Collineations in space of four dimensions.

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The object of this paper is to outline the theory of collineations in space of four dimensions: in particular to develop what Prof Newson calls the "normal forms," in his 'Theory of Collineations', and to deduce the twenty seven types of collineations which exist in the R_4.

The subject is introduced by a brief discussion of the ordinary analytic transformation in five variables with regard to its geometrical interpretation, and the dependent transformations which line plane and space coordinates undergo under a point transformation.

The geometrical terminology followed is that of M. Jouffret in his "Traite de Geometrie a quatre dimensions.

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Section 1

General Analytic form of Collineation in the R_4.

§ 1. Consider the substitution T:

$$x' = \frac{a_{11}x + a_{12}y + a_{13}z + a_{14}u + a_{15}}{a_{51}x + a_{52}y + a_{53}z + a_{54}u + a_{55}} = \frac{\bar{X}}{\bar{V}}$$

$$y' = \frac{a_{21}x + a_{22}y + a_{23}z + a_{24}u + a_{25}}{a_{51}x + a_{52}y + a_{53}z + a_{54}u + a_{55}} = \frac{\bar{Y}}{\bar{V}}$$

$$z' = \frac{a_{31}x + a_{32}y + a_{33}z + a_{34}u + a_{35}}{a_{51}x + a_{52}y + a_{53}z + a_{54}u + a_{55}} = \frac{\bar{Z}}{\bar{V}}$$

$$u' = \frac{a_{41}x + a_{42}y + a_{43}z + a_{44}u + a_{45}}{a_{51}x + a_{52}y + a_{53}z + a_{54}u + a_{55}} = \frac{\bar{U}}{\bar{V}}$$

This ~~set of~~ substitutions transforms the set of numbers x, y, z, u into the set x', y', z', u'.

If we take x, y, z, u to be the cartesian coordinates of a point in the R_4, referred to four mutually perpendicular spaces as spaces of reference, then the substitution transforms the point (x, y, z, u) into the point $(x,' y,' z,' u,')$

If we solve the four equations of T for x, y, z, u we get a transformation T^{-1}:

$$x = \frac{A_{11} x' + A_{21} y' + A_{31} z' + A_{41} u' + A_{51}}{A_{15} x' + A_{25} y' + A_{35} z' + A_{45} u' + A_{55}} = \frac{\bar{X}^{-1}}{\bar{V}^{-1}}$$

$$y = \frac{A_{12} x' + A_{22} y' + A_{32} z' + A_{42} u' + A_{52}}{A_{15} x' + A_{25} y' + A_{35} z' + A_{45} u' + A_{55}} \quad \frac{\bar{Y}^{-1}}{\bar{V}^{-1}}$$

$$z = \frac{A_{13} x' + A_{23} y' + A_{33} z' + A_{43} u' + A_{53}}{A_{15} x' + A_{25} y' + A_{35} z' + A_{45} u' + A_{55}} \quad \frac{\bar{Z}^{-1}}{\bar{V}^{-1}}$$

$$u = \frac{A_{14} x' + A_{24} y' + A_{34} z' + A_{44} u' + A_{45}}{A_{15} x' + A_{25} y' + A_{35} z' + A_{45} u' + A_{55}} = \frac{\bar{U}^{-1}}{\bar{V}^{-1}}$$

where A_{ij} is the cofactor of a_{ij} in the matrix (a)

$$(a) = \begin{Vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{Vmatrix}$$

This transformation turns the point (x', y', z', u') back into (x, y, z, u).

Evidently T and T^{-1} set up a one to one correspondence between the points of the R_4. T^{-1} always exists if $\Delta \neq 0$, Δ being the determinant of (a).

By T the point $X = \bar{Y} = \bar{U} = \bar{Z} = 0$ is sent into the origin $x = y = z = u = 0$. The origin itself goes over into the pt $\left(\frac{a_{15}}{a_{55}}, \frac{a_{25}}{a_{15}}, \frac{a_{35}}{a_{15}}, \frac{a_{45}}{a_{15}} \right)$

all points of the space $\bar{V} = 0$ go into points in the space at infinity.

Since T^{-1} evidently sends points in $\bar{V}' = 0$ to infinity, T must send points in the infinitely distant space into points in $\bar{V}' = 0$

T and T^{-1} may be written in homogeneous form as follows

1] T: $\quad \rho x_i' = a_{i1} x_1 + a_{i2} x_2 + a_{i3} x_3 + a_{i4} x_4 + a_{i5} x_5$

2] T^{-1}: $\quad \rho' x_i = A_{1i} x_1' + A_{2i} x_2' + A_{3i} x_3' + A_{4i} x_4' + A_{5i} x_5'$

$(i = 1, 2, 3, 4, 5)$

where the A_{ij} has the same meaning as before.

§ 2. T transforms lines into lines, for let x, y, and z be three collinear points, then a relation exists

$$z_i = c_1 x_i + c_2 y_i \quad (i = 1, 2, 3, 4, 5)$$

transform the three points by T

$$x_i' = a_{i1} x_1 + a_{i2} x_2 + a_{i3} x_3 + a_{i4} x_4 + a_{i5} x_5$$

$$y_i' = a_{i1} y_1 + a_{i2} y_2 + a_{i3} y_3 + a_{i4} y_4 + a_{i5} y_5$$

$$z_i' = a_{i1} z_1 + a_{i2} z_2 + a_{i3} z_3 + a_{i4} z_4 + a_{i5} z_5$$

$$= a_{i1}(c_1 x_1 + c_2 y_1) + a_{i2}(c_1 x_2 + c_2 y_2)$$
$$+ a_{i3}(c_1 x_3 + c_2 y_3) + a_{i4}(c_1 x_4 + c_2 y_4) + a_{i5}(c_1 x_5 + c_2 y_5)$$

$$z_i' = c_1 x_i' + c_2 y_i'$$

then x' y' and z' are also collinear and T is a collineation.

Two intersecting lines are sent into two intersecting lines by T, for the point of intersection of the original lines must go over into a point on both lines after the transformation, and as a point can go into but a single pt the transformed lines must intersect. Hence T sends planes into planes, and it can be easily shown that spaces go into spaces under T.

§. 3. By means of T we can transform any six points of the R_4 into any other six points, for if we lay down the conditions that a set of six given points be transformed into any six other points by T we obtain a set of thirty equations in thirty-one unknowns viz: the twenty-five parameters of (a) and six ρ's. It is not difficult to show that there is a solution for which no one of the ρ's is 0, and that all other solutions are proportional to this one. Hence any six points can be transformed into any other six points in one, and only one way.

§ $3\frac{1}{2}$. What points are left invariant by T. Any such points, if there are any, must satisfy the conditions

$$\rho x_i = a_{i1} x_1 + a_{i2} x_2 + a_{i3} x_3 + a_{i4} x_4 + a_{i5} x_5 -$$
$(i = 1 \ldots 5)$

or,

(1. Bochers Algebra)

III

$$
\begin{aligned}
(a_{11}-\rho)x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + a_{15}x_5 &= 0\\
a_{21}x_1 + (a_{22}-\rho)x_2 + a_{23}x_3 + a_{24}x_4 + a_{25}x_5 &= 0\\
a_{31}x_1 + a_{32}x_2 + (a_{33}-\rho)x_3 + a_{34}x_4 + a_{35}x_5 &= 0\\
a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + (a_{44}-\rho)x_4 + a_{45}x_5 &= 0\\
a_{51}x_1 + a_{52}x_2 + a_{53}x_3 + a_{54}x_4 + (a_{55}-\rho)x_5 &= 0
\end{aligned}
$$

The necessary and sufficient condition that this system of equations may have other solutions than $x_1 = x_2 = x_3 = x_4 = x_5 =$ is that their eliminant $\Delta(\rho) = 0$.

