Malliavin calculus for backward stochastic differential equations and stochastic differential equations driven by fractional Brownian motion and numerical schemes

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Abstract

In this dissertation, I investigate two types of stochastic differential equations driven by fractional Brownian motion and backward stochastic differential equations. Malliavin calculus is a powerful tool in developing the main results in this dissertation.

This dissertation is organized as follows.

In Chapter 1, I introduce some notations and preliminaries on Malliavin Calculus for both Brownian motion and fractional Brownian motion.

In Chapter 2, I study backward stochastic differential equations with general terminal value and general random generator. In particular, the terminal value has not necessary to be given by a forward diffusion equation. The randomness of the generator does not need to be from a forward equation neither. Motivated from applications to numerical simulations, first the L^p-Hölder continuity of the solution is obtained. Then, several numerical approximation schemes for backward stochastic differential equations are proposed and the rate of convergence of the schemes is established based on the obtained L^p-Hölder continuity results.

Chapter 3 is concerned with a singular stochastic differential equation driven by an additive one-dimensional fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. Under some assumptions on the drift, we show that there is a unique solution, which has moments of all orders. We also apply the techniques of Malliavin calculus to prove that the solution has an absolutely continuous law at any time $t > 0$.

In Chapter 4, I am interested in some approximation solutions of a type of stochastic differential equations driven by multi-dimensional fractional Brownian motion B^H with Hurst parameter $H > \frac{1}{2}$. In order to obtain an optimal rate of convergence, some techniques are developed in the deterministic case. Some work in progress is contained in this chapter.

The results obtained in Chapter 2 are accepted by the Annals of Applied Probability, and the material contained in Chapter 3 has been published in Statistics and Probability Letters 78 (2008) 2075-2085.

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Contents

Chapter 1

Introduction

1.1 Notations and preliminaries

1.1.1 Notations and preliminaries Malliavin calculus for Brownian motion

Let $W = \{W_t\}_{0\le t\le T}$ be a one-dimensional standard Brownian motion defined on some complete filtered probability space $(\Omega, \mathscr{F}, P, \{\mathscr{F}_t\}_{0\le t\le T})$. We assume that $\{\mathscr{F}_t\}_{0\le t\le T}$ is the filtration generated by the Brownian motion and the P-null sets, and $\mathscr{F} = \mathscr{F}_T$. We denote by $\mathscr{P}$ the progressive σ-field on the product space $[0,T]\times\Omega$.

For any $p \ge 1$ we consider the following classes of processes.

- $M^{2,p}$, for any $p \ge 2$, denotes the class of square integrable random variables F with a stochastic integral representation of the form

$$F = \mathbb{E}F + \int_0^T u_t dW_t,$$

where u is a progressively measurable process satisfying $\sup_{0\le t\le T}\mathbb{E}|u_t|^p < \infty$.

- $H^p_{\mathscr{F}}([0,T])$ denotes the Banach space of all progressively measurable processes $\varphi:\ ([0,T]\times\Omega,\ \mathscr{P})\to(\mathbb{R},\mathscr{B})$ with norm

$$\|\varphi\|_{H^p}=\left(\mathbb{E}\left(\int_0^T|\varphi_t|^2dt\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}<\infty.$$

- $S^p_{\mathscr{F}}([0,T])$ denotes the Banach space of all the RCLL (right continuous with left limits) adapted processes $\varphi:\ ([0,T]\times\Omega,\ \mathscr{P})\to(\mathbb{R},\mathscr{B})$ with norm

$$\|\varphi\|_{S^p}=\left(\mathbb{E}\sup_{0\le t\le T}|\varphi_t|^p\right)^{\frac{1}{p}}<\infty.$$

Next, we present some preliminaries on Malliavin calculus for the Brownian motion W and we refer the reader to the book by Nualart [31] for more details.

Let $\mathbf{H}=L^2([0,T])$ be the separable Hilbert space of all square integrable real-valued functions on the interval $[0,T]$ with scalar product denoted by $\langle\cdot,\cdot\rangle_{\mathbf{H}}$. The norm of an element $h\in\mathbf{H}$ will be denoted by $\|h\|_{\mathbf{H}}$. For any $h\in\mathbf{H}$ we put $W(h)=\int_0^T h(t)dW_t$.

We denote by $C_p^\infty(\mathbb{R}^n)$ the set of all infinitely continuously differentiable functions $g:\mathbb{R}^n\to\mathbb{R}$ such that g and all of its partial derivatives have polynomial growth. We make use of the notation $\partial_i g=\frac{\partial g}{\partial x_i}$ whenever $g\in C^1(\mathbb{R}^n)$.

Let $\mathscr{S}$ denote the class of smooth random variables such that a random variable $F\in\mathscr{S}$ has the form

$$F=g(W(h_1),\dots,W(h_n)),\tag{1.1.1}$$

where g belongs to $C_p^\infty(\mathbb{R}^n)$, $h_1,\dots,h_n$ are in $\mathbf{H}$, and $n\ge 1$.

The Malliavin derivative of a smooth random variable F of the form (1.1.1) is the $\mathbf{H}$-valued random variable given by

$$D_t F = \sum_{i=1}^{n} \partial_i g(W(h_1), \ldots, W(h_n)) h_i(t).$$

For any $p \geq 1$ we will denote the domain of D in $L^p(\Omega)$ by $\mathbb{D}^{1,p}$, meaning that $\mathbb{D}^{1,p}$ is the closure of the class of smooth random variables $\mathscr{S}$ with respect to the norm

$$\|F\|_{1,p} = \left(\mathbb{E}|F|^p + \mathbb{E}\|DF\|_{\mathbf{H}}^p\right)^{\frac{1}{p}}.$$

We can define the iteration of the operator D in such a way that for a smooth random variable F, the iterated derivative $D^k F$ is a random variable with values in $\mathbf{H}^{\otimes k}$. Then for every $p \geq 1$ and any natural number $k \geq 1$ we introduce the seminorm on $\mathscr{S}$ defined by

$$\|F\|_{k,p} = \left(\mathbb{E}|F|^p + \sum_{j=1}^{k} \mathbb{E}\|D^j F\|_{\mathbf{H}^{\otimes j}}^p\right)^{\frac{1}{p}}.$$

We will denote by $\mathbb{D}^{k,p}$ the completion of the family of smooth random variables $\mathscr{S}$ with respect to the norm $\|\cdot\|_{k,p}$.

Let μ be the Lebesgue measure on $[0,T]$. For any $k \geq 1$ and $F \in \mathbb{D}^{k,p}$, the derivative

$$D^k F = \{D^k_{t_1,\ldots,t_k} F, t_i \in [0,T], i = 1, \ldots, k\},$$

is a measurable function on the product space $[0,T]^k \times \Omega$, which is defined a.e. with respect to the measure $\mu^k \times P$.

We use $\mathbb{L}_a^{1,p}$ to denote the set of real-valued progressively measurable processes $u = \{u_t\}_{0 \leq t \leq T}$ such that

(1) For almost all $t \in [0,T]$, $u_t \in \mathbb{D}^{1,p}$.

(2) $\mathbb{E}\left(\left(\int_0^T |u_t|^2 dt\right)^{\frac{p}{2}} + \left(\int_0^T \int_0^T |D_\theta u_t|^2 d\theta dt\right)^{\frac{p}{2}}\right) < \infty.$

Notice that we can choose a progressively measurable version of the **H**-valued process $\{Du_t\}_{0\le t\le T}$.

1.1.2 Notations and preliminaries on Malliavin calculus for fractional Brownian motion

For any $a \le b$ and any $\beta \in (0,1)$, $C^\beta([a,b];\mathbb{R}^d)$ denotes the space of $\mathbb{R}^d$-valued β-Hölder continuous functions, and $C([a,b];\mathbb{R}^d)$ denotes the Banach space of $\mathbb{R}^d$-valued continuous functions equipped with the supremum norm on the interval $[a,b]$. We will make use of the notations

$$\|x\|_{a,b,\beta} = \sup_{a\le\theta<r\le b} \frac{|x(r)-x(\theta)|}{|r-\theta|^\beta},$$

if $x : [a,b] \to \mathbb{R}^d$ is in $C^\beta([a,b];\mathbb{R}^d)$, and

$$\|x\|_{a,b,\infty} = \sup_{a\le r\le b} |x(r)|,$$

if $x : [a,b] \to \mathbb{R}^d$ is in $C([a,b];\mathbb{R}^d)$.

If $d = 1$, then denote $C^\beta([a,b]) = C^\beta([a,b];\mathbb{R})$.

Let $B^H = \{B^H_t\}_{t\ge0}$ be a fractional Brownian motion with Hurst parameter $H \in (1/2,1)$, defined on a complete probability space $(\Omega,\mathscr{F},P)$. Namely, B^H is a mean zero Gaussian process with covariance

$$\mathbb{E}(B^H_t B^H_s) = R_H(s,t) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right). \tag{1.1.2}$$

Notice that for any $\beta \in (0,H)$, $T > 0$, the fractional Brownian motion $\{B_t^H\}_{0\le t\le T} \in C^{\beta}([0,T])$ a.s.

Next, we make some preliminaries on the Malliavin calculus for the fractional Brownian motion, and we refer to Decreusefond and Üstünel [10], Nualart [31] and Saussereau and Nualart [36] for a more complete treatment of this topic.

Fix a time interval $[0,T]$. Denote by $\mathscr{E}$ the set of real valued step functions on $[0,T]$ and let $\mathscr{H}$ be the Hilbert space defined as the closure of $\mathscr{E}$ with respect to the scalar product $\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathscr{H}} = R_H(t,s)$, where R_H is the covariance function of the fBm, given in (1.1.2). We know that

$$\begin{aligned} R_H(t,s) &= \alpha_H \int_0^t \int_0^s |r-u|^{2H} du dr \\ &= \int_0^{t\wedge s} K_H(t,r) K_H(s,r) dr, \end{aligned}$$

where $K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du \mathbf{1}_{\{s<t\}}$ with $c_H = \sqrt{\frac{H(2H-1)}{B(2-2H, H-\frac{1}{2})}}$ and B denotes the Beta function, and $\alpha_H = H(2H-1)$. In general, for any $\varphi, \psi \in \mathscr{E}$ we have

$$\langle \varphi, \psi \rangle_{\mathscr{H}} = \alpha_H \int_0^T \int_0^T |r-u|^{2H-2} \varphi_r \psi_u du dr \, .$$

The mapping $\mathbf{1}_{[0,t]} \longmapsto B_t^H$ can be extended to an isometry between $\mathscr{H}$ and the Gaussian space $\mathscr{H}_1$ spanned by B^H. We denote this isometry by $\varphi \longmapsto B^H(\varphi)$.

We consider the operator $K_H^* : \mathscr{E} \to L^2(0,T)$ defined by

$$(K_H^* \varphi)(s) = \int_s^T \varphi(t) \frac{\partial K_H}{\partial t}(t,s) dt. \tag{1.1.3}$$

Notice that $(K_H^*(\mathbf{1}_{[0,t]}))(s) = K_H(t,s)\mathbf{1}_{[0,t]}(s)$. For any $\varphi, \psi \in \mathscr{E}$ we have

$$\langle \varphi, \psi \rangle_{\mathscr{H}} = \langle K_H^* \varphi, K_H^* \phi \rangle_{L^2(0,T)} = \mathbb{E}(B^H(\varphi) B^H(\phi)), \tag{1.1.4}$$

and K_H^* provides an isometry between the Hilbert space $\mathscr{H}$ and a closed subspace of $L^2([0,T])$. We denote $K_H : L^2([0,T]) \to \mathscr{H}_H := K_H(L^2([0,T]))$ the operator defined by $(K_H h)(t) := \int_0^t K_H(t,s) h(s) ds$. The space $\mathscr{H}_H$ is the fractional version of the Cameron-Martin space. Finally, we denote by $R_H = K_H \circ K_H^* : \mathscr{H} \to \mathscr{H}_H$ the operator $R_H \varphi = \int_0^\cdot K_H(\cdot, s)(K_H^* \varphi)(s) ds$. For any $\varphi \in \mathscr{H}$, $R_H \varphi$ is Hölder continuous of order H. In fact,

$$R_H \varphi(t) = \langle K_H^* \mathbf{1}_{[0,t]}, K_H^* \varphi \rangle_{\mathscr{H}} = \mathbb{E}(B_t^H B^H(\varphi)),$$

which implies

$$|R_H \varphi(t) - R_H \varphi(s)| \leq \|\varphi\|_{\mathscr{H}} |t-s|^H.$$

If we assume that Ω is the canonical probability space $C_0([0,T])$, equipped with the Borel σ-field and the probability P is the law of the fBm. Then, the injection $R_H : \mathscr{H} \to \Omega$ embeds $\mathscr{H}$ densely into Ω and $(\Omega, \mathscr{H}, P)$ is an abstract Wiener space in the sense of Gross ([16] and [21]). In the sequel we will make this assumption on the underlying probability space.

Let $\mathscr{S}$ be the space of smooth and cylindrical random variables of the form

$$F = f(B^H(\varphi_1), \ldots, B^H(\varphi_n)), \tag{1.1.5}$$

where $f \in C_b^\infty(\mathbb{R}^n)$ (f and all its partial derivatives are bounded). For a random variable F of the form (1.1.5) we define its Malliavin derivative as the $\mathscr{H}$-valued random variable

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(B^H(\varphi_1), \dots, B^H(\varphi_n))\varphi_i.$$

We denote by $\mathbb{D}^{1,2}$ the Sobolev space defined as the completion of the class $\mathscr{S}$, with respect to the norm

$$\|F\|_{1,2} = \left[\mathbb{E}(F^2) + \mathbb{E}\left(\|DF\|_{\mathscr{H}}^2\right)\right]^{1/2}.$$

Since we shall deal with Brownian motion and fractional Brownian motion in separate chapters, it is not confusing if the same D is used to denote the corresponding Malliavin derivatives.

1.2 Introduction to main results

This dissertation is mainly based on three papers joint with Yaozhong Hu and David Nualart.

Chapter 2 is mainly from the paper *" Malliavin calculus for backward stochastic differential equations and application to numerical solutions"*, which is accepted by the Annals of Applied Probability.

In this chapter, we are concerned with the following backward stochastic differential equation (BSDE, for short):

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r)dr - \int_t^T Z_r dW_r, \quad 0 \le t \le T, \tag{1.2.6}$$

where $W = \{W_t\}_{0 \le t \le T}$ is a standard Brownian motion, the generator f is a measurable function $f : ([0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}, \mathscr{P} \otimes \mathscr{B} \otimes \mathscr{B}) \to (\mathbb{R}, \mathscr{B})$, and the terminal value ξ is an $\mathscr{F}_T$-measurable random variable.

Definition 1.2.1. *A solution to the BSDE (1.2.6) is a pair of progressively measurable processes* (Y,Z) *such that:* $\int_0^T |Z_t|^2 dt < \infty$, $\int_0^T |f(t,Y_t,Z_t)|dt < \infty$, *a.s., and*

$$Y_t = \xi + \int_t^T f(r,Y_r,Z_r)dr - \int_t^T Z_r dW_r, \quad 0 \leq t \leq T.$$

The most important result in this chapter is the L^p-Hölder continuity of the process Z. Here we emphasize that the main difficulty in constructing a numerical scheme for BSDEs is usually the approximation of the process Z. It is necessary to obtain some regularity properties for the trajectories of this process Z. The Malliavin calculus turns out to be a suitable tool to handle these problems because the random variable Z_t can be expressed in terms of the trace of the Malliavin derivative of Y_t, namely, $Z_t = D_t Y_t$. This relationship was proved in the paper by El Karoui, Peng and Quenez [13] and used by these authors to obtain estimates for the moments of Z_t. We shall further exploit this identity to obtain the L^p-Hölder continuity of the process Z, which is the critical ingredient for the rate estimate of our numerical schemes.

Assumption 1.2.1. *Fix* $2 \leq p < \frac{q}{2}$.

(A3) $\xi \in \mathbb{D}^{2,q}$, *and there exists* $L > 0$, *such that for all* θ, $\theta' \in [0,T]$,

$$\mathbb{E}|D_\theta \xi - D_{\theta'} \xi|^p \leq L|\theta - \theta'|^{\frac{p}{2}}, \tag{1.2.7}$$

$$\sup_{0 \leq \theta \leq T} \mathbb{E}|D_\theta \xi|^q < \infty, \tag{1.2.8}$$

and

$$\sup_{0 \leq \theta \leq T} \sup_{0 \leq u \leq T} \mathbb{E}|D_u D_\theta \xi|^q < \infty. \tag{1.2.9}$$

(A4) The generator $f(t,y,z)$ *has continuous and uniformly bounded first and second order partial derivatives with respect to y and z, and* $f(\cdot,0,0) \in H^q_{\mathscr{F}}([0,T])$.

(A5) Assume that ξ and f satisfy the above conditions (A3) and (A4). Let (Y,Z) be the unique solution to Equation (1.2.6) with terminal value ξ and generator f. For each $(y,z) \in \mathbb{R}\times\mathbb{R}$, $f(\cdot,y,z)$, $\partial_y f(\cdot,y,z)$, and $\partial_z f(\cdot,y,z)$ belong to $\mathbb{L}_a^{1,q}$, and the Malliavin derivatives $Df(\cdot,y,z)$, $D\partial_y f(\cdot,y,z)$, and $D\partial_z f(\cdot,y,z)$ satisfy

$$\sup_{0\le\theta\le T} \mathbb{E}\left(\int_\theta^T |D_\theta f(t,Y_t,Z_t)|^2 dt\right)^{\frac{q}{2}} < \infty, \tag{1.2.10}$$

$$\sup_{0\le\theta\le T} \mathbb{E}\left(\int_\theta^T |D_\theta \partial_y f(t,Y_t,Z_t)|^2 dt\right)^{\frac{q}{2}} < \infty, \tag{1.2.11}$$

$$\sup_{0\le\theta\le T} \mathbb{E}\left(\int_\theta^T |D_\theta \partial_z f(t,Y_t,Z_t)|^2 dt\right)^{\frac{q}{2}} < \infty, \tag{1.2.12}$$

and there exists $L>0$ such that for any $t\in(0,T]$, and for any $0\le\theta,\,\theta'\le t\le T$

$$\mathbb{E}\left(\int_t^T |D_\theta f(r,Y_r,Z_r) - D_{\theta'} f(r,Y_r,Z_r)|^2 dr\right)^{\frac{p}{2}} \le L|\theta-\theta'|^{\frac{p}{2}}. \tag{1.2.13}$$

For each $\theta\in[0,T]$, and each pair of (y,z), $D_\theta f(\cdot,y,z)\in\mathbb{L}_a^{1,q}$ and it has continuous partial derivatives with respect to y,z, which are denoted by $\partial_y D_\theta f(t,y,z)$and $\partial_z D_\theta f(t,y,z)$, and the Malliavin derivative $D_u D_\theta f(t,y,z)$ satisfies

$$\sup_{0\le\theta\le T}\sup_{0\le u\le T} \mathbb{E}\left(\int_{\theta\vee u}^T |D_u D_\theta f(t,Y_t,Z_t)|^2 dt\right)^{\frac{q}{2}} < \infty. \tag{1.2.14}$$

Under the above integrability conditions, we can obtain the regularity of Z in the L^p sense in the following theorem.

Theorem 1.2.2. *Let Assumpaion 1.2.1 be satisfied.*

(a) There exists a unique solution pair $\{(Y_t,Z_t)\}_{0\le t\le T}$ to the BSDE (1.2.6), and Y, Z are in $\mathbb{L}_a^{1,q}$. A version of the Malliavin derivatives $\{(D_\theta Y_t, D_\theta Z_t)\}_{0\le\theta,t\le T}$ of the

solution pair satisfies the following linear BSDE:

$$\begin{aligned} D_\theta Y_t &= D_\theta \xi + \int_t^T [\partial_y f(r, Y_r, Z_r) D_\theta Y_r + \partial_z f(r, Y_r, Z_r) D_\theta Z_r \\ &\quad + D_\theta f(r, Y_r, Z_r)] dr - \int_t^T D_\theta Z_r dW_r, \, 0 \le \theta \le t \le T\,; \end{aligned} \tag{1.2.15}$$

$$D_\theta Y_t = 0, \, D_\theta Z_t = 0, \, 0 \le t < \theta \le T. \tag{1.2.16}$$

Moreover, $\{D_t Y_t\}_{0 \le t \le T}$ *defined by (1.2.15) gives a version of* $\{Z_t\}_{0 \le t \le T}$*, namely,* $\mu \times P$ *a.e.*

$$Z_t = D_t Y_t\,. \tag{1.2.17}$$

(b) There exists a constant $K > 0$*, such that, for all* $s, t \in [0, T]$*,*

$$\mathbb{E}|Z_t - Z_s|^p \le K|t - s|^{\frac{p}{2}}. \tag{1.2.18}$$

Our first numerical scheme has been inspired by the paper of Zhang [40], where the author considers a class of BSDEs whose terminal value ξ takes the form $g(X_\cdot)$, where g satisfies a Lipschitz condition with respect to the L^∞ or L^1 norms (similar assumptions for f), and X is a forward diffusion of the following form

$$X_t = X_0 + \int_0^t b(r, X_r) dr + \int_0^t \sigma(r, X_r) dW_r\,.$$

Let $\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ be any partition of the interval $[0, T]$ and $|\pi| = \max_{0 \le i \le n-1}(t_{i+1} - t_i)$. Denote $\Delta_i = t_{i+1} - t_i$, $i = 0, 1, \ldots, n-1$.

The discretization scheme in [40] is based on the regularity of the process Z in the following mean square sense

$$\sum_{i=0}^{n-1} \mathbb{E} \int_{t_i}^{t_{i+1}} \left[|Z_t - Z_{t_i}|^2 + |Z_t - Z_{t_{i+1}}|^2 \right] dt \le K|\pi|\,, \tag{1.2.19}$$

where K is a constant independent of the partition π. Moreover, the following rate of convergence is proved in [40] for this approximation scheme

$$\max_{0\le i\le n} \mathbb{E}|Y_{t_i} - Y_{t_i}^{\pi}|^2 + \mathbb{E}\int_0^T |Z_t - Z_t^{\pi}|^2 dt \le K|\pi|\,. \tag{1.2.20}$$

We consider the case of a general terminal value ξ which is twice differentiable in the sense of Malliavin calculus and the first and second derivatives satisfy some integrability conditions and we also made similar assumptions for the generator f (see Assumption 1.2.1). In this sense our framework extends that of [40] and is also natural. In this framework, we are able to obtain an estimate of the form

$$\mathbb{E}|Z_t - Z_s|^p \le K|t-s|^{\frac{p}{2}}\,, \tag{1.2.21}$$

where K is a constant independent of s and t. Clearly, (1.2.21) with $p=2$ implies (1.2.19). Moreover, (1.2.21) implies the existence of a γ-Hölder continuous version of the process Z for any $\gamma < \frac{1}{2} - \frac{1}{p}$. Notice that, up to now the path regularity of Z has been studied only when the terminal value and the generator are functional of a forward diffusion.

After establishing the regularity of Z, we consider different types of numerical schemes. First we analyze a scheme similar to the one proposed in [40]:

$$
\begin{aligned}
&Y_{t_n}^{\pi} = \xi^{\pi}, \quad Z_{t_n}^{\pi} = 0, \\
&Y_t^{\pi} = Y_{t_{i+1}}^{\pi} + f\left(t_{i+1}, Y_{t_{i+1}}^{\pi}, \mathbb{E}\left(\frac{1}{\Delta_{i+1}} \int_{t_{i+1}}^{t_{i+2}} Z_r^{\pi} dr | \mathscr{F}_{t_{i+1}}\right)\right) \Delta_i \\
&\qquad - \int_t^{t_{i+1}} Z_r^{\pi} dW_r, t \in [t_i, t_{i+1}), i = n-1, n-2, \ldots, 0, \qquad (1.2.22)
\end{aligned}
$$

where, by convention, $\mathbb{E}\left(\frac{1}{\Delta_{i+1}} \int_{t_{i+1}}^{t_{i+2}} Z_r^{\pi} dr | \mathscr{F}_{t_{i+1}}\right) = 0$ when $i = n-1$, and ξ^{π} is an approximation of the terminal value ξ.

In this case we can improve Zhang's work in [40] to be of the following form.

Theorem 1.2.3. *Consider the approximation scheme (1.2.22). Let Assumption 1.2.1 be satisfied, and let the partition π satisfy $\max_{0 \le i \le n-1} \Delta_i / \Delta_{i+1} \le L_1$, where L_1 is a constant. Assume that a constant $L_2 > 0$ exists such that*

$$
|f(t_2, y, z) - f(t_1, y, z)| \le L_2 |t_2 - t_1|^{\frac{1}{2}}, \qquad (1.2.23)
$$

for all $t_1, t_2 \in [0, T]$, and $y, z \in \mathbb{R}$. Then there are positive constants K and δ, independent of the partition π, such that, if $|\pi| < \delta$, then

$$
\mathbb{E} \sup_{0 \le t \le T} |Y_t - Y_t^{\pi}|^2 + \mathbb{E} \int_0^T |Z_t - Z_t^{\pi}|^2 dt \le K\left(|\pi| + \mathbb{E}|\xi - \xi^{\pi}|^2\right). \qquad (1.2.24)
$$

We also propose and study an "implicit" numerical scheme of the following form:

$$
\begin{aligned}
Y_{t_n}^{\pi} &= \xi^{\pi}, \\
Y_t^{\pi} &= Y_{t_{i+1}}^{\pi} + f\left(t_{i+1}, Y_{t_{i+1}}^{\pi}, \frac{1}{\Delta_i} \int_{t_i}^{t_{i+1}} Z_r^{\pi} dr\right) \Delta_i - \int_t^{t_{i+1}} Z_r^{\pi} dW_r, \\
&\qquad t \in [t_i, t_{i+1}), i = n-1, n-2, \ldots, 0, \qquad (1.2.25)
\end{aligned}
$$

where ξ^π is an approximation of the terminal value ξ. For this scheme we obtain a much better result on the rate of convergence.

Theorem 1.2.4. *Let Assumption 1.2.1 be satisfied, and let π be any partition. Assume that $\xi^\pi \in L^p(\Omega)$ and there exists a constant $L_1 > 0$ such that, for all $t_1, t_2 \in [0,T]$,*

$$|f(t_2,y,z) - f(t_1,y,z)| \le L_1 |t_2 - t_1|^{\frac{1}{2}}.$$

Then, there are two positive constants δ and K independent of the partition π, such that, when $|\pi| < \delta$, we have

$$\mathbb{E} \sup_{0\le t\le T} |Y_t - Y_t^\pi|^p + \mathbb{E}\left(\int_0^T |Z_t - Z_t^\pi|^2 dt\right)^{\frac{p}{2}} \le K\left(|\pi|^{\frac{p}{2}} + \mathbb{E}|\xi - \xi^\pi|^p\right).$$

In both schemes, the integral of the process Z is used in each iteration, and for this reason they are not completely discrete schemes. In order to implement the scheme on computers, one must replace an integral of the form $\int_{t_i}^{t_{i+1}} Z_s^\pi ds$ by discrete sums, and then the convergence of the obtained scheme is hardly guaranteed. To avoid this discretization we propose a truly discrete numerical scheme using our representation of Z_t as the trace of the Malliavin derivative of Y_t:

$$\begin{aligned}
Y_{t_n}^\pi &= \xi, \qquad Z_{t_n}^\pi = D_T \xi, \\
Y_{t_i}^\pi &= \mathbb{E}\left(Y_{t_{i+1}}^\pi + f(t_{i+1}, Y_{t_{i+1}}^\pi, Z_{t_{i+1}}^\pi)\Delta_i \middle| \mathscr{F}_{t_i}\right), \\
Z_{t_i}^\pi &= \mathbb{E}\left(\rho_{t_{i+1},t_n}^\pi D_{t_i}\xi + \sum_{k=i}^{n-1} \rho_{t_{i+1},t_{k+1}}^\pi D_{t_i} f(t_{k+1}, Y_{t_{k+1}}^\pi, Z_{t_{k+1}}^\pi)\Delta_k \middle| \mathscr{F}_{t_i}\right), \\
&\quad i = n-1, n-2, \dots, 0,
\end{aligned} \tag{1.2.26}$$

where $\rho^{\pi}_{t_i,t_i} = 1$, $i = 0, 1, \ldots, n$, and for $0 \le i < j \le n$,

$$
\begin{aligned}
\rho^{\pi}_{t_i,t_j} &= \exp\left\{ \sum_{k=i}^{j-1} \int_{t_k}^{t_{k+1}} \partial_z f(r, Y^{\pi}_{t_k}, Z^{\pi}_{t_k}) dW_r \right. \\
&\quad \left. + \sum_{k=i}^{j-1} \int_{t_k}^{t_{k+1}} \left(\partial_y f(r, Y^{\pi}_{t_k}, Z^{\pi}_{t_k}) - \frac{1}{2} [\partial_z f(r, Y^{\pi}_{t_k}, Z^{\pi}_{t_k})]^2 \right) dr \right\}. \qquad (1.2.27)
\end{aligned}
$$

We make the following assumptions:

(B1) $f(t,y,z)$ is deterministic, which implies $D_\theta f(t,y,z) = 0$.

(B2) $f(t,y,z)$ is linear with respect to y and z, namely, there are three functions $g(t)$, $h(t)$ and $f_1(t)$ such that

$$
f(t,y,z) = g(t)y + h(t)z + f_1(t).
$$

Assume that g, h are bounded and $f_1 \in L^2([0,T])$. Moreover, there exists a constant $L_2 > 0$, such that, for all $t_1, t_2 \in [0,T]$,

$$
|g(t_2) - g(t_1)| + |h(t_2) - h(t_1) + |f_1(t_2) - f_1(t_1)| \le L|t_2 - t_1|^{\frac{1}{2}}.
$$

(B3) $\mathbb{E} \sup_{0 \le \theta \le T} |D_\theta \xi|^r < \infty$, for all $r \ge 1$.

For this new scheme, we obtain a rate of convergence result of the form

$$
\mathbb{E} \max_{0 \le i \le n} \left\{ |Y_{t_i} - Y^{\pi}_{t_i}|^p + |Z_{t_i} - Z^{\pi}_{t_i}|^p \right\} \le K|\pi|^{\frac{p}{2} - \varepsilon},
$$

for any $\varepsilon > 0$. In fact, we have a slightly better rate of convergence

Theorem 1.2.5. *Let Assumption 1.2.1* (A3) *and assumptions (B1)-(B3) be satisfied. Then there are positive constants K and δ independent of the partition π, such that,* *when $|\pi| < \delta$ we have*

$$\mathbb{E} \max_{0 \le i \le n} \left\{ |Y_{t_i} - Y_{t_i}^{\pi}|^p + |Z_{t_i} - Z_{t_i}^{\pi}|^p \right\} \le K |\pi|^{\frac{p}{2} - \frac{p}{2 \log \frac{1}{|\pi|}}} \left(\log \frac{1}{|\pi|} \right)^{\frac{p}{2}} .$$

Chapter 3 is based on the paper *"A singular stochastic differential equation driven by fractional Brownian motion". Statistics and Probability Letters 78 (2008), 2075-2085.*

In this chapter, we are interested in the following stochastic differential equation driven by an additive fractional Brownian motion (fBm) B^H with Hurst parameter $H > 1/2$

$$X_t = x_0 + \int_0^t f(s, X_s) ds + B_t^H , \tag{1.2.28}$$

where $x_0 \ge 0$ is a constant and $f(s,x)$ has a singularity at $x = 0$ of the form $x^{-\alpha}$ with $\alpha > \frac{1}{H} - 1$.

The study of this type of singular equations is partially motivated by the equation satisfied by the d-dimensional fractional Bessel process $R_t = |B_t^H|$, $d \ge 2$ (see Guerra and Nualart [17], and Hu and Nualart [18]):

$$R_t = Y_t + H(d-1) \int_0^t \frac{s^{2H-1}}{R_s} ds,$$

where the process Y_t is equal to a divergence integral, $Y_t = \int_0^t \sum_{i=1}^d \frac{B_s^{H,i}}{R_s} \delta B_s^{H,i}$. Except in the case $H = \frac{1}{2}$, the process Y is not a one-dimensional fractional Brownian motion (see Eisenbaum and Tudor [11] and Hu and Nualart [18] for some results in this direction), although it shares with the fBm similar properties of scaling and $\frac{1}{H}$-variation. Notice that here the initial condition is zero.

The aim of this chapter is to consider the case where $x_0 \geq 0$ and the drift $f(t,x)$ is nonnegative and it has a singularity at $x = 0$ of the form $x^{-\alpha}$, where $\alpha > \frac{1}{H} - 1$, and $x_0 \geq 0$. We impose some conditions on f as follows.

(i) $f : [0,\infty) \times (0,\infty) \to [0,\infty)$ is a nonnegative, continuous function which has a continuous partial derivative with respect to x such that $\partial_x f(t,x) \leq 0$ for all $t > 0$, $x > 0$.

(ii) There exists $x_1 > 0$ and $\alpha > \frac{1}{\beta} - 1$ such that $f(t,x) \geq g(t)x^{-\alpha}$, for all $t \geq 0$ and $x \in (0,x_1)$, where $g(t)$ is a nonnegative continuous function with $g(t) > 0$ for all $t > 0$.

(iii) $f(t,x) \leq h(t)\left(1 + \frac{1}{x}\right)$ for all $t \geq 0$ and $x > 0$, where $h(t)$ is a certain nonnegative locally bounded function.

Using arguments based on fractional calculus inspired by the estimates obtained by Hu and Nualart in [19] and under the above conditions (i)-(iii), we can show that there exists a unique global solution which has an estimate of this form

$$\|X\|_{0,T,\infty} \leq C_{1,\gamma,\beta,T}(|x_0| + 1)\exp\left\{C_{2,\gamma,\beta,T}\left(1 + \|B^H\|_{0,T,\beta}^{\frac{\gamma}{\beta(\gamma-1)}}\right)\right\}, \tag{1.2.29}$$

where β is a constant in $(\frac{1}{2}, H)$ and $T > 0$. If we choose γ such that $\gamma > \frac{2\beta}{2\beta-1}$, then $\frac{\gamma}{\beta(\gamma-1)} < 2$, and by Fernique's theorem (see [14], Theorem 1.3.2, p. 11), we obtain

$$\mathbb{E}\left(e^{C\|B^H\|_{0,T,\beta}^{\frac{\gamma}{\beta(\gamma-1)}}}\right) < \infty, \tag{1.2.30}$$

for all $C > 0$, which implies that $\mathbb{E}(\|X\|_{0,T,\infty}^p) < \infty$ for all $p \geq 1$.

Furthermore, we can obtain the absolute continuity of the law of the solution.

Theorem 1.2.6. *Suppose that f satisfies the assumptions (i)-(iii). Let X_t be the solution to Equation (1.2.28). Then for any $t \geq 0$, $X_t \in \mathbb{D}^{1,2}$. Furthermore, for any $t > 0$ the law of X_t is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.*

In the particular case $f(t,x) = Kx^{-1}$, if t is small enough, we are able to show show the existence of negative moments for the solution. We will also show that the solution has an absolutely continuous law with respect to the Lebesgue measure, using the techniques of Malliavin calculus for the fractional Brownian motion.

Chapter 4 is part of my current project *"Approximation schemes of the solution of a stochastic differential equation driven by fractional Brownian motion"*, which is in progress.

In this chapter, we consider approximation solutions of multidimensional stochastic differential equations of the form

$$X_t^i = X_0^i + \sum_{j=1}^{m} \int_0^t \sigma^{i,j}(X_s) dB_s^{H,j},\ i = 1, \ldots, d, \tag{1.2.31}$$

where the integral is a pathwise Riemann-Stieltjes integral.

Fix n, and set $\tau_k = \frac{kT}{n}$ for $k = 0, \ldots, n$. Set $\kappa_n(t) = \frac{kT}{n}$ if $\frac{kT}{n} \leq t < \frac{(k+1)T}{n}$, $k = 0, \ldots, n$. We will also set $\delta = \frac{T}{n}$. The aim of the this project is to establish an optimal rate of convergence of the Euler scheme of the form

$$X_t^{(n),i} = X_0^i + \sum_{j=1}^{m} \int_0^t \sigma^{i,j}(X_{\kappa_n(s)}^{(n)}) dB_s^{H,j},\ i = 1, \ldots, d,$$

or equivalently,

$$X_t^{(n),i} = X_{\kappa_n(t)}^{(n),i} + \sum_{j=1}^{m} \sigma^{i,j}(X_{\kappa_n(t)}^{(n)})(B_t^{H,j} - B_{\kappa_n(t)}^{H,j}),$$

for any $\frac{kT}{n} < t \leq \frac{(k+1)T}{n}$, $k = 0, \ldots, n$.

