A Review of Jacobi's Theoriae Functionum Ellipticarum

by Edward George Fischer

1912

Submitted to the Graduate School of the University of Kansas in partial fulfillment of the requirements for the Degree of Mathematics.

A Review of Jacobi's

"THEORIAE FUNCTIONUM ELLIPTICARUM".

Edward Fischer.

1913

Jacobi introduces the subject of elliptic transformations by stating the work already done in the field and what he expects to accomplish. The first type of elliptic integral $\int \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$ had already received considerable attention. Abel and Legendre investigated the change in value for various values of k and ϕ, and by comparison of these values endeavored to obtain a general relation. Abel also by such comparison of values worked on the addition and multiplication of the integrals. Jacobi accepted the results obtained by them and used them for his elliptic formulas.

Theproblem which confronted Jacobi was to obtain a general value of y as " a rational integral function of x so that

$$\frac{dy}{\sqrt{A'+B'y+C'y^2+D'y^3+E'y^4}} = \frac{dx}{\sqrt{A+Bx+Cx^2+Dx^3+Ex^4}}$$

That is the tranformation was to be of the form:

$$y = \frac{a + a'x + a''x^2 + \cdots\cdots + a^{(n)}x^n}{b + b'x + b''x^2 + \cdots\cdots + b^{(n)}x^n}$$

the coefficients a,a',a'',........,b,b',b''........... to be determined. Each different transformation would determine a different relation between the moduli (k).

The second degree transformation $y = \frac{a + a'x + a''x^2}{b + b'x + b''x^2}$ was already well known and Legendre had been able to work out a similar transformation of the third degree, and by combinations and repeated applications of these two substitutions a substitution of any degree which was product of a power of three by a power of two could be obtained. Jacobi's problem was to prove that such a substitution existed, whatever the degree of the transformation.

-The Beginnings of the Transformation.-

In this section it is shown that by the general substitution $y = \frac{U}{V}$ where U and V are rational integral functions of x the transformation may be so determined that

$$\frac{dy}{\sqrt{A' + B'y + C'y^2 + D'y^3 + E'y^4}} \text{ changes to } \frac{dx}{M\sqrt{A + Bx + Cx^2 + Dx^3 + Ex^4}}$$

(where M is a rational function of x). Later it is proved that M is a constant.

If we make the substitution $y = \frac{U}{V}$ the given expression becomes

$$\frac{V\,dU - U\,dV}{\sqrt{A'V^4 + B'V^3U + C'V^2U^2 + D'VU^3 + E'V^4}} \quad \text{since } dy = \frac{VdU - UdV}{V^2}$$

We must find out whether it is possible to so determined U and V that all but four of the factors under the radical will be equal in pairs so that they will come out of the radical.

Let the degree of U be p and of V be m so that $m \not> p$

The degree under the radical would be 4p. If all but four factors are to be in pairs it means that conditions will be imposed on 2p-2 factors, and if 2p-2 conditions are fulfilled it must be possible to obtain that many equations between the coefficients in U and V. The number of coefficients including the constant term in U will be p + 1 and the number in V will be m + 1 . The total number then will be m+p+2 but one of the constants may be determined as 1 without changing $\frac{U}{V}$, leaving m+p+1 undertermined constants. Since the number of conditions to be fulfilled is 2p-2, m might be equal to any one of the numbers p-3, p-2, p-1 and p, giving the numbers of undertermined coefficients as 2p-2, 2p-1, 2p, and 2p+1 respectively. The first two cases are untenable for when the functions U and V are found which are consistent with the form prescribed for the quanity under the radical, if we substitute in place of x, $\alpha + \beta x$ where α and β are arbitrary constants the degrees of U and V are not changed at all, nor is the number of duplicate factors under the radical changed. Since these are the only conditions imposed upon the expression we must take two more conditions to allow for the introduction of these arbitrary constants, and m must be either equal to p-1 or p. The former case may be reduced to the latter by the substitution in place of x $\frac{\alpha + \beta x}{1 + \gamma x}$. This substitution would make the degree of U the same as that of V if m+1 equals p but would not change the degree ofV; and it allows for the introduction of three constants; $\alpha, \beta,$ and γ . From these arguments we can draw the general conclusion that the form

$$y = \frac{a + a'x + a''x^2 + \cdots\cdots + a^{(p)}x^p}{b + b'x + b''x^2 + \cdots\cdots + b^{(p)}x^p}$$

whatever number p is, may be so determined that

$$\frac{dy}{\sqrt{A' + B'y + C'y^2 + D'y^3 + E'y^4}} = \frac{dx}{M\sqrt{A + Bx + Cx^2 + Dx^3 + Ex^4}}$$

Where M is a rational function of x, and that three of the coefficients of the x's may be arbitrarily chosen.

