A Simple Mixture Theory for ν Newtonian and Generalized Newtonian Constituents

By

Michael J. Powell

Submitted to the graduate degree program in Mechanical Engineering

and the Graduate Faculty of the University of Kansas

in partial fulfillment of requirements for the degree of

Master of Science.

Dr. Karan S. Surana, Chairperson

Dr. Peter W. Tenpas

Dr. Bedru Yimer

Date Defended: June 4, 2012

The Thesis Committee for Michael J. Powell certifies

that this is the approved version of the following thesis:

A Simple Mixture Theory for ν Newtonian and Generalized Newtonian Constituents

---

Dr. Karan S. Surana, Chairperson

Date Approved: June 7, 2012

Abstract

This work presents development of mathematical models based on conservation laws for a saturated mixture of ν homogeneous, isotropic, and incompressible constituents for isothermal flows. The constituents and the mixture are assumed to be Newtonian or generalized Newtonian fluids. Power law and Carreau-Yasuda models are considered for generalized Newtonian shear thinning fluids. The mathematical model is derived for a ν constituent mixture with volume fractions ϕ_α using principles of continuum mechanics: conservation of mass, balance of momenta, first and second laws of thermodynamics, and principles of mixture theory yielding continuity equations, momentum equations, energy equation, and constitutive theories for mechanical pressures and deviatoric Cauchy stress tensors in terms of the dependent variables related to the constituents. It is shown that for Newtonian fluids with constant transport properties, the mathematical models for constituents are decoupled. In this case one could use individual constituent models to obtain constituent deformation fields, and then use mixture theory to obtain the deformation field for the mixture. In the case of generalized Newtonian fluids, the dependence of viscosities on deformation field does not permit decoupling. Numerical studies are also presented to demonstrate this aspect. Using fully developed flow of Newtonian and generalized Newtonian fluids between parallel plates as a model problem, it is shown that partial pressures p_α

of the constituents must be expressed in terms of the mixture pressure p. In this work we propose $p_\alpha = \phi_\alpha p$ and $\sum_\alpha^\nu p_\alpha = p$ which implies $\sum_\alpha^\nu \phi_\alpha = 1$ which obviously holds. This rule for partial pressure is shown to be valid for a mixture of Newtonian and generalized Newtonian constituents yielding Newtonian and generalized Newtonian mixture. Modifications of the currently used constitutive theories for deviatoric Cauchy stress tensor are proposed. These modifications are demonstrated to be essential in order for the mixture theory for ν constituents to yield a valid mathematical model when the constituents are the same. Dimensionless form of the mathematical models are derived and used to present numerical studies for boundary value problems using finite element processes based on a residual functional i.e. least squares finite element processes in which local approximations are considered in $H^{k,p}\left(\bar{\Omega}^e\right)$ scalar product spaces. Fully developed flow between parallel plates and 1:2 asymmetric backward facing step are used as model problems for a mixture of two constituents.

Acknowledgments

I would like to thank my adviser Dr. Karan S. Surana for chairing my thesis committee. His support and advice has been extremely valuable during my time in the masters program and during the work on this thesis. Further, I would like to thank Dr. Surana for giving me the opportunity to work with Dr. J. N. Reddy at Texas A & M. I look forward to continuing my studies and to working with Dr. Surana and Dr. Reddy in the future. I would also like to thank Dr. Peter Tenpas and Dr. Bedru Yimer for serving on my committee. I would especially like to thank Dr. Albert Romkes for convincing to become a graduate student and for his support throughout my time at KU. Dr. Romkes is truly one of the greatest educators that I have ever known. Finally, I would like to thank my parents Mike and Ann Powell for everything that they have done to help get me to this point. My gratitude for their tireless and loving support cannot be overstated.

This research was supported by a grant from ARO, Mathematical sciences division under the grant number FED0061541 to the University of Kansas, Lawrence, Kansas and Texas A & M University, College Station, Texas. The authors are grateful to Dr. Joseph D. Myers, Program Manager, Scientific Computing, ARO.

Contents

List of Figures

Chapter 1

Introduction, Literature Review, and Scope of Work

1.1 Introduction and Literature Review

Most of the literature on mixture theories can be divided into two major categories: theories based on volume averaging and theories based on the principles of continuum mechanics. The primary focus of this thesis is on mixture theories based on principles of continuum mechanics. Theories based on volume averaging involve applying volume and/or time integrals over a heterogeneous mixture to obtain "averaged" properties of the mixture. While these techniques may be useful due to their ability to reduce the number of dependent variables for a given problem, they generally lack a mechanism to recover meaningful information about the behavior of individual constituents. Because of this shortcoming, the primary focus of the majority of recently published works has been on continuum mechanics based theories. Information on averaged theories can be found in papers by Drew [1], Rubinow and Keller [2], Enlwald and Almstedt [3,4], Terada et al [5], and Ahmadi et al [6].

Mixture theories based on continuum mechanics principles assume that each material point in the mixture is occupied simultaneously by each constituent [7]. This assumption is not physically accurate of course, but is necessary so that the quantities used to describe deformation are continuous and differentiable. This allows the development of the mathematical models that describe the behaviors of mixtures in a similar manner to those for homogeneous matter. One of the first authors to use this idea was Truesdell [8] who proposed a theory called a mechanical basis for diffusion. Author presents definitions for the basic kinematic relations as well as the continuity and momentum equations for mixtures of ν arbitrary constituents. This theory allows for the transfer of mass and momentum from one constituent to another, which is commonly referred to as the "interaction force" [7, 9, 10]. It is shown that Fick's Law of diffusion is a specific case of this theory.

Later Müller [11] presented the energy equation and entropy inequality for ν constituents, as well as a linear constitutive theory for a mixture of two Newtonian fluids. The author uses density gradients, the symmetric and anti-symmetric parts of the velocity gradient tensors, temperature gradient, and relative velocity between constituents as the arguments of the dependent variables in the constitutive theory. The author also shows that based on this theory, a mixture of two ideal gasses is still an ideal gas with properties that agree with the principle of partial pressures based classical thermodynamics. Green and Naghdi [12] propose a similar theory in which they use the energy equation and entropy inequality to derive the continuity and momentum equations. This is followed by the derivation of constitutive equations for the mixture of two Newtonian fluids including resulting thermodynamic restrictions. Atkin and Craine [13] derive continuity, momentum, and energy equations, and the entropy inequality for mixtures, as well as a constitutive theory for mixtures of ideal inviscid fluids. The authors show that the results agree with kinetic theory of gasses. Bedford and Drumheller [14] present a survey of continuum theories of mixtures. The authors include constitutive examples for mixtures of immiscible fluids, solid particles suspended in fluids, fluids flowing through porous media, chemically reacting fluids, and composite materials. The authors also provide an overview of volume averaged theories, and micro-structure theories. The theory for mixtures of two fluids is restricted to mixtures of an inviscid and a viscous fluid.

In [7] Rajagopal and Tao derive the conservation laws for mixtures and provide details for several example problems including: diffusion of a fluid through a solid experiencing finite deformation, steady state diffusion problems, a diffusing singular surface, wave propagation, mixtures of Newtonian fluids, and solid particle suspensions. The main difference in these is the constitutive theory used for the stress tensor and the “interaction force”. The authors derive the constitutive theory by selecting argument tensors based on the assumed physics of the problem and use the entropy inequality to determine appropriate restrictions on the material coefficients.

In [9] Rajagopal et al give a review of interaction force terms for fluid-solid mixtures. The authors compare constitutive theory for the interaction force to volume averaged theories based on results for single particle flows. The results include comparisons for drag, lift, buoyancy, and other effects. Johnson et al present numerical results for flow between parallel plates of solid particles suspended in a fluid [15]. The authors present a constitutive model for granular particles suspended in a fluid and simplify the governing equations to a system of ODE’s which are then solved using a collocation method. Results are presented showing the effect of varying the volume fraction of the constituents and the coefficients of the interaction force terms. Massoudi et al [16] present results for a similar problem using pipe flow assumptions, and Massoudi and Rao [17] give results for flow between parallel plates. In [18] Massoudi et al show results for particulate flow down an inclined plane.

Rajagopal et al [19–21] present a series of studies for mixtures of fluids in a bearing. In [19] an oil-water mixture is considered. The authors give a mathematical model and results for 2D non-isothermal flow in a bearing. Portions of this mathematical model are used in section 2.3.4. The authors use a constitutive theory that includes relative velocity, volume fraction gradients, temperature gradient, and the symmetric part of the velocity gradient tensor as argument tensors of the dependent variables in the constitutive theories. Results are given for different volume fractions. In [20] a "bubbly oil" mixture is considered, and in [21] an oil-water mixture is studied in an elastohydrodynamic bearing. Similar results are given and the mathematical models only vary because of the different constitutive theory used for the gas phase. In all of the published work, the authors use finite difference method to obtain numerical results.

Massoudi [10] shows how the constitutive theory for solid particles suspended in a fluid (given previously in [9]) can be derived using the theory of invariants and generators. In [22] the author gives a method for applying boundary conditions when computing solutions to mixture problems. Massoudi [23] also shows that the constitutive theory used for a mixture of two fluids must reduce to the theory for a single fluid as the volume fraction approaches the limiting case of 0 or 1. The author also notes that the best way to ensure this is to have viscosity terms that are weighted by volume fraction. For more information on mixture theories see reference [7].

1.2 Scope of Work

Mathematical models are derived based on mixture theory for ν homogeneous, isotropic, and incompressible constituents using conservation of mass, balance of momenta, and the first law of thermodynamics. For isothermal flows the constitutive theories for mechanical pressure and the deviatoric Cauchy stress tensor are presented for the constituents and the mixture based on the second law of thermodynamics. Currently used mixture theories are examined and essential modifications are suggested based on the physics. The resulting modified mixture theory is used in the numerical studies to demonstrate its validity. The mixture theory presented in the work considers Newtonian and generalized Newtonian fluids. Power law and Carreau-Yasuda models for shear thinning fluids are used for the generalized Newtonian fluids. Dimensionless forms of the mathematical models are derived and used in the numerical studies. Numerical studies are given for Newtonian, power law, and Carreau fluids using fully developed flow between parallel plates and 1:2 asymmetric sudden expansion as model problems for a saturated mixture of two constituents.

Numerical solutions of the BVPs are computed using finite element processes based on a residual functional, i.e. least squares finite element processes, that ensure unconditionally stable computation. Local approximations are considered in $H^{k,p}\left(\bar{\Omega}^e\right)$ scalar product spaces.

Chapter 2

Development of Mathematical Model for a Mixture of ν Fluids

2.1 Introduction

In this chapter we present derivations of continuity equation, momentum equations, energy equation, entropy inequality, and the constitutive theory derived from the entropy inequality for a saturated mixture of ν Newtonian and generalized Newtonian fluids. Some basic definitions of bulk densities of constituents, mixture density, mixture velocities, etc. are introduced based on basic physical principles that are used in the development of the mathematical model for the mixture. To avoid confusion in the notation used here and those commonly used in continuum mechanics we adopt the following convention. Greek letters such as α, β, γ, ν, etc. used as subscripts, superscripts, or indices refer to a quantity associated with an individual constituent and have no implied summation when the index is repeated. Any index using English letters i, j, k, etc. implies standard continuum mechanics summation conventions, i.e. summation over repeated indices. The derivation of the mathematical model presented in this chapter is strictly based on principles of continuum mechanics and thermodynamics.

2.2 Preliminary Definitions

In this section we present basic definitions of bulk densities of constituents, mixture density, mixture velocity, material derivative for the constituents and the mixture etc. These are subsequently used in the conservation laws. We consider a saturated mixture of ν constituents with ϕ_α; $\alpha = 1, 2, \ldots, \nu$ volume fraction, and $\rho^{(\alpha)}$; $\alpha = 1, 2, \ldots, \nu$ constituent densities. Following Truesdell [8] we can give the following definitions:

2.2.1 Definitions of densities

Consider an elemental volume dV of the mixture of Volume V. Then $\rho^{(\alpha)}\phi_\alpha dV$ is the mass of each constituent in the volume dV. If ρ_m is the bulk density of the mixture, then $\rho_m dV$ is also the total mass in the elemental volume dV. Hence, for volume V, we have

$$\int\limits_{V(t)} \rho_m dV = \sum_{\alpha=1}^{\nu} \int\limits_{V(t)} \rho^{(\alpha)}\phi_\alpha dV \tag{2.1}$$

or

$$\int\limits_{V(t)} \left(\rho_m - \sum_{\alpha=1}^{\nu} \rho^{(\alpha)}\phi_\alpha\right) dV = 0 \tag{2.2}$$

Since $V(t)$ is arbitrary, we have

$$\rho_m = \sum_{\alpha=1}^{\nu} \rho^{(\alpha)}\phi_\alpha \tag{2.3}$$

If we define bulk density of a constituent ρ_α as

$$\rho_\alpha = \rho^{(\alpha)}\phi_\alpha \tag{2.4}$$

Then 2.3 can be written as

$$\rho_m = \sum_{\alpha=1}^{\nu} \rho_\alpha \tag{2.5}$$

Additionally, for a saturated mixture, the volume additivity constraint must hold, i.e.

$$\sum_{\alpha=1}^{\nu} \phi_\alpha = 1 \tag{2.6}$$

2.2.2 Mixture velocities

Let $\mathbf{v}_\alpha$ be the velocities of the constituents at a material particle (simultaneously occupied by all constituents) and $\mathbf{v}$ the velocity of the mixture, then using the principle of balance of momentum, i.e. the momentum of the mixture must be equal to the sum of the momenta of the constituents, we have

$$\rho_m \mathbf{v} = \sum_{\alpha=1}^{\nu} \rho_\alpha \mathbf{v}_\alpha \tag{2.7}$$

Equation 2.7 defines the mixture velocity at a material particle in terms of bulk densities of the constituents, their velocities, and the mixture density.

2.2.3 Material derivative for the constituents and the mixture

Since the material derivative $\frac{D(\cdot)}{Dt}$ in Eulerian description uses the velocity of a material particle, it needs to be defined for each constituent. The material derivative of a dependent variable Q for constituent α is defined as

$$\frac{D_\alpha Q}{Dt} = \frac{\partial Q}{\partial t} + \mathbf{v}_\alpha \cdot \boldsymbol{\nabla} Q \tag{2.8}$$

The material derivative of Q for the mixture is defined as

$$\rho_m \frac{DQ}{Dt} = \sum_{\alpha=1}^{\nu} \rho_\alpha \frac{D_\alpha Q}{Dt} = \sum_{\alpha=1}^{\nu} \rho_\alpha \left(\frac{\partial Q}{\partial t} + \mathbf{v}_\alpha \cdot \boldsymbol{\nabla} Q \right)$$

or

$$\rho_m \frac{DQ}{Dt} = \left(\sum_{\alpha=1}^{\nu} \rho_\alpha\right) \frac{\partial Q}{\partial t} + \left(\sum_{\alpha=1}^{\nu} \rho_\alpha \mathbf{v}_\alpha\right) \cdot \boldsymbol{\nabla} Q$$

$\therefore$

$$\rho_m \frac{DQ}{Dt} = \rho_m \frac{\partial Q}{\partial t} + \rho_m \mathbf{v} \cdot \boldsymbol{\nabla} Q \tag{2.9}$$

2.3 Conservation Laws

We use the definitions presented in section 2.2 to derive details of the mathematical model for the mixture using conservation laws. We assume the constituents and the mixture to be incompressible and the flows to be isothermal. The constituents and the mixture are considered to be Newtonian and generalized Newtonian fluids. The viscosities of the constituents and the mixture are described using the Carreau-Yasuda model [24]. We present a general derivation which is made specific based on the assumptions stated above.

2.3.1 Conservation of Mass

If we apply conservation of mass to an arbitrary volume containing ν constituents with bulk densities ρ_α and velocities $\mathbf{v}_\alpha$, then for each constituent we obtain

$$\frac{\partial \rho_\alpha}{\partial t} + \boldsymbol{\nabla} \cdot (\rho_\alpha \mathbf{v}_\alpha) = 0 \tag{2.10}$$

Summing (2.10) for the constituents

$$\sum_{\alpha=1}^{\nu} \frac{\partial \rho_\alpha}{\partial t} + \sum_{\alpha=1}^{\nu} \boldsymbol{\nabla} \cdot (\rho_\alpha \mathbf{v}_\alpha) = 0 \tag{2.11}$$

or

$$\frac{\partial}{\partial t}\left(\sum_{\alpha=1}^{\nu} \rho_\alpha\right) + \boldsymbol{\nabla} \cdot \left(\sum_{\alpha=1}^{\nu} \rho_\alpha \mathbf{v}_\alpha\right) = 0 \tag{2.12}$$

Using (2.5) and (2.7), (2.12) can be written as

$$\frac{\partial \rho_m}{\partial t} + \boldsymbol{\nabla} \cdot (\rho_m \mathbf{v}) = 0 \tag{2.13}$$

For the incompressible case (2.10) and (2.13) reduce to

$$\rho_\alpha \left(\boldsymbol{\nabla} \cdot \mathbf{v}_\alpha \right) = 0 \tag{2.14}$$

$$\rho_m \left(\boldsymbol{\nabla} \cdot \mathbf{v} \right) = 0 \tag{2.15}$$

2.3.2 Balance of Momenta

Using the principle of balance of linear momentum to an arbitrary volume of mixture yields the following three equations for constituent α (in the absence of body forces)

$$\rho_\alpha \frac{D_\alpha \mathbf{v}_\alpha}{Dt} = \boldsymbol{\nabla} \cdot [\sigma_\alpha]^T + \boldsymbol{\pi}_\alpha \tag{2.16}$$

Where $[\sigma_\alpha]^T$ is the contra-variant Cauchy stress tensor and $\boldsymbol{\pi}_\alpha$ is the force exerted on the α^{th} constituent by each of the other constituents. In general

$$\sum_{\alpha=1}^{\nu} \boldsymbol{\pi}_\alpha = 0 \tag{2.17}$$

must hold. In the case of a mixture of two constituents, 2.17 reduces to:

$$\boldsymbol{\pi}_1 = -\boldsymbol{\pi}_2 \tag{2.18}$$

2.3.3 Energy equation

In the derivation of the energy equation we assume that the sum of the constituent energies is the total energy of the mixture. For a constituent α, the rate of change of the total energy must be equal to the rate of heat added and the rate of work done.

$$\frac{D_\alpha E_t^\alpha}{Dt} = \frac{D_\alpha Q^\alpha}{Dt} + \frac{D_\alpha W^\alpha}{Dt} \tag{2.19}$$

and for the mixture

$$\sum_{\alpha=1}^{\nu} \frac{D_\alpha E_t^\alpha}{Dt} = \sum_{\alpha=1}^{\nu} \frac{D_\alpha Q^\alpha}{Dt} + \sum_{\alpha=1}^{\nu} \frac{D_\alpha W^\alpha}{Dt} \tag{2.20}$$

where (in the absence of body forces)

$$E_t^\alpha = \int\limits_{V(t)} \rho_\alpha \left(e_\alpha + \frac{1}{2} \mathbf{v}_\alpha \cdot \mathbf{v}_\alpha \right) dV \tag{2.21}$$

$$\sum_{\alpha=1}^{\nu} \frac{D_\alpha Q^\alpha}{Dt} = - \int\limits_{\partial V} \mathbf{q} \cdot \mathbf{n} dS = - \int\limits_{V(t)} \boldsymbol{\nabla} \cdot \mathbf{q} dV \tag{2.22}$$

$\mathbf{q}$ is total heat flux and $\mathbf{n}$ is the outward unit normal to the boundary dV of volume $V(t)$ in the current configuration.

$$\begin{aligned} \frac{D_\alpha W^\alpha}{Dt} &= \int\limits_{\partial V} \mathbf{P} \cdot \mathbf{v}_\alpha dS = \int\limits_{\partial V} \left([\sigma_\alpha]^T \cdot \mathbf{n} \right) \cdot \mathbf{v}_\alpha dS \\ &= \int\limits_{V} \boldsymbol{\nabla} \cdot \left(\mathbf{v}_\alpha \cdot [\sigma_\alpha]^T \right) dV \end{aligned} \tag{2.23}$$

or

$$\frac{D_\alpha W^\alpha}{Dt} = \int\limits_{V} \left(\mathbf{v}_\alpha \cdot \left(\boldsymbol{\nabla} \cdot [\sigma_\alpha]^T \right) + (\sigma_\alpha)_{ij} \frac{\partial (v_\alpha)_i}{\partial x_j} \right) dV \tag{2.24}$$

$$\frac{D_\alpha E_t^\alpha}{Dt} = \frac{D_\alpha}{Dt} \int\limits_{V(t)} \rho_\alpha \left(e_\alpha + \frac{1}{2} \mathbf{v}_\alpha \cdot \mathbf{v}_\alpha \right) dV \tag{2.25}$$

for the α^{th} constituent

$$(\rho_\alpha)_0 \, dV_0 = (\rho_\alpha) \, dV \tag{2.26}$$

$(\rho_\alpha)_0$ and dV_0 are densities and volumes in the reference configuration. Hence

$$\frac{D_\alpha E_t^\alpha}{Dt} = \int\limits_{V_0} \frac{D_\alpha}{Dt} \left(\left(e_\alpha + \frac{1}{2} \mathbf{v}_\alpha \cdot \mathbf{v}_\alpha \right) (\rho_\alpha)_0 \right) dV_0 \tag{2.27}$$

Since $\frac{D(\rho_\alpha)_0}{Dt} = 0$, (2.27) reduces to

$$\begin{aligned}\frac{D_\alpha E_t^\alpha}{Dt} &= \int\limits_{V_0} \frac{D_\alpha}{Dt}(e_\alpha + \frac{1}{2}\mathbf{v}_\alpha \cdot \mathbf{v}_\alpha)\,(\rho_\alpha)_0\, dV_0 \\ &= \int\limits_{V(t)} \frac{D_\alpha}{Dt}(e_\alpha + \frac{1}{2}\mathbf{v}_\alpha \cdot \mathbf{v}_\alpha)\rho_\alpha dV \\ &= \int\limits_{V(t)} \left(\frac{D_\alpha e_\alpha}{Dt} + \frac{1}{2}\frac{D_\alpha}{Dt}\left(\mathbf{v}_\alpha \cdot \mathbf{v}_\alpha\right)\right)\rho_\alpha dV\end{aligned}$$

or

$$\frac{D_\alpha E_t^\alpha}{Dt} = \int\limits_{V(t)} \left(\frac{D_\alpha e_\alpha}{Dt} + \mathbf{v}_\alpha \cdot \frac{D_\alpha\left(\mathbf{v}_\alpha\right)}{Dt}\right)\rho_\alpha dV \tag{2.28}$$

Thus, the energy equation for the α^{th} constituent can be written as

$$\begin{aligned}\int\limits_{V(t)} \rho_\alpha \left(\frac{D_\alpha e_\alpha}{Dt} + \mathbf{v}_\alpha \cdot \frac{D_\alpha(\mathbf{v}_\alpha)}{Dt}\right) dV = -\int\limits_{V(t)} \boldsymbol{\nabla} \cdot \mathbf{q}_\alpha dV \\ + \int\limits_{V(t)} \left(\mathbf{v}_\alpha \cdot \left(\boldsymbol{\nabla} \cdot [\sigma_\alpha]^T\right) + (\sigma_\alpha)_{ij}\,\frac{\partial(v_\alpha)_i}{\partial x_j}\right) dV\end{aligned} \tag{2.29}$$

In (2.29) we have used

$$\mathbf{q} = \sum_{\alpha=1}^{\nu} \mathbf{q}_\alpha \tag{2.30}$$

Since the volume $V(t)$ is arbitrary, (2.29) reduces to

$$\rho_\alpha \frac{D_\alpha e_\alpha}{Dt} + \rho_\alpha \mathbf{v}_\alpha \cdot \frac{D_\alpha(\mathbf{v}_\alpha)}{Dt} + \boldsymbol{\nabla} \cdot \mathbf{q}_\alpha - \left(\mathbf{v}_\alpha \cdot \left(\boldsymbol{\nabla} \cdot [\sigma_\alpha]^T\right) + (\sigma_\alpha)_{ij}\,\frac{\partial(v_\alpha)_i}{\partial x_j}\right) = 0 \tag{2.31}$$

From the momentum equation for α^{th} constituent

$$\rho_\alpha \frac{D_\alpha \mathbf{v}_\alpha}{Dt} = \boldsymbol{\nabla} \cdot [\sigma_\alpha]^T + \boldsymbol{\pi}_\alpha \tag{2.32}$$

Substituting from (2.32) into (2.31)

$$\begin{aligned}\rho_\alpha \frac{D_\alpha e_\alpha}{Dt} + \mathbf{v}_\alpha \cdot \left(\boldsymbol{\nabla} \cdot [\sigma_\alpha]^T + \boldsymbol{\pi}_\alpha\right) + \boldsymbol{\nabla} \cdot \mathbf{q}_\alpha \\ - \left(\mathbf{v}_\alpha \cdot \left(\boldsymbol{\nabla} \cdot [\sigma_\alpha]^T\right) + (\sigma_\alpha)_{ij}\,\frac{\partial(v_\alpha)_i}{\partial x_j}\right) = 0\end{aligned} \tag{2.33}$$

or

$$\rho_\alpha \frac{D_\alpha e_\alpha}{Dt} + \mathbf{v}_\alpha \cdot \boldsymbol{\pi}_\alpha + \boldsymbol{\nabla} \cdot \mathbf{q}_\alpha - (\sigma_\alpha)_{ij} \frac{\partial (v_\alpha)_i}{\partial x_j} = 0 \tag{2.34}$$

Summing (2.34) over the constituents and using (2.30)

$$\sum_{\alpha=1}^{\nu} \rho_\alpha \frac{D_\alpha e_\alpha}{Dt} + \sum_{\alpha=1}^{\nu} \mathbf{v}_\alpha \cdot \boldsymbol{\pi}_\alpha + \boldsymbol{\nabla} \cdot \mathbf{q} - \sum_{\alpha=1}^{\nu} (\sigma_\alpha)_{ij} \frac{\partial (v_\alpha)_i}{\partial x_j} = 0 \tag{2.35}$$

If we assume that for the α^{th} constituent

$$e_\alpha = c_{p_\alpha} \theta \tag{2.36}$$

and further assume constant c_{p_α}, then (2.35) reduces to

$$\sum_{\alpha=1}^{\nu} \rho_\alpha c_{p_\alpha} \frac{D_\alpha \theta}{Dt} + \sum_{\alpha=1}^{\nu} \mathbf{v}_\alpha \cdot \boldsymbol{\pi}_\alpha + \boldsymbol{\nabla} \cdot \mathbf{q} - \sum_{\alpha=1}^{\nu} (\sigma_\alpha)_{ij} \frac{\partial (v_\alpha)_i}{\partial x_j} = 0 \tag{2.37}$$

This is the final form of the energy equation for ν constituents. If we consider only two constituents then (2.37) becomes

$$\left(\rho_1 c_{p_1} \frac{D_1 \theta}{Dt} + \rho_2 c_{p_2} \frac{D_2 \theta}{Dt} \right) + (\mathbf{v}_1 \cdot \boldsymbol{\pi}_1 + \mathbf{v}_2 \cdot \boldsymbol{\pi}_2) + \boldsymbol{\nabla} \cdot \mathbf{q} - (\sigma_1)_{ij} \frac{\partial (v_1)_i}{\partial x_j} - (\sigma_2)_{ij} \frac{\partial (v_2)_i}{\partial x_j} = 0 \tag{2.38}$$

The theories based on (2.37) and (2.38) are much simplified as some interaction effects [7] are neglected. But in view of the fact that we only consider incompressible constituents and isothermal flows, these derivations are adequate.