The numerical solution of stochastic differential equations (SDEs, for short) driven by Brownian motion is essentially based on the method of time discretization and has a long history. Difficulties appear in constructing numerical solutions of SDEs driven by fractional Brownian motion, because the fraction Brownian motion B^H is not a semi-martingale. Numerical schemes for SDEs driven by fractional Brownian motion are studies only in few works, see [29] and the references therein. The authors in [30] gave an exact rate of convergence of the Euler scheme in one-dimensional case by using a specific representation for the solution. However, new techniques are required in multi-dimensional case. In our work, we are searching for optimal estimates of the errors of Euler Scheme and Milstein scheme by using some different techniques such as the variation property of the fractional Brownian motion.

First, we investigate the following differential equation driven by a Hölder continuous function $g : [0,T] \to \mathbb{R}^m$ of order $\beta > \frac{1}{2}$:

$$X_t^i = X_0^i + \sum_{j=1}^{m} \int_0^t \sigma^{i,j}(X_s) dg_s^j,\ i = 1, \ldots, d, \tag{1.2.32}$$

where $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ is a continuously differentiable function whose partial derivatives are bounded and Hölder continuous of order $\gamma > \frac{1}{\beta} - 1$.

The Euler scheme is given by

$$X_t^{(n),i} = X_{\kappa_n(t)}^{(n),i} + \sum_{j=1}^{m} \sigma^{i,j}(X_{\kappa_n(t)}^{(n)})(g_t^j - g_{\kappa_n(t)}^j), \tag{1.2.33}$$

for any $\frac{kT}{n} < t \leq \frac{(k+1)T}{n}$, $k = 0, \ldots, n$.

We summarize the conditions on σ as follows.

(H1) $|\sigma(x)| \leq L_1(1 + |x|)$, for some positive constant L_1.

(H2) $|\sigma(x)-\sigma(y)| \leq L_2|x-y|, \forall x, y \in \mathbb{R}^d$, for some positive constant L_2.

(H3) $|\sigma_{x_i}(x)-\sigma_{x_i}(y)| \leq M|x-y|^{\gamma}, \forall x, y \in \mathbb{R}^d, i = 1,\dots,d$, for some positive constant M.

Theorem 1.2.7. *Suppose σ satisfies the conditions (**H1**) - (**H3**). Let X and $X^{(n)}$ be the solutions to equations (1.2.32) and (1.2.33) respectively. Then there exist two positive constants δ_0 and K such that*

$$\sup_{0\leq t\leq T}\left|X_t - X_t^{(n)}\right| \leq K\delta^{1-2\alpha},$$

for all $\delta \leq \delta_0$.

This project has not been completed yet, and it requires our further investigation.

Chapter 2

Malliavin calculus for backward stochastic differential equations and application to numerical solutions

2.1 Introduction

The backward stochastic differential equation (BSDE, for short) we shall consider in this chapter takes the following form:

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r) dr - \int_t^T Z_r dW_r, \quad 0 \le t \le T, \tag{2.1.1}$$

where $W = \{W_t\}_{0 \le t \le T}$ is a standard Brownian motion, ξ is the given terminal value, and f is the given (random) generator. To solve this equation is to find a pair of adapted processes $Y = \{Y_t\}_{0 \le t \le T}$ and $Z = \{Z_t\}_{0 \le t \le T}$ satisfying the above equation (2.1.1).

Linear backward stochastic differential equations were first studied by Bismut [3] in an attempt to solve some optimal stochastic control problem through the method of maximum principle. The general nonlinear backward stochastic differential equations were first studied by Pardoux and Peng [34]. Since then there have been extensive studies of this equation. We refer to the review paper El Karoui, Peng and Quenez

[13], and to the books of El Karoui and Mazliak [12] and of Ma and Yong [25] and the references therein for more comprehensive presentation of the theory.

A current important topic in the applications of BSDEs is the numerical approximation schemes. In most work on numerical simulations, a certain forward stochastic differential equation of the following form

$$X_t = X_0 + \int_0^t b(r, X_r, Y_r)dr + \int_0^t \sigma(r, X_r)dW_r \tag{2.1.2}$$

is needed. Usually it is assumed that the generator f in (2.1.1) depends on X_r at the time r: $f(r, Y_r, Z_r) = f(r, X_r, Y_r, Z_r)$, where $f(r,x,y,z)$ is a deterministic function of (r,x,y,z), and f is global Lipschitz in (x,y,z). If in addition the terminal value ξ is of the form $\xi = h(X_T)$, where h is a deterministic function, a so-called four step numerical scheme has been developed by Ma, Protter and Yong in [23]. A basic ingredient in this chapter is that the solution $\{Y_t\}_{0\le t\le T}$ to the BSDE is of the form $Y_t = u(t, X_t)$, where $u(t,x)$ is determined by a quasi-linear partial differential equation of parabolic type. Recently, Bouchard and Touzi [4] propose a Monte-Carlo approach which may be more suitable for high-dimensional problems. Again in this forward-backward setting, if the generator f has a quadratic growth in Z, a numerical approximation is developed by Imkeller and Dos Reis [20] in which a truncation procedure is applied.

In the case where the terminal value ξ is a functional of the path of the forward diffusion X, namely, $\xi = g(X_\cdot)$, different approaches to construct numerical methods have been proposed. We refer to Bally [1] for a scheme with a random time partition. In the work by Zhang [40], the L^2-regularity of Z is obtained, which allows the use of deterministic time partitions as well as to obtain the rate estimate (see Bender and Denk [2], Gobet, Lemor and Warin [15] and Zhang [40] for different algorithms). We should also mention the works by Briand, Delyon and Mémin [7] and Ma, Protter, San Martin and Torres [24], where the Brownian motion is replaced by a scaled random walk.

The purpose of the present chapter is to construct numerical schemes for the general BSDE (2.1.1), without assuming any particular form for the terminal value ξ and generator f. This means that ξ can be an arbitrary random variable and $f(r,y,z)$ can be arbitrary $\mathscr{F}_r$-measurable random variable (see Assumption 2.2.2 in Section 2.2 for precise conditions on ξ and f). The natural tool that we shall use is the Malliavin calculus. We emphasize that the main difficulty in constructing a numerical scheme for BSDEs is usually the approximation of the process Z. It is necessary to obtain some regularity properties for the trajectories of this process Z. The Malliavin calculus turns out to be a suitable tool to handle these problems because the random variable Z_t can be expressed in terms of the trace of the Malliavin derivative of Y_t, namely, $Z_t = D_t Y_t$. This relationship was proved in the paper by El Karoui, Peng and Quenez [13] and used by these authors to obtain estimates for the moments of Z_t. We shall further exploit this identity to obtain the L^p-Hölder continuity of the process Z, which is the critical ingredient for the rate estimate of our numerical schemes.

Our first numerical scheme has been inspired by the paper of Zhang [40], where the author considers a class of BSDEs whose terminal value ξ takes the form $g(X_\cdot)$, where X is a forward diffusion of the form (2.1.2), and g satisfies a Lipschitz condition with respect to the L^∞ or L^1 norms (similar assumptions for f). The discretization scheme is based on the regularity of the process Z in the mean square sense, that is, for any partition $\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$, one obtains

$$\sum_{i=0}^{n-1} \mathbb{E} \int_{t_i}^{t_{i+1}} \left[|Z_t - Z_{t_i}|^2 + |Z_t - Z_{t_{i+1}}|^2 \right] dt \le K|\pi| , \tag{2.1.3}$$

where $|\pi| = \max_{0 \le i \le n-1} (t_{i+1} - t_i)$ and K is a constant independent of the partition π.

We consider the case of a general terminal value ξ which is twice differentiable in the sense of Malliavin calculus and the first and second derivatives satisfy some integrability conditions and we also made similar assumptions for the generator f (see Assumption 2.2.2 in Section 2.2 for details). In this sense our framework extends that of [40] and is also natural. In this framework, we are able to obtain an estimate of the form

$$\mathbb{E}|Z_t - Z_s|^p \leq K|t-s|^{\frac{p}{2}}, \tag{2.1.4}$$

where K is a constant independent of s and t. Clearly, (2.1.4) with $p=2$ implies (2.1.3). Moreover, (2.1.4) implies the existence of a γ-Hölder continuous version of the process Z for any $\gamma < \frac{1}{2} - \frac{1}{p}$. Notice that, up to now the path regularity of Z has been studied only when the terminal value and the generator are functional of a forward diffusion.

After establishing the regularity of Z, we consider different types of numerical schemes. First we analyze a scheme similar to the one proposed in [40] (see (2.3.2)). In this case we obtain a rate of convergence of the following type

$$\mathbb{E} \sup_{0\leq t\leq T} |Y_t - Y_t^\pi|^2 + \int_0^T \mathbb{E}|Z_t - Z_t^\pi|^2 dt \leq K\left(|\pi| + \mathbb{E}|\xi - \xi^\pi|^2\right).$$

Notice that this result is stronger than that in [40] which can be stated as (when $\xi^\pi = \xi$)

$$\sup_{0\leq t\leq T} \mathbb{E}|Y_t - Y_t^\pi|^2 + \int_0^T \mathbb{E}|Z_t - Z_t^\pi|^2 dt \leq K|\pi|.$$

We also propose and study an "implicit" numerical scheme (see (2.4.1) in Section 2.4 for the details). For this scheme we obtain a much better result on the rate of convergence

$$\mathbb{E} \sup_{0\leq t\leq T} |Y_t - Y_t^\pi|^p + \mathbb{E}\left(\int_0^T |Z_t - Z_t^\pi|^2 dt\right)^{\frac{p}{2}} \leq K\left(|\pi|^{\frac{p}{2}} + \mathbb{E}|\xi - \xi^\pi|^p\right),$$

where $p > 1$ depends on the assumptions imposed on the terminal value and the coefficients.

In both schemes, the integral of the process Z is used in each iteration, and for this reason they are not completely discrete schemes. In order to implement the scheme on computers, one must replace an integral of the form $\int_{t_i}^{t_{i+1}} Z_s^\pi ds$ by discrete sums, and then the convergence of the obtained scheme is hardly guaranteed. To avoid this discretization we propose a truly discrete numerical scheme using our representation of Z_t as the trace of the Malliavin derivative of Y_t (see Section 2.5 for details). For this new scheme, we obtain a rate of convergence result of the form

$$\mathbb{E} \max_{0\le i\le n} \left\{ |Y_{t_i} - Y_{t_i}^\pi|^p + |Z_{t_i} - Z_{t_i}^\pi|^p \right\} \le K|\pi|^{\frac{p}{2}-\varepsilon},$$

for any $\varepsilon > 0$. In fact, we have a slightly better rate of convergence (see Theorem 2.5.2)

$$\mathbb{E} \max_{0\le i\le n} \left\{ |Y_{t_i} - Y_{t_i}^\pi|^p + |Z_{t_i} - Z_{t_i}^\pi|^p \right\} \le K|\pi|^{\frac{p}{2} - \frac{p}{2\log\frac{1}{|\pi|}}} \left(\log \frac{1}{|\pi|} \right)^{\frac{p}{2}}.$$

However, this type of result on the rate of convergence applies only to some classes of BSDEs and thus this scheme remains to be further investigated.

In the computer realization of our schemes or any other schemes, an extremely important procedure is to compute the conditional expectation of form $\mathbb{E}(Y|\mathscr{F}_{t_i})$. In this chapter we shall not discuss this issue but only mention the papers [2], [4] and [15].

This chapter is organized as follows. In Section 2.2 we obtain a representation of the martingale integrand Z in terms of the trace of the Malliavin derivative of Y. And then we get the L^p-Hölder continuity of Z by using this representation. The conditions that we assume on the terminal value ξ and the generator f are also specified in this section.

Some examples of application are presented to explain the validity of the conditions. Section 2.3 is devoted to the analysis of the approximation scheme similar to the one introduced in [40]. Under some differentiability and integrability conditions in the sense of Malliavin calculus on ξ and the nonlinear coefficient f, we establish a better rate of convergence for this scheme. In Section 2.4, we introduce an "implicit" scheme and obtain the rate of convergence in the L^p norm. A completely discrete scheme is proposed and analyzed in Section 2.5.

Throughout this chapter for simplicity we consider only scalar BSDEs. The results obtained in this chapter can be easily extended to multi-dimensional BSDEs.

2.2 The Malliavin calculus for BSDEs

2.2.1 Estimates on the solutions of BSDEs

The generator f in the BSDE (2.1.1) is a measurable function $f : ([0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}, \mathscr{P} \otimes \mathscr{B} \otimes \mathscr{B}) \to (\mathbb{R}, \mathscr{B})$, and the terminal value ξ is an $\mathscr{F}_T$-measurable random variable.

Definition 2.2.1. *A solution to the BSDE (2.1.1) is a pair of progressively measurable processes (Y,Z) such that: $\int_0^T |Z_t|^2 dt < \infty$, $\int_0^T |f(t,Y_t,Z_t)| dt < \infty$, a.s., and*

$$Y_t = \xi + \int_t^T f(r,Y_r,Z_r)dr - \int_t^T Z_r dW_r, \quad 0 \le t \le T.$$

The next lemma provides a useful estimate on the solution to the BSDE (2.1.1).

Lemma 2.2.2. *Fix $q \ge 2$. Suppose that $\xi \in L^q(\Omega)$, $f(t,0,0) \in H^q_{\mathscr{F}}([0,T])$, and f is uniformly Lipschitz in (y,z), namely, there exists a positive number L such that $\mu \times P$*

a.e.

$$|f(t,y_1,z_1)-f(t,y_2,z_2)|\le L(|y_1-y_2|+|z_1-z_2|)\,,$$

for all $y_1,y_2\in\mathbb{R}$ *and* $z_1,z_2\in\mathbb{R}$. *Then, there exists a unique solution pair* $(Y,Z)\in S^q_{\mathscr{F}}([0,T])\times H^q_{\mathscr{F}}([0,T])$ *to Equation (2.1.1). Moreover, we have the following estimate for the solution*

$$\mathbb{E}\sup_{0\le t\le T}|Y_t|^q+\mathbb{E}\left(\int_0^T|Z_t|^2dt\right)^{\frac{q}{2}}\le K\left(\mathbb{E}|\xi|^q+\mathbb{E}\left(\int_0^T|f(t,0,0)|^2dt\right)^{\frac{q}{2}}\right),\quad (2.2.1)$$

where K is a constant depending only on L, q and T.

Proof. The proof of the existence and uniqueness of the solution (Y,Z) can be found in [13, Theorem 5.1] with the local martingale $M\equiv 0$, since the filtration here is the filtration generated by the Brownian motion W. The estimate (2.2.1) can be easily obtained from Proposition 5.1 in [13] with $(f^1,\xi^1)=(f,\xi)$ and $(f^2,\xi^2)=(0,0)$. □

As we will see later, for a given BSDE the process Z will be expressed in terms of the Malliavin derivative of the solution Y, which will satisfy a linear BSDE with random coefficients. To study the properties of Z we need to analyze a class of linear BSDEs.

Let $\{\alpha_t\}_{0\le t\le T}$ and $\{\beta_t\}_{0\le t\le T}$ be two progressively measurable processes. We will make use of the following integrability conditions.

Assumption 2.2.1. *(A1) For any $\lambda>0$,*

$$C_\lambda:=\mathbb{E}\,\exp\left(\lambda\int_0^T\left(|\alpha_t|+\beta_t^2\right)dt\right)<\infty.$$

(A2) For any $p \geq 1$,

$$K_p := \sup_{0 \leq t \leq T} \mathbb{E}\left(|\alpha_t|^p + |\beta_t|^p\right) < \infty.$$

Under condition (A1), we denote by $\{\rho_t\}_{0 \leq t \leq T}$ the solution of the linear stochastic differential equation

$$\begin{cases} d\rho_t = \alpha_t \rho_t dt + \beta_t \rho_t dW_t, \, 0 \leq t \leq T \\ \rho_0 = 1\,. \end{cases} \tag{2.2.2}$$

The following theorem is a critical tool for the proof of the main theorem in this section, and it has also its own interest.

Theorem 2.2.3. *Let $q > p \geq 2$ and let $\xi \in L^q(\Omega)$ and $f \in H^q_{\mathscr{F}}([0,T])$. Assume that $\{\alpha_t\}_{0 \leq t \leq T}$ and $\{\beta_t\}_{0 \leq t \leq T}$ are two progressively measurable processes satisfying conditions (A1) and (A2) in Assumption 2.2.1. Suppose that the random variables $\xi \rho_T$ and $\int_0^T \rho_t f_t dt$ belong to $M^{2,q}$, where $\{\rho_t\}_{0 \leq t \leq T}$ is the solution to Equation (2.2.2). Then, the following linear BSDE*

$$Y_t = \xi + \int_t^T [\alpha_r Y_r + \beta_r Z_r + f_r] dr - \int_t^T Z_r dW_r, \, 0 \leq t \leq T \tag{2.2.3}$$

has a unique solution pair (Y, Z) and there is a constant $K > 0$ such that

$$\mathbb{E}|Y_t - Y_s|^p \leq K|t-s|^{\frac{p}{2}}, \tag{2.2.4}$$

for all $s, t \in [0,T]$.

We need the following lemma to prove the above result.

Lemma 2.2.4. *Let $\{\alpha_t\}_{0\le t\le T}$ and $\{\beta_t\}_{0\le t\le T}$ be two progressively measurable processes satisfying condition (A1) in Assumption 2.2.1, and $\{\rho_t\}_{0\le t\le T}$ be the solution of Equation (2.2.2). Then, for any $r\in\mathbb{R}$ we have*

$$\mathbb{E}\sup_{0\le t\le T}\rho_t^r<\infty. \tag{2.2.5}$$

Proof. Let $t\in[0,T]$. The solution to Equation (2.2.2) can be written as

$$\rho_t=\exp\left\{\int_0^t\left(\alpha_s-\frac{\beta_s^2}{2}\right)\,ds+\int_0^t\beta_s dW_s\right\}.$$

For any real number r, we have

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le T}\rho_t^r &= \mathbb{E}\sup_{0\le t\le T}\exp\left\{\int_0^t r\left(\alpha_s-\frac{\beta_s^2}{2}\right)\,ds+r\int_0^t\beta_s dW_s\right\}\\ &\le \mathbb{E}\left(\exp\left\{|r|\int_0^T|\alpha_s|\,ds+\frac{1}{2}(|r|+r^2)\int_0^T\beta_s^2 ds\right\}\right.\\ &\qquad\left.\times\sup_{0\le t\le T}\exp\left\{r\int_0^t\beta_s dW_s-\frac{r^2}{2}\int_0^t\beta_s^2 ds\right\}\right).\end{aligned}$$

Then, fixing any $p>1$ and using Hölder's inequality, we obtain

$$\mathbb{E}\sup_{0\le t\le T}\rho_t^r\le C\left(\mathbb{E}\sup_{0\le t\le T}\exp\left\{rp\int_0^t\beta_s dW_s-\frac{pr^2}{2}\int_0^t\beta_s^2 ds\right\}\right)^{\frac{1}{p}}, \tag{2.2.6}$$

where

$$C=\left(\mathbb{E}\,\exp\left\{q\,|r|\int_0^T|\alpha_s|\,ds+\frac{q}{2}(|r|+r^2)\int_0^T\beta_s^2 ds\right\}\right)^{\frac{1}{q}},$$

and $\frac{1}{p}+\frac{1}{q}=1$.

Set $M_t = \exp\left\{ r\int_0^t \beta_s dW_s - \frac{r^2}{2}\int_0^t \beta_s^2 ds \right\}$. Then, $\{M_t\}_{0\le t\le T}$ is a martingale due to (A1). We can rewrite (2.2.6) into

$$\mathbb{E} \sup_{0\le t\le T} \rho_t^r \le C\left(\mathbb{E} \sup_{0\le t\le T} M_t^p\right)^{\frac{1}{p}}. \tag{2.2.7}$$

By Doob's maximal inequality, we have

$$\mathbb{E} \sup_{0\le t\le T} M_t^p \le c_p \mathbb{E} M_T^p, \tag{2.2.8}$$

for some constant $c_p > 0$ depending only on p. Finally, choosing any $\gamma > 1$, $\lambda > 1$ such that $\frac{1}{\gamma} + \frac{1}{\lambda} = 1$ and applying again the Hölder inequality yield

$$\begin{aligned}
\mathbb{E} M_T^p &= \mathbb{E}\left(\exp\left\{ rp\int_0^T \beta_s dW_s - \frac{\gamma}{2} p^2 r^2 \int_0^T \beta_s^2 ds \right\}\right. \\
&\qquad \left. \times \exp\left\{ \frac{\gamma p - 1}{2}\, pr^2 \int_0^T \beta_s^2 ds \right\}\right) \\
&\le \left(\mathbb{E} \exp\left\{ rp\gamma\int_0^T \beta_s dW_s - \frac{1}{2}\gamma^2 p^2 r^2 \int_0^T \beta_s^2 ds \right\}\right)^{\frac{1}{\gamma}} \\
&\qquad \times \left(\mathbb{E} \exp\left\{ \frac{\lambda(\gamma p - 1)}{2}\, pr^2 \int_0^T \beta_s^2 ds \right\}\right)^{\frac{1}{\lambda}} \\
&= \left(\mathbb{E} \exp\left\{ \frac{\lambda(\gamma p - 1)}{2} pr^2 \int_0^T \beta_s^2 ds \right\}\right)^{\frac{1}{\lambda}} < \infty.
\end{aligned}$$

Combining this inequality with (2.2.7) and (2.2.8) we conclude the proof. □

Proof of Theorem 2.2.3. The existence and uniqueness is well-known. We are going to prove (2.2.4). Let $t \in [0,T]$. Denote $\gamma_t = \rho_t^{-1}$, where $\{\rho_t\}_{0\le t\le T}$ is the solution to Equation (2.2.2). Then $\{\gamma_t\}_{0\le t\le T}$ satisfies the following linear stochastic differential equation:

$$\begin{cases} d\gamma_t = (-\alpha_t + \beta_t^2)\gamma_t dt - \beta_t \gamma_t dW_t, \quad 0 \le t \le T \\ \gamma_0 = 1. \end{cases}$$

For any $0 \le s \le t \le T$ and any positive number $r \ge 1$, we have, using (A2), the Hölder inequality, the Burkholder-Davis-Gundy inequality and Lemma 2.2.4 applied to the process $\{\gamma_t\}_{0\le t\le T}$,

$$\begin{aligned} \mathbb{E}|\gamma_t - \gamma_s|^r &= \mathbb{E}\left|\int_s^t (-\alpha_u + \beta_u^2)\gamma_u du - \int_s^t \beta_u \gamma_u dW_u\right|^r \\ &\le 2^{r-1}\left[\mathbb{E}\left|\int_s^t (-\alpha_u+\beta_u^2)\gamma_u du\right|^r + C_r \mathbb{E}\left|\int_s^t \beta_u^2 \gamma_u^2 du\right|^{\frac{r}{2}}\right] \\ &\le C(t-s)^{\frac{r}{2}}, \end{aligned} \tag{2.2.9}$$

where C_r is a constant depending only on r and C is a constant depending on T, r, and the constants appearing in conditions (A1) and (A2).

From (2.2.3), (2.2.2), and by Itô's formula, we obtain

$$d(Y_t\rho_t) = -\rho_t f_t dt + (\beta_t \rho_t Y_t + \rho_t Z_t) dW_t\,.$$

As a consequence,

$$Y_t = \rho_t^{-1}\mathbb{E}\left(\xi\rho_T + \int_t^T \rho_r f_r dr \middle| \mathscr{F}_t\right) = \mathbb{E}\left(\xi\rho_{t,T} + \int_t^T \rho_{t,r} f_r dr \middle| \mathscr{F}_t\right), \tag{2.2.10}$$

where we write $\rho_{t,r} = \rho_t^{-1}\rho_r = \gamma_t \rho_r$ for any $0 \le t \le r \le T$.

Now, fix $0 \le s \le t \le T$. We have

$$\begin{aligned}
\mathbb{E}|Y_t - Y_s|^p &= \mathbb{E}\left|\mathbb{E}\left(\xi\rho_{t,T} + \int_t^T \rho_{t,r} f_r dr \middle| \mathscr{F}_t\right) - \mathbb{E}\left(\xi\rho_{s,T} + \int_s^T \rho_{s,r} f_r dr \middle| \mathscr{F}_s\right)\right|^p \\
&\le 2^{p-1}\Big[\mathbb{E}\left|\mathbb{E}\left(\xi\rho_{t,T}\middle|\mathscr{F}_t\right) - \mathbb{E}(\xi\rho_{s,T}\middle|\mathscr{F}_s)\right|^p \\
&\quad + \mathbb{E}\left|\mathbb{E}\left(\int_t^T \rho_{t,r} f_r dr \middle| \mathscr{F}_t\right) - \mathbb{E}\left(\int_s^T \rho_{s,r} f_r dr \middle| \mathscr{F}_s\right)\right|^p\Big] \\
&= 2^{p-1}(I_1 + I_2).
\end{aligned}$$

First we estimate I_1. We have

$$\begin{aligned}
I_1 &= \mathbb{E}\left|\mathbb{E}\left(\xi\rho_{t,T}\middle|\mathscr{F}_t\right) - \mathbb{E}\left(\xi\rho_{s,T}\middle|\mathscr{F}_s\right)\right|^p \\
&= \mathbb{E}\left|\mathbb{E}\left(\xi\rho_{t,T}\middle|\mathscr{F}_t\right) - \mathbb{E}\left(\xi\rho_{s,T}\middle|\mathscr{F}_t\right) + \mathbb{E}\left(\xi\rho_{s,T}\middle|\mathscr{F}_t\right) - \mathbb{E}\left(\xi\rho_{s,T}\middle|\mathscr{F}_s\right)\right|^p \\
&\le 2^{p-1}\left[\mathbb{E}\left|\mathbb{E}\left(\xi\rho_{t,T}\middle|\mathscr{F}_t\right) - \mathbb{E}\left(\xi\rho_{s,T}\middle|\mathscr{F}_t\right)\right|^p + \mathbb{E}\left|\mathbb{E}\left(\xi\rho_{s,T}\middle|\mathscr{F}_t\right) - \mathbb{E}\left(\xi\rho_{s,T}\middle|\mathscr{F}_s\right)\right|^p\right] \\
&\le 2^{p-1}\left[\mathbb{E}|\xi(\rho_{t,T} - \rho_{s,T})|^p + \mathbb{E}\left|\mathbb{E}\left(\xi\rho_{s,T}\middle|\mathscr{F}_t\right) - \mathbb{E}\left(\xi\rho_{s,T}\middle|\mathscr{F}_s\right)\right|^p\right] \\
&= 2^{p-1}(I_3 + I_4).
\end{aligned}$$

Using the Hölder inequality, Lemma 2.2.4, and the estimate (2.2.9) with $r = \frac{2pq}{q-p}$, the term I_3 can be estimated as follows

$$\begin{aligned}
I_3 &\le (\mathbb{E}|\xi|^q)^{\frac{p}{q}}\left(\mathbb{E}|\rho_{t,T} - \rho_{s,T}|^{\frac{pq}{q-p}}\right)^{\frac{q-p}{q}} \\
&\le (\mathbb{E}|\xi|^q)^{\frac{p}{q}}\left(\mathbb{E}|\gamma_t - \gamma_s|^{\frac{2pq}{q-p}}\right)^{\frac{q-p}{2q}}\left(\mathbb{E}\rho_T^{\frac{2pq}{q-p}}\right)^{\frac{q-p}{2q}} \le C|t-s|^{\frac{p}{2}},
\end{aligned}$$

where C is a constant depending only on p, q, T, $\mathbb{E}|\xi|^q$, and the constants appearing in conditions (A1) and (A2).

In order to estimate the term I_4 we will make use of the condition $\xi\rho_T \in M^{2,q}$. This condition implies that

$$\xi\rho_T = \mathbb{E}(\xi\rho_T) + \int_0^T u_r dW_r,$$

where u is a progressively measurable process satisfying $\sup_{0\le t\le T} \mathbb{E}|u_t|^q < \infty$. Therefore, by the Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned}
&\mathbb{E}\left|\mathbb{E}(\xi\rho_T|\mathscr{F}_t) - \mathbb{E}(\xi\rho_T|\mathscr{F}_s)\right|^q = \mathbb{E}\left|\int_s^t u_r dW_r\right|^q \\
\le\ & C_q\mathbb{E}\left|\int_s^t u_r^2 dr\right|^{\frac{q}{2}} \le C_q(t-s)^{\frac{q-2}{2}}\mathbb{E}\left(\int_s^t |u_r|^q dr\right) \\
\le\ & C_q(t-s)^{\frac{q}{2}} \sup_{0\le t\le T} \mathbb{E}|u_t|^q.
\end{aligned}$$

As a consequence, from the definition of I_4 we have

$$\begin{aligned}
I_4 &= \mathbb{E}|\gamma_s[\mathbb{E}(\xi\rho_T|\mathscr{F}_t) - \mathbb{E}(\xi\rho_T|\mathscr{F}_s)]|^p \\
&\le \left(\mathbb{E}\gamma_s^{\frac{pq}{q-p}}\right)^{\frac{q-p}{q}} (\mathbb{E}|\mathbb{E}(\xi\rho_T|\mathscr{F}_t) - \mathbb{E}(\xi\rho_T|\mathscr{F}_s)|^q)^{\frac{p}{q}} \le C|t-s|^{\frac{p}{2}},
\end{aligned}$$

where C is a constant depending on $p,q,T,$ $\sup_{0\le t\le T} \mathbb{E}|u_t|^q < \infty$, and the constants appearing in conditions (A1) and (A2).

The term I_2 can be decomposed as follows

$$\begin{aligned}
I_2 &= \mathbb{E}\left|\mathbb{E}\left(\int_t^T \rho_{t,r} f_r dr|\mathscr{F}_t\right) - \mathbb{E}\left(\int_s^T \rho_{s,r} f_r dr|\mathscr{F}_s\right)\right|^p \\
&\leq 3^{p-1}\Bigg[\mathbb{E}\left|\mathbb{E}\left(\int_t^T \rho_{t,r} f_r dr|\mathscr{F}_t\right) - \mathbb{E}\left(\int_t^T \rho_{s,r} f_r dr|\mathscr{F}_t\right)\right|^p \\
&\qquad +\mathbb{E}\left|\mathbb{E}\left(\int_t^T \rho_{s,r} f_r dr|\mathscr{F}_t\right) - \mathbb{E}\left(\int_s^T \rho_{s,r} f_r dr|\mathscr{F}_t\right)\right|^p \\
&\qquad +\mathbb{E}\left|\mathbb{E}\left(\int_s^T \rho_{s,r} f_r dr|\mathscr{F}_t\right) - \mathbb{E}\left(\int_s^T \rho_{s,r} f_r dr|\mathscr{F}_s\right)\right|^p\Bigg] \\
&= 3^{p-1}(I_5 + I_6 + I_7).
\end{aligned}$$

Let us first estimate the term I_5. Suppose that $p < p' < q$. Then, using (2.2.9) and the Hölder inequality, we can write

$$\begin{aligned}
I_5 &= \mathbb{E}\left|\mathbb{E}\left(\int_t^T \rho_{t,r} f_r dr|\mathscr{F}_t\right) - \mathbb{E}\left(\int_t^T \rho_{s,r} f_r dr|\mathscr{F}_t\right)\right|^p \\
&\leq \mathbb{E}\left|\int_t^T (\rho_{t,r} - \rho_{s,r}) f_r dr\right|^p = \mathbb{E}\left(|\gamma_t - \gamma_s|^p \left|\int_t^T \rho_r f_r dr\right|^p\right) \\
&\leq \left\{\mathbb{E}|\gamma_t - \gamma_s|^{\frac{pp'}{p'-p}}\right\}^{\frac{p'-p}{p'}} \left\{\mathbb{E}\left|\int_t^T \rho_r f_r dr\right|^{p'}\right\}^{\frac{p}{p'}} \\
&\leq C|t-s|^{\frac{p}{2}} \left\{\mathbb{E}\left(\int_t^T \rho_r^2 dr\right)^{\frac{p'q}{2(q-p')}}\right\}^{\frac{p(q-p')}{p'q}} \left\{\mathbb{E}\left(\int_t^T f_r^2 dr\right)^{\frac{q}{2}}\right\}^{\frac{p}{q}} \\
&\leq \widehat{C}|t-s|^{\frac{p}{2}} \|f\|_{H^q}^p,
\end{aligned}$$

where $\widehat{C}$ is a constant depending on p, p', q, T, and the constants appearing in conditions (A1) and (A2).

Now we estimate I_6. Suppose that $p < p' < q$. We have, as in the estimate of the term I_5,

$$\begin{aligned}
I_6 &= \mathbb{E}\left|\mathbb{E}\left(\int_t^T \rho_{s,r} f_r dr|\mathscr{F}_t\right) - \mathbb{E}\left(\int_s^T \rho_{s,r} f_r dr|\mathscr{F}_t\right)\right|^p \\
&\le \mathbb{E}\left|\int_s^t \rho_{s,r} f_r dr\right|^p = \mathbb{E}\left(\rho_s^{-p}\left|\int_s^t \rho_r f_r dr\right|^p\right) \\
&\le \left\{\mathbb{E}\rho_s^{-\frac{pp'}{p'-p}}\right\}^{\frac{p'-p}{p'}} \left\{\mathbb{E}\left|\int_s^t \rho_r f_r dr\right|^{p'}\right\}^{\frac{p}{p'}} \\
&= C\left\{\mathbb{E}\left|\int_s^t \rho_r f_r dr\right|^{p'}\right\}^{\frac{p}{p'}} \\
&\le C|t-s|^{\frac{p}{2}}\left\{\mathbb{E}\sup_{0\le t\le T} \rho_t^{\frac{p'q}{q-p'}}\right\}^{\frac{p(q-p')}{p'q}} \|f\|_{H^q}^p = \widehat{C}|t-s|^{\frac{p}{2}},
\end{aligned}$$

where $\widehat{C}$ is a constant depending on p, p', q, T, and the constants appearing in conditions (A1) and (A2).

The fact that $\int_0^T \rho_r f_r dr$ belongs to $M^{2,q}$ implies that

$$\int_0^T \rho_r f_r dr = \mathbb{E}\int_0^T \rho_r f_r dr + \int_0^T v_r dW_r,$$

where $\{v_t\}_{0\le t\le T}$ is a progressively measurable process satisfying $\sup_{0\le t\le T}\mathbb{E}|v_t|^q < \infty$. Then, by the Burkholder-Davis-Gundy inequality we have

$$\begin{aligned}
&\mathbb{E}\left|\mathbb{E}\left(\int_s^T \rho_r f_r dr|\mathscr{F}_t\right) - \mathbb{E}\left(\int_s^T \rho_r f_r dr|\mathscr{F}_s\right)\right|^q \\
&= \mathbb{E}\left|\mathbb{E}\left(\int_0^T \rho_r f_r dr|\mathscr{F}_t\right) - \mathbb{E}\left(\int_0^T \rho_r f_r dr|\mathscr{F}_s\right)\right|^q \\
&= \mathbb{E}\left|\int_s^t v_r dW_r\right|^q \le C_q (t-s)^{\frac{q}{2}} \sup_{0\le t\le T}\mathbb{E}|v_t|^q.
\end{aligned}$$

Finally, we estimate I_7 as follows

$$
\begin{aligned}
I_7 &= \mathbb{E}\left|\mathbb{E}\left(\int_s^T \rho_{s,r} f_r dr|\mathscr{F}_t\right) - \mathbb{E}\left(\int_s^T \rho_{s,r} f_r dr|\mathscr{F}_s\right)\right|^p \\
&= \mathbb{E}\left|\rho_s^{-1}\left(\mathbb{E}\left(\int_s^T \rho_r f_r dr|\mathscr{F}_t\right) - \mathbb{E}\left(\int_s^T \rho_r f_r dr|\mathscr{F}_s\right)\right)\right|^p \\
&\leq \left\{\mathbb{E}\rho_s^{-\frac{pq}{q-p}}\right\}^{\frac{q-p}{p}} \left\{\mathbb{E}\left|\mathbb{E}\left(\int_s^T \rho_r f_r dr|\mathscr{F}_t\right) - \mathbb{E}\left(\int_s^T \rho_r f_r dr|\mathscr{F}_s\right)\right|^q\right\}^{\frac{p}{q}} \\
&\leq C\left\{\mathbb{E}\left|\mathbb{E}\left(\int_s^T \rho_r f_r dr|\mathscr{F}_t\right) - \mathbb{E}\left(\int_s^T \rho_r f_r dr|\mathscr{F}_s\right)\right|^q\right\}^{\frac{p}{q}} \\
&\leq \widehat{C}|t-s|^{\frac{p}{2}}, \qquad (2.2.11)
\end{aligned}
$$

where $\widehat{C}$ is a constant depending on p, q, T, $\sup_{0\leq t\leq T}\mathbb{E}|v_t|^q$, and the constants appearing in conditions (A1) and (A2).