This is a quintic in ρ and in general has five distinct roots. For each value of ρ we get one solution of the system of equations. Hence in general T leaves invariant a figure consisting of five points and their joins which consist of ten lines, ten planes and five spaces. In a later section we shall investigate this characteristic equation more closely, to discover what variations may occur in this invariant configuration.

Section 2.

Dependent transformations

§4. By the same argument[1] which is used in space of fewer dimensions, we can show that the ten second order determinants of the the matrix

$$\left\|\begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \end{matrix}\right\|$$

determine uniquely the line through the two points x and y and can be used as the homogeneous coordinates of the line.

Likewise the ten third order determinants of the matrix

$$\left\|\begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \\ z_1 & z_2 & z_3 & z_4 & z_5 \end{matrix}\right\|$$

determine uniquely the plane through x y and z and may be taken as the homogeneous coordinates of the plane.

1. See Bôcher's Higher algebra.

and the five fourth order determinants of the matrix

$$\left\| \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \\ z_1 & z_2 & z_3 & z_4 & z_5 \\ u_1 & u_2 & u_3 & u_4 & u_5 \end{matrix} \right\|$$

determine uniquely the space through x, y, z, and u, and can be used as the homogeneous coordinates of the space.

Denote $\begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}$ by p_{ij} $\quad i \neq j$

$\begin{vmatrix} x_i & x_j & x_k \\ y_i & y_j & y_k \\ z_i & z_j & z_k \end{vmatrix}$ by q_{ijk} $\quad i \neq j \neq k$

and $\begin{vmatrix} x_i & x_j & x_k & x_l \\ y_i & y_j & y_k & y_l \\ z_i & z_j & z_k & z_l \\ u_i & u_j & u_k & u_l \end{vmatrix}$ by σ_{ijkl} $\quad i \neq j \neq k \neq l$

when our R_4 is transformed by T the p's, q's and s's undergo certain transformations dependent on the T. The form which these dependent transformations take, we shall now show

Suppose T sends the points x and y into X and Y respectively.

Then we have

$$P_{ij} = \begin{vmatrix} X_i & X_j \\ Y_i & Y_j \end{vmatrix} \quad \left\{ \begin{matrix} i = 1, 2, \cdots 5 \\ j = 1, 2, \cdots 5 \\ i \neq j \end{matrix} \right\}$$

$$= \begin{vmatrix} a_{i1} x_1 + a_{i2} x_2 + a_{i3} x_3 + a_{i4} x_4 + a_{i5} x_5 & a_{j1} x_1 + a_{j2} x_2 + a_{j3} x_3 + a_{j4} x_4 + a_{j5} x_5 \\ a_{i1} y_1 + a_{i2} y_2 + a_{i3} y_3 + a_{i4} y_4 + a_{i5} y_5 & a_{j1} y_1 + a_{j2} y_2 + a_{j3} y_3 + a_{j4} y_4 + a_{j5} y_5 \end{vmatrix}$$

which turns out to be

$$\cancel{P_{ij} = \sum \begin{matrix} a_{i1} & a_{j1} \\ a_{i2} & a_{j2} \end{matrix}}$$

$$[III] \quad P_{ij} = \sum \begin{vmatrix} a_{ir} & a_{jr} \\ a_{is} & a_{js} \end{vmatrix} p_{rs}$$

$$\left\{ \begin{matrix} i = 1 \cdots 5 \\ j = 1 \cdots 5 \\ i \neq j \end{matrix} \right. \qquad \begin{matrix} r = 1 \cdots 5 \\ s = 1 \cdots 5 \\ r \neq s \end{matrix}$$

a transformation consisting of ten equations on the p's, the determinant of its matrix being equal to Δ^4.

These ten equations are not independent, which is shown as follows.

If we expand the vanishing fourth order determinants of the matrix

$$\left\| \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \end{matrix} \right\|$$

we find the following five equations which must be satisfied by the p's

(1) $p_{12}\, p_{34} + p_{13}\, p_{42} + p_{14}\, p_{23} = 0$

(2) $p_{12}\, p_{35} + p_{13}\, p_{52} + p_{15}\, p_{23} = 0$

(3) $p_{12}\, p_{45} + p_{14}\, p_{52} + p_{15}\, p_{24} = 0$

(4) $p_{13}\, p_{45} + p_{14}\, p_{53} + p_{15}\, p_{34} = 0$

(5) $p_{23}\, p_{45} + p_{24}\, p_{53} + p_{25}\, p_{34} = 0$.

as it turns out any pair of these five equations is equivalent to any other pair. To show this multiply (1) by p_{35}, (2) by p_{34} and subtract. This gives

$$(I)\quad p_{13}\,p_{42}\,p_{35} + p_{14}\,p_{23}\,p_{35} + p_{13}\,p_{23}\,p_{34} + p_{15}\,p_{32}\,p_{34} = 0$$

then if we multiply (4) by p_{32}, (5) by p_{31} and subtract we get again identically (I) which shows that the two pairs of conditions are equivalent. Hence the five ~~(5)~~ equations are equivalent to three independent conditions on the p's.

§ 5. In the same manner it may be shown that we have a dependent transformation on the q's consisting of a set of ten equations whose coefficients are the third order minors of (a) and whose determinant equals Δ^6. These ten equations on the q's are not independent but are tied up to the same extent as the p's.

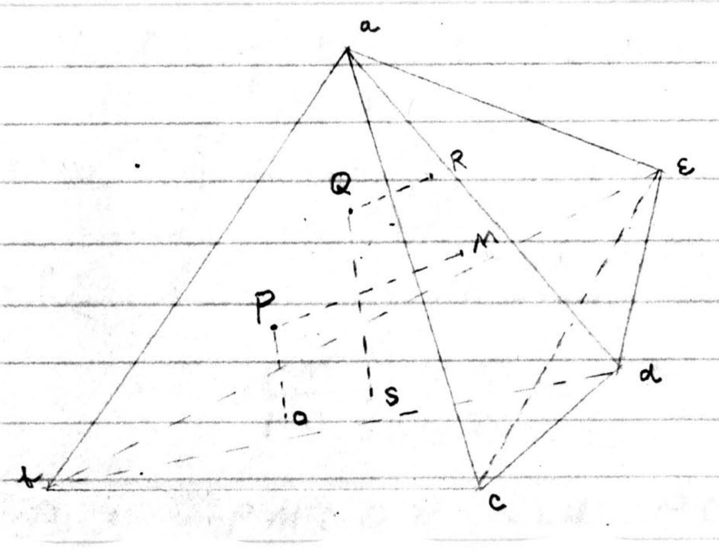
Likewise there is a dependent transformation on the σ's consisting of a set of five independent equations, whose coefficients are the fourth order minors of (a) and whose determinant $= \Delta^4$.

Section 3.

Collineation in Normal form.

§6 In this section we shall develop through geometrical considerations, our collineation in what Prof. Newson has called the "normal form." In this form the parameters of the collineation are explicitly the natural parameters of the geometrical transformation, namely: the coordinates of the invariant points and the essential cross ratios along the invariant lines.

Let a, b, c, d, and ε be the five invariant points under our collineation having coordinates $(a_1\ a_2\ a_3\ a_4)$, $(b_1\ b_2\ b_3\ b_4)$, etc.



