar{R}_m)^2 + \tilde{e}'_{i_t} .$$

Kraus and Litzenberger [18] have advanced this specification when investors prefer skewness and have separable utility functions. Moreover, data from the market model are reported herein since the results are virtually identical with either specification.

[4]The phrase "markedly right skewed" is used rather than "significantly right skewed" since, as a formal matter, tests for nonsymmetry are contingent upon some assumption about the population distribution. One test, proposed by Pearson [see 24 and 10, pp. 201-208 and 620-621], rejects the null hypothesis of normality on the basis of sample skew. For the residuals reported in Table 1, the computed standardized skew is 1.38 which far exceeds the extreme tabulated critical value of .081 (for a sample size of 5000 at the .01 significance level). Further, even when the four extremely positive (greater than +1.0) errors are ignored, the value is 1.10, which also far exceeds the critical value of .081.

The contention of Section III above (see especially footnote 1)--that the unsystematic skew would be "the major determinant of the change in portfolio skew"--can now be documented. The average measured unsystematic skew of the individual assets equals +.000358, while (three times (see equation (4))) the average of the cross-products terms equals +.000108 (with the cross-product of market deviations with squared residuals making up nearly 98 percent of the latter). Hence, the cross-products certainly are not precisely equal to zero, but the skews of the residuals are definitely dominant.

[5]The measured standard deviation of the errors is .063704. If the four extreme positive observations are ignored, the measured standard deviation is .063267. The former is used for Table 2.

Table 1

Distribution of Regression Errors

Range	Number of Errors	Range	Number of Errors	Range	Number of Errors	Range	Number of Errors
-1.00 to -.98	0	-.50 to -.48	2	0.0 to .02	20487	.50 to .52	3
-.98 to -.96	0	-.48 to -.46	2	.02 to .04	15413	.52 to .54	2
-.96 to -.94	0	-.46 to -.44	0	.04 to .06	10071	.54 to .56	6
-.94 to -.92	0	-.44 to -.42	0	.06 to .08	6445	.56 to .58	4
-.92 to -.90	0	-.42 to -.40	0	.08 to .10	4074	.58 to .60	3
-.90 to -.88	0	-.40 to -.38	1	.10 to .12	2551	.60 to .62	5
-.88 to -.86	0	-.38 to -.36	0	.12 to .14	1647	.62 to .64	3
-.86 to -.84	0	-.36 to -.34	1	.14 to .16	1028	.64 to .66	2
-.84 to -.82	1	-.34 to -.32	3	.16 to .18	672	.66 to .68	3
-.82 to -.80	0	-.32 to -.30	10	.18 to .20	438	.68 to .70	0
-.80 to -.88	0	-.30 to -.28	8	.20 to .22	284	.70 to .72	2
-.78 to -.76	0	-.28 to -.26	20	.22 to .24	246	.72 to .74	0
-.76 to -.74	0	-.26 to -.24	32	.24 to .26	159	.74 to .76	0
-.74 to -.72	0	-.24 to -.22	58	.26 to .28	111	.76 to .78	0
-.72 to -.70	0	-.22 to -.20	91	.28 to .30	87	.78 to .80	3
-.70 to -.68	0	-.20 to -.18	173	.30 to .32	63	.80 to .82	0
-.68 to -.66	0	-.18 to -.16	309	.32 to .34	47	.82 to .84	1
-.66 to -.64	0	-.16 to -.14	603	.34 to .36	35	.84 to .86	1
-.64 to -.62	0	-.14 to -.12	1276	.36 to .38	36	.86 to .88	1
-.62 to -.60	0	-.12 to -.10	2451	.38 to .40	27	.88 to .90	0
-.60 to -.58	2	-.10 to -.08	4680	.40 to .42	24	.90 to .92	0
-.58 to -.56	1	-.08 to -.06	8316	.42 to .44	12	.92 to .94	0
-.56 to -.54	0	-.06 to -.04	13875	.44 to .46	18	.94 to .96	1
-.54 to -.52	0	-.04 to -.02	19622	.46 to .48	9	.96 to .98	0
-.52 to -.50	0	-.02 to 0.00	22771	.48 to .50	12	.98 to 1.00	0
						greater than 1.00	4