2.3.4 Constitutive theory

We follow the derivations in reference [19] based on the following notations

$$\begin{aligned} &\mathbf{L}_{(\alpha)} = \operatorname{grad} \mathbf{v}_\alpha(x,t) && \mathbf{D}_{(\alpha)} = \frac{1}{2}\left(\mathbf{L}_{(\alpha)} + \mathbf{L}^{\mathbf{T}}_{(\alpha)}\right) && \mathbf{q} = \sum_{\alpha=1}^{\nu} \mathbf{q}_\alpha \\ &Q = \frac{1}{\rho_m} \sum_{\alpha=1}^{\nu} \rho_\alpha Q_\alpha && \eta = \frac{1}{\rho_m} \sum_{\alpha=1}^{\nu} \rho_\alpha \eta_\alpha(x,t) && \boldsymbol{\pi} = -\boldsymbol{\pi}_1 = \boldsymbol{\pi}_2 \end{aligned} \tag{2.39}$$

In which $\mathbf{q}$ is heat flux, Q is heat supply, η and η_α are entropy densities of the mixture and the constituents. We begin with the entropy inequality

$$\rho_m \frac{D\eta}{Dt} + \operatorname{div}\left(\frac{\mathbf{q}}{\theta}\right) - \rho\frac{Q}{\theta} \geq 0 \tag{2.40}$$

We have assumed that entropy due to heat flux $\mathbf{q}_\alpha$ is $\frac{\mathbf{q}_\alpha}{\theta}$ where θ is the common temperature of the constituents and the entropy due to heat supply Q_α is $\frac{Q_\alpha}{\theta}$.

Let the partial Helmholtz free energy Φ_α for the constituent α be

$$\Phi_\alpha = e_\alpha - \theta\eta_\alpha \tag{2.41}$$

Using (2.40) and (2.41) and the energy equation in e_α and the additivity constraint $\sum_{\alpha=1}^{\nu} \phi_\alpha = 1$ we can establish the following dependent variables in the constitutive theory for constituent α.

$$\Phi_\alpha, \eta_\alpha, \boldsymbol{\pi}, \mathbf{q}, \boldsymbol{\sigma}_\alpha \tag{2.42}$$

The following argument tensors of the dependent variables in the constitutive theory are considered in the development of the constitutive theory.

$$\mathbf{v}^{(12)}, \mathbf{g}, \mathbf{h}^{(\alpha)}, \mathbf{D}_{(\alpha)}, w_{(12)} \tag{2.43}$$

in which $\mathbf{v}^{(12)}$ is relative velocity, $\mathbf{h}^{(\alpha)} = \operatorname{grad}\phi_\alpha$, and $w_{(12)}$ is relative spin. We consider $\Phi_\alpha = \Phi_\alpha(\phi_\alpha, \theta)$, $\Phi = \Phi(\phi_\alpha, \theta)$. We have the following for the constitutive theory derived using the theory of generators and invariants [25, 26] based on the assumption of linear dependence of the constitutive variables on the argument tensors. We consider two constituents only.

$$
\begin{aligned}
\eta =& -\frac{\partial \Phi}{\partial \theta} \\
\pi =& \beta_1 \mathbf{v}^{(12)} + \beta_4 \mathbf{g} + \left(-\rho_2 \frac{\partial \Phi_2}{\partial \phi_1} + \frac{\rho_2}{\rho_m}\pi \right) \mathbf{h}^{(1)} \\
& + \left(\rho_1 \frac{\partial \Phi_1}{\partial \phi_2} + \frac{\rho_1}{\rho_m}\pi \right) \mathbf{h}^{(2)} \\
\mathbf{q} =& -k_1 \mathbf{g} - k_2 \mathbf{v}^{(12)} \\
\boldsymbol{\sigma}_1 =& -p_1 [I] +_d \boldsymbol{\sigma}_1 \\
\boldsymbol{\sigma}_2 =& -p_2 [I] +_d \boldsymbol{\sigma}_2
\end{aligned}
\tag{2.44}
$$

in which p_1 and p_2 are mechanical pressures and ${}_d\boldsymbol{\sigma}_1$ and ${}_d\boldsymbol{\sigma}_2$ are deviatoric contravariant Cauchy stress tensors for constituents one and two.

$$
\begin{aligned}
p_1 &= \phi_1 \left(\rho_1 \frac{\partial \Phi_1}{\partial \phi_1} + \rho_2 \frac{\partial \Phi_2}{\partial \phi_1} - \Pi \right) = p_1^s - \phi_1 \Pi \\
p_2 &= \phi_2 \left(\rho_1 \frac{\partial \Phi_1}{\partial \phi_2} + \rho_2 \frac{\partial \Phi_2}{\partial \phi_2} - \Pi \right) = p_2^s - \phi_2 \Pi \\
{}_d\boldsymbol{\sigma}_1 &= \left(\lambda_1 \operatorname{tr} \mathbf{D}_{(1)} + \lambda_3 \operatorname{tr} \mathbf{D}_{(2)} \right) [I] + 2\mu_1 \mathbf{D}_{(1)} + 2\mu_3 \mathbf{D}_{(2)} + \lambda_5 w_{(12)} \\
{}_d\boldsymbol{\sigma}_2 &= \left(\lambda_4 \operatorname{tr} \mathbf{D}_{(1)} + \lambda_2 \operatorname{tr} \mathbf{D}_{(2)} \right) [I] + 2\mu_4 \mathbf{D}_{(1)} + 2\mu_2 \mathbf{D}_{(2)} + \lambda_5 w_{(12)}
\end{aligned}
\tag{2.45}
$$

In which Π is a Lagrange multiplier [19] and

$$
\begin{aligned}
& \beta_1 \geq 0 \quad k_1 \geq 0 \quad \left(\rho_2 \left(\eta_2 + \frac{\partial \Phi_2}{\partial \theta} \right) + \beta_4 + \frac{1}{\theta} k_2 \right) \leq \frac{1}{\theta} 4 \beta_1 k_1 \\
& \lambda_5 \geq 0 \quad \mu_1 \geq 0 \quad \mu_2 \geq 0 \quad (\mu_3 + \mu_4)^2 \leq 4 \mu_1 \mu_2 \\
& \lambda_1 + \frac{2}{3}\mu_1 \geq 0 \quad \frac{2}{3}\mu_2 \geq 0 \\
& \left[\lambda_3 + \lambda_4 + \frac{2}{3}(\mu_3 + \mu_4) \right]^2 \leq 4 \left(\lambda_1 + \frac{2}{3}\mu_1 \right) \left(\lambda_2 + \frac{2}{3}\mu_2 \right)
\end{aligned}
\tag{2.46}
$$

The constitutive theory can be simplified for incompressible constituents and the mixture with further assumption of isothermal flow.

$$
\mathbf{h}^{(1)} = 0, \quad \mathbf{h}^{(2)} = 0, \quad \mathbf{g} = 0, \quad \operatorname{tr} \mathbf{D}_{(1)} = 0, \quad \operatorname{tr} \mathbf{D}_{(2)} = 0
$$

If we assume $\Phi_\alpha = \Phi_\alpha(\theta)$, then

$$\frac{\partial \Phi_\alpha}{\partial \phi_1} = 0, \quad \frac{\partial \Phi_\alpha}{\partial \phi_2} = 0$$

and if we ignore dependence of ${}_d\boldsymbol{\sigma}_\alpha$ on $w_{(12)}$, then the constitutive theory becomes

$$\begin{aligned}
\boldsymbol{\pi} &= \beta_1 \mathbf{v}^{(12)} \\
\boldsymbol{\sigma}_1 &= -p_1[I] +_d \boldsymbol{\sigma}_1 \\
\boldsymbol{\sigma}_2 &= -p_2[I] +_d \boldsymbol{\sigma}_2 \\
{}_d\boldsymbol{\sigma}_1 &= 2\mu_1 \mathbf{D}_{(1)} + 2\mu_3 \mathbf{D}_{(2)} \\
{}_d\boldsymbol{\sigma}_2 &= 2\mu_4 \mathbf{D}_{(1)} + 2\mu_2 \mathbf{D}_{(2)}
\end{aligned} \tag{2.47}$$

$\mathbf{q}$ is not a dependent variable in this constitutive theory due to the assumption of isothermal flow.

2.4 Complete mathematical model

If we consider two incompressible, homogeneous, and isotropic constituents with saturated mixture that is also incompressible, we have the following.

Continuity equations

$$\begin{aligned}
\rho_1 \boldsymbol{\nabla} \cdot \mathbf{v}_1 &= 0 \\
\rho_2 \boldsymbol{\nabla} \cdot \mathbf{v}_2 &= 0
\end{aligned} \tag{2.48}$$

Momentum equations (in the absence of body forces)

$$\rho_\alpha \left((v_\alpha)_1 \frac{\partial (v_\alpha)_1}{\partial x_1} + (v_\alpha)_2 \frac{\partial (v_\alpha)_1}{\partial x_2} \right) + \frac{\partial p_1}{\partial x_1} - \frac{\partial ({}_d\sigma_\alpha)_{11}}{\partial x_1} - \frac{\partial ({}_d\sigma_\alpha)_{21}}{\partial x_2} - (\pi_\alpha)_1 = 0$$
$$\rho_\alpha \left((v_\alpha)_1 \frac{\partial (v_\alpha)_2}{\partial x_1} + (v_\alpha)_2 \frac{\partial (v_\alpha)_2}{\partial x_2} \right) + \frac{\partial p_1}{\partial x_2} - \frac{\partial ({}_d\sigma_\alpha)_{12}}{\partial x_1} - \frac{\partial ({}_d\sigma_\alpha)_{22}}{\partial x_2} - (\pi_\alpha)_2 = 0$$
$$\alpha = 1, 2 \tag{2.49}$$

Constitutive equations

$$\begin{aligned} {}_d\boldsymbol{\sigma}_1 &= 2\mu_1 \mathbf{D}_{(1)} + 2\mu_3 \mathbf{D}_{(2)} \\ {}_d\boldsymbol{\sigma}_2 &= 2\mu_4 \mathbf{D}_{(1)} + 2\mu_2 \mathbf{D}_{(2)} \end{aligned} \tag{2.50}$$

Material coefficients $\mu_1,\ \mu_2,\ \mu_3$, and μ_4 are functions of η_α, viscosities of the constituents and the volume fractions ϕ_α. This mathematical model has closure, twenty equations in twenty variables for 3D case and twelve equations in twelve variables for 2D case: $\mathbf{v}_\alpha$, $\alpha = 1, 2$; p_1, p_2; ${}_d\boldsymbol{\sigma}_\alpha$, $\alpha = 1, 2$.

Material coefficients

Based on references [19, 27], we consider the following:

$$\begin{aligned} \mu_1 &= \phi_1^2 \eta_1 + \phi_1 \phi_2 \eta_{12} \\ \mu_2 &= \phi_2^2 \eta_2 + \phi_1 \phi_2 \eta_{12} \\ \mu_3 &= \mu_4 = \phi_1 \phi_2 \eta_{12} \\ \eta_{12} &= \sqrt{\eta_1 \eta_2} \end{aligned} \tag{2.51}$$

where η_1, η_2 are constituent viscosities. For Newtonian fluids these are constant. When the constituents are generalized Newtonian fluids, then $\eta_1 = \eta_1(I_2^1)$, $\eta_2 = \eta_2(I_2^2)$ in which $I_2^\alpha;\ \alpha = 1, 2$ are second invariants of the strain rate tensors $\mathbf{D}_{(\alpha)};\ \alpha = 1, 2$. Both Power Law and Carreau-Yasuda models are admissible in defining η_1 and η_2 when the constituents are generalized Newtonian fluids.

Remarks

1. We note that deviatoric Cauchy stress ${}_d\boldsymbol{\sigma}$ for the mixture is the sum of ${}_d\boldsymbol{\sigma}_1$ and ${}_d\boldsymbol{\sigma}_2$. The constitutive theories for ${}_d\boldsymbol{\sigma}_1$ and ${}_d\boldsymbol{\sigma}_2$ must satisfy this requirement. Using (2.50) and (2.51) we consider the following.

   Consider the two constituents to be the same (say constituent one), hence in this case $\eta_2 = \eta_1$. Thus

$$\begin{aligned}
\mu_1 &= \eta_1 \left(\phi_1^2 + \phi_1\phi_2\right) \\
\mu_2 &= \eta_1 \left(\phi_2^2 + \phi_1\phi_2\right) \\
\mu_3 &= \mu_4 = \phi_1\phi_2\eta_1
\end{aligned} \tag{2.52}$$

   Therefore

$${}_d\boldsymbol{\sigma}_1 = 2\eta_1 \left(\phi_1^2 + \phi_1\phi_2\right) \mathbf{D}_{(1)} + 2\phi_1\phi_2\eta_1\mathbf{D}_{(2)} \tag{2.53}$$

$${}_d\boldsymbol{\sigma}_2 = 2\eta_1 \left(\phi_2^2 + \phi_1\phi_2\right) \mathbf{D}_{(2)} + 2\phi_1\phi_2\eta_1\mathbf{D}_{(1)} \tag{2.54}$$

   Since constituent two is the same as constituent one

$$\rho^{(2)} = \rho^{(1)} \; , \quad \rho_1 = \phi_1\rho^{(1)} \; , \quad \rho_2 = \phi_2\rho^{(1)}$$

   Since $\rho_m\mathbf{v} = \rho_1\mathbf{v}_1 + \rho_2\mathbf{v}_2$ and $\rho_m = \rho^{(1)}$

$$\begin{aligned}
\rho^{(1)}\mathbf{v} &= \phi_1\rho^{(1)}\mathbf{v}_1 + \phi_2\rho^{(1)}\mathbf{v}_2 \\
\therefore \quad \mathbf{v} &= \phi_1\mathbf{v}_1 + \phi_2\mathbf{v}_2
\end{aligned}$$

   Thus for the mixture we have

$$\mathbf{D} = \phi_1\mathbf{D}_{(1)} + \phi_2\mathbf{D}_{(2)} \tag{2.55}$$

Now, going back to (2.53) and (2.54)

$$
\begin{aligned}
{}_d\boldsymbol{\sigma}_1 &= 2\eta_1\left(\phi_1+\phi_2\right)\phi_1\mathbf{D}_{(1)} + 2\eta_1\phi_1\phi_2\mathbf{D}_{(2)} \\
{}_d\boldsymbol{\sigma}_2 &= 2\eta_1\left(\phi_1+\phi_2\right)\phi_2\mathbf{D}_{(2)} + 2\eta_1\phi_1\phi_2\mathbf{D}_{(1)}
\end{aligned}
\tag{2.56}
$$

Since $\phi_1+\phi_2=1$, using (2.56) we can write

$$
{}_d\boldsymbol{\sigma} = {}_d\,\boldsymbol{\sigma}_1 + {}_d\,\boldsymbol{\sigma}_2 = 2\eta_1\left(\phi_1\mathbf{D}_{(1)}+\phi_2\mathbf{D}_{(2)}\right) + 2\eta_1\phi_1\phi_2\left(\mathbf{D}_{(1)}+\mathbf{D}_{(2)}\right) \tag{2.57}
$$

using (2.55) in (2.57), we can write

$$
{}_d\boldsymbol{\sigma} = 2\eta_1\mathbf{D} + 2\eta_1\phi_1\phi_2\left(\mathbf{D}_{(1)}+\mathbf{D}_{(2)}\right) \tag{2.58}
$$

But ${}_d\boldsymbol{\sigma} = 2\eta_1\mathbf{D}$ must hold regardless of ϕ_1 and ϕ_2, hence the second term in (2.58) must be zero which is only possible if $\mu_3=\mu_4=0$.

Thus for saturated Newtonian and generalized Newtonian mixtures of two Newtonian and generalized Newtonian fluids we have the following constitutive equations

$$
\begin{aligned}
{}_d\boldsymbol{\sigma}_1 &= 2\mu_1\mathbf{D}_{(1)} \\
{}_d\boldsymbol{\sigma}_2 &= 2\mu_2\mathbf{D}_{(2)}
\end{aligned}
\tag{2.59}
$$

2. Generalized Newtonian fluids

If we consider both constituents and the mixture to be generalized Newtonian fluids, then

$$
\eta_1 = \eta_1\left(I_2^1\right), \qquad \eta_2 = \eta_2\left(I_2^2\right) \tag{2.60}
$$

In which I_2^1 and I_2^2 are the second invariants of the tensors $\mathbf{D}_{(1)}$ and $\mathbf{D}_{(2)}$. We can use power law or Carreau-Yasuda model to define η_1 and η_2.

Power law

The viscosity of the α^{th} constituent is defined by

$$\eta_\alpha = \eta_\alpha^0 \left(I_2^\alpha\right)^{\frac{n_\alpha - 1}{2}}; \quad \alpha = 1, 2 \tag{2.61}$$

where η_α^0 is the zero shear rate viscosity, n_α is the power law index, and (I_2^α) is the second invariant of $\mathbf{D}_{(\alpha)}$. For example in $\mathbb{R}^2$ we have the following

$$I_2^\alpha = 2\left(\frac{\partial\left(v_\alpha\right)_1}{\partial x_1}\right)^2 + 2\left(\frac{\partial\left(v_\alpha\right)_2}{\partial x_2}\right)^2 + \left(\frac{\partial\left(v_\alpha\right)_1}{\partial x_2} + \frac{\partial\left(v_\alpha\right)_2}{\partial x_1}\right)^2; \quad \alpha = 1, 2 \tag{2.62}$$

and η_α^0 and n_α are given data for a fluid.

Carreau-Yasuda model

$$\eta_\alpha = \eta_\alpha^\infty + \left(\eta_\alpha^0 + \eta_\alpha^\infty\right)\left(1 + \lambda_\alpha^2 I_2^\alpha\right)^{\frac{m_\alpha - 1}{2}} \; ; \quad \alpha = 1, 2 \tag{2.63}$$

η_α^0 and η_α^∞ are zero and infinite shear rate viscosity. η_α^0, η_α^∞, λ_α, and m_α are constants of the α^{th} constituent.

3. Mixture viscosity

The mixture viscosity μ_m can be determined using $\mathbf{D}_{(1)}$, $\mathbf{D}_{(2)}$, μ_1, μ_2, and ρ_1, ρ_2, ρ_m. For an isotropic, homogeneous, saturated mixture (Newtonian or generalized Newtonian) we can write

$${}_d\boldsymbol{\sigma}_m = \mu_m \mathbf{D} \tag{2.64}$$

in which

$${}_d\boldsymbol{\sigma}_m = \sum_\alpha {}_d\boldsymbol{\sigma}_\alpha \tag{2.65}$$

and

$$ {}_d\boldsymbol{\sigma}_\alpha = \mu_\alpha \mathbf{D}_\alpha \tag{2.66} $$

using (2.7), we can write

$$ \mathbf{D} = \sum_\alpha \frac{\rho_\alpha}{\rho_m} \mathbf{D}_\alpha \tag{2.67} $$

using (2.67) and (2.64), we obtain

$$ {}_d\boldsymbol{\sigma}_m = \mu_m \left(\sum_\alpha \frac{\rho_\alpha}{\rho_m} \mathbf{D}_\alpha \right) \tag{2.68} $$

or

$$ \left({}_d\sigma_m\right)_{ij} = \mu_m \left(\sum_\alpha \frac{\rho_\alpha}{\rho_m} \left(D_\alpha\right)_{ij} \right) \tag{2.69} $$

also from (2.64)

$$ \left({}_d\sigma_m\right)_{ij} = \mu_m \left(D_{ij}\right) \tag{2.70} $$

The mixture viscosity μ_m is deterministic from (2.69) or (2.70). For known volume fractions and constituent viscosities it is shown that for fully developed flow between parallel plates (2.69) or (2.70) holds.

2.5 Dimensionless form of the mathematical models in $\mathbb{R}^2$

For convenience, we introduce more familiar notation. Let

$$ \left(v_\alpha\right)_1 = u_\alpha \quad , \quad \left(v_\alpha\right)_2 = v_\alpha \quad , \quad x_1 = x \quad , \quad x_2 = y $$

In $\left({}_d\sigma_\alpha\right)_{ij}$; $i, j = 1, 2$ correspond to x and y. Velocities u and v are x and y components of $\mathbf{v}$. Likewise, $\mathbf{v}_\alpha$ has components u_α and v_α in the x and y directions.

Using this notation, the mathematical model in $\mathbb{R}^2$ for a two constituent, saturated, incompressible mixture of Newtonian or generalized Newtonian fluids can be written as (for isothermal flows).

$$
\begin{aligned}
\rho_\alpha &= \phi_\alpha \rho^{(\alpha)} \\
\rho_m &= \sum_{\alpha=1}^{2} \rho_\alpha \\
\sum_{\alpha=1}^{2} \phi_\alpha &= 1 \\
\rho_m \mathbf{v} &= \sum_{\alpha=1}^{2} \rho_\alpha \mathbf{v}_\alpha
\end{aligned}
\tag{2.71}
$$

Continuity equations:

$$
\rho_\alpha \left(\frac{\partial u_\alpha}{\partial x} + \frac{\partial v_\alpha}{\partial y} \right) = 0 \; ; \quad \alpha = 1,2 \tag{2.72}
$$

Momentum equations:

$$
\begin{aligned}
\rho_\alpha \left(\frac{\partial u_\alpha}{\partial t} + u_\alpha \frac{\partial u_\alpha}{\partial x} + v_\alpha \frac{\partial u_\alpha}{\partial y} \right) + \frac{\partial p_\alpha}{\partial x} - \frac{\partial \left({}_d\sigma_\alpha \right)_{xx}}{\partial x} - \frac{\partial \left({}_d\sigma_\alpha \right)_{xy}}{\partial x} - \left(\pi_\alpha \right)_x = 0 \; ; \quad \alpha = 1,2 \\
\rho_\alpha \left(\frac{\partial v_\alpha}{\partial t} + u_\alpha \frac{\partial v_\alpha}{\partial x} + v_\alpha \frac{\partial v_\alpha}{\partial y} \right) + \frac{\partial p_\alpha}{\partial y} - \frac{\partial \left({}_d\sigma_\alpha \right)_{xy}}{\partial x} - \frac{\partial \left({}_d\sigma_\alpha \right)_{yy}}{\partial x} - \left(\pi_\alpha \right)_y = 0 \; ; \quad \alpha = 1,2
\end{aligned}
\tag{2.73}
$$

Constitutive equations:

$$
{}_d\boldsymbol{\sigma}_\alpha = \mu_\alpha \mathbf{D}_{(\alpha)} \; ; \quad \alpha = 1,2 \tag{2.74}
$$

where

$$
\mu_1 = \phi_1^2 \eta_1 + \phi_1 \phi_2 \eta_{12} \quad ; \quad \mu_2 = \phi_2^2 \eta_2 + \phi_1 \phi_2 \eta_{12} \quad ; \quad \eta_{12} = \sqrt{\eta_1 \eta_2} \tag{2.75}
$$

η_1 and η_2 are the viscosities of the two constituents.

Power Law model:

$$\eta_\alpha = \eta_\alpha^0 \left(I_2^\alpha\right)^{\frac{n_\alpha - 1}{2}} \quad ; \quad \alpha = 1, 2$$
$$I_2^\alpha = 2\left(\frac{\partial u_\alpha}{\partial x}\right)^2 + 2\left(\frac{\partial v_\alpha}{\partial y}\right)^2 + \left(\frac{\partial u_\alpha}{\partial y} + \frac{\partial v_\alpha}{\partial x}\right)^2 \quad ; \quad \alpha = 1, 2 \tag{2.76}$$

Carreau-Yasuda model:

$$\eta_\alpha = \eta_\alpha^\infty + \left(\eta_\alpha^0 + \eta_\alpha^\infty\right)\left(1 + \lambda_\alpha^2 I_2^\alpha\right)^{\frac{m_\alpha - 1}{2}} \quad ; \quad \alpha = 1, 2 \tag{2.77}$$

2.5.1 Dimensionless form

First we introduce ‘ ˆ ’ (hat) on all quantities in (2.71) – (2.77) indicating that the quantities have their usual dimensions or units and use the following reference quantities and the dimensionless variables

$$\begin{aligned} &\hat{x} = xL_0, \qquad \hat{y} = yL_0, \qquad \hat{u}_\alpha = u_\alpha u_0, \qquad \hat{v}_\alpha = v_\alpha u_0 \\ &\hat{\eta}_\alpha = \eta_\alpha \eta_0, \qquad \hat{p}_\alpha = p_\alpha p_0, \qquad {}_d\hat{\boldsymbol{\sigma}}_\alpha = {}_d\,\boldsymbol{\sigma}_\alpha \tau_0, \qquad \hat{\rho}_\alpha = \rho_\alpha \rho_0 \end{aligned} \tag{2.78}$$

In which L_0 is the reference length, u_0 is the reference velocity, η_0 is the reference viscosity, p_0 is the reference pressure, τ_0 is the reference stress, and ρ_0 is reference density. For consistency we must use $p_0 = \tau_0$. We can use either characteristic kinetic energy or characteristic viscous stress to choose reference value τ_0.

The reference time t_0 is given by

$$t_0 = \frac{L_0}{u_0} \tag{2.79}$$

Using (2.71) – (2.77) with ’ ˆ ’ (hat) on all quantities and using (2.78) and (2.79), we can obtain the following dimensionless form of the GDEs for the two constituent mathematical model in $\mathbb{R}^2$.