Suppose $P(x_1' x_2' x_3' x_4')$ and $Q(x_1 x_2 x_3 x_4)$ are two corresponding points under our collineation.
Draw perpendiculars from P and Q to the spaces bcde and acde respectively, and let these perpendiculars be PO, QS, PM, & QR,
Now acde and bcde are the two double spaces of a one dimensional projective pencil of spaces through the plane cde and Pcde and Qcde will be two corresponding ~~points~~ spaces of the pencil.
The cross ratio of this pencil is

$$\frac{PM}{PO} : \frac{QR}{QS}$$

Corresponding spaces of this pencil will cut the line ab in corresponding points of the projective range on ab and the cross ratio of the pencil of spaces is the same as the cross ratio along the line. Call this cross ratio K_{ab}

$$\therefore \quad \frac{PM}{PO} : \frac{QR}{QS} = K_{ab}$$

But PM : QR and PO : QS are the ratios of the volumes of the

Pentahedroides Pcdea : Qcdea
and Pbcde and Qbcde
then we have:

$$\frac{Pcdea}{Pbdea} : \frac{Qcdea}{Qbdea} = \frac{PM}{PO} : \frac{QR}{QS} = K_{ab}$$

or

$$[\mathrm{IV}] \quad \frac{\begin{vmatrix} x_1' & x_2' & x_3' & x_4' & 1 \\ c_1 & c_2 & c_3 & c_4 & 1 \\ d_1 & d_2 & d_3 & d_4 & 1 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & 1 \\ a_1 & a_2 & a_3 & a_4 & 1 \end{vmatrix}}{\begin{vmatrix} x_1' & x_2' & x_3' & x_4' & 1 \\ b_1 & b_2 & b_3 & b_4 & 1 \\ c_1 & c_2 & c_3 & c_4 & 1 \\ d_1 & d_2 & d_3 & d_4 & 1 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & 1 \end{vmatrix}} = K_{ab} \frac{\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & 1 \\ c_1 & c_2 & c_3 & c_4 & 1 \\ d_1 & d_2 & d_3 & d_4 & 1 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & 1 \\ a_1 & a_2 & a_3 & a_4 & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & 1 \\ b_1 & b_2 & b_3 & b_4 & 1 \\ c_1 & c_2 & c_3 & c_4 & 1 \\ d_1 & d_2 & d_3 & d_4 & 1 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & 1 \end{vmatrix}}$$

and we can deduce in the same manner three other equations:

$$\frac{\begin{vmatrix} x_1' & x_2' & x_3' & x_4' & 1 \\ d_1 & d_2 & d_3 & d_4 & 1 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & 1 \\ a_1 & a_2 & a_3 & a_4 & 1 \\ b_1 & b_2 & b_3 & b_4 & 1 \end{vmatrix}}{D} = K_{ac} \frac{\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & 1 \\ d_1 & d_2 & d_3 & d_4 & 1 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & 1 \\ a_1 & a_2 & a_3 & a_4 & 1 \\ b_1 & b_2 & b_3 & b_4 & 1 \end{vmatrix}}{D'}$$

(D and D' are the two denominators in equation I)

$$\frac{\begin{vmatrix} x_1' & x_2' & x_3' & x_4' & 1 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & 1 \\ a_1 & a_2 & a_3 & a_4 & 1 \\ b_1 & b_2 & b_3 & b_4 & 1 \\ c_1 & c_2 & c_3 & c_4 & 1 \end{vmatrix}}{D} = K_{ad} \frac{\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & 1 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & 1 \\ a_1 & a_2 & a_3 & a_4 & 1 \\ b_1 & b_2 & b_3 & b_4 & 1 \\ c_1 & c_2 & c_3 & c_4 & 1 \end{vmatrix}}{D'}$$

$$\frac{\begin{vmatrix} x_1' & x_2' & x_3' & x_4' & 1 \\ a_1 & a_2 & a_3 & a_4 & 1 \\ b_1 & b_2 & b_3 & b_4 & 1 \\ c_1 & c_2 & c_3 & c_4 & 1 \\ d_1 & d_2 & d_3 & d_4 & 1 \end{vmatrix}}{D} = K_{a\varepsilon} \frac{\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & 1 \\ a_1 & a_2 & a_3 & a_4 & 1 \\ b_1 & b_2 & b_3 & b_4 & 1 \\ c_1 & c_2 & c_3 & c_4 & 1 \\ d_1 & d_2 & d_3 & d_4 & 1 \end{vmatrix}}{D'}$$

We have four linear equations to determine x_1', x_2', x_3', x_4' in terms of x_1, x_2, x_3, x_4 which is the proper number to give us a unique solution.

Moreover we have the 24 essential parameters of our collineation, namely: the 20 coordinates of our invariant points and the four independent crossratios K_{ab}, K_{ac} K_{ad} $K_{a\varepsilon}$.

That there can be only four

independent crossratios is apparent at once from the theorem that the product of the cross ratios around an invariant is unity.
So the cross ratio along some other line of the figure than the four already used, K_{bd}, for example, is directly dependent on K_{ab} and K_{ad}.

The system of 4 equations just deduced, involving the variables in the manner shown is called the implicit normal form of the collineation.
We wish to express the 4 x''s explicitly in terms of x and the parameters. This necessitates a solution of the 4 equations simultaneously, which by the ordinary methods of elimination is very cumbersome. But since the form of the answer is known from the consideration of the form which the collineation takes in space of fewer dimensions, all that is necessary is to set up the explicit normal form and identify it with the implicit form that we are solving.

This form is

[V] $$x_i' = \frac{\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & 1 & 0 \\ a_1 & a_2 & a_3 & a_4 & 1 & a_i \\ b_1 & b_2 & b_3 & b_4 & 1 & K b_i \\ c_1 & c_2 & c_3 & c_4 & 1 & K' c_i \\ d_1 & d_2 & d_3 & d_4 & 1 & K'' d_i \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & 1 & K''' \varepsilon_i \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & 1 & 0 \\ a_1 & a_2 & a_3 & a_4 & 1 & 1 \\ b_1 & b_2 & b_3 & b_4 & 1 & K \\ c_1 & c_2 & c_3 & c_4 & 1 & K' \\ d_1 & d_2 & d_3 & d_4 & 1 & K'' \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & 1 & K''' \end{vmatrix}}$$

$i = 1, 2, 3, 4$

where k, k', k'', k''' are k_{ab}, k_{ac} k_{ad}, and $k_{a\varepsilon}$ respectively

In homogeneous form this becomes

[VI] $$\rho x_i' = \begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_i \\ b_1 & b_2 & b_3 & b_4 & b_5 & K b_i \\ c_1 & c_2 & c_3 & c_4 & c_5 & K' c_i \\ d_1 & d_2 & d_3 & d_4 & d_5 & K'' d_i \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & K''' \varepsilon_i \end{vmatrix}$$

$i = 1, 2, 3, 4, 5$

To show that [VI] and [IV] are identical collineations, make [IV] also homogeneous and take for the invariant pentahedroid the frame of reference, whose vertices are

a =	0 0 0 0 1
b,	1 0 0 0 0
c,	0 1 0 0 0
d,	0 0 1 0 0
ε,	0 0 0 1 0

Then IV becomes

$$\frac{x_1'}{x_5'} = \frac{K x_1}{x_5}$$

$$\frac{x_2'}{x_5'} = \frac{K' x_2}{x_5}$$

$$\frac{x_3'}{x_5'} = \frac{K'' x_3}{x_5}$$

$$\frac{x_4'}{x_5'} = \frac{K''' x_4}{x_5}$$

and VI becomes

$$[\mathrm{VII}] \qquad \begin{aligned} \rho x_1' &= \kappa x_1 \\ \rho x_2' &= \kappa' x_2 \\ \rho x_3' &= \kappa'' x_3 \\ \rho x_4' &= \kappa''' x_4 \\ \rho x_5' &= x_5 \end{aligned}$$

These two collineations evidently are identical, since both reduce to the same canonic form.

