# Continuous Groups of Projective Transformations in Two Dimensions

by Fred Keplinger

*1904*

Submitted to the Department of Mathematics of the University of Kansas in partial fulfillment of the requirements for the Degree of Master of Arts

Master Thesis

Mathematics

Keplinger, Fred 1904

"Continuous groups of projective transformations in two dimensions."

# Thesis



Continuous Groups of Projective Transformations in two Dimensions.

Fred Keplinger. 1904

In one dimensional transformations, by treating each complex parameter as a double parameter[1], Prof. Newson found several new groups. It is the purpose of this paper to attempt the same with transformations in two dimensions. In that paper it is shown that the relation $k_2 = kk_1$, which is equivalent to two parameters in the complex plane, by putting $k$ in the form $e^{(c+i)\theta}$ and making $c$ constant, represents only one parameter, and the group property is not destroyed. Likewise the relation $t_2 = t + t_1$ can be written $r_2 e^{i\theta_2} = re^{i\theta} + r_1 e^{i\theta_1}$ and $\theta$ made constant without disturbing the group property. In this way wherever either of these relations hold in two-

---

(1) "Projective Transformations in one Dimension and their Lie Groups", K. U. Science Bulletin, Vol. I No. 5.

dimensional transformations a new group is found, and where both these relations hold there are three new subgroups. By picking out the 'groups in which one or more of these relations exist,'[1] a list of more than 90 groups can readily be formed.

In the case of one-dimensional transformations, after this method was exhausted, three other groups were found by means of the path curves and invariant figures in the complex plane. It is certain that other groups also exist in two-dimensional transformations, but in this case a similar method for determining them, requires the consideration of path curves and invariant figures in four dimensions, which, if not impossible, is at least very difficult.

For this reason it is thought best to undertake this investigation by Lie's method of infinitesimal transformations. In order

1. See "A new Theory of Collineations and their Lie Groups". American Journal of Mathematics Vol. XXIV pp. 109-170

to test the adaquacy of the method and its adaptability to the problem in hand, and our ability to use it, we will first apply it to the known problem of one dimensional transformations.

The infinitesimal projective transformation in one dimension is represented by the equation[1]:

$$dx = (\alpha + \beta x + \gamma x^2)\, dt.$$

In this equation all of the quantities are in general complex. Writing them in the complex form: i.e. putting $x + iy$ for $x$, $a + ib$ for $\alpha$, $c + id$ for $\beta$, and $e + if$ for $\gamma$, the equation becomes:-

$$dx + i\,dy = [a + ib + (c+id)(x+iy) + (e+if)(x+iy)^2]\, dt$$

or $dx + i\,dy = [a + ib + cx + idx + ciy - dy + ex^2 + 2iexy - ey^2 + ifx^2 + 2fxy - ify^2]\, dt.$

Separating reals and imaginaries,

$$dx = [a + cx - dy + ex^2 - ey^2 - 2fxy]\, dt.$$

$$dy = [b + dx + cy + 2exy + fx^2 - fy^2]\, dt.$$

Putting $\frac{df}{dx} = p$ and $\frac{df}{dy} = q$ The infinitesimal transformation is,

---

(1) Lie "Continuierliche Gruppe" P. 119.

$Uf = [a + cx - dy + ex^2 + ey^2 - 2fxy]p + [b + dx + cy + 2exy + fx^2 + fy^2]q$

The coefficients of the constants a, b, c, etc are the infinitesimal transformations of which the general group is composed. They are:-

$$p,\ q,\ xp + yq,\ -yp + xq,\ (x^2 - y^2)p + 2xyq,\ -2xyp + (x^2 - y^2)q.$$

We will now prove the group property and determine the various sub-groups by applying Lie's[(1)] Hauptsatz: "r independent infinitesimal transformations, $U_1, U_2, U_3 \cdots U_r$, generate an r-parameter group, when, and only when each [(2)]Klammerausdruck $(U_iU_k)$ is a linear combination of $U_1, U_2, \cdots U_r$.

Let $U_1 = p$, $U_2 = q$, $U_3 = xp + yq$, $U_4 = -yp + xq$, $U_5 = (x^2 - y^2)p + 2xyq$, $U_6 = -2xyp + (x^2 - y^2)q$.

The Klammerausdrücken are as follows:-

| | | | | |
|---|---|---|---|---|
| $U_1U_2 = 0$ | $U_2U_3 = U_2$ | $U_3U_4 = 0$ | $U_4U_5 = U_6$ | $U_5U_6 = 0$ |
| $U_1U_3 = U_1$ | $U_2U_4 = U_1$ | $U_3U_5 = U_5$ | $U_4U_6 = U_5$ | |
| $U_1U_4 = U_2$ | $U_2U_5 = 2U_4$ | $U_3U_6 = U_5$ | | |
| $U_1U_5 = 2U_3$ | $U_2U_6 = 2U_3$ | | | |
| $U_1U_6 = 2U_4$ | | | | |

(1) Continuierliche Gruppen, P 211. (2) PP. 37-38.