Equations (2.71) and the continuity equations remain unchanged.

$$\rho_\alpha = \phi_\alpha \rho^{(\alpha)} \quad ; \quad \rho_m = \sum_{\alpha=1}^{2} \rho_\alpha \quad ; \quad \sum_{\alpha=1}^{2} \phi_\alpha = 1 \quad ; \quad \rho_m \mathbf{v} = \sum_{\alpha=1}^{2} \rho_\alpha \mathbf{v}_\alpha \tag{2.80}$$

Continuity equations:

$$\rho_\alpha \left(\frac{\partial u_\alpha}{\partial x} + \frac{\partial v_\alpha}{\partial y} \right) = 0 \; ; \quad \alpha = 1, 2 \tag{2.81}$$

Momentum equations:

$$\begin{aligned}
&\rho_\alpha \left(\frac{\partial u_\alpha}{\partial t} + u_\alpha \, \frac{\partial u_\alpha}{\partial x} + v_\alpha \frac{\partial u_\alpha}{\partial y} \right) + \left(\frac{p_0}{\rho_0 u_0^2} \right) \frac{\partial p_\alpha}{\partial x} \\
&\qquad - \left(\frac{\tau_0}{\rho_0 u_0^2} \right) \left(\frac{\partial \left({}_d\sigma_\alpha \right)_{xx}}{\partial x} + \frac{\partial \left({}_d\sigma_\alpha \right)_{xy}}{\partial y} \right) - \left(\frac{L_0}{\rho_0 u_0^2} \right) \left(\pi_\alpha \right)_x = 0 \; ; \quad \alpha = 1, 2 \\
&\rho_\alpha \left(\frac{\partial v_\alpha}{\partial t} + u_\alpha \, \frac{\partial v_\alpha}{\partial x} + v_\alpha \frac{\partial v_\alpha}{\partial y} \right) + \left(\frac{p_0}{\rho_0 u_0^2} \right) \frac{\partial p_\alpha}{\partial y} \\
&\qquad - \left(\frac{\tau_0}{\rho_0 u_0^2} \right) \left(\frac{\partial \left({}_d\sigma_\alpha \right)_{xy}}{\partial x} + \frac{\partial \left({}_d\sigma_\alpha \right)_{yy}}{\partial y} \right) - \left(\frac{L_0}{\rho_0 u_0^2} \right) \left(\pi_\alpha \right)_y = 0 \; ; \quad \alpha = 1, 2
\end{aligned} \tag{2.82}$$

2.5.2 Power law for constituents and mixture

$$\hat{\eta}_\alpha = \hat{\eta}_\alpha^0 \left(\hat{I}_2^\alpha \right)^{\frac{n_\alpha - 1}{2}} \quad ; \quad \alpha = 1, 2 \tag{2.83}$$

where $\hat{\eta}_\alpha$ are the viscosities of the constituents. $\hat{\eta}_\alpha^0$, $\hat{I}_2^\alpha$, and n_α are zero shear rate viscosity, second invariant of the strain rate tensor, and power law index for constituent α. Using (2.78), we can write (2.83) as

$$\hat{\eta}_\alpha = \eta_0 \eta_\alpha^0 \left(\frac{u_0}{L_0} \right)^{n_\alpha - 1} \left(I_2^\alpha \right)^{\frac{n_\alpha - 1}{2}} = \left(\eta_0 \left(\frac{u_0}{L_0} \right)^{n_\alpha - 1} \right) \eta_\alpha^0 \left(I_2^\alpha \right)^{\frac{n_\alpha - 1}{2}} \quad ; \quad \alpha = 1, 2 \tag{2.84}$$

η_α^0 is dimensionless zero shear rate viscosity and I_2^α is the dimensionless second invariant of the strain rate tensor for constituent α.

or

$$\hat{\eta}_\alpha = \left(\eta_0 \left(\frac{u_0}{L_0} \right)^{n_\alpha - 1} \right) \eta_\alpha \quad ; \quad \eta_\alpha = \eta_\alpha^0 \left(I_2^\alpha \right)^{\frac{n_\alpha - 1}{2}} \quad ; \quad \alpha = 1, 2 \tag{2.85}$$

in which η_α is the dimensionless viscosity of constituent α. Using (2.85) we can define $\hat{\mu}_1$ and $\hat{\mu}_2$ in (2.75).

$$\begin{aligned} \hat{\mu}_1 &= \phi_1^2 \hat{\eta}_1 + \phi_1 \phi_2 \sqrt{\hat{\eta}_1 \hat{\eta}_2} \\ \hat{\mu}_2 &= \phi_2^2 \hat{\eta}_2 + \phi_1 \phi_2 \sqrt{\hat{\eta}_1 \hat{\eta}_2} \end{aligned} \tag{2.86}$$

Consider $\hat{\mu}_1$. Substituting from (2.85) for $\alpha = 1$.

$$\hat{\mu}_1 = \phi_1^2 \eta_0 \left(\frac{u_0}{L_0} \right)^{n_1 - 1} \eta_1 + \phi_1 \phi_2 \sqrt{\eta_0 \left(\frac{u_0}{L_0} \right)^{n_1 - 1} \eta_0 \left(\frac{u_0}{L_0} \right)^{n_2 - 1} \eta_1 \eta_2} \tag{2.87}$$

Consider $\left({}_d\hat{\sigma}_1 \right)_{xx}$ in (2.74). Substituting from (2.87) and non-dimensionalizing gives

$$\tau_0 \left({}_d\sigma_1 \right)_{xx} = 2 \left(\phi_1^2 \eta_0 \left(\frac{u_0}{L_0} \right)^{n_1 - 1} \eta_1 + \phi_1 \phi_2 \sqrt{\eta_0 \left(\frac{u_0}{L_0} \right)^{n_1 - 1} \eta_0 \left(\frac{u_0}{L_0} \right)^{n_2 - 1} \eta_1 \eta_2} \right) \frac{u_0}{L_0} \frac{\partial u_1}{\partial x}$$

or

$$\begin{aligned} \left({}_d\sigma_1 \right)_{xx} = 2 \Bigg(& \phi_1^2 \left(\frac{u_0}{\tau_0 L_0} \right) \eta_0 \left(\frac{u_0}{L_0} \right)^{n_1 - 1} \\ & + \phi_1 \phi_2 \sqrt{ \left(\frac{u_0}{\tau_0 L_0} \eta_0 \left(\frac{u_0}{L_0} \right)^{n_1 - 1} \right) \left(\frac{u_0}{\tau_0 L_0} \eta_0 \left(\frac{u_0}{L_0} \right)^{n_2 - 1} \right) \eta_1 \eta_2 } \Bigg) \frac{\partial u_1}{\partial x} \end{aligned} \tag{2.88}$$

If we use $\tau_0 = \rho_0 u_0^2$ (characteristic kinetic energy), then

$$\frac{u_0}{\tau_0 L_0}\left(\eta_0\left(\frac{u_0}{L_0}\right)^{n_1-1}\right)=\frac{\eta_0 u_0}{\rho_0 u_0^2 L_0}\left(\eta_0\left(\frac{u_0}{L_0}\right)^{n_1-1}\right)=\frac{\eta_0}{\rho_0\left(L_0\right)^{n_1}\left(u_0\right)^{2-n_1}}=\frac{1}{\left(R_{en}\right)_1} \tag{2.89}$$

where $\left(R_{en}\right)_1$ is the Reynolds number for constituent one. Similarly

$$\frac{u_0}{\tau_0 L_0}\left(\eta_0\left(\frac{u_0}{L_0}\right)^{n_2-1}\right)=\frac{\eta_0}{\rho_0\left(L_0\right)^{n_2}\left(u_0\right)^{2-n_2}}=\frac{1}{\left(R_{en}\right)_2} \tag{2.90}$$

Hence, we can write the following for $\left({}_d\sigma_1\right)_{xx}$

$$\left({}_d\sigma_1\right)_{xx}=2\left(\phi_1^2\frac{\eta_1}{\left(R_{en}\right)_1}+\phi_1\phi_2\sqrt{\frac{1}{\left(R_{en}\right)_1\left(R_{en}\right)_2}\cdot\eta_1\eta_2}\right)\frac{\partial u_1}{\partial x} \tag{2.91}$$

or

$$\left({}_d\sigma_1\right)_{xx}=2\underset{\sim}{\mu}_1\frac{\partial u_1}{\partial x} \tag{2.92}$$

where

$$\underset{\sim}{\mu}_1=\phi_1^2\frac{\eta_1}{\left(R_{en}\right)_1}+\phi_1\phi_2\sqrt{\frac{1}{\left(R_{en}\right)_1\left(R_{en}\right)_2}\cdot\eta_1\eta_2} \tag{2.93}$$

Similarly for $\left({}_d\sigma_2\right)_{xx}$, we have

$$\left({}_d\sigma_2\right)_{xx}=2\underset{\sim}{\mu}_2\frac{\partial u_2}{\partial x} \tag{2.94}$$

where

$$\underset{\sim}{\mu}_2=\phi_2^2\frac{\eta_2}{\left(R_{en}\right)_2}+\phi_1\phi_2\sqrt{\frac{1}{\left(R_{en}\right)_1\left(R_{en}\right)_2}\cdot\eta_1\eta_2} \tag{2.95}$$

Similar derivation holds for the other components of the deviatoric Cauchy stress components. In summary we have the following for the constitutive equations

$$ {}_d\boldsymbol{\sigma}_\alpha = \underset{\sim}{\mu}_\alpha \mathbf{D}_{(\alpha)} \; ; \quad \alpha = 1, 2 \tag{2.96} $$

and

$$ {}_d\boldsymbol{\sigma}_m = \underset{\sim}{\mu}_m \mathbf{D} \tag{2.97} $$

Equations (2.80)–(2.82), (2.96), (2.93), (2.95), and (2.85) constitute the dimensionless form of the complete mathematical model in $\mathbb{R}^2$ for a power law mixture of two power law constituents.

2.5.3 Carreau model for constituents and mixture

In the case of the Carreau model, the definitions of μ_1 and μ_2 change compared to power law. We consider details in the following.

Using (2.77)

$$ \hat{\eta}_\alpha = \hat{\eta}^0_\alpha + \left(\hat{\eta}^0_\alpha - \hat{\eta}^\infty_\alpha\right)\left(1 + \lambda^2_\alpha \hat{I}^\alpha_2\right)^{\frac{m_\alpha - 1}{2}} \quad ; \quad \alpha = 1, 2 \tag{2.98} $$

Using (2.78) we can write the following for (2.98)

$$ \hat{\eta}_\alpha = \eta_0 \left(\eta^0_\alpha + \left(\eta^0_\alpha - \eta^\infty_\alpha\right)\left(1 + \lambda^2_\alpha \left(\frac{u_0}{L_0}\right)^2 I^\alpha_2\right)^{\frac{m_\alpha - 1}{2}} \right) \; ; \quad \alpha = 1, 2 \tag{2.99} $$

Let $\dfrac{\lambda_\alpha u_0}{L_0} = c_{u\alpha}$ be the Carreau number for constituent α.

$$ \therefore \quad \hat{\eta}_\alpha = \eta_0 \left(\eta^0_\alpha + \left(\eta^0_\alpha - \eta^\infty_\alpha\right)\left(1 + (c_{u1})^2 I^\alpha_2\right)^{\frac{m_\alpha - 1}{2}} \right) = \eta_0 \eta_\alpha \; ; \quad \alpha = 1, 2 \tag{2.100} $$

where

$$\eta_\alpha = \eta_\alpha^0 + \left(\eta_\alpha^0 - \eta_\alpha^\infty\right)\left(1 + (c_{u\alpha})^2 I_2^\alpha\right)^{\frac{m_\alpha - 1}{2}} \quad ; \quad \alpha = 1, 2 \tag{2.101}$$

Using (2.100) we can define $\hat{\mu}_1$ and $\hat{\mu}_2$ in (2.75).

$$\begin{aligned} \hat{\mu}_1 &= \phi_1^2 \hat{\eta}_1 + \phi_1 \phi_2 \sqrt{\hat{\eta}_1 \hat{\eta}_2} \\ \hat{\mu}_2 &= \phi_2^2 \hat{\eta}_2 + \phi_1 \phi_2 \sqrt{\hat{\eta}_1 \hat{\eta}_2} \end{aligned} \tag{2.102}$$

Consider $\hat{\mu}_1$. Substituting from (2.100) we obtain

$$\hat{\mu}_1 = \phi_1^2 \eta_0 \eta_1 + \phi_1 \phi_2 \sqrt{\eta_0 \eta_1 \eta_0 \eta_2} \tag{2.103}$$

Consider $\left({}_d\sigma_1\right)_{xx}$ in (2.74). Substituting from (2.103) and nondimensionalizing gives

$$\tau_0 \left({}_d\sigma_1\right)_{xx} = 2\left(\phi_1^2 \eta_0 \eta_1 + \phi_1 \phi_2 \sqrt{\eta_0 \eta_1 \eta_0 \eta_2}\right) \frac{u_0}{L_0} \frac{\partial u_1}{\partial x} \tag{2.104}$$

using $\tau_0 = \rho_0 u_0^2$ (characteristic kinetic energy)

$$\left({}_d\sigma_1\right)_{xx} = 2\left(\phi_1^2 \left(\frac{\eta_0}{L_0 \rho_0 u_0}\right) \eta_1 + \phi_1 \phi_2 \sqrt{\left(\frac{\eta_0}{L_0 \rho_0 u_0}\right) \eta_1 \left(\frac{\eta_0}{L_0 \rho_0 u_0}\right) \eta_2}\right) \frac{\partial u_1}{\partial x} \tag{2.105}$$

or

$$\left({}_d\sigma_1\right)_{xx} = 2\left(\frac{1}{Re} \phi_1^2 \eta_1 + \phi_1 \phi_2 \sqrt{\eta_1 \eta_2}\right) \frac{\partial u_1}{\partial x} = 2 \underset{\sim}{\mu}_1 \frac{\partial u_1}{\partial x} \tag{2.106}$$

where $Re = \frac{L_0 \rho_0 u_0}{\eta_0}$; Reynolds number

Similarly for constituent two we have

$$\left({}_d\sigma_2\right)_{xx} = 2\left(\frac{1}{Re} \phi_2^2 \eta_2 + \phi_1 \phi_2 \sqrt{\eta_1 \eta_2}\right) \frac{\partial u_2}{\partial x} = 2 \underset{\sim}{\mu}_2 \frac{\partial u_2}{\partial x} \tag{2.107}$$

In summary, we have the following for the constitutive equations

$$ {}_d\boldsymbol{\sigma}_\alpha = \underset{\sim}{\mu}_\alpha \mathbf{D}_{(\alpha)} \; ; \quad \alpha = 1, 2 \tag{2.108} $$

and

$$ {}_d\boldsymbol{\sigma}_m = \underset{\sim}{\mu}_m \mathbf{D} \tag{2.109} $$

Clearly, $\underset{\sim}{\mu}_1 = \dfrac{\mu_1}{Re}$ and $\underset{\sim}{\mu}_2 = \dfrac{\mu_2}{Re}$.

2.5.4 Newtonian constituents and mixture

For this case $\hat{\eta}_\alpha \; ; \; \alpha = 1, 2$ are constant, hence we have

$$ \begin{aligned} \hat{\mu}_1 &= \eta_0 \left(\phi_1^2 \eta_1 + \phi_1 \phi_2 \sqrt{\eta_1 \eta_2} \right) = \eta_0 \mu_1 \\ \hat{\mu}_2 &= \eta_0 \left(\phi_2^2 \eta_2 + \phi_1 \phi_2 \sqrt{\eta_1 \eta_2} \right) = \eta_0 \mu_2 \end{aligned} \tag{2.110} $$

where

$$ \mu_1 = \phi_1^2 \eta_1 + \phi_1 \phi_2 \sqrt{\eta_1 \eta_2} \quad ; \quad \mu_2 = \phi_2^2 \eta_2 + \phi_1 \phi_2 \sqrt{\eta_1 \eta_2} \tag{2.111} $$

Consider $({}_d\sigma_1)_{xx}$. Using (2.110) and nondimensionalizing $({}_d\sigma_1)_{xx}$

$$ \tau_0 \left({}_d\sigma_1 \right)_{xx} = 2 \eta_0 \mu_1 \frac{u_0}{L_0} \frac{\partial u_1}{\partial x} \tag{2.112} $$

or

$$ \left({}_d\sigma_1 \right)_{xx} = 2 \mu_1 \left(\frac{\eta_0 u_0}{\tau_0 L_0} \right) \frac{\partial u_1}{\partial x} \tag{2.113} $$

when $\tau_0 = \rho_0 u_0^2$ (characteristic kinetic energy), we have

$$({}_d\sigma_1)_{xx} = 2\mu_1 \left(\frac{\eta_0}{\rho_0 u_0 L_0}\right)\frac{\partial u_1}{\partial x} = 2\frac{\mu_1}{Re}\frac{\partial u_1}{\partial x} = 2\underset{\sim}{\mu}_1\frac{\partial u_1}{\partial x} \tag{2.114}$$

In summary, we have the following constitutive equations in the dimensionless form when the constituents and the mixture are Newtonian fluids.

$${}_d\boldsymbol{\sigma}_\alpha = \underset{\sim}{\mu}_\alpha \mathbf{D}_{(\alpha)}\ ; \quad \alpha = 1,2 \tag{2.115}$$

and

$${}_d\boldsymbol{\sigma}_m = \underset{\sim}{\mu}_m \mathbf{D} \tag{2.116}$$

2.6 Remarks

1. If the constituents are Newtonian fluids and the mixture is also a Newtonian fluid and if we neglect $(\pi_1)_x$, $(\pi_2)_x$, $(\pi_1)_y$, and $(\pi_2)_y$, then the mathematical model for the constituents is decoupled. In this case we can use the continuity equation, momentum equations, and the constitutive equations for each constituent to obtain deformation fields and then use (2.80) to obtain the mixture deformation field. The combined model will also function properly in the least squares computational process (see Chapter 3). In the following we present details of the decoupled mathematical models in $\mathbb{R}^2$ for a two constituent mixture. For partial pressures p_α of the constituents we assume $p_\alpha = \phi_\alpha p$ and $\sum_\alpha p_\alpha = p$ yielding $\sum_\alpha \phi_\alpha = 1$ which holds. Thus, for a two constituent mixture we can write

$$\begin{aligned} p_\alpha &= \phi_\alpha p \\ \frac{\partial p_\alpha}{\partial x_i} &= \phi_\alpha \frac{\partial p}{\partial x_i}\ ; \quad \alpha, i = 1,2 \end{aligned} \tag{2.117}$$

Constituent 1: Decoupled mathematical model (BVP)

Using (2.117) and (2.81), (2.82), and (2.115) we have

$$
\begin{aligned}
\rho_1\left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}\right)&=0\\
\rho_1\left(u_1\frac{\partial u_1}{\partial x}+v_1\frac{\partial u_1}{\partial y}\right)&+\left(\frac{p_0}{\rho_0 u_0^2}\right)\phi_1\frac{\partial p}{\partial x}\\
&-\left(\frac{\tau_0}{\rho_0 u_0^2}\right)\left(\frac{\partial\left({}_d\sigma_1\right)_{xx}}{\partial x}+\frac{\partial\left({}_d\sigma_1\right)_{xy}}{\partial y}\right)-\left(\frac{L_0}{\rho_0 u_0^2}\right)\left(\pi_1\right)_x=0\\
\rho_1\left(u_1\frac{\partial v_1}{\partial x}+v_1\frac{\partial v_1}{\partial y}\right)&+\left(\frac{p_0}{\rho_0 u_0^2}\right)\phi_1\frac{\partial p}{\partial y}\\
&-\left(\frac{\tau_0}{\rho_0 u_0^2}\right)\left(\frac{\partial\left({}_d\sigma_1\right)_{xy}}{\partial x}+\frac{\partial\left({}_d\sigma_1\right)_{yy}}{\partial y}\right)-\left(\frac{L_0}{\rho_0 u_0^2}\right)\left(\pi_1\right)_y=0\\
\left({}_d\sigma_1\right)_{xx}=2\underset{\sim}{\mu_1}\frac{\partial u_1}{\partial x}\quad;\quad\left({}_d\sigma_1\right)_{xy}&=\underset{\sim}{\mu_1}\left(\frac{\partial u_1}{\partial y}+\frac{\partial v_1}{\partial x}\right)\quad;\quad\left({}_d\sigma_1\right)_{yy}=2\underset{\sim}{\mu_1}\frac{\partial v_1}{\partial y}
\end{aligned}
\tag{2.118}
$$

Constituent 2: Decoupled mathematical model (BVP)

In this case also using (2.117), (2.81), (2.82), and (2.115) we obtain

$$
\begin{aligned}
\rho_2\left(\frac{\partial u_2}{\partial x}+\frac{\partial v_2}{\partial y}\right)&=0\\
\rho_2\left(u_1\frac{\partial u_2}{\partial x}+v_1\frac{\partial u_2}{\partial y}\right)&+\left(\frac{p_0}{\rho_0 u_0^2}\right)\phi_2\frac{\partial p}{\partial x}\\
&-\left(\frac{\tau_0}{\rho_0 u_0^2}\right)\left(\frac{\partial\left({}_d\sigma_2\right)_{xx}}{\partial x}+\frac{\partial\left({}_d\sigma_2\right)_{xy}}{\partial y}\right)-\left(\frac{L_0}{\rho_0 u_0^2}\right)\left(\pi_2\right)_x=0\\
\rho_2\left(u_2\frac{\partial v_2}{\partial x}+v_2\frac{\partial v_2}{\partial y}\right)&+\left(\frac{p_0}{\rho_0 u_0^2}\right)\phi_2\frac{\partial p}{\partial y}\\
&-\left(\frac{\tau_0}{\rho_0 u_0^2}\right)\left(\frac{\partial\left({}_d\sigma_2\right)_{xy}}{\partial x}+\frac{\partial\left({}_d\sigma_2\right)_{yy}}{\partial y}\right)-\left(\frac{L_0}{\rho_0 u_0^2}\right)\left(\pi_2\right)_y=0\\
\left({}_d\sigma_2\right)_{xx}=2\underset{\sim}{\mu_2}\frac{\partial u_2}{\partial x}\quad;\quad\left({}_d\sigma_2\right)_{xy}&=\underset{\sim}{\mu_2}\left(\frac{\partial u_2}{\partial y}+\frac{\partial v_2}{\partial x}\right)\quad;\quad\left({}_d\sigma_2\right)_{yy}=2\underset{\sim}{\mu_2}\frac{\partial v_2}{\partial y}
\end{aligned}
\tag{2.119}
$$

when using mathematical models (2.118) and (2.119) for constituents 1 and 2 the calculated p in (2.118) is p_1 and p from (2.119) is p_2 and the pressure field for the mixture is $p = p_1 + p_2$.

2. However, when the constituents are generalized Newtonian fluids and when the mixture is also a generalized Newtonian fluid, decoupling is not possible due to the fact that μ_1 and μ_2 are functions of deformation fields of both constituents.

3. In the numerical studies we neglect (π_1) and (π_2) in the momentum equations.

4. In section 2.7 that follows these remarks, we derive the mathematical model for fully developed flow between parallel plates. This model reveals some features that are not obvious from the mathematical model in $\mathbb{R}^2$.

2.7 Mathematical model for fully developed flow between parallel plates: mixture of two constituents

In this case the mathematical model describes a BVP. For fully developed flow between parallel plates we only need to consider the one dimensional case i.e. a typical section **A–A** (Figure 2.1) where the flow is fully developed. In this case

$$v_1 = 0 \quad , \quad u_1 \neq 0$$
$$\frac{\partial u_1}{\partial x} = 0 \quad , \quad \frac{\partial v_1}{\partial x} = 0 \quad , \quad \frac{\partial v_1}{\partial x} = 0 \quad , \quad \frac{\partial u_1}{\partial y} \neq 0 \quad , \quad ({}_d\sigma_1)_{xy} \neq 0 \; ,$$
$$\frac{\partial p_1}{\partial x} \neq 0 \quad , \quad \frac{\partial p_1}{\partial y} = 0 \quad , \quad ({}_d\sigma_1)_{xx} = 0 \quad , \quad ({}_d\sigma_1)_{yy} = 0$$

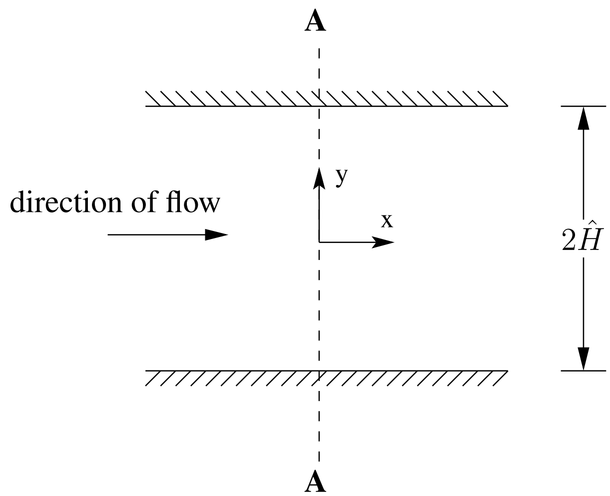

Figure 2.1: Flow between parallel plates

similarly

$$v_2 = 0 \quad , \quad u_2 \neq 0$$
$$\frac{\partial u_2}{\partial x} = 0 \quad , \quad \frac{\partial v_2}{\partial x} = 0 \quad , \quad \frac{\partial v_2}{\partial x} = 0 \quad , \quad \frac{\partial u_2}{\partial y} \neq 0 \quad , \quad ({}_d\sigma_2)_{xy} \neq 0 \; , \tag{2.120}$$
$$\frac{\partial p_2}{\partial x} \neq 0 \quad , \quad \frac{\partial p_2}{\partial y} = 0 \quad , \quad ({}_d\sigma_2)_{xx} = 0 \quad , \quad ({}_d\sigma_2)_{yy} = 0$$

Hence, continuity equations are identically satisfied. Using (2.120), the dimensionless forms of the momentum equations and the constitutive equations reduce to (neglecting (π_1) and (π_2))

$$\left(\frac{p_0}{\rho_0 u_0^2}\right)\frac{\partial p_\alpha}{\partial x} - \left(\frac{\tau_0}{\rho_0 u_0^2}\right)\frac{\partial \left({}_d\sigma_\alpha\right)_{xy}}{\partial y} = 0 \quad ; \quad ({}_d\sigma_\alpha)_{xy} = \underset{\sim}{\mu}_\alpha\left(\eta_1, \eta_2, \phi_1, \phi_2\right)\frac{\partial u_\alpha}{\partial y} \; ; \quad \alpha = 1, 2 \tag{2.121}$$

Details of $\underset{\sim}{\mu}_1$ and $\underset{\sim}{\mu}_2$ are given in the following.