Table 2

Error Distribution Summary

Standard Deviation Bounds	Frequency of Negative Errors	Frequency of Positive Errors
between 0 and 1 SDs	57974	47255
between 1 and 2 SDs	14279	12395
between 2 and 3 SDs	1744	2955
between 3 and 4 SDs	247	852
between 4 and 5 SDs	51	307
between 5 and 6 SDs	4	128
between 6 and 7 SDs	1	68
between 7 and 8 SDs	4	39
between 8 and 9 SDs	1	9
between 9 and 10 SDs	2	14
greater than 10 SDs	1	18

As can be seen from the Table 2 data, the skew of the errors is not the result of a small number of outliers. In other words, not only are there more "extreme" positive residuals (e.g., greater than six standard deviations), but more positive "moderate" (e.g., between two and four standard deviations) observations occur also.

To summarize, the analytics of Section III demonstrated that positively (negatively) skewed errors from equation (1) imply decreasing (increasing) portfolio skew with diversification. In this section the errors have been shown to be positively skewed in point of fact. In Part V, portfolios are constructed to confirm the implication.

V. The Behavior of Portfolio Skew with Increasing Diversification

Portfolios containing different numbers of stocks have been constructed in the following manner. First, 549 portfolios containing one stock each were built. (Stocks were selected randomly without replacement.) The 252 monthly holding period returns of each portfolio were used to compute the mean return and temporal variance and raw skew of the return. Additionally, the ratios of the percentage return (mean holding period return less unity) to the standard deviation, and cube root of the raw skew to the standard deviation were computed for each portfolio. The arithmetic averages of these five values were struck across the 549 portfolios and the results reported on the first line of Table 3 below. Next 274 (= 549 ÷ 2 truncated) portfolios, containing two randomly selected stocks each, were built. Each portfolio's statistics were computed, the averages struck across the portfolios, and the results reported on the second line of Table 3. The process was continued through 10 portfolios with 50 stocks each. Finally, one portfolio containing all 549 stocks was constructed.

Reading down Table 3, return remains, for all practical purposes, constant, as it should under a random selection scheme. As previous studies have shown, portfolio dispersion decreases with diversification. Nearly 80 percent (=(.00582320 - .00247009) ÷ (.00582320 - .00161643)) of the unsystematic risk is diversified away by the five-stock level.

Of particular interest here is the behavior of skewness as the degree of diversification increases. As predicted in Parts III and IV above, raw portfolio skew decreases as the number of assets in the portfolio increases.[6] Furthermore, skew is diversified away rapidly. Over 92 percent (=(.00046209 -

[6]Sampling error explains the nonmonotonic nature of the variance and skew as they approach their asymptotes. The same phenomenon can be seen in the Evans and Archer [11, p. 765] and Fielitz [13, p. 56] works.