From this table it is seen that the following combinations form groups:-

$U_1 U_2 U_3 U_4 U_5 U_6$

| | |
|---|---|
| $U_1 U_2 U_3 U_4$ | $U_1 U_2$ |
| $U_1 U_2 U_3$ | $U_1 U_3$ |
| $U_1 U_2 U_4$ | $U_2 U_3$ |
| $U_1 U_3 U_5$ | $U_3 U_4$ |
| $U_3 U_5 U_6$ | $U_3 U_5$ |
| $U_4 U_5 U_6$ | $U_3 U_6$ |
| | $U_5 U_6$ |

The first is evidently the group $H_6$ of Prof. Newson's table[1]. The character of the other groups may be determined by integrating the corresponding differential equations and thus finding the finite form of the transformation. The equations for the group $(U_1 U_2 U_3 U_4)$ are:-

$$\frac{dx}{a+cx-dy} = \frac{dy}{b+cy+dx} = dt$$

Multiply both terms of second ratio by $i$ and combine the two ratios, putting $x+iy=z_1$, $a+ib=\alpha$ and $c+id=\beta$. $\frac{dz_1}{\alpha+\beta z_1} = dt$. which integrated

---

(1) K. U. Sc. B. Vol. I, No. 5. pp. 141-142.

gives $\frac{1}{\beta}\log(\alpha+\beta z_1) = t + c$

and since $z_1 = z$ when $t=0$, $c = \frac{1}{\beta}\log(\alpha+\beta z)$

$$\log(\alpha+\beta z_1) - \log(\alpha+\beta z) = \beta t$$

$$\alpha+\beta z_1 = e^{\beta t}(\alpha+\beta z).$$

This transformation leaves invariant the infinitely distant point, and corresponds to $H_4(A)$ in the table above referred to.

$$K = e^{\beta t}.$$

By examining the differential equation, it is easily seen that the omission of $U_4$ from this group leaves $K$ real, while omitting $U_3$ gives $K$ pure imaginary; hence $(U_1 U_2 U_3)$ is $hH_3(A)$, and $(U_1 U_2 U_4)$ is $eH_3(A)$.

In the loxodromic group $H_3(A)_c$, $K$ has the form $e^{(c+i)\theta}$. This group is formed from $U_1 U_2$ and $(cU_3+U_4)$. That this combination forms a group is shown by forming the various Klammerausdrücken.

By integration $(U_3 U_5 U_6)$ is found to be $hH_3(A)$, and $(U_4 U_5 U_6)$ $eH_3(A)$ leaving the origin invariant instead of the infinite point. $(U_1 U_3 U_5)$ is the group $H_3(C)$.

Of the two-parameter groups $(U_1 U_2)$ and $(U_5 - U_6)$ correspond to $pH_2(A)$; $(U_1 U_3)$, $(U_2 U_3)$, $(U_3 U_{5'})$, and $(U_3 U_6)$ to $H_2(AC)$, and $(U_3 U_4)$ to $H_2(AA')$.

There are four kinds of one parameter groups,

$$(U_1) = pH_1(A)_0 \qquad (U_4) = eH_1(AA')$$
$$(U_3) = hH_1(AA') \qquad (cU_3 + U_4) = H_1(AA')_c\,.$$

There is also one more three parameter group,

$$[(U_1 + U_{5'}),\ U_4,\ (U_2 - U_6)] = H_3(iC)\,.$$

This completes the list of groups of one dimensional transformations, and as the groups here found correspond exactly to the list above referred to, which was obtained by a wholly different method, the present method is shown to be equally accurate and adaquate, and for the reasons above stated it seems better adapted to the problem under consideration.

Lie's[1] equation for the general projective infinitesimal transformation in two dimensions is:-

$$Uf \equiv (a+cx+dy+hx^2+kxy)p+(b+ex+gy+hxy+ky^2)q$$

or $dx=(a+cx+dy+hx^2+kxy)\,dt$

and $dy=(b+ex+gy+hxy+ky^2)\,dt.$

Changing to complex form, as in the previous problem, putting $x+iy$ for $x$, and $z+iw$ for $y$:

$$dx+idy=\big(a+ib+(c+id)(x+iy)+(e+if)(z+iw)+(g+ih)(x+iy)^2 + (j+ik)(x+iy)(z+iw)\big)\,dt$$

$$dz+idw=\big((l+im)+(n+io)(x+iy)+(u+iv)(z+iw)+(g+ih)(x+iy)(z+iw) + (j+ik)(z+iw)^2\big)\,dt.$$