Newtonian constituents and mixture

$$\underset{\sim}{\mu}_\alpha = \frac{\mu_\alpha}{Re} \tag{2.122}$$

μ_α is defined in (2.114). If we assume the mixture to be a Newtonian fluid, then using (2.69) or (2.70) we have the following for the dimensionless case

$$({}_d\sigma)_{xy} = \underset{\sim}{\mu}_m \frac{\partial u_m}{\partial y} \tag{2.123}$$

In which $({}_d\sigma)_{xy} = ({}_d\sigma_1)_{xy} + ({}_d\sigma_2)_{xy}$ and u_m is the mixture velocity in the x direction. Using (2.123) we can determine $\underset{\sim}{\mu}_m$ for the mixture. However, since $\frac{\partial u}{\partial y} = 0$ at the center-line it is better to use

$$\underset{\sim}{\mu}_m = \left(\frac{\frac{\partial ({}_d\sigma)_{xy}}{\partial y}}{\frac{\partial^2 u}{\partial y^2}} \right) \tag{2.124}$$

to determine $\underset{\sim}{\mu}_m$.

Power law model for constituents and mixture

In this case $\underset{\sim}{\mu}_1$ and $\underset{\sim}{\mu}_2$ are given by

$$\underset{\sim}{\mu}_\alpha = \phi_\alpha^2 \frac{\eta_\alpha}{(R_{en})_\alpha} + \phi_1 \phi_2 \sqrt{\frac{1}{(R_{en})_1 (R_{en})_2} \eta_1 \eta_2} \; ; \quad \alpha = 1, 2 \tag{2.125}$$

where

$$\eta_\alpha = \eta_\alpha^0 \left(I_2^\alpha \right)^{\frac{n_\alpha - 1}{2}} \; ; \quad \alpha = 1, 2 \tag{2.126}$$

and

$$I_2^\alpha = \left(\frac{\partial u_\alpha}{\partial y}\right)^2 \quad ; \quad \alpha = 1, 2 \tag{2.127}$$

For the mixture we can write

$$\left({}_d\sigma\right)_{xy} = \underset{\sim}{\mu}_m \frac{\partial u_m}{\partial y} \tag{2.128}$$

Using (2.128) we can determine $\underset{\sim}{\mu}_m$ for the mixture.

Carreau model for constituents and mixture

In this case $\underset{\sim}{\mu}_1$ and $\underset{\sim}{\mu}_2$ are given by (2.106) and (2.107) in which η_α are defined by (2.101). The definition of I_2^α remains the same as in (2.127). For the mixture we can write the following using (2.69) or (2.70).

$$\left({}_d\sigma\right)_{xy} = \underset{\sim}{\mu}_m \frac{\partial u_m}{\partial y} \tag{2.129}$$

In this case also we can determine $\underset{\sim}{\mu}_m$ for the mixture using (2.129).

Remarks:

1. We note that the mathematical model consists of four PDEs (2.121) in u_1, u_2, $\left({}_d\sigma_1\right)_{xy}$, $\left({}_d\sigma_1\right)_{xy}$, p_1, and p_2. Thus, the mathematical model does not have closure. However, for this case (fully developed flow), if we assume the flow to be pressure driven, then $\frac{\partial p_1}{\partial x}$ and $\frac{\partial p_2}{\partial x}$ are known. p_1 and p_2 are partial pressures of the constituents and hence must be related to the volume fractions of the constituents. We assume

$$\begin{gathered} p_1 = \phi_1 p \quad , \quad p_2 = \phi_2 p \\ \text{ie} \quad p_1 + p_2 = p \end{gathered} \tag{2.130}$$

Hence,

$$\frac{\partial p_1}{\partial x} = \phi_1 \frac{\partial p}{\partial x} \quad , \quad \frac{\partial p_2}{\partial x} = \phi_2 \frac{\partial p}{\partial x} \tag{2.131}$$

Thus, knowing volume fractions ϕ_1, ϕ_2 and $\frac{\partial p}{\partial x}$ for the mixture, $\frac{\partial p_1}{\partial x}$ and $\frac{\partial p_2}{\partial x}$ are defined and the mathematical model has closure. Based on this (as stated earlier), $p_\alpha = \phi_\alpha p$ and $\sum_{\alpha=1}^{\nu} p_\alpha = p$ which implies $\sum_{\alpha=1}^{\nu} \phi_\alpha = 1$ which obviously holds regardless of the model problem as long as the constituents and the mixture are Newtonian or generalized Newtonian fluids. Validity of this assumption is demonstrated for this model problem as well as the backward facing step.

2. The validity of the assumption in remark (1) can be verified using the model problem in $\mathbb{R}^2$ using the combined model in which p_1 and p_2 remain dependent variables in the mathematical model.

3. Using (2.130) and (2.131) the mathematical model given by (2.121) reduces to

$$\begin{gathered} \left(\frac{p_0}{\rho_0 u_0^2}\right) \phi_\alpha \frac{\partial p}{\partial x} - \left(\frac{\tau_0}{\rho_0 u_0^2}\right) \frac{\partial \left({}_d\sigma_\alpha\right)_{xy}}{\partial y} = 0 \\ \left({}_d\sigma_\alpha\right)_{xy} = \underset{\sim}{\mu}_\alpha \left(\eta_1, \eta_2, \phi_1, \phi_2\right) \frac{\partial u_\alpha}{\partial y} \ ; \quad \alpha = 1, 2 \end{gathered} \tag{2.132}$$

in which $\frac{\partial p}{\partial x}$ is known (pressure driven flow). This mathematical model has closure.

4. In the case of Newtonian constituents and mixture, $\underset{\sim}{\mu}_1$ and $\underset{\sim}{\mu}_2$ are not functions of the deformation field, hence the combined mathematical model can be decoupled for the constituents using (2.132) we can obtain mathematical models for each constituent $(\alpha = 1, 2)$.

Constituent 1: (decoupled model)

$$\left(\frac{p_0}{\rho_0 u_0^2}\right)\phi_1 \frac{\partial p}{\partial x} - \left(\frac{\tau_0}{\rho_0 u_0^2}\right)\frac{\partial\left({}_d\sigma_1\right)_{xy}}{\partial y} = 0 \tag{2.133}$$

$$\left({}_d\sigma_1\right)_{xy} = \underset{\sim}{\mu_1}\left(\eta_1, \eta_2, \phi_1, \phi_2\right)\frac{\partial u_1}{\partial y} \tag{2.134}$$

Constituent 2: (decoupled model)

$$\left(\frac{p_0}{\rho_0 u_0^2}\right)\phi_2 \frac{\partial p}{\partial x} - \left(\frac{\tau_0}{\rho_0 u_0^2}\right)\frac{\partial\left({}_d\sigma_2\right)_{xy}}{\partial y} = 0 \tag{2.135}$$

$$\left({}_d\sigma_2\right)_{xy} = \underset{\sim}{\mu_2}\left(\eta_1, \eta_2, \phi_1, \phi_2\right)\frac{\partial u_2}{\partial y} \tag{2.136}$$

Solutions for (2.132) ie the combined mathematical model must be the same as the combined solution obtained using decoupled models (2.133), (2.134) and (2.135), (2.136) for constituents 1 and 2.

Chapter 3

Numerical studies

3.1 Introduction

The mathematical models presented in Chapter 2 are a system of non-linear partial differential equations describing boundary value problems. Based on references [28–31] the finite element processes derived using the residual functional (least squares process) yield variationally consistent integral forms when the second variation of the residuals are neglected in the second variation of the residual functional. Justifications for doing so are given in the references by the authors. Variationally consistent integral forms yield unconditionally stable computations. Hence, in the present work we use this approach for obtaining numerical solutions of the mixtures of Newtonian and generalized Newtonian fluids. The local approximations are considered in $H^{k,p}(\bar{\Omega}^e)$ scalar product spaces in which k is the order of the space defining global differentiability of approximations and p is the degree of local approximations for all dependent variables. With this choice the least squares processes remain convergent [32].

We consider two model problems consisting of fully developed flow between parallel plates and an asymmetric backward facing step. In both model problems we only consider a saturated mixture of two fluids. Both Newtonian and generalized Newtonian fluids are considered. In the case of generalized Newtonian fluids we consider power law and Carreau-Yasuda models for shear thinning fluids. In all numerical studies (both $\mathbb{R}^1$ and $\mathbb{R}^2$) $p_0 = \tau_0 = \rho_0 u_0^2$ (characteristic kinetic energy) is used to choose reference pressure and reference stress.

3.2 Fully developed flow between parallel plates

In this model problem we consider fully developed flow between parallel plates. Figure 2.1 shows a schematic. We only need to consider a typical section **A–A**. Furthermore, due to symmetry considerations only half of the domain **A–A** is considered (consider $0 < y < 1$ at **A–A**). We consider the distance between the plates to be $2\hat{H} = 2$ cm and if we choose $L_0 = 0.01$ m then the dimensionless distance $2H$ between the plates is 2 and our computational domain is $0 \leq y \leq 1$ at **A–A**. We consider saturated mixtures of two constituents. The properties of the constituents are given in the following.

Newtonian constituents [19]

Fluid 1 (or constituent 1)

$\hat{\rho}^{(1)} = 900 \qquad \hat{\eta}_1 = 0.0267$

Fluid 2 (or constituent 2)

$\hat{\rho}^{(2)} = 1000 \qquad \hat{\eta}_2 = 0.0018$

Power law constituents [30]

Fluid 1 (or constituent 1)

$\hat{\rho}^{(1)} = 1001;\qquad \hat{\eta}_1^0 = 0.567$ (zero shear rate viscosity)

$n_1 = 0.854$ (power law index)

Fluid 2 (or constituent 2)

$\hat{\rho}^{(2)} = 1001;\qquad \hat{\eta}_2^0 = 0.332$ (zero shear rate viscosity)

$n_2 = 0.738$ (power law index)

Carreau model constituents [30]

Fluid 1 (or constituent 1)

$\hat{\rho}^{(1)} = 1001 \quad , \quad \hat{\eta}_1^0 = 0.18 \quad , \quad \hat{\eta}_1^\infty = 0.0 \quad , \quad \lambda_1 = 0.048 \quad , \quad m_1 = 0.729$

Fluid 2 (or constituent 2)

$\hat{\rho}^{(2)} = 1001 \quad , \quad \hat{\eta}_2^0 = 0.450 \quad , \quad \hat{\eta}_2^\infty = 0.0 \quad , \quad \lambda_2 = 2.28 \quad , \quad m_2 = 0.756$

We consider a 5 element discretization of the domain $0 \leq y \leq 1$ (at $\mathbf{A}$–$\mathbf{A}$) using 3-node p-version elements with local approximation in $H^{k,p}(\bar{\Omega}^e)$ scalar product spaces.

3.2.1 Newtonian constituents and Newtonian mixture

In this section we present a number of different numerical studies using the combined model for both constituents as well as using individual models for the constituents to demonstrate

1. that for Newtonian constituents and mixture the mathematical models for the constituents are decoupled

2. that the combined model produces exactly the same results as the individual models for the constituents.

In the numerical studies we choose $\dfrac{\partial p}{\partial x} = -0.2$, thus based on the assumption $p_1 = \phi_1 p$ and $p_2 = \phi_2 p$ we have

$$\begin{aligned} \frac{\partial p_1}{\partial x} &= \phi_1 \frac{\partial p}{\partial x} = -0.2\phi_1 \\ \frac{\partial p_2}{\partial x} &= \phi_2 \frac{\partial p}{\partial x} = -0.2\phi_2 \end{aligned} \tag{3.1}$$

We use (3.1) in the numerical studies using the combined model as well as the individual models for the constituents. The validity of assumption (3.1) is also verified numerically in the section containing numerical studies in $\mathbb{R}^2$. We consider and present results for various numerical studies using the combined mathematical model based on assumption (3.1) in the following. We consider a 5 element discretization using 3-node p-version elements. C^1 approximations at p-level 3 are used for the Newtonian studies, and C^2 approximations at p-level 9 are used for power law and Carreau model studies.

Case (a) when constituent 2 is the same as constituent 1 (combined model)

This is perhaps the simplest case for which the mixture theory must produce results that are obvious. We choose

$\eta_0 = \hat{\eta}_1^0 = 0.0267$; $\rho_0 = \hat{\rho}^{(1)} = 900$; and $\phi_1 = 0,\ 0.01,\ 0.1,\ 0.5,\ 0.9,\ 0.99$, and 1.

As expected the velocity u (figure 3.1) as a function of y is independent of volume fraction and the mixture velocity is the same as those of the constituents. Figure 3.2 shows plots of the mixture and constituent shear stresses for different volume fractions. $({}_d\sigma)_{xy} = ({}_d\sigma_1)_{xy} + ({}_d\sigma_2)_{xy}$ produces shear stress for the mixture that is in agreement with

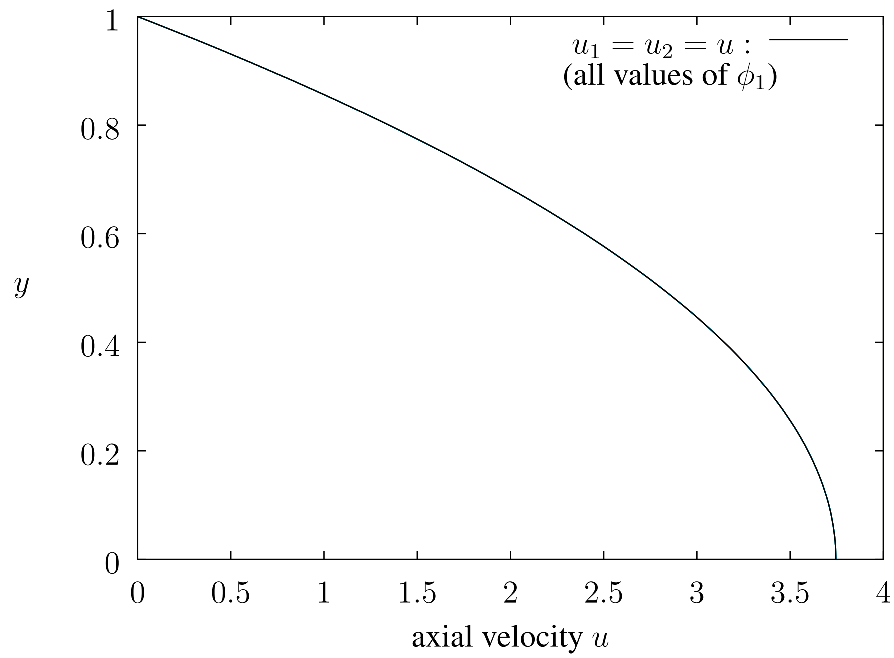

Figure 3.1: Velocity of constituents and mixture: Newtonian - fluid 2 same as fluid 1 (Combined Model)

the theoretical solution. Figure 3.3 shows plots of $\underset{\sim}{\mu}_1$, $\underset{\sim}{\mu}_2$, and $\underset{\sim}{\mu}_m$ versus volume fraction ϕ_1. With progressively increasing ϕ_1, $\underset{\sim}{\mu}_1$ increases linearly while $\underset{\sim}{\mu}_2$ decreases linearly such that $\underset{\sim}{\mu}_1 + \underset{\sim}{\mu}_2 = \underset{\sim}{\mu}_m =$ constant (corresponding to $\hat{\eta}_1$). This study shows the validity of mixture theory when the two constituents are the same.

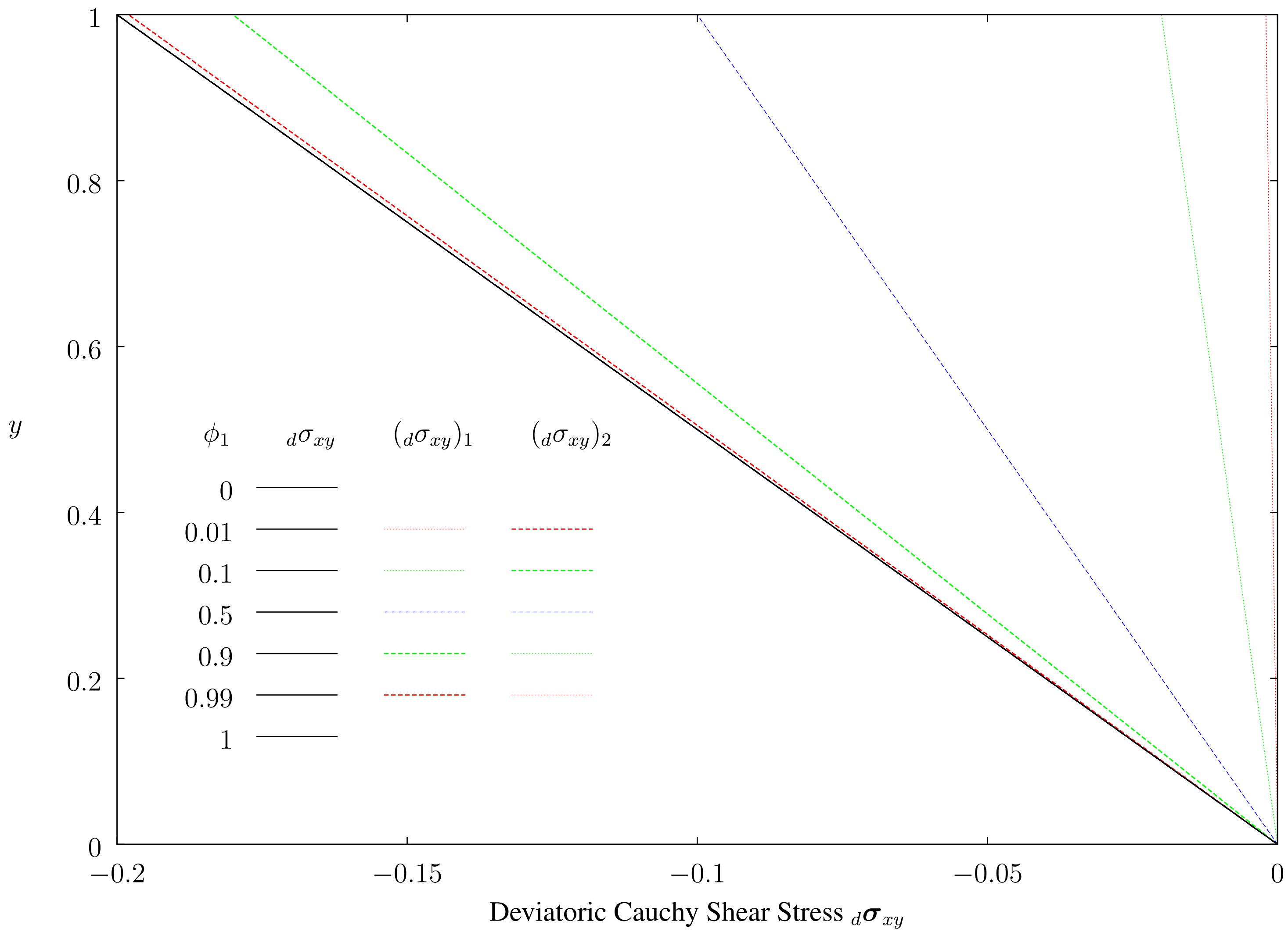

Figure 3.2: Deviatoric Cauchy shear stress for the constituents and the mixture:

Newtonian - fluid 2 same as fluid 1 (combined model)

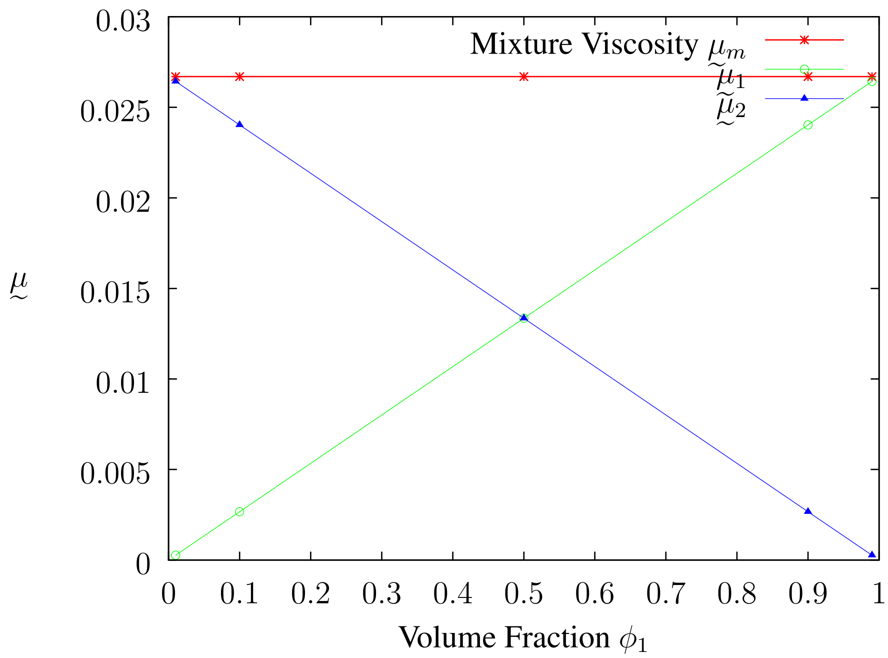

Figure 3.3: $\underset{\sim}{\mu}_1$, $\underset{\sim}{\mu}_2$, and $\underset{\sim}{\mu}_m$ versus y for different volume fractions

Case (b) mixture of constituent 1 and constituent 2 (combined model)

In this study we consider a saturated mixture of constituents one and two for different volume fractions. We choose $\rho_0 = \hat{\rho}^{(2)} = 1000$ and $\eta_0 = \hat{\eta}_1^0 = 0.0267$ as reference quantities. Figures (3.5) and (3.6) show plots of u_1, u_2, and u for different volume fractions. For $\phi_1 = 0$, the mixture consists of only constituent 2 and likewise for $\phi_1 = 1$, the mixture consists purely of constituent 1. The plots of u versus y for $\phi_1 = 0.0$ and $\phi_1 = 1.0$ confirm this. For $\phi_1 = 0.0$ and $\phi_1 = 1.0$, u versus y agrees precisely with the theoretical solutions for constituent 2 and constituent 1. u versus y for $\phi_1 = 0.0$ and $\phi_1 = 1.0$ obviously bracket the velocity profiles for different values of the volume fractions. Plots of shear stress for the constituents and the mixture are shown in figure 3.7. Plots of $\underset{\sim}{\mu}_1$, $\underset{\sim}{\mu}_2$, and $\underset{\sim}{\mu}_m$ for different volume fractions are shown in figure 3.4. For $\phi_1 = 1$ and $\phi_1 = 0$, $\underset{\sim}{\mu}_m$ corresponds to η_1

and η_2 as expected.

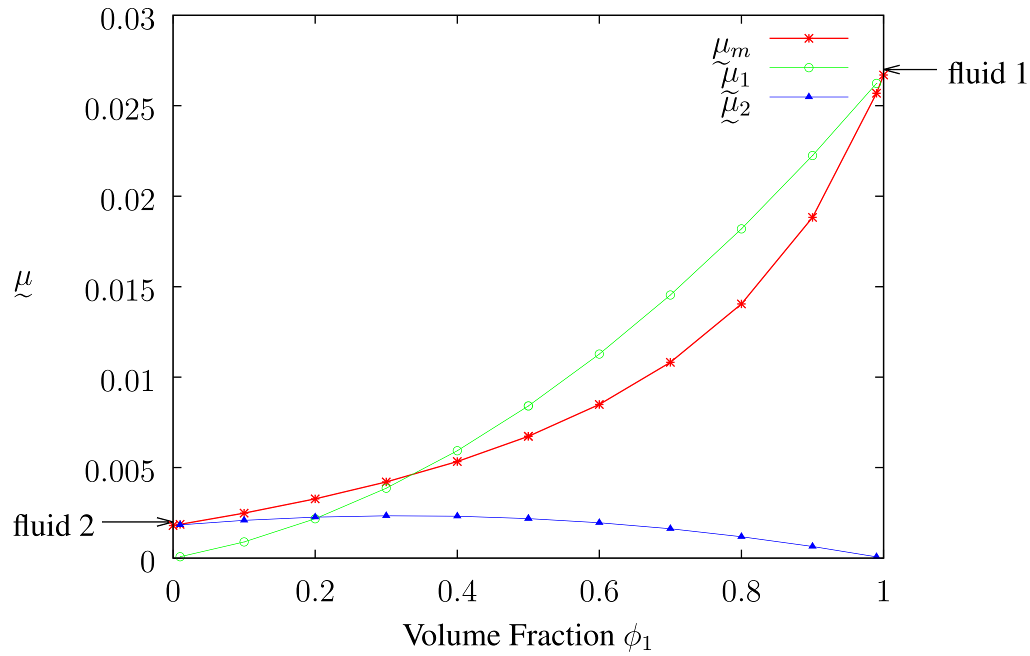

Figure 3.4: Mixture viscosity: Newtonian

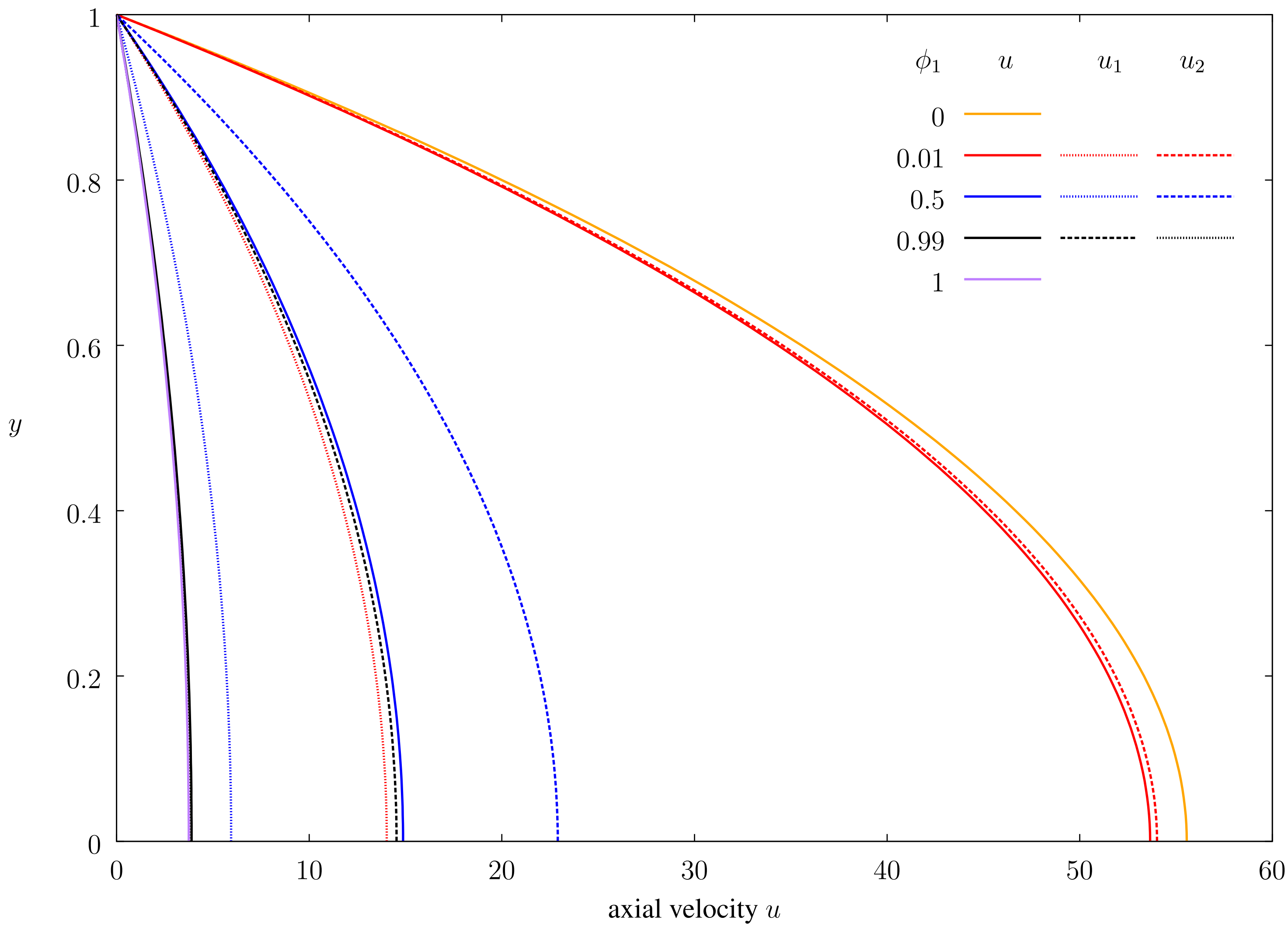

Figure 3.5: Velocity of constituents and mixture: Newtonian (combined model)

