Long time behavior and stability of special solutions of nonlinear partial differential equations.

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Abstract

LONG-TIME BEHAVIOR AND THE STABILITY OF SPECIAL SOLUTIONS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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This dissertation deals with a variety of problems concerning solutions of a large class of partial differential equations (PDEs) of mathematical physics, which can be viewed as dynamical systems on an infinite-dimensional space. Many PDEs support coherent structures like solitary waves (both ground states and bound states), as well as traveling wave solutions. These coherent structures are very important objects when modeling physical processes and their stability is essential in practical applications. Stable states of the system are key because they attract all nearby configurations, while the loss of stability or being able to control it is of practical importance as well. In this dissertation, I apply spectral and variational methods, evolution semigroups, as well as the techniques of Fourier analysis, to study some outstanding open problems in the theory of stability and long time behavior for solutions of nonlinear PDEs. The point of view is that of infinite-dimensional dynamical systems which takes advantage of the analogy between PDEs and ODEs by looking at systems whose time evolution occurs on ap-

propriately defined infinite-dimensional function spaces. In general, the main difficulty in the study of long time behavior of the solutions occurs in higher dimensional spaces and on unbounded domains. To overcome this difficulty, either the modified equations have been studied, or the initial data and the domain have been restricted. In the study of stability, one of the most interesting problems is the relation between the linear stability/instability and the nonlinear stability/instability. This question is more or less resolved in the ODE case, but it is much more complicated in the case of PDEs where infinite-dimensional function spaces and unbounded operators are needed to describe the situation. Based on the linear results, the challenge is to establish nonlinear stability/instability and complete invariant manifolds description for these equations. My contribution described in this dissertation can be divided in two parts.

In the first part, I study the long-time behavior of the solutions of the Kuramoto-Sivashinsky (KS) equation and the Burgers-Sivashinsky equation. KS equation has been widely studied and many results have been obtained for bounded domains in dimension one. However when the dimension is higher, the problem becomes much more challenging due to the nonlinear term. Previous results for dimension two have been obtained either for restricted initial data and a thin domain, or for a modified version of the KS equation. I work on a two-dimensional modified Kuramoto-Sivashinsky equation and prove the existence of a global attractor on a bounded domain. Next, I study the long-time behavior of the solutions of the one-dimensional Burgers-Sivashinsky equation for general initial data as opposed to the usually considered odd initial data. My main contribution is in the study of radially symmetric solutions of the KS equation in dimension two and higher. More precisely, I study the long-time behavior of radially symmetric solutions of the KS equation in a shell domain in three-dimensions and prove the existence of a time independent bound for the L^2 norm of the solution. I also show that similar results hold in any dimension n as long as we have the domain, which excludes the origin. We utilize various techniques from analysis and PDE such as energy estimates, coercivity and evolution semigroups .

In the second part, we deal with the conditional stability of radial steady state solutions for the one-dimensional Klein-Gordon equation. It is known that these solutions are linearly unstable and it has been proved that they are also nonlinearly unstable. Our results complement these. I consider the one-dimensional case and construct the infinite-dimensional invariant manifolds explicitly. The result is a precise center-stable manifold theorem, which includes the co-dimension of the manifolds and the decay rates. I use spectral theory, dynamical systems methods, functional analysis and Strichartz estimates to obtain this. The main difficulty in dimension one compared to higher dimensions is that the required decay of the Klein-Gordon semigroup does not follow from Strichartz estimates alone. Thus I apply additional weighted decay estimates in order to close the argument. In this part of my dissertation, the goal is to develop a systematic approach to study the fine properties of the solutions in the vicinity of the center-stable manifold and to apply the conditional stability results to control the perturbations in order to keep the stable configurations.

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I would like to thank all of my friends, it is almost impossible to name them all, who helped me directly or indirectly with my dissertation. I actually decided to write out all of the names even though it would have taken up many pages. However, I was scared to miss one name unintentionally. I am sorry not to include your names here now, but I would like you to know that you are all in my heart and you will always be.

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Contents

Introduction

Partial differential equations that can be studied as dynamical systems on an infinite-dimensional space describe many important physical phenomena. This point of view is very beneficial because it allows the generalization of finite dimensional notions and ideas to the infinite-dimensional systems through use of functional analysis, operator semigroups and spectral theory. Lately, the unprecedented expansion of this field of mathematics has found applications in areas as diverse as fluid dynamics, nonlinear optics and network communications, combustion and flame propagation.

This dissertation consists of two main parts. In the first part, we study the long-time behavior of the solutions of Kuramoto-Sivashinsky and Burger-Sivashinsky equations. The second part deals with the one-dimensional Klein-Gordon equation and the linear and nonlinear stability of its radially symmetric steady-state solutions. In Chapter 1, we describe some basic tools from differential equations and harmonic analysis that we use in the rest of this dissertation. We start with some definitions and elementary properties of semigroups, stability and attractors. Then we give some basic facts about function spaces and introduce the Littlewood-Paley operators. In the last part, we present Strichartz estimates for the Klein-Gordon solution semigroup operators.

Kuramoto has discovered the KS equation in the context of angular turbulence of a system of reaction-diffusion equations modeling Belousov-Zhabotinsky reaction in three dimensions. Sivashinsky discovered the equation working in combustion theory to model small thermal diffusive instabilities in laminar flame fronts in two space dimensions. The equation is also suitable for numerical work since it is one-dimensional and thus more tractable, but nevertheless exhibits complex dynamics. There has been a lot of work done on this equation in one space dimension in the last three decades which by now has become classical-the existence of solutions, the low dimensional global attractor asserted by the inertial manifold theorem of Nicholaenko, Scheurer and Temam [40], as well as Sell and Foias [23]. The problem of existence of solutions and their long time behavior for the Kuramoto-Sivashinsky equation in higher space dimensions is very difficult and still open. Some of the available results have restrictions on the domain [38] or work on a modified equation. One is tempted to compare the two-dimensional Kuramoto-Sivashinsky equation to other difficult equations like the Navier-Stokes equation where global existence can be proved via energy estimates that give control of the L^2-norm of the solution. On the other hand, in the Burgers-Sivashinsky equation no control of any L^p norm is possible but one can use the maximum principle to gain control of the L^∞-norm of the solution. In contrast, both of these are not available for the Kuramoto-Sivashinsky equation in higher space dimensions. Our interest in the equation was inspired by the recent progress on the long time behavior of the solution made by Bronski and Gambill [7], see also [25]. In Chapter 2, we study the long time behavior for the special solutions of Kuramoto-Sivashinsky and Burgers-Sivashinsky equations. Burgers-Sivashinsky equation,

$$\varphi_t = \Delta\varphi + \varphi - |\nabla\varphi|^2 \tag{0.0.1}$$

is related to the Kuramoto-Sivashinsky equation,

$$\varphi_t = -\Delta^2\varphi - \Delta\varphi - \frac{1}{2}|\nabla\varphi|^2 \tag{0.0.2}$$

and can be derived as model for flames propagation. Since the BS equation is a second order equation, it is considered a simpler model. The two equations are often compared because there are some similarities, but a lot of new phenomena appear in the fourth order case.

In Chapter 2, first we consider a modified version of Kuramoto-Sivashinsky Equation in space dimension two:

$$u_t = -\Delta^2 u - \Delta u - uu_x - uu_y + g(x) \tag{0.0.3}$$

on a bounded domain $[-L,L] \times [-L,L]$. With certain conditions on the boundary, initial value and the external force g, we have the following result:

Theorem 2.2.1: The dynamical system associated with the two-dimensional periodic Kuramoto-Sivashinsky type equation (2.2.1) with its boundary conditions is globally well-posed and possesses a global attractor.

