Enumerative and Algebraic Aspects of Slope Varieties

By

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Doctor of Philosophy

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Abstract

The slope variety of a graph G is an algebraic variety whose points correspond to the slopes arising from point-line configurations of G. We start by reviewing the background material necessary to understand the theory of slope varieties. We then move on to slope varieties over finite fields and determine the size of this set. We show that points in this variety correspond to graphs without an induced path on four vertices. We then establish a bijection between graphs without an induced path on four vertices and series-parallel networks. Next, we study the defining polynomials of the slope variety in more detail. The polynomials defining the slope variety are understood but we show that those of minimal degree suffice to define the slope variety set theoretically. We conclude with some remarks on how we would define the slope variety for point-line configurations in higher dimensions.

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Chapter 1

Introduction

The main object of study in this dissertation is the slope variety $\mathscr{S}(G)$ associated with a graph G. A graph can be realized as a point-line configuration in the plane; each edge is a line and contains the two points corresponding to its vertices. The slope vector of such a realization is the ordered list of the edge slopes. The slope variety of a graph is the set of edge slopes arising from these realizations. In [11] explicit polynomials are constructed for each graph that define its slope variety. First, we study these polynomials over finite fields and count the number of common zeroes. Second, we show that a certain subset of these polynomials are sufficient to define the slope variety for the complete graph over an algebraically closed field. Finally, we consider point-line configurations of graphs in arbitrary-dimensional space and give some of the defining polynomials.

Chapter 2 builds up the background needed to study slope varieties. We begin by reviewing basic definitions from graph theory and matroid theory. From there we may discuss combinatorial rigidity [8, 17]. We think of constructing a graph G using rods of fixed lengths for the edges and ball joints for the vertices. Then, G is rigid if it cannot change shape when constructed this way. For example, a 3-cycle is rigid (this is the side-side-side rule) but a 4-cycle is not (because we can pull apart two non-adjacent vertices while keeping all edge lengths fixed). An important result of combinatorial rigidity is Laman's Theorem [10], which says that (generic) planar rigidity is a graph invariant. One consequence of Laman's Theorem is that the geometric constraints on edge lengths correspond to certain graphs called *rigidity circuits* which can be described combinatorially in terms of their spanning trees.

Combinatorial rigidity plays a role in describing the defining polynomials of the slope variety [11]. We walk through their construction. Start with the equations in the coordinates of points and lines describing the point-line configurations of G. Eliminating the coordinates of the vertices gives a matrix in the slopes. The defining polynomials τ_C of the slope variety are given by certain minors of this matrix, which correspond to the rigidity circuits C. The polynomial τ_C is called a *tree polynomial* because it is a generating function for certain spanning trees of the rigidity circuit C. We need the determinant and generating function descriptions throughout the following chapters.

Previous work on tree polynomials was done over an algebraically closed field [11, 12]. In Chapter 3 we work instead over the finite field $\mathbb{F}_q$ with q elements. The set of common zeroes $S_q(G)$ is a finite analogue of the slope variety of G, raising the question of determining its cardinality. The main result of this chapter is a bijection between points of $S_2(K_n)$ and combinatorial objects known as series-parallel networks (for which see [14, Exercise 5.40], [4]). A natural question is whether or not there are combinatorial interpretations for other values of q. We give some data for larger q but the question remains open.

Finally, in Chapter 4 we study the tree polynomials for slope variety of K_n in more detail. Recall that the tree polynomials correspond to rigidity circuits. The most important rigidity circuits are the *wheels*. A k-wheel is a k-cycle together with a vertex that is adjacent to every vertex in the cycle. In [12] it is shown that the ideal I_n generated by the tree polynomials of the wheels in K_n is prime and its zero set is exactly the slope variety $\mathscr{S}(K_n)$. Experimental evidence shows that in fact $I_n = J_n$ for $n \leq 9$, where J_n is the ideal generated by the tree polynomials of the 3-wheels (that is, copies of K_4) in K_n. The main result in this chapter is Theorem 4.1.2, which shows that $\sqrt{J_n} = I_n$ for all n, that is, the zero set of J_n is the slope variety $\mathscr{S}(K_n)$. In the last section we sketch how to generalize the slope variety to higher dimensions by mimicking the techniques used in the plane, and explain why the tree polynomials alone do not determine all constraints on the edge directions.

Chapter 2

Background

2.1 Graph theory

We list some necessary notation here; for a general background on graph theory see [2] or [16]. A *graph* G is an ordered pair (V,E) of *vertices* and *edges* i.e., V is a finite set, and E is a set of 2-subsets of V. Two vertices $u,v \in V$ are *adjacent* if there is an edge $uv \in E$ between them. We use the notation $V(G)$ for the vertex set of G and $E(G)$ for the edge set. Throughout, we will assume graphs are connected unless otherwise stated, i.e., there is a path between any two vertices, and *simple*, i.e., no loops or parallel edges.

For $U \subseteq V$, the *induced subgraph* $G|_U$ of G on U, is the graph with vertex set U and edge set $\{uv \in E(G) \mid u,v \in U\}$. The *intersection* $G \cap H$ of two graphs G and H is the graph with vertex set $V(G) \cap V(H)$ and edge set $E(G) \cap E(H)$. The *complement* $\overline{G}$ of G is the graph on the same set of vertices as G whose edges are exactly the non-edges of G.

Though we will define the slope variety for any graph G, the main graph we are concerned with is the complete graph on n vertices K_n. An important set of graphs is the set of wheels in K_n. The *k-wheel* $W(v_0; v_1, \dots, v_k)$ is the graph on the vertices $\{v_0, \dots, v_k\}$ and whose edges are $v_0v_1, \dots, v_0v_k$, $v_1v_2, \dots, v_{k-1}v_k, v_kv_1$, where $k \geq 3$. The wheel $W(v_0; v_1, \ldots, v_k)$ is invariant up to dihedral permutations of $v_1, \ldots, v_k$. The vertex v_0 is called the *center*; the other vertices are called the *spokes*. The edges incident to the center are called the *radii*, and the other edges are *chords*. Note that a 3-wheel is the complete graph on four vertices and the designation of center is arbitrary.

Let P_n denote the path on n vertices, also called the n-path. A *complement-reducible graph*, or *cograph*, has no induced P_4. An important fact that we will need is that G is complement-reducible if and only if for every induced subgraph $H \subseteq G$, either H or the complement $\overline{H}$ is disconnected; see [5].

2.2 Matroids

There are many mathematical structures in which there is a notion of independence. Matroids are defined to give precise meaning to what it means to be independent.

Definition 2.2.1. A *matroid* M is a finite set E, together with a collection of subsets $\mathscr{I}$, called the *independent sets*, that satisfies the following conditions:

$M1$: $\emptyset \in \mathscr{I}$;

$M2$: if $A \subseteq B$ and $B \in \mathscr{I}$, then $A \in \mathscr{I}$;

$M3$: if $A, B \in \mathscr{I}$ and $|A| > |B|$, then there is some $x \in A \setminus B$ such that $B \cup \{x\} \in \mathscr{I}$.

Definition 2.2.2. Let M be a matroid on E with independence system $\mathscr{I}$.

- A maximal independent set $A \in \mathscr{I}$, with respect to set containment, is a *basis*.
- A set $A \subseteq E$ that is not in $\mathscr{I}$ is a *dependent set*.
- A minimal dependent set C is a *circuit*.
- The *rank function* $r : 2^E \to \mathbb{N}$ is $r(A) = \max\{|A \cap B| \mid B \in \mathscr{I}\}$.

A matroid can be described by any one of the following: its circuits, its bases, or its rank function. There are several other equivalent ways to describe a matroid see [1] but we are not concerned with those descriptions here. Much of the terminology used in matroid theory is motivated by the two basic examples given below: a finite set of vectors, and a graphic matroid.

Example 1. Consider a finite set of vectors $E = \{v_1, \ldots, v_n\}$ in a vector space V. Then $\mathscr{I} := \{A \subseteq E \mid A \text{ is linearly independent}\}$ is a matroid, called the vector matroid on E.

Example 2. A *graphic matroid* is the set of edges E in a graph G, together with the collection of acyclic subsets $\mathscr{I}$ of E. It is not hard to check that $\mathscr{I}$ satisfies $(M1)$ and $(M2)$.

We need to prove condition $(M3)$ holds. Let $A, B \in \mathscr{I}$ be acyclic sets with $|A| > |B|$. By way of contradiction, suppose $B \cup \{e\}$ contains a cycle for every edge $e \in A \setminus B$. Then, it must be the case that $V(A) \subseteq V(B)$ (the vertices of A are contained in the vertices of B) and hence $|V(A)| \leq |V(B)|$. It is a well known fact in graph theory that if c_A is the number of components in A and c_B is the number of components in B, then $|A| = |V(A)| - c_A$ and $|B| = |V(B)| - c_B$. Therefore, B must have more components than A since we are assuming $|A| > |B|$. Therefore, by the pigeonhole principle, there must be an edge $uv \in A$ whose vertices u and v are in different components in B. Hence, $B \cup \{uv\}$ is acyclic, contradicting our assumption that every edge in $A \setminus B$ completes a cycle in B.

The graphic matroid is the best known matroid on the edge set of a graph. However, there are several different ways to define an independence system. In particular, the next section will develop the rigidity matroid. We will assign a row vector to each edge in K_n that corresponds to the edge length. Then, obtain a vector matroid where a collection of edges is linearly independent if and only if their lengths are independent.

2.3 Combinatorial rigidity

The motivation for much of the work in this thesis comes from the study of combinatorial rigidity. The basic problem is to determine when the drawing of a graph $G = (V, E)$ is rigid in $\mathbb{R}^d$. Informally, suppose we construct G using rods of fixed length for the edges and ball joints for the vertices. If the edges are allowed to pivot about the vertices, will G keep its shape? Before making the notion of rigidity precise, we demonstrate that a graph may be constructed in a rigid or non-rigid fashion.

Example 3. Let $V = \{1,2,3,4,5,6\}$, $E = \{12,13,14,23,24,35,46,56\}$ be drawn, in $\mathbb{R}^2$, as shown in Figure 2.1. Triangles are rigid, therefore, the distance between vertices 3 and 4 is fixed in both frameworks. Since the vertices 3,4,5 and 6 are colinear in the first framework, it is impossible to keep the edge lengths fixed and have them pivot about the vertices. Informally, in the second framework vertices 5 and 6 can move without changing any edge length.

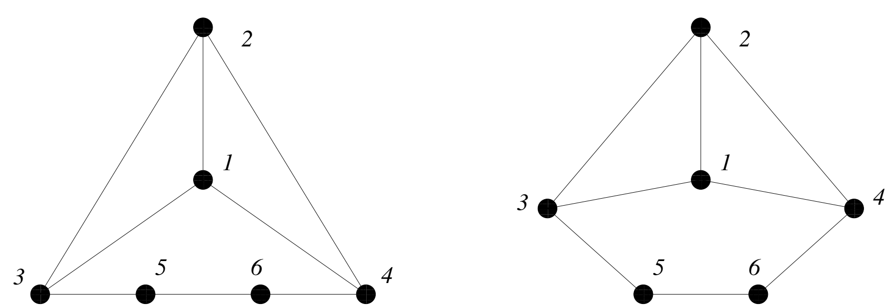

Figure 2.1: Rigid and non-rigid frameworks for the same graph

We introduce some terminology from [8] so that we can make the notion of rigidity more precise. Let $\mathbf{p} : V \to \mathbb{R}^d$ be a map and let $\mathbf{p}_i$ denote the image of vertex $i \in V$. The collection $(V, E, \mathbf{p})$ is called a *framework*, which is a geometric realization of G in $\mathbb{R}^d$ (the edges are line segments between the appropriate vertices). Any movement of a vertex $\mathbf{p}_i$ in $\mathbb{R}^d$ is assumed to be along a differentiable curve. A *motion* of the framework $(V, E, \mathbf{p})$ is given by a differentiable function $\mathbf{q} : [0,1] \to (\mathbb{R}^d)^{|V|}$ where $\mathbf{q}_i(0) = \mathbf{p}_i$ for all $i \in V$ and all edge-lengths are constant. The square of the distance between $\mathbf{q}_i(t)$ and $\mathbf{q}_j(t)$ is given by the dot product $(\mathbf{q}_i(t) - \mathbf{q}_j(t)) \cdot (\mathbf{q}_i(t) - \mathbf{q}_j(t))$. So if $ij \in E$ is an edge, then differentiating gives

$$(\mathbf{q}_i'(t) - \mathbf{q}_j'(t)) \cdot (\mathbf{q}_i(t) - \mathbf{q}_j(t)) = 0. \tag{2.1}$$

An *infinitesimal motion* of a framework is the initial velocity $\mathbf{q}'(0)$. That is, assign a vector $\mathbf{q}_i'(0)$, to each vertex $i \in V$, so that equation (2.1) is satisfied for each edge. We then express the system of equations (2.1) in the form of a matrix. Define the *rigidity matrix* $\mathbf{R}(\mathbf{p})$ to be the $\binom{|V|}{2} \times |V|$ matrix whose entry in row ij column k is

$$\begin{cases} \mathbf{p}_i - \mathbf{p}_j & \text{if } i = k \\ \mathbf{p}_j - \mathbf{p}_i & \text{if } j = k \\ 0 & \text{otherwise .} \end{cases}$$

The rigidity matrix is really an $\binom{|V|}{2} \times d|V|$ matrix. For example, let $\mathbf{p} : \{1,2,3\} \to \mathbb{R}^2$ be a map and let $(\{1,2,3\}, \{12, 13, 23\}, \mathbf{p})$ be the corresponding framework of K_3 in the plane $\mathbb{R}^2$, with $\mathbf{p}_i = (x_i, y_i)$. The rigidity matrix is

$$\mathbf{R}(\mathbf{p}) = \begin{bmatrix} \mathbf{p}_1 - \mathbf{p}_2 & \mathbf{p}_2 - \mathbf{p}_1 & 0 \\ \mathbf{p}_1 - \mathbf{p}_3 & 0 & \mathbf{p}_3 - \mathbf{p}_1 \\ 0 & \mathbf{p}_2 - \mathbf{p}_3 & \mathbf{p}_3 - \mathbf{p}_2 \end{bmatrix}$$
$$= \begin{bmatrix} x_1 - x_2 & y_1 - y_2 & x_2 - x_1 & y_2 - y_1 & 0 & 0 \\ x_1 - x_3 & y_1 - y_3 & 0 & 0 & x_3 - x_1 & y_3 - y_1 \\ 0 & 0 & x_2 - x_3 & y_2 - y_3 & x_3 - x_2 & y_3 - y_2 \end{bmatrix}.$$

Let $\mathbf{U} \in (\mathbb{R}^d)^{|V|}$ be a column vector. Then $\mathbf{U}$ is an infinitesimal motion of the framework $(V, E, \mathbf{p})$ if the ij entry of $\mathbf{R}(\mathbf{p})\mathbf{U}$ is 0 for all edges $ij \in E$. An infinitesimal motion $\mathbf{U}$ is rigid if $\mathbf{R}(\mathbf{p})\mathbf{U} = 0$, i.e., every distance between two vertices is constant.