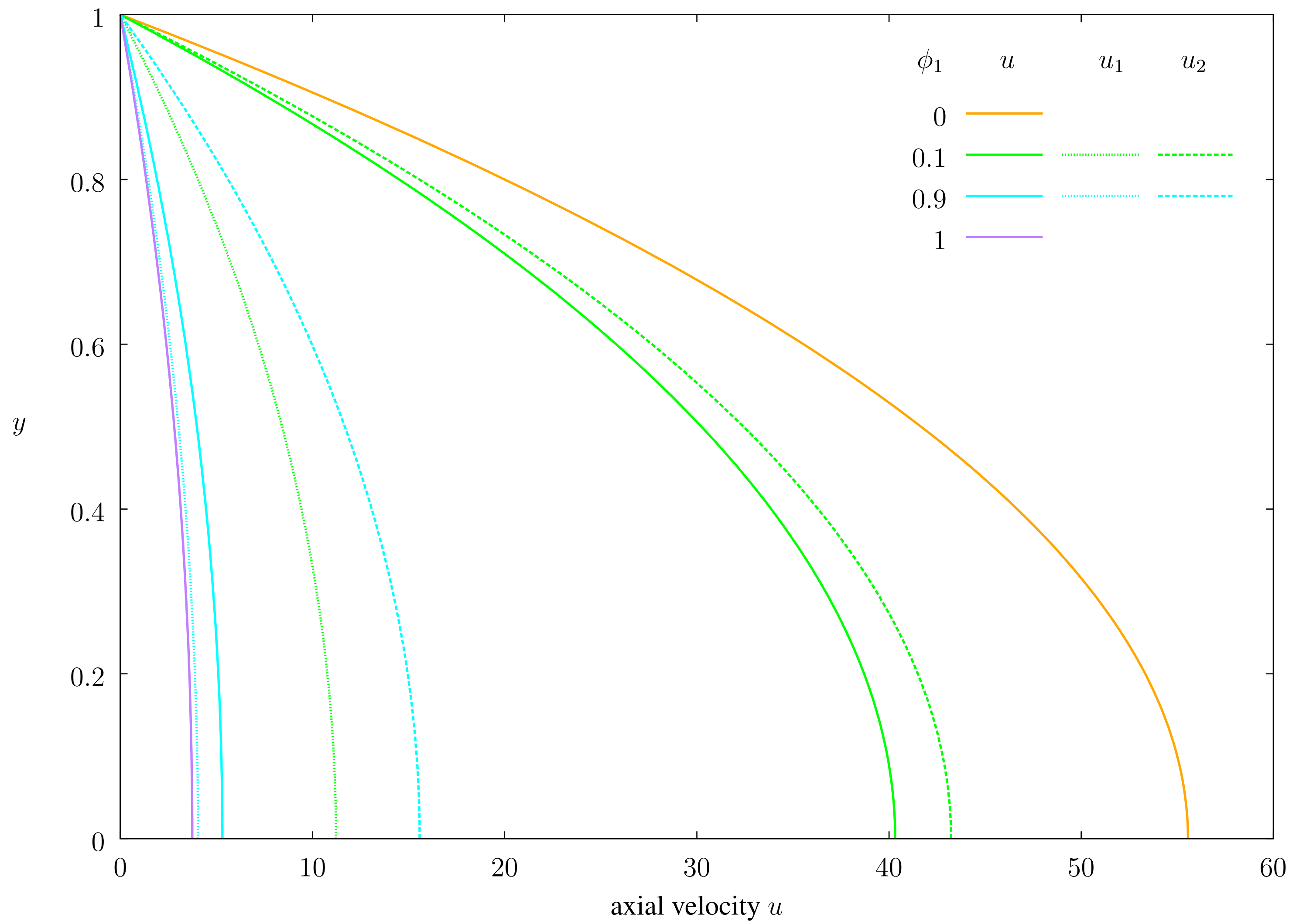

Figure 3.6: Velocity of constituents and mixture: Newtonian (combined model)

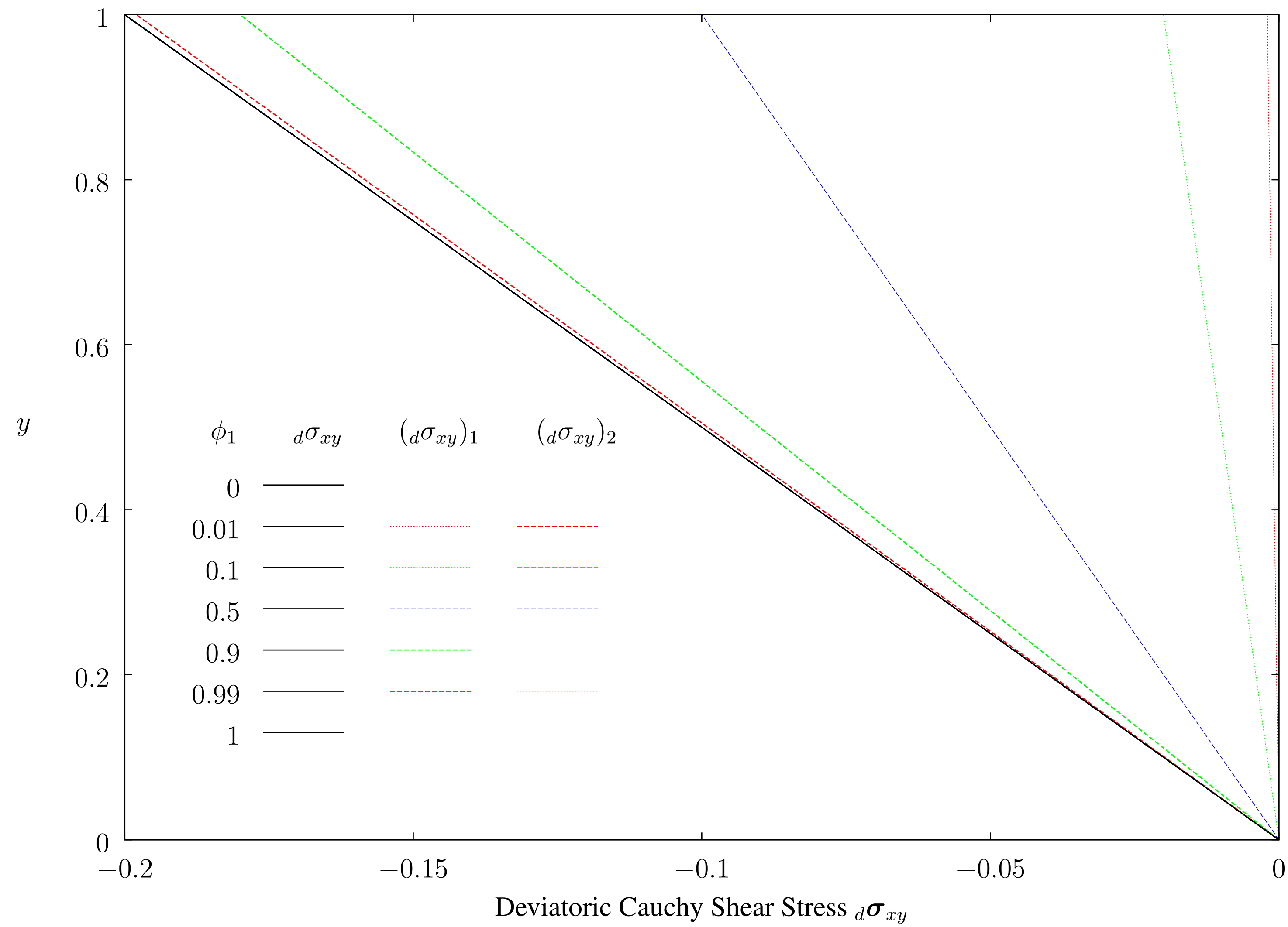

Figure 3.7: Deviatoric Cauchy shear stress for the constituents and the mixture: Newtonian (combined model)

Remarks

1. The same numerical studies were repeated using decoupled models for the constituents. The results are identical to those reported above using the combined model.

2. The assumption (3.1) regarding partial pressures p_1 and p_2 appears to work well. The validity of this assumption is further established numerically (see section 3.3).

3.2.2 Carreau model for constituents and the mixture (combined model)

As described earlier, for generalized Newtonian fluids the decoupled model can not be used due to the fact that viscosities are deformation field dependent. In this section we present numerical studies similar to those presented in section 3.2.1 for the Newtonian case. In these studies the local approximations (equal order, equal degree) for all variables are of class $C^2(\bar{\Omega}^e)$ with p-level of 9. For this choice, I is $O(10^-8)$ or lower. The uniform discretization consists of five 3-node p-version elements.

Case (a): when constituent 2 is the same as constituent 1

For this case we choose $\rho_0 = \hat{\rho}^{(1)} = 1001$ and $\eta_0 = \hat{\eta}_1^0 = 0.18$ as reference values for density and viscosity. The plot of axial velocity versus y (figure 3.8) confirms that $u_1 = u_2 = u$ holds for all volume fractions as expected. Figure 3.9 shows plots of shear stresses for constituents and the mixture for different volume fractions. For $\phi_1 = \phi_2 = 0.5$ we note that $({}_d\sigma_1)_{xy} = ({}_d\sigma_2)_{xy}$. For all volume fractions $({}_d\sigma_m)_{xy} = ({}_d\sigma_1)_{xy} + ({}_d\sigma_2)_{xy}$ holds. As expected, shear stresses are linear functions of the y coordinate. $\underset{\sim}{\mu}_m$ as a function of y (figure 3.10) is independent of the volume fraction due to the fact that the two constituents are the same. Graphs of $\underset{\sim}{\mu}_1$ and $\underset{\sim}{\mu}_2$ are shown in figures 3.10 and 3.11. For all volume fractions $\underset{\sim}{\mu}_m = \underset{\sim}{\mu}_1 + \underset{\sim}{\mu}_2$ holds as $\frac{\partial u_1}{\partial y} = \frac{\partial u_2}{\partial y} = \frac{\partial u}{\partial y}$.

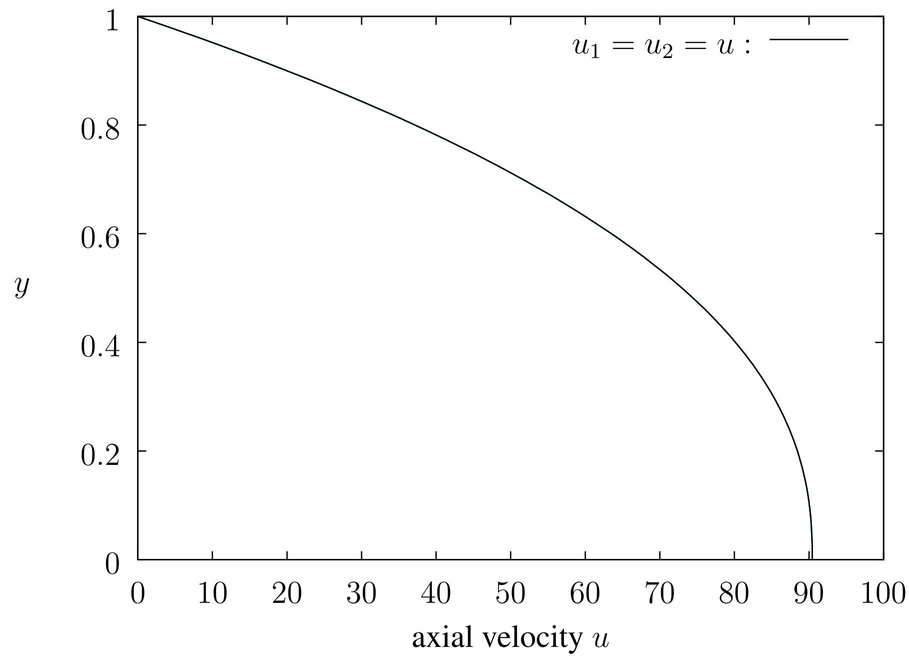

Figure 3.8: Velocity of constituents and mixture: Carreau - fluid 2 same as fluid 1 (combined model)

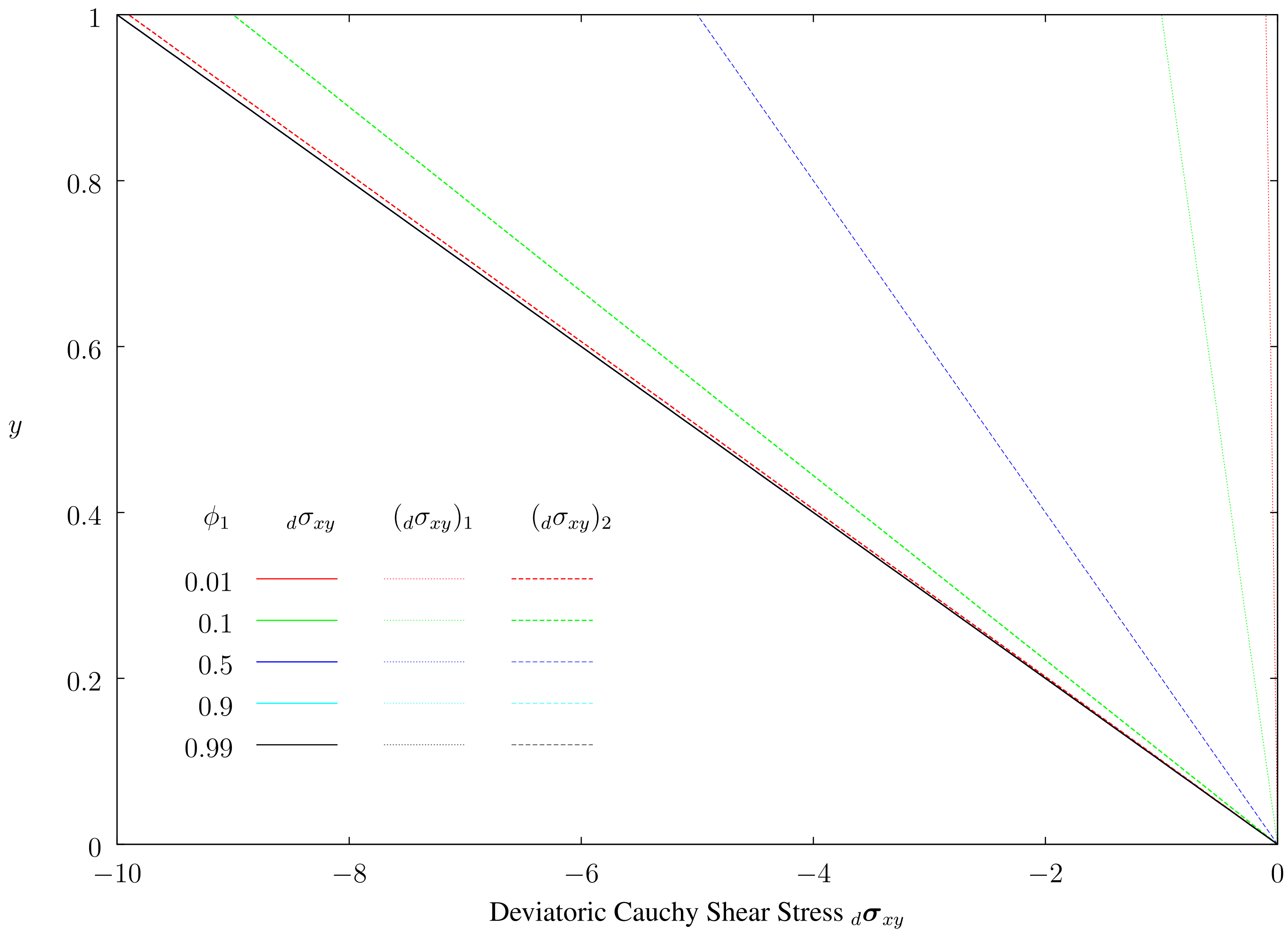

Figure 3.9: Deviatoric Cauchy shear stress for the constituents and the mixture:

Carreau - fluid 2 same as fluid 1 (combined model)

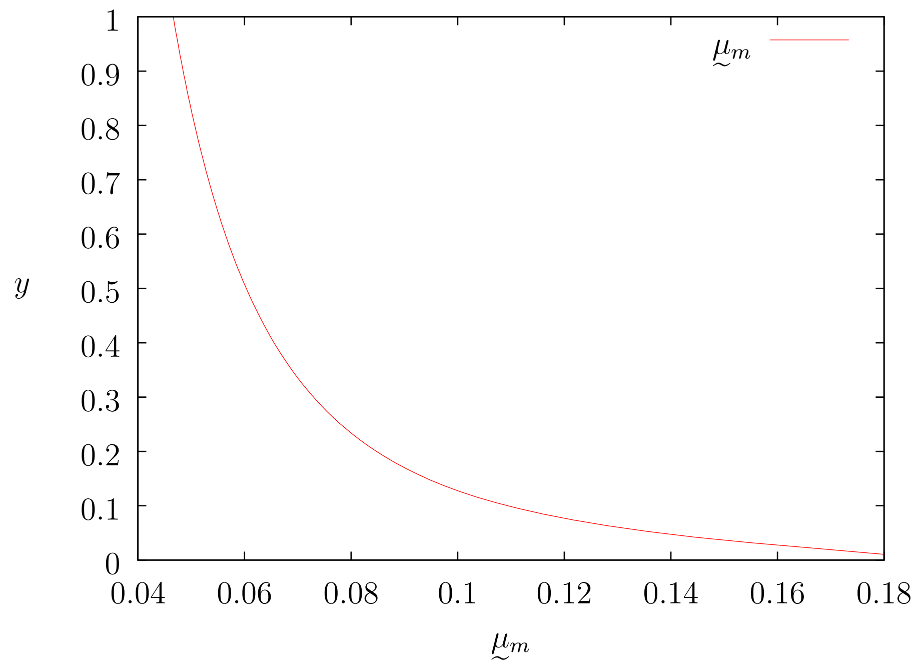

Figure 3.10: $\underset{\sim}{\mu}_m$ for the mixture versus y: Carreau

Case (b): Mixture of constituents 1 and 2 (combined model)

In this case we consider the same discretization with $k = 3$ (order of approximation space) and $p = 5$ as in case (a). We choose $\rho_0 = \hat{\rho}^{(1)} = 1001$ and $\eta_0 = \hat{\eta}_2^0 = 3.6$ as reference values of density and viscosity. Plots of velocities u_1, u_2, and u versus y for different volume fractions are shown in figures 3.13 and 3.14. Shear stresses for the constituents and the mixture as a function of y are shown in figure 3.15. These remain linear functions of y and are the same as those reported in case (a). Plots of μ_m as a function of I_2, second invariant of the strain rate tensor for different volume fractions are shown in figure 3.17. For $\phi_1 = 0.99$ and $\phi_2 = 0.01$, μ_m is close to η_1 and η_2 for constituents 1 and 2.

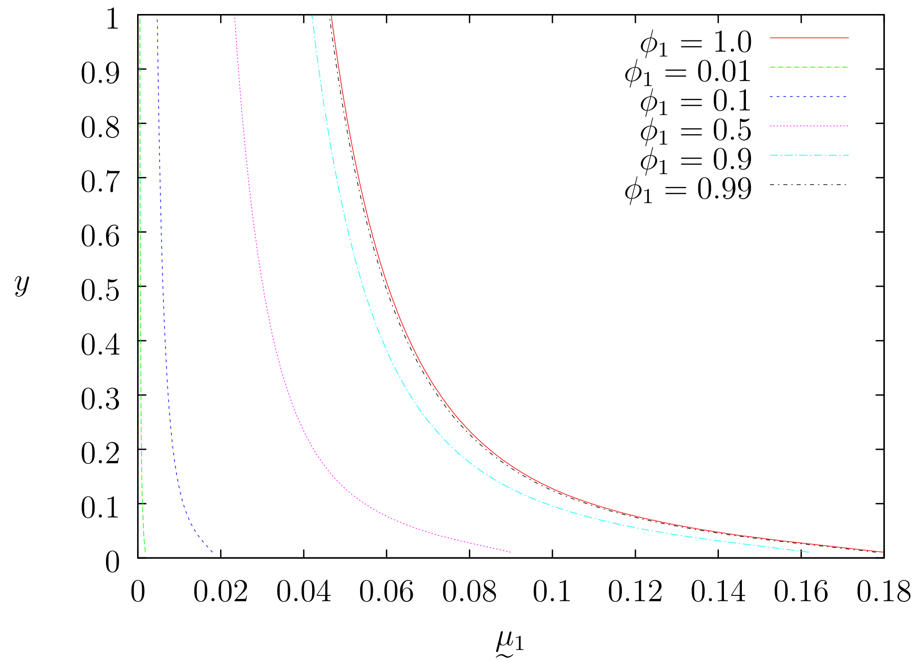

Figure 3.11: Viscosity fluid 1: Carreau

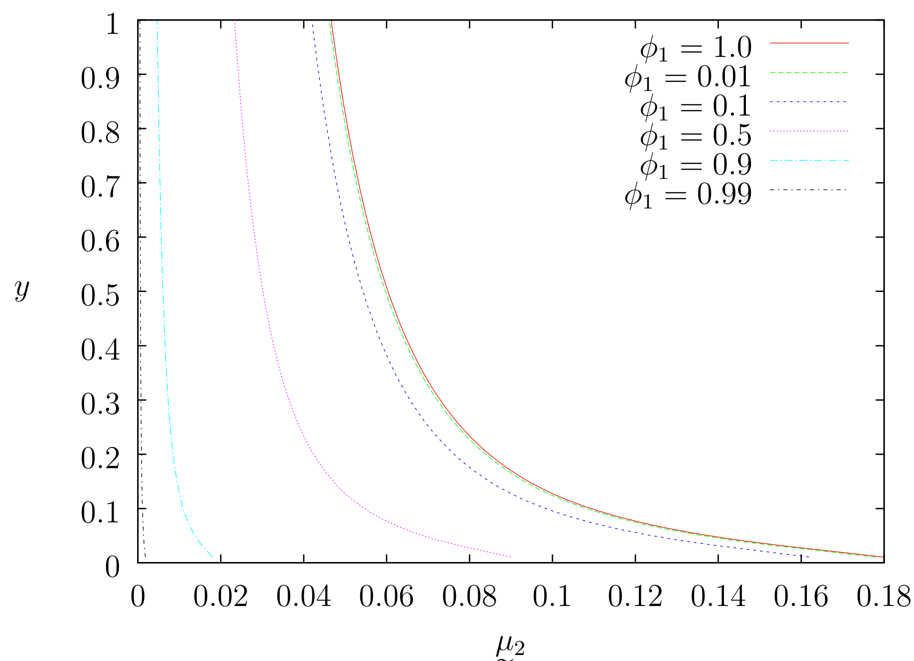

Figure 3.12: Viscosity fluid 2: Carreau

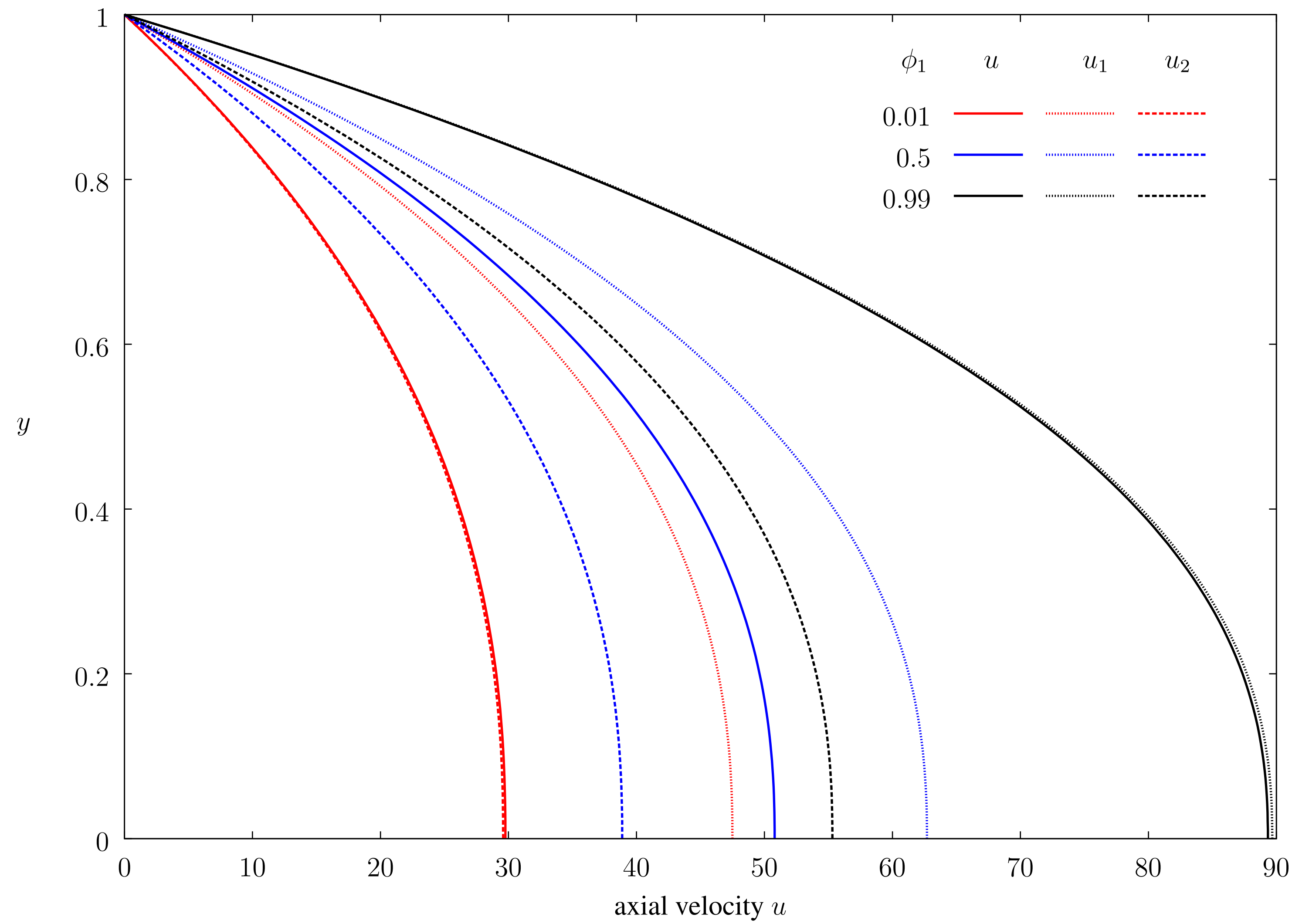

Figure 3.13: Velocity of constituents and mixture: Carreau fluid (combined model)