The analysis is based on the Lyapunov function approach, point dissipativeness and asymptotic compactness. Here the main difficulty is to prove the asymptotic compactness. In order to achieve this, using the techniques of Fourier analysis, we show $(a)-(c)$ of Proposition 1.1.20 where $P_{>N}$ are the Littlewood-Paley projections.

Since BS equation is a simpler model to the KS equation, in the second part of Chapter 2, we start with one-dimensional Burgers-Sivashinsky equation and prove the existence of a time independent bound for the L^2 norm of the solutions. The result is for a bounded domain [-L,L] and in the case of any *general initial data*. Since Lyapunov function methods rely strongly on the fact that odd solutions vanish at zero, the sharpest results were always obtained in the odd data case first. Collet in [16] resolved this obstacle by introducing a translation of the potential, governed by a solution-dependent gradient flow dynamics. We use similar ideas to give a simple proof in the case of

Burger-Sivashinsky equation in dimension one. Then we consider radially symmetric solutions of this equation in two and higher dimensions in a bounded domain $[0,R]$. After deriving results for the radially symmetric solutions of BS equation [19], we work on the harder model, KS equation in order to derive similar results. In the third part of Chapter 2, we consider the radially symmetric solutions of the KS equation in a shell domain $\Omega = \{x \in \mathbb{R}^n$ such that $0 < r_0 < \|x\| < R_0\}$ in any dimension n. We prove the existence of a time independent bound for the L^2 norm of the solution and show that in the three dimensional case this bound is given by $C(R_0 - r_0)^{3/2}$ and we give an estimate of the rate with which the constant C blows up when $r_0 \to 0$. Similar results hold for any n-dimensional shell domain which does not contain the origin. In particular we show that if the dimension is sufficiently high one can use the estimates for the constant $C(r_0)$ to prove that the radially symmetric solution does not blow up at the origin. More rigorously, assuming that the initial condition u_0 is a radial function and u solves the differentiated (2.1.1) and taking the boundary conditions (2.4.3) which are similar to the ones in [6] (these are the Neumann boundary conditions for a fourth order model) and using Lyapunov function approach, we prove the following theorem for the radial system (2.4.2)-(2.4.4).

Theorem 2.4.1: Consider the Kuramoto-Sivashinsky equation (2.4.2) with $0 < r_0 < R_0 < \infty$, subject to the boundary and initial conditions given by (2.4.3), (2.4.4). Assume also $(R_0 - r_0) \geq \alpha(1 + \frac{1}{r_0^2})^{-1/2}$ for some $\alpha > 0$. Then, there is constant $C = C_\alpha$, so that

$$\limsup_{t\to\infty} \|u(t)\|_{L^2[r_0,R_0]} \leqslant C_\alpha (R_0 - r_0)^{3/2} \left(1 + \frac{1}{r_0^2}\right)^3 . \tag{0.0.4}$$

If $(R_0 - r_0) \leq (1+\frac{1}{r_0^2})^{-1/2}$, then

$$\limsup_{t\to\infty} \|u(t)\|_{L^2[r_0,R_0]} \leq C\frac{(1+\frac{1}{r_0^2})^2}{\sqrt{R_0-r_0}}. \tag{0.0.5}$$

In Chapter 3, we study the stability for the radial steady-state solutions of the one dimensional Klein-Gordon equation. The Klein-Gordon equation is a relativistic version of the Schrödinger equation. It was named after Oscar Klein and Walter Gordon who proposed the Klein-Gordon equation to describe quantum particles in the framework of relativity. It describes the spinless composite particles. However Schrödinger was the first who considered this equation as a quantum wave equation. Klein-Gordon type equations are in the form:

$$u_{tt} - \Delta u + u - \mathcal{N}(u) = 0 \qquad (t,x) \in \mathbf{R}^+ \times \mathbf{R}^\mathbf{d} \tag{0.0.6}$$

where $\mathcal{N}(u)$ is the nonlinear term. With some assumptions on the nonlinear term, it has been proved by the authors of [32] that these solutions are in fact linearly and nonlinearly unstable. Our interest is the conditional stability of such steady state solutions. This kind of stability has been extensively studied recently. For example for the equation $u_{tt} - \Delta u = u^5$, in [32], the existence of steady state solutions, the linear and the nonlinear instability of such solutions have been proved. However it has been also proved in [33] that for the special perturbation to the steady state solution of $u_{tt} - \Delta u = u^5$, the solution exists globally and remains near the steady state. Thus, a center-stable manifold for the steady state in the sense of Bates and Jones [2] is described. In 1989, Bates and Jones [2], [3] proved that for a large class of semilinear equations, including the Klein-Gordon equation, the space of solutions decomposes into an unstable and center-stable manifold. Similar result was proved in [26] for the semilinear Schrödinger equation in any dimension. Both are abstract results and do not deal with the global in time behavior of the solutions, e.g. existence and asymptotic behavior. The first asymptotic stability result was obtained by Soffer and Weinstein, [52], [53] (see also [54]), followed by works of Pillet and Wayne [42], Buslaev, Perelman, Sulem [9], [10], [11], Rodnianski-Schlag-Soffer [45], [46] etc. In this context we would like to mention some recent work of Schlag [47] and Beceanu [4],[5] on the existence of center-stable manifold for the pulse solutions of the focusing cubic nonlinear Schr odinger equation in dimension three. It identifies a center-stable manifold in the critical for the equation space $H^{1/2}$ and shows that solutions starting on the manifold exist globally in time and remain on the manifold for all time answering an open question in [26]. Recently the authors of [57] proved a conditional stability of the steady state solutions of (3.1.1) with $\mathscr{N}(u) = |u|^{p-1}u$ for the dimension $d = 2,3$ and 4 where $p \geq 1+4/d$. In terms of center-stable manifold for the solution, their result shows the global in time behavior of the solutions and a precise description of the manifold which includes its co-dimension and decay rates. In these problems, since Strichartz estimates are key, the lower the dimension, the harder it is to close the argument. The main difficulty in the one-dimensional case is that the required decay of the Klein-Gordon semigroup does not follow from Strichartz estimates alone. One needs to further refine the function spaces and use additional decay estimates to resolve this issue. The techniques we use are similar to the ones used in [37]. We prove the following theorem:

Theorem 3.3.1: For (3.1.2) with $5 \leq p < \infty$, and $\mathscr{H}\psi = -\sigma^2\psi$ where $\sigma = \sigma(p)$, there exists $0 < \varepsilon = \varepsilon(p) << 1$ and $0 < \delta = \delta(p) << 1$, and a function

$$h : B_{H^1}(\delta\varepsilon) \times B_{L^2}(\delta\varepsilon) \cap \{(f,g) : \langle \sigma f + g, \psi \rangle = 0\} \to \mathbf{R}^1$$

so that whenever the initial data is even and

$$\begin{aligned} u(0) = & \quad \phi + f_1 + h(f_1, f_2)\psi \\ u_t(0) = & \qquad\qquad f_2 \end{aligned}$$

$$\langle \sigma f_1 + f_2, \psi \rangle = 0; \|(f_1, f_2)\|_{H^1 \times L^2} < \delta\varepsilon,$$

then

$$u(t,x) = \phi(x) + a(t)\psi + \mathbf{z}(t,x) \quad \text{ where } \mathbf{z} = P_{a.c.}(\mathscr{H})\mathbf{z} \tag{0.0.7}$$

and

$$\|\mathbf{z}\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1 \cap L_x^\infty(\mathbb{R};\langle x\rangle^{-3/2}dx)L_t^2} \leq \varepsilon, \qquad \|a\|_{L_t^3[0,\infty)\cap L_t^\infty[0,\infty)} \leq \varepsilon.$$