- To Determine M -

For the cases mentioned before where p is of the form $2^\alpha 3^\beta (2m+1)^2$ M has already been determined as a constant. We will now show that under the above general substitution M is still a constant.

For convenience let $\sqrt{A'V^4 + B'V^3U + C'V^2U^2 + D'VU^3 + E'U^4} = \sqrt{Y}$

We have shown just above that Y by the type of substitution used may be reduced to $(A + Bx + Cx^2 + Dx^3 + Ex^4)T^2$ where T is a rational function of x, and also we know that

$$\frac{VdU - UdV}{\sqrt{Y}\,(= T\sqrt{A + Bx + Cx^2 + Dx^3 + Ex^4})} = \frac{dx}{M\sqrt{A + Bx + Cx^2 + Dx^3 + Ex^4}}$$

From this we get $M = \dfrac{T\,dx}{VdU - UdV} = \dfrac{T}{V\frac{dU}{dx} - U\frac{dV}{dx}}$

The degree of T is 2p-2 and $V\frac{dU}{dx} - U\frac{dV}{dx}$ cannot be of greater degree since M is a rational ~~integral~~ function of x.

We may suppose U and V to have no common factor since if one existed we could remove it without altering the value of y in the substitution. Let the expression $A' + B'y + C'y^2 + D'y^3 + E'y^4$ be resolved into its linear factors $A'(1-\alpha' y)(1-\beta' y)(1-\gamma' y)(1-\delta' y)$ whence

$$Y = A'V^4 + B'V^3U + C'V^2U^2 + D'VU^3 + E'U^4 = A'(V-\alpha'U)(V-\beta'U)(V-\gamma'U)(V-\delta'U)$$

There exists no factor which is common to two of the quantities $(V-\alpha'U)$, $V-\beta'U$, $V-\gamma'U$, $V-\delta'U$ since it would then divide both U and V which is contrary to the supposition made that they had no common factor. And so when Y is divided by some pair of equal linear factors it is necessary that one of the quantities $V-\alpha'U$, $V-\beta'U$, $V-\gamma'U$, $V-\delta'U$ also contain it.

By the following equations:

$$(1)\ (V-\alpha'U)\frac{dU}{dx} - \frac{d(V-\alpha'U)}{dx}U = V\frac{dU}{dx} - U\frac{dV}{dx}$$

$$(2)\ (V-\beta'U)\frac{dU}{dx} - \frac{d(V-\beta'U)}{dx}U = V\frac{dU}{dx} - U\frac{dV}{dx}$$

$$(3)\ (V-\gamma'U)\frac{dU}{dx} - \frac{d(V-\gamma'U)}{dx}U = V\frac{dU}{dx} - U\frac{dV}{dx}$$

$$(4)\ (V-\delta'U)\frac{dU}{dx} - \frac{d(V-\delta'U)}{dx}U = V\frac{dU}{dx} - U\frac{dV}{dx}$$

Which are obtained by the addition and subtraction of $\alpha'U\frac{dU}{dx}$, etc. from the right hand member.

it follows that a factor which divides any of the four factors of Y and its differential divides also the quantity $V\frac{dU}{dx} - U\frac{dV}{dx}$. Since no two of these four quantities contain common factors both members of any pair of factors of Y must be in some one of these four quantities. If a factor is contained twice in a quantity it is also contained in its first derivative. Since the product of all these factors occuring twice in Y has been assumed to be T, T itself divides $V\frac{dU}{dx} - U\frac{dV}{dx}$ but we have seen before that the rank of T could not be less than that of $V\frac{dU}{dx} - U\frac{dV}{dx}$ whence it is evident that M is a constant for $M = \dfrac{T}{V\frac{dU}{dx} - U\frac{dV}{dx}}$ and M could then be put under the radical without increasing the degree of the radical. We can also see from this result a fact previously derived in a different way that the degree of V must be at least as large as p-1 since if it were less the degree of $V\frac{dU}{dx} - U\frac{dV}{dx}$ would be not greater than 2p-3 which would make the degree less than that of T (which is 2p-2).