The determinant of the explicit normal form is:

$$\Delta =$$

$$\begin{vmatrix} a_2 & a_3 & a_4 & a_5 & -a_1 \\ b_2 & b_3 & b_4 & b_5 & -kb_1 \\ c_2 & c_3 & c_4 & c_5 & -k'c_1 \\ d_2 & d_3 & d_4 & d_5 & -k''d_1 \\ \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & -k'''\varepsilon_1 \end{vmatrix} \begin{vmatrix} a_3 & a_4 & a_5 & -a_1 & a_1 \\ b_3 & b_4 & b_5 & -kb_1 & b_1 \\ c_3 & c_4 & c_5 & -k'c_1 & c_1 \\ d_3 & d_4 & d_5 & -k''d_1 & d_1 \\ \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & -k'''\varepsilon_1 & \varepsilon_1 \end{vmatrix} \begin{vmatrix} a_4 & a_5 & -a_1 & a_1 & a_2 \\ b_4 & b_5 & -kb_1 & b_1 & b_2 \\ c_4 & c_5 & -k'c_1 & c_1 & c_2 \\ d_4 & d_5 & -k''d_1 & d_1 & d_2 \\ \varepsilon_4 & \varepsilon_5 & -k'''\varepsilon_1 & \varepsilon_1 & \varepsilon_2 \end{vmatrix} \begin{vmatrix} a_5 & -a_1 & a_1 & a_2 & a_3 \\ b_5 & -kb_1 & b_1 & b_2 & b_3 \\ c_5 & -k'c_1 & c_1 & c_2 & c_3 \\ d_5 & -k''d_1 & d_1 & d_2 & d_3 \\ \varepsilon_5 & -k'''\varepsilon_1 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \end{vmatrix} \begin{vmatrix} a_1 & a_1 & a_2 & a_3 & a_4 \\ kb_1 & b_1 & b_2 & b_3 & b_4 \\ k'c_1 & c_1 & c_2 & c_3 & c_4 \\ k''d_1 & d_1 & d_2 & d_3 & d_4 \\ k'''\varepsilon_1 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \end{vmatrix}$$

$$\begin{vmatrix} a_2 & a_3 & a_4 & a_5 & -a_2 \\ b_2 & b_3 & b_4 & b_5 & -kb_2 \\ c_2 & c_3 & c_4 & c_5 & -k'c_2 \\ d_2 & d_3 & d_4 & d_5 & -k''d_2 \\ \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & -k'''\varepsilon_2 \end{vmatrix} \begin{vmatrix} a_3 & a_4 & a_5 & -a_2 & a_1 \\ b_3 & b_4 & b_5 & -kb_2 & b_1 \\ c_3 & c_4 & c_5 & k'c_2 & c_1 \\ d_3 & d_4 & d_5 & k''d_2 & d_1 \\ \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & -k'''\varepsilon_2 & \varepsilon_1 \end{vmatrix} \begin{vmatrix} a_4 & a_5 & -a_2 & a_1 & a_2 \\ b_4 & b_5 & -kb_2 & b_1 & b_2 \\ c_4 & c_5 & -k'c_2 & c_1 & c_2 \\ d_4 & d_5 & k''d_2 & d_1 & d_2 \\ \varepsilon_4 & \varepsilon_5 & -k'''\varepsilon_2 & \varepsilon_1 & \varepsilon_2 \end{vmatrix} \begin{vmatrix} a_5 & -a_2 & a_1 & a_2 & a_3 \\ b_5 & -kb_2 & b_1 & b_2 & b_3 \\ c_5 & -k'c_2 & c_1 & c_2 & c_3 \\ d_5 & -k''d_2 & d_1 & d_2 & d_3 \\ \varepsilon_5 & -k'''\varepsilon_2 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \end{vmatrix} \begin{vmatrix} a_2 & a_1 & a_2 & a_3 & a_4 \\ kb_2 & b_1 & b_2 & b_3 & b_4 \\ k'c_2 & c_1 & c_2 & c_3 & c_4 \\ k''d_2 & d_1 & d_2 & d_3 & d_4 \\ k'''\varepsilon_2 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \end{vmatrix}$$

$$\begin{vmatrix} a_2 & a_3 & a_4 & a_5 & -a_3 \\ b_2 & b_3 & b_4 & b_5 & -kb_3 \\ c_2 & c_3 & c_4 & c_5 & -k'c_3 \\ d_2 & d_3 & d_4 & d_5 & -k''d_3 \\ \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & -k'''\varepsilon_3 \end{vmatrix} \begin{vmatrix} a_3 & a_4 & a_5 & -a_3 & a_1 \\ b_3 & b_4 & b_5 & -kb_3 & b_1 \\ c_3 & c_4 & c_5 & -k'c_3 & c_1 \\ d_3 & d_4 & d_5 & -k''d_3 & d_1 \\ \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & -k'''\varepsilon_3 & \varepsilon_1 \end{vmatrix} \begin{vmatrix} a_4 & a_5 & -a_3 & a_1 & a_2 \\ b_4 & b_5 & -kb_3 & b_1 & b_2 \\ c_4 & c_5 & -k'c_3 & c_1 & c_2 \\ d_4 & d_5 & -k''d_3 & d_1 & d_2 \\ \varepsilon_4 & \varepsilon_5 & -k'''\varepsilon_3 & \varepsilon_1 & \varepsilon_2 \end{vmatrix} \begin{vmatrix} a_5 & -a_3 & a_1 & a_2 & a_3 \\ b_5 & -kb_3 & b_1 & b_2 & b_3 \\ c_5 & -k'c_3 & c_1 & c_2 & c_3 \\ d_5 & -k''d_3 & d_1 & d_2 & d_3 \\ \varepsilon_5 & -k'''\varepsilon_3 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \end{vmatrix} \begin{vmatrix} a_3 & a_1 & a_2 & a_3 & a_4 \\ kb_3 & b_1 & b_2 & b_3 & b_4 \\ k'c_3 & c_1 & c_2 & c_3 & c_4 \\ k''d_3 & d_1 & d_2 & d_3 & d_4 \\ k'''\varepsilon_3 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \end{vmatrix}$$

$$\begin{vmatrix} a_2 & a_3 & a_4 & a_5 & a_4 \\ b_2 & b_3 & b_4 & b_5 & -kb_4 \\ c_2 & c_3 & c_4 & c_5 & k'c_4 \\ d_2 & d_3 & d_4 & d_5 & -k''d_4 \\ \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & k'''\varepsilon_4 \end{vmatrix} \begin{vmatrix} a_3 & a_4 & a_5 & -a_4 & a_1 \\ b_3 & b_4 & b_5 & -kb_4 & b_1 \\ c_3 & c_4 & c_5 & k'c_4 & c_1 \\ d_3 & d_4 & d_5 & -k''d_4 & d_1 \\ \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & k'''\varepsilon_4 & \varepsilon_1 \end{vmatrix} \begin{vmatrix} a_4 & a_5 & -a_4 & a_1 & a_2 \\ b_4 & b_5 & -k''b_4 & b_1 & b_2 \\ c_4 & c_5 & k''c_4 & c_1 & c_2 \\ d_4 & d_5 & -k'''d_4 & d_1 & d_2 \\ \varepsilon_4 & \varepsilon_5 & -k'''\varepsilon_4 & \varepsilon_1 & \varepsilon_2 \end{vmatrix} \begin{vmatrix} a_5 & -a_4 & a_1 & a_2 & a_3 \\ b_5 & -kb_4 & b_1 & b_2 & b_3 \\ c_5 & -k'c_4 & c_1 & c_2 & c_3 \\ d_5 & -k''d_4 & d_1 & d_2 & d_3 \\ \varepsilon_5 & -k'''\varepsilon_4 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \end{vmatrix} \begin{vmatrix} a_4 & a_1 & a_2 & a_3 & a_4 \\ kb_4 & b_1 & b_2 & b_3 & b_4 \\ k'c_4 & c_1 & c_2 & c_3 & c_4 \\ k''d_4 & d_1 & d_2 & d_3 & d_4 \\ k'''\varepsilon_4 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \end{vmatrix}$$