To prove this result, we use the spectral decomposition of the linearized operator $\mathscr{H} = -\Delta + 1 - p\phi^{p-1}$ and set up an iteration scheme in the appropriate Strichartz spaces. The goal is to prove that the corresponding map is a contraction map. The spectral information for the linearized operator that we need is readily available, say in [14]. The main difference in $d = 1$ case compared to other cases like $d = 2,3,4$, is the need of decay estimates because the argument can not be closed with Strichartz estimates alone. In order to prove the decay estimates for the linearized operator $\mathscr{H}$, similar to [37], we work on two estimates, the high energy estimate and the low energy estimate. We use Green's functions for the first one, and Jost functions and scattering theory for the second one.

Chapter 1

Preliminaries

In this chapter, we present some tools from differential equations and harmonic analysis which we will use later on in this dissertation. In Section 1.1, we present some definitions and elementary properties of semigroups of linear operators and their use to describe solutions of evolution equations. Next, we give some preliminaries from the theory of spectral, linear and nonlinear stability for special solutions of nonlinear PDEs. We present some conditions that give the relations between these different notions of stability as well as some examples where one implies the other. Then we discuss special sets that help us to determine the long time behavior of dynamical systems. We start with absorbing sets, and then describe global and local attractors. We end this section with the existence theorem for a global attractor of a dynamical system. In Section 1.2, we start with the basic definitions and facts about function spaces and present some basic inequalities that we will need to use later. Then we introduce Littlewood-Paley projections, which we use to obtain the compactness results in Chapter 2. We end this section by giving the Strichartz estimates for the Klein-Gordon semigroup operators. These are key estimates in the proof of the conditional stability theorem in Chapter 3.

1.1 Differential Equations Tools

1.1.1 Semigroups of linear operators

In this dissertation, our interest is to study dynamical systems whose state is described by an element $u = u(t)$ of a metric space H. In most cases, particularly for the systems associated to ODE and PDE, the parameter, t (mostly time variable) varies continuously in $\mathbb{R}$. Usually the space H is a Hilbert or a Banach space.

Definition 1.1.1. *A family of operators $S(t), t \geq 0$ describing the evolution of the dynamical system*

$$\frac{d}{dt}u(t) = F(u(t)), \qquad u(0) = u_0$$

*is called the **solution semigroup operators** if the map $S(t)$ from H into itself enjoys the usual semigroup properties:*

$$\begin{aligned} S(t+s) &= S(t)S(s), \qquad \forall s,t \geq 0 \\ S(0) &= I \qquad \textit{(Identity in } H\textit{)} \end{aligned} \tag{1.1.1}$$

and

$$\begin{aligned} u(t) &= S(t)u(0) \\ u(t+s) &= S(t)u(s) = S(s)u(t), \qquad \forall s,t \geq 0 \end{aligned}$$

Remark 1. *The solution of a differential equation determines the solution semigroup $S(t)$, thus that $S(t)$ does not have to be linear. In the ODE case, the general theorems of existence of solutions provide the definition of the operators $S(t)$. However in the PDE case, there are no theorems of existence and uniqueness, so the first step is to prove the existence of such operators.*

Definition 1.1.2. *Let X be a Banach space. A one parameter family $\{T(t)\}_{t\geq 0}$ of bounded linear operators from X into itself is called as* ***strongly continuous*** *semigroup of bounded linear operators (C_0-semigroup) if $T(t)$ enjoys the usual semigroup properties:*

$$\begin{aligned} T(t+s) &= T(t)T(s), \qquad \forall s,t \geq 0 \\ T(0) &= I \qquad \textit{(Identity in X)} \end{aligned} \tag{1.1.2}$$

and

$$\lim_{t\downarrow 0} T(t)x = x \quad \textit{for every } x \in X.$$

Definition 1.1.3. *The linear operator A defined by*

$$\begin{aligned} D(A) &= \{x \in X : \lim_{t\downarrow 0} \tfrac{T(t)x-x}{t} \textit{ exists in X norm.}\} \\ Ax &= \lim_{t\downarrow 0} \tfrac{T(t)x-x}{t} \qquad \textit{for } x \in D(A) \end{aligned} \tag{1.1.3}$$

is the ***infinitesimal generator*** *of the semigroup $T(t)$, where $D(A)$ is the domain of A.*

Theorem 1.1.4. *([41]) Let $T(t)$ be a C_0-semigroup. There exist constants $\omega \in \mathbb{R}$ and $M \geq 0$ such that*

$$\|T(t)\| \leq Me^{\omega t} \textit{ for } 0 \leq t \leq \infty$$

Definition 1.1.5. *If $\omega = 0$, then $T(t)$ is called* ***uniformly bounded*** *and if, moreover, $M \leq 1$, $T(t)$ is called C_0-**semigroup of contractions**.*

Let X be a Banach space, and consider the linear Cauchy problem

$$\begin{aligned} u_t &= Au, \qquad \text{for } 0 < t < \infty \\ u(0) &= u_0 \end{aligned} \tag{1.1.4}$$

where A is a linear operator, which generates a C_0-semigroup, $S(t) = e^{At}$. If $u_0 \in X$, then the function $u(t) := e^{tA}u_0$ is called a mild solution of the differential equation (1.1.4). If $u_0 \in D(A)$, then $u(t) := e^{tA}u_0$ is called a classical solution.

Theorem 1.1.6. *(see [36]) Suppose $S(t)$ is a C_0-semigroup on a Banach space X, and $A : D \to X$ is defined by (1.1.3). Then the following hold.*

1. *The domain $D(A)$ is a dense subset of X.*

2. *$A : D(A) \to X$ is a closed operator.*

3. *For $u \in D(A)$, we have $S(t)u \in D$ for all $t \geq 0$ and $AS(t)u = S(t)Au$ for all $t > 0$.*

4. *For $g \in D(A)$, $u(t) = S(t)g$ is a classical solution of (1.1.4).*

Remark 2. *If $x \in D(A)$, then $e^{tA} \in D(A)$. The function $t \to e^{tA}$ is not only continuous, but also differentiable and $\frac{d}{dt}e^{tA}x = Ae^{tA}x = e^{tA}Ax$. Thus it makes sense to use the notation $T(t) = e^{tA}$ for the infinitesimal generator of the semigroup in Definition 1.1.3.*

Example: Consider the initial value for the wave equation in $\mathbb{R}^n$, that is,

$$\begin{cases} u_{tt} = \triangle u & \text{for } x \in \mathbb{R}^n, t > 0 \\ u(0,x) = u_1(x), \;\; u_t(0,x) = u_2(x) & \text{for } x \in \mathbb{R}^n \end{cases} \tag{1.1.5}$$

If we introduce a new variable $v := u_t$, this problem becomes equivalent to the first order system:

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 0 & I \\ \triangle & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \qquad \text{for } x \in \mathbb{R}^n, t > 0 \tag{1.1.6}$$

and

$$\begin{pmatrix} u(0,x) \\ v(0,x) \end{pmatrix} = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} \qquad \text{for } x \in \mathbb{R}^n \tag{1.1.7}$$

Theorem 1.1.7. *(see [41]) The operator* $A = \begin{pmatrix} 0 & I \\ \triangle & 0 \end{pmatrix}$ *is the infinitesimal generator of a* C_0-*semigroup of operators in the Hilbert space* $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ *equipped with the norm* $\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\| = \left(\int_{\mathbb{R}^n} (|u|^2 + |\nabla u|^2 + |v|^2) dx \right)^{1/2}$.