Expanding:-

$$dx+idy=[a+ib+cx+icy+idx-dy+ez+iew+ifz-fw+g(x^2-y^2) +2igxy+ih(x^2-y^2)-2hxy+jxz+ijxw+ijyz-jyw +ikxz-kxw-kyz-ikyw]\,dt$$

$$dz+idw=[l+im+nx+iny+iox-oy+uz+iuw+ivz-vw+gxz +igxw+igyz-gyw+ihxz-hxw-hyz-ihyw +j(z^2-w^2)+2ijzw+ik(z^2-w^2)-2kzw].$$

Separating reals and imaginaries:-

---

(1) "Continuierliche Gruppe" PP. 24-25.

$$dx = [a + cx - dy + ez - fw + g(x^2 - y^2) - 2hxy - kxw - kyz + jxz - jyw]\,dt$$

$$dy = [b + cy + dx + ew + fz + 2gxy + h(x^2 - y^2) + jxw + jyz + kxz - kyw]\,dt$$

$$dz = [l + nx - oy + uz - vw + gxz - gyw - hxw - hyz - 2kzw + j(z^2 - w^2)]\,dt.$$

$$dw = [m + ny + ox + uw + vz + gxw + gyz + hxz - hyw + 2jzw + k(z^2 - w^2)]\,dt$$

The coefficients of the constants, a, b. c, etc are the infinitesimal transformations of the group.

Putting $\frac{df}{dx} = p$, $\frac{df}{dy} = q$, $\frac{df}{dz} = r$, and $\frac{df}{dw} = s$, they are:-

(1) $p$, (2) $q$, (3) $r$, (4) $s$, (5) $xp + yq$, (6) $xr + ys$, (7) $-yp + xq$, (8) $-yr + xs$, (9) $zp + wq$,

(10) $zr + ws$, (11) $-wp + zq$, (12) $-wr + zs$, (13) $(x^2 - y^2)p + 2xyq + (xz - yw)r + (xw + yz)s$,

(14) $-2xyp + (x^2 - y^2)q - (xw + yz)r + (xz - yw)s$, (15) $(xz - yw)p + (xw + yz)q + (z^2 - w^2)r + 2zws$,

(16) $-(xw + yz)p + (xz - yw)q - 2zwr + (z^2 - w^2)s$.

In the following work these are designated $U_1, U_2, U_3$, etc. in the order here given.

It is now necessary to derive a formula for Klammerausdrücken in 4 dimensions.

As before $p = \frac{df}{dx}$, $q = \frac{df}{dy}$, $r = \frac{df}{dz}$, $s = \frac{df}{ds}$. [Lie pp. 37-38]

$$U_1 f = \xi_1 p + \eta_1 q + \zeta_1 r + y_1 s$$

$$U_2 f = \xi_2 p + \eta_2 q + \zeta_2 r + y_2 s$$

$$U_1(U_2 f) = \xi_1 \frac{\partial U_2 f}{\partial x} + \eta_1 \frac{\partial U_2 f}{\partial y} + \zeta_1 \frac{\partial U_2 f}{\partial z} + y_1 \frac{\partial U_2 f}{\partial s}$$

$$= \xi_1 \left[\frac{\partial \xi_2}{\partial x} p + \xi_2 \frac{\partial p}{\partial x} + \frac{\partial \eta_2}{\partial x} q + \eta_2 \frac{\partial q}{\partial x} + \frac{\partial \zeta_2}{\partial x} r + \zeta_2 \frac{\partial r}{\partial x} + \frac{\partial y_2}{\partial x} s + y_2 \frac{\partial s}{\partial x}\right] + \eta_1 \left[\frac{\partial \xi_2}{\partial y} p + \xi_2 \frac{\partial p}{\partial y} + \eta_2 \frac{\partial q}{\partial y} + \frac{\partial \eta_2}{\partial y} q + \frac{\partial \zeta_2}{\partial y} r + \zeta_2 \frac{\partial r}{\partial y} + \frac{\partial y_2}{\partial y} s + y_2 \frac{\partial s}{\partial y}\right]$$

$$+ \zeta_1 \left[\frac{\partial \xi_2}{\partial z} p + \xi_2 \frac{\partial p}{\partial z} + \eta_2 \frac{\partial q}{\partial z} + \frac{\partial \eta_2}{\partial z} q + \frac{\partial \zeta_2}{\partial z} r + \zeta_2 \frac{\partial r}{\partial z} + \frac{\partial y_2}{\partial z} s + y_2 \frac{\partial s}{\partial z}\right]$$

$$+ y_1 \left[\frac{\partial \xi_2}{\partial w} p + \xi_2 \frac{\partial p}{\partial w} + \frac{\partial \eta_2}{\partial w} q + \eta_2 \frac{\partial q}{\partial z} + \frac{\partial \zeta_2}{\partial w} r + \zeta_2 \frac{\partial r}{\partial z} + \frac{\partial y_2}{\partial z} s + y_2 \frac{\partial s}{\partial z}\right]$$