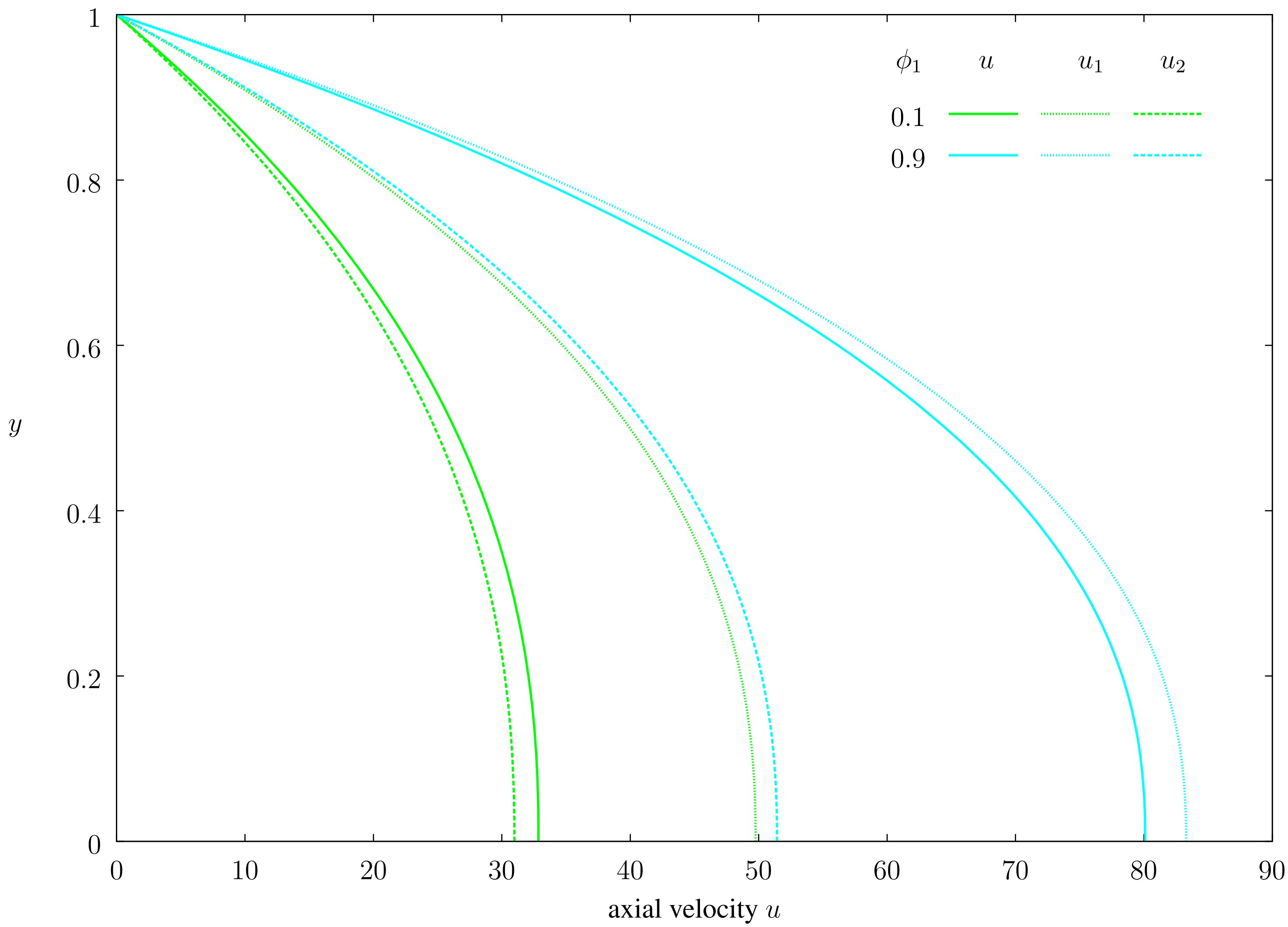

Figure 3.14: Velocity of constituents and mixture: Carreau fluid (combined model)

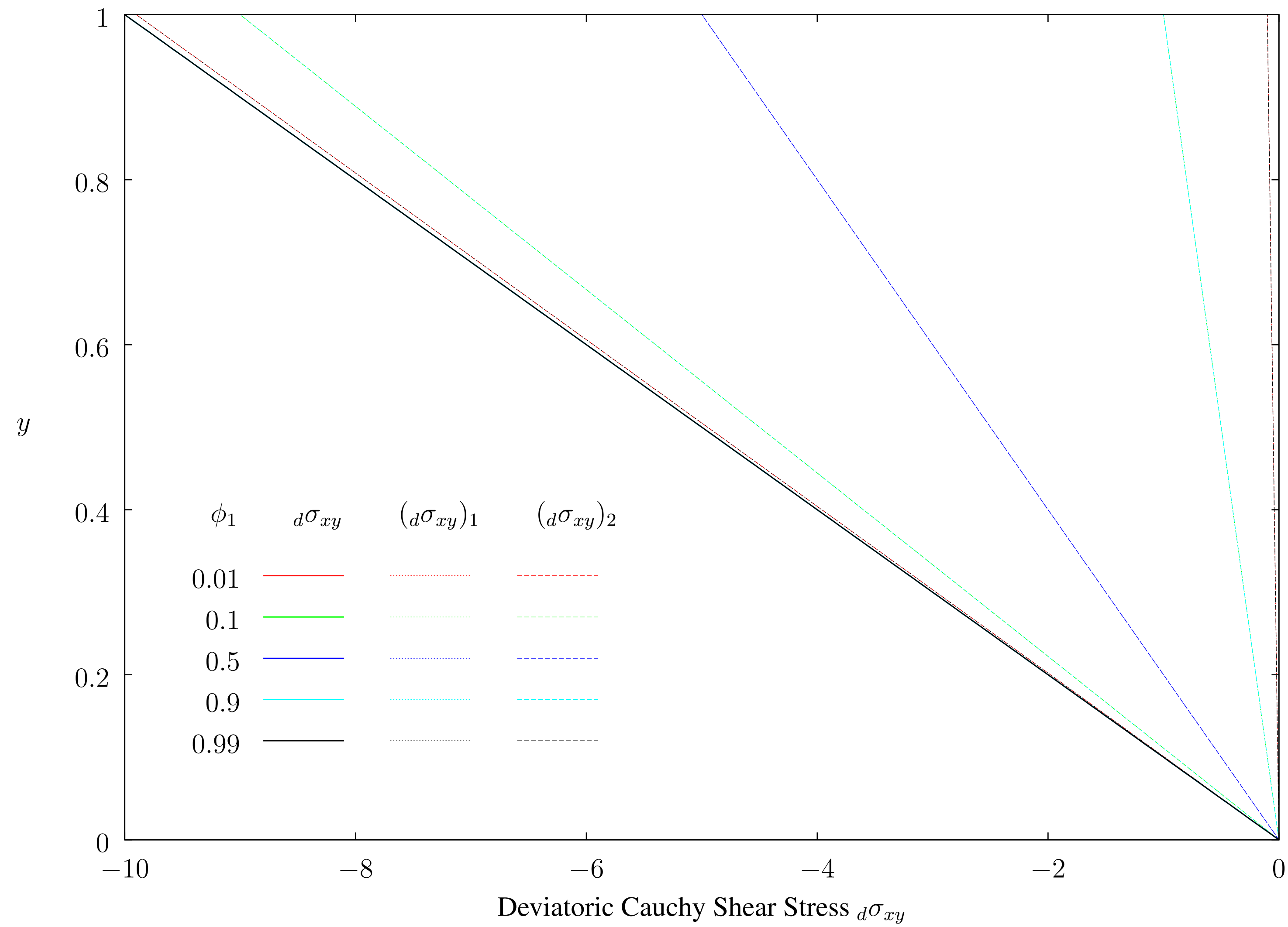

Figure 3.15: Deviatoric Cauchy shear stress for the constituents and the mixture:

Carreau fluid (combined model)

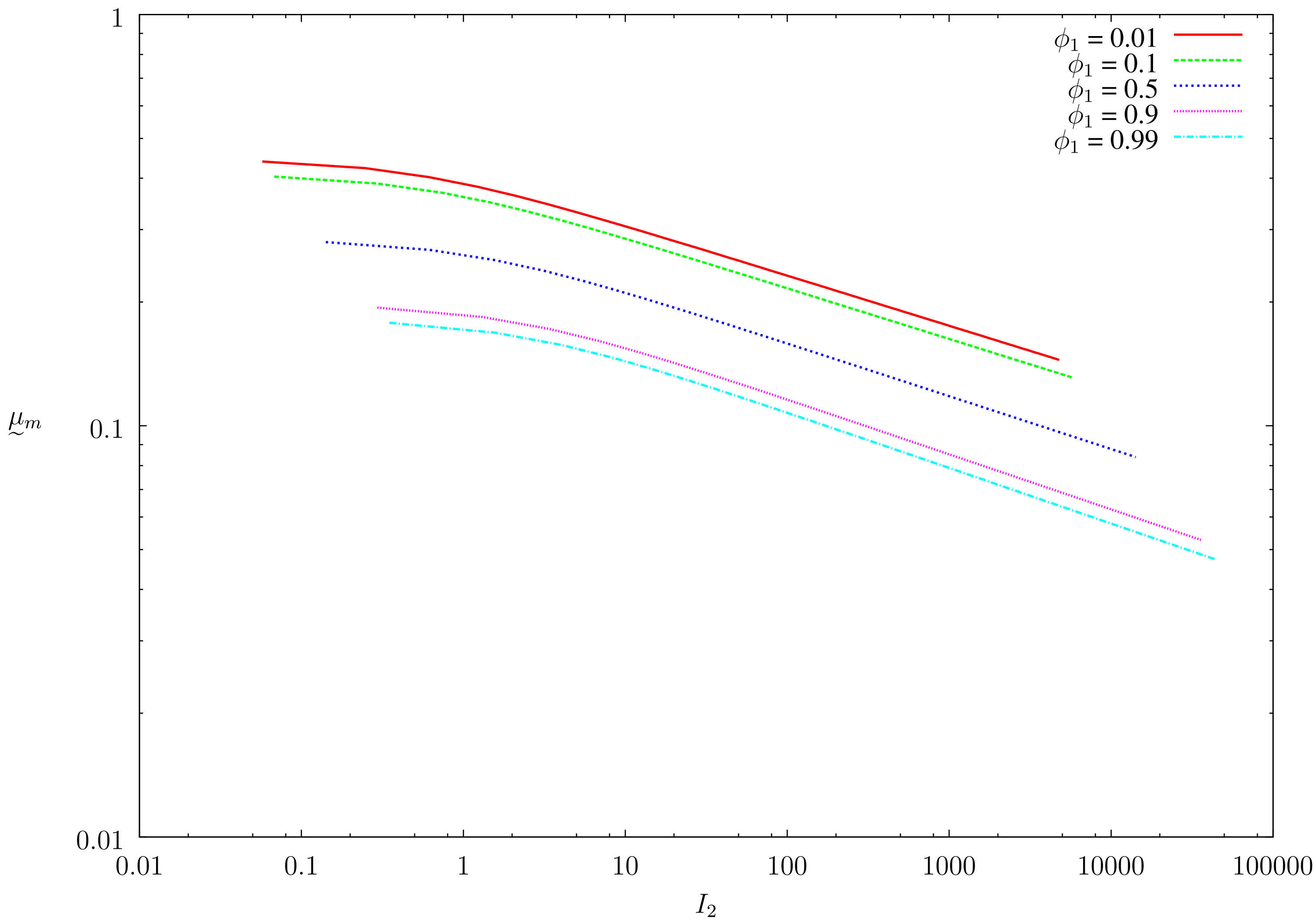

Figure 3.16: Viscosity of fluid mixtures: Carreau

3.2.3 Power law model for constituents and the mixture (combined model)

These studies are parallel to those for the Carreau model using the same discretization, k, and p.

Case (a): when constituent 2 is the same as constituent 1

We use $\rho_0 = \hat{\rho}^{(1)} = 1001$ and $\eta_0 = \hat{\eta}_1^0 = 0.332$ as reference values. Plots of $u_1 = u_2 = u$ versus y, shear stresses versus y, and μ_m as a function of y for different volume fractions are shown in figures 3.17 – 3.21. The results follow the same pattern and behaviors as explained for the Carreau model.

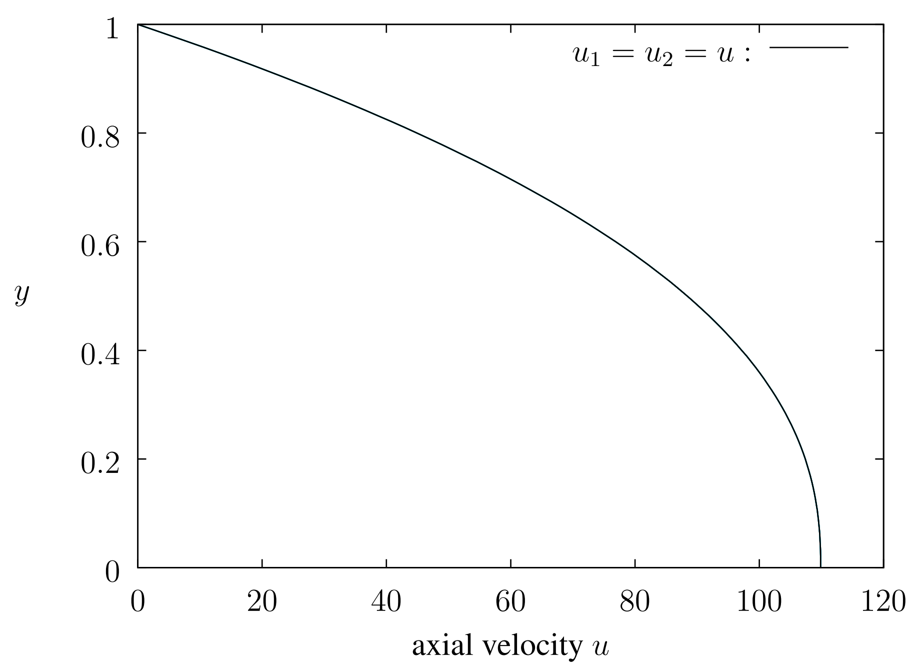

Figure 3.17: Velocity of constituents and mixture: Power Law - fluid 2 same as fluid 1

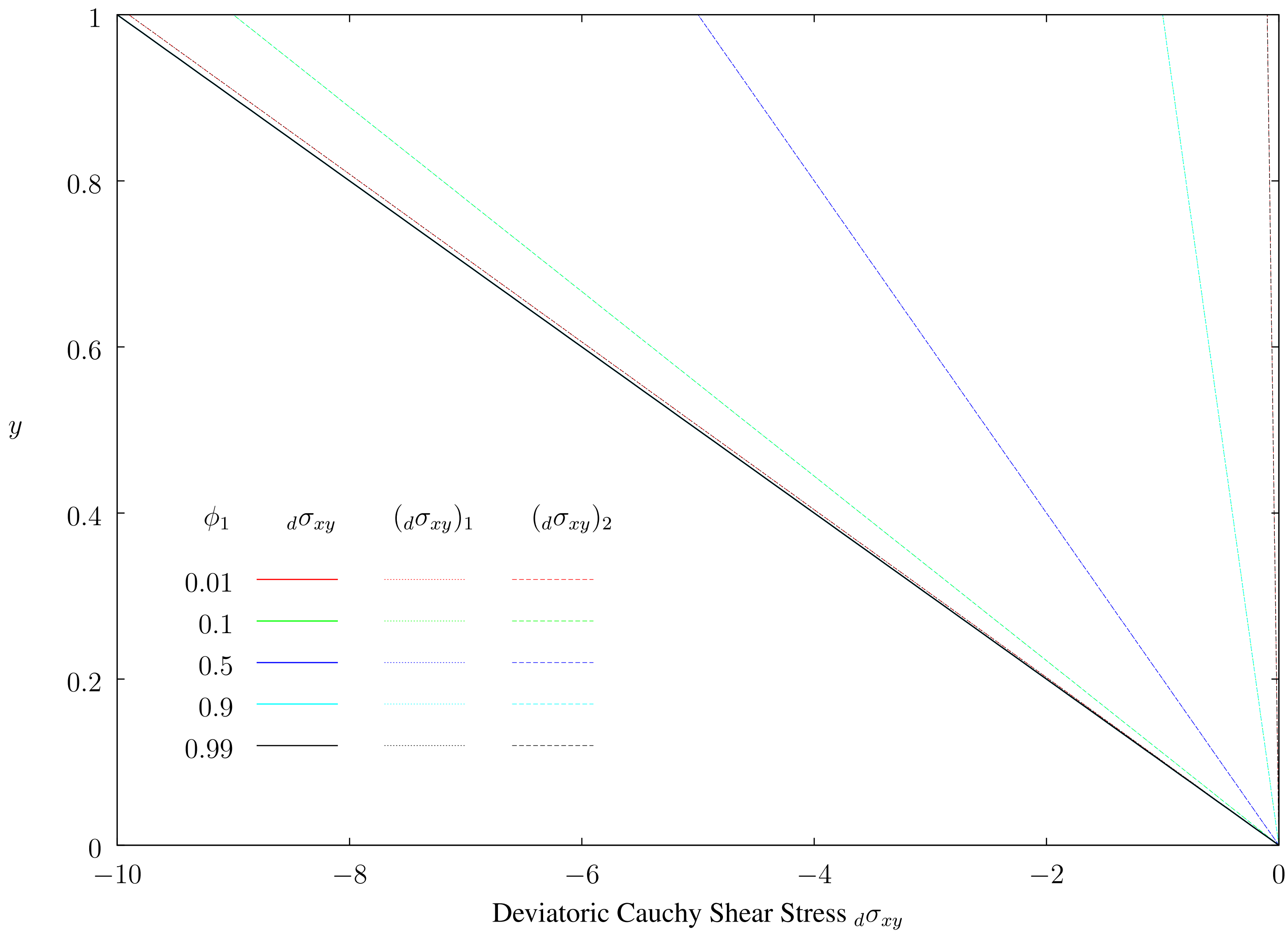

Figure 3.18: Deviatoric Cauchy shear stress of constituents and mixture: Power Law - fluid 2 same as fluid 1

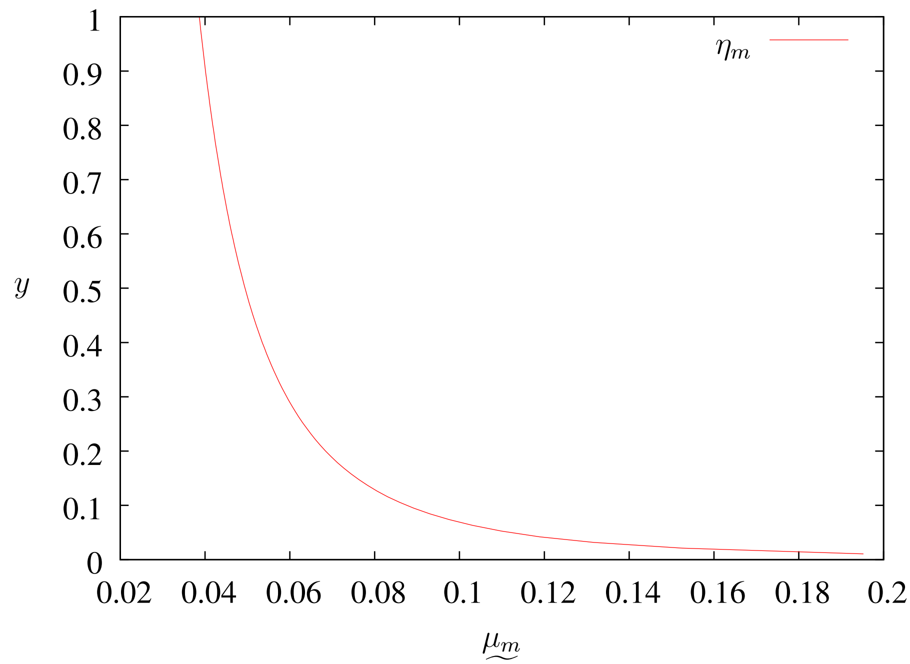

Figure 3.19: $\underset{\sim}{\mu}_m$ for the mixture versus y: Power Law

Case (b): mixture of constituents 1 and 2 (combined model)

For these numerical studies we choose $\rho_0 = \hat{\rho}^{(1)} = 1001$ and $\eta_0 = \hat{\eta}_2^0 = 2.04$ as reference values. Graphs of $u_1 = u_2 = u$ versus y, and μ_m as a function of y for different volume fractions are shown in figures 3.22 – 3.25. Behaviors are similar to the Carreau model.

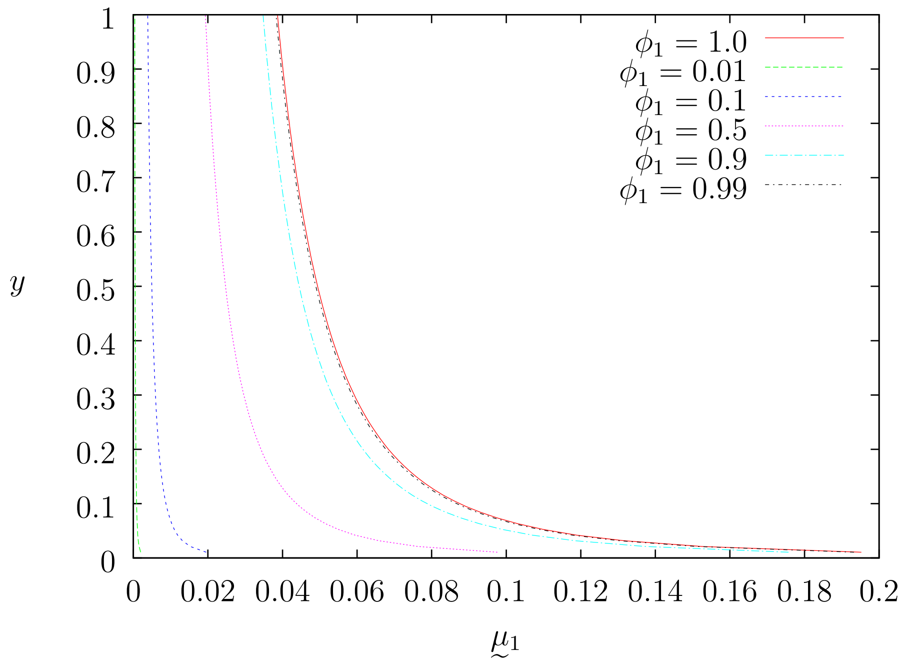

Figure 3.20: Viscosity of fluid 1: Power Law

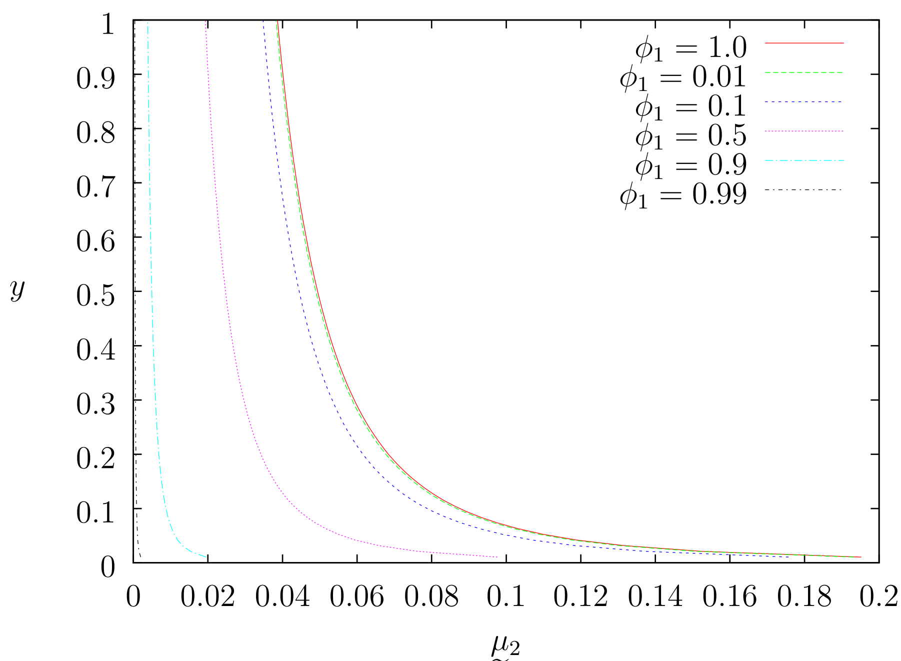

Figure 3.21: Viscosity of fluid 2: Power Law

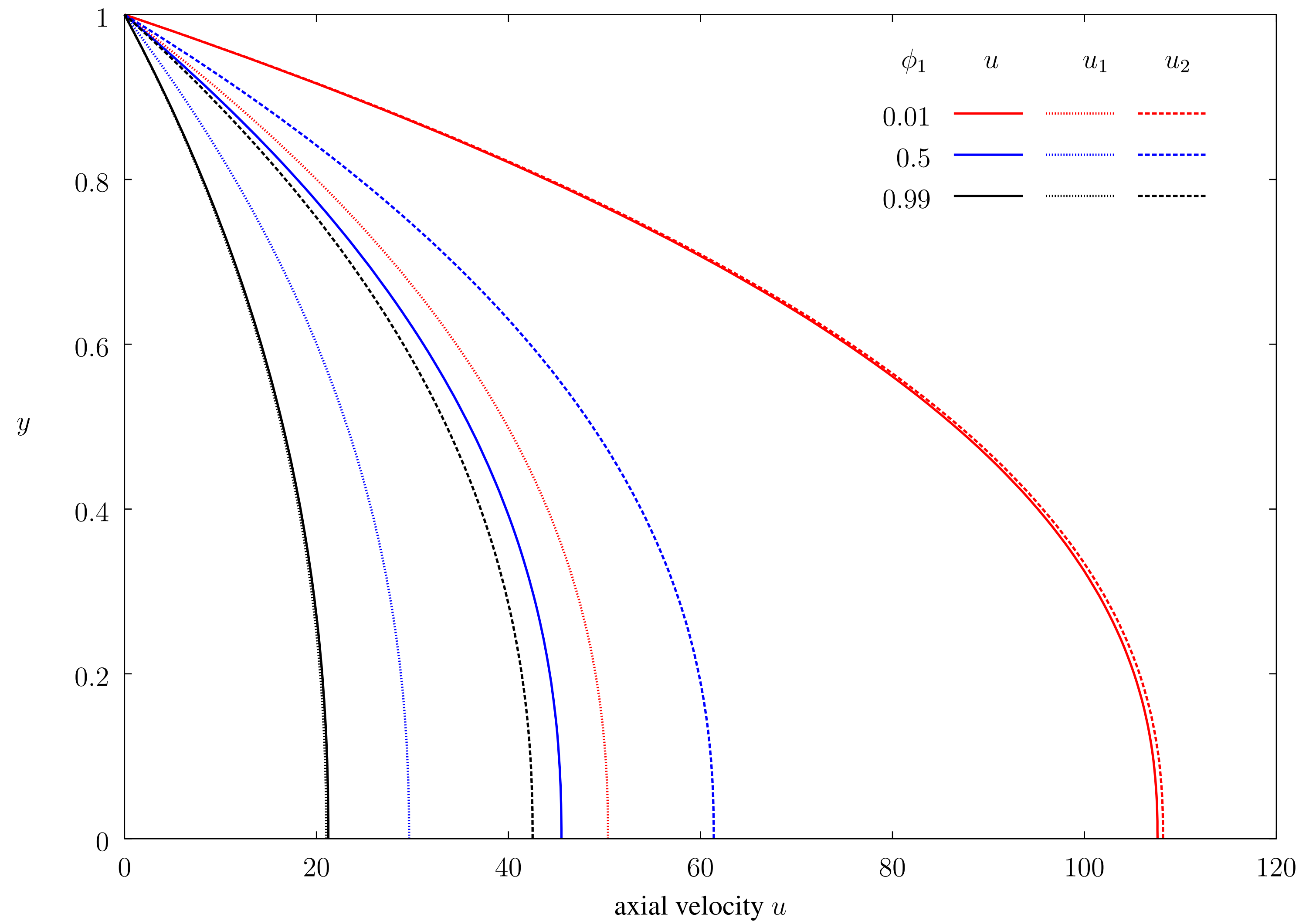

Figure 3.22: Velocity of constituents and mixture: Power Law fluid (combined model)