Definition 2.3.1. A framework $(V, E, \mathbf{p})$ is *infinitesimally rigid* if $\mathbf{R}(\mathbf{p})\mathbf{U} = 0$ for each infinitesimal motion $\mathbf{U}$.

Definition 2.3.2. A *general embedding* of a graph is a framework in which the vertices are in general position. A graph is *generically rigid* if there is a general embedding that is infinitesimally rigid.

Lemma 2.3.3. *If a framework $(V, E, \mathbf{p})$ is infinitesimally rigid for some general embedding $\mathbf{p}$ in $\mathbb{R}^d$, then $(V, E, \mathbf{q})$ is infinitesimally rigid for every general embedding $\mathbf{q}$.*

We will now only be concerned with generic rigidity, which is a combinatorial property. Pick a general framework $(V, E, \mathbf{p})$ in $\mathbb{R}^d$. It is not hard to check that a framework is infinitesimally rigid if and only if each row in $\mathbf{R}(\mathbf{p})$ is a linear combination of the rows $e_1, \ldots, e_r \in E$. The set of rows in $\mathbf{R}(\mathbf{p})$ is a finite set of vectors, hence they form a vector matroid, called the *d-rigidity matroid*. The matroid's ground set is the set of edges $E(K_V)$ in the complete graph on V. A graph $G \subseteq K_n$ is rigid if the rank of $E(G)$ equals the rank of $E(K_V)$.

There are no known conditions on a graph that determine rigidity in arbitrary dimensional space $\mathbb{R}^d$. The case $d = 2$ was solved by Laman [10].

Theorem 2.3.4. *A graph $G = (V, E)$ is rigid in the plane if and only if there is a subset $F \subseteq E$ such that*

1. $|F| = 2|V| - 3$,
2. *for any nonempty $F' \subseteq F$, $|F'| \leq 2|V(F')| - 3$.*

The idea of Laman's Theorem is that in order for a graph to be rigid, it needs "enough" edges and they have to be evenly distributed throughout the graph. Theorem 2.3.4 gives the independence system, for the 2-rigidity matroid, in terms of the number of edges. A set E is rigidity independent if $|F| \leq 2|V(F)| - 3$ for all $F \subseteq E$. A *rigidity circuit* F has $|F| = 2|V(F)| - 2$ and $|F'| \leq 2|V(F')| - 3$ for all $F' \subseteq F$. We get the following useful proposition from [8].

Proposition 2.3.5. *An edge set E is a rigidity circuit if and only if E is the disjoint union of two spanning trees, and no proper subset of E has this property.*

The edge set E is a *rigidity pseudocircuit* if it can be partitioned into two disjoint spanning trees. We define these spanning trees to be *coupled spanning trees*.

2.4 Grassmannians and algebraic geometry

As stated earlier, the goal of this chapter is to define the slope variety of a graph. We need Grassmann varieties to describe them. First, we list necessary definitions and facts from algebraic geometry. Fix a field $\mathbb{F}$. Given a set of polynomials $S \subseteq \mathbb{F}[x_1, \dots, x_n]$, the set of common zeroes is denoted as

$$Z(S) := \{\underline{a} = (a_1, \dots, a_n) \in \mathbb{F}^n \mid f(\underline{a}) = 0 \qquad \forall f \in S\}.$$

Let $I = \langle f_1, \dots, f_s \rangle$ be the ideal generated by $f_1, \dots, f_s$. Then, $Z(I) = Z(f_1, \dots, f_s)$. Given a set $T \subseteq \mathbb{F}^n$, the set of polynomials that vanish on T is denoted as

$$I(T) := \{f \in \mathbb{F}[x_1, \dots, x_n] \mid f(a) = 0 \ \forall a \in T\}.$$

This set $I(T)$ is in fact an ideal.

Definition 2.4.1. A set $T \subseteq \mathbb{F}^n$ is called an *algebraic set* if there is an ideal I such that $T = Z(I)$.

It is not necessarily the case that $I(Z(I)) = I$. For example, let $I = \langle x^2 \rangle \in \mathbb{F}[x]$. Then $Z(I) = \{0\}$, so $x \in I(Z(I)) \setminus I$.

Definition 2.4.2. For an ideal $I \subseteq R$ in a commutative ring R, the *radical ideal* $\sqrt{I}$ is

$$\sqrt{I} := \{a \in R \mid a^t \in I \text{ for some positive } t \in \mathbb{Z}\}.$$

Equivalently, the radical of I is the intersection of all prime ideals containing I.

Proposition 2.4.3. *Let I be an ideal. Then the following hold:*

1. $I \subseteq \sqrt{I}$,

2. $Z(I) = Z(\sqrt{I})$,

3. $I(Z(I)) = \sqrt{I}$.

Definition 2.4.4. The *Zariski topology* on $\mathbb{F}^n$ is the topology where the basic closed sets are of the form $Z(I)$, for some ideal $I \subseteq \mathbb{F}[x_1,\ldots,x_n]$. The *Zariski closure* of a set $T \subseteq \mathbb{F}^n$ is $\overline{T} := Z(I(T))$.

Definition 2.4.5. An algebraic set Z is *reducible* if there are proper algebraic subsets X and Y with $Z = X \cup Y$. It is *irreducible* if it is not reducible.

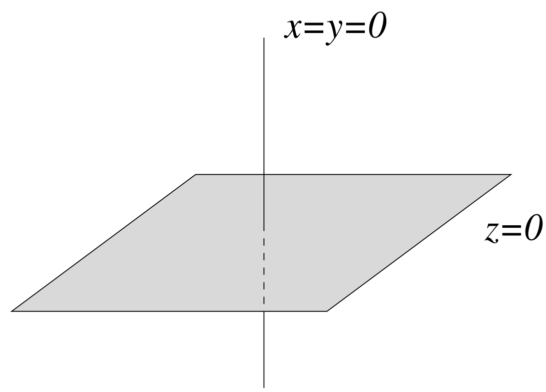

Figure 2.2: The algebraic set $Z(xz,yz) = Z(x,y) \cup Z(z)$ is reducible.

Definition 2.4.6. Let I,J be ideals in a commutative ring R. The *ideal quotient*, or *colon ideal* is

$$I : J := \{r \in R \mid rJ \subseteq I\}.$$

The colon ideal is an ideal. We can think of the colon ideal $I : J$ as the ideal I divided by the ideal J. In the special case that $I = \langle fg \rangle$ and $J = \langle g \rangle$, then the colon $I : J = \langle f \rangle$ is fg divided by g. On the geometric level, this amounts to removing the algebraic set defined by J from $Z(I)$.

Theorem 2.4.7. *If $\mathbb{F}$ is algebraically closed and $I = \sqrt{I}$, then*

$$Z(I : J) = \overline{Z(I) \setminus Z(J)}.$$

Proposition 2.4.8. *If I, J and K are ideals, then $I : (J + K) = (I : J) \cap (I : J)$.*

See [6].

2.4.1 Grassmannians

Given a field $\mathbb{F}$, the Grassmannian $Gr(k,n)$ is the set of k-dimensional subspaces of $\mathbb{F}^n$. The set of lines through the origin $Gr(1,n)$ is projective $(n-1)$-space, the set of planes through the origin $Gr(2,n)$ is the set of lines in projective $(n-1)$-space, etc. For a more extensive coverage of Grassmannians and Plücker relations, see [3] or [7].

We now walk through the construction of the Plücker relations. A point $W \in Gr(k,n)$ can be represented as a full-rank $n \times k$ matrix M whose column vectors form a basis of W (in the previous section we were concerned with the row vectors of a matrix not the column vectors). This matrix is not unique. However, for any other representation M', there is an invertible $k \times k$ matrix U such that $M = M'U$. Define the notation $M_{a_1,\ldots,a_k}$ be the submatrix with rows $a_1, \ldots, a_k$, let $\lambda := \det(U) \neq 0$ and, following the notation of [3], define $M[a_1, \ldots, a_k] := \det(M_{a_1,\ldots,a_k})$. Then $M[a_1, \ldots, a_k] = \lambda M'[a_1, \ldots, a_k]$ for each k-element subset $\{a_1, \ldots, a_k\} \subseteq [n]$. If the matrix M is understood, we use $[a_1, \ldots, a_k] = M[a_1, \ldots, a_k]$. Each $W \in Gr(k,n)$ is realized, not uniquely, as a point in $\mathbb{F}^{\binom{n}{k}}$, with coordinates indexed by k-element subsets of $[n]$; the entry in the $\{a_1, \ldots, a_k\}$-coordinate is $[a_1, \ldots, a_k]$. Two of these representations are equivalent if and only if one is a scalar multiple of the other. Therefore, $Gr(k,n) \subseteq \mathbb{P}^{\binom{n}{k}-1}$.

Not every point in $\mathbb{P}^{\binom{n}{k}-1}$ is in $Gr(k,n)$; its coordinates must satisfy the *Plücker relations*. This is the complete set of polynomial relations among the coordinates. Therefore, Grassmannians are algebraic sets.

Let $\mathfrak{S}_n$ be the permutation group on $[n]$ and define

$$S(t,n) := \{\sigma \in \mathfrak{S}_n \mid \sigma_1 < \sigma_2 < \cdots < \sigma_t \text{ , } \sigma_{t+1} < \cdots < \sigma_n\}.$$

Lemma 2.4.9 (Plücker Relations)**.** *Let M be an $n \times k$ matrix, $k \leq n$, of indeterminates over $\mathbb{Z}$, let $1 \leq t \leq k$ and let $a_1, \ldots, a_m$, $c_1, \ldots, c_s$, $b_1, \ldots, b_\ell \in [n]$ with $t + m = k$ and $s - t + \ell = k$. Then*

$$\sum_{\sigma \in S(t,s)} sgn(\sigma)[a_1, \ldots, a_m, c_{\sigma_1}, \ldots, c_{\sigma_t}][c_{\sigma_{t+1}}, \ldots, c_{\sigma_s}, b_1, \ldots, b_\ell] = 0. \qquad (2.2)$$

The first nontrivial example is a 4×2 matrix, say

$$M = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \\ x_{41} & x_{42} \end{bmatrix}.$$

The only Plücker relation with $a_1 = 1, c_1 = 2, c_2 = 3, c_3 = 4$ $s = 3$ and $t = 1$ is

$$[1,2][3,4] - [1,3][2,4] + [1,4][2,3] = 0,$$

i.e.,

$$\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \begin{vmatrix} x_{31} & x_{32} \\ x_{41} & x_{42} \end{vmatrix} - \begin{vmatrix} x_{11} & x_{12} \\ x_{31} & x_{32} \end{vmatrix} \begin{vmatrix} x_{21} & x_{22} \\ x_{41} & x_{42} \end{vmatrix} + \begin{vmatrix} x_{11} & x_{12} \\ x_{41} & x_{42} \end{vmatrix} \begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix} = 0$$

2.4.2 Schubert cells

We can partition $Gr(k,n)$ into *Schubert Cells*. Given $W \in Gr(k,n)$, there is a unique matrix representation, after reducing W by only performing column operations. For example, every element of $Gr(3,4)$ is represented by a matrix in one of the following forms:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & * \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} * & * & * \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let M be the unique matrix representation for $W \in Gr(k,n)$. Let p_i be the row in which column i has a 1 with only 0's below it. Then, $\mu = (p_k - k, p_{k-1} - (k-1), \ldots, p_1 - 1)$ is an integer partition of $|\mu| = p_k + \cdots + p_1 - \binom{k+1}{2}$. The value $p_i - i$ equals the number of $*$'s in column i. Say that μ is its *type*. Let $\mu = (\mu_1, \ldots, \mu_t) \vdash m \leq n$ be a partition of m with $t \leq k$ and $\mu_1 \leq n-k$. Define the Schubert cell

$$X_\mu := \{W \in Gr(k,n) \mid W \text{ has type } \mu\}.$$

Proposition 2.4.10. *Let $Gr(k,n)$ be given. We have the following:*

1. *The set of all X_μ forms a partition of $Gr(k,n)$.*

2. *X_μ is a $|\mu|$-dimensional affine space.*

3. $\bigcup_{\mu<\nu} X_\mu = \overline{X_\nu}$.