$$U_2(U_1 f) = \xi_2 \left[\xi_1 \frac{\partial p}{\partial x} + p \frac{\partial \xi_1}{\partial x} + \eta_1 \frac{\partial q}{\partial x} + q \frac{\partial \eta_1}{\partial x} + \zeta_1 \frac{\partial r}{\partial x} + r \frac{\partial \zeta_1}{\partial x} + y_1 \frac{\partial s}{\partial x} + s \frac{\partial y_1}{\partial x}\right]$$

$$+ \eta_2 \left[\xi_1 \frac{\partial p}{\partial y} + p \frac{\partial \xi_1}{\partial y} + \eta_1 \frac{\partial q}{\partial y} + q \frac{\partial \eta_1}{\partial x} + \zeta_1 \frac{\partial r}{\partial y} + r \frac{\partial \zeta_1}{\partial y} + y_1 \frac{\partial s}{\partial y} + s \frac{\partial y_1}{\partial y}\right]$$

$$+ \zeta_2 \left[\xi_1 \frac{\partial p}{\partial z} + p \frac{\partial \xi_1}{\partial z} + \eta_1 \frac{\partial q}{\partial z} + q \frac{\partial \eta_1}{\partial z} + \zeta_1 \frac{\partial r}{\partial z} + r \frac{\partial \zeta_1}{\partial z} + y_1 \frac{\partial s}{\partial z} + s \frac{\partial y_1}{\partial z}\right]$$

$$+ y_2 \left[\xi_1 \frac{\partial p}{\partial w} + p \frac{\partial \xi_1}{\partial w} + \eta_1 \frac{\partial q}{\partial w} + q \frac{\partial \eta_1}{\partial w} + \zeta_1 \frac{\partial r}{\partial w} + r \frac{\partial \zeta_1}{\partial w} + y_1 \frac{\partial s}{\partial w} + s \frac{\partial y_1}{\partial w}\right]$$

$$U_1(U_2 f) - U_2(U_1 f) = U_1 U_2$$

Subtracting and collecting, [$\frac{\partial p}{\partial x} = \frac{\partial q}{\partial y}$ etc. from def. of p, q, etc.]

$$U_1 U_2 = \left[\xi_1 \frac{\partial \xi_2}{\partial x} + \eta_1 \frac{\partial \xi_2}{\partial y} + \zeta_1 \frac{\partial \xi_2}{\partial z} + y_1 \frac{\partial \xi_2}{\partial w} - \xi_2 \frac{\partial \xi_1}{\partial x} - \eta_2 \frac{\partial \xi_1}{\partial y} - \zeta_2 \frac{\partial \xi_1}{\partial z} - y_2 \frac{\partial \xi_1}{\partial w}\right] p$$

$$+ \left[\xi_1 \frac{\partial \eta_2}{\partial x} + \eta_1 \frac{\partial \eta_2}{\partial y} + \zeta_1 \frac{\partial \eta_2}{\partial z} + y_1 \frac{\partial \eta_2}{\partial w} - \xi_2 \frac{\partial \eta_1}{\partial x} - \eta_2 \frac{\partial \eta_1}{\partial y} - \zeta_2 \frac{\partial \eta_1}{\partial z} - y_2 \frac{\partial \eta_1}{\partial w}\right] q$$

$$+ \left[\xi_1 \frac{\partial \zeta_2}{\partial x} + \eta_1 \frac{\partial \zeta_2}{\partial y} + \zeta_1 \frac{\partial \zeta_2}{\partial z} + y_1 \frac{\partial \zeta_2}{\partial w} - \xi_2 \frac{\partial \zeta_1}{\partial x} - \eta_2 \frac{\partial \zeta_1}{\partial y} - \zeta_2 \frac{\partial \zeta_1}{\partial z} - y_2 \frac{\partial \zeta_1}{\partial w}\right] r$$

$$+ \left[\xi_1 \frac{\partial y_2}{\partial x} + \eta_1 \frac{\partial y_2}{\partial y} + \zeta_1 \frac{\partial y_2}{\partial z} + y_1 \frac{\partial y_2}{\partial w} - \xi_2 \frac{\partial y_1}{\partial x} - \eta_2 \frac{\partial y_1}{\partial y} - \zeta_2 \frac{\partial y_1}{\partial z} - y_2 \frac{\partial y_1}{\partial w}\right] s$$

# Table of Klammerausdrücken.

$U_1 U_2 = 0$
$U_1 U_3 = 0$
$U_1 U_4 = 0$
$U_1 U_5 = U_1$
$U_1 U_6 = U_3$
$U_1 U_7 = U_2$
$U_1 U_8 = U_4$
$U_1 U_9 = 0$
$U_1 U_{10} = 0$
$U_1 U_{11} = 0$
$U_1 U_{12} = 0$
$U_1 U_{13} = U_5 + U_{10}$
$U_1 U_{14} = U_{12} + 2U_7$
$U_1 U_{15} = U_9$
$U_1 U_{16} = U_{11}$