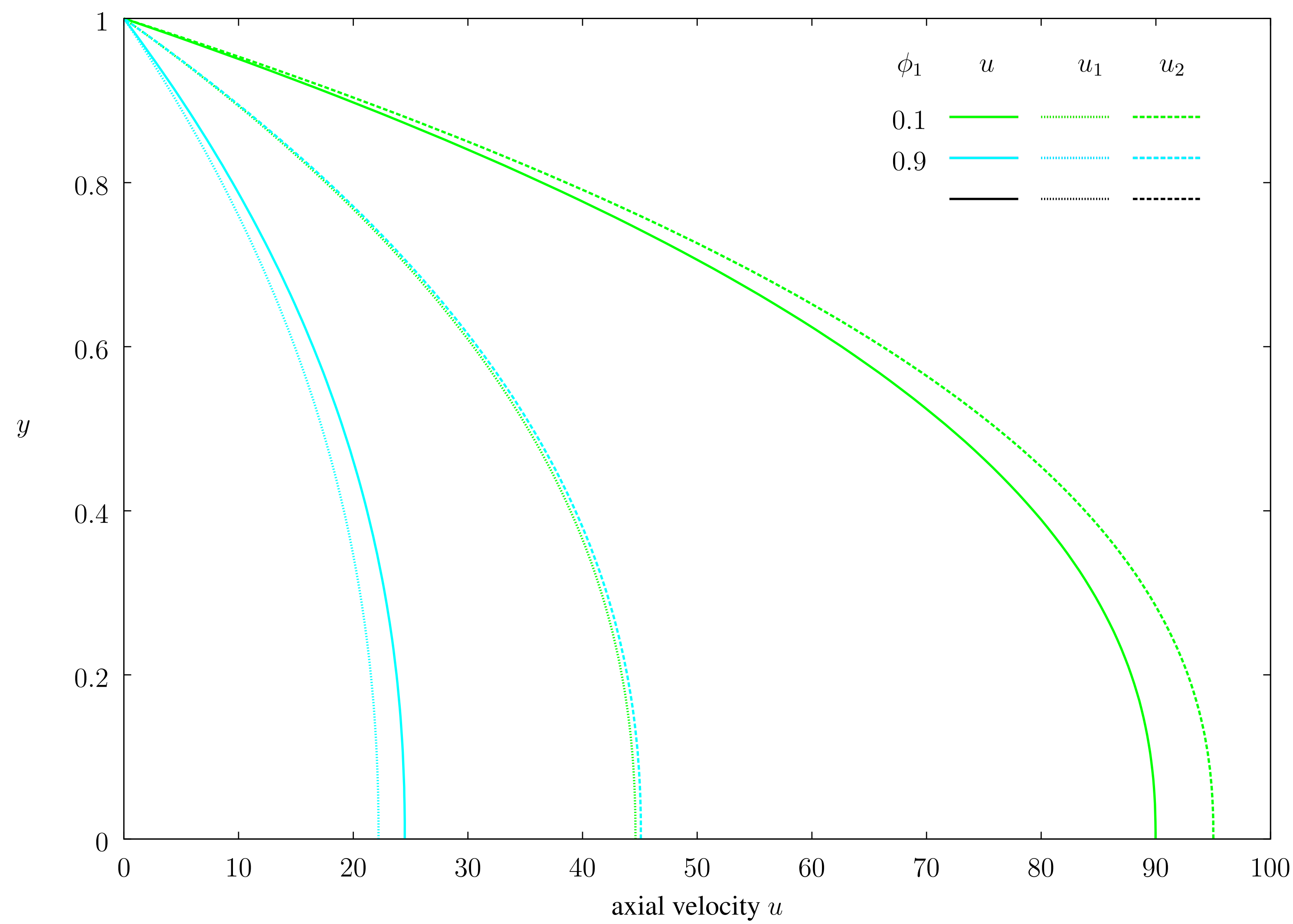

Figure 3.23: Velocity of constituents and mixture: Power Law fluid (combined model)

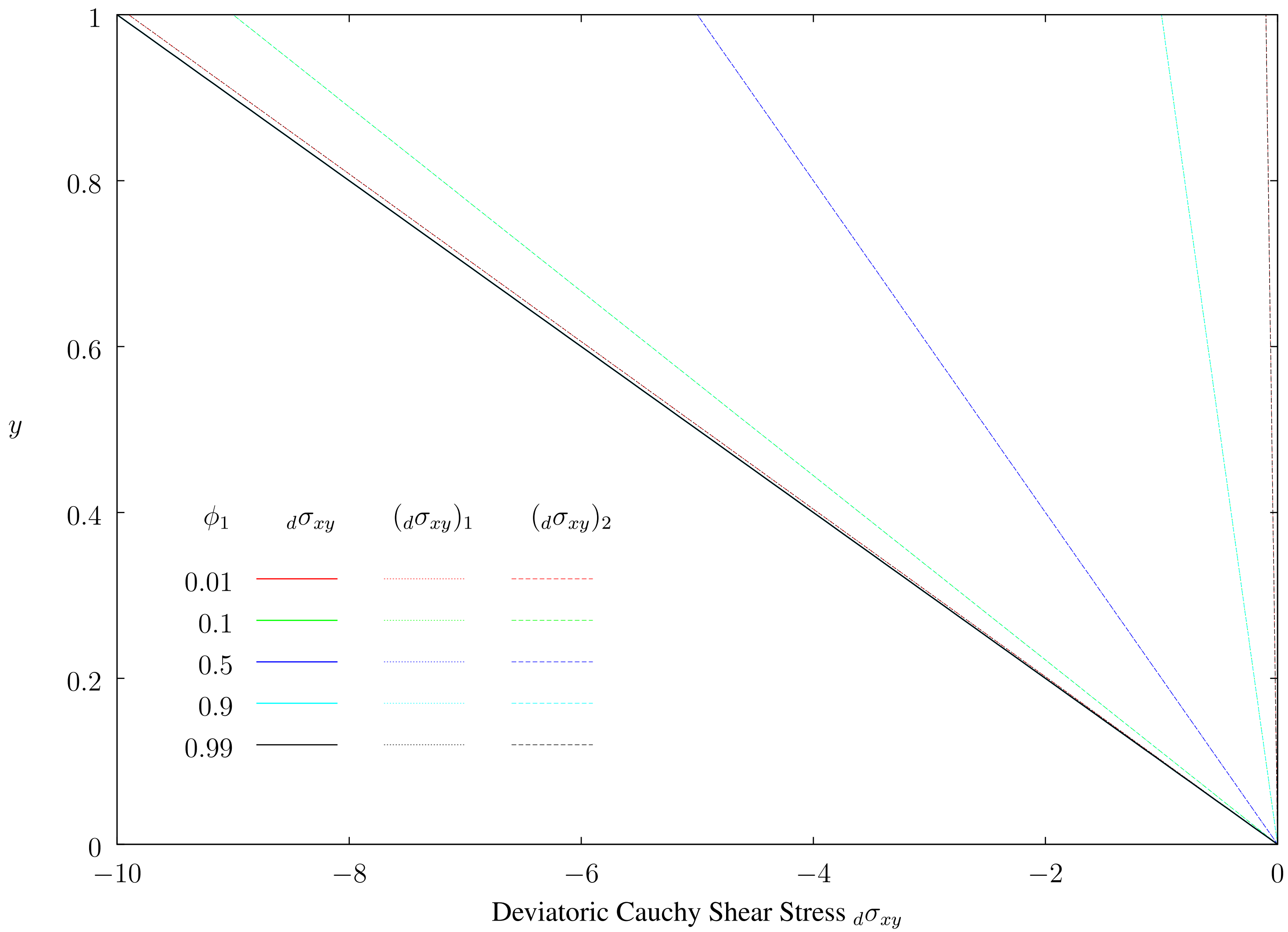

Figure 3.24: Deviatoric Cauchy shear stress of constituents and mixture: Power Law (combined model)

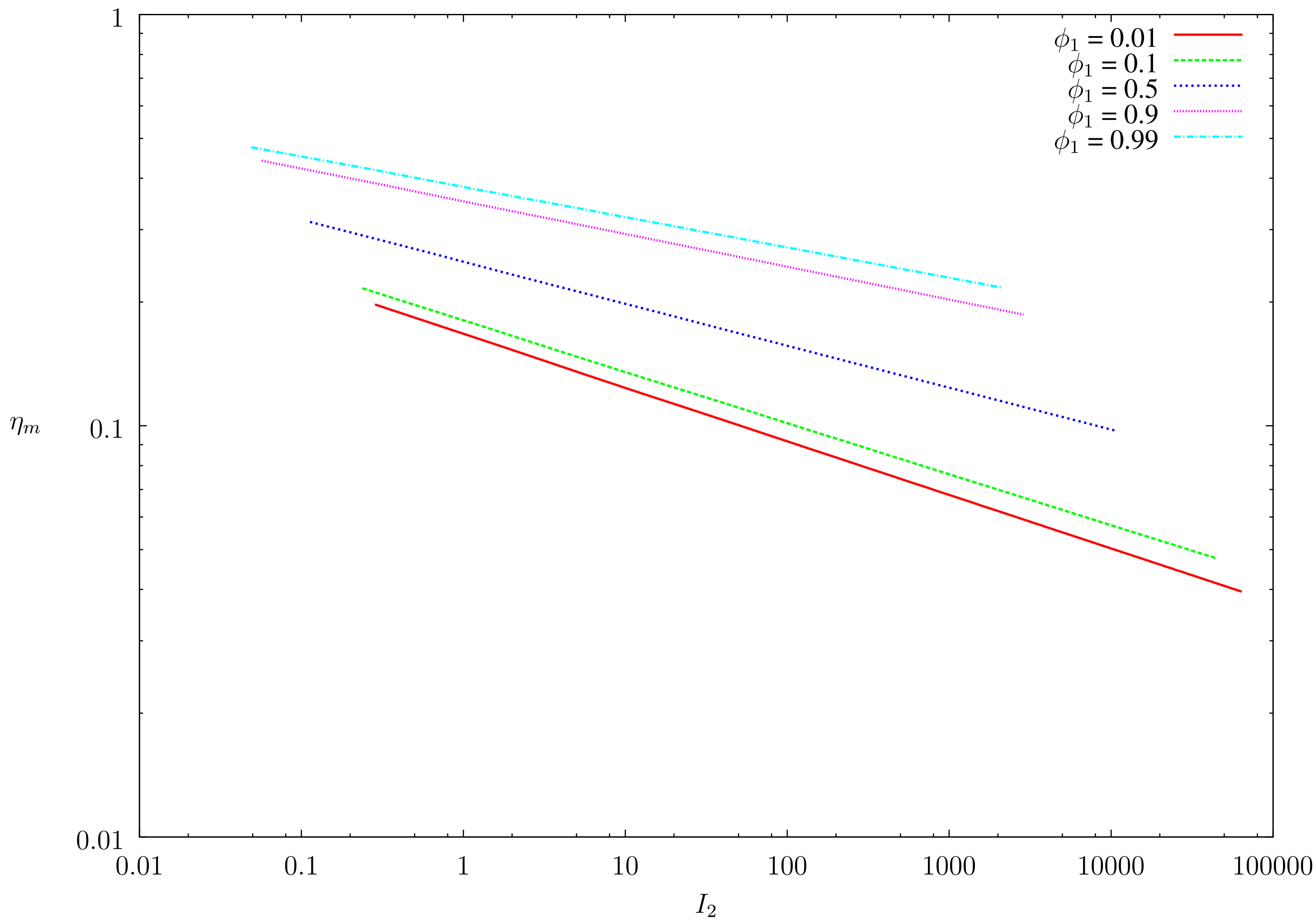

Figure 3.25: Viscosity of mixture of Power Law fluids

3.3 1:2 backward facing asymmetric expansion

We consider a 1:2 backward facing asymmetric expansion. A schematic and the boundary conditions are shown in figure 3.26. This problem has been experimentally investigated by Patrick and Denham [33]. More recently Winterscheidt and Surana [31] presented numerical simulations using p-version least squares finite element method. Figure 3.27 shows a graded twenty element discretization using nine node p-version elements. In the numerical studies we only consider the constituents and the mixture to be Newtonian and use the same properties as listed for the Newtonian constituents for fully developed flow between parallel plates (section 3.2). At the inlet, the flow is assumed to be fully developed with a parabolic velocity field with maximum value of one (figure 3.26). C^{00} local approximations at p-level 9 are used for all variables. For this choice, I values are $O(10^{-8})$ or lower confirming good accuracy of the solution. Characteristic kinetic energy is used for reference pressure and reference stress.

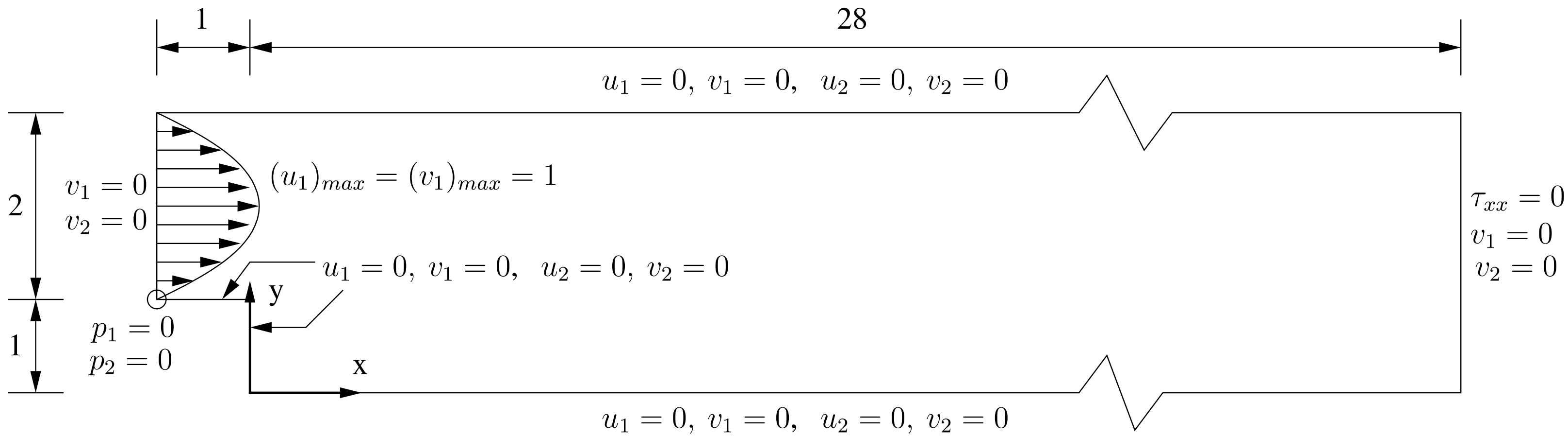

Figure 3.26: Schematic of boundary conditions

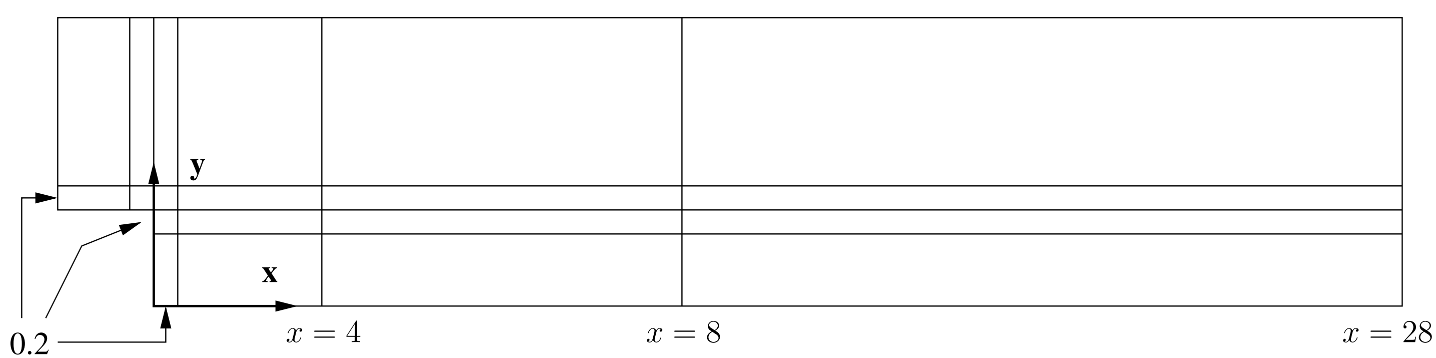

Figure 3.27: Graded discretization: 20 nine-node p-version elements

Case (a): constituent 2 same as constituent 1 (coupled model)

We choose $\rho_0 = \hat{\rho}^{(2)} = 1000$ and $\eta_0 = \hat{\eta}_1^0 = 0.0267$ as reference values. We consider two combinations of volume fractions, $\phi_1 = \phi_2 = 0.5$ and $\phi_1 = 1.0$, $\phi_2 = 0.0$. When $\phi_1 = \phi_2 = 0.5$ we expect the two constituent behaviors to be the same. The mixture response in this case is obviously the same as when $\phi_1 = 1.0$, $\phi_2 = 0.0$. As obvious in this case the mixture behavior is independent of the volume fractions. In this study p_1 and p_2, the constituent partial pressures, are dependent variables. Figures 3.28 and 3.29 show plots of pressures p_1, p_2, and $p\ (= p_1 + p_2)$ for $\phi_1 = 0.5$, $\phi_2 = 0.5$ and $\phi_1 = 1.0$, $\phi_2 = 0.0$ at the top and bottom boundaries (or plates). Results for pressure for volume fraction $\phi_1 = 0.2$ and $\phi_2 = 0.8$ and comparisons with $\phi_1 = 1.0$, $\phi_2 = 0.0$ are shown in figures 3.30 and 3.31. Plots of representative u_1, u_2, and u versus y at $x = 0.0$ and $x = 2.0$ are shown in figures 3.32 and 3.33. These are obviously independent of the volume fractions for the case when both constituents are the same.

Numerical studies were also conducted using decoupled models for the constituents using $p_1 = \phi_1 p$ and $p_2 = \phi_2 p$. The results obtained from these studies are identical to those presented here using combined models in which volume fractions are not used to describe partial pressures of the constituents. These studies confirm that (2.118) and (2.119) used in chapter 2 and in the studies for fully developed flow between parallel plates is justified.

Case (b): mixture of constituents 1 and 2

In this case we choose volume fractions $\phi_1 = 0.8$ and $\phi_2 = 0.2$. Figures 3.34 and 3.35 show plots of u_1, u_2, and u versus y at $x = 0.0$ and $x = 2.0$. Differences in u_1, u_2, and u are quite clear in figure 3.35. Figures 3.36 and 3.37 show plots of pressures p_1, p_2, and p at

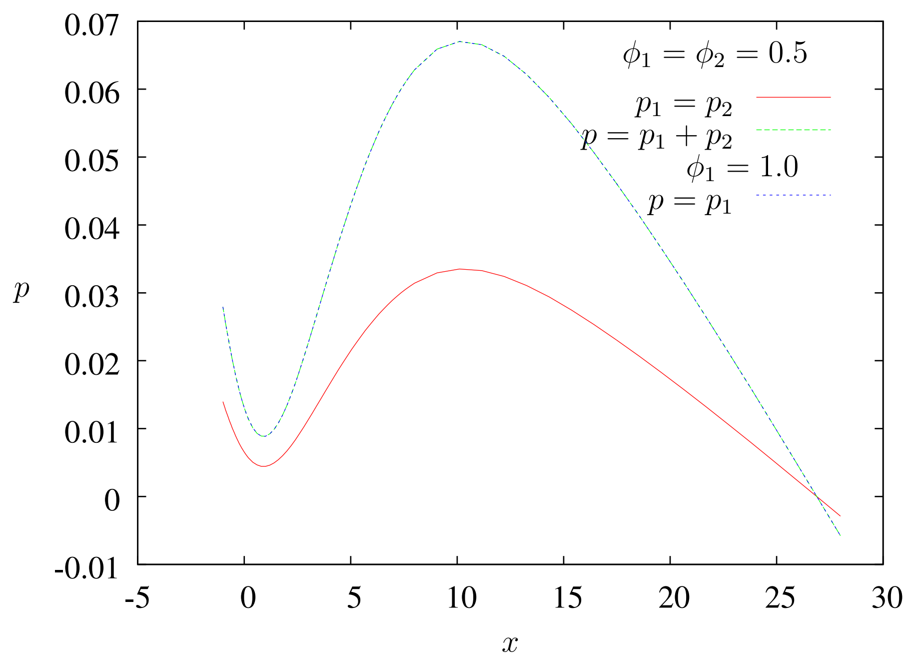

Figure 3.28: Pressure at top boundary ($y = 3, -1 \leq x \leq 28$): fluid 2 same as fluid 1, $\phi_1 = \phi_2 = 0.5$

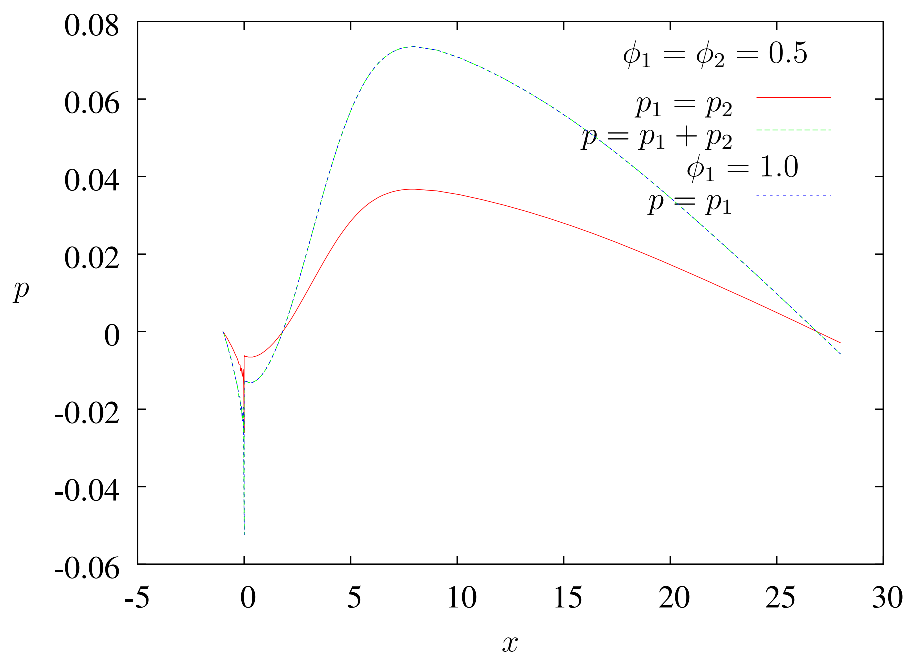

Figure 3.29: Pressure at bottom boundary ($y = 1, -1 \leq x \leq 0$; $y = 0, 0 \leq x \leq 28$): fluid 2 same as fluid 1, $\phi_1 = \phi_2 = 0.5$