1.1.2 Stability of special solutions

In this section we give the basic definitions and results related to stability of special solutions of nonlinear PDEs. An equilibrium solutions of a PDE is stable if any orbit that starts nearby will stay close or will approach these states as time grows to infinity. Similar to the finite dimensional case, the stability can be inferred by investigating the spectrum of the linearization around the special solutions. There are some complications that arise in the case of infinite-dimensional systems (PDEs).

Spectral Stability

Definition 1.1.8. *Assume A is a linear, not necessarily bounded, operator in a Banach space* X. *Then the resolvent set of A, denoted by* $\rho(A)$, *is,*

$$\rho(A) = \{\lambda \in \mathbb{C} : (\lambda I - A)^{-1} : X \to X \text{ is bounded.}\}$$

The complement of the resolvent is called the spectrum, denoted by $\sigma(A)$. *The complex number* λ *is in spectum if* $\lambda I - A$ *is not invertible, i.e.,* $(\lambda I - A)^{-1}$ *is not a bounded linear operator in* X.

The notion of the spectrum extends to densely-defined unbounded operators, which will be investigated in this dissertation. In this case a complex number λ is in $\sigma(A)$ where $A : D(A) \to X$ (where D(A) is dense in X) if there is no bounded inverse $(\lambda I - A)^{-1} : X \to D(A)$.

There are several ways to classify the points in the spectrum of an operator A based on the reasons behind the non-existence of the resolvent as a bounded operator. The classification that we will use is

$$\sigma(A) = \sigma_{pt} \cup \sigma_{ess}$$

where σ_{pt}, the point spectrum, contains all the isolated eigenvalues with finite multiplicity. The rest of the spectrum is the essential spectrum, denoted by σ_{ess}.

Definition 1.1.9. *We call the operator A spectrally stable if its spectrum is to the left of the imaginary axis,*

$$\sigma(A) \subset \{Re(\lambda) < 0\}.$$

Linear Stability

Example 1**:** For the equation $u_t = u_{xx} + f(u)$ on $\mathbb{R}$, one has $F(u) := u_{xx} + f(u)$. If $\phi(x)$ is a steady state solution, one has $F(\phi) = 0$. We will also assume that $\phi \to 0$ as $x \to \pm\infty$, and $|f'(z)| \sim O(z)$. Thus the linearized operator is $\mathscr{L}v = v_{xx} + f'(\phi)v$ with the domain $H^2(\mathbb{R})$.

Example 2**:** Consider the same equation $u_t = u_{xx} + f(u)$, but look for the traveling wave solutions, which are in the form $u(x,t) = u(x+ct,t)$, where c is a scalar. Define the moving variable $\xi = x + ct$ to get the equation

$$u_t = u_{\xi\xi} + cu_\xi + f(u)$$

Assume that for fixed $c = c^*$ there exists an equilibrium solution $u(x,t) = \varphi(x + c^*t) = \varphi(\xi)$, such that

$$\varphi_{\xi\xi} + c^*\varphi_\xi + f(\varphi) = 0$$

Then define $F(u) := (\partial_{xx} + c^*)u + f(u)$ and linearize the equation $u_t = F(u)$ about the traveling wave. We get the linearized operator $\mathscr{L}v = \partial_{xx}v + c^*v + f'(\varphi)v$ with the domain $H^2(\mathbb{R})$. As in the previous example, we assume that $\phi \to 0$ as $x \to \pm\infty$, and $|f'(z)| \sim O(z)$.

Consider the nonlinear Cauchy problem:

$$u_t = F(u), \quad u(0) = u_0 \tag{1.1.8}$$

where F is nonlinear.

Definition 1.1.10. *Assume Q solves the problem (1.1.8) and $\mathscr{L}$ is the linearized operator of the Cauchy problem and $u(t,\cdot) = e^{\mathscr{L}t}u_0$. Then we say Q is linear stable if*

$$\lim_{t\to\infty} e^{-\delta t}\|u(t,\cdot)\| = 0$$

for every $\delta > 0$.

It is important to know whether spectral stability implies linear stability, as in the case of ODEs. This holds true if the spectrum is mapped correctly by the exponential map, that is,

$$\sigma(e^{t\mathscr{L}})\backslash\{0\} = e^{t\sigma(\mathscr{L})} \tag{1.1.9}$$

as in the finite-dimensional case. (1.1.9) holds for matrices, analytic semigroups and parabolic equations. In the infinite dimensional case, this amounts to the spectral mapping property, that is, for every $t > 0$, (1.1.9) holds.

Nonlinear Stability

Finally, one would like to know whether the the special solution is stable in terms of the full nonlinear equation rather than the linearized one. In this case, the solution can be asymptotically or orbitally stable.

Definition 1.1.11. *Assume Q solves the problem (1.1.8) and let $U_\delta(Q)$ be the open ball centered at Q with the radius δ. Then Q is nonlinearly stable if $\forall \varepsilon > 0$, $\exists \delta > 0$, such that if u_0 is an initial condition in $U_\delta(Q)$, then the associated solution $u(x,t)$ satisfies:*

$$u(\cdot,t) \in U_\varepsilon(Q\{(\cdot+\tau);\tau \in R)\} \text{ for } t > 0$$

Definition 1.1.12. *Q is nonlinearly stable with asymptotic phase if for each u_0 in $U_\delta(Q)$, there exists $\tau^* = \tau^*(u_0)$ such that*

$$\|u(\cdot,t) - Q(\cdot+\tau^*)\| \to 0 \text{ as } t \to \infty$$

Remark 3. *One may look for the answers to the following questions. In which cases does spectral stability/linear stability imply nonlinear stability? Suppose we are studying the stability of a special solutions Q of the PDE : $u_t = Au + N(u)$ and assume the linearized operator about Q is $\mathscr{L} = A + \partial_u N(Q)$, we will describe several scenarios in which one can claim nonlinear stability.*

- Case 1. If A is a sectorial operator ($\sigma(A) \setminus \{0\} \subset \{\lambda \in \mathbb{R}$ s.t. $\lambda < -\delta, \delta > 0\}$ and $\lambda = 0$ is a simple eigenvalue), then $\|(A - \lambda I)^{-}1\| \leq \frac{K}{|\lambda - a|}$ in a sector. D. Henry proved using center manifold reduction that in this case the linearly stable special solution is nonlinearly stable with asymptotic phase.

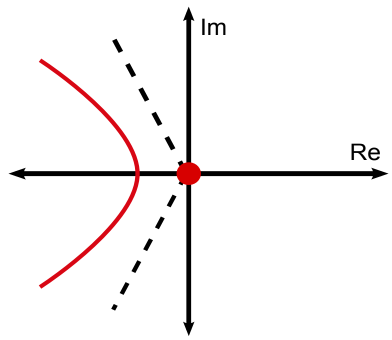

Figure 1.1: Example for Case 1. 0 is the simple eigenvalue of A, the rest of the spectrum lies inside the region enclosed by the dashed lines.