Proposition 2.4.10 shows that Schubert cells decompose the complicated Grassmannian space $Gr(k,n)$ into much nicer spaces (specifically CW-complexes) and shows how they fit together.

2.5 Graph varieties

2.5.1 Picture space

We return to the topic of graph drawings, but consider them in a different light than in Section 2.3. Fix a field $\mathbb{F}$ and a graph G with $V(G) = \{v_1, \dots, v_n\}$ and $E(G) = \{e_1, \dots, e_r\}$. Draw G in the projective plane $\mathbb{P}^2 = Gr(1,3)$. Each vertex is an element of $Gr(1,3)$ and each edge is an element of $Gr(2,3)$. A *picture* of G is a point

$$P = (p(v_1), \dots, p(v_n), p(e_1), \dots, p(e_r)) \in \prod_{v\in V} Gr(1,3) \times \prod_{e\in E} Gr(2,3) \tag{2.3}$$

such that if $v_i \in e_j$, then $p(v_i) \in p(e_j)$. The *picture space* $\mathscr{X}(G)$ is the set of all pictures of G.

Analogous to the partition of Grassmannians into Schubert cells, the picture space can be partitioned into *cellules*. Instead of integer partitions, cellules are indexed by set partitions. Let π be a partition of $[n]$ and $\sim_\pi$ the corresponding equivalence relation. Define the cellules as the sets

$$\mathscr{X}_\pi(G) = \{P \in \mathscr{X}(G) \mid p(v_i) = p(v_j) \iff v_i \sim_\pi v_j\}$$

Each picture belongs to exactly one cellule, so the cellules partition the picture space. The most natural cellule is the discrete cellule $\mathscr{X}_D(G)$ where D is the discrete partition;

every block has one element. How these cellules fit together was studied by Martin, [11, Theorem 6.3].

Theorem 2.5.1. *Let $G = (V,E)$ be a graph, and let π, σ be partitions of V. Then $\mathscr{X}_\sigma(G) \subseteq \overline{\mathscr{X}_\pi(G)}$ if and only if the following conditions hold:*

1. *$\pi \leq \sigma$ (every block of π is in a block of σ);*

2. *no rigidity circuit of G/π is collapsed by σ/π (i.e., if C is a rigidity circuit in G/π, then not all the vertices of C are in the same block of σ);*

3. *If A_i and A_j are distinct blocks of π contained in the same block of σ, then E contains at most one edge between A_i and A_j.*

Consider the picture space for the complete graph on two vertices K_2. There are only two partitions of [2]: $\pi = \{\{1\},\{2\}\}$ and $\sigma = \{\{12\}\}$. Let $P \in \mathscr{X}_\sigma(K_2)$ be a

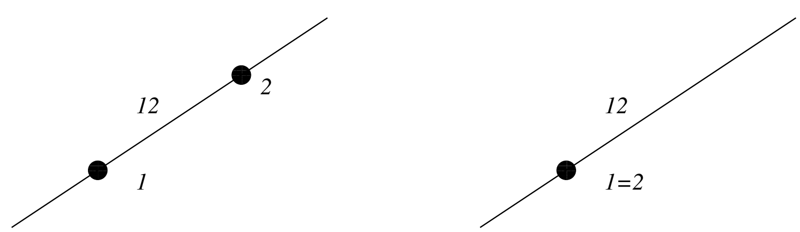

Figure 2.3: Two ways to draw K_2

picture in which the two vertices coincide. Define $Y \subseteq \mathscr{X}_\pi(K_2)$ to be the set of pictures Q with $P(12) = Q(12)$ and $P(1) = Q(1)$. Then, P is an accumulation point of Y. We can find pictures in Y arbitrarily close to P by taking $Q(2)$ arbitrarily close to $Q(1)$. It would be nice to generalize this result to any graph G and get that if a partition π, on $[n]$ is a refinement of a partition σ, then $\mathscr{X}_\sigma(G) \subseteq \overline{\mathscr{X}_\pi(G)}$. However, we cannot do this for arbitrary graphs because we need to worry about rigidity circuits.

In general, the space $\mathscr{X}(G)$ is not irreducible. The cellule that we are most concerned with is the *discrete cellule*. The partition is the discrete partition, so no two vertices have the same coordinates. Call a picture in the discrete cellule *generic*.

2.5.2 Defining ideal for the slope variety

We now consider combinatorial rigidity in a different light. Suppose we construct the graph G in $\mathbb{R}^2$ using lines with fixed slopes, instead of line segments with fixed lengths, for the edges. We allow the vertices to move in the plane, as long as the edge slopes remain constant. This point-line configuration of G is *rigid* if it keeps its shape under these movements of the vertices. That is, for any two vertices, the slope of the line through them remains constant. Define a matroid on $E(K_n)$ by taking the independent sets to be the sets $A \subseteq E(K_n)$ such that no line slope in A depends on any other slope. This matroid is equivalent to the length rigidity matroid of Section 2.3, see [11] [17]. Though length rigidity is a more natural construction, the equivalence of these two matroids justifies the study of point-line configurations in rigidity theory.

In this section, we work in affine space instead of projective space. For a field $\mathbb{F}$ and graph G, let $\mathscr{X}_D(G)$ be the set of generic pictures in $\mathbb{F}^2$ and let $\widetilde{\mathscr{X}_D(G)}$ be such that no edge is parallel to the y-axis. Define the *slope variety* $\mathscr{S}(G) := \overline{S(G)}$ to be the Zariski closure of the set

$$S(G) := \{(m_e)_{e\in E} \in \mathbb{F}^{|E|} \mid \exists P \in \widetilde{\mathscr{X}_D(G)} \text{ with edge slopes } m_e \qquad \forall e \in E\}.$$

We will see that the only relations among the coordinates of slope vectors are determined by the rigidity circuits, and given by *tree polynomials*. Since much of the work in the following chapters deals with these polynomials, we will walk through their construction.

Fix a graph $G = (V,E)$ with $|E| = 2|V| - 2$. Label the coordinates of $v_i \in V$ as (x_i, y_i) and the coordinates of the edge-slope for $v_i v_{i+1}$ as $m_{i,i+1}$. Let C be a cycle in G. To make notation easier, relabel the vertices so that $C = v_0, v_1, \ldots, v_k, v_0$. For each $i = 0, 1, \ldots, k$, the point-slope formula gives

$$y_i - y_{i+1} = m_{i,i+1}(x_i - x_{i+1}) \tag{2.4}$$

so

$$\sum_{i=0}^{k} m_{i,i+1}(x_i - x_{i+1}) = 0. \tag{2.5}$$

take $i+1 \pmod{k+1}$ and assume that no two vertices have the same x-coordinate. By substituting the identity $(x_k - x_0) = -\sum_{i=0}^{k-1}(x_i - x_{i+1})$, equation (2.5) can be expressed as

$$\sum_{i=0}^{k-1} m_{i,i+1}(x_i - x_{i+1}) - m_{k,0}\sum_{i=0}^{k-1}(x_i - x_{i+1}) = 0 \tag{2.6}$$

so

$$\sum_{i=0}^{k-1}(m_{i,i+1} - m_{k,0})(x_i - x_{i+1}) = 0. \tag{2.7}$$

We get a system of equations from (2.7), one for each cycle in G. Fix a spanning tree T of G. Let m_e be the slope of edge e and let $\bar{x}_e = x_v - x_u$ for $e = uv$. For any edge $f \notin T$, there is a unique cycle $C_T(f)$ in $T \cup \{f\}$. These cycles form a cycle basis for the graph. So we only need the equations (2.7) for these cycles. Rewrite (2.7), for each edge $f \notin T$:

$$\sum_{e \in E(T)} c_{e,f}(m_e - m_f)\bar{x}_e = 0, \tag{2.8}$$

$$c_{e,f} = \begin{cases} \pm 1 & \text{if } e \text{ is in the cycle of } T \cup \{f\} \\ 0 & \text{otherwise} \end{cases}. \tag{2.9}$$

Let M_T be the matrix with columns indexed by the edges of T, the rows indexed by the edges of $G \setminus T$ and the entry in row f column e is $c_{f,e}(m_e - m_f)$. Let X_T be the column vector whose entries $\bar{x}_e$ are indexed by the edges of T. The equations arising from (2.8) can be rewritten as $M_T X_T = 0$. The matrix M_T depends on the choice of spanning tree T. However, the determinant $|\det(M_T)|$ is independent of T.

Definition 2.5.2. Let $G = (V,E)$ be a graph with $|E| = 2|V| - 2$. The *tree polynomial* is $\tau_G := \det(M_G)$ (this is a polynomial in the slopes m_e).

Recall from Proposition 2.3.5 that rigidity circuits can be partitioned into two disjoint spanning trees.

Proposition 2.5.3. *Let $G = (V,E)$ be a graph with $|E| = 2|V| - 2$. The following hold:*

1. *τ_G vanishes on the slope variety $\mathscr{S}(G)$.*

2. *If G does not contain a rigidity circuit, then $\tau_G = 0$.*

3. *If $H \subseteq G$ is a rigidity circuit, then $\tau_H \big| \tau_G$.*

4. *If G is a rigidity circuit, then* $\deg(\tau_G) = |V| - 1$.

5. *For $A \subseteq E$, let $m_A = \prod_{e \in A} m_E$. Then*

$$\tau_G = \sum_{T \in Cpl(G)} sgn(T) m_T.$$

where $Cpl(G)$ is the set of coupled spanning trees of G, and $sgn(T) \in \{+1,-1\}$ coming from the determinant.

Remark 2.5.4. Let $G = (V,E)$ be a graph with $|E| = 2|V|-2$ and suppose there is a vertex v of degree 2. If e and f are the two edges containing v, then every coupled spanning tree has exactly one of those two edges and the tree polynomial factors:

$$\tau_G = \pm(m_e - m_f)\tau_{G\setminus\{e,f\}}.$$

Theorem 2.5.5. *Let G be a graph. The slope variety $\mathscr{S}(G)$ is defined by the prime ideal*

$$I(G) := \langle \tau_H \mid H \subseteq G \textit{ is a rigidity circuit}\rangle.$$

Theorem 2.5.6. *The slope variety $\mathscr{S}(K_n)$ of the complete graph K_n is defined by the ideal*

$$I(K_n) := \langle \tau_W \mid W \subseteq K_n \textit{ is a wheel}\rangle.$$

2.5.3 Gröbner bases

Before we say more about the tree polynomials we must give some background on Gröbner bases. Fix a polynomial ring $\mathbb{F}[x_1,\ldots,x_n]$ over a field $\mathbb{F}$. It is notationally convenient to use the abbreviation $x^a = x_1^{a_1}\cdots x_n^{a_n}$, where $a = (a_1,\ldots,a_n) \in \mathbb{Z}^n$.

Definition 2.5.7. A *term order* on $\mathbb{F}[x]$ is an ordering $>$ of the monomials $x^a \in \mathbb{F}[x]$ satisfying:

1. $>$ is a total ordering.

2. If $x^a > x^b$ and x^c is a monomial, then $x^{a+c} > x^{b+c}$.

3. $>$ is a well ordering.

A natural term order on $\mathbb{F}[x]$ is the *lexicographical term order*, denoted $>_{lex}$. Order the variables $x_1 > x_2 > \cdots > x_n$ and declare $x^a > x^b$ if $a_i > b_i$ for the first i in which $a_i \neq b_i$. A problem with this order is that it does not take into account the total degree of a monomial. For example, take $x > y$ in $\mathbb{F}[x,y]$, then $x^2y > xy^6$.

It is often convenient to use term orders that refine the total degree partial order. One particular example we use is the *reverse lexicographical (revlex)* term order. This is denoted $>_{revlex}$. Order the variables $x_1 > x_2 > \cdots > x_n$ and declare $x^a > x^b$ if

1. $a_1 + \cdots + a_n > b_1 + \cdots + b_n$, or

2. $a_1 + \cdots + a_n = b_1 + \cdots + b_n$ and $a_i < b_i$ for the largest i for which $a_i \neq b_i$.

Definition 2.5.8. Fix a term order $>$ on $\mathbb{F}[x]$. The *initial term* $in_>(f)$ of a polynomial $f \in \mathbb{F}[x]$ is the largest term of f with respect to $>$. The *initial ideal* of an ideal $I \subseteq \mathbb{F}[x]$ is the ideal

$$in_>(I) := \langle in_>(f) \mid f \in I \rangle.$$

When it is understood what the order is, we simply write $in(f)$ and $in(I)$. Suppose the ideal $I \subseteq \mathbb{F}[x]$ is generated by polynomials $f_1, \dots, f_t \in \mathbb{F}[x]$. Then,

$$\langle in(f_1), \dots, in(f_t) \rangle \subseteq in(I).$$

These ideals are not equal, in general.