$U_2 U_3 = 0$
$U_2 U_4 = 0$
$U_2 U_5 = U_2$
$U_2 U_6 = U_4$
$U_2 U_7 = U_1$
$U_2 U_8 = U_3$
$U_2 U_9 = 0$
$U_2 U_{10} = 0$
$U_2 U_{11} = 0$
$U_2 U_{12} = 0$
$U_2 U_{13} = 2U_7 + U_{12}$
$U_2 U_{14} = U_{12} - 2U_5$
$U_2 U_{15} = U_{11}$
$U_2 U_{16} = U_9$

$U_3 U_4 = 0$
$U_3 U_5 = 0$
$U_3 U_6 = 0$
$U_3 U_7 = 0$
$U_3 U_8 = 0$
$U_3 U_9 = U_1$
$U_3 U_{10} = U_3$
$U_3 U_{11} = U_2$
$U_3 U_{12} = U_4$
$U_3 U_{13} = U_6$
$U_3 U_{14} = U_8$
$U_3 U_{15} = U_5 + 2U_{10}$
$U_3 U_{16} = U_7 + 2U_{12}$

$U_4 U_5 = 0$
$U_4 U_6 = 0$
$U_4 U_7 = 0$
$U_4 U_8 = 0$
$U_4 U_9 = U_2$
$U_4 U_{10} = U_4$
$U_4 U_{11} = U_1$
$U_4 U_{12} = U_3$
$U_4 U_{13} = U_8$
$U_4 U_{14} = U_6$
$U_4 U_{15} = U_7 + 2U_{12}$
$U_4 U_{16} = U_5 + 2U_{10}$

$U_5 U_6 = U_6$
$U_5 U_7 = 0$
$U_5 U_8 = U_8$
$U_5 U_9 = U_9$
$U_5 U_{10} = 0$
$U_5 U_{11} = U_{11}$
$U_5 U_{12} = U_{12}$
$U_5 U_{13} = U_{13}$
$U_5 U_{14} = U_{14}$
$U_5 U_{15} = 0$
$U_5 U_{16} = 0$

$U_6 U_7 = U_8$
$U_6 U_8 = 0$
$U_6 U_9 = U_5 - U_{10}$
$U_6 U_{10} = U_6$
$U_6 U_{11} = U_7 - U_{12}$
$U_6 U_{12} = U_8$
$U_6 U_{13} = 0$
$U_6 U_{14} = 0$
$U_6 U_{15} = U_{13}$
$U_6 U_{16} = U_{14}$

$U_7 U_8 = U_6$

$U_7 U_9 = U_{11}$

$U_7 U_{10} = 0$

$U_7 U_{11} = U_9$

$U_7 U_{12} = 0$

$U_7 U_{13} = U_{14}$

$U_7 U_{14} = U_{13}$

$U_7 U_{15} = 0$

$U_7 U_{16} = 0$

$U_8 U_9 = U_7 - U_{12}$

$U_8 U_{10} = U_8$

$U_8 U_{11} = U_{10} - U_5$

$U_8 U_{12} = U_6$

$U_8 U_{13} = 0$

$U_8 U_{14} = 0$

$U_8 U_{15} = U_{14}$

$U_8 U_{16} = U_{13}$

$U_9 U_{10} = U_9$

$U_9 U_{11} = 0$

$U_9 U_{12} = U_{11}$

$U_9 U_{13} = U_{15}$

$U_9 U_{14} = U_{16}$

$U_9 U_{15} = 0$

$U_9 U_{16} = 0$

$U_{10} U_{11} = U_{11}$

$U_{10} U_{12} = 0$

$U_{10} U_{13} = 0$

$U_{10} U_{14} = 0$

$U_{10} U_{15} = U_{15}$

$U_{10} U_{16} = U_{16}$

$U_{11} U_{12} = U_9$

$U_{11} U_{13} = U_{16}$

$U_{11} U_{14} = U_{15}$

$U_{11} U_{15} = 0$

$U_{11} U_{16} = 0$

$U_{12} U_{13} = 0$

$U_{12} U_{14} = 0$

$U_{12} U_{15} = U_6$

$U_{12} U_{16} = U_{15}$

$U_{13} U_{14} = 0$

$U_{13} U_{15} = 0$

$U_{13} U_{16} = 0$

$U_{14} U_{15} = 0$

$U_{14} U_{16} = 0$

$U_{15} U_{16} = 0$

Other infinitesimal transformations used in this paper are:-

$U_5 - U_{10} = U_{18}$

$U_7 - U_{12} = U_{19}$

$U_5 + 2U_{10} = U_{22}$

$U_7 + 2U_{12} = U_{23}$

$U_1 + U_6 = U_{24}$

$U_2 + U_8 = U_{25}$

$U_1 + U_{10} = U_{26}$

$U_2 + U_{12} = U_{27}$

$U_1 + U_{13} = U_{28}$

$U_2 + U_{14} = U_{29}$

$U_3 + U_{15} = U_{30}$

$U_4 + U_{16} = U_{31}$

$U_9 - U_6 = U_{32}$

$U_{11} - U_8 = U_{33}$

$U_2 - U_{14} = U_{34}$

$U_4 - U_{16} = U_{35}$

$U_{11} + U_8 = U_{37}$

$U_{13} - U_9 = U_{38}$

$U_{14} - U_{11} = U_{39}$

Hereafter these will be referred to merely by number.