- Case 2. If A generates a strongly continuous semigroup $S(t) = e^{At}$. In this case we have to check that

$$\|(A - \lambda I)^{-1}\| \leq K$$

for all λ with $\Re\lambda \geq \eta$. Then by Gearhart-Prüss Theorem $\|e^{At}\| \leq Ce^{\eta t}$. If, in addition the nonlinearity is differentiable, then spectral stability implies nonlinear stability [2]. This has been used to prove the existence of invariant stable, unstable and center manifolds for a large class of dissipative and conservative equations like the Fitzhugh-Nagumo equation.

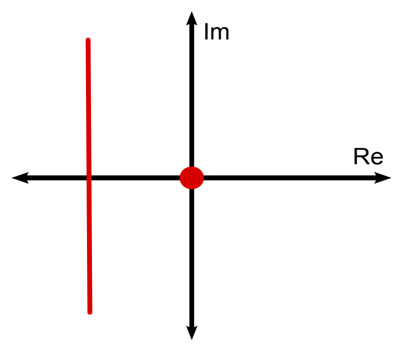

Figure 1.2: Example for Case 2. 0 is the simple eigenvalue of A, the rest of the spectrum has negative real part.

- Case 3. Essential spectrum up to the imaginary axis. This case was treated by using exponentially weighted spaces or spaces with polynomial weights plus resolvent estimates. An example of these are the KPP equation and the real Gintburg-Landau equation, where there is a continuum of waves for every wave speed $c > c^*$. The nonlinear stability of these waves has been studied in such spaces.

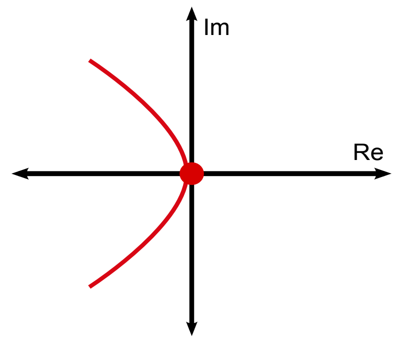

Figure 1.3: Example for Case 3: The essential spectrum of A has negative real part, but it is tangent at 0.

- Case 4. **Hamiltonian PDE's** have essential spectrum on the imaginary axis and point spectrum, which is symmetric as in the case of the bound states for the Nonlinear Schrödinger equation. The authors of [28][29] have developed deep theory to treat these cases by using the second variation of the reduced Hamiltonian at the wave. The wave is then nonlinearly stable if this second variation is sign definite. The method known as Grillakis-Shatah-Strauss method has been used in a variety of problems.

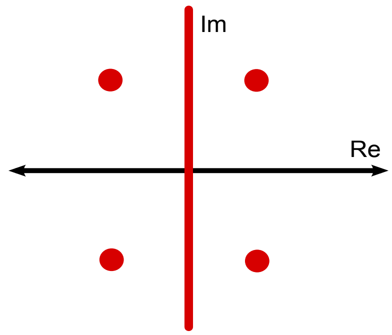

Figure 1.4: Example for Case 4. Point spectrum is symmetric to both axes, essential spectrum lies on the imaginary axis.

The special solution Q of a PDE can be spectrally, linearly and nonlinearly unstable. However if the initial values are chosen dynamically in a special manner, then the nearby solution may stay asymptotically close to Q as time grows. We call this **conditional stability**. It has been studied by Krieger and Schlag ([33]). They showed that all steady state solutions $\varphi_\lambda(x) = \frac{(3\lambda^2)^{1/4}}{\sqrt{\lambda^2+|x|^2}}$, $\lambda > 0$ of the equation $u_{tt} - \Delta u = u^5, x \in R^3$ are nonlinearly unstable, but one can construct a manifold Σ such that if the *radial* perturbation to φ_1, $(\psi_0, \psi_1) \in \Sigma$, then

- The solution exists globally
- $\lim_{t\to\infty}[\|u(t,x) - \varphi_1\| + \|u_t\|] = 0$.
- The tangent plane to Σ is given by $\sigma \int_{R^n} \xi(x)\psi_o(x)dx + \int_{R^n} \xi(x)\psi_1(x)dx = 0$.

1.1.3 Attractors

Definition 1.1.13. *An attractor is a set $\mathscr{A} \in H$ that satisfies the following properties:*

- $S(t)\mathscr{A} = \mathscr{A}$
- $\mathscr{A}$ *possesses an open neighborhood* U *such that, for every* u_0 *in* U*,* $S(t)u_0$ *converges to* $\mathscr{A}$ *as* $t \to \infty$.

$$\inf_{y \in \mathscr{A}} d(S(t)u_0, y) \to 0 \quad t \to \infty$$

where $d(x,y)$ *denotes the distance of* x *to* y *in* H.

Definition 1.1.14. $\mathscr{A} \in H$ *is called a* ***global attractor*** *for the semigroup* $\{S(t)\}_{t \geq 0}$ *if it is a compact attractor and attracts every bounded set H, i.e., for any bounded set* $B \in H$*, it satisfies the following*

$$d(S(t)B, \mathscr{A}) \to 0 \quad \text{as } t \to \infty.$$

Definition 1.1.15. *Let* B *be a subset of* H *and* U *an open set containing* B*.* B *is called an* ***absorbing set*** *in* U *if any bounded set of* U *enters into* B *after a certain time, i.e.,for every bounded* $B_0 \subset U$ *there exists* $t_1(B_0)$ *such that*

$$S(t)B_0 \subset B \quad \text{for} \geq t_1(B_0)$$

Definition 1.1.16. *Let* $S(t)$ *be a solution semigroup, acting on a normed space H. Then* $S(t)$ *is called* ***point dissipative*** *if there is a bounded set* $B \subset H$ *such that for any* $u_0 \in H$*,* $S(t)u_0 \in B$ *for all sufficiently large* $t \geq 0$*, i.e.,*

$$\sup_{u_0 \in H} \limsup_{t \to \infty} \|S(t)u_0\|_H < \infty$$

Remark 4. *In the ODE case, the point dissipativeness of $S(t)$ implies the existence of an absorbing set. However, in the PDE case, there might be dissipative systems for which the existence of an absorbing set is unknown.(e.g., the Navier-Stokes equations in dimension 3.)*

Remark 5. *A classical result in dynamical systems is that the existence of an attractor implies the existence of an absorbing set. However the converse is not true. To guarantee the existence of an attractor, one needs an additional compactness result.([48]).*

Definition 1.1.17. *The semigroup $\{S(t)\}_{t\geq 0}$ is asymptotically compact if for every bounded sequence $\{x_n\}$ in H and every sequence $t_n \to \infty$, $\{S(t_n)x_n\}_n$ is relatively compact in H.*

Next, we recall the Riesz-Rellich Criteria for precompactness.

Theorem 1.1.18. *(Rellich's criterion, Theorem XIII.65 in[44]) Let F and G be two functions on $\mathbb{R}^n$ so that $F \to 0$ and $G \to 0$. Then*

$$S = \{\psi \mid \int |\psi(x)|^2 dx \leq 1, \int F(x)|\psi(x)|^2 dx \leq 1, \int G(p)|\hat{\psi}(p)|^2 dp \leq 1\}$$

is a compact subset of $L^2(\mathbb{R}^n)$.