$$\begin{vmatrix} a_2 & a_3 & a_4 & a_5 & -a_5- \\ b_2 & b_3 & b_4 & b_5 & -kb_5- \\ c_2 & c_3 & c_4 & c_5 & -k'c_5- \\ d_2 & d_3 & d_4 & d_5 & -k''d_5- \\ \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & -k'''\varepsilon_5- \end{vmatrix} \begin{vmatrix} a_3 & a_4 & a_5 & a_5 & -a_1 \\ b_3 & b_4 & b_5 & -kb_5 & -b_1 \\ c_3 & c_4 & c_5 & -k'c_5 & c_1 \\ d_3 & d_4 & d_5 & k''d_5 & -d_1 \\ \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & -k'''\varepsilon_5 & -\varepsilon_1 \end{vmatrix} \begin{vmatrix} a_4 & a_5 & -a_5 & a_1 & a_2 \\ b_4 & b_5 & -kb_5 & b_1 & b_2 \\ c_4 & c_5 & -k'c_5 & c_1 & c_2 \\ d_4 & d_5 & -k''d_5 & -d_1 & d_2 \\ \varepsilon_4 & \varepsilon_5 & -k'''\varepsilon_5 & \varepsilon_1 & \varepsilon_2 \end{vmatrix} \begin{vmatrix} a_5 & -a_5 & -a_1 & a_2 & a_3 \\ b_5 & -kb_5 & b_1 & b_2 & b_3 \\ c_5 & -k'c_5 & -c_1 & c_2 & c_3 \\ d_5 & k''d_5 & d_1 & d_2 & d_3 \\ \varepsilon_5 & k'''\varepsilon_5 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \end{vmatrix} \begin{vmatrix} a_5- & a_1 & a_2 & a_3 & a_4 \\ kb_5- & b_1 & b_2 & b_3 & b_4 \\ k'c_5- & c_1 & c_2 & c_3 & c_4 \\ k''d_5 & d_1 & d_2 & d_3 & d_4 \\ k'''\varepsilon_5 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \end{vmatrix}$$

This determinant factors into

$$
\begin{vmatrix}
a_1 & a_2 & a_3 & a_4 & a_5 \\
Kb_1 & Kb_2 & Kb_3 & Kb_4 & Kb_5 \\
K'c_1 & K'c_2 & K'c_3 & K'c_4 & K'c_5 \\
K''d_1 & K''d_2 & K''d_3 & K''d_4 & K''d_5 \\
K'''\varepsilon_1 & K'''\varepsilon_2 & K'''\varepsilon_3 & K'''\varepsilon_4 & K'''\varepsilon_5
\end{vmatrix}
\quad \times
$$

$$
\begin{vmatrix}
\begin{vmatrix} b_2 & b_3 & b_4 & b_5 \\ c_2 & c_3 & c_4 & c_5 \\ d_2 & d_3 & d_4 & d_5 \\ \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 \end{vmatrix} &
\begin{vmatrix} b_3 & b_4 & b_5 & b_1 \\ c_3 & c_4 & c_5 & c_1 \\ d_3 & d_4 & d_5 & d_1 \\ \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & \varepsilon_1 \end{vmatrix} &
\begin{vmatrix} b_4 & b_5 & b_1 & b_2 \\ c_4 & c_5 & c_1 & c_2 \\ d_4 & d_5 & d_1 & d_2 \\ \varepsilon_4 & \varepsilon_5 & \varepsilon_1 & \varepsilon_2 \end{vmatrix} &
\begin{vmatrix} b_5 & b_1 & b_2 & b_3 \\ c_5 & c_1 & c_2 & c_3 \\ d_5 & d_1 & d_2 & d_3 \\ \varepsilon_5 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \end{vmatrix} &
\begin{vmatrix} b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \end{vmatrix} \\
\begin{vmatrix} c_2 & c_3 & c_4 & c_5 \\ d_2 & d_3 & d_4 & d_5 \\ \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 \\ a_2 & a_3 & a_4 & a_5 \end{vmatrix} &
\begin{vmatrix} c_3 & c_4 & c_5 & c_1 \\ d_3 & d_4 & d_5 & d_1 \\ \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & \varepsilon_1 \\ a_3 & a_4 & a_5 & a_1 \end{vmatrix} &
\begin{vmatrix} c_4 & c_5 & c_1 & c_2 \\ d_4 & d_5 & d_1 & d_2 \\ \varepsilon_4 & \varepsilon_5 & \varepsilon_1 & \varepsilon_2 \\ a_4 & a_5 & a_1 & a_2 \end{vmatrix} &
\begin{vmatrix} c_5 & c_1 & c_2 & c_3 \\ d_5 & d_1 & d_2 & d_3 \\ \varepsilon_5 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \\ a_5 & a_1 & a_2 & a_3 \end{vmatrix} &
\begin{vmatrix} c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \\ a_1 & a_2 & a_3 & a_4 \end{vmatrix} \\
\begin{vmatrix} d_2 & d_3 & d_4 & d_5 \\ \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 \\ a_2 & a_3 & a_4 & a_5 \\ b_2 & b_3 & b_4 & b_5 \end{vmatrix} &
\begin{vmatrix} d_3 & d_4 & d_5 & d_1 \\ \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & \varepsilon_1 \\ a_3 & a_4 & a_5 & a_1 \\ b_3 & b_4 & b_5 & b_1 \end{vmatrix} &
\begin{vmatrix} d_4 & d_5 & d_1 & d_2 \\ \varepsilon_4 & \varepsilon_5 & \varepsilon_1 & \varepsilon_2 \\ a_4 & a_5 & a_1 & a_2 \\ b_4 & b_5 & b_1 & b_2 \end{vmatrix} &
\begin{vmatrix} d_5 & d_1 & d_2 & d_3 \\ \varepsilon_5 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \\ a_5 & a_1 & a_2 & a_3 \\ b_5 & b_1 & b_2 & b_3 \end{vmatrix} &
\begin{vmatrix} d_1 & d_2 & d_3 & d_4 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{vmatrix} \\
\begin{vmatrix} \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 \\ a_2 & a_3 & a_4 & a_5 \\ b_2 & b_3 & b_4 & b_5 \\ c_2 & c_3 & c_4 & c_5 \end{vmatrix} &
\begin{vmatrix} \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & \varepsilon_1 \\ a_3 & a_4 & a_5 & a_1 \\ b_3 & b_4 & b_5 & b_1 \\ c_3 & c_4 & c_5 & c_1 \end{vmatrix} &
\begin{vmatrix} \varepsilon_4 & \varepsilon_5 & \varepsilon_1 & \varepsilon_2 \\ a_4 & a_5 & a_1 & a_2 \\ b_4 & b_5 & b_1 & b_2 \\ c_4 & c_5 & c_1 & c_2 \end{vmatrix} &
\begin{vmatrix} \varepsilon_5 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \\ a_5 & a_1 & a_2 & a_3 \\ b_5 & b_1 & b_2 & b_3 \\ c_5 & c_1 & c_2 & c_3 \end{vmatrix} &
\begin{vmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix} \\
\begin{vmatrix} a_2 & a_3 & a_4 & a_5 \\ b_2 & b_3 & b_4 & b_5 \\ c_2 & c_3 & c_4 & c_5 \\ d_2 & d_3 & d_4 & d_5 \end{vmatrix} &
\begin{vmatrix} a_3 & a_4 & a_5 & a_1 \\ b_3 & b_4 & b_5 & b_1 \\ c_3 & c_4 & c_5 & c_1 \\ d_3 & d_4 & d_5 & d_1 \end{vmatrix} &
\begin{vmatrix} a_4 & a_5 & a_1 & a_2 \\ b_4 & b_5 & b_1 & b_2 \\ c_4 & c_5 & c_1 & c_2 \\ d_4 & d_5 & d_1 & d_2 \end{vmatrix} &
\begin{vmatrix} a_5 & a_1 & a_2 & a_3 \\ b_5 & b_1 & b_2 & b_3 \\ c_5 & c_1 & c_2 & c_3 \\ d_5 & d_1 & d_2 & d_3 \end{vmatrix} &
\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}
\end{vmatrix}
$$