In the one dimensional group $H_c$ there are two 3-parameter groups, $H_3(C)$ and $H_3(iC)$. Corresponding to these in two dimensions there are two 8-parameter groups, (1, 3, 5, 6, 9, 10, 13, 15) and (28, 34, 30, 35, 32, 37, 7, 12). The first of these, which corresponds to $H_3(C)$, includes as a special case, when $y = w = 0$, the group of real transformations, which with its subgroups corresponds exactly to Lie's table of groups, when all quantities in it are considered real. That these transformations form a group may be seen from the preceding table of Klammerausdrücken.

The Klammerausdrücken for the set (28, 34, 30, 35, 32, 37, 7, 12) are:–

| | | | |
|---|---|---|---|
| $U_{28}U_{34} = 4U_7 + 2U_{12}$ | $U_{34}U_{30} = 2U_{37}$ | $U_{30}U_{35} = 2U_7 + 4U_{12}$ | $U_{35}U_{32} = U_{34}$ |
| $U_{28}U_{30} = U_{32}$ | $U_{34}U_{35} = U_{32}$ | $U_{30}U_{32} = U_{28}$ | $U_{35}U_{37} = U_{28}$ |
| $U_{28}U_{35} = U_{37}$ | $U_{34}U_{32} = U_{35}$ | $U_{30}U_{37} = U_{34}$ | $U_{35}U_7 = 0$ |
| $U_{28}U_{32} = U_{30}$ | $U_{34}U_{37} = U_{30}$ | $U_{30}U_7 = 0$ | $U_{35}U_{12} = U_{30}$ |
| $U_{28}U_{37} = U_{34}$ | $U_{34}U_7 = U_{28}$ | $U_{30}U_{12} = U_{35}$ | |
| $U_{28}U_7 = 35$ | $U_{34}U_{12} = 0$ | | |
| $U_{28}U_{12} = 0$ | | | |

| | | |
|---|---|---|
| $U_{32}U_{37} = 2(U_7 - U_{12})$ | $U_{37}U_7 = U_{32}$ | $U_7U_{12} = 0$ |
| $U_{32}U_7 = U_{37}$ | $U_{37}U_{12} = U_{32}$ | |
| $U_{32}U_{12} = U_{37}$ | | |

This group is perfectly symetrical with regard to the $xy$ and $zw$ planes, and reduces to $H_3(iC)$ in either of them, i.e. if $z$ and $w$ vanish there is left the group $H_3(iC)$ in $xy$ and if $x$ and $y$ vanish there remains $H_3(iC)$ in $zw$. The Klammerausdrücken show that this is a group with subgroups which arranged symetrically are as follows:–

32, 37, 7, 12

28, 34, 7, 12 || 30, 35, 7, 12
28, 30, 32 || 35, 34, 32
28, 35, 37 || 30, 34, 37

7, 12

28, 12 || 30, 7
34, 12 || 35, 7

32
37

7 || 12
28 || 30
34 || 35

In the following integrations we have put, $u = x + iy$, and $v = z + iw$.

Integrating (7) gives $u_1 = e^{it}u$.
This leaves invariant all points on the y axes and all lines through the infinite point on the x axis, i.e. all lines parallel to the x axis, and it is therefore a perspective transformation.
(12) gives $v_1 = e^{it}v$; hence it is the same kind as (7) with the x axis as the line of invariant points.
The finite equations for (32) are:-

$$u_1 = \frac{e^{it}+e^{-it}}{2}u + \frac{e^{it}-e^{-it}}{2i}v, \quad v_1 = -\frac{e^{it}-e^{-it}}{2i}u + \frac{e^{it}+e^{-it}}{2}v$$

or $u_1 = u\cos t + v\sin t$, $v_1 = -u\sin t + v\cos t$.
Which are equations for rotation about the origin. The invariant figure is a triangle with vertices at the origin and the circular points at infinity, and the path curves are circles about the origin.

The equations to (37) may be reduced to the form,

$$u_1 = u\cos t + vi\sin t$$
$$v_1 = ui\sin t + v\cos t$$

This is like (32) except that the invariant triangle is real in all its parts, being composed of the infinite line and two lines through the origin making angles $\pm 45^\circ$ with the x axis. The path curves are rectangular hyperbolas instead of circles as in (32).