As a consequence, we obtain for all $s, t \in [0,T]$

$$
\mathbb{E}|Y_t - Y_s|^p \leq K|t-s|^{\frac{p}{2}},
$$

where K is a constant independent of s and t. □

2.2.2 The Malliavin calculus for BSDEs

We return to the study of Equation (2.1.1). The main assumptions we make on the terminal value ξ and generator f are the following.

Assumption 2.2.2. *Fix* $2 \leq p < \frac{q}{2}$.

(A3) $\xi \in \mathbb{D}^{2,q}$*, and there exists* $L > 0$*, such that for all* $\theta, \theta' \in [0,T]$*,*

$$
\mathbb{E}|D_\theta \xi - D_{\theta'}\xi|^p \leq L|\theta - \theta'|^{\frac{p}{2}}, \qquad (2.2.12)
$$

$$\sup_{0\le\theta\le T} \mathbb{E}|D_\theta\xi|^q < \infty, \tag{2.2.13}$$

and

$$\sup_{0\le\theta\le T}\sup_{0\le u\le T} \mathbb{E}|D_u D_\theta\xi|^q < \infty. \tag{2.2.14}$$

(A4) The generator $f(t,y,z)$ has continuous and uniformly bounded first and second order partial derivatives with respect to y and z, and $f(\cdot,0,0)\in H^q_{\mathscr{F}}([0,T])$.

(A5) Assume that ξ and f satisfy the above conditions (A3) and (A4). Let (Y,Z) be the unique solution to Equation (2.1.1) with terminal value ξ and generator f. For each $(y,z)\in\mathbb{R}\times\mathbb{R}$, $f(\cdot,y,z)$, $\partial_y f(\cdot,y,z)$, and $\partial_z f(\cdot,y,z)$ belong to $\mathbb{L}_a^{1,q}$, and the Malliavin derivatives $Df(\cdot,y,z)$, $D\partial_y f(\cdot,y,z)$, and $D\partial_z f(\cdot,y,z)$ satisfy

$$\sup_{0\le\theta\le T} \mathbb{E}\left(\int_\theta^T |D_\theta f(t,Y_t,Z_t)|^2 dt\right)^{\frac{q}{2}} < \infty, \tag{2.2.15}$$

$$\sup_{0\le\theta\le T} \mathbb{E}\left(\int_\theta^T |D_\theta \partial_y f(t,Y_t,Z_t)|^2 dt\right)^{\frac{q}{2}} < \infty, \tag{2.2.16}$$

$$\sup_{0\le\theta\le T} \mathbb{E}\left(\int_\theta^T |D_\theta \partial_z f(t,Y_t,Z_t)|^2 dt\right)^{\frac{q}{2}} < \infty, \tag{2.2.17}$$

and there exists $L>0$ such that for any $t\in(0,T]$, and for any $0\le\theta,\ \theta'\le t\le T$

$$\mathbb{E}\left(\int_t^T |D_\theta f(r,Y_r,Z_r) - D_{\theta'} f(r,Y_r,Z_r)|^2 dr\right)^{\frac{p}{2}} \le L|\theta-\theta'|^{\frac{p}{2}}. \tag{2.2.18}$$

For each $\theta\in[0,T]$, and each pair of (y,z), $D_\theta f(\cdot,y,z)\in\mathbb{L}_a^{1,q}$ and it has continuous partial derivatives with respect to y, z, which are denoted by $\partial_y D_\theta f(t,y,z)$and $\partial_z D_\theta f(t,y,z)$, and the Malliavin derivative $D_u D_\theta f(t,y,z)$ satisfies

$$\sup_{0\le\theta\le T}\sup_{0\le u\le T} \mathbb{E}\left(\int_{\theta\vee u}^T |D_u D_\theta f(t,Y_t,Z_t)|^2 dt\right)^{\frac{q}{2}} < \infty. \tag{2.2.19}$$

The following property is easy to check and we omit the proof.

Remark 2.2.5. *Conditions (2.2.16) and (2.2.17) imply*

$$\sup_{0\le\theta\le T} \mathbb{E}\left(\int_\theta^T |\partial_y D_\theta f(t,Y_t,Z_t)|^2 dt\right)^{\frac{q}{2}} < \infty,$$

and

$$\sup_{0\le\theta\le T} \mathbb{E}\left(\int_\theta^T |\partial_z D_\theta f(t,Y_t,Z_t)|^2 dt\right)^{\frac{q}{2}} < \infty,$$

respectively.

The following is the main result of this section.

Theorem 2.2.6. *Let Assumption 2.2.2 be satisfied.*

(a) There exists a unique solution pair $\{(Y_t,Z_t)\}_{0\le t\le T}$ to the BSDE (2.1.1), and Y, Z are in $\mathbb{L}_a^{1,q}$. A version of the Malliavin derivatives $\{(D_\theta Y_t, D_\theta Z_t)\}_{0\le\theta,t\le T}$ of the solution pair satisfies the following linear BSDE:

$$\begin{aligned} D_\theta Y_t &= D_\theta\xi + \int_t^T [\partial_y f(r,Y_r,Z_r)D_\theta Y_r + \partial_z f(r,Y_r,Z_r)D_\theta Z_r \\ &\quad + D_\theta f(r,Y_r,Z_r)]dr - \int_t^T D_\theta Z_r dW_r, \, 0\le\theta\le t\le T; \end{aligned} \tag{2.2.20}$$

$$D_\theta Y_t = 0, D_\theta Z_t = 0, 0\le t<\theta\le T. \tag{2.2.21}$$

Moreover, $\{D_t Y_t\}_{0\le t\le T}$ defined by (2.2.20) gives a version of $\{Z_t\}_{0\le t\le T}$, namely, $\mu\times P$ a.e.

$$Z_t = D_t Y_t\,. \tag{2.2.22}$$

(b) There exists a constant $K > 0$, such that, for all $s, t \in [0,T]$,

$$\mathbb{E}|Z_t - Z_s|^p \leq K|t-s|^{\frac{p}{2}}. \tag{2.2.23}$$

Proof. Part (a): The proof of the existence and uniqueness of the solution (Y,Z), and $Y, Z \in \mathbb{L}_a^{1,2}$ is similar to that of Proposition 5.3 in [13], and also the fact that $(D_\theta Y_t, D_\theta Z_t)$ is given by (2.2.20) and (2.2.21). In Proposition 5.3 in [13] the exponent q is equal to 4, and one assumes that $\int_0^T \|D_\theta f(\cdot,Y,Z)\|_{H^2}^2 d\theta < \infty$, which is a consequence of (2.2.15) and the fact that $Y, Z \in \mathbb{L}_a^{1,2}$.

Furthermore, from conditions (2.2.13) and (2.2.15) and the estimate in Lemma 2.2.2, we obtain

$$\sup_{0\leq\theta\leq T}\left\{\mathbb{E}\sup_{\theta\leq t\leq T}|D_\theta Y_t|^q + \mathbb{E}\left(\int_\theta^T |D_\theta Z_t|^2 dt\right)^{\frac{q}{2}}\right\} < \infty. \tag{2.2.24}$$

Hence, by Proposition 1.5.5 in [31], Y and Z belong to $\mathbb{L}_a^{1,q}$.

Part (b): Let $0 \leq s \leq t \leq T$. In this proof, $C > 0$ will be a constant independent of s and t, and may vary from line to line.

By the representation (2.2.22) we have

$$Z_t - Z_s = D_t Y_t - D_s Y_s = (D_t Y_t - D_s Y_t) + (D_s Y_t - D_s Y_s). \tag{2.2.25}$$

From Lemma 2.2.2 and Equation (2.2.20) for $\theta = s$ and $\theta' = t$ respectively, we obtain, using conditions (2.2.12) and (2.2.18),

$$
\begin{aligned}
& \mathbb{E}|D_tY_t - D_sY_t|^p + \mathbb{E}\left(\int_t^T |D_tZ_r - D_sZ_r|^2 dr\right)^{\frac{p}{2}} \\
\leq \;& C\left[\mathbb{E}|D_t\xi - D_s\xi|^p + \mathbb{E}\left(\int_t^T |D_tf(r,Y_r,Z_r) - D_sf(r,Y_r,Z_r)|^2 dr\right)^{\frac{p}{2}}\right] \\
\leq \;& C|t-s|^{\frac{p}{2}}. \qquad (2.2.26)
\end{aligned}
$$

Denote $\alpha_u = \partial_y f(u,Y_u,Z_u)$ and $\beta_u = \partial_z f(u,Y_u,Z_u)$ for all $u \in [0,T]$. Then, by Assumption 2.2.2 $(A4)$, the processes α and β satisfy conditions (A1) and (A2) in Assumption 2.2.1, and from (2.2.20) we have for $r \in [s,T]$

$$D_sY_r = D_s\xi + \int_r^T [\alpha_u D_sY_u + \beta_u D_sZ_u + D_sf(u,Y_u,Z_u)]du - \int_r^T D_sZ_u dW_u.$$

Next, we are going to use Theorem 2.2.3 to estimate $\mathbb{E}|D_sY_t - D_sY_s|^p$. Fix p' with $p < p' < \frac{q}{2}$ (notice that $p' < \frac{q}{2}$ is equivalent to $\frac{p'}{q-p'} < 1$). From conditions (2.2.13) and (2.2.15), it is obvious that $D_s\xi \in L^q(\Omega) \subset L^{p'}(\Omega)$ and $D_sf(\cdot,Y,Z) \in H^q([0,T]) \subset H^{p'}([0,T])$ for any $s \in [0,T]$. We are going to show that, for any $s \in [0,T]$, $\rho_T D_s\xi$ and $\int_s^T \rho_u D_sf(u,Y_u,Z_u)du$ are elements in $M^{2,p'}$, where

$$\rho_r = \exp\left\{\int_0^r \beta_u dW_u + \int_0^r \left(\alpha_u - \frac{1}{2}\beta_u^2\right)du\right\}.$$

For any $0 \leq \theta \leq r \leq T$, let us compute

$$\begin{aligned}
D_\theta \rho_r &= \rho_r \Big\{ \int_\theta^r [\partial_{yz} f(u, Y_u, Z_u) D_\theta Y_u + \partial_{zz} f(u, Y_u, Z_u) D_\theta Z_u \\
&\qquad + D_\theta \partial_z f(u, Y_u, Z_u)] dW_u + \partial_z f(\theta, Y_\theta, Z_\theta) \\
&\quad + \int_\theta^r (\partial_{yy} f(u, Y_u, Z_u) - \partial_{yz} f(u, Y_u, Z_u) \beta_u) D_\theta Y_u du \\
&\quad + \int_\theta^r (\partial_{yz} f(u, Y_u, Z_u) - \partial_{zz} f(u, Y_u, Z_u) \beta_u) D_\theta Z_u du \\
&\quad + \int_\theta^r (D_\theta \partial_y f(u, Y_u, Z_u) - \beta_u D_\theta \partial_z f(u, Y_u, Z_u)) du \Big\}.
\end{aligned}$$

By the boundedness of the first and second order partial derivatives of f with respect to y and z, (2.2.16), (2.2.17), (2.2.24), Lemma 2.2.4, the Hölder inequality and the Burkholder-Davis-Gundy inequality, it is easy to show that for any $p'' < q$,

$$\sup_{0 \leq \theta \leq T} \mathbb{E} \sup_{\theta \leq r \leq T} |D_\theta \rho_r|^{p''} < \infty. \tag{2.2.27}$$

By the Clark-Ocone-Haussman formula, we have

$$\begin{aligned}
\rho_T D_s \xi &= \mathbb{E}(\rho_T D_s \xi) + \int_0^T \mathbb{E}(D_\theta (\rho_T D_s \xi) | \mathscr{F}_\theta) dW_\theta \\
&= \mathbb{E}(\rho_T D_s \xi) + \int_0^T \mathbb{E}(D_\theta \rho_T D_s \xi + \rho_T D_\theta D_s \xi | \mathscr{F}_\theta) dW_\theta \\
&= \mathbb{E}(\rho_T D_s \xi) + \int_0^T u_\theta^s dW_\theta,
\end{aligned}$$

and

$$
\begin{aligned}
& \int_s^T \rho_r D_s f(r,Y_r,Z_r)dr \\
= \ & \mathbb{E}\int_s^T \rho_r D_s f(r,Y_r,Z_r)dr + \int_0^T \mathbb{E}\left(D_\theta \int_s^T \rho_r D_s f(r,Y_r,Z_r)dr \Big| \mathscr{F}_\theta\right) dW_\theta \\
= \ & \mathbb{E}\int_s^T \rho_r D_s f(r,Y_r,Z_r)dr \\
& + \int_0^T \mathbb{E}\bigg(\int_s^T [D_\theta \rho_r D_s f(r,Y_r,Z_r) + \rho_r \partial_y D_s f(r,Y_r,Z_r) D_\theta Y_r \\
& \quad + \rho_r \partial_z D_s f(r,Y_r,Z_r) D_\theta Z_r + \rho_r D_\theta D_s f(r,Y_r,Z_r)] dr \Big| \mathscr{F}_\theta \bigg) dW_\theta \\
= \ & \mathbb{E}\int_s^T \rho_r D_s f(r,Y_r,Z_r)dr + \int_0^T v_\theta^s dW_\theta.
\end{aligned}
$$

We claim that $\sup_{0\le\theta\le T}\mathbb{E}|u_\theta^s|^{p'} < \infty$ and $\sup_{0\le\theta\le T}\mathbb{E}|v_\theta^s|^{p'} < \infty$. In fact,

$$
\begin{aligned}
\mathbb{E}|u_\theta^s|^{p'} &= \mathbb{E}|\mathbb{E}(D_\theta \rho_T D_s \xi + \rho_T D_\theta D_s \xi | \mathscr{F}_\theta)|^{p'} \\
&\le 2^{p'-1}\left(\mathbb{E}|D_\theta \rho_T D_s \xi|^{p'} + \mathbb{E}|\rho_T D_\theta D_s \xi|^{p'}\right) \\
&\le 2^{p'-1}\left(\left(\mathbb{E}|D_\theta \rho_T|^{\frac{p'q}{q-p'}}\right)^{\frac{q-p'}{q}} (\mathbb{E}|D_s \xi|^q)^{\frac{p'}{q}} + \left(\mathbb{E}\rho_T^{\frac{p'q}{q-p'}}\right)^{\frac{q-p'}{q}} (\mathbb{E}|D_\theta D_s \xi|^q)^{\frac{p'}{q}}\right).
\end{aligned}
$$

By (2.2.13)-(2.2.14), (2.2.27), and Lemma 2.2.4, we have $\sup_{0\le s\le T}\sup_{0\le\theta\le T}\mathbb{E}|u_\theta^s|^{p'} < \infty$. On the other hand,

$$
\begin{aligned}
\mathbb{E}|v_\theta^s|^{p'} &= \mathbb{E}\bigg|\mathbb{E}\bigg(\int_s^T \big[D_\theta \rho_r D_s f(r,Y_r,Z_r) + \rho_r \partial_y D_s f(r,Y_r,Z_r) D_\theta Y_r \\
& \quad + \rho_r \partial_z D_s f(r,Y_r,Z_r) D_\theta Z_r + \rho_r D_\theta D_s f(r,Y_r,Z_r)\big] dr \Big| \mathscr{F}_\theta \bigg)\bigg|^{p'} \\
&\le 4^{p'-1}[J_1 + J_2 + J_3 + J_4],
\end{aligned}
$$

where

$$J_1 = \mathbb{E}\left|\int_s^T D_\theta \rho_r D_s f(r,Y_r,Z_r)dr\right|^{p'},$$

$$J_2 = \mathbb{E}\left|\int_s^T \rho_r \partial_y D_s f(r,Y_r,Z_r) D_\theta Y_r dr\right|^{p'},$$

$$J_3 = \mathbb{E}\left|\int_s^T \rho_r \partial_z D_s f(r,Y_r,Z_r) D_\theta Z_r dr\right|^{p'},$$

and

$$J_4 = \mathbb{E}\left|\int_s^T \rho_r D_\theta D_s f(r,Y_r,Z_r)dr\right|^{p'}.$$

For J_1, we have

$$\begin{aligned}
J_1 &\leq \mathbb{E}\left(\sup_{\theta\leq r\leq T}|D_\theta\rho_r|^{p'}\left|\int_s^T D_s f(r,Y_r,Z_r)dr\right|^{p'}\right) \\
&\leq \left(\mathbb{E}\sup_{\theta\leq r\leq T}|D_\theta\rho_r|^{\frac{p'q}{q-p'}}\right)^{\frac{q-p'}{q}}\left(\mathbb{E}\left|\int_s^T D_s f(r,Y_r,Z_r)dr\right|^q\right)^{\frac{p'}{q}} \\
&\leq T^{\frac{p'}{2}}\left(\mathbb{E}\sup_{\theta\leq r\leq T}|D_\theta\rho_r|^{\frac{p'q}{q-p'}}\right)^{\frac{q-p'}{q}}\left(\mathbb{E}\left(\int_0^T |D_s f(r,Y_r,Z_r)|^2 dr\right)^{\frac{q}{2}}\right)^{\frac{p'}{q}}.
\end{aligned}$$

For J_2, we have

$$
\begin{aligned}
J_2 &\leq \mathbb{E}\left(\sup_{\theta\leq r\leq T}|D_\theta Y_r|^{p'}\left(\sup_{0\leq r\leq T}\rho_r\int_s^T|\partial_y D_s f(r,Y_r,Z_r)|dr\right)^{p'}\right)\\
&\leq \left(\mathbb{E}\sup_{\theta\leq r\leq T}|D_\theta Y_r|^q\right)^{\frac{p'}{q}}\left(\mathbb{E}\left(\sup_{0\leq r\leq T}\rho_r\int_s^T|\partial_y D_s f(r,Y_r,Z_r)|dr\right)^{\frac{p'q}{q-p'}}\right)^{\frac{q-p'}{q}}\\
&\leq \left(\mathbb{E}\sup_{\theta\leq r\leq T}|D_\theta Y_r|^q\right)^{\frac{p'}{q}}\left(\mathbb{E}\sup_{0\leq r\leq T}\rho_r^{\frac{p'q}{q-2p'}}\right)^{\frac{q-2p'}{q}}\\
&\quad\times\left(\mathbb{E}\left(\int_s^T|\partial_y D_s f(r,Y_r,Z_r)|dr\right)^q\right)^{\frac{p'}{q}}\\
&\leq T^{\frac{p'}{2}}\left(\mathbb{E}\sup_{\theta\leq r\leq T}|D_\theta Y_r|^q\right)^{\frac{p'}{q}}\left(\mathbb{E}\sup_{0\leq r\leq T}\rho_r^{\frac{p'q}{q-2p'}}\right)^{\frac{q-2p'}{q}}\\
&\quad\times\left(\mathbb{E}\left(\int_0^T|\partial_y D_s f(r,Y_r,Z_r)|^2dr\right)^{\frac{q}{2}}\right)^{\frac{p'}{q}}.
\end{aligned}
$$

Using a similar techniques as before, we obtain that

$$
\begin{aligned}
J_3 &\leq T^{\frac{p'}{2}}\left(\mathbb{E}\left(\int_0^T|D_\theta Z_r|^2dr\right)^{\frac{q}{2}}\right)^{\frac{p'}{q}}\left(\mathbb{E}\sup_{0\leq r\leq T}\rho_r^{\frac{p'q}{q-2p'}}\right)^{\frac{q-2p'}{q}}\\
&\quad\times\left(\mathbb{E}\left(\int_0^T|\partial_z D_s f(r,Y_r,Z_r)|^2dr\right)^{\frac{q}{2}}\right)^{\frac{p'}{q}},
\end{aligned}
$$

and

$$
J_4 \leq T^{\frac{p'}{2}}\left(\mathbb{E}\sup_{0\leq r\leq T}\rho_r^{\frac{p'q}{q-p'}}\right)^{\frac{q-p'}{q}}\left(\mathbb{E}\left(\int_0^T|D_\theta D_s f(r,Y_r,Z_r)|^2dr\right)^{\frac{q}{2}}\right)^{\frac{p'}{q}}.
$$

By (2.2.15), (2.2.16)-(2.2.19), (2.2.27), and Lemma 2.2.4, we obtain that

$$\sup_{0\le s\le T}\sup_{0\le\theta\le T}\mathbb{E}|v_\theta^s|^{p'}<\infty.$$

Therefore, $\rho_T\xi$ and $\int_0^T\rho_u D_s f(u,Y_u,Z_u)du$ belong to $M^{2,p'}$.

Thus by Theorem 2.2.3 with $p<p'$, there is a constant $C(s)>0$, such that

$$\mathbb{E}|D_sY_t-D_sY_s|^p\le C(s)|t-s|^{\frac{p}{2}},$$

for all $t\in[s,T]$. Furthermore, taking into account the proof of the estimates I_k ($k=3,4,\cdots,7$) in the proof of Theorem 2.2.3, we can show that $\sup_{0\le s\le T}C(s)=:C<\infty$. Thus we have

$$\mathbb{E}|D_sY_t-D_sY_s|^p\le C|t-s|^{\frac{p}{2}},\tag{2.2.28}$$

for all $s,t\in[0,T]$. Combining (2.2.28) with (2.2.25) and (2.2.26), we obtain that there is a constant $K>0$ independent of s and t, such that,

$$\mathbb{E}|Z_t-Z_s|^p\le K|t-s|^{\frac{p}{2}},$$

for all $s,t\in[0,T]$. $\square$

Corollary 2.2.7. *Under the assumptions in Theorem 2.2.2, let $(Y,Z)\in S^q_{\mathscr{F}}([0,T])\times H^q_{\mathscr{F}}([0,T])$ be the unique solution pair to Equation (2.1.1). If $\sup_{0\le t\le T}\mathbb{E}|Z_t|^q<\infty$, then there exists a constant C, such that, for any $s,t\in[0,T]$,*

$$\mathbb{E}|Y_t-Y_s|^q\le C|t-s|^{\frac{q}{2}}.\tag{2.2.29}$$

Proof. Without loss of generality we assume $0 \leq s \leq t \leq T$. $C > 0$ is a constant independent of s and t, which may vary from line to line. Since

$$Y_s = Y_t + \int_s^t f(r, Y_r, Z_r) dr - \int_s^t Z_r dW_r,$$

we have, by the Lipschitz condition on f,

$$\begin{aligned}
\mathbb{E}|Y_t - Y_s|^q &= \mathbb{E}\left|\int_s^t f(r, Y_r, Z_r) dr - \int_s^t Z_r dW_r\right|^q \\
&\leq 2^{q-1}\left(\mathbb{E}\left|\int_s^t f(r, Y_r, Z_r) dr\right|^q + \mathbb{E}\left|\int_s^t Z_r dW_r\right|^q\right) \\
&\leq C_q\left(|t-s|^{\frac{q}{2}}\mathbb{E}\left(\int_s^t |f(r, Y_r, Z_r)|^2 dr\right)^{\frac{q}{2}} + \mathbb{E}\left(\int_s^t |Z_r|^2 dr\right)^{\frac{q}{2}}\right) \\
&\leq C\Bigg\{|t-s|^{\frac{q}{2}}\left[\mathbb{E}\left(\int_s^t |Y_r|^2 dr\right)^{\frac{q}{2}} + \mathbb{E}\left(\int_s^t |Z_r|^2 dr\right)^{\frac{q}{2}}\right. \\
&\qquad \left. + \mathbb{E}\left(\int_s^t |f(r,0,0)|^2 dr\right)^{\frac{q}{2}}\right] + |t-s|^{\frac{q}{2}} \sup_{0 \leq r \leq T} \mathbb{E}|Z_r|^q\Bigg\} \\
&\leq C|t-s|^{\frac{q}{2}}.
\end{aligned}$$

The proof is completed. □

Remark 2.2.8. *From Theorem 2.2.6 we know that $\{(D_\theta Y_t, D_\theta Z_t)\}_{0 \leq \theta \leq t \leq T}$ satisfies Equation (2.2.20) and $Z_t = D_t Y_t$, $\mu \times P$ a.e. Moreover, since (2.2.13) and (2.2.15) hold, we can apply the estimate (2.2.1) in Lemma 2.2.2 to the linear BSDE (2.2.20) and deduce $\sup_{0 \leq t \leq T} \mathbb{E}|Z_t|^q < \infty$. Therefore, by Lemma 2.2.7, the process Y satisfies the inequality (2.2.29). By Kolmogorov's continuity criterion this implies that Y has Hölder continuous trajectories of order γ for any $\gamma < \frac{1}{2} - \frac{1}{q}$.*

2.2.3 Examples

In this section we discuss three particular examples where Assumption 2.2.2 is satisfied.

Example 2.2.9. *Consider Equation (2.1.1). Assume that:*

(a1) $f(t,y,z) : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a deterministic function that has uniformly bounded first and second order partial derivatives with respect to* y *and* z*, and* $\int_0^T f(t,0,0)^2 dt < \infty$.

(a2) The terminal value ξ *is a multiple stochastic integral of the form*

$$\xi = \int_{[0,T]^n} g(t_1, \dots, t_n) dW_{t_1} \dots dW_{t_n}, \tag{2.2.30}$$

where $n \geq 2$ *is an integer and* $g(t_1, \dots, t_n)$ *is a symmetric function in* $L^2([0,T]^n)$*, such that,*

$$\sup_{0 \leq u \leq T} \int_{[0,T]^{n-1}} g(t_1, \dots, t_{n-1}, u)^2 dt_1 \dots dt_{n-1} < \infty,$$

$$\sup_{0 \leq u,v \leq T} \int_{[0,T]^{n-2}} g(t_1, \dots, t_{n-2}, u, v)^2 dt_1 \dots dt_{n-2} < \infty,$$

and, there exists a constant $L > 0$ *such that for any* $u, v \in [0,T]$

$$\int_{[0,T]^{n-1}} |g(t_1, \dots, t_{n-1}, u) - g(t_1, \dots, t_{n-1}, v)|^2 dt_1 \dots dt_{n-1} < L|u - v|.$$

From (2.2.30)*, we know that*

$$D_u \xi = n \int_{[0,T]^{n-1}} g(t_1, \dots, t_{n-1}, u) dW_{t_1} \dots dW_{t_{n-1}}.$$

The above assumption implies Assumption 2.2.2, and therefore, Z *satisfies the Hölder continuity property (2.2.23).*

Example 2.2.10. *Let* $\Omega = C_0([0,1])$ *equipped with the Borel* σ*-field and Wiener measure. Then,* Ω *is a Banach space with supremum norm* $\| \cdot \|_\infty$ *and* $W_t = \omega(t)$ *is the* *canonical Wiener process. Consider Equation (2.1.1) on the interval* $[0,1]$. *Assume that:*

(g1) $f(t,y,z): [0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *is a deterministic function that has uniformly bounded first and second order partial derivatives with respect to y and z, and* $\int_0^1 f(t,0,0)^2 dt < \infty$.

(g2) $\xi = \varphi(W)$, *where* $\varphi: \Omega\to\mathbb{R}$ *is twice Fréchet differentiable and the first and second order Fréchet derivatives* $\delta\varphi$ *and* $\delta^2\varphi$ *satisfy*

$$|\varphi(\omega)| + \|\delta\varphi(\omega)\| + \|\delta^2\varphi(\omega)\| \le C_1 \exp\{C_2\|\omega\|_\infty^r\},$$

for all $\omega\in\Omega$ *and some constants* $C_1>0$, $C_2>0$ *and* $0<r<2$, *where* $\|\cdot\|$ *denotes the operator norm (total variation norm).*

(g3) If λ *denotes the signed measure on* $[0,1]$ *associated with* $\delta\varphi$, *there exists a constant* $L>0$ *such that for all* $0\le\theta\le\theta'\le 1$,

$$\mathbb{E}|\lambda((\theta,\theta'])|^p \le L|\theta-\theta'|^{\frac{p}{2}},$$

for some $p\ge 2$.

It is easy to show that $D_\theta\xi = \lambda((\theta,1])$ *and* $D_uD_\theta\xi = \nu((\theta,1]\times(u,1])$, *where* ν *denotes the signed measure on* $[0,1]\times[0,1]$ *associated with* $\delta^2\varphi$. *From the above assumptions and Fernique's theorem, we can get Assumption 2.2.2, and therefore, the Hölder continuity property (2.2.23) of Z.*

Example 2.2.11. *Consider the following forward-backward stochastic differential equation (FBSDE for short)*

$$\begin{cases} X_t = X_0 + \int_0^t b(r, X_r)dr + \int_0^t \sigma(r, X_r)dW_r, \\ Y_t = \varphi\left(\int_0^T X_r^2 dr\right) + \int_t^T f(r, X_r, Y_r, Z_r)dr - \int_t^T Z_r dW_r, \end{cases} \tag{2.2.31}$$

where b, σ, φ and f are deterministic functions, and $X_0 \in \mathbb{R}$.

We make the following assumptions.

(h1) *b and σ has uniformly bounded first and second order partial derivatives with respect to x, and there is a constant $L > 0$, such that, for any $s, t \in [0, T]$, $x \in \mathbb{R}$,*

$$|\sigma(t, x) - \sigma(s, x)| \le L|t - s|^{\frac{1}{2}}.$$

(h2) $\sup_{0 \le t \le T}\{|b(t, 0)| + |\sigma(t, 0)|\} < \infty$.

(h3) *φ is twice differentiable, and there exist a constant $C > 0$ and a positive integer n such that*

$$\left|\varphi\left(\int_0^T X_t^2 dt\right)\right| + \left|\varphi'\left(\int_0^T X_t^2 dt\right)\right| + \left|\varphi''\left(\int_0^T X_t^2 dt\right)\right| \le C\left(1 + \|X\|_\infty\right)^n,$$

where $\|x\|_\infty = \sup\{|x(t)|, 0 \le t \le T\}$ for any $x \in C([0, T])$.

(h4) *$f(t, x, y, z)$ has continuous and uniformly bounded first and second order partial derivatives with respect to x, y and z, and $\int_0^T f(t, 0, 0, 0)^2 dt < \infty$.*

Notice that in this example, $\Phi(X) = \varphi\left(\int_0^T X_t^2 dt\right)$ is not necessarily globally Lipschitz in X and the results of [40] cannot be applied directly.

Under the above assumptions (h1) and (h4), Equation (2.2.31) has a unique solution triple (X,Y,Z), and we have the following classical results: for any real number $r > 0$, there exists a constant $C > 0$ such that

$$\mathbb{E} \sup_{0 \le t \le T} |X_t|^r < \infty, \quad \mathbb{E}|X_t - X_s|^r \le C|t-s|^{\frac{r}{2}},$$

for any $t,s \in [0,T]$. For any fixed $(y,z) \in \mathbb{R} \times \mathbb{R}$, we have $D_\theta f(t,X_t,y,z) = \partial_x f(t,X_t,y,z) D_\theta X_t$. Then, under all the assumptions in this example, by Theorem 2.2.1 and Lemma 2.2.2 in [31] and the results listed above, we can verify Assumption 2.2.2. Therefore, Z has the Hölder continuity property (2.2.23).

Note that in the multidimensional case we do not require the matrix $\sigma\sigma^T$ to be invertible.

2.3 An explicit scheme for BSDEs

In the remaining part of this chapter, we let $\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ be a partition of the interval $[0,T]$ and $|\pi| = \max_{0 \le i \le n-1} |t_{i+1} - t_i|$. Denote $\Delta_i = t_{i+1} - t_i$, $0 \le i \le n-1$.

From equation (2.1.1), we know that, when $t \in [t_i, t_{i+1}]$,

$$Y_t = Y_{t_{i+1}} + \int_t^{t_{i+1}} f(r,Y_r,Z_r)dr - \int_t^{t_{i+1}} Z_r dW_r. \tag{2.3.1}$$

Comparing with the numerical schemes for forward stochastic differential equations, we could introduce a numerical scheme of the form

$$
\begin{aligned}
Y_{t_n}^{1,\pi} &= \xi^\pi, \\
Y_{t_i}^{1,\pi} &= Y_{t_{i+1}}^{1,\pi} + f\left(t_{i+1}, Y_{t_{i+1}}^{1,\pi}, Z_{t_{i+1}}^{1,\pi}\right)\Delta_i - \int_{t_i}^{t_{i+1}} Z_r^{1,\pi} dW_r\,, \\
&\qquad t \in [t_i, t_{i+1}), i = n-1, n-2, \ldots, 0,
\end{aligned}
$$

where $\xi^\pi \in L^2(\Omega)$ is an approximation of the terminal condition ξ. This leads to a backward recursive formula for the sequence $\{Y_{t_i}^{1,\pi}, Z_{t_i}^{1,\pi}\}_{0\le i\le n}$. In fact, once $Y_{t_{i+1}}^{1,\pi}$ and $Z_{t_{i+1}}^{1,\pi}$ are defined, then we can find $Y_{t_i}^{1,\pi}$ by

$$
Y_{t_i}^{1,\pi} = \mathbb{E}\left(Y_{t_{i+1}}^{1,\pi} + f\left(t_{i+1}, Y_{t_{i+1}}^{1,\pi}, Z_{t_{i+1}}^{1,\pi}\right)\Delta_i \middle| \mathscr{F}_{t_i}\right)
$$

and $\{Z_r^{1,\pi}\}_{t_i\le r<t_{i+1}}$ is determined by the stochastic integral representation of the random variable

$$
Y_{t_i}^{1,\pi} - Y_{t_{i+1}}^{1,\pi} - f\left(t_{i+1}, Y_{t_{i+1}}^{1,\pi}, Z_{t_{i+1}}^{1,\pi}\right)\Delta_i.
$$

Although $\{Z_r^{1,\pi}\}_{t_i\le r<t_{i+1}}$ can be expressed explicitly by Clark-Ocone-Haussman formula, its computation is a hard problem in practice. On the other hand, there are difficulties to study the convergence of the above scheme.

An alternative scheme is introduced in [40], where the approximating pairs (Y^π, Z^π) are defined recursively by

$$
\begin{aligned}
&Y_{t_n}^\pi = \xi^\pi, \quad Z_{t_n}^\pi = 0, \\
&Y_t^\pi = Y_{t_{i+1}}^\pi + f\left(t_{i+1}, Y_{t_{i+1}}^\pi, \mathbb{E}\left(\frac{1}{\Delta_{i+1}}\int_{t_{i+1}}^{t_{i+2}} Z_r^\pi dr | \mathscr{F}_{t_{i+1}}\right)\right)\Delta_i \\
&\qquad - \int_t^{t_{i+1}} Z_r^\pi dW_r\,, t \in [t_i, t_{i+1}), i = n-1, n-2, \ldots, 0,
\end{aligned} \tag{2.3.2}
$$

where, by convention, $\mathbb{E}\left(\frac{1}{\Delta_{i+1}}\int_{t_{i+1}}^{t_{i+2}} Z_r^\pi dr|\mathscr{F}_{t_{i+1}}\right) = 0$ when $i = n-1$. In [40] the following rate of convergence is proved for this approximation scheme, assuming that the terminal value ξ and the generator f are functionals of a forward diffusion associated with the BSDE,

$$\max_{0\le i\le n} \mathbb{E}|Y_{t_i} - Y_{t_i}^\pi|^2 + \mathbb{E}\int_0^T |Z_t - Z_t^\pi|^2 dt \le K|\pi|\,. \tag{2.3.3}$$

The main result of this section is the following, which on one hand improves the above rate of convergence and on the other hand extends terminal value ξ and generator f to more general situation.