We have now proven "the fundamental theorem in the theory of transformation of elliptic functions" namely:

It is possible so to determine the form

$$y = \frac{a + a'x + \cdots\cdots + a^{(p)}x^p}{b + b'x + \qquad + b^{(p)}x^p}$$ whatever number p is

~~so~~ that

$$\frac{dy}{\sqrt{A' + B'y + C'y^2 + D'y^3 + E'y^4}} = \frac{dx}{\sqrt{A + Bx + Cx^2 + Dx^3 + Ex^4}}$$

- - - - - - - - - - - - - - -

To reduce the expression $\dfrac{dy}{\sqrt{\pm(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)}}$ into the more simple form $\dfrac{dx}{M\sqrt{(1-x^2)(1-k^2x^2)}}$ by means of the substitution $y = \dfrac{a + a'x + a''x^2}{b + b'x + b''x^2}$

We must determine the modulus k and constant M in terms of the given constants $\alpha, \beta, \gamma, \delta$ since in this case p is 2 when we place

$$a + a'x + a''x^2 = U \quad \text{and}$$
$$b + b'x + b''x^2 = V$$

$(U - \alpha V)(U - \beta V)(U - \gamma V)(U - \delta V)$ may be made to equal

$$\overline{K}(1-x^2)(1-k^2x^2)(1+nx)^2(1+nx)^2.$$

(Designating by $\overline{K}$ an arbitrary constant).

We have seen that two of the factors $U-\alpha V$, $U-\beta V$, $U-\gamma V$, $U-\delta V$ would have to be perfect squares. Then let

$$U-\gamma V = C(1+mx)^2 \text{ and } U-\delta V = D(1+nx)^2$$

For the other two factors $U-\alpha V$ and $U-\beta V$ apparently either of two cases would be possible.

$$\text{1st}\begin{cases} U-\alpha V = A(1-x^2) \\ U-\beta V = B(1-\kappa^2x^2) \end{cases} \text{ or } \text{2nd}\begin{cases} U-\alpha V = A(1-x)(1-\kappa x) \\ U-\beta V = B(1+x)(1+\kappa x) \end{cases}$$

(where A,B,C & D are constants)

but the 1st must be rejected for that would make

$$\frac{U-\alpha V}{U-\beta V} = \frac{y-\alpha}{y-\beta} = \frac{A(1-x^2)}{B(1-\kappa^2x^2)}$$

which would leave y unchanged when x changes to -x. This is shown to be absurd from these equations

$$\frac{U-\alpha V}{U-\gamma V} = \frac{y-\alpha}{y-\gamma} = \frac{A}{C}\,\frac{1-x^2}{(1+mx)^2} \text{ and}$$

$$\frac{U-\alpha V}{U-\delta V} = \frac{y-\alpha}{y-\delta} = \frac{A}{D}\,\frac{1-x^2}{(1+nx)^2}$$

in which the value of y does change for the change of x into -x. Therefore we will use the following equations:

(1) $U-\alpha V = A(1-x)(1-\kappa x)$

(2) $U-\beta V = B(1+x)(1+\kappa x)$

(3) $U-\gamma V = C(1+mx)^2$

(4) $U-\delta V = D(1+nx)^2$

It may be noted that any one of the constants A,B,C,D may be arbitrarily chosen.

If in (I) we set $x = 1$ or $\frac{1}{\sqrt{k}}$ U becomes equal to αV

Dividing equation (3) by (2)

$$\frac{U-\gamma V}{U-\beta V} = \frac{C}{B}\,\frac{(1+mx)^2}{(1+x)(1+kx)}$$

Letting $x = 1$ this becomes

$$\frac{\alpha V-\gamma V}{\alpha V-\beta V} = \frac{\alpha-\gamma}{\alpha-\beta} = \frac{C}{B}\,\frac{(1+m)^2}{2(1+k)}$$

Letting $x = \frac{1}{\sqrt{k}}$ the quotient becomes

$$\frac{\alpha-\gamma}{\alpha-\beta} = \frac{C}{B}\,\frac{\left(1+\frac{m}{k}\right)^2}{2\left(1+\frac{1}{k}\right)} = \frac{C}{B}\,\frac{k\left(1+\frac{m}{k}\right)^2}{2(1+k)}$$

Whence $(1+m)^2 = k\left(1+\frac{m}{k}\right)^2$

or $1 + 2m + m^2 = k + 2m + \frac{m^2}{k}$

$$m^2\left(1-\frac{1}{k}\right) = k-1$$

$$m^2 = \frac{k-1}{\frac{k-1}{k}} = k$$

$$m = \pm\sqrt{k}$$

Similarly

$$(1+n)^2 = k\left(1+\frac{n}{k}\right)^2$$

$$n = \pm\sqrt{k}$$

If we select the positive value for m we must take the negative value for n since if $m = n$