Definition 2.5.9. A set of polynomials $\{g_1,\dots,g_t\} \subseteq \mathbb{F}[x]$ is a *Gröbner basis* if

$$\langle in(g_1),\dots,in(g_t)\rangle = in\langle g_1,\dots,g_t\rangle.$$

Proposition 2.5.10. *If $I \subseteq J$ and $in(I) = in(J)$, then $I = J$.*

We now tie in Gröbner bases with slope varieties. We work in the polynomial ring $\mathbb{F}[m_{i,j} \mid 1 \le i < j \le n]$ where the variables correspond to the slopes of the edges of K_n. Order the variables $m_{1,2} > m_{1,3} > \cdots > m_{n-1,n}$ and take the revlex term order on the monomials. As stated earlier, the defining ideal for the slope variety $\mathscr{S}(K_n)$ of K_n is generated by the set

$$\{\tau_W \mid W \text{ is a wheel in } K_n\}.$$

It was shown in [12] that this set is a Gröbner basis with respect to this term order.

Since the terms of the tree polynomials correspond to spanning trees, we will describe initial terms via the corresponding spanning trees. Say that a tree T is the *initial tree* of W if $m_T = in(\tau_W)$.

Theorem 2.5.11. *[12, Theorem 4.3] Let T be a tree with $V(T) \subseteq [n]$. Then the following are equivalent:*

1. *There exists a wheel $W \subseteq K_n$ such that $m_T = in(\tau_W)$.*

2. *T contains a path $(v_1,\dots,v_k)$ satisfying the conditions*

$$\begin{aligned}
&k \geq 4,\\
&\max(v_1,\dots,v_k) = v_1,\\
&\max(v_2,\dots,v_k) = v_k,\\
&v_2 > v_{k-1}.
\end{aligned} \tag{2.10}$$

Remark 2.5.12. Fix a path $T = (v_1,\dots,v_k)$ satisfying conditions (2.10). In the proof of Theorem 2.5.11, Martin shows that the wheel $W(v_k; v_1,\dots,v_{k-1})$ has T as its initial tree.

The tree polynomials for wheels have another representation that we will need throughout. Let T be the spanning tree, of $W := W(v_0; v_1,\dots,v_k)$, consisting of the edges $\{v_0v_1, v_0v_2,\dots,v_0v_k\}$. Construct the matrix M with columns indexed by the edges in T and rows indexed by the remaining edges as above:

$$M = \begin{bmatrix}
m_{0,1}-m_{1,2} & m_{1,2}-m_{0,2} & 0 & \cdots & 0\\
0 & m_{0,2}-m_{2,3} & m_{2,3}-m_{0,3} & \cdots & 0\\
0 & 0 & m_{0,3}-m_{3,4} & \cdots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
m_{1,k}-m_{0,1} & 0 & 0 & \cdots & m_{0,k}-m_{1,k}
\end{bmatrix}$$

Take the determinant to get

$$\tau_W = \underbrace{\prod_{i=1}^{k}(m_{0,i}-m_{i,i+1})}_{\tau_1} - \underbrace{\prod_{i=1}^{k}(m_{0,i}-m_{i-1,i})}_{\tau_2} \tag{2.11}$$

where $m_{k,k+1} = m_{1,k}$ [12, eqn. (6)]. We note that each factor in the first term τ_1 is a radius minus an adjacent chord pointing in the clockwise direction (assuming that

W is drawn so that the vertices of the cycle $(v_1, v_2, \ldots, v_k)$ increase in the clockwise direction) and the factors in the term τ_2 are a radius minus an adjacent chord in the counter-clockwise direction. This form of the tree polynomial for wheels will be useful in Chapter 3.

Chapter 3

Slope varieties over finite fields

The tree polynomials have integer coefficients, which raises the question of counting their solutions over a finite field. Let $\mathbb{F}_q$ be the field with q elements. In this chapter, we count the solutions of the tree polynomials over $\mathbb{F}_2$ and give some generalizations for $q > 2$. When the slope variety is considered over $\mathbb{F}_q$, the points correspond to drawings in $\overline{\mathbb{F}_q}^2$ whose slopes are in $\mathbb{F}_q$. These drawings need not be in $\mathbb{F}_q^2$. If $q = 2$, then there is no way to draw K_n, for $n \geq 3$, so that the vertices have distinct x-coordinates. When considered over $\mathbb{Z}$, the tree polynomials are in the kernel of the map from $\mathbb{Z}[\{m_{i,j}\}] \to \mathbb{Q}[\{x_i, y_i, \frac{1}{x_i - x_j}\}]$ defined by $m_{i,j} \mapsto \frac{y_i - y_j}{x_i - x_j}$. The main result of this chapter is the following theorem:

Theorem 3.0.13. *Let n be a positive integer and let $\mathbb{F}_2[K_n] := \mathbb{F}_2[m_{1,2}, \ldots, m_{n-1,n}]$. Let I_n denote the ideal of $\mathbb{F}_2[K_n]$ generated by the tree polynomials of wheel subgraphs of K_n, and let J_n denote the ideal generated by the tree polynomials of K_4-subgraphs of K_n (so $J_n \subseteq I_n$).*

Then the following sets are equinumerous:

1. *the zeroes of I_n, i.e., the points in $\mathbb{F}_2^{\binom{n}{2}}$ on which all tree polynomials vanish;*

2. *the zeroes of J_n, i.e., the points in $\mathbb{F}_2^{\binom{n}{2}}$ on which all tree polynomials of 3-wheels vanish;*

3. *complement-reducible graphs (or "cographs") on vertex set $[n] = \{1,2,\ldots,n\}$, that is, graphs on $[n]$ having no induced subgraph isomorphic to a four-vertex path;*

4. *switching-equivalence classes of graphs on vertex set $[n+1]$ such that no member of the class contains an induced 5-cycle.*

We will explain all these combinatorial interpretations below. The following Theorem appears in [15, Exercise 5.40] and is credited to Cameron [4].

Theorem 3.0.14. *The following sets are equinumerous:*

1. *switching-equivalence classes of graphs on vertex set $[n+1]$ such that no member of the class contains an induced 5-cycle;*

2. *series-parallel posets with n labeled vertices;*

3. *series-parallel networks with n labeled edges.*

In this chapter, we use the special structure of tree polynomials to prove first the equality of (1), (2) and (3) of Theorem 3.0.13 (Proposition 3.4.3), and then a bijection between (3) and (4) of Theorem 3.0.13 (Proposition 3.5.1).

We note that a bijection between *unlabeled* complement-reducible graphs and *unlabeled* series-parallel networks was given by Sloane, see sequence A000084 [13]. We have not found in the literature an explicit bijection for the corresponding labeled objects. Let $S_2(K_n)$ be the zero set of I_n. The numbers of points in $S_2(K_1)$, $S_2(K_2)$, $\ldots$, are

$$1,\ 2,\ 8,\ 52,\ 472,\ 5504,\ 78416,\ \ldots$$

which is sequence A006351 in [13].

3.1 Series-parallel networks

A *network* is a graph G with two vertices s_G, t_G designated as the *source* and *sink*, respectively. Two networks G and H can be connected in *series* or *parallel*. The *series connection* $G \oplus H$ is defined by identifying t_G with s_H, and designating s_G as the source and t_H as the sink. The *parallel connection* $G + H$ is defined by identifying s_G with s_H and t_G with t_H.

A *series-parallel network* is a graph obtained from the following rules:

1. a graph with one edge st is a series-parallel network;

2. if G and H are series-parallel networks, then $G \oplus H$ and $G + H$ are series-parallel networks.

One can define series and parallel connections for posets in a similar fashion; see [14, Section 3.2]. Two posets P and Q are connected in series by taking their *ordinal sum* $P \oplus Q$: declaring that all elements of Q are larger than all elements of P (or vice versa) leaving all other relations unchanged. The two posets are connected in parallel by taking the disjoint union. A *series-parallel poset* is a poset built up from single-element posets by series and parallel extensions.

Let $s(n)$ be the number of labeled series-parallel networks on n vertices. The sequence begins

$$s(1) = 1, \quad s(2) = 2, \quad s(3) = 8, \quad s(4) = 52, \quad s(5) = 472, \quad s(6) = 5504, \quad \ldots$$

This is sequence A006351 in the On-Line Encyclopedia of Integer Sequences [13].

3.2 Switching equivalence

Let G be a graph on $[n+1]$ and let $X \subseteq [n]$. The *switch* of G with respect to X is the graph $s_X(G)$ on $[n+1]$ whose edges e satisfy one of two conditions:

1. $e \in E(G)$ and either both vertices of e belong to X or neither do;

2. $e \notin E(G)$ and exactly one vertex of e belongs to X.

This operation is also referred to as graph switching or Seidel switching [18]. Let $\mathscr{G}_{n+1}$ be the set of graphs on $[n+1]$. Then switching defines an action of $\mathbb{Z}_2^n$ on $\mathscr{G}_{n+1}$. For $x = (x_1, \ldots, x_n) \in \mathbb{Z}_2^n$, let $X = \{i \mid x_i = 1\} \subset [n]$. Then the group action is $xG = s_X(G)$. This action is free because $s_X(G) = G$ if and only if $X = \emptyset$. The orbits are called *switching classes*, denoted by $[G]$. To see that each orbit contains exactly one graph in which the vertex $n+1$ is isolated, let $G \in \mathscr{G}_{n+1}$ and let $X = N(n+1)$ be the set of neighbors of $n+1$. Then the graph $s_X(G)$ has $n+1$ as an isolated vertex. On the other hand if X is any other subset of $[n]$, then $n+1$ will be adjacent to some vertex of $s_X(G)$. The number of switching classes on $[n+1]$ is $s(n)$, the number of labeled series-parallel networks [15, Exercise 5.40(b)], [4].

3.3 Tree polynomials

We revisit the tree polynomials constructed in Section 2.5.3. A *(rigidity) pseudocircuit* is a graph H whose edge set can be partitioned into two spanning trees. A *coupled spanning tree* of H is a tree whose complement is also a spanning tree; the set of all coupled spanning trees of H is denoted $Cpl(H)$. For each pseudocircuit $H \subseteq G$, there is a polynomial

$$\tau_H = \sum_{T \in Cpl(H)} \varepsilon(H,T) m_T \tag{3.1}$$

that vanishes on the slope variety of G; where each $\varepsilon(H,T) \in \{1,-1\}$. Because the tree polynomials have integer coefficients, it makes sense to consider these polynomials inside the polynomial ring

$$\mathbb{F}_q[G] := \mathbb{F}_q[m_e \mid e \in E(G)].$$

Define the *q-slope variety* $S_q(G)$ to be the zero set of the ideal generated by the tree polynomials of all pseudocircuit subgraphs of G. The main concern of this chapter is $S_2(K_n)$, the set of zeroes of the complete graph over $\mathbb{F}_2$.

Suppose we draw the wheel $W(v_0; v_1, \ldots, v_k)$ with v_0 in the center and the indices of the spokes increasing as we travel clockwise around the perimeter. Each binomial factor in τ_1 is a radius minus the adjacent chord pointing in the clockwise direction, whereas each binomial factor in τ_2 is a radius minus the adjacent chord pointing in the counter-clockwise direction. Therefore, if we expand the expression (2.11) for τ_W, then the claw subgraph (i.e., the graph consisting of three edges that meet at a point) and the cycle of all the chords each occur twice, and with opposite signs. The only remaining terms are coupled spanning trees, which are obtained by picking a nontrivial subset of radii along with all chords pointing clockwise or counterclockwise, but not both. See Figure 3.1.

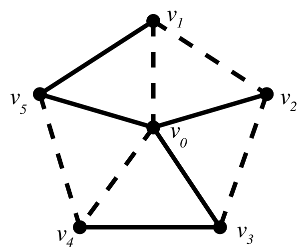

Figure 3.1: Two complementary spanning trees of a 5-wheel

The tree polynomials of all the wheels in K_n generate the ideal of tree polynomials of all rigidity pseudocircuits in K_n [12]. Define ideals $I_n, J_n \subseteq \mathbb{F}_2[G]$ as follows:

$$I_n = \langle \tau_W \mid W \text{ is a wheel in } K_n \rangle, \tag{3.2}$$

$$J_n = \langle \tau_Q \mid Q \subseteq K_n \text{ is isomorphic to } K_4 \rangle. \tag{3.3}$$

It was conjectured in [12] that $I_n = J_n$ when considered as ideals over $\mathbb{C}$. Using the computer algebra system Macaulay [9] this conjecture has been verified for $n \leq 9$.

3.4 A bijection between slope vectors and complement-reducible graphs

In this section we count the points of $S_2(K_n)$, the slope variety of K_n over $\mathbb{F}_2$. The points of $\mathbb{F}_2^{\binom{n}{2}}$ have their coordinates indexed by the edges of K_n and have value either 0 or 1, which motivates the following notation:

Definition 3.4.1. Let $a = (a_{1,2}, a_{1,3}, \ldots, a_{n-1,n}) \in \mathbb{F}_2^{\binom{n}{2}}$. We define the graph G_a to be the graph on $[n]$ with edge set $E(G_a) = \{ij \mid a_{i,j} = 1\}$.

Proposition 3.4.2. *Let $W = W(v_0; v_1, \dots, v_k)$ be a wheel and $a \in \mathbb{F}_2^{\binom{n}{2}}$. Then $\tau_W(a) \neq 0$ if and only if $H_a := G_a \cap W$ is a coupled spanning tree of W.*

Proof. ($\Leftarrow$) Suppose that H_a is a coupled spanning tree of W. When τ_W is written in the form of equation (3.1), over $\mathbb{F}_2$, it is the sum of all coupled spanning trees of W. Evaluating τ_W at a gives exactly one non-zero term, hence $\tau_W(a) \neq 0$.