Theorem 1.1.19. *(M. Riesz's criterion, Theorem XIII.66 in[44]) Let $p < \infty$. Let $S \subset L^p(\mathbb{R}^n)_1$, the unit ball of L^p. A necessary and sufficient condition that the norm closure of S be norm compact is that:*

1. *$f \to 0$ in L^p sense at infinity uniformly in S, i.e., for any ε, there is a bounded set $K \in \mathbb{R}^n$ so that*

$$\int_{\mathbb{R}^n \setminus K} |f(x)|^p dx \leq \varepsilon^p$$

for all $f \in S$;

2. *$f(\cdot - y) \to f$ uniformly in S as $y \to 0$, i.e., for any ε, there is a δ so that $f \in S$ and $|y| < \delta$ imply that*

$$\int_{\mathbb{R}^n} |f(x-y) - f(x)|^p dx \leq \varepsilon^p$$

Remark 6. *As shown in [55] and [56], we may replace condition (2) in the Riesz-Rellich Criteria above by an equivalent condition, which basically says that the mass of the high-frequency component has to go uniformly to zero. The following proposition is the exact formulation.*

Proposition 1.1.20. *Assume that*

- *(a)* $\sup_n \|u_n(t_n, \cdot)\|_{L^2} \leqslant C$
- *(b)* $\lim_{N\to\infty} \limsup_n \|P_{>N} u_n(t_n, \cdot)\|_{L^2} = 0$ *as* $N \to \infty$
- *(c)* $\lim_{N\to\infty} \limsup_{n\to\infty} \|u_n(t_n, \cdot)\|_{L^2(|x|>N)} = 0$

Then the sequence $\{u_n(t_n, \cdot)\}$ is relatively compact in $L^2(\mathbb{R}^n)$.

Remark 7. *If we are in a bounded domain, then (c) is automatically satisfied.*

Theorem 1.1.21. *[1][35][48] Assume that H is a metric space and the operator $S(t)$ is the solution semigroup and it is asymptotically compact. Also assume that there exists an open set U and a bounded set B of U such that B is absorbing in U. Then $\bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} S(t)B}$ is a compact attractor which attracts the bounded sets of U. It is the maximal bounded attractor in U.*

1.2 Harmonic Analysis Tools

1.2.1 Function spaces and Littlewood-Paley projections

In this section, we define L^p spaces, Fourier transform, and introduce Littlewood-Paley operators which we will use in Chapter 2 in order to prove the asymptotic compactness of the solution semigroup operators.

ℓ^p and L^p Spaces

ℓ^p is the subspace of the set of all sequences of scalars, consisting of all sequences $x = (x_n)$ satisfying

$$\sum_n |x_n|^p < \infty$$

where $0 < p < \infty$. If $p \geq 1$, then $\|x\|_p = \left(\sum_n |x_n|^p\right)^{1/p}$ defines a norm on ℓ^p. In fact ℓ^p is a complete metric space with respect to this norm, and therefore is a Banach space.

Assume (X, M, μ) is a measure space and f is a measurable function on X and $0 < p < \infty$, then the L^p space defined on this measure space is defined by

$$L^p(X, M, \mu) = \{f : X \to \mathbb{C} : \text{f is measurable and} \|f\|_p < \infty\}$$

where $\|f\|_p = \left(\int |f|^p d\mu\right)^{1/p}$. In general $L^p(X, M, \mu)$ is abbreviated by $L^p(X)$, or simply L^p.

Hölder's Inequality:

Assume $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ and f, g are measurable functions on X, then

$$\|fg\|_1 \leq \|f\|_p \|g\|_q. \tag{1.2.1}$$

Minkowski's Inequality:

If $1 \leq p < \infty$ and $f, g \in L^p$, then

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p \tag{1.2.2}$$

Fourier Transform

Fourier Transform is defined on $L^2([-L,L]^d) \to l^2(\mathbb{Z}^d)$ by $f \to \{a_k\}_{k\in\mathbb{Z}^d}$, where

$$a_k = \frac{1}{(2L)^{d/2}} \int_{[-L,L]^d} f(x) e^{-2\pi i k\cdot x/L} dx.$$

The inverse Fourier transform is the Fourier expansion

$$f(x) = \frac{1}{(2L)^{d/2}} \sum_{k\in\mathbb{Z}^d} a_k e^{2\pi i k\cdot x/L}.$$

For $f \in L^1(\mathbb{R}^d)$, the Fourier transform of f is defined as

$$\hat{f}(\xi) = \mathscr{F}(f)(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x\dot{\xi}} dx.$$

The inverse transform is given by

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(x) e^{2\pi i x\dot{\xi}} d\xi.$$

Plancherel Theorem:

If f is a square integrable function, then the following statements hold. In the case of bounded domain $[-L,L]^d$:

$$\int_{[-L,L]^d} |f(x)|^2 dx = \sum_{k\in\mathbb{Z}^d} |a_k|^2. \tag{1.2.3}$$

and in the case of $\mathbb{R}^d$,

$$\int_{\mathbb{R}^d} |f(x)|^2 dx = \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 d\xi. \tag{1.2.4}$$

Hausdorff-Young Inequality:

Let $1 < p \leq 2$ and let $f \in L^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Then with q satisfying $\frac{1}{p} + \frac{1}{q} = 1$,

$$\left(\int |\hat{f}(\xi)|^q d\xi\right)^{1/q} \leq c_p \left(\int |f(x)|^p dx\right)^{1/p}. \tag{1.2.5}$$

Note that a special case of Hausdorf-Young inequality, $p = q = 2$ is the Plancherel formula.

Littlewood-Paley operators

We will define the *Littlewood-Paley operators* acting on $L^2([-L,L]^d)$ via Fourier transform. The projection operator $P_{\leq n}$ truncates the terms in the Fourier series expansion with frequencies $k : |k| > 2^n L$. The **Littlewood-Paley operators** on $L^2([-L,L]^d)$ for a function f are

$$P_{\leq n} f(x) = \frac{1}{(2L)^{d/2}} \sum_{k:|k|\leq 2^n L} a_k e^{2\pi i k\cdot x/L}.$$

More generally, we may define for all $0 \leq n < m \leq \infty$

$$P_{n\leq\cdot\leq m} f(x) = \frac{1}{(2L)^{d/2}} \sum_{k:2^n L \leq |k| \leq 2^m L} a_k e^{2\pi i k\cdot x/L}.$$

Lemma 1.2.1. *[59] For the Littlewood-Paley operator P_k defined by*

$$P_k f(x) = \frac{1}{(2L)^{d/2}} \sum_{|n|\sim 2^k L} a_n e^{2\pi i n\cdot x/L}$$

we have

$$\|P_k f\|_2 \lesssim 2^k \|f\|_{L^1([-L,L]^2)}. \tag{1.2.6}$$

Sobolev Spaces

Definition 1.2.2. *The Sobolev space $W^{s,p}(\mathbb{R}^n)$ where $1 \leq p \leq \infty$, and $s \in \mathbb{N}$ is defined to be the set of all functions $f \in L^p(\mathbb{R}^n)$, that is,*

$$W^{s,p}(\mathbb{R}^n) = \{f \in L^p(\mathbb{R}^n) : \partial^\alpha f \in L^p(\mathbb{R}^n) : \forall |\alpha| \leq s\}$$

with the norm

$$\|f\|_{W^{s,p}(\mathbb{R}^n)} = \sum_{|\alpha|\leq s} \|\partial^\alpha f\|_{L^p(\mathbb{R}^n)} \tag{1.2.7}$$

where ∂^α is the weak partial derivative.