Theorem 2.3.1. *Consider the approximation scheme (2.3.2). Let Assumption 2.2.2 be satisfied, and let the partition π satisfy $\max_{0\le i\le n-1}\Delta_i/\Delta_{i+1} \le L_1$, where L_1 is a constant. Assume that a constant $L_2 > 0$ exists such that*

$$|f(t_2,y,z) - f(t_1,y,z)| \le L_2|t_2 - t_1|^{\frac{1}{2}}\,, \tag{2.3.4}$$

for all $t_1, t_2 \in [0,T]$, and $y, z \in \mathbb{R}$. Then there are positive constants K and δ, independent of the partition π, such that, if $|\pi| < \delta$, then

$$\mathbb{E}\sup_{0\le t\le T} |Y_t - Y_t^\pi|^2 + \mathbb{E}\int_0^T |Z_t - Z_t^\pi|^2 dt \le K\left(|\pi| + \mathbb{E}|\xi - \xi^\pi|^2\right)\,. \tag{2.3.5}$$

Proof. In this proof, $C > 0$ will denote a constant independent of the partition π, which may vary from line to line. The inequality (2.2.23) in Theorem 2.2.6(b) yields the following estimate (Theorem 3.1 in [40]) with $p = 2$

$$\sum_{i=0}^{n-1}\mathbb{E}\int_{t_i}^{t_{i+1}}(|Z_t - Z_{t_i}|^2 + |Z_t - Z_{t_{i+1}}|^2)dt \le C|\pi|.$$

Using this estimate and following the same argument as the proof of Theorem 5.3 in [40], we can obtain the following result

$$\max_{0\le i\le n}\mathbb{E}|Y_{t_i}-Y_{t_i}^{\pi}|^2+\mathbb{E}\int_0^T|Z_t-Z_t^{\pi}|^2dt\le C\left(|\pi|+\mathbb{E}|\xi-\xi^{\pi}|^2\right). \tag{2.3.6}$$

Denote

$$\widetilde{Z}_{t_i}^{\pi}=\begin{cases}0 & \text{if } i=n;\\ \mathbb{E}\left(\frac{1}{\Delta_i}\int_{t_i}^{t_{i+1}}Z_r^{\pi}dr\Big|\mathscr{F}_{t_i}\right) & \text{if } i=n-1,n-2,\dots,0.\end{cases} \tag{2.3.7}$$

If $t_i\le t<t_{i+1}$, $i=n-1,n-2,\dots,0$, then, by iteration, we have

$$\begin{aligned}Y_t^{\pi} &= Y_{t_{i+1}}^{\pi}+f\left(t_{i+1},Y_{t_{i+1}}^{\pi},\widetilde{Z}_{t_{i+1}}^{\pi}\right)\Delta_i-\int_t^{t_{i+1}}Z_r^{\pi}dW_r\\ &= \xi^{\pi}+\sum_{k=i+1}^{n}f\left(t_k,Y_{t_k}^{\pi},\tilde{Z}_{t_k}^{\pi}\right)\Delta_{k-1}-\int_t^TZ_r^{\pi}dW_r.\end{aligned} \tag{2.3.8}$$

Therefore,

$$Y_t^{\pi}=\mathbb{E}\left(\xi^{\pi}+\sum_{k=i+1}^{n}f\left(t_k,Y_{t_k}^{\pi},\widetilde{Z}_{t_k}^{\pi}\right)\Delta_{k-1}\Bigg|\mathscr{F}_t\right),t\in[t_i,t_{i+1}).$$

We rewrite the BSDE (2.1.1) as follows

$$\begin{aligned}Y_t &= \xi+\int_t^Tf(r,Y_r,Z_r)dr-\int_t^TZ_rdW_r\\ &= \xi+\sum_{k=i+1}^{n}f(t_k,Y_{t_k},Z_{t_k})\Delta_{k-1}-\int_t^TZ_rdW_r+R_t^{\pi},\end{aligned} \tag{2.3.9}$$

where

$$
\begin{aligned}
|R_t^\pi| &= \left| \int_t^T f(r, Y_r, Z_r) dr - \sum_{k=i+1}^{n} f(t_k, Y_{t_k}, Z_{t_k}) \Delta_{k-1} \right| \\
&= \left| \sum_{k=i+1}^{n} \int_{t_{k-1}}^{t_k} [f(r, Y_r, Z_r) - f(t_k, Y_{t_k}, Z_{t_k})] \, dr - \int_{t_i}^{t} f(r, Y_r, Z_r) dr \right| \\
&\leq \sum_{k=i+1}^{n} \int_{t_{k-1}}^{t_k} |f(r, Y_r, Z_r) - f(t_k, Y_{t_k}, Z_{t_k})| \, dr + \int_{t_i}^{t_{i+1}} |f(r, Y_r, Z_r)| dr.
\end{aligned}
$$

By Lemma 2.2.2 and the Lipschitz condition on f, we have

$$
\mathbb{E} \left(\int_0^T |f(r, Y_r, Z_r)|^2 dr \right)^{\frac{p}{2}} < \infty,
$$

and hence,

$$
\mathbb{E} \max_{0 \leq i \leq n-1} \left(\int_{t_i}^{t_{i+1}} |f(r, Y_r, Z_r)| dr \right)^p \leq |\pi|^{\frac{p}{2}} \mathbb{E} \left(\int_0^T |f(r, Y_r, Z_r)|^2 dr \right)^{\frac{p}{2}}. \qquad (2.3.10)
$$

Define a function $\{t(r)\}_{0 \leq r \leq T}$ by

$$
t(r) = \begin{cases} T & \text{if } r = T, \\ t_{i+1} & \text{if } t_i \leq r < t_{i+1},\ i = n-1, \ldots, 0. \end{cases}
$$

By the Hölder inequality, the boundedness of the first order partial derivatives of f, (2.3.4), (2.2.23), Remark 2.2.8 and (2.3.10), it is easy to see that

$$
\begin{aligned}
\mathbb{E}\sup_{0\le t\le T}|R_t^\pi|^p &\le 2^{p-1}\left[\mathbb{E}\left(\int_0^T \left|f(r,Y_r,Z_r)-f\left(t(r),Y_{t(r)},Z_{t(r)}\right)\right| dr\right)^p \right.\\
&\qquad \left. +\mathbb{E}\max_{0\le i\le n-1}\left(\int_{t_i}^{t_{i+1}} |f(r,Y_r,Z_r)|dr\right)^p\right]\\
&\le (2T)^{p-1}\mathbb{E}\int_0^T \left|f(r,Y_r,Z_r)-f\left(t(r),Y_{t(r)},Z_{t(r)}\right)\right|^p dr\\
&\qquad +2^{p-1}|\pi|^{\frac{p}{2}}\mathbb{E}\left(\int_0^T |f(r,Y_r,Z_r)|^2 dr\right)^{\frac{p}{2}}\\
&\le C|\pi|^{\frac{p}{2}},
\end{aligned}
\tag{2.3.11}
$$

where, by convention, $R_T=0$. In particular, we obtain

$$
\mathbb{E}\sup_{0\le t\le T}|R_t^\pi|^2 \le C|\pi|\,. \tag{2.3.12}
$$

To simplify the notation we denote

$$
\delta Y_t^\pi = Y_t - Y_t^\pi\,, \quad \delta Z_t^\pi = Z_t - Z_t^\pi\,, \quad \text{for all } t\in[0,T],
$$

and

$$
\widehat{Z}_{t_i}^\pi = Z_{t_i} - \widetilde{Z}_{t_i}^\pi, \qquad \text{for } i=n,n-1,\dots,0.
$$

Then, when $t_i\le t<t_{i+1}$, by (2.3.8) and (2.3.9) we can write

$$
\delta Y_t^\pi = \sum_{k=i+1}^n \left[f(t_k,Y_{t_k},Z_{t_k}) - f(t_k,Y_{t_k}^\pi,\widetilde{Z}_{t_k}^\pi)\right]\Delta_{k-1} - \int_t^T \delta Z_r^\pi dW_r + R_t^\pi + \delta\xi^\pi\,,
$$

where $\delta\xi^\pi = \xi - \xi^\pi$. Therefore, we obtain

$$\delta Y_t^\pi = \mathbb{E}\left(\sum_{k=i+1}^{n}\left[f(t_k, Y_{t_k}, Z_{t_k}) - f(t_k, Y_{t_k}^\pi, \widetilde{Z}_{t_k}^\pi)\right]\Delta_{k-1} + R_t^\pi + \delta\xi^\pi \middle| \mathscr{F}_t\right). \qquad (2.3.13)$$

Denote $\widetilde{f}_{t_k}^\pi = f(t_k, Y_{t_k}, Z_{t_k}) - f(t_k, Y_{t_k}^\pi, \widetilde{Z}_{t_k}^\pi)$. From the equality (2.3.13) for $t_j \le t < t_{j+1}$, where $i \le j \le n-1$, and taking into account that $\delta Y_T^\pi = \delta Y_{t_n}^\pi = \delta\xi^\pi$, we obtain

$$\sup_{t_i \le t \le T} |\delta Y_t^\pi| \le \sup_{t_i \le t \le T} \mathbb{E}\left(\sum_{k=i+1}^{n} |\tilde{f}_{t_k}^\pi| \Delta_{k-1} + \sup_{0 \le r \le T} |R_r^\pi| + |\delta\xi^\pi| \middle| \mathscr{F}_t\right).$$

The above conditional expectation is a martingale if it is considered as a process indexed by $t \in [t_i, T]$. Thus, using Doob's maximal inequality, we obtain

$$\begin{aligned}
\mathbb{E} \sup_{t_i \le t \le T} |\delta Y_t^\pi|^2 &\le \mathbb{E} \sup_{t_i \le t \le T}\left[\mathbb{E}\left(\sum_{k=i+1}^{n}\left|\widetilde{f}_{t_k}^\pi\right|\Delta_{k-1} + \sup_{0 \le r \le T}|R_r^\pi| + |\delta\xi^\pi| \middle| \mathscr{F}_t\right)\right]^2 \\
&\le C\mathbb{E}\left(\sum_{k=i+1}^{n}\left|\widetilde{f}_{t_k}^\pi\right|\Delta_{k-1} + \sup_{0 \le r \le T}|R_r^\pi| + |\delta\xi^\pi|\right)^2 \\
&\le C\left\{\mathbb{E}\left(\sum_{k=i+1}^{n}\left|\widetilde{f}_{t_k}^\pi\right|\Delta_{k-1}\right)^2 + \mathbb{E}\sup_{0 \le r \le T}|R_r^\pi|^2 + \mathbb{E}|\delta\xi^\pi|^2\right\}.
\end{aligned}$$

From (2.3.12), we deduce

$$\mathbb{E} \sup_{t_i \le t \le T} |\delta Y_t^\pi|^2 \le C\left\{\mathbb{E}\left(\sum_{k=i+1}^{n}\left|\widetilde{f}_{t_k}^\pi\right|\Delta_{k-1}\right)^2 + \mathbb{E}|\delta\xi^\pi|^2 + |\pi|\right\}.$$

Using the Lipschitz condition on f, we obtain

$$\begin{aligned}
\mathbb{E}\sup_{t_i\le t\le T}|\delta Y_t^\pi|^2 &\le C\left\{(T-t_i)^2\mathbb{E}\sup_{i+1\le k\le n}|\delta Y_{t_k}^\pi|^2+\mathbb{E}\left(\sum_{k=i+1}^{n-1}|\widehat{Z}_{t_k}^\pi|\Delta_{k-1}\right)^2\right.\\
&\qquad\left.+\mathbb{E}|\widehat{Z}_{t_n}|^2\Delta_{n-1}^2\right\}+C\left(\mathbb{E}|\delta\xi^\pi|^2+|\pi|\right). \qquad (2.3.14)
\end{aligned}$$

Notice that

$$\begin{aligned}
\mathbb{E}\left(\sum_{k=i+1}^{n-1}|\widehat{Z}_{t_k}^\pi|\Delta_{k-1}\right)^2 &= \mathbb{E}\left(\sum_{k=i+1}^{n-1}\left|Z_{t_k}-\frac{1}{\Delta_k}\int_{t_k}^{t_{k+1}}\mathbb{E}(Z_u^\pi|\mathscr{F}_{t_k})du\right|\Delta_{k-1}\right)^2\\
&\le \mathbb{E}\left(\sum_{k=i+1}^{n-1}\frac{\Delta_{k-1}}{\Delta_k}\int_{t_k}^{t_{k+1}}\mathbb{E}\left(|Z_{t_k}-Z_u^\pi|\Big|\mathscr{F}_{t_k}\right)du\right)^2\\
&\le L_1^2\mathbb{E}\left(\sum_{k=i+1}^{n-1}\int_{t_k}^{t_{k+1}}\mathbb{E}\left(|Z_{t_k}-Z_u^\pi|\Big|\mathscr{F}_{t_k}\right)du\right)^2\\
&\le 2L_1^2\left\{\mathbb{E}\left(\sum_{k=i+1}^{n-1}\int_{t_k}^{t_{k+1}}\mathbb{E}\left(|Z_{t_k}-Z_u|\Big|\mathscr{F}_{t_k}\right)du\right)^2\right.\\
&\qquad\left.+\mathbb{E}\left(\sum_{k=i+1}^{n-1}\int_{t_k}^{t_{k+1}}\mathbb{E}\left(|Z_u-Z_u^\pi|\Big|\mathscr{F}_{t_k}\right)du\right)^2\right\}\\
&= 2L_1^2(I_1+I_2). \qquad (2.3.15)
\end{aligned}$$

Now the Minkowsk and the Hölder inequalities yield

$$\begin{aligned}
I_1 &\le \mathbb{E}\left(\sum_{k=i+1}^{n-1}\left\{\int_{t_k}^{t_{k+1}}\left(\mathbb{E}\left(|Z_{t_k}-Z_u|\Big|\mathscr{F}_{t_k}\right)\right)^2du\right\}^{1/2}\Delta_k^{1/2}\right)^2\\
&\le (T-t_i)\sum_{k=i+1}^{n-1}\int_{t_k}^{t_{k+1}}\mathbb{E}\left(\mathbb{E}\left(|Z_{t_k}-Z_u|\Big|\mathscr{F}_{t_k}\right)\right)^2du\\
&\le (T-t_i)\sum_{k=i+1}^{n-1}\int_{t_k}^{t_{k+1}}\mathbb{E}|Z_{t_k}-Z_u|^2du\\
&\le C(T-t_i)\sum_{k=i+1}^{n-1}\int_{t_k}^{t_{k+1}}|t_k-u|du\le C|\pi|. \qquad (2.3.16)
\end{aligned}$$

In a similar way and by (2.3.6), we obtain

$$
\begin{aligned}
I_2 &\leq (T-t_i)\sum_{k=i+1}^{n-1}\int_{t_k}^{t_{k+1}}\mathbb{E}|Z_u - Z_u^{\pi}|^2 du \\
&= (T-t_i)\int_{t_{i+1}}^{T}\mathbb{E}|\delta Z_u^{\pi}|^2 du \leq C|\pi|.
\end{aligned}
\tag{2.3.17}
$$

On the other hand,

$$
\mathbb{E}\left(\widehat{Z}_{t_n}^{\pi}\Delta_{n-1}\right)^2 = \mathbb{E}|Z_{t_n}|^2|\Delta_{n-1}|^2 \leq C|\pi|^2. \tag{2.3.18}
$$

From (2.3.14)-(2.3.18), we have

$$
\mathbb{E}\sup_{t_i\leq t\leq T}|\delta Y_t^{\pi}|^2 \leq C_1(T-t_i)^2\mathbb{E}\sup_{i+1\leq k\leq n}|\delta Y_{t_k}^{\pi}|^2 + C_2\left(\mathbb{E}|\delta\xi^{\pi}|^2 + |\pi|\right). \tag{2.3.19}
$$

where C_1 and C_2 are two positive constants independent of the partition π.

We can find a constant $\delta > 0$ independent of the partition π, such that, $C_1(3\delta)^2 < \frac{1}{2}$ and $T > 2\delta$. Denote $l = [\frac{T}{2\delta}]$ ($[x]$ means the greatest integer no larger than x). Then $l \geq 1$ is an integer independent of the partition π. If $|\pi| < \delta$, then for the partition π we can choose $n-1 > i_1 > i_2 > \cdots > i_l \geq 0$, such that, $T-2\delta \in (t_{i_1-1}, t_{i_1}]$, $T-4\delta \in (t_{i_2-1}, t_{i_2}]$, $\ldots$, $T-2\delta l \in [0, t_{i_l}]$ (with $t_{-1} = 0$).

For simplicity, we denote $t_{i_0} = T$ and $t_{i_{l+1}} = 0$. Each interval $[t_{i_{j+1}}, t_{i_j}]$, $j = 0, 1, \ldots, l$, has length less than 3δ, that is, $|t_{i_j} - t_{i_{j+1}}| < 3\delta$. On each interval $[t_{i_{j+1}}, t_{i_j}]$, $j = 0, 1, \ldots, l$, we consider the recursive formula (2.3.2), and (2.3.19) becomes

$$
\mathbb{E}\sup_{t_{i_{j+1}}\leq t\leq t_{i_j}}|\delta Y_t^{\pi}|^2 \leq C_1(t_{i_j} - t_{i_{j+1}})^2\mathbb{E}\sup_{i_{j+1}+1\leq k\leq i_j}|\delta Y_{t_k}^{\pi}|^2 + C_2\left(\mathbb{E}|\delta Y_{t_{i_j}}^{\pi}|^2 + |\pi|\right). \tag{2.3.20}
$$

Using (2.3.20), we can obtain inductively

$$\begin{aligned}
& \mathbb{E} \sup_{t_{i_{j+1}} \le t \le t_{i_j}} |\delta Y_t^\pi|^2 \\
\le\ & C_1(t_{i_j} - t_{i_{j+1}})^2 \mathbb{E} \sup_{i_{j+1}+1 \le k \le i_j} |\delta Y_{t_k}^\pi|^2 + C_2 \left(\mathbb{E}|\delta Y_{t_{i_j}}^\pi|^2 + |\pi| \right) \\
\le\ & C_1(t_{i_j} - t_{i_{j+1}})^2 \ldots C_1(t_{i_j} - t_{i_j-1})^2 \mathbb{E}|\delta Y_{t_{i_j}}^\pi|^2 + C_2 \left(\mathbb{E}|\delta Y_{t_{i_j}}^\pi|^2 + |\pi| \right) \\
& \times \Big(1 + C_1(t_{i_j} - t_{i_{j+1}})^2 + C_1(t_{i_j} - t_{i_{j+1}})^2 C_1(t_{i_j} - t_{i_{j+1}+1})^2 \\
& + \cdots + C_1(t_{i_j} - t_{i_{j+1}})^2 C_1(t_{i_j} - t_{i_{j+1}+1})^2 \ldots C_1(t_{i_j} - t_{i_j-1})^2 \Big) \\
\le\ & (C_1(3\delta)^2)^{i_j - i_{j+1}} \mathbb{E}|\delta Y_{t_{i_j}}^\pi|^2 \\
& + C_2 \left(\mathbb{E}|\delta Y_{t_{i_j}}^\pi|^2 + |\pi| \right) \left(1 + C_1(3\delta)^2 + (C_1(3\delta)^2)^2 + \cdots + (C_1(3\delta)^2)^{i_j - i_{j+1}} \right) \\
\le\ & \mathbb{E}|\delta Y_{t_{i_j}}^\pi|^2 + \frac{C_2}{1 - C_1(3\delta)^2} \left(\mathbb{E}|\delta Y_{t_{i_j}}^\pi|^2 + |\pi| \right) \\
\le\ & \mathbb{E}|\delta Y_{t_{i_j}}^\pi|^2 + 2C_2 \left(\mathbb{E}|\delta Y_{t_{i_j}}^\pi|^2 + |\pi| \right) \\
=\ & (2C_2 + 1) \mathbb{E}|\delta Y_{t_{i_j}}^\pi|^2 + 2C_2 |\pi|.
\end{aligned} \tag{2.3.21}$$

By recurrence, we obtain

$$\begin{aligned}
& \mathbb{E} \sup_{t_{i_{j+1}} \le t \le t_{i_j}} |\delta Y_t^\pi|^2 \\
\le\ & (2C_2 + 1)^{j+1} \mathbb{E}|\delta \xi^\pi|^2 + C_2 |\pi| \left(1 + (2C_2 + 1) + \cdots + (2C_2 + 1)^j \right) \\
\le\ & (2C_2 + 1)^{l+1} \mathbb{E}|\delta \xi^\pi|^2 + C_2 |\pi| \left(1 + (2C_2 + 1) + \cdots + (2C_2 + 1)^l \right) \\
\le\ & \frac{3(2C_2 + 1)^{l+1}}{2} \left(\mathbb{E}|\delta \xi^\pi|^2 + |\pi| \right).
\end{aligned} \tag{2.3.22}$$

Therefore, taking $C = \frac{3(2C_2+1)^{l+1}}{2}$, we obtain

$$\mathbb{E} \sup_{0 \le t \le T} |\delta Y_t^\pi|^2 \le \max_{0 \le j \le l} \mathbb{E} \sup_{t_{i_{j+1}} \le t \le t_{i_j}} |\delta Y_t^\pi|^2 \le C \left(|\pi| + \mathbb{E}|\xi - \xi^\pi|^2 \right).$$

Combining the above estimate with (2.3.6), we know that there exists a constant $K > 0$ independent of the partition π, such that,

$$\mathbb{E}\sup_{0\le t\le T}|Y_t - Y_t^\pi|^2 + \mathbb{E}\int_0^T |Z_t - Z_t^\pi|^2 dt \le K\left(|\pi| + \mathbb{E}|\xi - \xi^\pi|^2\right).$$

$\square$

Remark 2.3.2. *The numerical scheme introduced before, as other similar schemes, involves the computation of conditional expectations with respect to the σ-field $\mathscr{F}_{t_{i+1}}$. To implement this scheme in practice we need to approximate these conditional expectations. Some work has been done to solve this problem, and we refer the reader to the references [2], [4] and [15].*

2.4 An implicit scheme for BSDEs

In this section, we propose an implicit numerical scheme for the BSDE (2.1.1). Define the approximating pairs (Y^π, Z^π) recursively by

$$\begin{aligned} Y_{t_n}^\pi &= \xi^\pi, \\ Y_t^\pi &= Y_{t_{i+1}}^\pi + f\left(t_{i+1}, Y_{t_{i+1}}^\pi, \frac{1}{\Delta_i}\int_{t_i}^{t_{i+1}} Z_r^\pi dr\right)\Delta_i - \int_t^{t_{i+1}} Z_r^\pi dW_r, \\ &\qquad t\in[t_i,t_{i+1}),\ i = n-1, n-2, \ldots, 0, \end{aligned} \tag{2.4.1}$$

where the partition π and Δ_i, $i = n-1,\ldots,0$ are defined in Section 2.3, and ξ^π is an approximation of the terminal value ξ. In this recursive formula (2.4.1), on each subinterval $[t_i,t_{i+1})$, $i = n-1,\ldots,0$, the nonlinear "generator" f contains the information of Z^π on the same interval. In this sense, this formula is different from formula (2.3.2), and (2.4.1) is an equation for $\{(Y_t^\pi, Z_t^\pi)\}_{t_i\le t<t_{i+1}}$. When $|\pi|$ is sufficiently small, the

existence and uniqueness of the solution to the above equation can be established. In fact, Equation (2.4.1) is of the following form:

$$Y_t = \xi + g\left(\int_a^b Z_r dr\right) - \int_t^b Z_r dW_r\,, \quad t \in [a,b] \quad \text{and} \quad 0 \le a < b \le T \tag{2.4.2}$$

For the BSDE (2.4.2), we have the following theorem.

Theorem 2.4.1. *Let $0 \le a < b \le T$ and $p \ge 2$. Let ξ be $\mathscr{F}_b$-measurable and $\xi \in L^p(\Omega)$. If there exists a constant $L > 0$ such that $g : (\Omega \times \mathbb{R}, \mathscr{F}_b \otimes \mathscr{B}) \to (\mathbb{R}, \mathscr{B})$ satisfies*

$$|g(z_1) - g(z_2)| \le L|z_1 - z_2|,$$

for all $z_1, z_2 \in \mathbb{R}$, and $g(0) \in L^p(\Omega)$, then there is a constant $\delta(p,L) > 0$, such that, when $b - a < \delta(p,L)$, Equation (2.4.2) has a unique solution $(Y,Z) \in S^p_{\mathscr{F}}([a,b]) \times H^p_{\mathscr{F}}([a,b])$.

Proof. We shall use the fixed point theorem for the mapping from $H^p_{\mathscr{F}}([a,b])$ into $H^p_{\mathscr{F}}([a,b])$ which maps z to Z, where (Y,Z) is the solution of the following BSDE

$$Y_t = \xi + g\left(\int_a^b z_r dr\right) - \int_t^b Z_r dW_r, \quad t \in [a,b]. \tag{2.4.3}$$

In fact, by the martingale representation theorem, there exist a progressively measurable process $Z = \{Z_t\}_{a \le t \le b}$ such that $\mathbb{E}\int_a^b Z_t^2 dt < \infty$ and

$$\xi + g\left(\int_a^b z_r dr\right) = \mathbb{E}\left(\xi + g\left(\int_a^b z_r dr\right) \middle| \mathscr{F}_a\right) + \int_a^b Z_t dW_t.$$

By the integrability properties of ξ, $g(0)$ and z, one can show that $Z \in H^p_{\mathscr{F}}([a,b])$. Define $Y_t = \mathbb{E}\left(\xi + g\left(\int_a^b z_r dr\right) |\mathscr{F}_t\right)$, $t \in [a,b]$. Then (Y,Z) satisfies Equation (2.4.3). Notice that Y is a martingale. Then by the Lipschitz condition on g, the integrability of ξ, $g(0)$ and z, and Doob's maximal inequality, we can prove that $Y \in S^p_{\mathscr{F}}([a,b])$.

Let z^1, z^2 be two elements in the Banach space $H^p_{\mathscr{F}}([a,b])$, and let (Y^1,Z^1), (Y^2,Z^2) be the associated solutions, i.e,

$$Y^i_t = \xi + g\left(\int_a^b z^i_r dr\right) - \int_t^b Z^i_r dW_r, t \in [a,b], i = 1,2.$$

Denote

$$\bar{Y} = Y^1 - Y^2, \quad \bar{Z} = Z^1 - Z^2, \quad \bar{z} = z^1 - z^2.$$

Then

$$\bar{Y}_t = g\left(\int_a^b z^1_r dr\right) - g\left(\int_a^b z^2_r dr\right) - \int_t^b \bar{Z}_r dW_r, \tag{2.4.4}$$

for all $t \in [a,b]$. So

$$\bar{Y}_t = \mathbb{E}\left(g\left(\int_a^b z^1_r dr\right) - g\left(\int_a^b z^2_r dr\right) |\mathscr{F}_t\right),$$

for all $t \in [a,b]$. Thus by Doob's maximal inequality, we have

$$\begin{aligned}
\mathbb{E}\sup_{a\le t\le b}|\bar{Y}_t|^p &= \mathbb{E}\sup_{a\le t\le b}\left|\mathbb{E}\left(g\left(\int_a^b z^1_r dr\right) - g\left(\int_a^b z^2_r dr\right)|\mathscr{F}_t\right)\right|^p \\
&\le C\mathbb{E}\left|g\left(\int_a^b z^1_r dr\right) - g\left(\int_a^b z^2_r dr\right)\right|^p \\
&\le C\mathbb{E}\left|\int_a^b z^1_r dr - \int_a^b z^2_r dr\right|^p \\
&\le C(b-a)^{\frac{p}{2}}\mathbb{E}\left(\int_a^b |\bar{z}_r|^2 dr\right)^{\frac{p}{2}}, \qquad (2.4.5)
\end{aligned}$$

where $C > 0$ is a generic constant depending on L and p, which may vary from line to line. From Equation (2.4.4), it is easy to see

$$\bar{Y}_t = \bar{Y}_a + \int_a^t \bar{Z}_r dW_r,$$

for all $t \in [a,b]$. Therefore, by the Burkholder-Davis-Gundy inequality and (2.4.5), we have

$$\begin{aligned}\mathbb{E}\left(\int_a^b |\bar{Z}_r|^2 dr\right)^{\frac{p}{2}} &\leq C\mathbb{E}\sup_{a\leq t\leq b}\left|\int_a^t \bar{Z}_r dW_r\right|^p \\ &\leq C\left[\mathbb{E}|\bar{Y}_a|^p + \mathbb{E}\sup_{a\leq t\leq b}|\bar{Y}_t|^p\right] \\ &\leq C(b-a)^{\frac{p}{2}}\mathbb{E}\left(\int_a^b |\bar{z}_r|^2 dr\right)^{\frac{p}{2}}, \qquad (2.4.6)\end{aligned}$$

that is,

$$\|\bar{Z}\|_{H^p} \leq C_1(b-a)^{\frac{1}{2}}\|\bar{z}\|_{H^p},$$

where C_1 is a positive constant depending only on L and p.

Take $\delta(p,L) = 1/C_1^2$. It is obvious that the mapping is a contraction when $b-a < \delta(p,L)$, and hence there exists a unique solution $(Y,Z) \in S^p_{\mathscr{F}}([a,b]) \times H^p_{\mathscr{F}}([a,b])$ to the BSDE (2.4.2). □

Now we begin to study the convergence of the scheme (2.4.1).

Theorem 2.4.2. *Let Assumption 2.2.2 be satisfied, and let π be any partition. Assume that $\xi^\pi \in L^p(\Omega)$ and there exists a constant $L_1 > 0$ such that, for all $t_1, t_2 \in [0,T]$,*

$$|f(t_2,y,z) - f(t_1,y,z)| \leq L_1|t_2 - t_1|^{\frac{1}{2}}.$$

Then, there are two positive constants δ and K independent of the partition π, such that, when $|\pi| < \delta$, we have

$$\mathbb{E}\sup_{0\leq t\leq T}|Y_t - Y_t^\pi|^p + \mathbb{E}\left(\int_0^T |Z_t - Z_t^\pi|^2 dt\right)^{\frac{p}{2}} \leq K\left(|\pi|^{\frac{p}{2}} + \mathbb{E}|\xi - \xi^\pi|^p\right).$$

Proof. If $|\pi| < \delta(p,L)$, where $\delta(p,L)$ is the constant in Theorem 2.4.1, then Theorem 2.4.1 guarantees the existence and uniqueness of (Y^π, Z^π). Denote, for $i = n-1, n-2, \ldots, 0$,

$$\widetilde{Z}^\pi_{t_{i+1}} = \frac{1}{t_{i+1} - t_{t_i}} \int_{t_i}^{t_{i+1}} Z^\pi_r dr.$$

Notice that $\{\widetilde{Z}^\pi_{t_i},\}_{i=n-1,n-2,\ldots,0}$ here is different from that in Section 2.3. Then

$$Y^\pi_{t_i} = Y^\pi_{t_{i+1}} + f\left(t_{i+1}, Y^\pi_{t_{i+1}}, \widetilde{Z}^\pi_{t_{i+1}}\right)\Delta_i - \int_{t_i}^{t_{i+1}} Z^\pi_r dW_r,\, i = n-1, n-2, \ldots, 0.$$

Recursively, we obtain

$$Y^\pi_{t_i} = \xi^\pi + \sum_{k=i+1}^{n} f\left(t_k, Y^\pi_{t_k}, \widetilde{Z}^\pi_{t_k}\right)\Delta_{k-1} - \int_{t_i}^{T} Z^\pi_r dW_r,\, i = n-1, n-2, \ldots, 0.$$

Denote

$$\delta\xi^\pi = \xi - \xi^\pi, \quad \delta Y^\pi_t = Y_t - Y^\pi_t, \quad \delta Z^\pi_t = Z_t - Z^\pi_t, \quad t \in [0,T],$$

and

$$\widehat{Z}^\pi_{t_i} = Z_{t_i} - \widetilde{Z}^\pi_{t_i} \;\; i = n-1, \ldots, 0.$$

If $t \in [t_i, t_{i+1})$, $i = n-1, n-2, \ldots, 0$, then by iteration, we have

$$\begin{aligned} \delta Y^\pi_t &= \delta\xi^\pi + \sum_{k=i+1}^{n}\left[f\left(t_k, Y_{t_k}, Z_{t_k}\right) - f\left(t_k, Y^\pi_{t_k}, \widetilde{Z}^\pi_{t_k}\right)\right]\Delta_{k-1} \\ &\quad - \int_{t_i}^{T} \delta Z^\pi_r dW_r + R^\pi_t, \end{aligned} \tag{2.4.7}$$

where R^π_t is exactly the same as that in Section 2.3.