($\Rightarrow$) Suppose $\tau_W(a) \neq 0$. First we show that H_a is a spanning tree of W. Over $\mathbb{F}_2$ exactly one of $\tau_1(a)$ or $\tau_2(a)$ has value 1, say $\tau_1(a) = 1$. Each binomial factor of $\tau_1(a)$ must contain exactly one variable with value 1. Therefore, H_a contains exactly k edges, which is the number of edges of a spanning tree of W. In order to show that H_a is a spanning tree it is enough to show that it is acyclic. If H_a contains a cycle C, then either there exist i and j, $1 \leq i < j \leq k$, such that $v_0v_i, v_iv_{i+1}, v_{j-1}v_j, v_0v_j \in E(C)$, or C is the set of chords of W. In the first case, both terms $\tau_1(a)$ and $\tau_2(a)$ have value 0 because $m_{0i} - m_{i(i+1)}$ is a factor of τ_1 and $m_{0j} - m_{(j-1)j}$ is a factor of τ_2. In the second case, formula (2.11) will be

$$\tau_W(a) = \prod_{i=1}^{k}(a_{0i} - 1) - \prod_{i=1}^{k}(a_{0i} - 1) = 0.$$

Now we show that H_a is in fact a *coupled* spanning tree of W. Define $\overline{a} \in \mathbb{F}_2^{\binom{n}{2}}$ by $\overline{a_{ij}} = 1 - a_{ij}$ for all $1 \leq i < j \leq n$. Therefore, $G_{\overline{a}} = \overline{G_a}$ is the complement of G_a. If $\tau_W(a) \neq 0$ then $\tau_W(\overline{a}) \neq 0$ because each binomial factor of $\tau_i(\overline{a})$ will have the same value as in $\tau_i(a)$, for $i = 1, 2$. Therefore $H_{\overline{a}} = \overline{H_a} \cap W$ is a spanning tree of W, hence H_a is a coupled spanning tree of W. $\square$

The following proposition gives additional evidence to suggest that I_n and J_n are equal over $\mathbb{C}$, but whether or not they are equal is still unknown.

Proposition 3.4.3. *Let $a \in \mathbb{F}_2^{\binom{n}{2}}$. The following are equivalent:*

1. *a is a zero of I_n;*

2. *a is a zero of J_n;*

3. *G_a is a complement-reducible graph.*

Proof. $(1 \Rightarrow 2)$ This implication follows from the containment $J_n \subseteq I_n$.

$(2 \Rightarrow 3)$ Suppose $a \in \mathbb{F}_2^{\binom{n}{2}}$ is a zero of J_n. By Proposition 3.4.2, if $W \subseteq K_n$ is any 3-wheel (and hence isomorphic to K_4), then $G_a \cap W$ is not a coupled spanning tree of W. Since every coupled spanning tree of K_4 is isomorphic to P_4 (the only spanning trees of K_4 are isomorphic to P_4 or the three edge claw), G_a does not contain an induced P_4.

$(3 \Rightarrow 1)$ Let $a \in \mathbb{F}_2^{\binom{n}{2}}$ be such that G_a is a complement-reducible graph. Let $W \subseteq K_n$ be a wheel with $V = V(W)$. Either $G_a|_V$ or $\overline{G_a}|_V$ is disconnected, because G_a is a complement-reducible graph. Therefore either $G_a \cap W$ or $G_{\overline{a}} \cap W$ is disconnected. Since these two graphs are complementary subgraphs of W, neither one is a coupled spanning tree. Therefore, by Proposition 3.4.2, $\tau_W(a) = 0$ for every wheel $W \subseteq K_n$. □

3.5 A bijection between complement-reducible graphs and switching classes

In this section, we establish a bijection (Proposition 3.5.1) between the set of graphs on n labeled vertices with an induced P_4, and the switching classes on $n+1$ labeled vertices containing a graph with an induced 5-cycle. Recall from Section 3.2 that each switching class contains exactly one graph in which the vertex $n+1$ is isolated. Therefore the bijection from the set of graphs on $[n]$ to the switching classes on $[n+1]$ is given by sending $G \subseteq K_n$ to $[G]$, the orbit containing G with isolated vertex $n+1$.

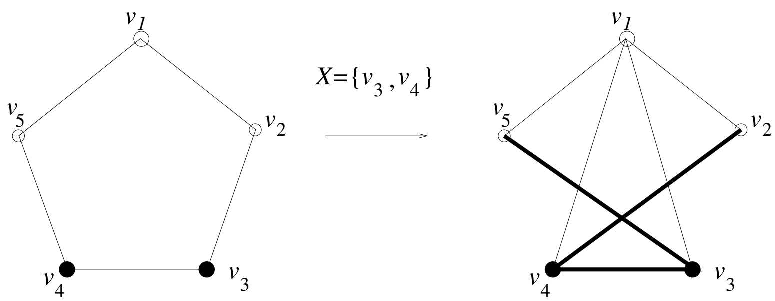

Figure 3.2: The action with $X = \{v_3, v_4\}$

Proposition 3.5.1. *Let the additive group $\mathbb{Z}_2^n$ act on $\mathscr{G}_{n+1}$ by switching as described in Section 3.2. Then:*

1. *If $G \in \mathscr{G}_{n+1}$ has an induced 5-cycle, then every $H \in [G]$ has an induced 4-path.*

2. *If $G \in \mathscr{G}_n$ has an induced 4-path, then, regarding G as a graph on $[n+1]$ by introducing $n+1$ as an isolated vertex, there is an $H \in \mathscr{G}_{n+1}$ such that $G \in [H]$ and H has an induced 5-cycle.*

Proof. (1) Let $G \in \mathscr{G}_{n+1}$ have an induced 5-cycle $C = \{v_1, \ldots, v_5\}$, and let $X \subseteq [n]$. If $|V(C) \cap X| < 2$, then four of the vertices, say $U = \{v_1, v_2, v_3, v_4\}$, are in $[n] \setminus X$. Switching by X does not affect the induced subgraph on U. Similarly, if $|V(C) \cap X| > 3$, then $(s_X(G) \mid_U) \cong P_4$.

Suppose $|V(C) \cap X| = 2$. Without loss of generality we may assume either $X = \{v_2, v_5\}$ or $X = \{v_3, v_4\}$. In both cases $v_5v_3v_4v_2$ is an induced 4-path in $s_X(G)$, as shown in the figure. If $|V(C) \cap X| = 3$, then $|V(C) \cap ([n+1] \setminus X)| = 2$. The same results as above will hold for this case, therefore $s_X(G)$ has an induced P_4.

(2) Suppose $v_5v_3v_4v_2$ is an induced P_4 in $G \subseteq K_n$ and $X = \{v_2, v_5\}$. Then $s_X(G)$ has the induced 5-cycle $C = \{v_1, \ldots, v_5\}$ with $v_1 = n+1$. $\square$

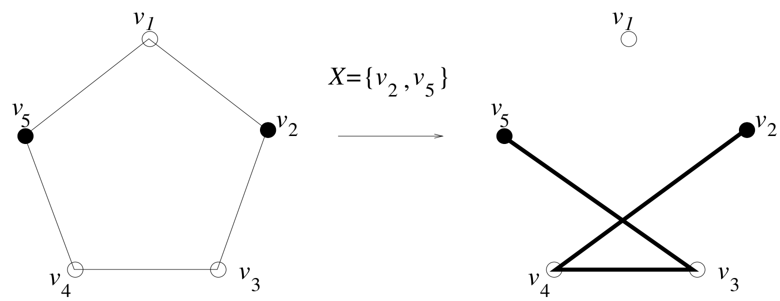

Figure 3.3: The action with $X = \{v_2, v_5\}$

3.6 Counting points over other finite fields

It is natural to ask whether these techniques can be extended to enumerate points of the slope variety $S_q(K_n)$ over $\mathbb{F}_q$. This problem appears to be difficult, because the zeroes of a tree polynomial over an arbitrary field do not seem to admit a uniform graph-theoretic description as they do over $\mathbb{F}_2$. In this section, we describe some partial progress in this direction, and explicitly work out the simplest nontrivial case ($n = 4$, $q = 3$) to illustrate the kinds of difficulties involved.

A point in $\mathbb{F}_q^{\binom{n}{2}}$ corresponds to an $\mathbb{F}_q$-*weighted* K_n, that is, a copy of K_n whose edges are assigned weights in $\mathbb{F}_q$. For $a \in \mathbb{F}_q^{\binom{n}{2}}$ define G_a to be the $\mathbb{F}_q$-weighted K_n where edge ij is given weight a_{ij}. We say that G_a has a *weight-induced subgraph* H if there is some value $\alpha \in \mathbb{F}_q$ such that

$$E(H) = \{e \in E(K_n) \mid a_e = \alpha\}.$$

One possible approach to generalizing the previous results would be to define a q-analogue to switching. Let the additive group $\mathbb{F}_q^n$ act on $\mathbb{F}_q^{\binom{n}{2}}$ by

$$((x_1, \ldots, x_n) \cdot a))_{ij} = (a_{ij} + x_i + x_j).$$

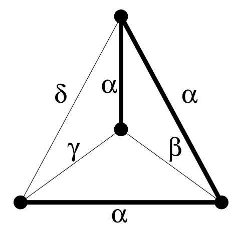

Figure 3.4: A weight induced P_4

If $q = 2$, then this is exactly the switching action described in Section 3.2. Note that this is not the same definition of q-switching given by Zaslavsky [18]. One would hope to generalize the $q = 2$ case by describing the points of $S_q(K_n)$ in terms of forbidden weight-induced subgraphs. It is not clear how to generalize the definition over an arbitrary field, or what the forbidden weight-induced subgraphs should be. However, some facts do carry over to the setting of an arbitrary finite field.

Proposition 3.6.1. *Let $W = W(v_0; v_1, v_2, v_3)$ be a 3-wheel, and let $a \in \mathbb{F}_q^{\binom{4}{2}}$ be a point whose coordinates correspond to assigning weights to the edges of W. Then:*

1. *If G_a has a weight-induced P_4, then a is not a zero of τ_W.*
2. *If G_a has a weight-induced claw (that is, a star with three edges), then a is a zero of τ_W.*
3. *If G_a has a weight-induced cycle, then a is a zero of τ_W.*

Proof. (1) Suppose that G_a has a weight-induced P_4. The induced subgraph on the vertices of that P_4 can be drawn as in Figure 3.4. where $\alpha, \beta, \gamma, \delta \in \mathbb{F}_q$, and α does not equal any of the other values. Then,

$$\tau_W(a) = (\alpha - \alpha)(\beta - \alpha)(\gamma - \delta) - (\alpha - \delta)(\beta - \alpha)(\gamma - \alpha) \neq 0.$$

(2) Suppose that G_a has a weight-induced claw, whose edges have the weight $\alpha \in \mathbb{F}_q$. If we draw W so that the center of the claw is the center of the wheel, then

$$\tau_W(a) = (\alpha-\beta)(\alpha-\gamma)(\alpha-\delta) - (\alpha-\gamma)(\alpha-\delta)(\alpha-\beta) = 0,$$

for some $\beta, \gamma, \delta \in \mathbb{F}_q$.

(3) Suppose that G_a has a weight-induced cycle C. The graph W can be drawn so that C contains the vertex v_0. Then, for some $1 \leq i < j \leq 3$, the edges v_0v_i, v_iv_{i+1}, v_0v_j, v_jv_{j-1} all have the same weight α. (Note that if the cycle is a 3-cycle then $v_{j-1} = v_i$.) Then both $\tau_1(a)$ and $\tau_2(a)$ contain the factor $\alpha - \alpha$, so $\tau_W(a) = 0$. $\square$

Corollary 3.6.2. *Let $a \in \mathbb{F}_q^{\binom{n}{2}}$. If G_a contains a weight-induced P_4, then a is not a zero of I_n over $\mathbb{F}_q$. On the other hand, if every 4-clique of G_a contains a weight-induced cycle or a weight-induced claw, then a is a zero of J_n over $\mathbb{F}_q$.*

Example 4. Let $W = W(v_0; v_1, v_2, v_3)$ be a 3-wheel. We use Proposition 3.6.1 to count the number of zeroes of τ_W over $\mathbb{F}_3$.

If some value occurs at least four times in a, then $\tau_W(a) = 0$ because G_a has a weight-induced cycle. If some value α occurs exactly three times in a, then $\tau_W(a) \neq 0$ if and only if the weight-induced graph on α is a 4-path. The cases where each value of a occurs two times are not covered by Proposition 3.6.1, so we must consider them separately. For distinct $\alpha, \beta, \gamma \in \mathbb{F}_3$ there are three possibilities, up to a relabeling of the vertices; see Figure 3.5.

Define the *type* of $a \in \mathbb{F}_q^d$ to be the partition whose parts are the numbers of occurrences of each element of $\mathbb{F}_q$ among the entries of a. Some simple counting gives the following table:

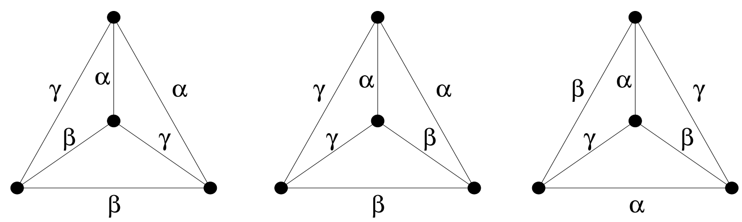

Figure 3.5: The first weight corresponds to a zero of τ_W, but the second two do not.