$$= K K' K'' K''' \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 \end{vmatrix} \begin{vmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ B_1 & B_2 & B_3 & B_4 & B_5 \\ C_1 & C_2 & C_3 & C_4 & C_5 \\ D_1 & D_2 & D_3 & D_4 & D_5 \\ E_1 & E_2 & E_3 & E_4 & E_5 \end{vmatrix}$$

where A_i is the cofactor of a_i in the first determinant of the product.

We may write this product in the following way if we chose

$$\Delta = \begin{vmatrix} A_1 & B_1 & C_1 & D_1 & E_1 \\ A_2 & B_2 & C_2 & D_2 & E_2 \\ A_3 & B_3 & C_3 & D_3 & E_3 \\ A_4 & B_4 & C_4 & D_4 & E_4 \\ A_5 & B_5 & C_5 & D_5 & E_5 \end{vmatrix} \times \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & K & 0 & 0 & 0 \\ 0 & 0 & K' & 0 & 0 \\ 0 & 0 & 0 & K'' & 0 \\ 0 & 0 & 0 & 0 & K''' \end{vmatrix} \times \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 \end{vmatrix}$$

In this form the matrix of Δ is equal to the product of the matrices of the three determinants, and therefore our transformation T_2 has been factored into three transformations, with matrices (A), (K), and (a). Since the transformation (A) is the inverse of (a) we can write our result

$$T_2 = T_1^{-1} K T_1$$

Hence: Any transformation in the normal

form can be written as the transform of its canonic form by a collineation whose coefficients are the coordinates of the invariant points of the original transformation.

§8 Inverse of Normal form.

A glance at [IV] shows that the inverse of T is obtained from T by replacing x_i by x_i' and $K^{(i)}$ by $\frac{1}{K^{(i)}}$.
Hence

$$T^{-1}: \quad \rho x_i = \begin{vmatrix} x_1' & x_2' & x_3' & x_4' & x_5' & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_i \\ b_1 & b_2 & b_3 & b_4 & b_5 & \frac{1}{K} b_i \\ c_1 & c_2 & c_3 & c_4 & c_5 & \frac{1}{K'} c_i \\ d_1 & d_2 & d_3 & d_4 & d_5 & \frac{1}{K''} d_i \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & \frac{1}{K'''} \varepsilon_i \end{vmatrix}$$

§9 Resultant of two collineations.
Let T and T_1 be two collineations with matrices (a) and (a')

$$T: \quad \rho x_i' = a_{i1} x_1 + a_{i2} x_2 + a_{i3} x_3 + a_{i4} x_4 + a_{i5} x_5$$

$$T_1: \quad \rho' x_i'' = a_{i1}' x_1' + a_{i2}' x_2' + a_{i3}' x_3' + a_{i4}' x_4' + a_{i5}' x_5'$$

We can obtain $T_2 = T\,T_1$ as follows

$$T^{-1} = \frac{1}{\rho} \Delta x_i = A_{1i} x_1' + A_{2i} x_2' + A_{3i} x_3' + A_{4i} x_4' + A_{5i} x_5'$$

Now take the five equations of T^{-1} with one equation of T_1 and eliminate x_i'. The eliminant is

$$\begin{vmatrix} -\frac{\Delta}{\rho} x_1 & A_{11} & A_{21} & A_{31} & A_{41} & A_{51} \\ -\frac{\Delta}{\rho} x_2 & A_{12} & A_{22} & A_{32} & A_{42} & A_{52} \\ -\frac{\Delta}{\rho} x_3 & A_{13} & A_{23} & A_{33} & A_{43} & A_{53} \\ -\frac{\Delta}{\rho} x_4 & A_{14} & A_{24} & A_{34} & A_{44} & A_{54} \\ -\frac{\Delta}{\rho} x_5 & A_{15} & A_{25} & A_{35} & A_{45} & A_{55} \\ -\rho' x_i'' & a_{i1}' & a_{i2}' & a_{i3}' & a_{i4}' & a_{i5}' \end{vmatrix} = 0$$

which solved for x_i'' gives

$$\rho'' x_i'' = \begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & 0 \\ A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & a_{i1}' \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & a_{i2}' \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & a_{i3}' \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} & a_{i4}' \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} & a_{i5}' \end{vmatrix}$$

This collineation is T_2 the resultant of T and T_1

Cross ratios of the resultant.

§10 The form of the cross ratio of the resultant of two transformations can easily be shown by a consideration of the canonic forms.

Suppose T and T_1 in canonic form are

$$\rho x_i' = k^{(i)} x_i$$

and $$\rho' x_i'' = k_1^{(i)} x_i'$$

Then by eliminating x_i' we have

$$T_2: \quad \rho'' x_i'' = k_1^{(i)} k^{(i)} x_i$$

That is to say: The cross ratio of the resultant of two collineations along an invariant line of the figure is equal to the product of the corresponding cross ratios of the two collineations.

§11. Roots of the characteristic equation.

It was shown in §4 that there are five values of ρ for which a point is left invariant under T. Our normal form of T gives an easy solution for these five roots of the characteristic equation.

The five roots must satisfy identities of the following form, where ρ_a is the value which leaves a invariant, ρ_b leaves b, etc.

$$\rho_a a_i = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 - & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 - & a_i \\ b_1 & b_2 & b_3 & b_4 & b_5 - & k b_i \\ c_1 & c_2 & c_3 & c_4 & c_5 - & k' c_i \\ d_1 & d_2 & d_3 & d_4 & d_5 - & k'' d_i \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 - & k''' \varepsilon_i \end{vmatrix}$$

replace the second row of this determinant by the second minus the first and solve we have

$$\rho_a a_i = a_i \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 - \\ b_1 & b_2 & b_3 & b_4 & b_5 - \\ c_1 & c_2 & c_3 & c_4 & c_5 - \\ d_1 & d_2 & d_3 & d_4 & d_5 - \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 - \end{vmatrix}$$

$\therefore \rho_a = \Delta'$ (where Δ' is the det formed by the coordinates of the invariant pts.)