The integration of (28) gives the equations,

$$\frac{u_1 - i}{u_1 + i} = e^{2it}\frac{u-i}{u+i}, \text{ or } u_1 = \frac{i(u+i) + ie^{2it}(u-i)}{(u+i) - e^{2it}(u-i)}$$

$$v_1 = \frac{2ie^{it}v}{(u+i) - e^{2it}(u-i)}$$

This leaves invariant a triangle composed of the x axis and the lines $x \pm i = 0$. The cross ratios along the sides are $\kappa = e^{2it}$, $\kappa' = \sqrt{\kappa} = e^{it}$, hence the path curves are conics through the invariant points on the x axis, i.e. conics about the opposite vertex $(0, \infty)$.

The finite equations for (34) are:-

$$\frac{u_1 + 1}{u_1 + 1} = e^{-2it}\frac{u-1}{u+1} \text{ or } u_1 = \frac{(u+1) + e^{-2it}(u-1)}{(u+1) - e^{-2it}(u-1)}$$

$$v_1 = \frac{2e^{-it}v}{(u+1) - e^{-2it}(u-1)}$$

The invariant figure is a real triangle composed of the x axis and the lines $x \pm i = 0$. The cross ratios are, $e^{-2it} = \kappa$, $e^{-it} = \kappa' = \sqrt{\kappa}$. The path curves are conics through the invariant points on the x-axis.

In (28, 12) the equation for $u_1$ is the same as in (28) and for $v_1$,

$$v_1 = \frac{2ie^{(a+1)it}v}{(u+i) - e^{2it}(u-i)}$$

The invariant triangle is the same as in (28) and the cross ratios have the relation $K' = K^r$ [$r$ = any number]. (28) is a special case with $r = \frac{1}{2}$.

(34, 12) gives the equations,

$$u_1 = \frac{(u+1) + e^{-2it}(u-1)}{(u+1) - e^{-2it}(u-1)}, \quad v_1 = \frac{2e^{(a-1)it}v}{(u+1) - e^{-2it}(u-1)}$$

The invariant figure is the same as in (34) and the cross ratios $K' = K^r$, which when $r = \frac{1}{2}$ gives (34).

Since (7) leaves invariant the x axis and all points on the y axis, and 12 leaves the y axis and all points on the x axis, and both (7) and (12) leave the line at infinity unchanged, the group (7, 12) has for its invariant figure a triangle composed of the axes and the line at infinity.

(32, 37, 7, 12) has for its finite equations,

$$U_1 = \frac{-(ic-m)e^{[i(b+c)+m]t} + (ic+m)e^{[i(b+c)-m]t}}{2m}U + \frac{(1+ai)\left[e^{[i(b+c)+m]t} - e^{[i(b+c)-m]t}\right]}{2m}V$$

$$V_1 = \frac{(ic-m)(ic+m)\left[e^{[i(b+c)-m]t} - e^{[i(b+c)+m]t}\right]}{-2(1+ai)m}U + \frac{(ic+m)e^{[i(b+c)+m]t} - (ic-m)e^{[i(b+c)-m]t}}{-2m}V.$$

$$m = \sqrt{-(b-c)^2 - 4(1+a^2)}$$

These equations have the form of equations for rotation. The cross ratios are, $K = e^{[i(b+c)-m]t}$, $K' = e^{[i(b+c)+m]t}$ The tangents of angles made with x axis by invariant lines of component transformations are, $T = \frac{ic+m}{1+ai}$, and $T' = \frac{ic-m}{1+ai}$. The invariant figure for the group consists of the origin and infinite line.

(28, 34, 7, 12) integrates to;

$$\frac{U_1 + \frac{bi-n}{2(1-ai)}}{U_1 + \frac{bi+n}{2(1-ai)}} = e^{nt}\frac{U + \frac{bi-n}{2(1-ai)}}{U + \frac{bi+n}{2(1-ai)}}, \quad \left[n = \sqrt{-b^2 - 4(1+a^2)}\right]$$

$$V_1 = \frac{\frac{n}{1-ai}e^{(ci-\frac{n}{2})t}V}{\left(U + \frac{bi+n}{2(1-ai)}\right) - e^{nt}\left(U + \frac{bi-n}{2(1-ai)}\right)}$$

This is the same kind of group as (32, 37, 7, 12) with the invariant elements the x axis and point $(0, \infty)$ instead of origin and line at infinity.

Another very interesting group is (24,25; 22,23,38,39) [$H_6(K)$] which leaves invariant a conic section; It corresponds to (23) in Lie's[1] list and to $G_3(K)$ in Newson's[2]. This group can be derived from the one dimensional group in the following manner. If we take the equations for transformation of points on a line, $x_1 = \frac{ax+b}{cx+d}$, or in homogeneous form

$x_1 = ax+by$, $y_1 = cx+dy$.