Type	Number of zeroes	Number of non-zeroes
(6)	3	0
$(5,1)$	36	0
$(4,2)$	90	0
$(4,1,1)$	90	0
$(3,3)$	24	36
$(3,2,1)$	144	216
$(2,2,2)$	36	54
Total	423	306

If $q > 3$, then there are more cases to check which are not covered by Proposition 3.6.1. Using the computer algebra software Maple, one can check that over $\mathbb{F}_3$ the number of zeroes of I_4 and I_5 are 423 and 9243, respectively. Over $\mathbb{F}_5$ the numbers are 4909, 262645, respectively. It is not clear what combinatorial structure (analogous to complement-reducible graphs) might count these points; for instance, these numbers do not appear in the Encyclopedia of Integer Sequences [13].

Chapter 4

The defining ideal for slope variety

4.1 The ideal generated by tree polynomials of 3-wheels

The goal of this section is to better understand the slope variety $\mathscr{S}(K_n)$ of the complete graph K_n. The defining ideal $I(\mathscr{S}(K_n))$ is generated by the set $\{\tau_W \mid W \subseteq K_n \text{ is a wheel}\}$ [12] (recall τ_G denotes the tree polynomial of G, see Section 2.5.3). More precisely, let $R = \mathbb{F}[m_{1,2}, m_{1,3}, \ldots, m_{n-1,n}]$ be the polynomial ring in $\binom{n}{2}$ variables over an algebraically closed field $\mathbb{F}$. Define a class of ideals generated by wheel tree polynomials:

$$I_{k,n} := \langle \tau_W \mid W \subseteq K_n \text{ is a wheel with } |V(W)| \leq k \rangle \tag{4.1}$$

Then, $I_{n,n}$ is the defining ideal of the slope variety $\mathscr{S}(K_n)$ and $I_{4,n} \subseteq I_{5,n} \subseteq \cdots \subseteq I_{n,n}$. (In the notation from Chapter 3, $I_{4,n} = J_n$ and $I_{n,n} = I_n$.) The motivating problem for this chapter is the following conjecture from [12]:

Conjecture 4.1.1. *For all $n \geq 4$, $I_{4,n} = I_{n,n}$.*

Though we do not prove Conjecture 4.1.1, we give evidence to suggest it is true. If $n = 4$, then it is trivially true. The case for $n = 5$ is more involved. Consider the

4-wheel $W = W(1;2,3,4,5)$ and let $\tau_{i,j,k,\ell}$ denote the tree polynomial of the complete graph on vertices i, j, k, ℓ. The computer algebra software Macaulay 2 gives

$$\begin{aligned} 2\tau_W &= (m_{1,2} - m_{1,3} + m_{1,4} - m_{1,5})\tau_{2,3,4,5} \\ &+(-m_{1,2} + m_{2,3} - m_{2,4} + m_{2,5})\tau_{1,3,4,5} \\ &+(-m_{1,3} + m_{2,3} + m_{3,4} - m_{3,5})\tau_{1,2,4,5} \\ &+(-m_{1,4} - m_{2,4} + m_{3,4} + m_{4,5})\tau_{1,2,3,5} \\ &+(-m_{1,5} + m_{2,5} - m_{3,5} + m_{4,5})\tau_{1,2,3,4} \in I_{4,5}, \end{aligned} \tag{4.2}$$

(see Appendix A for the source code). Relabeling the vertices, we see that every 4-wheel tree polynomial is in $I_{4,n}$. Hence, $I_{4,n} = I_{5,n}$ for all n. We would like a similar expression for any k-wheel, $k \geq 5$. Using Macaulay 2, we can check that any 5-wheel tree polynomial is in $I_{4,n}$. However, it is not clear how equation (4.2) generalizes to k-wheels for $k \geq 5$. The hope is that there is a representation with a combinatorial interpretation.

Theorem 4.1.2. *For all $n \geq 4$, $\sqrt{I_{4,n}} = I_{n,n}$.*

The algebraic set $Z(I_{n,n})$ is the slope variety of the complete graph K_n. Theorem 4.1.2 says that the slope variety is defined, set-theoretically, by the tree polynomials of the 3-wheels. This section is devoted to proving this theorem. To do this, we need several lemmata and we may assume throughout that $n \geq 6$.

Lemma 4.1.3. *Let $W \subseteq K_n$ be a wheel on n vertices. Let i, j and ℓ be vertices of W such that ℓ is the center and ij is a non-edge of W. Then there are wheels $A, B \subseteq K_n$, on fewer than n vertices, such that $(m_{\ell,i} - m_{i,j})\tau_W \in \langle \tau_A, \tau_B \rangle$.*

Proof. We will construct a matrix M, as we did in Section 2.5.3, whose maximal minors are tree polynomials. The Plücker relations, Lemma 2.4.9, on these maximal minors will give the desired relations on the tree polynomials. We will now construct M. Fix the wheel $W = W(1;2,3,\ldots,n)$ on $[n]$, let k be an integer, $3 \leq k \leq n-2$, and take $G = ([n],E)$ to be the graph W with the two edges $2(n-1)$ and kn adjoined. To simplify notation, relabel the edges $e_1 = (n-1)n$, $e_2 = 2n$, $e_3 = 2(n-1)$ and $e_4 = kn$.

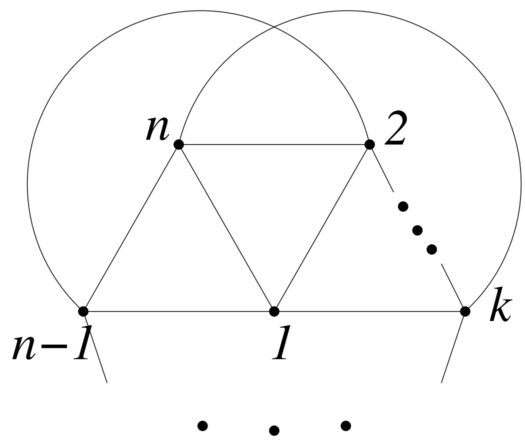

Figure 4.1: The graph G with $E(G) = E(W) \cup \{2(n-1), kn\}$

Let T be the spanning tree of W in which every edge contains vertex 1. For each edge $e \notin T$, there is a unique cycle $C(e)$ in $T \cup \{e\}$. Construct an $(n+1) \times (n-1)$ matrix M, as we did in Section 2.5.1: the columns are indexed by the edges of T and the rows are indexed by the edges of $G \setminus T$. The (e,f)-coordinate of M is

$$M_{e,f} = \begin{cases} \pm 1(m_e - m_f) & \text{if } f \in C(e), \\ 0 & \text{otherwise.} \end{cases}$$

Let M_{ij} denote the the maximal square submatrix where rows e_i and e_j are deleted and define $X_{ij} := \det(M_{ij})$. Let G_{ij} be the graph obtained from G by deleting edges e_i and e_j. Then X_{ij} is the tree polynomial of G_{ij}. By the Plücker relations, we have

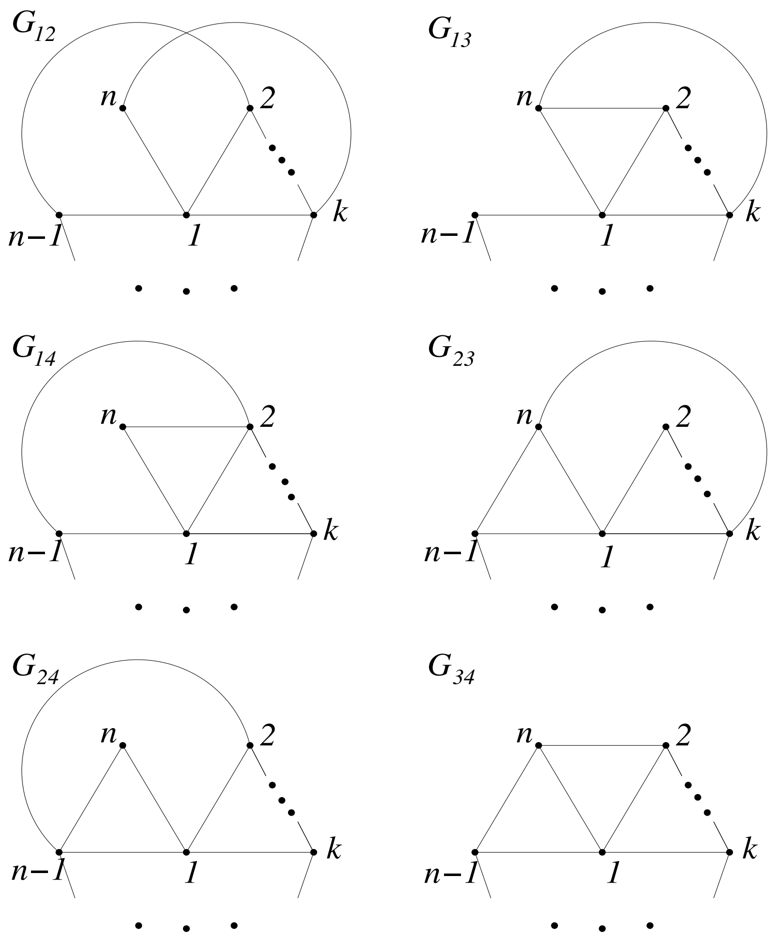

Figure 4.2: The graphs corresponding to the maximal submatrices

$$X_{12}X_{34} - X_{13}X_{24} + X_{14}X_{23} = 0. \tag{4.3}$$

Each graph G_{ij} contains one of the following wheels as a proper subgraph:

$$A := W(1; 2, 3, \ldots, k, n),$$

$$B := W(1; k, k+1, \ldots, n),$$

$$C := W(1; 2, 3, \ldots, n-1).$$

By repeated iterations of Remark 2.5.4, factor the tree polynomials:

$$X_{12} = (m_{1,n} - m_{k,n})\tau_C$$
$$X_{13} = (m_{1,k+1} - m_{k,k+1})(m_{1,k+2} - m_{k+1,k+2})\cdots(m_{1,n-1} - m_{n-2,n-1})\tau_A$$
$$X_{14} = (m_{1,n} - m_{2,n})\tau_C$$
$$X_{23} = (m_{1,2} - m_{2,3})\cdots(m_{1,k-1} - m_{k-1,k})\tau_B$$
$$X_{24} = (m_{1,n} - m_{n-1,n})\tau_C$$
$$X_{34} = \tau_W$$

Each term of (4.3) has τ_C as a factor. Cancel it to get

$$(m_{1,n} - m_{k,n})\tau_W = (m_{1,n} - m_{n-1,n})X_{13} - (m_{1,n} - m_{2,n})X_{23}.$$

Therefore, $(m_{1,n} - m_{k,n})\tau_W \in \langle X_{13}, X_{23}\rangle \subseteq \langle \tau_A, \tau_B\rangle$. The lemma follows, because vertex k can be any spoke not adjacent to n. $\square$

We will generalize Lemma 4.1.3 so that $(m_e - m_f)\tau_W \in I_{n-1,n}$ for any edges e and f. To do so, break the problem into cases, depending on the edges e and f.

Proposition 4.1.4. *If W is a wheel on n vertices and neither e nor f are chords of W, then*

$$(m_e - m_f)\tau_W \in I_{n-1,n}.$$

Proof. Say $W = W(1; 2, \ldots, n)$. Let F be the set of edges $f \in K_n$ that are non-chords of W. Let U be the vector space spanned by $\{m_f - m_{f'} \mid f, f' \in F\}$. The proposition is equivalent to the statement that $u\tau_W \in I_{n-1,n}$ for all $u \in U$. Fix a radius $r = 1i$. Then the set $\{m_r - m_f \mid f \in F\}$ spans U. Hence, it suffices to prove that $(m_r - m_f)\tau_W \in I_{n-1,n}$ for every $f \in F$. There are four cases to consider:

1. $f = 1j$ is a radius and $ij \in E(W)$;

2. $f = 1j$ is a radius and $ij \notin E(W)$;

3. $f = ij \notin E(W)$ where i and j are distinct spokes;

4. $f = jk \notin E(W)$ where i, j and k are distinct spokes.

We have already proven Case 3 in Lemma 4.1.3. We will prove the remaining cases in order.

Case 1: Since $n \geq 6$, there is a spoke k such that $ik, jk \in F$. Case 3 gives

$$m_{1,i} - m_{1,j} = (m_{1,i} - m_{i,k}) + (m_{i,k} - m_{1,k}) + (m_{1,k} - m_{j,k}) + (m_{j,k} - m_{1,j}).$$

Case 2: Take a sequence of adjacent spokes $i = i_0, i_1, \ldots, i_s = j$. Then

$$m_{1,i} - m_{1,j} = (m_{1,i} - m_{1,i_1}) + (m_{1,i_1} - m_{1,i_2}) + \cdots + (m_{1,i_{s-1}} - m_{1,j}).$$

Each binomial is of the form covered in Case 1.

Case 4: We have

$$m_{1,i} - m_{j,k} = (m_{1,i} - m_{1,j}) + (m_{1,j} - m_{j,k}).$$

The two binomials on the right are covered by Case 2 and Case 3, respectively. $\square$

Lemma 4.1.5. *Let e and f be edges in K_n that share an endpoint. Then*

$$I_{n-1,n} : (m_e - m_f) = I_{n,n}.$$

Proof. Each of the ideals $I_{k,n}$ is fixed by the action of permuting the labels of the vertices. Therefore, it suffices to prove this lemma for a particular pair of edges e and f: $e = 1(n-1)$ and $f = 2(n-1)$.