Table 3

Distributional Measures Averaged Over Portfolios of Varying Sizes

Portfolio Size	Number of Portfolios	Holding Period Return	Variance (X100)	Raw Skewness (X100)	Percentage Return to Standard Deviation	Cube Root of Raw Skew to Standard Deviation
1	549	1.012858	.582320	.046209	.184332	.743956
2	274	1.012864	.373904	.013092	.220674	.608219
3	183	1.012858	.301102	.005399	.241560	.463321
4	137	1.012854	.265995	.002614	.255659	.323158
5	109	1.012888	.247009	.001339	.265002	.165966
6	91	1.012851	.230559	.000571	.272274	.033614
7	78	1.012863	.220846	.000159	.278376	-.036336
8	68	1.012852	.213705	-.000253	.282876	-.112328
9	61	1.012858	.207547	-.000607	.285756	-.262522
10	54	1.012859	.204138	-.000663	.288350	-.291118
11	49	1.012828	.199236	-.000827	.290385	-.328297
12	45	1.012883	.197638	-.000927	.293570	-.366999
13	42	1.012852	.194718	-.001058	.295091	-.428894
14	39	1.012863	.191679	-.001095	.297142	-.438736
15	36	1.012854	.189271	-.001168	.298658	-.472699
16	34	1.012862	.187970	-.001300	.300001	-.493856
17	32	1.012863	.183224	-.001358	.301469	-.504947
18	30	1.012873	.185016	-.001392	.301468	-.535639
19	28	1.012886	.182831	-.001351	.304458	-.526371
20	27	1.012871	.182894	-.001406	.303385	-.536918
21	26	1.012856	.181388	-.001555	.304237	-.559794
22	24	1.012901	.179137	-.001566	.306789	-.569712
23	23	1.012893	.178154	-.001582	.307511	-.570548
24	22	1.012910	.179782	-.001579	.306357	-.563376
25	21	1.012854	.176274	-.001569	.308253	-.593971
26	21	1.012842	.177991	-.001575	.306297	-.582409
27	20	1.012853	.176704	-.001665	.307606	-.598178
28	19	1.012849	.176957	-.001652	.307070	-.569961
29	18	1.012884	.177486	-.001647	.307835	-.597023
30	18	1.012859	.175716	-.001704	.308587	-.606409
31	17	1.012847	.176699	-.001773	.307827	-.613128
32	17	1.012869	.175197	-.001714	.308862	-.613229
33	16	1.012858	.175679	-.001728	.308572	-.611186
34	16	1.012877	.174641	-.001747	.309457	-.617575
35	15	1.012892	.174225	-.001778	.310022	-.620476
36	15	1.012853	.173763	-.001753	.309922	-.620478
37	14	1.012888	.174442	-.001765	.310414	-.619350
38	14	1.012880	.171290	-.001818	.312455	-.629580
39	14	1.012841	.171549	-.001866	.311292	-.636196
40	13	1.012898	.171499	-.001765	.312528	-.626326

Table 3 (continued)

Distributional Measures Averaged Over Portfolios of Varying Sizes

Portfolio Size	Number of Portfolios	Holding Period Return	Variance (X100)	Raw Skewness (X100)	Percentage Return to Standard Deviation	Cube Root of Raw Skew to Standard Deviation
41	13	1.012886	.172450	-.001884	.312189	-.635222
42	13	1.012851	.171626	-.001840	.311447	-.635579
43	12	1.012811	.172197	-.001822	.309618	-.632309
44	12	1.012903	.170868	-.001861	.313384	-.637473
45	12	1.012884	.171171	-.001827	.313048	-.634564
46	11	1.012867	.168132	-.001769	.315047	-.635801
47	11	1.012872	.168668	-.001830	.314610	-.640067
48	11	1.012858	.171437	-.001859	.311550	-.638227
49	11	1.012867	.169788	-.001889	.313483	-.645510
50	10	1.012810	.171265	-.001856	.310755	-.638454
549	1	1.012858	.161643	-.002133	.319813	-.689813

N.B.: The tabulated values are figures averaged across all portfolios of a particular size. Thus, the last two columns cannot be computed from the other tabulated data. For example, for the 10 portfolios with 50 securities, the values below are relevant:

Variance (X100)	Raw Skewness (X100)	Cube Root of Raw Skew to Standard Deviation
.1877612	-.0027752	-.6987055
.1694743	-.0017051	-.6252167
.1517732	-.0015653	-.6420987
.1614906	-.0018250	-.6551643
.1703259	-.0024849	-.7070764
.2033049	-.0011344	-.4983359
.1648817	-.0020168	-.6703517
.1628568	-.0018550	-.6559637
.1940222	-.0014429	-.5527041
.1467557	-.0017593	-.6789214

The average of the third column here equals the average of the last column of the table for the 50 stock portfolio size (with slight rounding errors).

(.00001339)) ÷ (.00046209 - (-.00002133))) of diversifiable skew is eliminated by the five-stock level.