Denote $\widetilde{f}^{\pi}_{t_k} = f(t_k, Y_{t_k}, Z_{t_k}) - f(t_k, Y^{\pi}_{t_k}, \widetilde{Z}^{\pi}_{t_k})$. Then for $t \in [t_i, t_{i+1})$, $i = n-1, n-2, \ldots, 0$, we have

$$\delta Y^{\pi}_t = \mathbb{E}\left(\delta \xi^{\pi} + \sum_{k=i+1}^{n} \widetilde{f}^{\pi}_{t_k} \Delta_{k-1} + R^{\pi}_t \middle| \mathscr{F}_t \right). \tag{2.4.8}$$

From the equality (2.4.8) for $t_j \le t < t_{j+1}$, where $i \le j \le n-1$, and taking into account that $\delta Y^{\pi}_T = \delta Y^{\pi}_{t_n} = \delta \xi^{\pi}$, we obtain

$$\sup_{t_i \le t \le T} |\delta Y^{\pi}_t| \;\; \le \;\; \sup_{t_i \le t \le T} \mathbb{E}\left(\sum_{k=i+1}^{n} \left| \widetilde{f}^{\pi}_{t_k} \right| \Delta_{k-1} + \sup_{0 \le r \le T} |R^{\pi}_r| + |\delta \xi^{\pi}| \middle| \mathscr{F}_t \right).$$

The above conditional expectation is a martingale if it is considered as a process indexed by t for $t \in [t_i, T]$. Using Doob's maximal inequality, (2.3.11), and the Lipschitz condition on f, we have

$$\begin{aligned}
& \mathbb{E} \sup_{t_i \le t \le T} |\delta Y^{\pi}_t|^p \\
\le \;& \mathbb{E} \sup_{t_i \le t \le T} \left[\mathbb{E}\left(\sum_{k=i+1}^{n} \left| \widetilde{f}^{\pi}_{t_k} \right| \Delta_{k-1} + \sup_{0 \le r \le T} |R^{\pi}_r| + |\delta \xi^{\pi}| \middle| \mathscr{F}_t \right) \right]^p \\
\le \;& C\mathbb{E}\left(\sum_{k=i+1}^{n} \left| \widetilde{f}^{\pi}_{t_k} \right| \Delta_{k-1} + \sup_{0 \le r \le T} |R^{\pi}_r| + |\delta \xi^{\pi}| \right)^p \\
\le \;& C\left\{ \mathbb{E}\left(\sum_{k=i+1}^{n} \left| \widetilde{f}^{\pi}_{t_k} \right| \Delta_{k-1} \right)^p + \mathbb{E} \sup_{0 \le r \le T} |R^{\pi}_r|^p + \mathbb{E}|\delta \xi^{\pi}|^p \right\} \\
\le \;& C\left\{ \mathbb{E}\left(\sum_{k=i+1}^{n} \left| \delta Y^{\pi}_{t_k} \right| \Delta_{k-1} \right)^p + \mathbb{E}\left(\sum_{k=i+1}^{n} \left| \widehat{Z}^{\pi}_{t_k} \right| \Delta_{k-1} \right)^p + |\pi|^{\frac{p}{2}} + \mathbb{E}|\delta \xi^{\pi}|^p \right\} \\
\le \;& C\left\{ (T - t_i)^p \mathbb{E} \sup_{i+1 \le k \le n} \left| \delta Y^{\pi}_{t_k} \right|^p + \mathbb{E}\left(\sum_{k=i+1}^{n} \left| \widehat{Z}^{\pi}_{t_k} \right| \Delta_{k-1} \right)^p + |\pi|^{\frac{p}{2}} + \mathbb{E}|\delta \xi^{\pi}|^p \right\},
\end{aligned}$$

where and in the following $C > 0$ denotes a generic constant independent of the partition π and may vary from line to line. On the other hand, we have, by the Hölder continuity of Z given by (2.2.23),

$$\begin{aligned}
\mathbb{E}\left(\sum_{k=i+1}^{n}\left|\widehat{Z}_{t_k}^{\pi}\right|\Delta_{k-1}\right)^p &= \mathbb{E}\left(\sum_{k=i+1}^{n}\left|Z_{t_k}-\frac{1}{\Delta_{k-1}}\int_{t_{k-1}}^{t_k} Z_r^{\pi}dr\right|\Delta_{k-1}\right)^p \\
&\leq \mathbb{E}\left(\sum_{k=i+1}^{n}\int_{t_{k-1}}^{t_k}|Z_{t_k}-Z_r|dr+\sum_{k=i+1}^{n}\int_{t_{k-1}}^{t_k}|Z_r-Z_r^{\pi}|dr\right)^p \\
&\leq C|\pi|^{\frac{p}{2}}+2^{p-1}\mathbb{E}\left(\int_{t_i}^{T}|Z_r-Z_r^{\pi}|dr\right)^p \\
&\leq C|\pi|^{\frac{p}{2}}+2^{p-1}(T-t_i)^{\frac{p}{2}}\mathbb{E}\left(\int_{t_i}^{T}|Z_r-Z_r^{\pi}|^2dr\right)^{\frac{p}{2}} \\
&= C|\pi|^{\frac{p}{2}}+2^{p-1}(T-t_i)^{\frac{p}{2}}\mathbb{E}\left(\int_{t_i}^{T}|\delta Z_r^{\pi}|^2dr\right)^{\frac{p}{2}}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\mathbb{E}\sup_{t_i\leq t\leq T}|\delta Y_t^{\pi}|^p \quad\leq\quad & C_1\Bigg\{(T-t_i)^p\mathbb{E}\sup_{i+1\leq k\leq n}|\delta Y_{t_k}|^p+(T-t_i)^{\frac{p}{2}}\mathbb{E}\left(\int_{t_i}^{T}|\delta Z_r^{\pi}|^2dr\right)^{\frac{p}{2}} \\
&+|\pi|^{\frac{p}{2}}+\mathbb{E}|\delta\xi^{\pi}|^p\Bigg\},
\end{aligned}\tag{2.4.9}$$

where C_1 is a constant independent of the partition π. By the Burkholder-Davis-Gundy inequality, we have

$$\mathbb{E}\left(\int_{t_i}^{T}|\delta Z_r^{\pi}|^2dr\right)^{\frac{p}{2}}\leq c_p\mathbb{E}\left|\int_{t_i}^{T}\delta Z_r^{\pi}dW_r\right|^p.\tag{2.4.10}$$

From (2.4.7), we obtain

$$\int_{t_i}^{T}\delta Z_r^{\pi}dW_r=\delta\xi^{\pi}+\sum_{k=i+1}^{n}\widetilde{f}_{t_k}^{\pi}\Delta_{k-1}+R_{t_i}^{\pi}-\delta Y_{t_i}^{\pi}.\tag{2.4.11}$$

Thus, from (2.4.10) and (2.4.11), we obtain

$$\mathbb{E}\left(\int_{t_i}^{T}|\delta Z_r^{\pi}|^2dr\right)^{\frac{p}{2}} \le C_p\left\{\mathbb{E}\left|\sum_{k=i+1}^{n}\widetilde{f}_{t_k}^{\pi}\Delta_{k-1}\right|^p+\mathbb{E}\,|\delta\xi^{\pi}|^p+\mathbb{E}\left|R_{t_i}^{\pi}\right|^p+\mathbb{E}\left|\delta Y_{t_i}^{\pi}\right|^p\right\}.$$

As in the proof of (2.4.9), we have

$$\begin{aligned}\mathbb{E}\left(\int_{t_i}^{T}|\delta Z_r^{\pi}|^2dr\right)^{\frac{p}{2}} \quad \le \quad & C_2\Bigg\{(T-t_i)^p\mathbb{E}\sup_{i+1\le k\le n}|\delta Y_{t_k}|^p+(T-t_i)^{\frac{p}{2}}\mathbb{E}\left(\int_{t_i}^{T}|\delta Z_r^{\pi}|^2dr\right)^{\frac{p}{2}} \\ & +|\pi|^{\frac{p}{2}}+\mathbb{E}|\delta\xi^{\pi}|^p\Bigg\},\end{aligned}$$

where C_2 is a constant independent of the partition π.

If $C_2(T-t_i)^{\frac{p}{2}}<\frac{1}{2}$, then we have

$$\mathbb{E}\left(\int_{t_i}^{T}|\delta Z_r^{\pi}|^2dr\right)^{\frac{p}{2}} \le 2C_2(T-t_i)^p\mathbb{E}\sup_{i+1\le k\le n}|\delta Y_{t_k}|^p+2C_2\left(|\pi|^{\frac{p}{2}}+\mathbb{E}|\delta\xi^{\pi}|^p\right). \tag{2.4.12}$$

Substituting (2.4.12) into (2.4.9), we have

$$\begin{aligned}\mathbb{E}\sup_{t_i\le t\le T}|\delta Y_t^{\pi}|^p \quad \le \quad & C_1\left(1+2C_2(T-t_i)^{\frac{p}{2}}\right)(T-t_i)^p\mathbb{E}\sup_{i+1\le k\le n}|\delta Y_{t_k}|^p \\ & +C_1\left(1+2C_2(T-t_i)^{\frac{p}{2}}\right)\left(|\pi|^{\frac{p}{2}}+\mathbb{E}|\delta\xi^{\pi}|^p\right) \\ \le \quad & 2C_1(T-t_i)^p\mathbb{E}\sup_{i+1\le k\le n}|\delta Y_{t_k}|^p+2C_1\left(|\pi|^{\frac{p}{2}}+\mathbb{E}|\delta\xi^{\pi}|^p\right)\end{aligned} \tag{2.4.13}$$

We can find a positive constant $\delta<\delta(p,L)$ independent of the partition π, such that,

$$C_2(3\delta)^{\frac{p}{2}}<\frac{1}{2}, \tag{2.4.14}$$

$$2C_1(3\delta)^p<\frac{1}{2}, \tag{2.4.15}$$

and $T > 2\delta$. Denote $l = [\frac{T}{2\delta}]$. Then $l \geq 1$ is an integer independent of the partition π. If $|\pi| < \delta$, then for the partition π we can choose $n-1 > i_1 > i_2 > \cdots > i_l \geq 0$, such that, $T-2\delta \in (t_{i_1-1}, t_{i_1}]$, $T-4\delta \in (t_{i_2-1}, t_{i_2}]$, ..., $T-2\delta l \in [0, t_{i_l}]$ (with $t_{-1} = 0$). For simplicity, we denote $t_{i_0} = T$ and $t_{i_{l+1}} = 0$. Each interval $[t_{i_{j+1}}, t_{i_j}]$, $j = 0, 1, \ldots, l$, has length less than 3δ, that is, $|t_{i_j} - t_{i_{j+1}}| < 3\delta$. On $[t_{i_{j+1}}, t_{i_j}]$, we consider the recursive formula (2.4.1). Then (2.4.13)-(2.4.15) yield

$$\begin{aligned} \mathbb{E} \sup_{t_{i_{j+1}} \leq t \leq t_{i_j}} |\delta Y_t^\pi|^p &\leq 2C_1 (t_{i_j} - t_{i_{j+1}})^p \mathbb{E} \sup_{i_{j+1}+1 \leq k \leq i_j} |\delta Y_{t_k}|^p + 2C_1 \left(|\pi|^{\frac{p}{2}} + \mathbb{E}|\delta Y^\pi_{t_{i_j}}|^p \right) \\ &\leq 2C_1 (3\delta)^p \mathbb{E} \sup_{i_{j+1}+1 \leq k \leq i_j} |\delta Y_{t_k}|^p + 2C_1 \left(|\pi|^{\frac{p}{2}} + \mathbb{E}|\delta Y^\pi_{t_{i_j}}|^p \right) \\ &\leq \frac{1}{2} \sup_{i_{j+1}+1 \leq k \leq i_j} |\delta Y_{t_k}|^p + 2C_1 \left(|\pi|^{\frac{p}{2}} + \mathbb{E}|\delta Y^\pi_{t_{i_j}}|^p \right). \end{aligned} \tag{2.4.16}$$

As in the proof of (2.3.21) and (2.3.22), we have

$$\mathbb{E} \sup_{t_{i_{j+1}} \leq t \leq t_{i_j}} |\delta Y_t^\pi|^p \leq (4C_1 + 1) \mathbb{E}|\delta Y^\pi_{t_{i_j}}|^p + 4C_1 |\pi|^{\frac{p}{2}},$$

and

$$\mathbb{E} \sup_{t_{i_{j+1}} \leq t \leq t_{i_j}} |\delta Y_t^\pi|^p \leq \frac{3(4C_1+1)^{l+1}}{2} \left(\mathbb{E}|\delta \xi^\pi|^2 + |\pi|^{\frac{p}{2}} \right).$$

Therefore, we obtain

$$\mathbb{E} \sup_{0 \leq t \leq T} |\delta Y_t^\pi|^p \leq \max_{0 \leq j \leq l} \mathbb{E} \sup_{t_{i_{j+1}} \leq t \leq t_{i_j}} |\delta Y_t^\pi|^p \leq \frac{3(4C_1+1)^{l+1}}{2} \left(\mathbb{E}|\delta \xi^\pi|^p + |\pi|^{\frac{p}{2}} \right). \tag{2.4.17}$$

On $[t_{i_{j+1}}, t_{i_j}]$, $j = 0, 1, \ldots, l$, based on the recursive formula (2.4.1) and (2.4.17), inequality (2.4.12) becomes

$$
\begin{aligned}
\mathbb{E}\left(\int_{t_{i_{j+1}}}^{t_{i_j}} |\delta Z_r^\pi|^2 dr\right)^{\frac{p}{2}} &\leq 2C_2(t_{i_j} - t_{i_{j+1}})^p \mathbb{E} \sup_{i_{j+1}+1 \leq k \leq i_j} |\delta Y_{t_k}|^p + 2C_2\left(|\pi|^{\frac{p}{2}} + \mathbb{E}|\delta\xi^\pi|^p\right) \\
&\leq 2C_2(3\delta)^p \mathbb{E} \sup_{i_{j+1}+1 \leq k \leq i_j} |\delta Y_{t_k}|^p + 2C_2\left(|\pi|^{\frac{p}{2}} + \mathbb{E}|\delta\xi^\pi|^p\right) \\
&\leq \frac{1}{2}\mathbb{E} \sup_{i_{j+1}+1 \leq k \leq i_j} |\delta Y_{t_k}|^p + 2C_2\left(|\pi|^{\frac{p}{2}} + \mathbb{E}|\delta\xi^\pi|^p\right) \\
&\leq \left(\frac{3(4C_1+1)^{l+1}}{4} + 2C_2\right)\left(|\pi|^{\frac{p}{2}} + \mathbb{E}|\delta\xi^\pi|^p\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\mathbb{E}\left(\int_0^T |\delta Z_t^\pi|^2 dt\right)^{\frac{p}{2}} &= \mathbb{E}\left(\sum_{j=0}^{l} \int_{t_{i_{j+1}}}^{t_{i_j}} |\delta Z_t^\pi|^2 dt\right)^{\frac{p}{2}} \\
&\leq (l+1)^{\frac{p}{2}-1} \sum_{j=0}^{l} \mathbb{E}\left(\int_{t_{i_{j+1}}}^{t_{i_j}} |\delta Z_t^\pi|^2 dt\right)^{\frac{p}{2}} \\
&\leq (l+1)^{\frac{p}{2}}\left(\frac{3(4C_1+1)^{l+1}}{4} + 2C_2\right)\left(|\pi|^{\frac{p}{2}} + \mathbb{E}|\delta\xi^\pi|^p\right) \qquad (2.4.18)
\end{aligned}
$$

Combining (2.4.17) and (2.4.18), we know that there exists a constant

$$
K = (l+1)^{\frac{p}{2}}\left(\frac{3(4C_1+1)^{l+1}}{2} + 4C_2\right)
$$

independent of the partition π, such that

$$
\mathbb{E} \sup_{0 \leq t \leq T} |Y_t - Y_t^\pi|^p + \mathbb{E}\left(\int_0^T |Z_t - Z_t^\pi|^2 dt\right)^{\frac{p}{2}} \leq K\left(|\pi|^{\frac{p}{2}} + \mathbb{E}|\xi - \xi^\pi|^p\right).
$$

□

Remark 2.4.3. *The advantages of this implicit numerical scheme are:*

(i) *we can obtain the rate of convergence in L^p sense;*

(ii) *the partition π can be arbitrary ($|\pi|$ should be small enough) without assuming* $\max_{0\le i\le n-1}\Delta_i/\Delta_{i+1}\le L_1$.

2.5 A New Discrete Scheme

For all the numerical schemes considered in Sections 2.3 and 2.4, one needs to evaluate processes $\{Z_t^\pi\}_{0\le t\le T}$ with continuous index t. In this section, we use the representation of Z in terms of the Malliavin derivative of Y to derive a completely discrete scheme.

From Equation (2.2.20), $\{D_\theta Y_t\}_{0\le\theta\le t\le T}$ can be represented as

$$D_\theta Y_t = \mathbb{E}\left(\rho_{t,T}D_\theta\xi + \int_t^T \rho_{t,r}D_\theta f(r,Y_r,Z_r)dr\middle|\mathscr{F}_t\right), \tag{2.5.1}$$

where

$$\rho_{t,r} = \exp\left\{\int_t^r \beta_s dW_s + \int_t^r\left(\alpha_s - \frac{1}{2}\beta_s^2\right)ds\right\} \tag{2.5.2}$$

with $\alpha_s = \partial_y f(s,Y_s,Z_s)$ and $\beta_s = \partial_z f(s,Y_s,Z_s)$.

Using that $Z_t = D_tY_t$, $\mu\times P$ a.e., from Equations (2.1.1), (2.5.1) and (2.5.2), we propose the following numerical scheme. We define recursively

$$\begin{aligned}
Y_{t_n}^\pi &= \xi, \qquad Z_{t_n}^\pi = D_T\xi,\\
Y_{t_i}^\pi &= \mathbb{E}\left(Y_{t_{i+1}}^\pi + f(t_{i+1},Y_{t_{i+1}}^\pi,Z_{t_{i+1}}^\pi)\Delta_i\middle|\mathscr{F}_{t_i}\right),\\
Z_{t_i}^\pi &= \mathbb{E}\left(\rho_{t_{i+1},t_n}^\pi D_{t_i}\xi + \sum_{k=i}^{n-1}\rho_{t_{i+1},t_{k+1}}^\pi D_{t_i}f(t_{k+1},Y_{t_{k+1}}^\pi,Z_{t_{k+1}}^\pi)\Delta_k\middle|\mathscr{F}_{t_i}\right),\\
&\quad i = n-1,n-2,\ldots,0,
\end{aligned} \tag{2.5.3}$$

where $\rho^{\pi}_{t_i,t_i} = 1$, $i = 0,1,\dots,n$, and for $0 \le i < j \le n$,

$$\begin{aligned}\rho^{\pi}_{t_i,t_j} &= \exp\left\{\sum_{k=i}^{j-1}\int_{t_k}^{t_{k+1}} \partial_z f(r,Y^{\pi}_{t_k},Z^{\pi}_{t_k})dW_r \right.\\ &\left. +\sum_{k=i}^{j-1}\int_{t_k}^{t_{k+1}}\left(\partial_y f(r,Y^{\pi}_{t_k},Z^{\pi}_{t_k}) - \frac{1}{2}[\partial_z f(r,Y^{\pi}_{t_k},Z^{\pi}_{t_k})]^2\right)dr\right\}. \end{aligned} \tag{2.5.4}$$

An alternative expression for $\rho^{\pi}_{t_i,t_j}$ is given by the following formula

$$\begin{aligned}\rho^{\pi}_{t_i,t_j} &= \exp\left\{\sum_{k=i}^{j-1}\partial_z f(t_k,Y^{\pi}_{t_k},Z^{\pi}_{t_k})(W_{t_{k+1}} - W_{t_k}) \right.\\ &\left. +\sum_{k=i}^{j-1}\left(\partial_y f(t_k,Y^{\pi}_{t_k},Z^{\pi}_{t_k}) - \frac{1}{2}[\partial_z f(t_k,Y^{\pi}_{t_k},Z^{\pi}_{t_k})]^2\right)\Delta_k\right\}. \end{aligned} \tag{2.5.5}$$

However, we will only consider the scheme (2.5.3) with $\rho^{\pi}_{t_i,t_j}$ given by (2.5.4).

We make the following assumptions:

(B1) $f(t,y,z)$ is deterministic, which implies $D_\theta f(t,y,z) = 0$.

(B2) $f(t,y,z)$ is linear with respect to y and z, namely, there are three functions $g(t)$, $h(t)$ and $f_1(t)$ such that

$$f(t,y,z) = g(t)y + h(t)z + f_1(t)\,.$$

Assume that g, h are bounded and $f_1 \in L^2([0,T])$. Moreover, there exists a constant $L_2 > 0$, such that, for all t_1, $t_2 \in [0,T]$,

$$|g(t_2) - g(t_1)| + |h(t_2) - h(t_1) + |f_1(t_2) - f_1(t_1)| \le L|t_2 - t_1|^{\frac{1}{2}}.$$

(B3) $\mathbb{E}\sup_{0\le\theta\le T}|D_\theta\xi|^r < \infty$, for all $r \ge 1$.

Notice that (B1)-(B2) imply $(A4)$-$(A5)$ in Assumption 2.2.2.

Remark 2.5.1. *We propose condition (B1) in order to simplify* $\{Z^{\pi}_{t_i}\}_{i=n-1,\dots,0}$ *in formula (2.5.3). In fact, there are some difficulties in generalizing the conditions (B)'s, especially (B1), to a forward-backward stochastic differential equation (FBSDE, for short) case*

If we consider a FBSDE

$$\begin{cases} X_t = X_0 + \int_0^t b(r,X_r)dr + \int_0^t \sigma(r,X_r)dW_r, \\ Y_t = \xi + \int_t^T f(r,X_r,Y_r,Z_r)dr - \int_t^T Z_r dW_r, \end{cases}$$

where $X_0 \in \mathbb{R}$*, and the functions* b, σ, f *are deterministic, then under some appropriate conditions (for instance, (h1)-(h4) in Example 2.2.11)* $Z^{\pi}_{t_i}$ *for* $i = n-1,\dots,0$ *in (2.5.3) is of the form*

$$Z^{\pi}_{t_i} = \mathbb{E}\left(\rho^{\pi}_{t_{i+1},t_n} D_{t_i}\xi + \sum_{k=i}^{n-1} \rho^{\pi}_{t_{i+1},t_{k+1}} \partial_x f(t_{k+1}, X^{\pi}_{t_{k+1}}, Y^{\pi}_{t_{k+1}}, Z^{\pi}_{t_{k+1}}) D_{t_i} X^{\pi}_{t_{k+1}} \Delta_k \middle| \mathscr{F}_{t_i}\right),$$

where $(X^{\pi}, Y^{\pi}, Z^{\pi})$ *is a certain numerical scheme for* (X,Y,Z)*. It is hard to guarantee the existence and the convergence of Malliavin derivative of* X^{π}*, and therefore, the convergence of* Z^{π} *is difficult to derive.*

Theorem 2.5.2. *Let Assumption 2.2.2* $(A3)$ *and assumptions (B1)-(B3) be satisfied. Then there are positive constants* K *and* δ *independent of the partition* π*, such that, when* $|\pi| < \delta$ *we have*

$$\mathbb{E} \max_{0\le i\le n} \left\{ |Y_{t_i} - Y^{\pi}_{t_i}|^p + |Z_{t_i} - Z^{\pi}_{t_i}|^p \right\} \le K|\pi|^{\frac{p}{2} - \frac{p}{2\log\frac{1}{|\pi|}}} \left(\log\frac{1}{|\pi|}\right)^{\frac{p}{2}}.$$

Proof. In the proof, $C > 0$ will denote a constant independent of the partition π, which may vary from line to line. Under the assumption (B1), we can see that

$$Z_{t_i}^{\pi} = \mathbb{E}\left(\rho_{t_{i+1},t_n}^{\pi} D_{t_i}\xi \middle| \mathscr{F}_{t_i}\right), \; i = n-1, n-2, \ldots, 0. \tag{2.5.6}$$

Denote, for $i = n-1, n-2, \ldots, 0$,

$$\delta Z_{t_i}^{\pi} = Z_{t_i} - Z_{t_i}^{\pi}, \quad \delta Y_{t_i}^{\pi} = Y_{t_i} - Y_{t_i}^{\pi}.$$

Since $|e^x - e^y| \leq (e^x + e^y)|x - y|$, we deduce, for all $i = n-1, n-2, \ldots, 0$,

$$\begin{aligned}
|\delta Z_{t_i}^{\pi}| &= \left| \mathbb{E}\left(\rho_{t_i,t_n} D_{t_i}\xi \middle| \mathscr{F}_{t_i}\right) - \mathbb{E}\left(\rho_{t_{i+1},t_n}^{\pi} D_{t_i}\xi \middle| \mathscr{F}_{t_i}\right) \right| \\
&\leq \mathbb{E}\left(\left| \rho_{t_i,t_n} - \rho_{t_{i+1},t_n}^{\pi} \right| |D_{t_i}\xi| \middle| \mathscr{F}_{t_i} \right) \\
&\leq \mathbb{E}\Bigg(|D_{t_i}\xi| \left(\rho_{t_i,t_n} + \rho_{t_{i+1},t_n}^{\pi}\right) \Bigg| \int_{t_i}^{T} h(r) dW_r + \int_{t_i}^{T} g(r) dr \\
&\quad - \frac{1}{2}\int_{t_i}^{T} h(r)^2 dr - \sum_{k=i+1}^{n-1} \int_{t_k}^{t_{k+1}} h(r) dW_r - \sum_{k=i+1}^{n-1} \int_{t_k}^{t_{k+1}} g(r) dr \\
&\quad + \frac{1}{2}\sum_{k=i+1}^{n-1} \int_{t_k}^{t_{k+1}} h(r)^2 dr \Bigg| \Bigg| \mathscr{F}_{t_i} \Bigg) \\
&\leq \mathbb{E}\Bigg(|D_{t_i}\xi| \left(\rho_{t_i,t_n} + \rho_{t_{i+1},t_n}^{\pi}\right) \Bigg[\left| \int_{t_i}^{t_{i+1}} h(r) dW_r \right| + \int_{t_i}^{t_{i+1}} |g(r)| dr \\
&\quad + \frac{1}{2}\int_{t_i}^{t_{i+1}} h(r)^2 dr \Bigg] \Bigg| \mathscr{F}_{t_i} \Bigg).
\end{aligned}$$

From (B2), we have

$$
\begin{aligned}
|D_{t_i}\xi|\rho^{\pi}_{t_{i+1},t_n} &\leq |D_{t_i}\xi|\exp\left\{\int_{t_{i+1}}^{T} h(r)dW_r + \sum_{k=i+1}^{n-1}\int_{t_k}^{t_{k+1}} g(r)dr - \frac{1}{2}\int_{t_{i+1}}^{T} h(r)^2 dr\right\} \\
&\leq C_1\left(\sup_{0\leq\theta\leq T}|D_\theta\xi|\right)\left(\sup_{0\leq t\leq T}\exp\left\{\int_t^T h(r)dW_r\right\}\right),
\end{aligned}
$$

where $C_1 > 0$ is a constant independent of the partition π.

In the same way, we obtain

$$
|D_{t_i}\xi|\rho_{t_i,t_n} < C_1\left(\sup_{0\leq\theta\leq T}|D_\theta\xi|\right)\left(\sup_{0\leq t\leq T}\exp\left\{\int_t^T h(r)dW_r\right\}\right).
$$

Thus for $i = n-1, n-2, \ldots, 0$,

$$
\begin{aligned}
|\delta Z^{\pi}_{t_i}| &\leq 2C_1\mathbb{E}\Bigg(\left(\sup_{0\leq\theta\leq T}|D_\theta\xi|\right)\left(\sup_{0\leq t\leq T}\exp\left\{\int_t^T h(r)dW_r\right\}\right)\Bigg[\left|\int_{t_i}^{t_{i+1}} h(r)dW_r\right| \\
&\qquad + \int_{t_i}^{t_{i+1}}|g(r)|dr + \frac{1}{2}\int_{t_i}^{t_{i+1}} h(r)^2 dr\Bigg]\Bigg|\mathscr{F}_{t_i}\Bigg) \\
&\leq 2C_1\mathbb{E}\Bigg(\left(\sup_{0\leq\theta\leq T}|D_\theta\xi|\right)\left(\sup_{0\leq t\leq T}\exp\left\{\int_t^T h(r)dW_r\right\}\right)\Bigg[\sup_{0\leq k\leq n-1}\left|\int_{t_k}^{t_{k+1}} h(r)dW_r\right| \\
&\qquad + \sup_{0\leq k\leq n-1}\int_{t_k}^{t_{k+1}}|g(r)|dr + \frac{1}{2}\sup_{0\leq k\leq n-1}\int_{t_k}^{t_{k+1}} h(r)^2 dr\Bigg]\Bigg|\mathscr{F}_{t_i}\Bigg).
\end{aligned}
$$

The right-hand side of the above inequality is a martingale as a process indexed by $i = n-1, n-2, \ldots, 0$.

Let $\eta_t = \exp\left\{-\int_0^t h(u)dW_u\right\}$. Then, η_t satisfies the following linear stochastic differential equation

$$
\begin{cases}
d\eta_t = -h(t)\eta_t dW_t + \frac{1}{2}h(t)^2\eta_t dt, \\
\eta_0 = 1.
\end{cases}
$$

By (B1), (B2), the Hölder inequality and Lemma 2.2.4, it is easy to show that, for any $r \geq 0$,

$$
\begin{aligned}
& \mathbb{E}\left(\sup_{0\leq t\leq T} \exp\left\{\int_t^T h(u)dW_u\right\}\right)^r \\
= \; & \mathbb{E}\left(\exp\left\{\int_0^T h(u)dW_u\right\}\sup_{0\leq t\leq T}\exp\left\{-\int_0^t h(u)dW_u\right\}\right)^r \\
\leq \; & \left(\mathbb{E}\exp\left\{2r\int_0^T h(u)dW_u\right\}\right)^{\frac{1}{2}}\left(\mathbb{E}\sup_{0\leq t\leq T}\exp\left\{-2r\int_0^t h(u)dW_u\right\}\right)^{\frac{1}{2}} \\
= \; & \exp\left\{r^2\int_0^T h(u)^2 dr\right\}\left(\mathbb{E}\sup_{0\leq t\leq T}\eta_t^{2r}\right)^{\frac{1}{2}} < \infty. \qquad (2.5.7)
\end{aligned}
$$

For any $p' \in (p, \frac{q}{2})$, by Doob's maximal inequality and the Hölder inequality, (B3) and (2.5.7), we have

$$
\begin{aligned}
& \mathbb{E} \sup_{0\le i\le n} |\delta Z^{\pi}_{t_i}|^p \\
\le\ & C\mathbb{E}\Bigg(\left(\sup_{0\le\theta\le T} |D_\theta \xi|\right)^p \left(\sup_{0\le t\le T} \exp\left\{\int_t^T h(r)dW_r\right\}\right)^p \Bigg[\sup_{0\le k\le n-1}\left|\int_{t_k}^{t_{k+1}} h(r)dW_r\right| \\
& \quad + \sup_{0\le k\le n-1}\int_{t_k}^{t_{k+1}} |g(r)|dr + \frac{1}{2}\sup_{0\le k\le n-1}\int_{t_k}^{t_{k+1}} h(r)^2 dr\Bigg]^p\Bigg) \\
\le\ & C\left[\mathbb{E}\left(\left(\sup_{0\le\theta\le T} |D_\theta \xi|\right)^{\frac{pp'}{p'-p}} \left(\sup_{0\le t\le T} \exp\left\{\int_t^T h(r)dW_r\right\}\right)^{\frac{pp'}{p'-p}}\right)\right]^{\frac{p'-p}{p'}} \\
& \times\Bigg[\mathbb{E}\Bigg(\sup_{0\le k\le n-1}\left|\int_{t_k}^{t_{k+1}} h(r)dW_r\right| + \sup_{0\le k\le n-1}\int_{t_k}^{t_{k+1}} |g(r)|dr \\
& \qquad + \frac{1}{2}\sup_{0\le k\le n-1}\int_{t_k}^{t_{k+1}} h(r)^2 dr\Bigg)^{p'}\Bigg]^{\frac{p}{p'}} \\
\le\ & C\left[\mathbb{E}\left(\sup_{0\le\theta\le T} |D_\theta \xi|\right)^{\frac{2pp'}{p'-p}}\right]^{\frac{p'}{2(p'-p)}} \left[\mathbb{E}\left(\sup_{0\le t\le T} \exp\left\{\int_t^T h(r)dW_r\right\}\right)^{\frac{2pp'}{p'-p}}\right]^{\frac{p'}{2(p'-p)}} \\
& \times\Bigg[\mathbb{E}\sup_{0\le k\le n-1}\left|\int_{t_k}^{t_{k+1}} h(r)dW_r\right|^{p'} + \mathbb{E}\sup_{0\le k\le n-1}\left(\int_{t_k}^{t_{k+1}} |g(r)|dr\right)^{p'} \\
& \quad + \mathbb{E}\sup_{0\le k\le n-1}\left(\int_{t_k}^{t_{k+1}} h(r)^2 dr\right)^{p'}\Bigg]^{\frac{p}{p'}} \\
=\ & C[I_1 + I_2 + I_3]^{\frac{p}{p'}}.
\end{aligned}
$$

For any $r > 1$, by the Hölder inequality we can obtain

$$
\begin{aligned}
I_1 &= \mathbb{E} \sup_{0 \le k \le n-1} \left| \int_{t_k}^{t_{k+1}} h(r) dW_r \right|^{p'} \le \left\{ \mathbb{E} \sup_{0 \le k \le n-1} \left| \int_{t_k}^{t_{k+1}} h(r) dW_r \right|^{p'r} \right\}^{\frac{1}{r}} \\
&\le \left\{ \mathbb{E} \sum_{k=0}^{n-1} \left| \int_{t_k}^{t_{k+1}} h(r) dW_r \right|^{p'r} \right\}^{\frac{1}{r}} .
\end{aligned}
$$

For any centered Gaussian variable X, and any $\gamma \ge 1$, we know that

$$
\mathbb{E}|X|^{\gamma} \le \tilde{C}^{\gamma} \gamma^{\frac{\gamma}{2}} (\mathbb{E}|X|^2)^{\frac{\gamma}{2}},
$$

where $\tilde{C}$ is a constant independent of γ. Thus, we can see that

$$
I_1 \le \left(\tilde{C}^{p'r} (p'r)^{\frac{p'r}{2}} \sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} h(r)^2 dr \right)^{\frac{p'r}{2}} \right)^{\frac{1}{r}} \le C r^{\frac{p'}{2}} |\pi|^{\frac{p'}{2} - \frac{1}{r}} .
$$

Take $r = \frac{2 \log \frac{1}{|\pi|}}{p'}$. Assume $|\pi|$ is small enough, then we have

$$
I_1 \le C |\pi|^{\frac{p'}{2} - \frac{p'}{2 \log \frac{1}{|\pi|}}} \left(\log \frac{1}{|\pi|} \right)^{\frac{p'}{2}} .
$$

It is easy to see that

$$
I_2 = \mathbb{E} \sup_{0 \le k \le n-1} \left(\int_{t_k}^{t_{k+1}} |g(r)| dr \right)^{p'} \le C |\pi|^{p'},
$$

and

$$
I_3 = \mathbb{E} \sup_{0 \le k \le n-1} \left(\int_{t_k}^{t_{k+1}} h(r)^2 dr \right)^{p'} \le C |\pi|^{p'} .
$$

Consequently, we obtain

$$\mathbb{E}\sup_{0\le i\le n}|\delta Z^{\pi}_{t_i}|^p \quad\le\quad C|\pi|^{\frac{p}{2}-\frac{p}{2\log\frac{1}{|\pi|}}}\left(\log\frac{1}{|\pi|}\right)^{\frac{p}{2}}. \tag{2.5.8}$$

Applying recursively the scheme given by (2.5.3), we obtain

$$Y^{\pi}_{t_i}=\mathbb{E}\left(\xi+\sum_{k=i+1}^{n}f(t_k,Y^{\pi}_{t_k},Z^{\pi}_{t_k})\Delta_{k-1}|\mathscr{F}_{t_i}\right),\ i=n-1,n-2,\ldots,0.$$

Therefore, for $i=n-1,n-2,\ldots,0$,

$$|\delta Y^{\pi}_{t_i}|\le\mathbb{E}\left(\sum_{k=i+1}^{n}\left|f(t_k,Y_{t_k},Z_{t_k})-f(t_k,Y^{\pi}_{t_k},Z^{\pi}_{t_k})\right|\Delta_{k-1}+|R^{\pi}_{t_i}|+|\delta\xi^{\pi}|\big|\mathscr{F}_{t_i}\right),$$

where R^{π}_{t} is exactly the same as in Section 2.3, and $\delta\xi^{\pi}=\xi-\xi=0$. In fact, we keep the term $\delta\xi^{\pi}$ to indicate the role it plays as the terminal value .

For $j=n-1,n-2,\ldots,i$, we have

$$|\delta Y^{\pi}_{t_j}|\le\mathbb{E}\left(\sum_{k=i+1}^{n}\left|f(t_k,Y_{t_k},Z_{t_k})-f(t_k,Y^{\pi}_{t_k},Z^{\pi}_{t_k})\right|\Delta_{k-1}+\sup_{0\le t\le T}|R^{\pi}_{t}|+|\delta\xi^{\pi}|\big|\mathscr{F}_{t_j}\right).$$

By Doob's maximal inequality and (2.5.8), we obtain

$$\begin{aligned}
&\mathbb{E}\sup_{i\le j\le n}|\delta Y^{\pi}_{t_j}|^p\\
\le\ & C\mathbb{E}\left(\sum_{k=i+1}^{n}\left|f(t_k,Y_{t_k},Z_{t_k})-f(t_k,Y^{\pi}_{t_k},Z^{\pi}_{t_k})\right|\Delta_{k-1}\right)^p+C\left(|\pi|^{\frac{p}{2}}+\mathbb{E}|\delta\xi^{\pi}|^p\right)\\
\le\ & C\left\{\mathbb{E}\left(\sum_{k=i+1}^{n}\left|Y_{t_k}-Y^{\pi}_{t_k}\right|\Delta_{k-1}\right)^p+\mathbb{E}\left(\sum_{k=i+1}^{n}\left|Z_{t_k}-Z^{\pi}_{t_k}\right|\Delta_{k-1}\right)^p\right\}\\
&+C\left(|\pi|^{\frac{p}{2}}+\mathbb{E}|\delta\xi^{\pi}|^p\right)\\
\le\ & C_2(T-t_i)^p\mathbb{E}\sup_{i+1\le k\le n}\left|Y_{t_k}-Y^{\pi}_{t_k}\right|^p+C_3\left(|\pi|^{\frac{p}{2}-\frac{p}{2\log\frac{1}{|\pi|}}}\left(\log\frac{1}{|\pi|}\right)^{\frac{p}{2}}+\mathbb{E}|\delta\xi^{\pi}|^p\right),
\end{aligned}$$

where C_2 and C_3 are constants independent of the partition π.

We can obtain the estimate for $\mathbb{E}\max_{0\le i\le n}|Y_{t_i}-Y_{t_i}^{\pi}|^p$ by using similar arguments to analyze (2.4.13) in Theorem 2.4.2 to get the estimate for $\mathbb{E}\sup_{0\le t\le T}|Y_t-Y_t^{\pi}|$. $\square$

Chapter 3

A singular stochastic differential equation driven by fractional Brownian motion

3.1 Introduction

Consider the following stochastic differential equation, driven by an additive fractional Brownian motion (fBm) B^H with Hurst parameter H

$$X_t = x_0 + \int_0^t f(s, X_s)ds + B_t^H . \tag{3.1.1}$$

The existence of weak and strong solutions for this type of equation has been proved under different hypotheses on the function f. In [32], using a Girsanov transformation for the fBm, Nualart and Ouknine proved the existence of a unique strong solution assuming that $f(s,x)$ satisfies the linear growth condition $|f(s,x)| \leq C(1+|x|)$ if $H \leq \frac{1}{2}$, and that $f(s,x)$ is Hölder continuous of order $\alpha > 1 - \frac{1}{2H}$ in x and of order $\gamma > H - \frac{1}{2}$ in s if $H > \frac{1}{2}$. This result was extended by Boufoussi and Ouknine in [5] to the case where we add to the drift a bounded non-decreasing left-(or right-) continuous function, in the case $H > \frac{1}{2}$. The existence of weak solutions assuming that the drift might have some jump-discontinuities was derived also in the paper by Mishura and Nualart [28], assuming $H \in (\frac{1}{2}, H_0)$, for some $H_0 > \frac{1}{2}$, by means of Girsanov theorem.