Define the ideal

$$J := \langle \tau_W \mid W \subseteq K_n \text{ is a wheel and neither } e \text{ nor } f \text{ are chords in } W \rangle.$$

We have

$$I_{n-1,n} + J \subseteq I_{n-1,n} : (m_e - m_f) \subseteq I_{n,n} : (m_e - m_f) = I_{n,n} \tag{4.4}$$

where the first containment follows from Proposition 4.1.4 and the final equality comes from the fact that $I_{n,n}$ is prime [12, Theorem 1.1] (see also Section 2.5.3). We will show that $in(I_{n-1,n} + J) \supseteq in(I_{n,n})$. By Proposition 2.5.10, the ideals in equation (4.4) are all equal, and the result will follow.

Let W be a wheel with $V(W) = [n]$ and let T be the initial tree of W. Let $P := (v_1, \dots, v_k)$ be the path in T satisfying conditions (2.10) in Theorem 2.5.11. If $k < n$, then there is a wheel W' on $\{v_1, \dots, v_k\}$ with initial tree P. The tree polynomial $\tau_{W'}$ is in $I_{n-1,n}$ and $in(\tau_{W'}) \big| in(\tau_W)$, so we are done. If P contains a proper subpath P' such that either P' or its reverse satisfies (2.10), then there is a wheel W' whose initial tree is P'. Again, $\tau_{W'}$ is in $I_{n-1,n}$ and $in(\tau_{W'}) \big| in(\tau_W)$. The only wheels left are the ones whose initial trees are paths on the full vertex set, and no proper subpath satisfies (2.10).

Let W be a wheel with initial tree $T = (v_1, \dots, v_n)$. Suppose T satisfies (2.10) but no proper subpath does. In particular $v_1 = n$ and $v_n = n-1$. In the proof of Theorem 2.5.11, a wheel W' with center $n-1$ and initial tree T is constructed; see

2.5.12. The edges e and f are not chords of W'. Therefore, by Proposition 4.1.4, $m_T = in(\tau_W) = in(\tau_{W'}) \in in(J)$.

Thus, $in(I_{n,n}) \subseteq in(I_{n-1,n} + J)$ as desired. □

Lemma 4.1.6. *Define the ideal* $L := \langle m_e - m_f \mid e, f \in E(K_n)\rangle$. *Then,* $I_{n-1,n} : L = I_{n,n}$.

Proof. For any edges $e, f, g \in E(K_n)$, the equality $m_e - m_f = (m_e - m_g) + (m_g - m_f)$ holds. Therefore, L is generated by all the linear forms $m_e - m_f$ such that e and f share an endpoint. Proposition 2.4.8 gives

$$\begin{aligned} I_{n-1,n} : L = I_{n-1,n} : &\sum_{1 \leq i < j < k \leq n} \langle m_{i,j} - m_{j,k} \rangle \\ = &\bigcap_{1 \leq i < j < k \leq n} I_{n-1,n} : \langle m_{i,j} - m_{j,k} \rangle \\ = &\bigcap_{1 \leq i < j < k \leq n} I_{n,n} = I_{n,n}. \end{aligned}$$

Each ideal in the first intersection equals $I_{n,n}$, by Lemma 4.1.5. □

Lemma 4.1.7. $\sqrt{I_{n-1,n}} = I_{n,n}$.

Proof. Recall from Section 2.4.2 that $\sqrt{I_{n-1,n}}$ is the intersection of all prime ideals containing $I_{n-1,n}$. To prove this lemma, it suffices to show that $I_{n,n}$ is contained in each such prime. Suppose P is a prime containing $I_{n-1,n}$ and let $a \in I_{n,n}$. Then $(m_e - m_f)a \in I_{n-1,n}$ for all edges e and f, by Lemma 4.1.6. Therefore, either $a \in P$ or $L \subseteq P$. But $I_{n,n} \subseteq L$. Thus, $I_{n,n} \subseteq P$. □

Let $A \subseteq [n]$ and take K_A to be the complete graph on vertex set A. Define the ideal

$$I_{k,A} := \langle \tau_W \mid W \subseteq K_A \text{ is a wheel with } |V(W)| \leq k \rangle.$$

Let $\binom{[n]}{k}$ denote the set of subsets of $[n]$ of cardinality k. By restricting the vertex set, Lemma 4.1.7 generalizes:

Corollary 4.1.8. *For any* $A \in \binom{[n]}{k}$, $4 < k \leq n$, $\sqrt{I_{k-1,A}} = I_{k,A}$.

Lemma 4.1.9. *For all* $4 < k \leq n$, $\sqrt{I_{k-1,n}} = \sqrt{I_{k,n}}$.

Proof. We have the containment $\sqrt{I_{k-1,n}} \subseteq \sqrt{I_{k,n}}$ because $I_{k-1,n} \subseteq I_{k,n}$. To prove this lemma we must show the reverse containment $\sqrt{I_{k,n}} \subseteq \sqrt{I_{k-1,n}}$. For each $A \in \binom{[n]}{k}$, $\sqrt{I_{k-1,A}} \subseteq \sqrt{I_{k-1,n}}$ because $I_{k-1,A} \subseteq I_{k-1,n}$. Therefore,

$$\sum_{A \in \binom{[n]}{k}} \sqrt{I_{k-1,A}} \subseteq \sqrt{I_{k-1,n}}.$$

Hence, by Corollary 4.1.8 $\sum_{A \in \binom{[n]}{k}} I_{k,A} \subseteq \sqrt{I_{k-1,n}}$. By definition,

$$\sum_{A \in \binom{[n]}{k}} I_{k,A} = I_{k,n}$$

Thus, $I_{k,n} \subseteq \sqrt{I_{k-1,n}}$. □

Thus, we have finally proven Theorem 4.1.2.

4.2 Higher-dimensional analogue of the slope variety

We conclude with some observations about generalizing slope varieties to higher dimensions. That is, consider the point-line configurations for graphs in $\mathbb{F}^d$ for $d \geq 3$. As in the $d = 2$ case, we would like to eliminate the coordinates of the vertices in order to obtain the relations on the directions of the edges. If we project the point-line configuration for G onto any plane in $\mathbb{F}^d$, then the tree polynomial relations hold. The relations on the lines include the pullbacks of these tree polynomials. However, there are other relations that are not obtained from the tree polynomials.

For example, consider the case $G = K_3$. For a picture of G in the plane, there are no constraints on the edge slopes. However, in 3-dimensional space there is the non-trivial constraint that the lines must be coplanar. Since the graph K_3 does not contain a rigidity circuit, this constraint cannot come from any tree polynomial. In general, the edges of an r-cycle must lie in a common $(r-1)$-dimensional space.

Fix a field $\mathbb{F}$ and graph G. Let $\widetilde{\mathscr{X}_D^d(G)}$ be the set of generic pictures in $\mathbb{F}^d$ such that no two vertices have the same x_1-coordinate. Let $a_e \in \mathbb{F}^d$ be a direction vector of edge e. Since the first coordinate is nonzero, we can take $a_e \in \mathbb{F}^d$ to be the unique direction vector with 1 in the first coordinate. Let m_e be the $(d-1)$ vector consisting of the last $d-1$ coordinates of a_e. For a picture $P \in \mathscr{X}_D^d(G)$ of G, define the *slope-vector* $\mathbf{m} = (m_e)_{e \in E} \in \mathbb{F}^{d|E|}$ to be the concatenation of its edge direction vectors. Let

$$S^d(G) := \{(m_e)_{e\in E} \in \mathbb{F}^{|E|(d-1)} \mid \exists P \in \mathscr{X}_D^d(G) \text{with edge slopes } m_e \forall e \in E\}.$$

Define the *d-slope variety* $\mathscr{S}^d(G) := \overline{S^d(G)}$ to be the Zariski closure of the set $S^d(G)$.

The goal is to determine the defining ideal for $\mathscr{S}^d(G)$ in any dimension d. Label the coordinates of vertex v_i by $(x_i^1, x_i^2, \ldots, x_i^d)$ and label the unique direction vector for the edge v_iv_j by $[1, a_{ij}^2, \ldots, a_{ij}^d]$. For each edge v_iv_j there is a scalar λ_{ij} such that

$$[x_i^1 - x_j^1, x_i^2 - x_j^2, \ldots, x_i^d - x_j^d] = \lambda_{ij}[1, a_{ij}^2, \ldots, a_{ij}^d.$$

For each coordinate we get the equations

$$x_i^k - x_j^k = a_{ij}^k (x_i^1 - x_j^1).$$

We mimic the construction in Section 2.5.3. Let $G = (V,E)$ be a graph with $|E| = 2|V| - 2$ and let T be a spanning tree. Then, for any scalars $\alpha_2, \ldots, \alpha_d \in \mathbb{F}$, and any edge $f \notin T$ there is a unique cycle $C(f)$ in $T \cup \{f\}$ and

$$\sum_{e=ij \in C(f)} (\alpha_2 a_e^2 + \cdots + \alpha_d a_e^d)(x_i^1 - x_j^1) = 0. \tag{4.5}$$

This is exactly Equation (2.5), but with $\alpha_2 a_e^2 + \cdots + \alpha_d a_e^d$ substituted in for m_{ij}. Set up matrices, as in Section 2.5.3, and take their determinants to get the corresponding tree polynomial

$$\tau_G(\alpha_2, \ldots, \alpha_d) = \sum_{T \in Cpl(G)} sgn(T) \prod_{e \in T} (\alpha_2 a_e^2 + \cdots + \alpha_d a_e^d). \tag{4.6}$$

The same results from Proposition 2.5.3 will hold. Therefore, if G is a graph, then the defining ideal for $\mathscr{S}^3(G)$ must contain $\tau_G(\alpha_2, \ldots, \alpha_d)$ for every rigidity circuit $H \subseteq G$ and every $\alpha_2, \ldots, \alpha_d \in \mathbb{F}$.

We now generalize the previous to any graph in higher dimension d. Fix a graph G on n vertices. Let $R = \mathbb{F}[a_e^k \mid e \in E,\ 1 \leq k \leq d-1]$ be a polynomial ring in $(d-1)|E|$ variables. Let $\alpha_2, \ldots, \alpha_d \in \mathbb{F}$, and let $H \subseteq G$ be a rigidity circuit. Then the tree polynomial

$$\tau_H(\alpha_2, \ldots, \alpha_d) = \sum_{T \in Cpl(H)} sgn(T) \prod_{e \in T} (r_1 a_e^1 + \cdots r_{d-1} a_e^{d-1})$$

vanishes on the slope variety $\mathscr{S}^d(G)$. If we set $r_i = 1$ and $r_j = 0$, for $j \neq i$, then this generalized tree polynomial

$$\tau_H = \sum_{T \in Cpl(H)} sgn(T) a_T^i$$

is the tree polynomial in the plane. This amounts to projecting the graph onto the $x_i x_d$-plane.

These generalized tree polynomials are not all of the generators for the ideal defining $\mathscr{S}^d(G)$. Suppose $G = C_3$ is the 3-cycle, with vertices 1, 2 and 3. When drawn in $\mathbb{F}^2$, the slopes can be anything and there is a generic picture of C_3. However, in $\mathbb{F}^3$, there is another constraint: the edge direction vectors must be coplanar. Therefore, the matrix of direction vectors

$$\begin{bmatrix} 1 & 1 & 1 \\ a_{12} & a_{23} & a_{34} \\ b_{12} & b_{23} & b_{34} \end{bmatrix}$$

does not have full rank. Therefore, the polynomial

$$a_{12}b_{23} - a_{12}b_{34} - a_{23}b_{12} + a_{23}b_{34} + a_{23}b_{12} - a_{34}b_{23}$$

vanishes on $\mathscr{S}^3(C_3)$. This is a degree-2 polynomial, hence it is not in the ideal generated by the tree polynomials. In general, the edge-vectors of an n-cycle cannot span a d-dimensional space if $d \geq n$.

Let $C = (v_1, v_2, \ldots, v_t) \subseteq G$ be a cycle (in particular $t \geq 3$) and define the $d \times t$ matrix of edge direction vectors

$$M_C := \begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_{12}^2 & a_{23}^2 & \cdots & a_{t1}^2 \\ a_{12}^3 & a_{23}^3 & \cdots & a_{t1}^3 \\ \vdots & \vdots & \ddots & \vdots \\ a_{12}^d & a_{23}^d & \cdots & a_{t1}^d \end{bmatrix}$$

Then, every $t \times t$ minor of M_C vanishes on the slope variety $\mathscr{S}^d(G)$. Note that if $d = 2$, then there are no such $t \times t$. Therefore, for each cycle C in G there are $\binom{d}{|C|}$ such polynomials that vanish on $\mathscr{S}^d(G)$.