Finally, the ratios of mean return to standard deviation and cube root of skew to standard deviation, averaged across the portfolios, are presented in Table 3.[7] Conventional wisdom suggests investment evaluation be conducted by comparing the ratio of the first and second moments. Such a ratio is, of course, everywhere increasing with diversification. However, Arditti [3, p. 911] has noted that the ratio of skew to dispersion provides further insights into investment performance. As can be seen from Table 3, this ratio is everywhere *decreasing* with diversification. The behavior of this ratio suggests that skew per unit of risk decreases with increasing diversification.[8]

These results pose an obvious dilemma. Consider individual i's utility function (U_i) with return, dispersion, and skew as arguments:

$$U_i = a_i R - b_i(\sigma_R^2) + c_i({}_R\mu^3) \tag{7}$$

and let $\theta_i \triangleq c/b$ measure the individual's concern for skewness vis-à-vis dispersion (here dubbed "skewness/variance awareness").[9] Traditionally c_i has been set equal to zero (quadratic utility functions have been used) and/or

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[7] The last ratio is in point of fact simply the cube root of the popular relative skewness measure, defined as the (here estimated) third central moment of the distribution divided by the cube of the (estimated) standard deviation.

[8] In addition to the reported results, several modifications of the empirics were investigated. First, in an attempt to capture a multi-period characterization of the tests, natural logs of portfolio holding period returns were taken before computing the relevant statistics. The only change of importance occurred in the return statistics. Portfolio dispersion was of course diversified away, and since the geometric mean is inversely related to the standard deviation, return increased with increasing N. The returns of these portfolios converged to their asymptotes (the return on the 549 stock portfolio) quickly, and importantly, skew computed around the geometric mean decreased with diversification.

Further, the same procedures outlined above were conducted for 12 subperiods, 10 two-year subperiods (1946-1947; 1948-1949;. . . .; 1964-1965), and two 10-year subperiods, 1946-1955 and 1956-1965. Once again, the generalizations remain unchanged. In other words, the fact that skew was diversified away was consistently observed. Furthermore, the speed of diversification was very similar in the shorter subperiods.

[9] A decision rule for investors with cubic utility functions, which equation (7) essentially represents, has been developed by Hanoch and Levy [15]. However, as they point out, the criterion is never satisfied when the mean return does not change [15, p. 189]. As noted above, the random selection construction scheme adopted herein results in constant (aside from sampling error) returns between portfolios.

$_{R}\mu^3$ has been ignored or set to zero (symmetric distributions have been assumed). Hence, since the fact is well established that dispersion is everywhere decreasing with the number of assets in a portfolio, total diversification has been suggested as optimal for all investors. At the other (admittedly unlikely) extreme would be $b_i = 0$, i.e., total concern for return and skew and neutrality to dispersion. Thus, θ_i would be undefined and no diversification would be appropriate. A more likely situation is for $b_i, c_i > 0$, such that values of θ_i small in magnitude denote intense concern for dispersion vis-à-vis skew and large θ_i's correspond to relatively more concern for skew. Some of these intermediate values of θ_i suggest some, but *not* total, diversification.

VI. Commentary

Many investors hold less than perfectly diversified portfolios [8], a phenomenon in contradiction with frequently offered advice. The results reported in this essay may be interpreted as suggesting that the contradiction may be the result of the inadequacy of the traditional two-parameter normative framework. In particular, if positive skewness is a desirable characteristic of return distributions, then the fact that the simple act of diversification destroys skew is a likely explanation of observed behavior. Moreover, the results presented herein suggest that even in a perfect, frictionless market, some investors *should* hold a limited number of assets in their portfolios, the exact number being a function of each individual's skewness/variance awareness. Those who are most concerned with skew (dispersion) should hold a relatively small (large) number of assets in their portfolios.

Also, these results have heavy implications for the builders of asset pricing models. The generalization that all investors individually hold combinations of a reference portfolio will be unambiguously appropriate only if the assumption that those investors have separable utility is invoked.

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