Note that Sobolev spaces with the defined norm (1.2.7) are Banach spaces. When $p = 2$, it becomes a Hilbert space and is denoted by $H^s(\mathbb{R}^n)$. There is also an equivalent definition using Fourier transform:

$$H^s(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \left(\int (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{1/2} < \infty\}.$$

This definition is used when s is a non-integer. For $p = 2$, the homogeneous Sobolev space is defined as

$$\dot{H}^s(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \left(\int |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi \right)^{1/2} < \infty\}.$$

For a bounded domain $[-L, L]^d \subset \mathbb{R}^d$, the homogeneous Sobolev space is defined as

$$\dot{H}^s([-L,L]^d) = \{f : [-L,L]^d \to \mathscr{C} : \left(\sum_{k \in Z^d} |a_k|^2 \left(\frac{|k|}{L} \right)^{2s} \right)^{1/2} < \infty\}. \tag{1.2.8}$$

Since we need Littlewood-Paley operators in order to obtain the compactness result in Chapter 2, we find convenient to work with the equivalent norm:

$$\|f\|_{\dot{H}^s([-L,L]^2)} \sim \left(\sum_{j \in \mathscr{Z}} 2^{2sj} (\sum_{|k| \sim 2^j L} |a_k|^2) \right)^{1/2} \sim \left(\sum_{j \in \mathscr{Z}} 2^{2sj} \|P_j f\|^2_{L^2([-L,L]^2)} \right)^{1/2}. \tag{1.2.9}$$

Gagliardo-Nirenberg-Sobolev Inequality:

Assume $1 \leq p < n$. Then there exists a constant C depending only on p and n, such that

$$\|u\|_{L^{\frac{np}{n-p}}(R^n)} \leq C \|\nabla u\|_{L^p(R^n)} \tag{1.2.10}$$

Sobolev Embedding:

Let $W^{k,p}(R^n)$ denote the Sobolev space consisting of all real-valued functions on $\mathbb{R}^n$ whose first k weak derivatives are functions in L^p. Assume k is a non-negative integer and $1 \leq p \leq \infty$. If $k > l$ and $1 \leq p < q \leq \infty$ satisfying $(k-l)p < n$ and $\frac{1}{q} = \frac{1}{p} - \frac{k-l}{n}$, then

$$W^{k,p}(R^n) \subseteq W^{l,q}(R^n). \tag{1.2.11}$$

Log-convexity of L^p norms:

Assume $0 < p_0 < p_1 \leq \infty$ and $f \in L^{p_0}(X) \cap L^{p_1}(X)$, then $f \in L^p(X)$ for all $p_0 \leq p \leq p_1$ and we have

$$\|f\|_{L^{p_\theta}(X)} \leq \|f\|^{1-\theta}_{L^{p_0}(X)} \|f\|^{\theta}_{L^{p_1}(X)} \tag{1.2.12}$$

for all $0 \leq \theta \leq 1$, where the exponent p_θ is defined by $\dfrac{1}{p_\theta} = \dfrac{1-\theta}{p_0} + \dfrac{\theta}{p_1}$.

1.2.2 Strichartz Estimates

Strichartz estimates are space-time estimates on wave equations and dispersive equations, like Klein-Gordon, Kortewegde Vries, Boussinesq, nonlinear Schrödinger equations. These estimates are needed if one wants to perturb linear wave and dispersive equations to nonlinear equations, because they are very helpful in order to control the space-time norm of solutions to the linear problem in terms of the norm of the initial datum. Space-time norms are defined as follows:

$$\|f\|_{L^r_t L^p_x} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(t,x)|^p dx \right)^{r/p} dt \right)^{1/r}$$
$$\|f\|_{L^p_x L^r_t} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(t,x)|^r dt \right)^{p/r} dx \right)^{1/p}$$

Note that these two norms are not equivalent.

We present the Strichartz estimates for the Klein-Gordon equation since we will need them in order to get the proper estimates in Chapter 3.

Definition 1.2.3. *We say that a pair (q,r) is KG admissible (sharp KG admissible respectively), if $q, r \geq 2 : 2/q + d/r \leq d/2$ ($q, r \geq 2 : 2/q + d/r = d/2$ respectively) and $(q,r,d) \neq (2,\infty,2)$.*

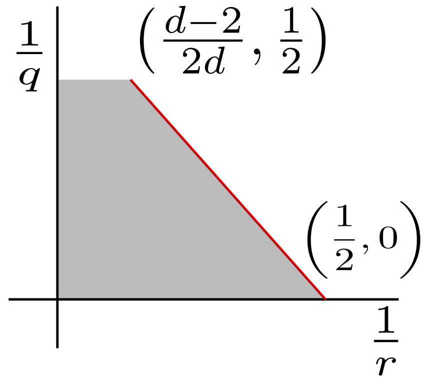

Figure 1.5: The region of KG admissible pairs.

Lemma 1.2.4. *(Lemma 2.1 in [39] with $\sigma = d, \lambda = (d+2)/2$). Let (q,r), (q_1,r_1) be both KG admissible pairs and $s \geq 0$. Then, for $\mathscr{H}_0 = -\Delta + 1$,*

$$\|e^{it\sqrt{\mathscr{H}_0}} f\|_{L_t^q W_x^{s,r}} \leq C\|f\|_{H^{s+\frac{d+2}{2}(\frac{1}{2}-\frac{1}{r})}}$$
$$\left\|\int_0^t \frac{\sin((t-s)\sqrt{\mathscr{H}_0})}{\sqrt{\mathscr{H}_0}} G(s,\cdot)ds\right\|_{L_t^q W_x^{s,r}} \leq C\|G\|_{L_t^{q_1'} W^{s-1+\frac{d+2}{2}(\frac{1}{r_1'}-\frac{1}{r}),r_1'}}$$

Chapter 2

Long-time behavior for the solutions of the Kuramoto-Sivashinsky Equation

2.1 Introduction and Previous Results

The Kuramoto-Sivashinsky equation,

$$\varphi_t = -\Delta^2\varphi - \Delta\varphi - \frac{1}{2}|\nabla\varphi|^2 \tag{2.1.1}$$

has been studied extensively in one space dimension. It is interesting mathematically because the linearization about the zero state has a large number of exponentially growing modes, whose growth corresponds to the development of nontrivial structures. The Kuramoto-Sivashinsky equation has become a canonical model for spatio-temporal chaos in $1+1$ dimensions. In [58], the instability of the travelling waves is a hint of the complexity of the dynamics of the equation if the domain is $\mathbb{R}$. When considered on a bounded domain with appropriate initial and boundary conditions, there are many important results, some of which we will explain here briefly. In this case it is convenient to work with the differentiated form of the equation, where $u = d\phi/dx$ and the equation becomes

$$u_t = -u_{xxxx} - u_{xx} - uu_x.$$

Using Lyapunov function approach, the authors of [40] gave the first long-time behavior result showing that $\limsup_{t\to\infty} \|u\|_2 \le CL^{5/2}$ for odd initial data. In [16], the exponent was improved to $\frac{8}{5}$ for any mean-zero initial data. Most recently, the authors of [7] improved the exponent from $\frac{8}{5}$ to $\frac{3}{2}$ for any mean-zero initial data. While all of the above results used the Lyapunov function framework, there are recent results in [25], [30] that do not use this approach. Our main goal is to treat the case of higher space dimensions, in particular in the case of two and three dimensional spaces. This problem is difficult and even the global regularity on unbounded domain and in the periodic case is still open. Some of the available results have restrictions on the domain or work on a modified equation. In the two dimensional case, defining $U = (u_1, u_2) = \nabla\varphi$, the differentiated KS equation becomes,

$$\begin{aligned} \partial_t u_1 + \triangle^2 u_1 + \triangle u_1 + u_1 \partial_x u_1 + u_2 \partial_x u_2 &= 0 \\ \partial_t u_2 + \triangle^2 u_2 + \triangle u_2 + u_1 \partial_y u_1 + u_2 \partial_y u_2 &= 0 \\ \partial_y u_1 &= \partial_x u_2 \end{aligned} \tag{2.1.2}$$