The aim of this chapter is to consider the case where $H > \frac{1}{2}$, $x_0 \geq 0$, and the drift $f(t,x)$ is nonnegative and it has a singularity at $x = 0$ of the form $x^{-\alpha}$, where $\alpha > \frac{1}{H} - 1$, and $x_0 \geq 0$. This singular drift cannot be covered by the above previous results and requires new techniques.

The study of this type of singular equations is partially motivated by the equation satisfied by the d-dimensional fractional Bessel process $R_t = |B_t^H|$, $d \geq 2$ (see Guerra and Nualart [17], and Hu and Nualart [18]):

$$R_t = Y_t + H(d-1) \int_0^t \frac{s^{2H-1}}{R_s} ds,$$

where the process Y_t is equal to a divergence integral, $Y_t = \int_0^t \sum_{i=1}^d \frac{B_s^{H,i}}{R_s} \delta B_s^{H,i}$. Except in the case $H = \frac{1}{2}$, the process Y is not a one-dimensional fractional Brownian motion (see Eisenbaum and Tudor [11] and Hu and Nualart [18] for some results in this direction), although it shares with the fBm similar properties of scaling and $\frac{1}{H}$-variation. Notice that here the initial condition is zero.

Using arguments based on fractional calculus inspired by the estimates obtained by Hu and Nualart in [19], we will show that there exists a unique global solution which has moments of all orders, and even negative moments, in the particular case $f(t,x) = Kx^{-1}$, if t is small enough. We will also show that the solution has an absolutely continuous law with respect to the Lebesgue measure, using the techniques of Malliavin calculus for the fractional Brownian motion. As an application we obtain the existence of a unique solution with moments of all orders for a fractional version of the CIR model in mathematical finance ([9]), which is a singular stochastic differential equation driven by fractional Brownian motion with the diffusion coefficient being $\sqrt{x}$.

This chapter is organized as follows. In Section 3.2 we will consider the case of a deterministic differential equation driven by a Hölder continuous function, and with singular drift. The case of the fractional Brownian motion is developed in Section 3.3.

3.2 Singular equations driven by rough paths

Fix $\beta \in (1/2, 1)$. Suppose that $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is a function such that $\varphi(0) = 0$, and $\varphi \in C^\beta([0,T])$ for all $T > 0$. Consider the following deterministic differential equation driven by the rough path φ

$$x_t = x_0 + \int_0^t f(s, x_s) ds + \varphi(t), \tag{3.2.1}$$

where $x_0 \geq 0$ is a constant. We are going to impose the following assumptions on the coefficient f:

(i) $f : [0,\infty) \times (0,\infty) \to [0,\infty)$ is a nonnegative, continuous function which has a continuous partial derivative with respect to x such that $\partial_x f(t,x) \leq 0$ for all $t > 0$, $x > 0$.

(ii) There exists $x_1 > 0$ and $\alpha > \frac{1}{\beta} - 1$ such that $f(t,x) \geq g(t)x^{-\alpha}$, for all $t \geq 0$ and $x \in (0, x_1)$, where $g(t)$ is a nonnegative continuous function with $g(t) > 0$ for all $t > 0$.

(iii) $f(t,x) \leq h(t)\left(1 + \frac{1}{x}\right)$ for all $t \geq 0$ and $x > 0$, where $h(t)$ is a certain nonnegative locally bounded function.

Theorem 3.2.1. *Under the assumptions (i)-(ii), there exists a unique solution x_t to Equation (3.2.1) such that $x_t > 0$ on $(0,\infty)$.*

Proof. Assume first that $x_0 > 0$. It is easy to see that there exists a continuous local solution x_t to Equation (3.2.1) on some interval $[0,T)$, where T satisfies $T = \inf\{t > 0 :$

$x_t = 0\}$. Then it suffices to show that $T = \infty$. Suppose that $T < \infty$. Then, then $x_t \to 0$, as $t \uparrow T$. Since $\varphi \in C^{\beta}([0,T])$, there exists a constant $C > 0$, such that $|\varphi(t) - \varphi(s)| \leq C|t-s|^{\beta}$, for all $s,t \in [0,T]$. Since x_t satisfies Equation (3.2.1), for all $t \in [0,T]$ we have

$$0 = x_T = x_t + \int_t^T f(s,x_s)ds + \varphi(T) - \varphi(t).$$

Since $f(s,x_s)$ is positive, for all $t \in [0,T]$ we have

$$x_t \leq x_t + \int_t^T f(s,x_s)ds = \varphi(t) - \varphi(T) \leq C(T-t)^{\beta}.$$

From the assumption (ii), there exists $t_0 \in (0,T)$ and a constant $K > 0$, such that $g(t) \geq K$ and $x_s \in (0,x_1)$ for all $t \in [t_0,T)$. Then, for all $t \in [t_0,T)$ we have

$$f(t,x_t) \geq \frac{g(t)}{x_t{}^{\alpha}} \geq \frac{K}{x_t{}^{\alpha}} \geq \frac{K}{C^{\alpha}(T-t)^{\alpha\beta}}.$$

Consequently, for all $t \in [t_0,T)$ we obtain

$$\frac{K(T-t)^{1-\alpha\beta}}{C^{\alpha}(1-\alpha\beta)} = \int_t^T \frac{K}{C^{\alpha}(T-s)^{\alpha\beta}}ds \leq \int_t^T f(s,x_s)ds \leq C(T-t)^{\beta},$$

which is a contradiction because $1 - \alpha\beta - \beta < 0$ and t can be arbitrarily close to T. Therefore, $T = \infty$. This proves the existence of the solution for all t.

To handle the case $x_0 = 0$, let us denote by x_t^n the solution to Equation (3.2.1) with initial condition $x_0 = \frac{1}{n}$. The sequence $(x_t^n, n \geq 1)$ is non increasing and positive, so it has a limit, denoted by x_t. By the monotone convergence theorem (putting $f(t,0) = +\infty$) we obtain

$$x_t = \int_0^t f(s,x_s)ds + \varphi(t).$$

Hence, $f(t,x_t) < \infty$ for almost all $t \geq 0$, and this implies that $x_t > 0$ for almost all $t \geq 0$. By the previous arguments, if $x_t > 0$, then $x_s > 0$ for all $s > t$. As a consequence, $x_t > 0$ for all $t > 0$.

Now we show the uniqueness. If $x_{1,t}$ and $x_{2,t}$ are two positive solutions to Equation (3.2.1), then

$$x_{1,t} - x_{2,t} = \int_0^t [f(s,x_{1,s}) - f(s,x_{2,s})] ds .$$

Because $\partial_x f(t,x) \leq 0$ for all $t > 0$, $x > 0$, we deduce

$$(x_{1,t} - x_{2,t})^2 = 2\int_0^t (x_{1,s} - x_{2,s})[f(s,x_{1,s}) - f(s,x_{2,s})] ds \leq 0.$$

So $x_{1,t} = x_{2,t}$.

Thus we conclude that there exists a unique solution x_t to Equation (3.2.1) such that $x_t > 0$ on $(0,\infty)$. □

Remark 3.2.2. *From the continuity of x_t and $f(t,x)$ and the Hölder continuity of $\varphi(t)$, we obtain that for any $T > 0$, $x \in C^\beta([0,T])$.*

The next result provides an estimate on the supremum norm of the solution in terms of the Hölder norm of the driving function φ.

Theorem 3.2.3. *Let the assumptions (i)-(iii) be satisfied. If x_t is the solution to Equation(3.2.1), then for any $\gamma > 2$, and for any $T > 0$,*

$$\|x\|_{0,T,\infty} \leq C_{1,\gamma,\beta,T}(|x_0| + 1) \exp\left\{ C_{2,\gamma,\beta,T} \left(1 + \|\varphi\|_{0,T,\beta}^{\frac{\gamma}{\beta(\gamma-1)}} \right) \right\}, \tag{3.2.2}$$

where $C_{1,\gamma,\beta,T}$ and $C_{2,\gamma,\beta,T}$ are constants depending on β, γ, $\|h\|_{0,T,\infty}$ and T.

Proof. Fix a time interval $[0,T]$. Let $y_t = x_t^\gamma$. Then the chain rule applied to x_t^γ yields

$$y_t = x_0^\gamma + \gamma \int_0^t f(s, y_s^{\frac{1}{\gamma}}) y_s^{1-\frac{1}{\gamma}} ds + \gamma \int_0^t y_s^{1-\frac{1}{\gamma}} d\varphi(s). \tag{3.2.3}$$

The second integral in (3.2.3) is a Riemann-Stieltjes integral (see Young [38]). From Assumption (iii), we have

$$\begin{aligned} |y_t - y_s| &= \gamma \left| \int_s^t f(u, y_u^{\frac{1}{\gamma}}) y_u^{1-\frac{1}{\gamma}} du + \int_s^t y_u^{1-\frac{1}{\gamma}} d\varphi(u) \right| \\ &\leq K_T \gamma \int_s^t \left[y_u^{1-\frac{2}{\gamma}} + y_u^{1-\frac{1}{\gamma}} \right] du + \gamma \left| \int_s^t y_u^{1-\frac{1}{\gamma}} d\varphi(u) \right|, \end{aligned} \tag{3.2.4}$$

where $K_T = \sup_{t \in [0,T]} h(t)$. Since $\gamma > 2$, we have

$$\int_s^t y_u^{1-\frac{2}{\gamma}} du \leq \left[\|y\|_{s,t,\infty}^{1-\frac{2}{\gamma}} + \|y\|_{s,t,\infty}^{1-\frac{1}{\gamma}} \right] (t-s). \tag{3.2.5}$$

Since $\alpha > \frac{1}{\beta} - 1$, we have $\alpha > \alpha\beta > 1 - \beta$. Thus $1 - \alpha < \beta$. From Remark 1.1, we know that $y \in C^\beta([0,T])$, for any $T > 0$. A fractional integration by parts formula (see Zähle [39]) yields

$$\int_s^t y_u^{1-\frac{1}{\gamma}} d\varphi(u) = (-1)^{-\alpha} \int_s^t D_{s+}^\alpha y_u^{1-\frac{1}{\gamma}} D_{t-}^{1-\alpha} \varphi_{t-}(u) du, \tag{3.2.6}$$

where $\varphi_{t-}(u) = \varphi(u) - \varphi(t)$, and D_{s+}^α and $D_{t-}^{1-\alpha}$ denote the left and right-sided fractional derivatives of orders α and $1-\alpha$, respectively (see [35]), defined by

$$D_{s+}^\alpha y_u^{1-\frac{1}{\gamma}} = \frac{1}{\Gamma(1-\alpha)} \left(\frac{y_u^{1-\frac{1}{\gamma}}}{(u-s)^\alpha} + \alpha \int_s^u \frac{y_u^{1-\frac{1}{\gamma}} - y_r^{1-\frac{1}{\gamma}}}{(u-r)^{\alpha+1}} dr \right), \tag{3.2.7}$$

and

$$D_{t-}^{1-\alpha}\varphi_{t-}(u)=\frac{(-1)^{1-\alpha}}{\Gamma(\alpha)}\left(\frac{\varphi(u)-\varphi(t)}{(t-u)^{1-\alpha}}+(1-\alpha)\int_u^t\frac{\varphi(u)-\varphi(r)}{(r-u)^{2-\alpha}}dr\right). \tag{3.2.8}$$

From (3.2.7), and using the Hölder continuity of y we obtain

$$\begin{aligned}|D_{s+}^{\alpha}y_u^{1-\frac{1}{\gamma}}| &\leq C\left(\|y\|_{s,t,\infty}^{1-\frac{1}{\gamma}}(u-s)^{-\alpha}+\int_s^u\frac{|y_u^{1-\frac{1}{\gamma}}-y_r^{1-\frac{1}{\gamma}}|}{(u-r)^{\alpha+1}}dr\right)\\ &\leq C\left(\|y\|_{s,t,\infty}^{1-\frac{1}{\gamma}}(u-s)^{-\alpha}+\int_s^u\frac{|y_u-y_r|^{1-\frac{1}{\gamma}}}{(u-r)^{\alpha+1}}dr\right)\\ &\leq C\left(\|y\|_{s,t,\infty}^{1-\frac{1}{\gamma}}(u-s)^{-\alpha}+\|y\|_{s,t,\beta}^{1-\frac{1}{\gamma}}\int_s^u(u-r)^{\beta(1-\frac{1}{\gamma})-\alpha-1}dr\right)\\ &\leq C\left(\|y\|_{s,t,\infty}^{1-\frac{1}{\gamma}}(u-s)^{-\alpha}+\|y\|_{s,t,\beta}^{1-\frac{1}{\gamma}}(u-s)^{\beta(1-\frac{1}{\gamma})-\alpha}\right), \end{aligned} \tag{3.2.9}$$

where and in what follows, C denotes a generic constant depending on α, β and T. On the other hand, from (3.2.8) we have

$$|D_{t-}^{1-\alpha}\varphi_{t-}(u)|\leq C\|\varphi\|_{0,T,\beta}(t-u)^{\alpha+\beta-1}. \tag{3.2.10}$$

Substituting (3.2.9) and (3.2.10) into (3.2.6) yields

$$\begin{aligned}\left|\int_s^t y_u^{1-\frac{1}{\gamma}}d\varphi(u)\right| &\leq C\int_s^t\left(\|y\|_{s,t,\infty}^{1-\frac{1}{\gamma}}(u-s)^{-\alpha}+\|y\|_{s,t,\beta}^{1-\frac{1}{\gamma}}(u-s)^{\beta(1-\frac{1}{\gamma})-\alpha}\right)\\ &\quad\times\|\varphi\|_{0,T,\beta}(t-u)^{\alpha+\beta-1}du\\ &\leq C\|\varphi\|_{0,T,\beta}\\ &\quad\times\left(\|y\|_{s,t,\infty}^{1-\frac{1}{\gamma}}(t-s)^{\beta}+\|y\|_{s,t,\beta}^{1-\frac{1}{\gamma}}(t-s)^{\beta(2-\frac{1}{\gamma})}\right). \end{aligned} \tag{3.2.11}$$

Substituting (3.2.11) and (3.2.5) into (3.2.4) we obtain

$$\begin{aligned} |y_t - y_s| \quad \le \quad & K_T \gamma \left[\|y\|_{s,t,\infty}^{1-\frac{2}{\gamma}} + \|y\|_{s,t,\infty}^{1-\frac{1}{\gamma}} \right] (t-s) + C\gamma \|\varphi\|_{0,T,\beta} \\ & \times \left(\|y\|_{s,t,\infty}^{1-\frac{1}{\gamma}} (t-s)^{\beta} + \|y\|_{s,t,\beta}^{1-\frac{1}{\gamma}} (t-s)^{\beta(2-\frac{1}{\gamma})} \right). \end{aligned}$$

Consequently, using the estimate $x^{1-\frac{1}{\gamma}} \le 1 + x$ for all $x > 0$, we obtain

$$\begin{aligned} \|y\|_{s,t,\beta} \quad \le \quad & K_T \gamma \left[\|y\|_{s,t,\infty}^{1-\frac{2}{\gamma}} + \|y\|_{s,t,\infty}^{1-\frac{1}{\gamma}} \right] (t-s)^{1-\beta} + C\gamma \|\varphi\|_{0,T,\beta} \\ & \times \left(\|y\|_{s,t,\infty}^{1-\frac{1}{\gamma}} + (1 + \|y\|_{s,t,\beta})(t-s)^{\beta(1-\frac{1}{\gamma})} \right), \end{aligned}$$

which implies

$$\begin{aligned} \left[1 - C\gamma \|\varphi\|_{0,T,\beta} (t-s)^{\beta(1-\frac{1}{\gamma})} \right] \|y\|_{s,t,\beta} \le K_T \gamma \left[\|y\|_{s,t,\infty}^{1-\frac{2}{\gamma}} + \|y\|_{s,t,\infty}^{1-\frac{1}{\gamma}} \right] \\ \times (t-s)^{1-\beta} + C\gamma \|\varphi\|_{0,T,\beta} \left(\|y\|_{s,t,\infty}^{1-\frac{1}{\gamma}} + (t-s)^{\beta(1-\frac{1}{\gamma})} \right). \end{aligned}$$

Suppose that Δ satisfies

$$\Delta \le \left(\frac{1}{2C\gamma \|\varphi\|_{0,T,\beta}} \right)^{\frac{\gamma}{\beta(\gamma-1)}}. \tag{3.2.12}$$

Then for all $s, t \in [0,T]$, $s \le t$, such that $t - s \le \Delta$, we have

$$\|y\|_{s,t,\beta} \le 2K_T \gamma \left[\|y\|_{s,t,\infty}^{1-\frac{2}{\gamma}} + \|y\|_{s,t,\infty}^{1-\frac{1}{\gamma}} \right] (t-s)^{1-\beta} + 2C\gamma \|\varphi\|_{0,T,\beta} \|y\|_{s,t,\infty}^{1-\frac{1}{\gamma}} + 1,$$

and this implies

$$\begin{aligned}\|y\|_{s,t,\infty} &\leq |y_s| + \|y\|_{s,t,\beta}(t-s)^{\beta} \\ &\leq |y_s| + 2K_T\gamma\left[\|y\|_{s,t,\infty}^{1-\frac{2}{\gamma}} + \|y\|_{s,t,\infty}^{1-\frac{1}{\gamma}}\right](t-s) \\ &\quad +2C\gamma\|\varphi\|_{0,T,\beta}\|y\|_{s,t,\infty}^{1-\frac{1}{\gamma}}(t-s)^{\beta} + (t-s)^{\beta}.\end{aligned}$$

Using again the inequality $x^{\alpha} \leq 1+x$ for all $x>0$ and $\alpha \in (0,1)$, we have

$$\begin{aligned}\|y\|_{s,t,\infty} &\leq |y_s| + 4K_T\gamma(1+\|y\|_{s,t,\infty})(t-s) \\ &\quad +2C\gamma\|\varphi\|_{0,T,\beta}(1+\|y\|_{s,t,\infty})(t-s)^{\beta} + (t-s)^{\beta},\end{aligned}$$

which can be written as

$$\begin{aligned}&\|y\|_{s,t,\infty}\Big(1-2C\gamma\|\varphi\|_{0,T,\beta}(t-s)^{\beta} - 4K_T\gamma(t-s)\Big) \\ \leq\ & |y_s| + 4K_T\gamma(t-s) + 2(t-s)^{\beta}. \qquad (3.2.13)\end{aligned}$$

Now we choose Δ such that

$$\Delta = \left(\frac{1}{2C\gamma\|\varphi\|_{0,T,\beta}}\right)^{\frac{\gamma}{\beta(\gamma-1)}} \wedge \left(\frac{1}{16K_T\gamma}\right) \wedge \left(\frac{1}{8C\gamma\|\varphi\|_{0,T,\beta}}\right)^{\frac{1}{\beta}}. \qquad (3.2.14)$$

Then, for all $s,t \in [0,T]$, $s<t$, such that $t-s \leq \Delta$, the inequality (3.2.13) implies

$$\|y\|_{s,t,\infty} \leq 2|y_s| + C_{\gamma,\beta,T}, \qquad (3.2.15)$$

where $C_{\gamma,\beta,T} = 8K_T\gamma T + 4T^{\beta}$. Take $n = [\frac{T}{\Delta}]+1$ (where $[a]$ denotes the largest integer bounded by a). Divide the interval $[0,T]$ into n subintervals. Applying the inequality

(3.2.15) for $s=0$ and $t=\Delta$, we have for all $t\in[0,\Delta]$

$$\|y\|_{0,t,\infty}\le 2|y_0|+C_{\gamma,\beta,T}\ . \tag{3.2.16}$$

Applying the inequality (3.2.16) on the intervals $[\Delta,2\Delta],\dots,[(n-2)\Delta,(n-1)\Delta],[(n-1)\Delta,T]$ recursively, we obtain

$$\begin{aligned}\|y\|_{0,T,\infty} &\le 2^n|y_0|+2^{n-1}C_{\gamma,\beta,T}+\cdots+C_{\gamma,\beta,T}\\ &\le 2^{[\frac{T}{\Delta}]+1}\left(|y_0|+C_{\gamma,\beta,T}\right)\\ &\le 2^{T\left(2C\gamma\|\varphi\|_{0,T,\beta}\right)^{\frac{\gamma}{\beta(\gamma-1)}}\vee(16K_T\gamma)\vee\left(8C\gamma\|\varphi\|_{0,T,\beta}\right)^{\frac{1}{\beta}}+1}\left(|y_0|+C_{\gamma,\beta,T}\right).\end{aligned}$$

Therefore, we obtain

$$\|x\|_{0,T,\infty}\le C_{1,\gamma,\beta,T}\left(|x_0|+1\right)\exp\left\{C_{2,\gamma,\beta,T}\left(1+\|\varphi\|_{0,T,\beta}^{\frac{\gamma}{\beta(\gamma-1)}}\right)\right\},$$

which concludes the proof of the theorem. □

3.3 Singular equations driven by fBm

Let $B^H=\{B^H_t\}_{t\ge0}$ be a fractional Brownian motion with Hurst parameter $H\in(1/2,1)$.We are interested in the following singular stochastic differential equation

$$X_t=x_0+\int_0^t f(s,X_s)ds+B^H_t\,, \tag{3.3.1}$$

where $x_0\ge0$, and the function $f(s,x)$ has a singularity at $x=0$ and satisfies the assumptions (i) to (iii). As an immediate consequence of Theorem 3.2.3 we have the following result.

Theorem 3.3.1. *Under the assumptions (i)-(iii), there is a unique pathwise solution $X = (X_t, t \geq 0)$ to Equation (3.3.1), such that $X_t > 0$ on $(0, \infty)$ and for any $T > 0$, $\|X\|_{0,T,\infty} \in L^p(\Omega)$, for all $p > 0$.*

Proof. Fix $\beta \in (\frac{1}{2}, H)$ and $T > 0$. Applying Theorem 3.2.3, we obtain that there is a unique pathwise solution $X = (X_t, t \geq 0)$ to Equation (3.3.1), such that $X_t > 0$ on $(0, \infty)$ and

$$\|X\|_{0,T,\infty} \leq C_{1,\gamma,\beta,T}(|x_0|+1)\exp\left\{C_{2,\gamma,\beta,T}\left(1+\|B^H\|_{0,T,\beta}^{\frac{\gamma}{\beta(\gamma-1)}}\right)\right\}. \tag{3.3.2}$$

If we choose γ such that $\gamma > \frac{2\beta}{2\beta-1}$, then $\frac{\gamma}{\beta(\gamma-1)} < 2$, and by Fernique's theorem (see [14], Theorem 1.3.2, p. 11), we obtain

$$\mathbb{E}(e^{C\|B^H\|_{0,T,\beta}^{\frac{\gamma}{\beta(\gamma-1)}}}) < \infty, \tag{3.3.3}$$

for all $C > 0$, which implies that $\mathbb{E}(\|X\|_{0,T,\infty}^p) < \infty$ for all $p \geq 1$. $\square$

Note that in the case $f(s,x) = \frac{1}{x}$ and $x_0 = 0$, the solution to Equation (3.3.1) is positive for any $t > 0$, as in the case of the standard Brownian motion.

Theorem 3.3.1 implies the existence of a unique solution to the following stochastic differential equation with non Lipschitz diffusion coefficient:

$$Y_t = y_0 + \int_0^t f(s, Y_s)ds + \int_0^t \sqrt{Y_s}dB_s^H, \tag{3.3.4}$$

where y_0 is a nonnegative constant and f is a nonnegative continuous function satisfying the following conditions:

(i') There exists $x_1 > 0$ such that $f(t,x) \geq g(t)$ for all $t > 0$ and $x \in (0, x_1)$, where g is a continuous function such that $g(t) > 0$ if $t > 0$.

(ii') $f(t,x) \geq x\partial_x f(t,x)$ for all $t > 0$ and $x > 0$.

(iii') $f(t,x) \leq h(t)(x+1)$ for all $t \geq 0$ and $x > 0$, where h is a nonnegative locally bounded function.

The stochastic integral in Equation (3.3.4) is a path-wise Riemann-Stieltjes integral, which exists by the results of Young [38]. The term $\sqrt{Y_s}$ appears in a fractional version of the CIR process in financial mathematics (see [9]) and cannot be treated directly by the approaches in Lyons [22], Nualart and Răşcanu [33], since function $g(x) = \sqrt{x}$ does not satisfy the usual Lipschitz conditions commonly imposed. We make the change of variables $X_t = 2\sqrt{Y_t}$. Then, from the chain rule for the Young integral, it follows that a positive stochastic process $Y = (Y_t, t \geq 0)$ satisfies (3.3.4) if and only if X_t satisfies the following equation:

$$X_t = 2\sqrt{y_0} + \int_0^t \frac{2f(s,X_s)}{X_s} ds + B_t^H. \tag{3.3.5}$$

Let $f_1(t,x) = 2f(t,x)x^{-1}$. Then $f_1(t,x)$ satisfies all assumptions (i)-(iii), and hence from Theorem 3.3.1, we know that there exists a unique positive solution X_t to Equation (3.3.5) with all positive moments. So $Y_t = X_t^2/4$ is the unique positive solution to Equation (3.3.4), and it has finite moments of all orders.

The next result states the scaling property of the solution to Equation (3.3.1), when the coefficient $f(s,x)$ satisfies some homogeneity condition on the variable x.

Proposition 3.3.2. *(Scaling Property) We denote by $Eq(x_0, f)$ Equation (3.3.1). Suppose that $x_0 \geq 0$, and $f(t,x)$ satisfies assumptions (i)-(iii), and $f(t,x)$ is homogeneous, that is, $f(st,yx) = s^m y^n f(t,x)$ for some constants m,n. Then, the process $\left(a^H X_{\frac{t}{a}}, t \geq 0\right)$ has the same law as the solution to the Equation $Eq(a^H x_0, a^{H-nH-m-1} f)$.*

Proof. For each $a > 0$, we know that $\{a^{-H}B^H_{at}, t \geq 0\}$ is a fractional Brownian motion. We denote $X_{a,t}$ the solution to the following equation:

$$X_{a,t} = x_0 + \int_0^t f(s, X_{a,s})ds + a^{-H}B^H_{at}.$$

So $(X_t, t \geq 0)$ (the solution to $Eq(x_0, f)$) has the same distribution as $(X_{a,t}, t \geq 0)$. Then

$$\begin{aligned} a^H X_{a,\frac{t}{a}} &= a^H x_0 + \int_0^{\frac{t}{a}} a^H f(s, X_{a,s})ds + B^H_t \\ &= a^H x_0 + \int_0^t a^{H-1-m-nH} f(r, a^H X_{a,\frac{r}{a}})dr + B^H_t, \end{aligned}$$

which implies the result. $\square$

As an example, we can consider the function $f(t,x) = s^\gamma x^{-\alpha}$, where $\alpha > \frac{1}{H} - 1$, and $\gamma > 0$. Then, if $(X_t, t \geq 0)$ is the solution to Equation

$$X_t = x_0 + \int_0^t s^\gamma X_s^{-\alpha} ds + B^H_t$$

(3.3.1), then $\left(a^H X_{\frac{t}{a}}, t \geq 0\right)$ has the same law as the solution to the Equation

$$X_t = a^H x_0 + a^{H-\alpha H-\gamma-1} \int_0^t s^\gamma X_s^{-\alpha} ds + B^H_t.$$

3.3.1 Absolute continuity of the law of the solution

In this subsection we will apply the Malliavin calculus to the solution to Equation (3.3.1) in order to study the absolute continuity of the law of the solution at a fixed time $t > 0$. The basic criterion for the existence of densities (see Bouleau and Hirsch [6]), says that if $F \in \mathbb{D}^{1,2}$, and $\|DF\|_{\mathscr{H}} > 0$ almost surely, then the law of F has a density with respect to the Lebesgue measure on the real line. Using this criterion we can show the following result.

Theorem 3.3.3. *Suppose that f satisfies the assumptions (i)-(iii). Let X_t be the solution to Equation (3.3.1). Then for any $t \geq 0$, $X_t \in \mathbb{D}^{1,2}$. Furthermore, for any $t > 0$ the law of X_t is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.*

Proof. Fix a time interval $[0,T]$, and let $\beta \in (\frac{1}{2}, H)$. We want to compute the directional derivative $\langle DX_t, \varphi \rangle_{\mathscr{H}}$, for some $\varphi \in \mathscr{H}$. The function $h = R_H \varphi$ belongs to $C^{\beta}([0,T])$ and $h_0 = 0$. Taking into account the embedding given by $R_H : \mathscr{H} \to \Omega$ mentioned before, we will have

$$\langle DX_t, \varphi \rangle_{\mathscr{H}} = \frac{dX_t^{\varepsilon}}{d\varepsilon}|_{\varepsilon=0}, \tag{3.3.6}$$

where X_t^{ε} is the solution to the following equation

$$X_t^{\varepsilon} = x_0 + \int_0^t f(s, X_s^{\varepsilon}) ds + B_t^H + \varepsilon h_t, \tag{3.3.7}$$

$t \in [0,T]$, where $\varepsilon \in [0,1]$.

From the estimate (3.3.2) replacing B^H by $B^H + \varepsilon h$ it follows that there is a constant C independent of ε such that

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} |X^{\varepsilon}|_t^p \right) \leq C < \infty,$$

for all $p \geq 1$. From Equations (3.3.1) and (3.3.7), we deduce

$$X_t^{\varepsilon} - X_t = \int_0^t (f(s, X_s^{\varepsilon}) - f(s, X_s)) ds + \varepsilon h_t. \tag{3.3.8}$$

By using Taylor expansion, Equation (3.3.8) becomes:

$$X_t^\varepsilon - X_t = \int_0^t \Theta_s (X_s^\varepsilon - X_s) ds + \varepsilon h_t \,, \tag{3.3.9}$$

where $\Theta_s = \partial_x f(s, X_s + \theta_s (X_s^\varepsilon - X_s))$ for some θ_s^ε between 0 and 1. By using (1.1.3) the solution to Equation (3.3.9) is given by

$$\begin{aligned}
X_t^\varepsilon - X_t &= \varepsilon \int_0^t \exp\left(\int_s^t \Theta_r dr\right) d(R_H \varphi)(s) \\
&= \varepsilon \int_0^t \exp\left(\int_s^t \Theta_r dr\right) \left(\int_0^s \frac{\partial K_H(s,u)}{\partial s} (K_H^* \varphi)(u) du\right) ds \\
&= \varepsilon \int_0^t \left(\int_u^t \exp\left(\int_s^t \Theta_r dr\right) \frac{\partial K_H(s,u)}{\partial s} ds\right) (K_H^* \varphi)(u) du.
\end{aligned}$$

Using (1.1.3) and (1.1.4) we can write

$$\begin{aligned}
X_t^\varepsilon - X_t &= \varepsilon \int_0^t (K_H^* \varphi)(u) \left(K_H^* \left(\exp\left(\int_\cdot^t \Theta_r dr\right)\right)\right)(u) du \\
&= \varepsilon \left\langle \varphi, \exp\left(\int_\cdot^t \Theta_r dr\right)\right\rangle_{\mathscr{H}} \\
&= \varepsilon \alpha_H \int_0^t \int_0^t \varphi(s) \exp\left(\int_u^t \Theta_r dr\right) |s-u|^{2H-2} du ds
\end{aligned}$$

Since $\partial_x f(t,x)$ is continuous and $\partial_x f(t,x) \leq 0$ for all $t > 0$ and $x > 0$, we have $\exp\left(\int_u^t \Theta_r dr\right) \leq 1$. As a consequence,

$$\begin{aligned}
\lim_{\varepsilon \to 0} \frac{X_t^\varepsilon - X_t}{\varepsilon} &= \alpha_H \int_0^t \int_0^t \varphi(s) \exp\left(\int_u^t \partial_x f(r, X_r) dr\right) |s-u|^{2H-2} du ds \\
&= \left\langle \varphi, \exp\left(\int_\cdot^t \partial_x f(r, X_r) dr\right) \mathbf{1}_{[0,t]} \right\rangle_{\mathscr{H}},
\end{aligned}$$

where the limit holds almost surely and in $L^2(\Omega)$. Then, taking into account (3.3.6), by the results of Sugita [37], we have $X_t \in \mathbb{D}^{1,2}$, and

$$DX_t = \exp\left(\int_{\cdot}^{t} \partial_x f(r, X_r) dr\right) \mathbf{1}_{[0,t]}. \tag{3.3.10}$$

Finally,

$$\begin{aligned} \|DF\|_{\mathscr{H}}^2 &= \alpha_H \int_0^t \int_0^t \exp\left(\int_s^t \partial_x f(r, X_r) dr\right) \\ &\quad \times \exp\left(\int_u^t \partial_x f(r, X_r) dr\right) |s-u|^{2H-2} du ds > 0. \end{aligned}$$

□

In the next proposition we will show the existence of negative moments for the solution to Equation (3.3.1). The proof is based again on the techniques of Malliavin calculus.

Proposition 3.3.4. *Let $(X_t, t \geq 0)$ be the solution to Equation (3.3.1), where f satisfies conditions (i)-(iii), and $x_0 > 0$. Suppose that $f(s,x)x \geq (p+1)Hs^{2H-1}$for some $p \geq 1$ and any $s \in [0,t]$ and $x > 0$. Then*

$$\mathbb{E}(X_t^{-p}) \leq x_0^{-p}. \tag{3.3.11}$$

In particular, for the function $f(t,x) = \frac{K}{x}$, we obtain that (3.3.11) holds if $t \leq \left(\frac{K}{(p+1)H}\right)^{\frac{1}{2H-1}}$

.

Proof. For any fixed $p \geq 1$, we construct the family of functions $\varphi_\varepsilon(x) = \frac{1}{(\varepsilon+x)^p}, x > 0$. Then $\varphi_\varepsilon \uparrow x^{-p}$, as $\varepsilon \downarrow 0$. For each $\varepsilon > 0$, φ_ε is a bounded continuously differentiable function and all its derivatives are bounded.

By the chain rule we obtain,

$$
\begin{aligned}
\varphi_\varepsilon(X_t) &= \varphi_\varepsilon(x_0) + \int_0^t \varphi_\varepsilon'(X_s) f(s, X_s) ds + \int_0^t \varphi_\varepsilon'(X_s) dB_s^H \\
&= \varphi_\varepsilon(x_0) - p \int_0^t \frac{f(s, X_s)}{(\varepsilon + X_s)^{p+1}} - p \int_0^t \frac{1}{(\varepsilon + X_s)^{p+1}} dB_s^H \qquad (3.3.12)
\end{aligned}
$$

Then, Proposition 5.3.2 in [31] implies that

$$
\begin{aligned}
\int_0^t \frac{1}{(\varepsilon + X_s)^{p+1}} dB_s^H &= \delta \left(\frac{1}{(\varepsilon + X_s)^{p+1}} \mathbf{1}_{[0,t]}(s) \right) - (p+1)\alpha_H \\
&\quad \times \int_0^t \int_0^t \frac{D_r X_s}{(\varepsilon + X_s)^{p+2}} \, |s - r|^{2H-2} dr ds, \qquad (3.3.13)
\end{aligned}
$$

where δ is the divergence operator with respect to fractional Brownian motion. Using Equation (3.3.10) we obtain

$$
\alpha_H \int_0^t \int_0^t \frac{D_r X_s}{(\varepsilon + X_s)^{p+2}} \, |s - r|^{2H-2} dr ds \leq H \int_0^t \frac{s^{2H-1}}{(\varepsilon + X_s)^{p+2}} ds. \qquad (3.3.14)
$$

From (3.3.14), (3.3.13), and (3.3.12) we get

$$
\begin{aligned}
\varphi_\varepsilon(X_t) &\leq \varphi_\varepsilon(x_0) - p \int_0^t \frac{f(s, X_s)}{(\varepsilon + X_s)^{p+1}} ds - p\delta \left(\frac{1}{(\varepsilon + X_s)^{p+1}} \mathbf{1}_{[0,t]}(s) \right) \\
&\quad + p(p+1)H \int_0^t \frac{s^{2H-1}}{(\varepsilon + X_s)^{p+2}} ds \\
&\leq \varphi_\varepsilon(x_0) - p \int_0^t \frac{f(s, X_s) X_s - (p+1)H s^{2H-1}}{(\varepsilon + X_s)^{p+2}} ds \\
&\quad - p\delta \left(\frac{1}{(\varepsilon + X_s)^{p+1}} \mathbf{1}_{[0,t]}(s) \right).
\end{aligned}
$$

Fix some t, such that $f(s,x)x \geq (p+1)Hs^{2H-1}$ for all $s \in [0,t]$ and $x > 0$. Taking expectation on above inequality, we have

$$\mathbb{E}(\varphi_\varepsilon(X_t)) \leq \varphi_\varepsilon(x_0) \leq \; x_0^{-p}.$$

Let ε tends to 0. Applying monotone convergence theorem we obtain

$$\mathbb{E}(X_t^{-p}) \leq \; x_0^{-p},$$

which completes the proof of the proposition. $\square$

Chapter 4

Approximation schemes of the solution of a stochastic differential equation driven by fractional Brownian motion

4.1 Introduction

Let $B^H = \{(B_t^{H,1}, B_t^{H,2}, \cdots, B_t^{H,m})\}_{t\in[0,T]}$ be an m-dimensional fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$.