$x_1^2 = a^2x^2+2abxy+b^2y^2$

$y_1^2 = c^2x^2+2cdxy+d^2y^2$

$x_1y_1 = acx^2+(ab+cd)xy+bdy^2$

Let $x' = x^2$, $y' = y^2$, and $z' = xy$

then $x_1'z_1'-y_1'^2 = (ad-bc)^2x'z'-y'^2$

The conic $y'^2-x'z'=0$ is invariant.

or writing the coefficients in the complex form,

$x_1 = (a+ib)x+(c-id)y$

$y_1 = (c+id)x+(a-ib)y$

Then let $x^2 = x'+iz'$, $y^2 = x'-iz'$, and $xy = iy'$

and $x_1'^2+y_1'^2+z_1'^2 = [(a^2+b^2)+c^2+d^2]^2(x'^2+y'^2+z'^2)$.

The conic $x'^2+y'^2+z'^2=0$ is invariant.

---

(1) "Continuierliche Gruppe" pp. 288-291. (2) Am. Journal of Math. Vol. XXIV, p. 170.

Since this group $H_6(K)$ is connected in this way with the group $H_6$ of one dimensional transformations the two groups must be simply isomorphic. The Klammerausdrücken show this to be true. If the infinitesimal transformations of $H_6(K)$ be numbered in this order $\left(\begin{array}{cccccc} 24, & 25; & 22, & 23, & 38, & 39 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array}\right)$ the table of Klammerausdrücken on page 4 is also the table for this group, hence $H_6(K)$ and $H_6$ break into subgroups in exactly the same way. To every subgroup of $H_6$ there is a corresponding one in $H_6(K)$. Corresponding to $H_3(C)$ is $H_3(K)$ which leaves a real conic invariant. $H_3(K)$ is also a subgroup of the 8-parameter group $(1, 3, 5, 6, 9, 10, 13, 15)$. To $H_3(iC)$ corresponds $H_3(iK)$ which leaves invariant an imaginary conic.

There is another 6-parameter group, (28, 29, 30, 31, 32, 33). The Klammerausdrücken are:-

| | | | |
|---|---|---|---|
| $U_{28}U_{29} = 0$ | $U_{29}U_{30} = U_{33}$ | $U_{30}U_{31} = 0$ | $U_{31}U_{32} = U_{29}$ |
| $U_{28}U_{30} = U_{32}$ | $U_{29}U_{31} = U_{32}$ | $U_{30}U_{32} = U_{28}$ | $U_{31}U_{33} = U_{28}$ |
| $U_{28}U_{32} = U_{30}$ | $U_{29}U_{32} = U_{31}$ | $U_{30}U_{33} = U_{29}$ | $U_{32}U_{33} = 0$ |
| $U_{28}U_{33} = U_{31}$ | $U_{29}U_{33} = U_{30}$ | | |
| $U_{28}U_{31} = U_{33}$ | | | |

The subgroups, arranged symetrically are:-

$$\begin{array}{c}
32 \\
33 \\
\begin{array}{c|c} 28 & 30 \\ 29 & 31 \end{array} \\
32,\ 33 \\
\begin{array}{c|c} 28, 29 & 30, 31 \end{array} \\
28, 30, 32 \\
29, 31, 32 \\
\begin{array}{c|c} 28, 31, 33 & 30, 29, 33 \end{array}
\end{array}$$

This group has one 3-parameter subgroup (28, 30, 32) in common with the 8-parameter group (28, 34, 30, 35, 32, 37, 7, 12).

In Lie's "Continuierliche Gruppe" [pp. 409–412] it is shown that if the coefficients of p, q, r, s, in the infinitesimal transformations of the group are taken as a matrix and all the determinants formed from it have a common factor, that factor is an invariant of the group. For this group the matrix is as follows:-

$$\left\|\begin{matrix} 1+x^2-y^2 & 2xy & xz-yw & xw+yz \\ -2xy & 1+x^2-y^2 & -xw-yz & xz-yw \\ xz-yw & xw+yz & 1+z^2-w^2 & 2zw \\ -xw-yz & xz-yw & -2zw & 1+z^2-w^2 \\ z & w & -x & -y \\ -w & z & y & -x \end{matrix}\right\|$$

Let $u = x+iy$, $u_1 = x-iy$, $v = z+iw$, $v_1 = z-iw$. Then the determinant formed from the last four rows is equal to $vv_1(1+v^2+u^2)(1+v_1^2+u_1^2)$. That formed from the first 2 and last 2 rows equals $uu_1(1+v^2+u^2)(1+v_1^2+u_1^2)$. The determinant of the 2nd and 3d rows with the last two also contains $(1+v^2+u^2)(1+v_1^2+u_1^2)$ as a factor. In fact it is evident that this factor is common to all of the determinants containing the last two rows. That this factor is contained in the other determinants of the matrix is not so easily seen, but is at least very probable. This seems to indicate that the invariant figure of the group is a pair of conjugate imaginary surfaces of the second order.