In the same way we obtain for the other roots

$$\rho_b = k\,\Delta'$$
$$\rho_c = k'\,\Delta'$$
$$\rho_d = k''\,\Delta'$$
$$\rho_\varepsilon = k'''\,\Delta'$$

From which it is evident that the roots of the characteristic equation are proportional to the cross ratios of the collineation.

Section 4

Types of Collineations.

§ 11. Collineations are classified by types according to the geometrical figure that they leave invariant. These geometrical configurations can be deduced from the number and character of solutions of the characteristic equation:

$$\begin{vmatrix} (a_{11}-\rho) & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & (a_{22}-\rho) & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & (a_{33}-\rho) & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & (a_{44}-\rho) & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & (a_{55}-\rho) \end{vmatrix} = 0$$

We shall consider first the solutions when our matrix is of rank 4. There are then five roots which may be

5 distinct roots.
3 single roots, 1 double root
1 " " , 2 " "
1 triple " , 1 double "
1 " " , 2 single "
1 quadruple " , 1 " "
1 quintuple root.

Multiple roots show multiple points for the invariant figure

What sort of a configuration do we have at such a point?
Let two points a and b approach coincidence along a curve on a hyper quadric surface with no singularities along the path of approach. When b is near to a the coordinates of the line ab become

$$\begin{vmatrix} a_i & a_j \\ a_i + \Delta a_i & a_j + \Delta a_j \end{vmatrix} = \begin{vmatrix} a_i & a_j \\ \Delta a_i & \Delta a_j \end{vmatrix}$$

$$= \rho \begin{vmatrix} a_i & a_j \\ \frac{\Delta a_i}{\Delta s} & \frac{\Delta a_j}{\Delta s} \end{vmatrix}$$

at the limit these coordinates become

$$\begin{vmatrix} a_i & a_j \\ \frac{d a_i}{ds} & \frac{d a_j}{ds} \end{vmatrix}$$

But these are the coordinates of a line through two consecutive pts of the curve, that is a tangent to the curve at a. So that at a double pt of our invariant figure we have a lineal element. Similar considerations show that at a triple pt we have an invariant plane tangent to our surface, and at a quadruple point a tangent space.

what happens when the rank of the matrix is less than 4.

In the first place, if the rank is of less than four for some value of ρ, ρ_1 say, we know that ρ_1 is at least a double root, for if the first minors of $\Delta(\rho_1)$ become 0, $\Delta'(\rho_1) = 0$, which is the usual criterion for a double root. Should the rank be less than 3, $\Delta''(\rho_1)$ will be 0 and we have a triple root at least, and so on.

What is our invariant figure in these cases?

For the case where the matrix is of rank 3, the solution of 3 of the equations of III of section I gives us the point or points corresponding to the ρ_1 for which the matrix is of rank 3. In the solution of these equations we can give any values we please to two of the variables and determine the others uniquely. Hence we get a single infinity of solutions for the ratios $\frac{x_1}{x_5}, \frac{x_2}{x_5}, \frac{x_3}{x_5}, \frac{x_4}{x_5}$, and our

invariant figure corresponding to this ρ_1 is a line of all invariant points, the line of intersection of

the three spaces given by the three equations which we solved.
For a root for which the matrix is of rank 2 we have two of the equations to solve, and get a plane of all invariant points.
For a root of rank 1, we have a space of all invariant points.
If the matrix is of rank 0, all points are invariant under T, and we have the identical collineation.
In deducing our complete invariant figure, we are aided by the principle of duality.
The figure must be completely self dualistic with regard to points and spaces, and with respect to lines and planes.
~~The~~ In addition, we know that the collineation in any invariant plane of the figure must be one of the five well known types of plane collineations.

The considerations given above are sufficient to determine completely the invariant figure for every case except when we have a quintuple root for which the matrix is of rank 3 or 2.

The first of these gives us three different configurations: the plane axis of the space pencil may intersect the plane π (1) in a pt, (2) in a line, (3) or may coincide with π.

If the rank is 2, the line axis of the two dimensional space pencil may intersect the plane of all invariant points or may lie in that plane.

There are 27 types of collineation in the R_4 if we include the identical collineation, and following this is a table in which invariant elements of ~~the~~ each kind is given for each type.

The difficulties of making a picture of a four ~~ge~~ dimensional geometric figure are obvious, but the figures as drawn will serve to differentiate the types, and assist in counting the invariant elements.

It is sometimes the case that a point in a range is the vertex of a pencil, or a line of a pencil, is the axis of a pencil of planes. In these cases the point or line is counted separately.

The first seven types occur when the rank of the matrix of our characteristic equation is of rank 4.

I. 5 pts — all distinct
10 lines
10 planes
5 spaces

II. 4 pts — one double pt, 3 single
7 lines
7 planes
4 spaces

III. 3 points — 2 double pts a + b, one single pt.
5 lines
5 planes
3 spaces

IV 2 pts — one double pt. a, one triple pt b,
3 lines
3 planes
2 spaces

V
3 pts. — one triple pt a, 2 single pts
4 lines
4 planes
3 spaces

VI. 2 pts. — one quad pt, 1 single pt.
2 lines
2 planes
2 spaces

VII 1 pt.
1 line
1 plane
1 space

3 single roots, 1 double root matrix rank 3.

VIII 3 + ∞ pts — 3 single pt. 1 range.
4 + 3 ∞ lines — 1 line containing range
of pt pts, 3 lines, l_1 l_2 l_3 axes
of pencils of planes.
~~4 + 3 ∞ planes~~ 3 plane pencils of lines
4 + 3 ∞ planes — 1 planes $l_1 l_2$, axis
of pencil of spaces. 3 planes π_1,
π_2, π_3, containing pencils
of lines. 3 pencils of planes
through l_1, l_2, + l_3,
3 + ∞ spaces — $l_1 l_4$, $l_2 l_4$, $l_3 l_4$.
pencil of spaces through $l_1 l_2$.

IX Two double roots rank 3, one single root.

$1 + 2\infty$ pts. — 1 single pt. a, 2 ranges.

$2 + 2\infty + \infty^2$ lines — 2 lines $l_1\ l_2$ containing ranges of pts, and pencils of lines. 2∞ lines from a, ∞^2 lines joining points on l_1 & l_2.

$2 + 2\infty + \infty^2$ planes — 2 planes π_1, π_2, axes of pencils of spaces, and containing pencils of lines. 2∞ planes through l_1 & l_2, ∞^2 planes through lines of π_1 and π_2.

$1 + 2\infty$ spaces — 1 single space $l_1 l_2$, 2 pencils of spaces through π_1 & π_2.

X One double root rank 4, one rank 3, one single root.

$\infty + 2$ pts — 1 single pt a, 1 double pt b, 1 range

$3 + 2\infty$ lines — l_1 & l_2 containing pencils of planes
l_3 containing ranges of pts.
2 pencils of lines in π_2 & π_3.

$3 + 2\infty$ planes π_2 & π_3 containing pencils of lines
π_1 axis of pencil of spaces
2 pencils of planes on l_1 and l_2

$2 + \infty$ spaces — $l_1 l_3$ & $l_2 l_3$, pencil of spaces through π_1

XI. One double root rank 3 one triple root rank 4.

1 pt + ∞ pts — triple pt a, range of pts.

2 + ∞ lines — l_1 axis of pencil of planes, l_2 containing range of pts, pencil of ∞ lines in π_2

2 + ∞ planes — π_2 containing pencil of pts & lines

π_1 axis of pencil of spaces.

pencil of planes through l_1 ~~π_1~~

1 + ∞ spaces — $l_1 l_2$ and pencil through π_1

XII. One double root rank 4, triple root rank 3.

1 + ∞ pts — 1 double pt a, one range.