In this chapter we are interested in approximation solutions of multidimensional stochastic differential equations of the form

$$X_t = X_0 + \int_0^t \sigma(X_s) dB_s^H, \tag{4.1.1}$$

or

$$X_t^i = X_0^i + \sum_{j=1}^m \int_0^t \sigma^{i,j}(X_s) dB_s^{H,j}, \; i = 1, \dots, d, \tag{4.1.2}$$

where the integral is a pathwise Riemann-Stieltjes integral.

Fix n, and set $\tau_k = \frac{kT}{n}$ for $k = 0, \dots, n$. Set $\kappa_n(t) = \frac{kT}{n}$ if $\frac{kT}{n} \le t < \frac{(k+1)T}{n}$, $k = 0, \dots, n$. We will also set $\delta = \frac{T}{n}$. The aim of the this project is to establish an optimal rate of convergence of the Euler scheme of the form

$$X_t^{(n),i} = X_0^i + \sum_{j=1}^{m} \int_0^t \sigma^{i,j}(X_{\kappa_n(s)}^{(n)}) dB_s^{H,j}, \, i = 1, \dots, d.$$

The numerical solution of stochastic differential equations (SDEs, for short) driven by Brownian motion is essentially based on the method of time discretization and has a long history. Difficulties appear in constructing numerical solutions of SDEs driven by fractional Brownian motion, because the fraction Brownian motion B^H is not a semi-martingale. Numerical schemes for SDEs driven by fractional Brownian motion are studies only in few works, see [29] and the references therein. The authors in [30] gave an exact rate of convergence of the Euler scheme in one-dimensional case by using a specific representation for the solution. However, new techniques are required in multi-dimensional case. One result for the rate of convergence can be found in Mishura's book [27]. In our work, we are searching for optimal estimates of the errors of Euler Scheme and Milstein scheme by using some different techniques such as the variation property of the fractional Brownian motion.

Throughout this chapter for simplicity we consider one-dimensional fractional Brownian motion B^H. The results obtained in this chapter can be easily extended to multi-dimensional case.

4.2 Fractional integrals and derivatives

Let $a, b \in \mathbb{R}$ with $a < b$. Denote by $L^p(a,b)$, $p \geq 1$, the usual space of Lebesgue measurable functions $f : [a,b] \to \mathbb{R}$ for which $\|f\|_{L^p} < \infty$, where

$$\|f\|_{L^p} = \begin{cases} \left(\int_a^b |f(t)|^p dt\right)^{1/p}, \text{ if } 1 \leq p < \infty \\ \operatorname{ess\,sup}\{|f(t)| : t \in [a,b]\}, \text{ if } p = \infty. \end{cases}$$

Let $f \in L^1(a,b)$ and $\alpha > 0$. The left-sided and right-sided fractional Riemann-Liouville integrals of f of order α are defined for almost all $x \in (a,b)$ by

$$I_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, ds$$

and

$$I_{b-}^{\alpha} f(t) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s)\, ds,$$

respectively, where $(-1)^{-\alpha} = e^{-i\pi\alpha}$ and $\Gamma(\alpha) = \int_0^\infty r^{\alpha-1} e^{-r} dr$ is the Euler gamma function. Let $I_{a+}^{\alpha}(L^p)$ (resp. $I_{b-}^{\alpha}(L^p)$) be the image of $L^p(a,b)$ by the operator I_{a+}^{α} (resp. I_{b-}^{α}). If $f \in I_{a+}^{\alpha}(L^p)$ (resp. $f \in I_{b-}^{\alpha}(L^p)$) and $0 < \alpha < 1$ then the Weyl derivatives are defined as

$$D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(t-a)^{\alpha}} + \alpha \int_a^t \frac{f(t) - f(s)}{(t-s)^{\alpha+1}} ds \right) 1_{(a,b)}(t) \tag{4.2.1}$$

and

$$D_{b-}^{\alpha} f(t) = \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(b-t)^{\alpha}} + \alpha \int_t^b \frac{f(t) - f(s)}{(s-t)^{\alpha+1}} ds \right) 1_{(a,b)}(t) \tag{4.2.2}$$

for almost all $t \in (a,b)$ (the convergence of the integrals at the singularity $s = t$ holds point-wise for almost all $t \in (a,b)$ if $p = 1$ and moreover in L^p-sense if $1 < p < \infty$).

Recall from [35] that we have:

- If $\alpha < \frac{1}{p}$ and $q = \frac{p}{1-\alpha p}$ then

$$I_{a+}^{\alpha}(L^p) = I_{b-}^{\alpha}(L^p) \subset L^q(a,b).$$

- If $\alpha > \frac{1}{p}$ then

$$I_{a+}^{\alpha}(L^p) \cup I_{b-}^{\alpha}(L^p) \subset C^{\alpha-\frac{1}{p}}(a,b).$$

The following inversion formulas hold:

$$\begin{aligned} I_{a+}^{\alpha}\left(D_{a+}^{\alpha} f\right) &= f, \qquad \forall f \in I_{a+}^{\alpha}(L^p) \qquad (4.2.3) \\ I_{b-}^{\alpha}\left(D_{b-}^{\alpha} f\right) &= f, \qquad \forall f \in I_{b-}^{\alpha}(L^p) \qquad (4.2.4) \end{aligned}$$

and

$$D_{a+}^{\alpha}\left(I_{a+}^{\alpha} f\right) = f, \quad D_{b-}^{\alpha}\left(I_{b-}^{\alpha} f\right) = f, \quad \forall f \in L^1(a,b). \qquad (4.2.5)$$

On the other hand, for any $f, g \in L^1(a,b)$ we have

$$\int_a^b I_{a+}^{\alpha} f(t) g(t) dt = (-1)^{\alpha} \int_a^b f(t) I_{b-}^{\alpha} g(t) dt, \qquad (4.2.6)$$

and for $f \in I_{a+}^{\alpha}(L^p)$ and $g \in I_{b-}^{\alpha}(L^p)$ we have

$$\int_a^b D_{a+}^{\alpha} f(t) g(t) dt = (-1)^{-\alpha} \int_a^b f(t) D_{b-}^{\alpha} g(t) dt. \qquad (4.2.7)$$

4.3 Generalized Lebesgue-Stieltjes integration

Following [39] we can give the definition of the generalized (fractional) Lebesgue-Stieltjes integral of f with respect to g. Let $f(a+) = \lim_{\varepsilon \to 0} f(a+\varepsilon)$, $g(b-) = \lim_{\varepsilon \to 0} g(b-\varepsilon)$ (supposing that the limits exist and are finite) and define

$$\begin{aligned} f_{a+}(t) &= (f(t) - f(a+))1_{(a,b)}(t), \\ g_{b-}(t) &= (g(t) - g(b-))1_{(a,b)}(t). \end{aligned}$$

Definition 4.3.1. *(Generalized (fractional) Lebesgue-Stieltjes Integral). Suppose that f and g are functions such that $f(a+)$, $g(a+)$ and $g(b-)$exist, $f_{a+} \in I^{\alpha}_{a+}(L^p)$ and $g_{b-} \in I^{1-\alpha}_{b-}(L^q)$ for some $p, q \geq 1$, $1/p + 1/q \leq 1$, $0 < \alpha < 1$. Then the integral of f with respect to g is defined by*

$$\int_a^b f dg = (-1)^{\alpha} \int_a^b D^{\alpha}_{a+} f_{a+}(t) D^{1-\alpha}_{b-} g_{b-}(t) dt + f(a+)(g(b-) - g(a+)). \tag{4.3.1}$$

Remark 4.3.2. *If $\alpha p < 1$, then we have $f_{a+} \in I^{\alpha}_{a+}(L^p)$ if and only if $f \in I^{\alpha}_{a+}(L^p)$. In this case, under the assumptions of the preceding definition (4.3.1) can be rewritten as*

$$\int_a^b f dg = (-1)^{\alpha} \int_a^b D^{\alpha}_{a+} f(t) D^{1-\alpha}_{b-} g_{b-}(t) dt. \tag{4.3.2}$$

Remark 4.3.3. *Suppose that $f \in C^{\lambda}([a,b])$ and $g \in C^{\mu}([a,b])$ with $\lambda + \mu > 1$. Then, from the classical paper by Young [38], the Riemann-Stieltjes integral $\int_a^b f dg$ exists. It is also proved in [39] that the conditions of the above definition and remark are fulfilled and we may choose $p = q = \infty$ and $\alpha < \lambda, 1 - \alpha < \mu$. Moreover, the generalized Lebesgue-Stieltjes integral $\int_a^b f dg$ coincides with the Riemann-Stieltjes integral.*

The linear spaces $I^{\alpha}_{a+}(L^p)$ are Banach spaces with respect to the norms

$$\|f\|_{I^{\alpha}_{a+}(L^p)} = \|f\|_{L^p} + \|D^{\alpha}_{a+}f\|_{L^p},$$

and the same is true for $I^{\alpha}_{b-}(L^p)$. If $0 < \alpha < 1/p$, then the norms of the spaces $I^{\alpha}_{a+}(L^p)$ and $I^{\alpha}_{b-}(L^p)$ are equivalent, and for $a \le c < d \le b$ the restriction of $f \in I^{\alpha}_{a+}(L^p(a,b))$ to (c,d) belongs to $I^{\alpha}_{c+}(L^p(c,d))$ and the continuation of $f \in I^{\alpha}_{c+}(L^p(c,d))$ by zero beyond (c,d) belongs to $I^{\alpha}_{a+}(L^p(a,b))$. As a consequence, if $f \in I^{\alpha}_{a+}(L^p(a,b))$ and $g_{b-} \in I^{1-\alpha}_{b-}(L^q(a,b))$ then the integral $\int_a^b 1_{(c,d)} f dg$ exists in the sense of (4.3.2) for any $a \le c < d \le b$ and we have

$$\int_c^d f dg = \int_a^b 1_{(c,d)} f dg,$$

whenever the left-hand side is determined in the sense of (4.3.2).

For a matrix $A = (a^{i,j})_{d\times m}$ and a vector $y = (y^i)_{d\times a}$ denote $|A| = \sqrt{\sum_{i,j} |a^{i,j}|^2}$ and $y = \sqrt{\sum_i |y^i|^2}$. For fixed $0 < \alpha < 1$, $\psi^{\alpha}_f(t) = |f(t)| + \int_0^t |f(t) - f(s)|(t-s)^{-\alpha-1} ds$. Consider the following functional spaces. Let $W^{\alpha}_0(0,T;\mathbb{R}^d)$ be the space of $\mathbb{R}^d$-valued measurable functions $f : [0,T] \to \mathbb{R}^d$ such that

$$\|f\|_{0,\alpha} = \sup_{0\le t\le T} \psi^{\alpha}_f(t) < \infty.$$

Let $W^{\alpha}_1(0,T;\mathbb{R}^d)$ be the space of $\mathbb{R}^d$-valued measurable functions $f : [0,T] \to \mathbb{R}^d$ such that

$$\|f\|_{1,\alpha} = \sup_{0\le s<t\le T} \left(\frac{|f(t) - f(s)|}{(t-s)^{\alpha}} + \int_s^t \frac{|f(u) - f(s)|}{(u-s)^{\alpha+1}} du \right) < \infty,$$

and $W_2^{\alpha}(0,T;\mathbb{R}^d)$ be the space of $\mathbb{R}^d$-valued measurable functions $f:[0,T]\to\mathbb{R}^d$ such that

$$\|f\|_{2,\alpha}=\int_0^T\frac{|f(t)|}{t^{\alpha}}dt+\int_0^T\int_0^t\frac{|f(t)-f(s)|}{(t-s)^{\alpha+1}}ds<\infty.$$

Note that the spaces $W_i^{\alpha}(0,T;\mathbb{R}^d)$, $i=0,2$ are Banach spaces with respect to the corresponding norms, and $\|f\|_{1,\alpha}$ is not a norm in the usual sense. Moreover, for any $0<\varepsilon<\alpha$

$$C^{\alpha+\varepsilon}(0,T;\mathbb{R}^d)\subset W_i^{\alpha}(0,T;\mathbb{R}^d)\subset C^{\alpha-\varepsilon}(0,T;\mathbb{R}^d),\ i=0,\,1,$$

and

$$C^{\alpha+\varepsilon}(0,T;\mathbb{R}^d)\subset W_2^{\alpha}(0,T;\mathbb{R}^d).$$

Therefore, the trajectories of a d-dimensional fBm B^H for a.a. $\omega\in\Omega$, any $T>0$ and any $0<\beta<H$ belong to $W_1^{\beta}(0,T;\mathbb{R}^d)$.

If $d=1$, then denote $W_i^{\alpha}(0,T)=W_i^{\alpha}(0,T;\mathbb{R})$, $i=0,\,1,\,2$.

Let $f\in W_1^{\alpha}(0,T)$. Then its restriction to $[0,t]\subset[0,T]$ belongs to $I_{t-}^{\alpha}(L^{\infty}(0,t))$ for all t and define

$$\Lambda_{\alpha}(f)=\sup_{0\le s<t\le T}|D_{t-}^{\alpha}f_{t-}(s)|\le\frac{1}{\Gamma(1-\alpha)}\|f\|_{1,\alpha}<\infty.$$

If $f\in W_1^{\alpha}(0,T;\mathbb{R}^d)$, then define $\Lambda_{\alpha}(f)=\max_{i=1,\dots,d}\Lambda_{\alpha}(f^i)$, and $\Lambda_{\alpha}(f)\le\frac{1}{\Gamma(1-\alpha)}\|f\|_{1,\alpha}<\infty$.

The restriction of $f\in W_2^{\alpha}(0,T)$ to $[s,t]\subset[0,T]$ belongs to $I_{s+}^{\alpha}(L^1(s,t))$ for all s,t.

Notice that if f is a function in the space $W_2^{\alpha}(0,T)$, and g belongs to the space $W_1^{1-\alpha}(0,T)$, then by (4.3.2) the integral $\int_s^t fdg$ exists for all $0 \le s < t \le T$ and we have

$$\begin{aligned}\left|\int_s^t fdg\right| &= \left|(-1)^{\alpha}\int_0^t D_{s+}^{\alpha}f(u)D_{t-}^{1-\alpha}g_{t-}(u)du\right| \\ &\le \sup_{0\le u\le t}\left|D_{t-}^{1-\alpha}g_{t-}(u)\right|\int_s^t\left|D_{s+}^{\alpha}f(u)\right|du \\ &\le \frac{\Lambda_{1-\alpha}(g)}{\Gamma(1-\alpha)}\int_s^t\left(\frac{|f(u)|}{(u-s)^{\alpha}}+\alpha\int_s^u\frac{|f(u)-f(r)|}{(u-r)^{\alpha+1}}dr\right)du. \quad (4.3.3)\end{aligned}$$

If $f \in W_2^{\alpha}(0,T;\mathbb{R}^d)$ and $g \in W_1^{1-\alpha}(0,T;\mathbb{R}^d)$, we have the following generalization of the above inequality

$$\left|\int_s^t fdg\right| = \left|\sum_{i=1}^d\int_s^t f^i dg^i\right| \le \Lambda_{1-\alpha}(g)c_1\int_s^t\left(\frac{|f(u)|}{(u-s)^{\alpha}}+\int_s^u\frac{|f(u)-f(r)|}{(u-r)^{\alpha+1}}dr\right)du, \quad (4.3.4)$$

where $c_1 > 0$ is a constant depending only on α and d.

4.4 Deterministic differential equations

Consider the following differential equation driven by a Hölder continuous function $g : [0,T] \to \mathbb{R}^m$ of order $\beta > \frac{1}{2}$:

$$X_t = X_0 + \int_0^t \sigma(X_s)dg_s, \quad (4.4.1)$$

or

$$X_t^i = X_0^i + \sum_{j=1}^m \int_0^t \sigma^{i,j}(X_s)dg_s^j,\ i = 1,\dots,d, \quad (4.4.2)$$

where $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ is a continuously differentiable function whose partial derivatives are bounded and Hölder continuous of order $\gamma > \frac{1}{\beta} - 1$.

We summarize the conditions on σ as follows:

(H1) $|\sigma(x)| \leq L_1(1+|x|)$, for some positive constant L_1.

(H2) $|\sigma(x)-\sigma(y)| \leq L_2|x-y|$, $\forall x, y \in \mathbb{R}^d$, for some positive constant L_2.

(H3) $|\sigma_{x_i}(x)-\sigma_{x_i}(y)| \leq M|x-y|^{\gamma}$, $\forall x, y \in \mathbb{R}^d$, $i=1,\dots,d$, for some positive constant M.

Notice that condition **(H2)** implies condition **(H1)** since σ is a deterministic function. However, we still list them for distinguishing the two constants L_1 and L_2.

Fix $\alpha < \frac{1}{2}$, such that $\alpha > 1-\beta$ and $\gamma > \frac{\alpha}{1-\alpha}$. By Theorem 5.1 in [33], there exists a unique solution $X \in W_0^{\alpha}([0,T];\mathbb{R}^d)$ to equation (4.4.1), and moreover, the solution is $(1-\alpha)$-Hölder continuous..

Fix n, and set $\tau_k = \frac{kT}{n}$ for $k=0,\dots,n$. Set $\kappa_n(t) = \frac{kT}{n}$ if $\frac{kT}{n} \leq t < \frac{(k+1)T}{n}$, $k = 0,\dots,n$. We will also set $\delta = \frac{T}{n}$. Consider the Euler approximation scheme defined by

$$X_t^{(n)} = X_0 + \int_0^t \sigma(X_{\kappa_n(s)}^{(n)})dg_s, \tag{4.4.3}$$

or equivalently,

$$X_t^{(n)} = X_{\kappa_n(t)}^{(n)} + \sigma(X_{\kappa_n(t)}^{(n)})(g_t - g_{\kappa_n(t)}), \tag{4.4.4}$$

for any $\frac{kT}{n} < t \leq \frac{(k+1)T}{n}$, $k=0,\dots,n$.

Given a multidmensional stochastic process $\{Y_t, t\in[0,T]\}$, we will make use of the following notation

$$\begin{aligned}
Y_t^* &= \sup_{0\leq s\leq t} |Y_s|, \\
\Delta_t(Y) &= \int_0^t \frac{|Y_t-Y_s|}{(t-s)^{\alpha+1}}ds, \\
\Psi_t(Y) &= Y_t^* + \Delta_t(Y).
\end{aligned}$$

Lemma 7.6 in [33] gives the following version of the Gronwall lemma.

Lemma 4.4.1. *Fix $\nu \in [0,1), A, B \geq 0$. Let $x : [0,\infty) \to [0,\infty)$ be a continuous function such that for each $t \in [0,\infty)$*

$$x_t \leq A + Bt^{\nu} \int_0^t x_s (t-s)^{-\nu} s^{-\nu} ds.$$

Then

$$x_t \leq A\Gamma(1-\nu) \sum_{n=0}^{\infty} \frac{(B\Gamma(1-\nu)t^{1-\nu})^n}{\Gamma((n+1)(1-\nu))} \leq Ad_{\nu} \exp\{c_{\nu} t B^{\frac{1}{1-\nu}}\},$$

where c_{ν} and d_{ν} are positive constants depending only on ν.

Theorem 4.4.2. *Suppose σ satisfies the conditions (**H1**) - (**H3**). Let X and $X^{(n)}$ be the solutions to equations (4.4.1) and (4.4.3) respectively. Then there exist two positive constants δ_0 and K such that*

$$\sup_{0\leq t\leq T} \left| X_t - X_t^{(n)} \right| \leq K\delta^{1-2\alpha},$$

for all $\delta \leq \delta_0$.

Proof. We will prove the theorem in three steps.

Step 1 Define the following modification of the above seminorm:

$$\Psi_t^{(n)} = X_t^{(n),*} + \int_0^{\kappa_n(t)} \frac{|X_{\kappa_n(t)}^{(n)} - X_s^{(n)}|}{(t-s)^{\alpha+1}} ds,$$

where $X_t^{(n),*} = \sup_{0\leq s\leq t} |X_s^{(n)}|$. By means of a suitable generalization of Gronwall's lemma we will show that $\Psi_t^{(n)}$ is uniformly bounded by a constant. To do this we need some estimates on $\left|X_t^{(n)}\right|$ and on the increments $\left|X_{\kappa_n(t)}^{(n)} - X_s^{(n)}\right|$. First we have, using the estimate (4.3.4) for Riemann-Stieltjes integrals,

$$
\begin{aligned}
\left|X_t^{(n)}\right| &\leq |X_0| + \left|\int_0^t \sigma(X_{\kappa_n(s)}^{(n)}) dg_s\right| \\
&\leq |X_0| + \Lambda_{1-\alpha}(g) c_1 \\
&\quad \times \left(\int_0^t \left|\sigma(X_{\kappa_n(s)}^{(n)})\right| s^{-\alpha} ds + \int_0^t \int_0^{\kappa_n(s)} \left|\sigma(X_{\kappa_n(s)}^{(n)}) - \sigma(X_{\kappa_n(u)}^{(n)})\right| (s-u)^{-\alpha-1} du\, ds \right).
\end{aligned}
$$

The linear growth and Lipschitz properties of σ imply that

$$
\begin{aligned}
\left|X_t^{(n)}\right| &\leq |X_0| + \Lambda_{1-\alpha}(g) c_1 \\
&\quad \times \left(\frac{L_1 T^{1-\alpha}}{1-\alpha} + L_1 \int_0^t \left|X_s^{(n),*}\right| s^{-\alpha} ds \right. \\
&\quad \left. + L_2 \int_0^t \int_0^{\kappa_n(s)} \left(\left|X_{\kappa_n(s)}^{(n)} - X_u^{(n)}\right| + \left|X_u^{(n)} - X_{\kappa_n(u)}^{(n)}\right| \right) (s-u)^{-\alpha-1} du\, ds \right)
\end{aligned}
\tag{4.4.5}
$$

By the definition of the Euler scheme (4.4.3)

$$
\left|X_u^{(n)} - X_{\kappa_n(u)}^{(n)}\right| = \left|\sigma(X_{\kappa_n(u)}^{(n)}) \left(g_u - g_{\kappa_n(u)}\right)\right| \leq L_1 \|g\|_\beta \left(1 + \left|X_{\kappa_n(u)}^{(n)}\right|\right) (u - \kappa_n(u))^\beta .
\tag{4.4.6}
$$

As a consequence,

$$
\begin{aligned}
& \int_0^t \int_0^{\kappa_n(s)} \left(1 + \left|X_{\kappa_n(u)}^{(n)}\right|\right) (u-\kappa_n(u))^\beta (s-u)^{-\alpha-1} du\, ds \\
\leq\ & \left(1 + X_t^{(n),*}\right) \int_0^t \int_0^{\kappa_n(s)} (u-\kappa_n(u))^\beta (s-u)^{-\alpha-1} du\, ds \\
\leq\ & \frac{\delta^\beta}{\alpha} \left(1 + X_t^{(n),*}\right) \int_0^t (s - \kappa_n(s))^{-\alpha} ds \\
\leq\ & \frac{\delta^\beta}{\alpha} \left(1 + X_t^{(n),*}\right) \int_0^T (s - \kappa_n(s))^{-\alpha} ds \\
\leq\ & \frac{n\delta^\beta}{\alpha} \left(1 + X_t^{(n),*}\right) \int_0^{t_1} (s - \kappa_n(s))^{-\alpha} ds \\
=\ & \frac{T\delta^{\beta-\alpha}}{\alpha(1-\alpha)} \left(1 + X_t^{(n),*}\right).
\end{aligned}
\tag{4.4.7}
$$

Substituting (4.4.7) into (4.4.5) yields

$$X_t^{(n),*} \le C_1 + C_2 X_t^{(n),*} \delta^{\beta-\alpha} + C_3 \int_0^t X_s^{(n),*} s^{-\alpha} ds + C_4 \int_0^t \int_0^{\kappa_n(s)} \frac{\left| X_{\kappa_n(s)}^{(n)} - X_u^{(n)} \right|}{(s-u)^{\alpha+1}} du\, ds, \tag{4.4.8}$$

with

$$\begin{aligned}
C_1 &= |X_0| + \Lambda_{1-\alpha}(g) c_1 L_1 \left(\frac{T^{1-\alpha}}{1-\alpha} + L_2 \|g\|_\beta \frac{T^{1+\beta-\alpha}}{\alpha(1-\alpha)} \right), \\
C_2 &= \Lambda_{1-\alpha}(g) c_1 L_1 L_2 \|g\|_\beta \frac{T}{\alpha(1-\alpha)}, \\
C_3 &= \Lambda_{1-\alpha}(g) c_1 L_1, \\
C_4 &= \Lambda_{1-\alpha}(g) c_1 L_2.
\end{aligned}$$

On the other hand, for $s \le \kappa_n(t)$, using (4.3.4) we have

$$\begin{aligned}
& \left| X_{\kappa_n(t)}^{(n)} - X_s^{(n)} \right| \\
\le\ & \Lambda_{1-\alpha}(g) c_1 \left(\int_s^{\kappa_n(t)} \left| \sigma(X_{\kappa_n(v)}^{(n)}) \right| (v-s)^{-\alpha} dv \right. \\
& \left. + \int_s^{\kappa_n(t)} \int_s^{\kappa_n(v)} \left| \sigma(X_{\kappa_n(v)}^{(n)}) - \sigma(X_{\kappa_n(z)}^{(n)}) \right| (v-z)^{-\alpha-1} dz dv \right) \\
\le\ & \Lambda_{1-\alpha}(g) c_1 \left(L_1 \frac{(\kappa_n(t)-s)^{1-\alpha}}{1-\alpha} + L_1 \int_s^{\kappa_n(t)} X_v^{(n),*} (v-s)^{-\alpha} dv \right. \\
& \left. + L_2 \int_s^{\kappa_n(t)} \int_s^{\kappa_n(v)} \left(\left| X_{\kappa_n(v)}^{(n)} - X_z^{(n)} \right| + \left| X_z^{(n)} - X_{\kappa_n(z)}^{(n)} \right| \right) (v-z)^{-\alpha-1} dz dv \right).
\end{aligned} \tag{4.4.9}$$

Using again the estimate (4.4.6) we obtain

$$\begin{aligned}
& \int_s^{\kappa_n(t)} \int_s^{\kappa_n(v)} \left| X_z^{(n)} - X_{\kappa_n(z)}^{(n)} \right| (v-z)^{-\alpha-1} dz dv \\
\leq\ & L_1 \|g\|_\beta \int_s^{\kappa_n(t)} \int_s^{\kappa_n(v)} \left(1 + \left| X_{\kappa_n(z)}^{(n)} \right| \right) (z - \kappa_n(z))^\beta (v-z)^{-\alpha-1} dz dv \\
\leq\ & L_1 \|g\|_\beta \delta^\beta \left(1 + X_t^{(n),*}\right) \int_s^{\kappa_n(t)} \int_s^{\kappa_n(v)} (v-z)^{-\alpha-1} dz dv \\
\leq\ & L_1 \|g\|_\beta \delta^\beta \left(1 + X_t^{(n),*}\right) \frac{1}{\alpha} \int_s^{\kappa_n(t)} (v - \kappa_n(v))^{-\alpha} dv. \qquad (4.4.10)
\end{aligned}$$

Substituting (4.4.10) into (4.4.9) yields

$$\begin{aligned}
& \left| X_{\kappa_n(t)}^{(n)} - X_s^{(n)} \right| \\
\leq\ & C_5 (\kappa_n(t) - s)^{1-\alpha} + C_6 \int_s^{\kappa_n(t)} X_v^{(n),*} (v-s)^{-\alpha} dv \\
& + C_7 \int_s^{\kappa_n(t)} \int_s^{\kappa_n(v)} \left| X_{\kappa_n(v)}^{(n)} - X_z^{(n)} \right| (v-z)^{-\alpha-1} dz dv \\
& + C_8 \left(1 + X_t^{(n),*}\right) \delta^\beta \int_s^{\kappa_n(t)} (v - \kappa_n(v))^{-\alpha} dv, \qquad (4.4.11)
\end{aligned}$$

where

$$\begin{aligned}
C_5 &= \Lambda_{1-\alpha}(g) c_1 L_1 \frac{1}{1-\alpha}, \\
C_6 &= \Lambda_{1-\alpha}(g) c_1 L_1, \\
C_7 &= \Lambda_{1-\alpha}(g) c_1 L_2, \\
C_8 &= \Lambda_{1-\alpha}(g) c_1 L_1 L_2 \|g\|_\beta \frac{1}{\alpha}.
\end{aligned}$$

Now we multiply each of the terms on the right-hand side of (4.4.11) by $(t-s)^{-\alpha-1}$ and integrate in s over the interval $[0, \kappa_n(t)]$. In this way we can obtain the following estimates.

$$\int_0^{\kappa_n(t)} (t-s)^{-\alpha-1} (\kappa_n(t)-s)^{1-\alpha} ds \leq \int_0^{\kappa_n(t)} (t-s)^{-2\alpha} ds \leq \frac{T^{1-2\alpha}}{1-2\alpha}. \qquad (4.4.12)$$

The second estimate is as follows

$$\begin{aligned} & \int_0^{\kappa_n(t)} (t-s)^{-\alpha-1} \left(\int_s^{\kappa_n(t)} X_v^{(n),*} (v-s)^{-\alpha} dv \right) ds \\ = \ & \int_0^{\kappa_n(t)} X_v^{(n),*} \left(\int_0^v (t-s)^{-\alpha-1} (v-s)^{-\alpha} ds \right) dv \\ \leq \ & c \int_0^{\kappa_n(t)} X_v^{(n),*} (t-v)^{-2\alpha} dv, \qquad (4.4.13) \end{aligned}$$

where $c = \int_0^\infty x^{-\alpha}(1+x)^{-\alpha-1}dx$. Then the third estimate is

$$\begin{aligned} & \int_0^{\kappa_n(t)} (t-s)^{-\alpha-1} \int_s^{\kappa_n(t)} \int_s^{\kappa_n(v)} \left| X_{\kappa_n(v)}^{(n)} - X_z^{(n)} \right| (v-z)^{-\alpha-1} dz dv ds \\ = \ & \int_0^{\kappa_n(t)} \int_0^{\kappa_n(v)} \int_0^{z \wedge v} \left| X_{\kappa_n(v)}^{(n)} - X_z^{(n)} \right| (v-z)^{-\alpha-1} (t-s)^{-\alpha-1} ds dz dv \\ \leq \ & \frac{1}{\alpha} \int_0^{\kappa_n(t)} (t-v)^{-\alpha} \int_0^{\kappa_n(v)} \left| X_{\kappa_n(v)}^{(n)} - X_z^{(n)} \right| (v-z)^{-\alpha-1} dz dv. \qquad (4.4.14) \end{aligned}$$

For the fourth estimate, let $\kappa_n(t) = k\delta$ for some $0 < k \le n$. Then, the interval $[0, \kappa_n(t)]$ can be decomposed as $[0, (k-1)\,\delta) \cup \big[(k-1)\delta, (k-\frac{1}{2})\delta\big) \cup \big[(k-\frac{1}{2}, k\delta\big]$ and we have

$$\begin{aligned}
& \int_0^{\kappa_n(t)} (t-s)^{-\alpha-1} \int_s^{\kappa_n(t)} (v-\kappa_n(v))^{-\alpha} dv ds \\
= & \int_0^{\kappa_n(t)} (v-\kappa_n(v))^{-\alpha} \int_0^{v} (t-s)^{-\alpha-1} ds dv \\
\le & \frac{1}{\alpha} \int_0^{\kappa_n(t)} (t-v)^{-\alpha} (v-\kappa_n(v))^{-\alpha} dv \\
= & \frac{1}{\alpha} \sum_{i=0}^{k-2} \int_{i\delta}^{(i+1)\delta} (t-v)^{-\alpha} (v-\kappa_n(v))^{-\alpha} dv + \frac{1}{\alpha} \int_{(k-1)\delta}^{(k-\frac{1}{2})\delta} (t-v)^{-\alpha} (v-\kappa_n(v))^{-\alpha} dv \\
& + \frac{1}{\alpha} \int_{(k-\frac{1}{2})\delta}^{k\delta} (t-v)^{-\alpha} (v-\kappa_n(v))^{-\alpha} dv \\
= & \frac{1}{\alpha} (I_1^{(n)} + I_2^{(n)} + I_3^{(n)}). \qquad (4.4.15)
\end{aligned}$$

$$\begin{aligned}
I_1^{(n)} & \le \sum_{i=0}^{k-2} (t-(i+1)\delta)^{-\alpha} \int_{i\delta}^{(i+1)\delta} (v-\kappa_n(v))^{-\alpha} dv \\
& = \frac{1}{1-\alpha} \sum_{i=0}^{k-2} (t-(i+1)\delta)^{-\alpha} \delta^{1-\alpha} \\
& \le \frac{1}{1-\alpha} \delta^{1-\alpha} \int_0^{(k-1)\delta} (t-v)^{-\alpha} dv \\
& \le \frac{1}{(1-\alpha)^2} \delta^{1-\alpha} T^{1-\alpha}. \qquad (4.4.16)
\end{aligned}$$

$$I_2^{(n)} \le \left(\frac{\delta}{2}\right)^{-\alpha} \int_{(k-1)\delta}^{(k-\frac{1}{2})\delta} (v-\kappa_n(v))^{-\alpha} dv = \frac{1}{1-\alpha} \left(\frac{\delta}{2}\right)^{1-2\alpha}. \qquad (4.4.17)$$

$$I_3^{(n)} \le \left(\frac{\delta}{2}\right)^{-\alpha} \int_{(k-\frac{1}{2})\delta}^{k\delta} (t-v)^{-\alpha} dv \le \frac{1}{1-\alpha} \left(\frac{\delta}{2}\right)^{-\alpha} \left(\frac{3\delta}{2}\right)^{1-\alpha}. \qquad (4.4.18)$$

Therefore, taking $C_9 = \frac{1}{\alpha(1-\alpha)^2}T^{2-\alpha} + \frac{1}{\alpha(1-\alpha)}\left(\frac{1}{2}\right)^{1-2\alpha}T^{1-\alpha} + \frac{1}{\alpha(1-\alpha)}\left(\frac{1}{2}\right)^{1-2\alpha}3^{1-\alpha}T^{1-\alpha}$, we obtain the fourth estimate

$$\delta^{\beta}\int_0^{\kappa_n(t)}(t-s)^{-\alpha-1}\int_s^{\kappa_n(t)}(v-\kappa_n(v))^{-\alpha}dvds \le C_9\delta^{\beta-\alpha}. \tag{4.4.19}$$

From (4.4.11)-(4.4.19), we get

$$\begin{aligned}\int_0^{\kappa_n(t)}\frac{\left|X^{(n)}_{\kappa_n(t)}-X^{(n)}_s\right|}{(t-s)^{\alpha+1}}ds \quad\le\quad & C_{10}+C_{11}X_t^{(n),*}\delta^{\beta-\alpha}+C_{12}\int_0^{\kappa_n(t)}X_v^{(n),*}(t-v)^{-2\alpha}dv \\ & +C_{13}\int_0^{\kappa_n(t)}(t-v)^{-\alpha}\int_0^{\kappa_n(v)}\frac{\left|X^{(n)}_{\kappa_n(v)}-X^{(n)}_z\right|}{(v-z)^{\alpha+1}}dzdv,\end{aligned} \tag{4.4.20}$$

where

$$\begin{aligned}C_{10} &= C_5\frac{T^{1-2\alpha}}{1-2\alpha}+C_8C_9T^{\beta-\alpha}, \\ C_{11} &= C_8C_9, \\ C_{12} &= C_6c, \\ C_{13} &= \frac{1}{\alpha}C_7.\end{aligned}$$

Finally, adding (4.4.20) and (4.4.8) yields

$$\Psi_t^{(n)} \quad\le\quad C_{14}+C_{15}X_t^{(n),*}\delta^{\beta-\alpha}+C_{16}\int_0^t\Psi_s^{(n)}\left(s^{-\alpha}+(t-s)^{-2\alpha}\right)ds, \tag{4.4.21}$$

where

$$
\begin{aligned}
C_{14} &= C_1 + C_{10}, \\
C_{15} &= C_2 + C_{11}, \\
C_{16} &= C_3 + C_4 T^{\alpha} + C_{12} + C_{13} T^{\alpha}.
\end{aligned}
$$

Note that all constants C's are independent of δ. Thus, we can choose a δ_0 such that $C_{15}\delta_0^{\beta-\alpha} \leq \frac{1}{2}$, then for all $\delta \leq \delta_0$, we have

$$
\begin{aligned}
\Psi_t^{(n)} &\leq C\left(1 + \int_0^t \Psi_s^{(n)} \left(s^{-\alpha} + (t-s)^{-2\alpha}\right) ds\right) \\
&\leq C\left(1 + t^{2\alpha} \int_0^t \Psi_s^{(n)} s^{-2\alpha} (t-s)^{-2\alpha} ds\right),
\end{aligned}
$$

where C is a generic constant independent of δ. Therefore, by Lemma 4.4.1

$$
\sup_n \sup_{0 \leq t \leq T} \Psi_t^{(n)} < K_1. \tag{4.4.22}
$$

Step 2 We will obtain the Hölder continuity of $X^{(n)}$, that is, there exists a positive constant K_3 such that

$$
\left| X_t^{(n)} - X_s^{(n)} \right| \leq K_3 (t-s)^{1-\alpha}, \tag{4.4.23}
$$

for all $0 \leq s \leq t \leq T$.