Appendix A

Source code for tree polynomials

This appendix provides the code used to test ideal containment in Chapter 4. We define the polynomial ring in $\binom{5}{2}$ variables over the rationals $\mathbb{Q}$ and construct the ideal of all 3-wheel tree polynomials in K_5. Call this ideal I, and we construct the matrix M of 3-wheel tree polynomials. The code given is in the Macaulay 2 format.

```
R = QQ[m_{1,2},m_{1,3},m_{1,4},m_{1,5},
    m_{2,3},m_{2,4},m_{2,5},m_{3,4},m_{3,5},m_{4,5}];
I = ideal(0_R);        --initialize the ideal of all 3-wheels
M = matrix{{0_R}};        --initialize the matrix of all 3-wheels
--the following will pick 4 vertices from 1 to 5
i = 1;
j = 2;
k = 3;
l = 4;
```

```
while i < j do(
    j = i+1;
    while j < k do(
        k = j+1;
        while k < l do(
            l = k+1;
            while l < 6 do(
--this is the tree polynomial for the wheel on verticesi, j, k, l
                T_{i, j,k,l} = (m_{i, j} - m_{j,k}) * (m_{i,k} - m_{k,l})*
                    (m_{i,l} - m_{j,l}) - (m_{i,k} - m_{j,k})*
                    (m_{i,l} - m_{k,l}) * (m_{i, j} - m_{j,l});
--add this tree polynomial to the ideal I
                I = I + ({T_{i, j,k,l}});
--add this tree polynomial to the matrix M
                M = M|matrix{{T_{i, j,k,l}}};
                l = l+1);
            k = k+1);
        j = j+1);
    i = i+1);
```

We test to see if the tree polynomial for a 4-wheel is in the ideal generated by the 3-wheel tree polynomials, then get the relations:

--Input the tree polynomial for the wheel W(1;2,3,4,5)

$$W = (m\_\{1,2\} - m\_\{2,3\}) * (m\_\{1,3\} - m\_\{3,4\}) * (m\_\{1,4\} - m\_\{4,5\})$$
$$* (m\_\{1,5\} - m\_\{2,5\}) - (m\_\{1,2\} - m\_\{2,5\}) * (m\_\{1,3\} - m\_\{2,3\})$$
$$* (m\_\{1,4\} - m\_\{3,4\}) * (m\_\{1,5\} - m\_\{4,5\});$$

$I : ideal(W);$ --Test for containment

$W//M;$ --Obtain the coefficients

The output is given in Chapter 4.

The following code is the same as above, but shows that the tree polynomial for a 5-wheel in K_6 is in the ideal of all 3-wheel polynomials.

$$R = \mathrm{QQ}[m\_\{1,2\}, m\_\{1,3\}, m\_\{1,4\}, m\_\{1,5\}, m\_\{1,6\}, m\_\{2,3\}, m\_\{2,4\}, m\_\{2,5\},$$
$$m\_\{2,6\}, m\_\{3,4\}, m\_\{3,5\}, m\_\{3,6\}, m\_\{4,5\}, m\_\{4,6\}, m\_\{5,6\}];$$

$I = \mathrm{ideal}(0\_R);$ --initialize the ideal of all 3-wheels

$M = \mathrm{matrix}\{\{0\_R\}\};$ --initialize the matrix of all 3-wheels

--the following will pick 4 vertices from 1 to 6

$i = 1;$

$j = 2;$

$k = 3;$

$l = 4;$

```
while i < j do(
    j = i+1;
    while j < k do(
        k = j+1;
        while k < l do(
            l = k+1;
            while l < 7 do(
--this is the tree polynomial for the wheel on verticesi, j,k,l
                T_{i, j,k,l} = (m_{i, j} − m_{j,k}) ∗ (m_{i,k} − m_{k,l})∗
                    (m_{i,l} − m_{j,l}) − (m_{i,k} − m_{j,k})∗
                    (m_{i,l} − m_{k,l}) ∗ (m_{i, j} − m_{j,l});
--add a new column to M this polynomial in that coordinate
                M = M|matrix{{T_{i, j,k,l}}};
                l = l+1);
            k = k+1);
        j = j+1);
    i = i+1);
```

--Input the tree polynomial for the wheel W(1;2,3,4,5,6)

$W = (m\_\{1,2\} - m\_\{2,3\}) * (m\_\{1,3\} - m\_\{3,4\}) * (m\_\{1,4\} - m\_\{4,5\})$

$* (m\_\{1,5\} - m\_\{5,6\}) * (m\_\{1,6\} - m\_\{2,6\})$

$- (m\_\{1,2\} - m\_\{2,5\}) * (m\_\{1,3\} - m\_\{2,3\})$

$* (m\_\{1,4\} - m\_\{3,4\}) * (m\_\{1,5\} - m\_\{4,5\}) * (m\_\{1,6\} - m\_\{5,6\})$

$I : ideal(W);$ --Test for containment

$W//M;$ --Obtain the coefficients

We can explicitly write the 5-wheel tree polynomial in terms of all the 3-wheel tree polynomials:

$$
\begin{aligned}
4\tau_{W(1;2,3,4,5,6)} = {} & (-2m_{1,2}m_{1,5} + 2m_{1,2}m_{1,6} - 2m_{1,2}m_{2,3} + 2m_{1,2}m_{2,4} + m_{1,3}m_{2,3} \\
& - m_{1,3}m_{2,4} + m_{1,4}m_{2,3} - m_{1,4}m_{2,4} - m_{1,5}m_{2,3} + m_{1,5}m_{2,4} \\
& + 2m_{1,5}m_{2,6} + m_{1,6}m_{2,3} - m_{1,6}m_{2,4} - 2m_{1,6}m_{2,6})\tau_{3,4,5,6} \\
& + (-m_{1,2}m_{2,3} - m_{1,2}m_{3,4} + m_{1,2}m_{3,5} + m_{1,2}m_{3,6} + m_{1,4}m_{2,3} \\
& + m_{1,4}m_{3,4} - m_{1,4}m_{3,5} - m_{1,4}m_{3,6} - m_{1,5}m_{2,3} - m_{1,5}m_{3,4} \\
& + m_{1,5}m_{3,5} + m_{1,5}m_{3,6} + m_{1,6}m_{2,3} + m_{1,6}m_{3,4} - m_{1,6}m_{3,5} \\
& - m_{1,6}m_{3,6})\tau_{2,4,5,6} \\
& + (m_{1,2}m_{2,4} - m_{1,2}m_{3,4} - m_{1,2}m_{4,5} + m_{1,2}m_{4,6} - m_{1,3}m_{2,4} \\
& + m_{1,3}m_{3,4} + m_{1,3}m_{4,5} - m_{1,3}m_{4,6} + m_{1,5}m_{2,4} - m_{1,5}m_{3,4} \\
& - m_{1,5}m_{4,5} + m_{1,5}m_{4,6} - m_{1,6}m_{2,4} + m_{1,6}m_{3,4} + m_{1,6}m_{4,5} \\
& - m_{1,6}m_{4,6})\tau_{2,3,5,6}
\end{aligned}
$$

$$
\begin{aligned}
&+(-2m_{1,2}m_{1,5}+m_{1,2}m_{3,5}-m_{1,2}m_{4,5}+2m_{1,2}m_{5,6}-m_{1,3}m_{3,5}\\
&+m_{1,3}m_{4,5}-m_{1,4}m_{3,5}+m_{1,4}m_{4,5}+2m_{1,5}m_{1,6}+2m_{1,5}m_{3,5}\\
&-2m_{1,5}m_{4,5}-m_{1,6}m_{3,5}+m_{1,6}m_{4,5}-2m_{1,6}m_{5,6})\tau_{2,3,4,6}\\
&+(2m_{1,2}m_{1,6}-m_{1,2}m_{3,6}+m_{1,2}m_{4,6}-2m_{1,2}m_{5,6}+m_{1,3}m_{3,6}\\
&-m_{1,3}m_{4,6}+m_{1,4}m_{3,6}-m_{1,4}m_{4,6}-2m_{1,5}m_{1,6}+2m_{1,5}m_{2,6}\\
&-m_{1,5}m_{3,6}+m_{1,5}m_{4,6}-2m_{1,6}m_{2,6}+2m_{1,6}m_{5,6})\tau_{2,3,4,5}\\
&+(-m_{1,2}m_{2,3}-m_{1,2}m_{3,4}+m_{1,2}m_{3,5}+m_{1,2}m_{3,6}-m_{1,3}m_{2,3}\\
&+m_{1,3}m_{2,4}-m_{2,3}m_{2,4}+m_{2,3}m_{2,5}+m_{2,3}m_{2,6}-m_{2,3}m_{3,4}\\
&+m_{2,3}m_{3,5}+m_{2,3}m_{3,6}+m_{2,5}m_{3,4}-m_{2,5}m_{3,5}-m_{2,5}m_{3,6}\\
&+m_{2,6}m_{3,4}-m_{2,6}m_{3,5}-m_{2,6}m_{3,6})\tau_{1,4,5,6}\\
&+(m_{1,2}m_{2,4}-m_{1,2}m_{3,4}-m_{1,2}m_{4,5}+m_{1,2}m_{4,6}+m_{1,4}m_{2,3}\\
&-m_{1,4}m_{2,4}+m_{2,3}m_{2,4}-2m_{2,3}m_{4,5}-m_{2,4}m_{2,5}-m_{2,4}m_{2,6}\\
&-m_{2,4}m_{3,4}+m_{2,4}m_{4,5}+m_{2,4}m_{4,6}+m_{2,5}m_{3,4}+m_{2,5}m_{4,5}\\
&-m_{2,5}m_{4,6}+m_{2,6}m_{3,4}+m_{2,6}m_{4,5}-m_{2,6}m_{4,6})\tau_{1,3,5,6}\\
&+(m_{1,2}m_{3,5}-m_{1,2}m_{4,5}-m_{1,5}m_{2,3}+m_{1,5}m_{2,4}+m_{2,3}m_{5,6}\\
&-m_{2,4}m_{5,6}-m_{2,6}m_{3,5}+m_{2,6}m_{4,5})\tau_{1,3,4,6}\\
&+(-m_{1,2}m_{3,6}+m_{1,2}m_{4,6}+m_{1,6}m_{2,3}-m_{1,6}m_{2,4}-m_{2,3}m_{5,6}\\
&+m_{2,4}m_{5,6}+m_{2,5}m_{3,6}-m_{2,5}m_{4,6})\tau_{1,3,4,5}
\end{aligned}
$$

$$
\begin{aligned}
&+(-m_{1,3}m_{2,4}+m_{1,3}m_{3,4}+m_{1,3}m_{4,5}-m_{1,3}m_{4,6}+m_{1,4}m_{2,3}\\
&+m_{1,4}m_{3,4}-m_{1,4}m_{3,5}-m_{1,4}m_{3,6}-3m_{2,3}m_{3,4}+2m_{2,3}m_{4,6}\\
&+m_{2,4}m_{3,4}+m_{3,4}m_{3,5}+m_{3,4}m_{3,6}-3m_{3,4}m_{4,5}+m_{3,4}m_{4,6}\\
&+2m_{3,6}m_{4,5}-2m_{3,6}m_{4,6})\tau_{1,2,5,6}\\
&+(-m_{1,3}m_{3,5}+m_{1,3}m_{4,5}-m_{1,5}m_{2,3}-m_{1,5}m_{3,4}+m_{1,5}m_{3,5}\\
&+m_{1,5}m_{3,6}+m_{2,3}m_{2,5}+m_{2,3}m_{3,5}-2m_{2,3}m_{4,5}+m_{2,3}m_{5,6}\\
&+m_{2,5}m_{3,4}-m_{2,5}m_{3,5}-m_{2,5}m_{3,6}-m_{3,4}m_{3,5}+m_{3,4}m_{5,6}\\
&+m_{3,5}m_{3,6}+m_{3,5}m_{4,5}-m_{3,5}m_{5,6}-m_{3,6}m_{5,6})\tau_{1,2,4,6}\\
&+(m_{1,3}m_{3,6}-m_{1,3}m_{4,6}+m_{1,6}m_{2,3}+m_{1,6}m_{3,4}-m_{1,6}m_{3,5}\\
&-m_{1,6}m_{3,6}-m_{2,3}m_{2,6}-m_{2,3}m_{3,6}+2m_{2,3}m_{4,6}-m_{2,3}m_{5,6}\\
&-m_{2,6}m_{3,4}+m_{2,6}m_{3,5}+m_{2,6}m_{3,6}+m_{3,4}m_{3,6}-m_{3,4}m_{5,6}\\
&-m_{3,5}m_{3,6}+m_{3,5}m_{5,6}-m_{3,6}m_{4,6}+m_{3,6}m_{5,6})\tau_{1,2,4,5}\\
&+(m_{1,4}m_{3,5}-m_{1,4}m_{4,5}+m_{1,5}m_{2,4}-m_{1,5}m_{3,4}-m_{1,5}m_{4,5}\\
&+m_{1,5}m_{4,6}-m_{2,4}m_{2,5}+m_{2,4}m_{4,5}-m_{2,4}m_{5,6}+m_{2,5}m_{3,4}\\
&+m_{2,5}m_{4,5}-m_{2,5}m_{4,6}-m_{3,4}m_{4,5}+m_{3,4}m_{5,6}-m_{3,5}m_{4,5}\\
&+m_{4,5}m_{4,6}+m_{4,5}m_{5,6}-m_{4,6}m_{5,6})\tau_{1,2,3,6}
\end{aligned}
$$

$$+(-m_{1,4}m_{3,6}+m_{1,4}m_{4,6}-m_{1,6}m_{2,4}+m_{1,6}m_{3,4}+m_{1,6}m_{4,5}$$

$$-m_{1,6}m_{4,6}+m_{2,4}m_{2,6}-m_{2,4}m_{4,6}+m_{2,4}m_{5,6}-m_{2,6}m_{3,4}$$

$$-m_{2,6}m_{4,5}+m_{2,6}m_{4,6}+m_{3,4}m_{4,6}-m_{3,4}m_{5,6}+2m_{3,6}m_{4,5}$$

$$-m_{3,6}m_{4,6}-m_{4,5}m_{4,6}-m_{4,5}m_{5,6}+m_{4,6}m_{5,6})\tau_{1,2,3,5}$$

$$+(m_{1,5}m_{3,6}-m_{1,5}m_{4,6}-m_{1,6}m_{3,5}+m_{1,6}m_{4,5}-m_{2,5}m_{3,6}$$

$$+m_{2,5}m_{4,6}+m_{2,6}m_{3,5}-m_{2,6}m_{4,5})\tau_{1,2,3,4}$$

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