Substituting these values of m and n in this equation we get

$$\frac{U-\gamma V}{U-\delta V}=\frac{y-\gamma}{y-\delta}=\frac{C}{D}\left(\frac{1+\sqrt{K}\cdot x}{1-\sqrt{K}\cdot x}\right)^2$$

If we let x= + 1 we get from (1)U= V whence

$$\text{(I)}\qquad \frac{\alpha-\gamma}{\alpha-\delta}=\frac{C}{D}\left\{\frac{1+\sqrt{K}}{1-\sqrt{K}}\right\}^2$$

If we let x := -1 we get from (2) U= V whence

$$\text{(II)}\qquad \frac{\beta-\gamma}{\beta-\delta}=\frac{C}{D}\left[\frac{1-\sqrt{K}}{1+\sqrt{K}}\right]^2$$

By multiplying equations (I) & (II) together we get

$$\frac{C}{D}=\sqrt{\frac{(\alpha-\gamma)(\beta-\gamma)}{(\alpha-\delta)(\beta-\delta)}}$$

We may put C= $\sqrt{(\alpha-\gamma)(\beta-\gamma)}$ for we could determine one of the quantities A,B,C & D arbitrarily and in that case D will equal $\sqrt{(\alpha-\delta)(\beta-\delta)}$ By dividing (I) by (II) we obtain

$$\frac{1+\sqrt{K}}{1-\sqrt{K}}=\sqrt[4]{\frac{(\alpha-\gamma)(\beta-\delta)}{(\alpha-\delta)(\beta-\gamma)}}$$

whence

$$\sqrt{K}\left(\sqrt[4]{(\alpha-\delta)(\beta-\gamma)}+\sqrt[4]{(\alpha-\gamma)(\beta-\delta)}\right)=\sqrt[4]{(\alpha-\gamma)(\beta-\delta)}-\sqrt[4]{(\alpha-\delta)(\beta-\gamma)}$$

or

$$\sqrt{K}=\frac{\sqrt[4]{(\alpha-\gamma)(\beta-\delta)}-\sqrt[4]{(\alpha-\delta)(\beta-\gamma)}}{\sqrt[4]{(\alpha-\gamma)(\beta-\delta)}+\sqrt[4]{(\alpha-\delta)(\beta-\gamma)}}$$

To determine A & B let $x=\frac{1}{\sqrt{k}}$ in (1),(2),(3),(4) then from (3)

$$U = \delta V$$

From (1) $U-\alpha V = \delta V - \alpha V = A\left(1-\frac{1}{\sqrt{K}}\right)\left(1-\sqrt{K}\right)$

From (2) $U-\beta V = \delta V - \beta V = B\left(1+\frac{1}{\sqrt{K}}\right)\left(1+\sqrt{K}\right)$

From (3) $U-\gamma V = \delta V - \gamma V = C(1+1)^2 = 4C$.

Then $$\frac{\delta-\alpha}{\delta-\gamma} = \frac{A\left(1-\frac{1}{\sqrt{K}}\right)\left(1-\sqrt{K}\right)}{4C} = \frac{A\left(1-\frac{1}{K}\right)\left(1-\sqrt{K}\right)}{4\sqrt{(\alpha-\gamma)(\beta-\gamma)}} \quad (a)$$

And $$\frac{\delta-\beta}{\delta-\gamma} = \frac{B\left(1+\sqrt{K}\right)\left(1+\frac{1}{\sqrt{K}}\right)}{4\sqrt{(\alpha-\gamma)(\beta-\gamma)}} \quad (b)$$

But $$\left(1-\frac{1}{\sqrt{K}}\right)\left(1-\sqrt{K}\right) = 1-\frac{1}{\sqrt{K}}-\sqrt{K}+1 = 2-\left(\sqrt{K}+\frac{1}{\sqrt{K}}\right)$$

$$= 2 - 2\,\frac{\sqrt{(\alpha-\gamma)(\beta-\delta)}+\sqrt{(\alpha-\delta)(\beta-\gamma)}}{\sqrt{(\alpha-\gamma)(\beta-\delta)}-\sqrt{(\alpha-\delta)(\beta-\gamma)}}$$

$$= \frac{-4\sqrt{(\alpha-\delta)(\beta-\gamma)}}{\sqrt{(\alpha-\gamma)(\beta-\delta)}-\sqrt{(\alpha-\delta)(\beta-\gamma)}}$$

Similarly
$$\left(1+\frac{1}{\sqrt{K}}\right)\left(1+\sqrt{K}\right) = 2+\left(\sqrt{K}+\frac{1}{\sqrt{K}}\right) = \frac{+4\sqrt{(\alpha-\gamma)(\beta-\delta)}}{\sqrt{(\alpha-\gamma)(\beta-\delta)}-\sqrt{(\alpha-\delta)(\beta-\gamma)}}$$