First, we give the following inequalities

$$
(y-a)^{1-\alpha} - (x-a)^{1-\alpha} \leq (y-x)^{1-\alpha}, \forall y \geq x \geq a \geq 0, \tag{4.4.24}
$$

and for any integer $n \geq 1$

$$x_1^{1-\alpha} + x_2^{1-\alpha} + \cdots + x_n^{1-\alpha} \leq n^{\alpha}(x_1 + x_2 + \cdots + x_n)^{1-\alpha}, \forall x_1, \ldots x_n \in \mathbb{R}. \tag{4.4.25}$$

Using the above inequalities, we get for any $0 < s < \kappa_n(t) \leq t \leq T$

$$\begin{aligned}
& \delta^{\beta} \int_s^{\kappa_n(t)} (v - \kappa_n(v))^{-\alpha} dv \\
= \; & \delta^{\beta} \int_{\kappa_n(s)+\delta}^{\kappa_n(t)} (v - \kappa_n(v))^{-\alpha} dv + \delta^{\beta} \int_s^{\kappa_n(s)+\delta} (v - \kappa_n(v))^{-\alpha} dv \\
= \; & \delta^{\beta} \left(\frac{\kappa_n(t) - \kappa_n(s) - \delta}{\delta} \right) \frac{\delta^{1-\alpha}}{1-\alpha} + \frac{\delta^{\beta}}{1-\alpha} \left(\delta^{1-\alpha} - (s - \kappa_n(s))^{1-\alpha} \right) \\
\leq \; & \frac{\delta^{\beta-\alpha}}{1-\alpha} (\kappa_n(t) - \kappa_n(s) - \delta) + \frac{\delta^{\beta}}{1-\alpha} (\kappa_n(s) + \delta - s)^{1-\alpha} \\
\leq \; & \frac{\delta^{\beta-\alpha} T^{\alpha}}{1-\alpha} (\kappa_n(t) - \kappa_n(s) - \delta)^{1-\alpha} + \frac{\delta^{\beta}}{1-\alpha} (\kappa_n(s) + \delta - s)^{1-\alpha} \\
\leq \; & \frac{T^{\beta}}{1-\alpha} \left[(\kappa_n(t) - \kappa_n(s) - \delta)^{1-\alpha} + (\kappa_n(s) + \delta - s)^{1-\alpha} \right] \\
\leq \; & \frac{2^{\alpha} T^{\beta}}{1-\alpha} (\kappa_n(t) - s)^{1-\alpha}.
\end{aligned} \tag{4.4.26}$$

Therefore, (4.4.11), (4.4.22) and (4.4.26) imply that for any $0 < s < \kappa_n(t) \leq t \leq T$ we have

$$\left| X_{\kappa_n(t)}^{(n)} - X_s^{(n)} \right| \leq K_2 (\kappa_n(t) - s)^{1-\alpha}, \tag{4.4.27}$$

where $K_2 = C_5 + \frac{C_6 K_1}{1-\alpha} + C_7 K_1 T^{\alpha} + C_8 (1 + K_1) \frac{2^{\alpha} T^{\beta}}{1-\alpha}$.

For any $0 \leq s \leq t \leq T$, if $s \geq \kappa_n(t)$, then it is easy to obtain $\left| X_t^{(n)} - X_s^{(n)} \right| \leq K_3 (t - s)^{1-\alpha}$ for some constant $K_3 > 0$. If $0 \leq s < \kappa_n(t) \leq t \leq T$, then by (4.4.6), (4.4.22) and (4.4.27), we deduce that there exists a positive constant K_3 such that

$$\left| X_t^{(n)} - X_s^{(n)} \right| \leq |X_t^{(n)} - X_{\kappa_n(t)}^{(n)}| + \left| X_{\kappa_n(t)}^{(n)} - X_s^{(n)} \right| \leq K_3 (t - s)^{1-\alpha}. \tag{4.4.28}$$

Step 3 We will complete our proof in this step. We denote the error in the Euler approximation by $Z_t^{(n)} = X_t - X_t^{(n)}$ and define

$$Z_t^{(n),*} = \sup_{0\le s\le t} \left| Z_s^{(n)} \right|.$$

Then,

$$\begin{aligned} Z_t^{(n),*} &\le \sup_{0\le s\le t} \left| \int_0^s \left[\sigma(X_u) - \sigma(X_u^{(n)}) \right] dg_u \right| + \sup_{0\le s\le t} \left| \int_0^s \left[\sigma(X_u^{(n)}) - \sigma(X_{\kappa_n(u)}^{(n)}) \right] dg_u \right| \\ &= J_t^{(n)} + R_t^{(n)}. \end{aligned} \tag{4.4.29}$$

The term $R_t^{(n)}$ is a residual term and it will provide the order of the error. In fact, this term can be estimated as follows

$$\begin{aligned} R_t^{(n)} &\le \Lambda_{1-\alpha}(g) c_1 \left(\int_0^t \left| \sigma(X_s^{(n)}) - \sigma(X_{\kappa_n(s)}^{(n)}) \right| s^{-\alpha} ds \right. \\ &\quad \left. + \int_0^t \int_0^s \left| \sigma(X_s^{(n)}) - \sigma(X_u^{(n)}) - \sigma(X_{\kappa_n(s)}^{(n)}) + \sigma(X_{\kappa_n(u)}^{(n)}) \right| (s-u)^{-\alpha-1} du ds \right) \\ &= G_t^{(n),1} + G_t^{(n),2}. \end{aligned} \tag{4.4.30}$$

The estimate (4.4.6) implies

$$G_t^{(n),1} \le K_4 \delta^\beta, \tag{4.4.31}$$

with $K_4 = \Lambda_{1-\alpha}(g) c_1 L_2 L_1 \|g\|_\beta (1+K_1)\frac{T^{1-\alpha}}{1-\alpha}$. In order to estimate $G_t^{(n),2}$, by (4.4.6), (4.4.7) and (4.4.28) we write

$$\begin{aligned}
& \int_0^t \int_0^s \left|\sigma(X_s^{(n)}) - \sigma(X_u^{(n)}) - \sigma(X_{\kappa_n(s)}^{(n)}) + \sigma(X_{\kappa_n(u)}^{(n)})\right| (s-u)^{-\alpha-1}\, du\, ds \\
= & \int_0^t \int_0^{\kappa_n(s)} \left|\sigma(X_s^{(n)}) - \sigma(X_u^{(n)}) - \sigma(X_{\kappa_n(s)}^{(n)}) + \sigma(X_{\kappa_n(u)}^{(n)})\right| (s-u)^{-\alpha-1}\, du\, ds \\
& + \int_0^t \int_{\kappa_n(s)}^s \left|\sigma(X_s^{(n)}) - \sigma(X_u^{(n)}) - \sigma(X_{\kappa_n(s)}^{(n)}) + \sigma(X_{\kappa_n(u)}^{(n)})\right| (s-u)^{-\alpha-1}\, du\, ds \\
= & \int_0^t \int_0^{\kappa_n(s)} \left|\left(\sigma(X_s^{(n)}) - \sigma(X_{\kappa_n(s)}^{(n)})\right) - \left(\sigma(X_u^{(n)}) - \sigma(X_{\kappa_n(u)}^{(n)})\right)\right| (s-u)^{-\alpha-1}\, du\, ds \\
& + \int_0^t \int_{\kappa_n(s)}^s \left|\sigma(X_s^{(n)}) - \sigma(X_u^{(n)})\right| (s-u)^{-\alpha-1}\, du\, ds \\
\leq & K_5 \delta^\beta \int_0^t \int_0^{\kappa_n(s)} (s-u)^{-\alpha-1}\, du\, ds + L_2 K_3 \int_0^t \int_{\kappa_n(s)}^s (s-u)^{-2\alpha}\, du\, ds \\
\leq & \frac{K_5 T}{\alpha(1-\alpha)} \delta^{\beta-\alpha} + \frac{L_2 K_3 T}{(1-2\alpha)(2-2\alpha)} \delta^{1-2\alpha} \\
\leq & \frac{K_5 T^{\beta+\alpha}}{\alpha(1-\alpha)} \delta^{1-2\alpha} + \frac{L_2 K_3 T}{(1-2\alpha)(2-2\alpha)} \delta^{1-2\alpha} \\
\leq & K_6 \delta^{1-2\alpha},
\end{aligned} \tag{4.4.32}$$

because $\beta - \alpha > 1 - 2\alpha$, where $K_5 = 2L_1 L_2 \|g\|_\beta (1+K_1)$ and $K_6 = \frac{K_5 T^{\beta+\alpha}}{\alpha(1-\alpha)} + \frac{L_2 K_3 T}{(1-2\alpha)(2-2\alpha)}$.

The term $J_t^{(n)}$ can be estimated as follows

$$\begin{aligned}
J_t^{(n)} \leq & \Lambda_{1-\alpha}(g) c_1 \left(\int_0^t \left|\sigma(X_s) - \sigma(X_s^{(n)})\right| s^{-\alpha} ds \right. \\
& \left. + \int_0^t \int_0^s \left|\sigma(X_s) - \sigma(X_u) - \sigma(X_s^{(n)}) + \sigma(X_u^{(n)})\right| (s-u)^{-\alpha-1}\, du\, ds \right) \\
= & J_t^{(n),1} + J_t^{(n),2}.
\end{aligned} \tag{4.4.33}$$

Clearly,

$$J_t^{(n),1} \leq \Lambda_{1-\alpha}(g) c_1 L_2 \int_0^t Z_s^{(n),*} s^{-\alpha} ds. \tag{4.4.34}$$

It is easy to check via the Taylor formula that the function σ satisfying the conditions **(H2)** and **(H3)** admits the following bound: for all $x_1, x_2, x_3, x_4 \in \mathbb{R}^d$,

$$|\sigma(x_1) - \sigma(x_2) - \sigma(x_3) + \sigma(x_4)| \le L_2 |x_1 - x_2 - x_3 + x_4| + M |x_1 - x_3| \left(|x_1 - x_2|^\gamma + |x_3 - x_4|^\gamma\right). \tag{4.4.35}$$

The above inequality yields

$$\begin{aligned} & \left|\sigma(X_s) - \sigma(X_u) - \sigma(X_s^{(n)}) + \sigma(X_u^{(n)})\right| \\ \le \;& L_2 \left|X_s - X_u - X_s^{(n)} + X_u^{(n)}\right| + M \left|X_s - X_s^{(n)}\right| \left(|X_s - X_u|^\gamma + \left|X_s^{(n)} - X_u^{(n)}\right|^\gamma\right). \end{aligned}$$

Hence,

$$J_t^{(n),2} \le \sum_{i=3}^{5} J_t^{(n),i}, \tag{4.4.36}$$

where

$$\begin{aligned} J_t^{(n),3} &= \Lambda_{1-\alpha}(g) c_1 L_2 \int_0^t \int_0^s \left|X_s - X_u - X_s^{(n)} + X_u^{(n)}\right| (s-u)^{-\alpha-1} du ds, \\ J_t^{(n),4} &= \Lambda_{1-\alpha}(g) c_1 M \int_0^t \int_0^s \left|X_s - X_s^{(n)}\right| |X_s - X_u|^\gamma (s-u)^{-\alpha-1} du ds, \\ J_t^{(n),5} &= \Lambda_{1-\alpha}(g) c_1 M \int_0^t \int_0^s \left|X_s - X_s^{(n)}\right| \left|X_s^{(n)} - X_u^{(n)}\right|^\gamma (s-u)^{-\alpha-1} du ds. \end{aligned}$$

We know from (4.4.28) in Step 2 that $\left|X_s^{(n)} - X_u^{(n)}\right|^\gamma \le K_3^\gamma (s-u)^{\gamma(1-\alpha)}$, and also $|X_s - X_u|^\gamma \le K_7^\gamma (s-u)^{\gamma(1-\alpha)}$ for some constant $K_7 > 0$ by Theorem 5.1 in [33]. Therefore,

$$\begin{aligned} J_t^{(n),4} + J_t^{(n),5} &\le \Lambda_{1-\alpha}(g) c_1 M \left(K_3^\gamma + K_7^\gamma\right) \int_0^t Z_s^{(n),*} \int_0^s (s-u)^{-\alpha-1+\gamma(1-\alpha)} du ds \\ &\le \frac{\Lambda_{1-\alpha}(g) c_1 M \left(K_3^\gamma + K_7^\gamma\right) T^{\gamma(1-\alpha)-\alpha}}{\gamma(1-\alpha)-\alpha} \int_0^t Z_s^{(n),*} ds, \end{aligned} \tag{4.4.37}$$

because $\gamma > \frac{\alpha}{1-\alpha}$. The term $J_t^{(n),3}$ involves an increment of the error process $\{X_s - X_s^{(n)}\}_{0\le s\le T}$, and it requires a further analysis. Define

$$\Delta_t^{(n)}(Z) = \int_0^t \frac{\left|Z_t^{(n)} - Z_u^{(n)}\right|}{(t-u)^{\alpha+1}} du.$$

Then,

$$\begin{aligned}
\Delta_t^{(n)}(Z) &\le \int_0^t \left| \int_u^t \left(\sigma(X_r) - \sigma(X_r^{(n)}) \right) dg_r \right| (t-u)^{-\alpha-1} du \\
&\quad + \int_0^t \left| \int_u^t \left(\sigma(X_r^{(n)}) - \sigma(X_{\kappa_n(r)}^{(n)}) \right) dg_r \right| (t-u)^{-\alpha-1} du \\
&\le \sum_{i=1}^4 \theta_t^{(n),i}, \qquad (4.4.38)
\end{aligned}$$

where

$$\begin{aligned}
\theta_t^{(n),1} &= \Lambda_{1-\alpha}(g) c_1 \int_0^t \int_u^t \left| \sigma(X_r) - \sigma(X_r^{(n)}) \right| (r-u)^{-\alpha} (t-u)^{-\alpha-1} drdu, \\
\theta_t^{(n),2} &= \Lambda_{1-\alpha}(g) c_1 \int_0^t \int_u^t \left| \sigma(X_r^{(n)}) - \sigma(X_{\kappa_n(r)}^{(n)}) \right| (r-u)^{-\alpha} (t-u)^{-\alpha-1} drdu, \\
\theta_t^{(n),3} &= \Lambda_{1-\alpha}(g) c_1 \int_0^t \int_u^t \int_u^s \left| \sigma(X_s) - \sigma(X_s^{(n)}) - \sigma(X_r) + \sigma(X_r^{(n)}) \right| \\
&\quad \times (s-r)^{-\alpha-1} (t-u)^{-\alpha-1} drdsdu, \\
\theta_t^{(n),4} &= \Lambda_{1-\alpha}(g) c_1 \int_0^t \int_u^t \int_u^s \left| \sigma(X_s^{(n)}) - \sigma(X_{\kappa_n(s)}^{(n)}) - \sigma(X_r^{(n)}) + \sigma(X_{\kappa_n(r)}^{(n)}) \right| \\
&\quad \times (s-r)^{-\alpha-1} (t-u)^{-\alpha-1} drdsdu.
\end{aligned}$$

It is clear that

$$\begin{aligned}
\theta_t^{(n),1} &\le \Lambda_{1-\alpha}(g) c_1 L_2 \int_0^t |Z_r^{(n)}| \left(\int_0^r (r-u)^{-\alpha} (t-u)^{-\alpha-1} du \right) dr \\
&\le K_8 \int_0^t |Z_r^{(n)}| (t-r)^{-2\alpha} dr, \qquad (4.4.39)
\end{aligned}$$

where $K_8 = \Lambda_{1-\alpha}(g)c_1L_2\int_0^\infty x^{-\alpha}(1+x)^{-\alpha-1}dx$. On the other hand, by (4.4.6) we have

$$\theta_t^{(n),2} \leq K_9\delta^\beta, \tag{4.4.40}$$

with $K_9 = \Lambda_{1-\alpha}(g)c_1L_1L_2\left\|g\right\|_\beta(1+K_1)\sup_{0\leq t\leq T}\int_0^t\int_u^t(r-u)^{-\alpha}(t-u)^{-\alpha-1}drdu.$

For $0 \leq u < \kappa_n(t) \leq t < T$ we have the following estimate by (4.4.24) and (4.4.25)

$$\begin{aligned}
& \delta^\beta\int_u^t(s-\kappa_n(s))^{-\alpha}ds \\
= \;& \delta^\beta\int_u^{\kappa_n(u)+\delta}(s-\kappa_n(s))^{-\alpha}ds+\delta^\beta\int_{\kappa_n(u)+\delta}^{\kappa_n(t)}(s-\kappa_n(s))^{-\alpha}ds+\delta^\beta\int_{\kappa_n(t)}^t(s-\kappa_n(s))^{-\alpha}ds \\
= \;& \frac{\delta^\beta}{1-\alpha}\left[(\kappa_n(u)+\delta-\kappa_n(u))^{1-\alpha}-(u-\kappa_n(u))^{1-\alpha}\right]+\frac{\delta^\beta}{1-\alpha}\left(\frac{\kappa_n(t)-\kappa_n(u)-\delta}{\delta}\right)\delta^{1-\alpha} \\
& +\frac{\delta^\beta}{1-\alpha}(t-\kappa_n(t))^{1-\alpha} \\
\leq \;& \frac{\delta^\beta}{1-\alpha}(\kappa_n(u)+\delta-u)^{1-\alpha}+\frac{\delta^{\beta-\alpha}T^\alpha}{1-\alpha}(\kappa_n(t)-\kappa_n(u)-\delta)^{1-\alpha}+\frac{\delta^\beta}{1-\alpha}(t-\kappa_n(t))^{1-\alpha} \\
\leq \;& \frac{3^\alpha T^\alpha\delta^{\beta-\alpha}}{1-\alpha}(t-u)^{1-\alpha},
\end{aligned}$$

and, for $0 \leq \kappa_n(t) \leq u \leq t \leq T$ we get by (4.4.24)

$$\delta^\beta\int_u^t(s-\kappa_n(s))^{-\alpha}ds = \frac{\delta^\beta}{1-\alpha}\left[(t-\kappa_n(t))^{1-\alpha}-(u-\kappa_n(t))^{1-\alpha}\right] \leq \frac{T^\alpha\delta^{\beta-\alpha}}{1-\alpha}(t-u)^{1-\alpha}.$$

Therefore, for any $0 \leq u \leq t \leq T$, the following estimate holds

$$\delta^\beta\int_u^t(s-\kappa_n(s))^{-\alpha}ds \leq \frac{3^\alpha T^\alpha\delta^{\beta-\alpha}}{1-\alpha}(t-u)^{1-\alpha}.$$

Then, using the above inequality, the term $\theta_t^{(n),4}$ can be estimated as follows by using the same techniques in handling (4.4.32)

$$\begin{aligned}
\theta_t^{(n),4} &\leq \Lambda_{1-\alpha}(g)c_1K_5\delta^\beta \int_0^t\int_u^t\int_u^{\kappa_n(s)}(s-r)^{-\alpha-1}(t-u)^{-\alpha-1}\,drdsdu \\
&\quad +\Lambda_{1-\alpha}(g)c_1L_2K_3\int_0^t\int_u^t\int_{\kappa_n(s)}^s(s-r)^{-2\alpha}(t-u)^{-\alpha-1}\,drdsdu \\
&\leq \frac{\Lambda_{1-\alpha}(g)c_1K_5}{\alpha}\delta^\beta\int_0^t\int_u^t(s-\kappa_n(s))^{-\alpha}(t-u)^{-\alpha-1}dsdu \\
&\quad +\frac{\Lambda_{1-\alpha}(g)c_1L_2K_3}{1-2\alpha}\int_0^t\int_u^t(s-\kappa_n(s))^{1-2\alpha}(t-u)^{-\alpha-1}dsdu \\
&\leq \frac{\Lambda_{1-\alpha}(g)c_1K_5 3^\alpha T^\alpha}{\alpha(1-\alpha)}\delta^{\beta-\alpha}\int_0^t(t-u)^{-2\alpha}dsdu \\
&\quad +\frac{\Lambda_{1-\alpha}(g)c_1L_2K_3}{1-2\alpha}\delta^{1-2\alpha}\int_0^t(t-u)^{-\alpha}dsdu \\
&\leq \frac{\Lambda_{1-\alpha}(g)c_1K_5 3^\alpha T^{1-\alpha}}{\alpha(1-\alpha)(1-2\alpha)}\delta^{\beta-\alpha}+\frac{\Lambda_{1-\alpha}(g)c_1L_2K_3T^{1-\alpha}}{(1-\alpha)(1-2\alpha)}\delta^{1-2\alpha} \\
&\leq K_{10}\delta^{1-2\alpha}, \qquad (4.4.41)
\end{aligned}$$

where $K_{10}=\frac{\Lambda_{1-\alpha}(g)c_1K_5 3^\alpha T^\beta}{\alpha(1-\alpha)(1-2\alpha)}+\frac{\Lambda_{1-\alpha}(g)c_1L_2K_3T^{1-\alpha}}{(1-\alpha)(1-2\alpha)}$, because $\beta-\alpha>1-2\alpha$.

Next, let us estimate $\theta_t^{(n),3}$. By (4.4.35) we can obtain

$$\begin{aligned}
\theta_t^{(n),3} &\leq \Lambda_{1-\alpha}(g)c_1L_2\int_0^t\int_u^t\int_u^s\left|Z_s^{(n)}-Z_r^{(n)}\right|(s-r)^{-\alpha-1}(t-u)^{-\alpha-1}drdsdu \\
&\quad+\Lambda_{1-\alpha}(g)c_1M\int_0^t\int_u^t\int_u^s\left|X_s-X_s^{(n)}\right||X_s-X_r|^{\gamma}(s-r)^{-\alpha-1}(t-u)^{-\alpha-1}drdsdu \\
&\quad+\Lambda_{1-\alpha}(g)c_1M\int_0^t\int_u^t\int_u^s\left|X_s-X_s^{(n)}\right|\left|X_s^{(n)}-X_r^{(n)}\right|^{\gamma}(s-r)^{-\alpha-1}(t-u)^{-\alpha-1}drdsdu \\
&\leq \Lambda_{1-\alpha}(g)c_1L_2\int_0^t\int_0^s\int_0^r\left|Z_s^{(n)}-Z_r^{(n)}\right|(s-r)^{-\alpha-1}(t-u)^{-\alpha-1}dudrds \\
&\quad+\Lambda_{1-\alpha}(g)c_1MK_7^{\gamma}\int_0^t\int_u^t\int_u^s\left|X_s-X_s^{(n)}\right|(s-r)^{\gamma(1-\alpha)-\alpha-1}(t-u)^{-\alpha-1}drdsdu \\
&\quad+\Lambda_{1-\alpha}(g)c_1MK_3^{\gamma}\int_0^t\int_u^t\int_u^s\left|X_s-X_s^{(n)}\right|(s-r)^{\gamma(1-\alpha)-\alpha-1}(t-u)^{-\alpha-1}drdsdu \\
&\leq \frac{\Lambda_{1-\alpha}(g)c_1L_2}{\alpha}\int_0^t\int_0^s\left|Z_s^{(n)}-Z_r^{(n)}\right|(s-r)^{-\alpha-1}(t-r)^{-\alpha}drds \\
&\quad+\frac{\Lambda_{1-\alpha}(g)c_1M(K_3^{\gamma}+K_7^{\gamma})}{\gamma(1-\alpha)-\alpha}\int_0^t\int_u^t\left|Z_s^{(n)}\right|(s-u)^{\gamma(1-\alpha)-\alpha}(t-u)^{-\alpha-1}dsdu
\end{aligned}$$

As a consequence,

$$\begin{aligned}
\theta_t^{(n),3} &\leq \frac{\Lambda_{1-\alpha}(g)c_1L_2}{\alpha}\int_0^t(t-s)^{-\alpha}\Delta_s^{(n)}(Z)ds \\
&\quad+\frac{\Lambda_{1-\alpha}(g)c_1M(K_3^{\gamma}+K_7^{\gamma})T^{\gamma(1-\alpha)-\alpha}}{\gamma(1-\alpha)-\alpha}\int_0^t\int_u^t\left|Z_s^{(n)}\right|(t-u)^{-\alpha-1}dsdu \\
&= \frac{\Lambda_{1-\alpha}(g)c_1L_2}{\alpha}\int_0^t(t-s)^{-\alpha}\Delta_s^{(n)}(Z)ds \\
&\quad+\frac{\Lambda_{1-\alpha}(g)c_1M(K_3^{\gamma}+K_7^{\gamma})T^{\gamma(1-\alpha)-\alpha}}{\gamma(1-\alpha)-\alpha}\int_0^t\int_0^s\left|Z_s^{(n)}\right|(t-u)^{-\alpha-1}duds \\
&\leq \frac{\Lambda_{1-\alpha}(g)c_1L_2}{\alpha}\int_0^t(t-s)^{-\alpha}\Delta_s^{(n)}(Z)ds \\
&\quad+\frac{\Lambda_{1-\alpha}(g)c_1M(K_3^{\gamma}+K_7^{\gamma})T^{\gamma(1-\alpha)-\alpha}}{\alpha(\gamma(1-\alpha)-\alpha)}\int_0^t\left|Z_s^{(n)}\right|(t-s)^{-\alpha}ds. \qquad (4.4.42)
\end{aligned}$$

Define $\Theta_t(Z) = Z_t^{(n),*} + \Delta_t^{(n)}(Z)$. Then from (4.4.29)-(4.4.42) we obtain

$$\begin{aligned}\Theta_t(Z) &\leq C\left(\delta^{1-2\alpha} + \int_0^t \Theta_s(Z)\left[s^{-\alpha} + (t-s)^{-2\alpha}\right] ds\right) \\ &\leq C\left(\delta^{1-2\alpha} + t^{2\alpha}\int_0^t \Theta_s(Z)\left[s^{-2\alpha}(t-s)^{-2\alpha}\right] ds\right),\end{aligned}$$

where $C > 0$ is a generic constants independent of δ.

Therefore, by Lemma 4.4.1 we can show that

$$\sup_{0 \leq t \leq T} \left| X_t - X_t^{(n)} \right| \leq K\delta^{1-2\alpha},$$

where $K > 0$ is a constant independent of δ. □

Bibliography

[1] Bally, V. Approximation scheme for solutions of BSDEs, in El Karoui, N.; Mazliak, L., Backward stochastic differential equations (Paris, 1995–1996) Pitman Research Notes in Mathematics Series, Vol. 364. Longman, Harlow, 1997, 177–191. Cited on 29

[2] Bender, C.; Denk, R. A forward scheme for backward SDEs. *Stochastic Process. Appl.* **117** (2007), no. 12, 1793–1812. Cited on 29, 32, 67

[3] Bismut, J.-M. Conjugate convex functions in optimal stochastic control. *J. Math. Anal. Appl.* **44** (1973), 384–404. Cited on 28

[4] Bouchard, B.; Touzi, N. Discrete-time approximation and Monte-Carlo simulation of backward stochastic differential equations. *Stochastic Process. Appl.* **111** (2004), no. 2, 175–206. Cited on 29, 32, 67

[5] Boufoussi, B.; Ouknine, Y. On a SDE driven by a fractional Brownian motion and with monotone drift. *Electron. Comm. Probab.* **8** (2003), 122–134. Cited on 87

[6] Bouleau, N.; Hirsch, F. *Dirichlet forms and analysis on Wiener space.* de Gruyter Studies in Mathematics, 14. Walter de Gruyter & Co., Berlin, 1991. Cited on 99

[7] Briand, P.; Delyon, B.; Mémin, J. Donsker-type theorem for BSDEs. *Electron. Comm. Probab.* **6** (2001), 61–14. Cited on 30

[8] Cherny, A. S.; Engelbert, H.-J. *Singular stochastic differential equations.* Lecture Notes in Mathematics **1858**. Springer-Verlag, Berlin, 2005. Cited on

[9] Cox, J. C.; Ingersoll Jr., J. E.; Ross, S.A. A theory of term structure of interest rates. *Econometrica* **53** (1985), 385–407. Cited on 88, 98

[10] Decreusefond, L.; Üstünel; A.S. Stochastic Analysis of the Fractional Brownian Motion. *Potential Analysis* **10** (1998), 177–214. Cited on 13

[11] Eisenbaum, N.; Tudor, C. A. On squared fractional Brownian motions. In: Séminaire de Probabilitś XXXVIII, *Lecture Notes in Math.* **1857** 2005, 282–289. Cited on 23, 88

[12] El Karoui, N.; Mazliak, L. Backward stochastic differential equations. Pitman Research Notes in Mathematics Series, Vol. 364. Longman, Harlow, 1997. Cited on 29

[13] El Karoui, N.; Peng, S.; Quenez, M. C. Backward stochastic differential equations in finance. *Math. Finance* **7** (1997), no. 1, 1–71. Cited on 16, 29, 30, 34, 46

[14] Fernique, X. Regularité des trajectoires des fonctions aléatoires gaussiennes. In: École d'Été de Probabilités de Saint-Flour, IV-1974. *Lecture Notes in Math.* **480** (1975), 1–96. Cited on 24, 97

[15] Gobet, E.; Lemor, J. P.; Warin, X. A regression-based Monte Carlo method to solve backward stochastic differential equations. *Ann. Appl. Probab.* **15** (2005), no. 3, 2172–2202. Cited on 29, 32, 67

[16] Gross, L. Abstract Wiener spaces. 1967 Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965/66), Vol. II: Contributions to Probability Theory, Part 1 pp. 31–42 Cited on 14

[17] Guerra, J. M. E.; Nualart, D. The $1/H$-variation of the divergence integral with respect to the fractional Brownian motion for $H > 1/2$ and fractional Bessel processes. *Stochastic Process. Appl.* **115** (2005), no. 1, 91–115. Cited on 23, 88

[18] Hu, Y.; Nualart, D. Some processes associated with fractional Bessel processes. *J. Theoret. Probab.* **18** (2005), no. 2, 377–397. Cited on 23, 88

[19] Hu, Y.; Nualart, D. Differential equations driven by Hölder continuous functions of order greater than 1/2. In *The Abel Symposium on Stochastic Analysis*, 399-423. Springer, 2007. Cited on 24, 88

[20] Imkeller, P.; Dos Reis, G. Path regularity and explicit convergence rate for BSDE with truncated quadratic growth. *Stochastic Process. Appl.* **120** (2010), no. 3, 348–379. Cited on 29

[21] Kuo, H. H. Gaussian measures in Banach spaces. Lecture Notes in Mathematics, Vol. 463. Springer-Verlag, Berlin-New York, 1975. Cited on 14

[22] Lyons, T. Differential equations driven by rough signals (I): An extension of an inequality of L.C. Young. *Mathematical Research Letters* **1** (1994), 451–464. Cited on 98

[23] Ma, J.; Protter, P.; Yong, J. Solving forward-backward stochastic differential equations explicitly–a four step scheme. *Probab. Theory Related Fields* **98** (1994), no. 3, 339–359. Cited on 29

[24] Ma, J.; Protter, P.; San Martin, J.; Torres, S. Numerical method for backward stochastic differential equations. *Ann. Appl. Probab.* **12** (2002), no. 1, 302–316. Cited on 30

[25] Ma, J.; Yong, J. Forward-backward stochastic differential equations and their applications. Lecture Notes in Mathematics, 1702. Springer-Verlag, Berlin, 1999. Cited on 29

[26] Ma, J.; Zhang, J. F. Path regularity for solutions of backward stochastic differential equations. *Probab. Theory Relat. Fields* **122** (2002), 163–190. Cited on

[27] Mishura, Yu. *Stochastic Calculus for Fractional Brownian Motion and Related Processes.* Lecture Notes in Mathematics, 1929. Springer-Verlag, Berlin Heidelberg, 2008. Cited on 106

[28] Mishura, Yu.; Nualart, D. Weak solutions for stochastic differential equations with additive fractional noise. *Statist. Probab. Lett.* **70** (2004), no. 4, 253–261. Cited on 88

[29] Neuenkirch, A. Optimal approximation of SDE's with additive fractional noise. *J. Complex.* **22** (2006) no. 4, 459-474. Cited on 26, 106

[30] Neuenkirch, A.; Nourdin I. Exact Rate of Convergence of Some Approximation Schemes Associated to SDEs Driven by a Fractional Brownian Motion. *J. Thoeret. Probab.* **20** (2007), no. 4, 871-899. Cited on 26, 106

[31] Nualart, D. *The Malliavin calculus and related topics.* Second edition. Springer, 2006. Cited on 10, 13, 46, 57, 103

[32] Nulalart, D.; Ouknine, Y. Regularization of differential equations by fractional noise. *Stochastic Process. Appl.* **102** (2002), no. 1, 103–116. Cited on 87

[33] Nualart, D; Răşcanu, A. Differential equations driven by fractional Brownian motion. *Collect. Math.* **53** (2002), no. 1, 55–81. Cited on 98, 113, 114, 125

[34] Pardoux, É.; Peng, S. G. Adapted solution of a backward stochastic differential equation. *Systems Control Lett.* **14** (1990), no. 1, 55–61. Cited on 28

[35] Samko S. G.; Kilbas A. A.; Marichev O. I. *Fractional Integrals and Derivatives. Theory and Applications*. Gordon and Breach, 1993. Cited on 92, 108

[36] Saussereau, B.; Nualart, D. Malliavin calculus for stochastic differential equations driven by a fractional Brownian motion. Preprint. Cited on 13

[37] Sugita, H. On a characterization of the Sobolev spaces over an abstract Wiener space. *J. Math. Kyoto Univ.* **25** (1985), no. 4, 717–725. Cited on 102

[38] Young, L. C. An inequality of the Hölder type connected with Stieltjes integration. *Acta Math.* **67** (1936), 251–282. Cited on 92, 98, 109

[39] Zähle, M. Integration with respect to fractal functions and stochastic calculus, I. *Prob. Theory Related Fields* **111** (1998), 333–374. Cited on 92, 109

[40] Zhang, J. F. A numerical scheme for BSDEs. *Ann. Appl. Probab.* **14** (2004), no. 1, 459–488. Cited on 18, 19, 20, 29, 30, 31, 33, 56, 58, 59, 60