2 + 2∞ lines — l_1 axis of pencil of planes, l_2 containing range of pts. 2 plane pencils in π_1 and π_2.

2 + 2∞ planes — π_1 axis of pencil of spaces, and containing pencil of ~~space~~ lines,

π_2 containing pencil of lines

2 pencils of planes through l_1, and ab.

1 + ∞ spaces — $l_1 l_2$, and pencil through π_1

XIII One double root rank ~~3~~ 1, triple root rank 2.

$1. + \infty^2$ pts — ~~single~~ double pt a and field of pts Π_1

$1 + 2\infty^2$ lines — axis of pencil of spaces and pencils of planes, ∞^2 lines on a, and ∞^2 lines in Π_1.

$1 + 2\infty^2$ planes — 1 plane Π_1 containing ∞^2 lines and ∞^2 points., ∞^2 planes by a with lines in Π, ∞^2 planes by l_1 with lines through a.

$1 + \infty^2$ spaces — single space a Π_1, ∞^2 spaces through l_1.

XIV 1 double root rank 3, 1 triple root rank 2.

$\infty + \infty^2$ pts — 1 range, 1 plane of mult pts.

$\infty^3 + \infty^2$ lines — ∞^3 lines from l_1 to Π_1, ∞^2 lines in Π_1,

$\infty^2 + \infty^3$ planes — ∞^2 planes by l_1 and pts in Π_1, ∞^3 planes by l_1 and lines from l_1 to Π_1,

$\infty + \infty^2$ spaces — ∞ spaces through Π_1, ∞^2 spaces through l_1, XV One triple root rank 3, 2 single roots,

3 + ∞ pts — 2 single pts a, & b, and 1 pt of l_2 1 range.

4 + 3 ∞ lines — l, axis of pencil of planes, l_2 and bc same, l_2 range of pts., 3 pencils in Π_1, Π_2, Π_3

4 + 3 ∞ planes — Π_1, Π_2, Π_3 containing pencils of lines. abc, axis of pencil of spaces.

3 + ∞ spaces — $a\Pi_3$, $b\Pi_3$, l, l_2, and pencil through abc.

XVI 1 Triple root rank 2, two single roots.

$2 + \infty^2$ pts — 2 single pts ∞^2 pt in Π

$1 + \infty^2 + 2\infty^2$ lines — l the axis of pencil of ∞^2 spaces and axis of plane pencils. ∞^2 lines in Π, and $2\infty^2$ lines through a & b.

$1 + \infty^2 + 2\infty^2$ planes — Π containing ∞^2 invariant pts ∞^2 planes by l and pts of Π, $2\infty^2$ planes by l and lines through a and b.

$2 + \infty^2$ spaces — $a\Pi$, $b\Pi$, and ∞^2 spaces through l.

XVII Triple root rank 3, double root rank 3.

$1 + 2\infty$ pts – 2 ranges l_1 and l_2 and 1 pt of inter.

$2 + 2\infty + \infty^2$ lines – l_1 and l_2 ranges of pts., a pencil in π_1 and in π_2, ∞^2 lines from range l_1 to l_2.

$2 + 2\infty + \infty^2$ planes π_1, π_2, containing pencils of lines. a pencil through l_1 and through l_2. ∞^2 planes through lines joining ~~l_1 to~~ l_2 ~~a to~~ l_2, ~~and~~ b to l_1.

$1 + 2\infty$ spaces – a π_2 and b π_1, pencil of spaces through π_1 and 1 through π_2

XVIII – Quadruple rank 3, single root

$1 + \infty$ pts – 1 single pt a, ~~1 point of range~~ l_1, range l_1.

$1 + \infty + \infty^2$ lines – l_1 containing range of pts.

∞ lines in π_1 each being axis of a pencil of planes.

∞ lines in each of the ∞ planes through l_1

$1 + \infty + \infty^2$ planes – π_1 containing a pencil of lines and being axis of pencil of spaces.

∞ planes through l_1 each containing a pencil of lines.

∞^2 planes through lines of π_1

$1 + \infty$ spaces – the pencil of spaces being through π_1

XIX Quadruple root rank 2, single root.

$1 + \infty^2$ pts - single pt a, ∞^2 pts in Π,

$1 + \infty^2 + \infty^2$ lines - l, axis of pencil of ∞^2 spaces,
∞^2 lines through a, ∞^2 in Π,

$1 + \infty^2 + \infty^2$ planes - Π, containing ∞^2 pts.
∞^2 planes by a and ~~pts~~ lines of Π,
∞^2 " " l, and lines through a.

$1 + \infty^2$ spaces - $a\,\Pi$, , and ∞^2 spaces through l..

XX. quadruple root rank 1, single root.

$1 + \infty^3$ pts - ~~quadruple~~ single pt a, ∞^3 pts in S.
$\infty^3 + \infty^4$ ~~p~~lines - ∞^3 lines through a, ∞^4 lines in S.
$\infty^4 + \infty^3$ planes - ∞^4 through a, ∞^3 in S.
$1 + \infty^3$ spaces - S, and ∞^3 spaces through a.

XXI Quintuple root rank 3
∞ pts on l_1
$3 + 2\infty$ lines: l_2 + x each containing
∞ planes l_1 being range of pts.
$3 + 2\infty$ planes Π_3 Π_2 containing pencils
of lines. Π_1 axis of ∞ spaces
∞ spaces through Π_1

XXII quintuple root rank 3.

One range of pts l_1

$\infty + 2$ lines. l_1 containing range of pts l_2 containing pencil of spaces ∞ lines in Π_1

$\infty + 2$ planes, Π_1 containing pencil of lines

Π_2 containing pencil of spaces

∞ planes through l_2.

1 pencil of spaces through Π_2

XXIII quintuple root rank 3.

1 range of pts l_1

$1 + \infty + \infty^2$ lines. ∞ lines in ~~p~~ Π, axes of ∞^2 planes. ∞ lines in ∞ planes through l_1

$1 + \infty + \infty^2$ planes: Π, axis of space pencil, ∞ planes through l_1 and ∞ planes through ~~xxxx.~~ line of Π.

1 pencil of spaces through Π.

XXIV quintuple root rank 2.

∞^2 pts in Π

$1 + \infty^2 + \infty^3$ lines: ∞^2 lines in Π, ∞^3 lines through a, 1 line l containing ∞^2 spaces.

$1 + \infty^2 + \infty^3$ planes: Π containing ∞^2 lines ∞^2 planes through l', ∞^3 planes through lines in Π and lines through a.

∞^2 spaces through ~~Π~~ l

XXV Quintuple root, rank 2.

∞^2 pts in Π
$1+\infty^2+\infty^3$ lines: l containing ∞^2 spaces
∞^2 in Π and ∞^3 in pencil through a
$1+\infty^2+\infty^3$ planes: Π containing ∞^2 pts.
∞^2 planes through l, ∞^3 through lines of
Π through a and pencil.
∞^2 spaces through l.

XXVI Quintuple root rank 1

$1+\infty^3$ pts: ∞^3 in space, 1 pt with ∞^3 spaces.
$\infty^4+\infty^3$ lines: ∞^4 lines in S, ∞^3 in pencil.
$\infty^4+\infty^3$ planes ∞^3 in S, ∞^4 determined
by lines of pencil and pts in space.
$1+\infty^3$ spaces, S containing ∞^3 pts
∞^3 spaces through pt. a.

XXVII Identity. leaving all points of the R_4 invariant. matrix rank 0.