Substituting these values in (a) & (b) and solving for A & B we obtain -

$$A = -\frac{\sqrt{(\alpha-\gamma)(\alpha-\delta)}}{\gamma-\delta}\left(\sqrt{(\alpha-\gamma)(\beta-\delta)}-\sqrt{(\alpha-\delta)(\beta-\gamma)}\right)$$

and $$B = \frac{\sqrt{(\beta-\gamma)(\beta-\delta)}}{\gamma-\delta}\left(\sqrt{(\alpha-\gamma)(\beta-\delta)}-\sqrt{(\alpha-\delta)(\beta-\gamma)}\right)$$

Which give us the values of the unknown constants A,B,C,D,& k

To Transform the Expression $\frac{dy}{\sqrt{(1-y^2)(1-\lambda^2y^2)}}$ into $\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$

It has already been shown that a substitution of the type

$$y = \frac{a + a'x + a''x^2 + \ldots + a^{(p)}x^p}{b + b'x + b''x^2 + \ldots + b^p x^p} = \frac{U}{V}$$

will transform $\frac{dy}{\sqrt{A' + B'y + C'y^2 + D'y^3 + E'y^4}}$ into $\frac{dx}{\sqrt{A + Bx + Cx^2 + Dx^3 + Ex^4}}$
We will now find the transformation which will change the more simple form $\frac{dy}{\sqrt{(1-y^2)(1-\lambda^2y^2)}}$ into the form $\frac{dx}{M\sqrt{(1-x^2)(1-k^2x^2)}}$. The value of k and M depend on the degree of the transformation as well as upon the given coefficients A',B',C',D', & E'.

There are two types of transformations which will satisfy the requirements; one in which the denominator is of 1 larger degree than the numerator (Jacobi calls this a transformation of even degree) and one in which the degree of the numerator exeeeds that of the denominator by 1 (or odd degree). In both cases the terms of the denominator contain only even powers of x, and the numerator only odd powers.

In the transformation of even degree y has the form

$$\frac{x(a + a'x^2 + a''x^4 + \ldots + a^{(m-1)}x^{2m-2})}{1 + b'x^2 + b''x^4 + \ldots + b^{m-1}x^{2m-2} + b^m x^{2m}}$$

The quantities V + U, V-U, V + U, V- U are all of even degree (since the denominator V is of higher degree than the numerator) and we may place

$$(1)\ V + U = (1 + x)(1+kx)\,A^2$$
$$(2)\ V - U = (1 - x)(1-kx)\,B^2$$
$$(3)\ V + \lambda U = C^2$$
$$(4)\ V - \lambda U = D^2$$

(A,B,C, & D are rational integral functions of x)

It is necessary to be able to determine A,B,C,D and k from the equations between the coefficients. Since V is unaltered when x changes sign, but U is changed to -U equation (2) will follow immediately, from (1) and (4) from (3). So equations (1) and(3) are all that may be used for the determination of these quantities. If it is possible to determine A,B,C & D as rational functions of x our problem is solved for this case, since

$$\frac{dy}{\sqrt{(1-y^2)(1-\lambda^2y^2)}} = \frac{V\,dU - U\,dV}{\sqrt{(V^2-U^2)(V^2-\lambda U^2)}}$$

The numerator is already a rational function of x and the denominator will have the form ABCD $\sqrt{(1-x^2)(1-k^2x^2)}$

WE have imposed on $V + \lambda U$ the condition that it have m pairs of equal linear factors, and upon V+U that it have m-1 pairs of equal factors, and also the factor (1 + x), giving in all 2m conditions to be fulfilled but this also is the number of undetermined coefficients in y; which shows that the problem may be solved.

The substitution of odd degree is shown to be possible in a similar manner. Here y has the form

$$\frac{x(a + a'x^2 + a''x^4 + \dots + a^{(m)}x^{2m})}{1 + b'x^2 + b''x^4 + \dots + b^{(m)}x^{2m}}$$

In this case $V + U$, $V - U$, $V + \lambda U$, $V - \lambda U$ are of odd degree and so we place

(1) $V + U = (1 + x)A^2$

(2) $V - U = (1 - x)B^2$

(3) $V + U = (1 + kx)C^2$

(4) $V - U = (1 - kx)D^2$

As before equations 1 and 3 are all that must be satisfied. Each of these requires conditions imposed upon m linear factors, and that one of the factors of $V + U$ be $(1 + x)$, making altogether $2m + 1$ conditions which again is just the number of undetermined coefficients.