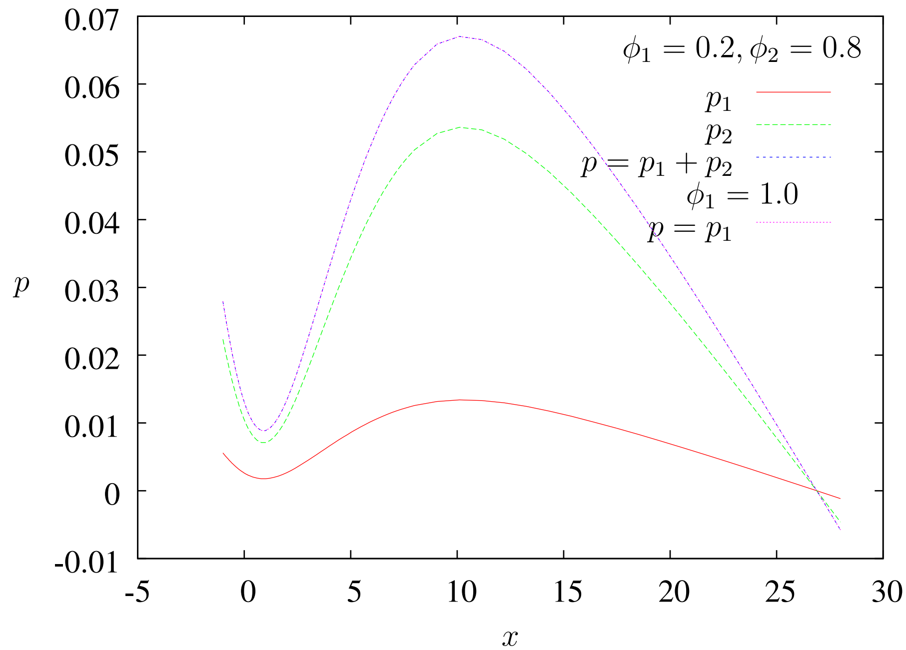

Figure 3.30: Pressure at top boundary ($y = 3, -1 \leq x \leq 28$): fluid 2 same as fluid 1, $\phi_1 = 0.2, \phi_2 = 0.8$

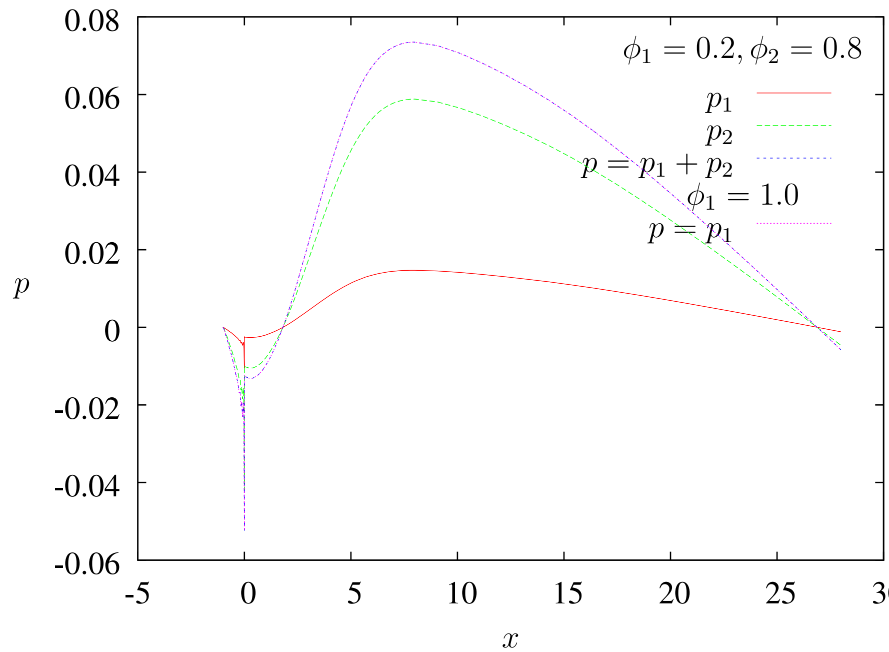

Figure 3.31: Pressure at bottom boundary ($y = 1, -1 \leq x \leq 0$; $y = 0, 0 \leq x \leq 28$): fluid 2 same as fluid 1, $\phi_1 = 0.2, \phi_2 = 0.8$

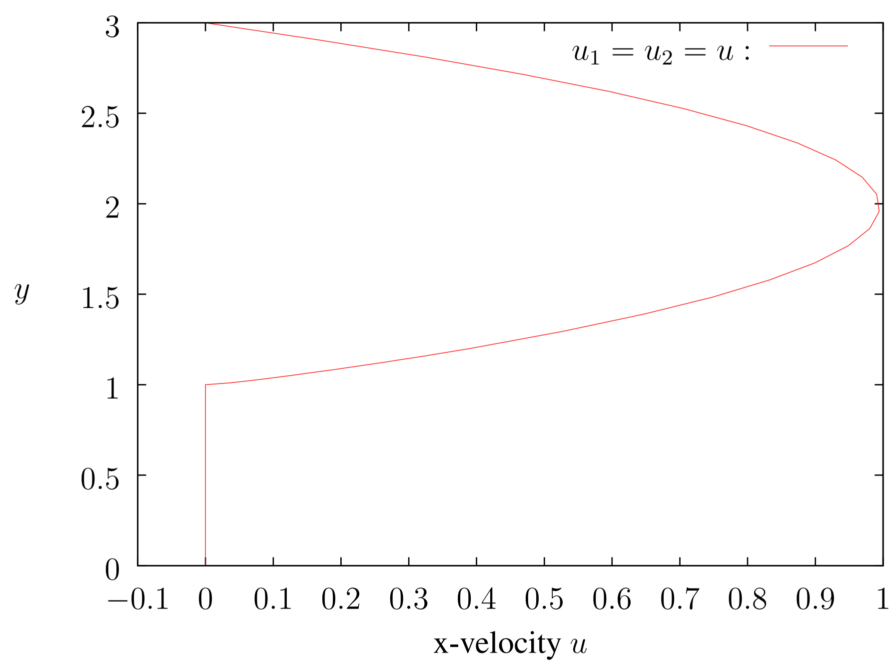

Figure 3.32: Velocity at $x = 0$: fluid 2 same as fluid 1

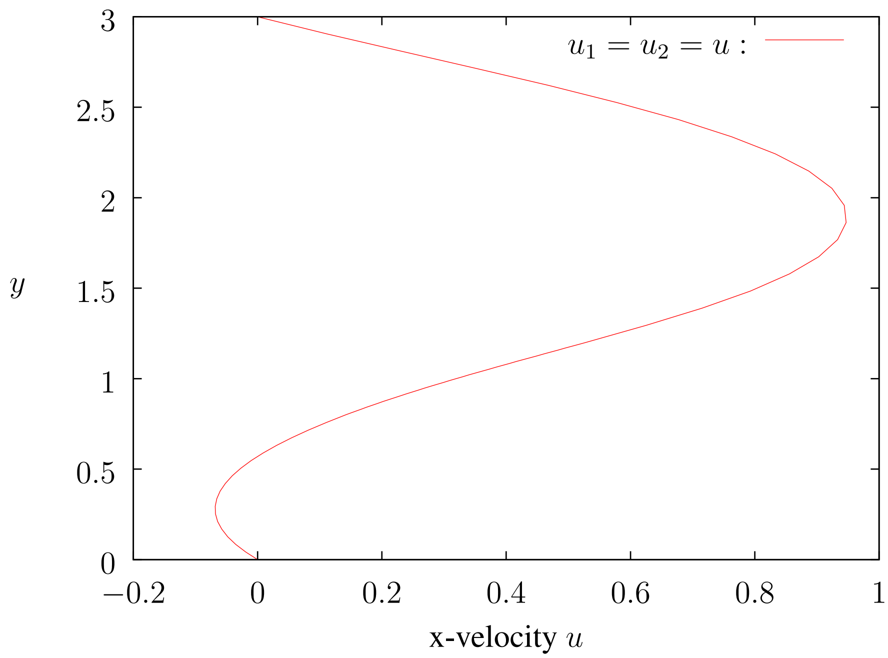

Figure 3.33: Velocity at $x = 2.0$: fluid 2 same as fluid 1

$y = 3$ and at $y = 0$.

Numerical studies were also conducted using decoupled models for the constituents using $p_1 = \phi_1 p$ and $p_2 = \phi_2 p$. The results obtained from these studies are identical to those presented in figures 3.34 – 3.37 using the combined model in which volume fractions are not used to define partial pressures of the constituents. These studies once again confirm the validity of (2.118) and (2.119) .

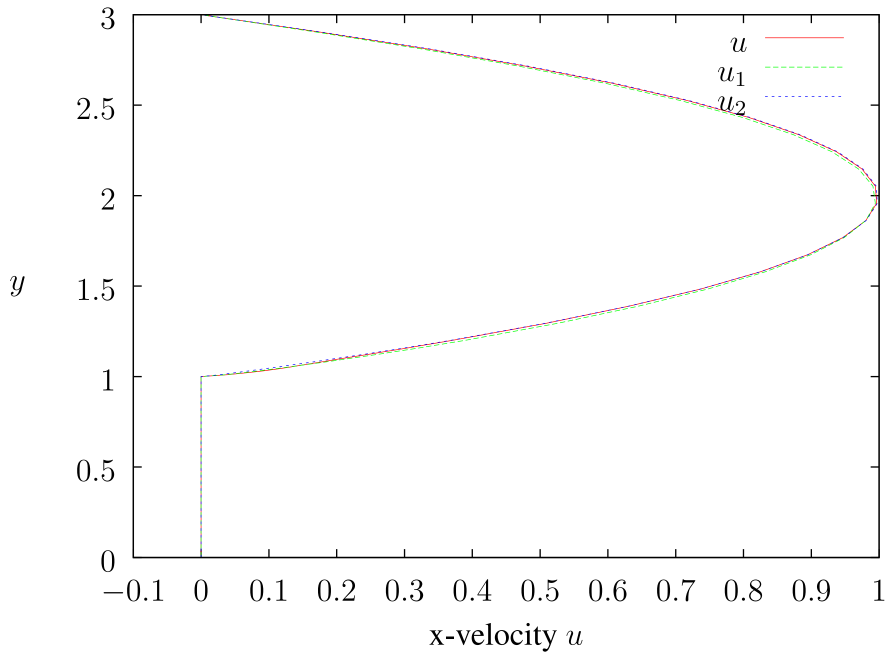

Figure 3.34: Velocity at $x = 0.0$: mixture of fluid 1 and fluid 2, $\phi_1 = 0.8, \phi_2 = 0.2$

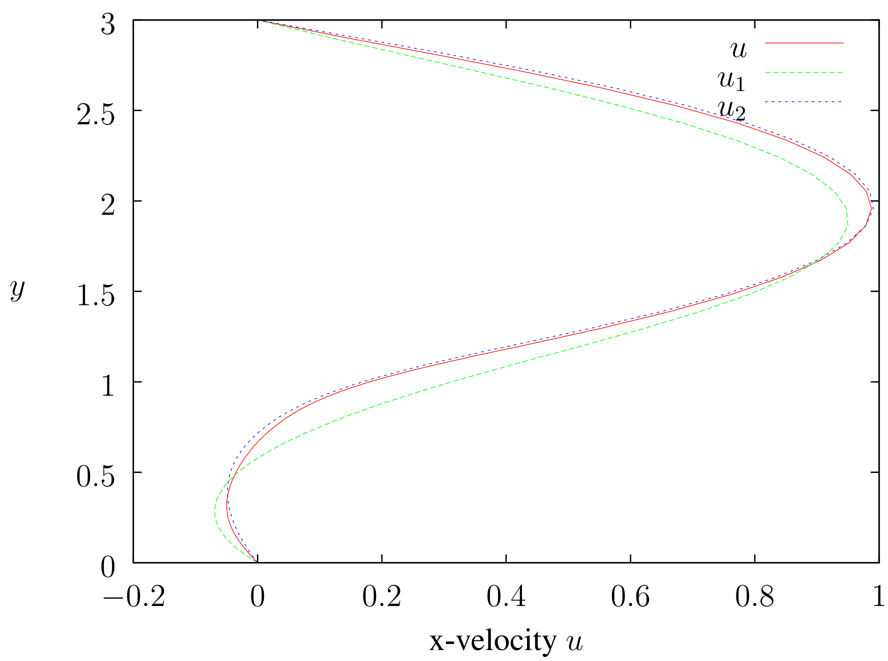

Figure 3.35: Velocity at $x = 2.0$: mixture of fluid 1 and fluid 2, $\phi_1 = 0.8, \phi_2 = 0.2$

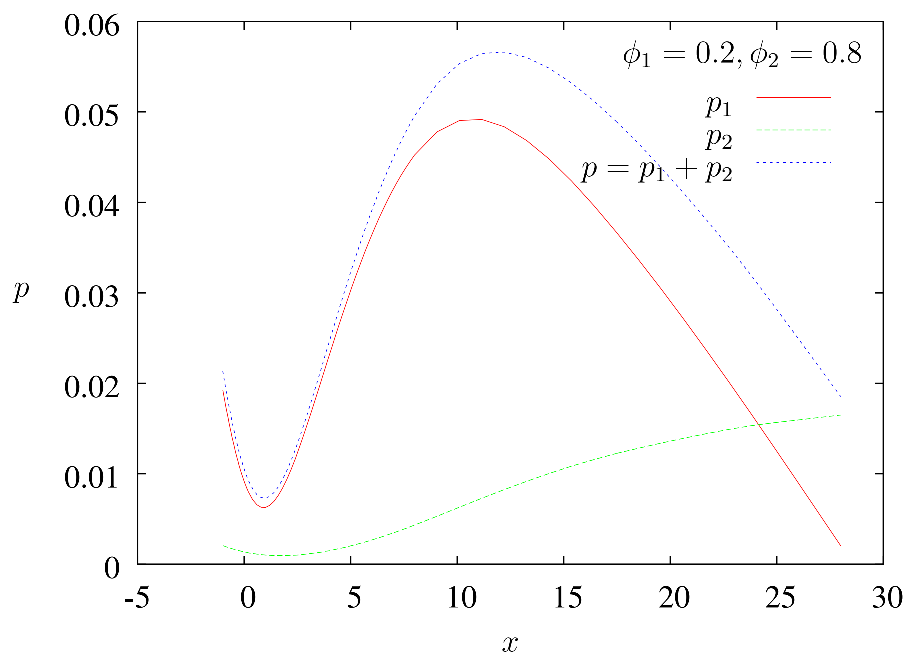

Figure 3.36: Pressure at top boundary ($y = 3, -1 \leq x \leq 28$): mixture of fluid 1 and fluid 2, $\phi_1 = 0.8, \phi_2 = 0.2$

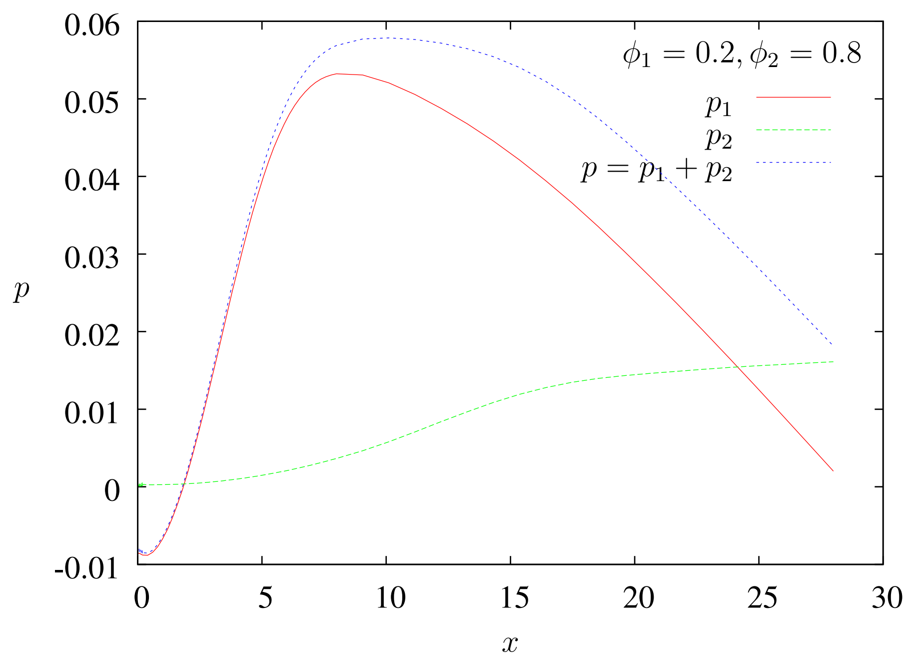

Figure 3.37: Pressure at bottom boundary ($y = 0, 0 \leq x \leq 28$): mixture of fluid 1 and fluid 2, $\phi_1 = 0.8, \phi_2 = 0.2$

Chapter 4

Summary and Conclusions

In this work, derivation of the mathematical model(s) for a homogeneous, isotropic, incompressible mixture of ν homogeneous, isotropic, and incompressible constituents using basic principles of mixture theory and continuum mechanics is presented. The deformation process is assumed to be isothermal, hence temperature effects due to viscous dissipation are assumed to be negligible. The basic definition of densities of the constituents, density of the mixture, mixture velocities, and the material derivative for the constituents and the mixture are presented and are utilized in the conservation laws: conservation of mass, balance of momenta for the constituents, and the energy equation for the mixture based on the first law of thermodynamics. The second law of thermodynamics (entropy inequality) and the theory of generators and invariants is used as a basis for the constitutive theories for the mechanical pressure and deviatoric Cauchy stress tensors for the constituents and the mixture. The constitutive theories borrow basic derivations from references [7, 19]; these are modified to account for the correct physics of the mixture for the constituents used in the present work. Specific forms of the complete mathematical models are presented in $\mathbb{R}^1$ and $\mathbb{R}^2$ using x–frame (x, y orthogonal coordinate system). The constituents and the mixture are assumed to be Newtonian or generalized Newtonian (power law and Carreau models). In $\mathbb{R}^2$, the mathematical model for two constituents indicated by subscripts 1 and 2 is presented in terms of velocities u_1, v_1, u_2, v_2, pressures p_1, p_2, and the deviatoric Cauchy stress tensors $({}_d\sigma_1)_{ij}$, $({}_d\sigma_2)_{ij}$; $i, j = x, y$ (total of 12 dependent variables). This constitutive model consists of twelve first order partial differential equations in twelve variables. The force π_α exerted on the α^{th} constituent by the other constituents are considered in the derivation of the momentum equations for the constituents but are neglected in the numerical studies and decoupled models. The constitutive theories presented here are based on [7, 19] and utilize material coefficients λ_i, $i = 1, 2, \ldots, 5$ and μ_i, $i = 1, 2, \ldots, 4$ which are shown to reduce to a much simplified form containing material coefficients $\mu_1, \mu_2, \ldots, \mu_4$ for the Newtonian and generalized Newtonian constituents and the mixture considered in the work.

The interaction forces π_α are much more significant in the case of liquid and solid particulate constituents, but are neglected in the present work. This mathematical model in various forms is commonly used for mixture theory in which the constituents are homogeneous, isotropic, incompressible fluids. In the present work we have shown that for the degenerated case when the two constituents in a mixture are the same, μ_3 and μ_4 must be zero for the mixture constitutive theory to be meaningful. Hence, in the constitutive theory used in the present work we use $\mu_3 = \mu_4 = 0$. The final mathematical model in $\mathbb{R}^2$ with u_1, v_1, u_2, v_2, p_1, p_2, and $({}_d\sigma_1)_{ij}$, $({}_d\sigma_2)_{ij}$; $i, j = x, y$ as dependent variables with only $\mu_1(\phi_1, \phi_2, \eta_1, \eta_2)$ and $\mu_2(\phi_1, \phi_2, \eta_1, \eta_2)$ as material coefficients in the constitutive theory has closure and is used for numerical studies in $\mathbb{R}^2$. This model requires no assumptions regarding p_1 and p_2 and is used to compute numerical results for 1:2 backward facing step.

From the mathematical model presented in $\mathbb{R}^1$ for fully developed flow between parallel plates, it is obvious that p_1 and p_2 for the constituents must be expressed in terms of the pressure p for the mixture. In the present work we propose $p_\alpha = \phi_\alpha p$, $\sum_{\alpha=1}^{\nu} p_\alpha = p$, which implies $\sum_{\alpha=1}^{\nu} \phi_\alpha = 1$ which obviously holds; hence this was used to compute numerical results for fully developed flow between parallel plates. This assumption is verified using the second model problem in which the combined model is used to compute constituent pressure p_1 and p_2 and then compared with p_1 and p_2 obtained using the decoupled model to demonstrate that p_1 and p_2 obtained from this mode are in precise agreement with those from the coupled model.

It is shown that the combined mathematical model proposed in this work can be decoupled when the constituents for the mixture are Newtonian fluids as for this case the viscosities are constant. However when the constituents and the mixture are generalized Newtonian fluids (power law and Carreau-Yasuda), the viscosities of the constituents are functions of the corresponding second invariant of the symmetric part of the velocity gradient tensors, hence the combined model can not be decoupled. The numerical studies presented for fully developed flow between parallel plates and 1:2 asymmetric backward facing step confirm the validity of the proposed mathematical model using $p_1 = \phi_1 p$, $p_2 = \phi_2 p$, and $p_1 + p_2 = p$ and the modifications proposed in the constitutive theory for the constituents.

Bibliography

[1] D. A. Drew. Mathematical modeling of two-phase flow. *Annual Review of Fluid Mechanics*, 15:261–291, 1983.

[2] S. I. Rubinow and Joseph B. Keller. The transverse force on a spinning sphere moving in a viscous fluid. *Journal of Fluid Mechanics*, 11(3):447–459, 1961.

[3] H. Enwald, E. Peirano, and A. E. Almstedt. Eulerian two-phase flow theory applied to fluidization. *International Journal of Multiphase Flow*, 22:21–66, 1996.

[4] H. Enwald and A. E. Almstedt. Fluid dynamics of a pressurized fluidized bed: comparison between numerical solutions from two-fluid models and experimental results. *Chemical Engineering Science*, 54:329–342, 1999.

[5] K. Terada, T. Ito, and N. Kikuchi. Characterization of the mechanical behaviors of solid-fluid mixture by the homogenization method. *Computer Methods in Applied Mechanics and Engineering*, 153:223–257, 1998.

[6] Goodarz Ahmadi, Jianfa Cao, Lukas Schneider, and Amsini Sadiki. A thermodynamical formulation for chemically active multiphase turbulent flows. *International Journal of Engineering Science*, 44:699–720, 2006.

[7] K. R. Rajagopal and L. Tao. *Mechanics of Mixtures*. World Scientific, River Edge, NJ, 1995.

[8] C. Truesdell. Mechanical basis of diffusion. *Journal of Chemical Physics*, 37(10):2336–2344, 1962.

[9] K. R. Rajagopal, Mehrdad Massoudi, and G. Johnson. A review of interaction mechanisms in fluid-solid flows. *DOE Report*, 90/9, 1991.

[10] Mehrdad Massoudi. Constitutive relations for the interaction force in multicomponent particulate flows. *International Journal of Non-Linear Mechanics*, 38:313–336, 2003.

[11] Ingo Müller. A thermodynamic theory of mixtures of fluids. *Archive for Rational Mechanics and Analysis*, 28(1):1–39, 1968.

[12] A. E. Green and P. M. Naghdi. A theory of mixtures. *Archive for Rational Mechanics and Analysis*, 24(4):243–263, 1967.

[13] R.J. Atkin and R.E. Craine. Continuum theories of mixtures: basic theory and historical development. *Quarterly Journal of Mechanics and Applied Mathematics*, 29(2):209–243, 1976.

[14] A. Bedford and D.S. Drumheller. Theories of immiscible and structured mixtures. *International Journal of Engineering Science*, 21(8):863–960, 1983.

[15] G. Johnson, M. Massoudi, and K.R. Rajagopal. Flow of a fluid-solid mixture between flat plates. *Chemical Engineering Science*, 46(7):1713–1723, 1991.

[16] M. Massoudi, K.R. Rajagopal, and T.X. Phuoc. On the fully developed flow of a dense particulate mixture in a pipe. *Powder Technology*, 104:258–268, 1999.

[17] Mehrdad Massoudi and C. Lakshmana Rao. Vertical flow of a mutiphase mixture in a channel. *Mathematical Problems in Engineering*, 6:505–526, 2001.

[18] Mehrdad Massoudi, P. Ravindran, and N. K. Anand. Steady free surface flow of a fluid-solid mixture down and inclined plane. *Particulate Science and Technology*, 22:253–273, 2004.

[19] K.R. Rajagopal, A. Al-Sharif, K. Chamniprasart, and A.Z. Szeri. Lubrication with binary mixtures: liquid-liquid emulsion. *Journal of Tribology*, 115(1):46–55, 1993.

[20] K.R. Rajagopal, K. Chamniprasart, A. Al-Sharif, and A.Z. Szeri. Lubrication with binary mixtures: bubbly oil. *Journal of Tribology*, 115(2):253–260, 1993.

[21] K.R. Rajagopal, S.H. Wang, A. Al-Sharif, and A.Z. Szeri. Lubrication with binary mixtures: liquid-liquid emulsion in an ehl conjunction. *Journal of Tribology*, 115(3):515–522, 1993.

[22] Mehrdad Massoudi. Boundary conditions in mixture theory and in cfd applications of higher order models. *Computers and Mathematics with Applications*, 53:156–167, 2007.

[23] Mehrdad Massoudi. A note on the meaning of mixture viscosity using the classical continuum theories of mixtures. *International Journal of Engineering Science*, 46:677–689, 2008.

[24] R. B. Bird, R. C. Armstrong, and O. Hassager. *Dynamics of polymeric liquids*. John Wiley and Sons, New York, NY, 2001.

[25] A. C. Eringen. *Nonlinear Theory of Continuous Media*. McGraw-Hill, 1962.

[26] A. C. Eringen. *Mechanics of Continua*. John Wiley and Sons, 1967.

[27] R. Sampaio and W.O. Williams. On the viscosities of liquid mixtures. *Zeitschrift für angewandte Mathematik und Physik (Journal of Applied Mathematics and Physics)*, 28(4):607–614, 1977.

[28] K. S. Surana and J. N. Reddy. *Mathematics of computations and the finite element method for Boundary Value Problems*. Manuscript of textbook in preparation, 2012.

[29] K. S. Surana and J. N. Reddy. *Mathematics of computations and the finite element method for Initial Value Problems*. Manuscript of textbook in preparation, 2012.

[30] B. Bell and K. S. Surana. p-version least squares finite element formulation for two-dimensional, incompressible, non-newtonian isothermal and non-isothermal fluid flow. *International Journal for Numerical Methods in Fluids*, 18(2):127–162, 1994.

[31] D. Winterscheidt and K. S. Surana. p-version least squares finite element formulation for two-dimensional, incompressible fluid flow. *International Journal for Numerical Methods in Fluids*, 18(1):43–69, 1994.

[32] Bo-Nan Jiang. *The Least-Squares Finite Element Method: Theory and Applications in Computational Fluid Dynamics and Electromagnetics*. Springer, River Edge, NJ, 1998.

[33] M. K. Denham and M. A. Patrick. Laminar flow over a downstream-facing step in a two-dimensional flow channel. *Transactions of the Institution of Chemical Engineers*, 52(4):361, 1974.