The authors of [49] showed the existence of a bounded local absorbing set and an attractor in thin two-dimensional domain, but with restricted initial data. Later in [38] this result was made sharper and more transparent. Molinet showed that there exist positive constants $C_0, K \geqslant 1$ such that for any $L_x \geqslant 2\pi$, if $0 < L_y < 2\pi$ satisfies

$$\left(1 - \left(\frac{L_y}{2\pi}\right)^2\right)^{-4/9} L_y \leqslant (K^2 C_0^3)^{-4/7} L_x^{-67/35} \tag{2.1.3}$$

then the solution satisfies

$$\limsup_{t\to\infty} \|u_1\|_2 \le K L_x^{8/5} L_y^{1/2}, \qquad \lim_{t\to\infty} \|u_2\|_2 = 0 \tag{2.1.4}$$

provided

$$\|u_{1_0}\|_2 \leqslant C_0^{-1} \left(1 - \left(\frac{L_y}{2\pi}\right)^2\right) L_x^{-1/4} L_y^{-7/4}, \quad \|u_{1_0}\|_2 \leqslant C_0^{-1} L_x^{-1/4} L_y^{1/4} \tag{2.1.5}$$

Using the results in [38] and [7] and assuming $L_y \le C L_x^{13/7}$, one gets a better bound

$$\limsup_{t\to\infty} \|\vec{u}\|_2 \le C L_x^{3/2} L_y^{1/2}.$$

If one is willing to modify the equation, as in [43], where the equation

$$u_t = -\Delta^2 u - u_{xx} - u u_x$$

with periodic boundary conditions is studied, then the existence of an attractor can be proved. In Section 2.2, we will also obtain results for a modified Kuramoto-Sivashinsky equation in space dimension 2.

One can also study the long-time behavior of some special solutions to the original equation. In our recent work, Section 2.4, inspired by the paper [6], we study the long-time behavior of the radially-symmetric solutions of the Kuramoto-Sivashinsky equation in space dimension 3. The authors of [6] worked on the radially symmetric solutions of

$$\varphi_t + \Delta^2 \varphi = |\nabla \varphi|^2 \tag{2.1.6}$$

in an annulus $\Omega = \{x \in \mathbb{R}^2$ such that $0 < r_0 < \|x\| < R_0\}$ with Neumann boundary conditions:

$$\frac{\partial \varphi}{\partial r} = \frac{\partial \triangle \varphi}{\partial r} = 0 \quad \text{on} \quad \Gamma_\infty \tag{2.1.7}$$

Assuming that the initial condition ϕ_0 is radially symmetric, they proved the existence of radially symmetric solution $\varphi(r,t)$ such that

$$\varphi \in L^\infty_{loc}([0,\infty); W^{1,2}(\Omega)) \cap L^2_{loc}([0,\infty); W^{3,2}(\Omega)) \tag{2.1.8}$$

Furthermore φ satisfies an exponentially growing with time bound on the norm of the solution as follows

$$\int_{r_0}^{R_0} \varphi^2(r,t)dr \leqslant e^t \frac{R_0}{r_0} \int_{r_0}^{R_0} \varphi^2(r,0)dr + (te^t+1)\frac{16c^2R_0^2}{r_0^2} e^{4ct} \left(\int_{r_0}^{R_0} \varphi_r^2(r,0)dr \right)^3$$

This global existence result is remarked there to be also true in space dimension 3 in a shell domain between two concentric spheres.

This chapter is organized as follows. In Section 2.2, work on the long-time behavior of the solutions of the modified Kuramoto-Sivashinsky equation (2.2.1) in two-dimensional space. In Section 2.3, we work on a simpler model, Burger-Sivashinsky equation (2.3.2), first in one-dimensional space, then in two-dimensional space. In Section 2.4, we study the three-dimensional radially-symmetric Kuramoto-Sivashinksy equation 2.4.2. We close the chapter by remarks and open questions.

Note that throughout this chapter, we use $\|\cdot\|_2$ to denote $\|\cdot\|_{L^2(\Omega)}$, where in Section 2.2, we take $\Omega = [-L,L]\times[-L,L]$. Then in the first part of Section 2.3, while working on one dimension, Ω will be $[-L,L]$, but in the second part, when we work on radially symmetric solutions, we will take Ω as $[0,R]$. Finally, we will take $\Omega = [r_0,R_0]$ in Section 2.4. While working on the polar coordinates, we will use the the following norm:

$\|\cdot\|_2 = \left(\int_0^R (\cdot)^2 dr\right)^{1/2}$, instead of the actual L^2 norm, that is, $\|\cdot\|_2 = \left(\int_0^R (\cdot)^2 r dr\right)^{1/2}$. We also use $\bar{H}^s(\Omega)$ to denote the Sobolev space obtained by taking the completion with respect to the norm $\|\cdot\|_{H^s}$ of smooth functions satisfying the given boundary conditions. As introduced in [40], we take a dot above any space to denote the subspace of the functions of zero mean, that is, $\phi \in \dot{\bar{H}}^s(\Omega)$ if and only if $\phi \in \bar{H}^s(\Omega)$ and $\int_\Omega \phi(x)dx = 0$.

2.2 Kuramoto-Sivashinsky type Equation in 2D

2.2.1 Formulation of the Problem

We consider the following variation of the Kuramoto-Sivashinsky equation in 2D:

$$u_t = -\Delta^2 u - \Delta u - uu_x - uu_y + g(\vec{x}) \tag{2.2.1}$$

$$u(0;x,y) = u_0(x,y) \tag{2.2.2}$$

$$u(t;x,y) = u(t;x+2L,y) = u(t;x,y+2L) \quad \forall (x,y) \in R^2, \ t \geq 0 \tag{2.2.3}$$

We assume that u is a mean-zero solution satisfying the boundary conditions:

$$\frac{d^k u}{dx^i dy^j}(x, \pm L) = \frac{d^k u}{dx^i dy^j}(\pm L, y) = \frac{d^k u}{dx^i dy^j}(L, L) \quad k = i + j = 0, 1, 2, 3 \tag{2.2.4}$$

where $(x,y) \in (-L,L) \times (-L,L)$. We also assume that the external force $g(x)$ is a mean zero function which is in $L^2([-L,L]^2)$.

2.2.2 Results

Theorem 2.2.1. *The dynamical system associated with the two-dimensional periodic Kuramoto-Sivashinsky type equation (2.2.1) with its boundary conditions is globally well-posed and possesses a global attractor in in* $L^2([-L,L]^2)$.

Proof. First we will prove the local well-posedness in the periodic case. In order to prove the global well-posedness in the periodic case, we will use the potential function ϕ_x introduced in [7], which gives the following result:

$$\limsup_{t\to\infty} \|u\|_{L^2([-L,L]^2)} \leqslant CL^2. \tag{2.2.5}$$

Then we will show that the solution u is point dissipative and asymptotically compact in the periodic case with the assumption of the initial solution u_0 being in the class of L^2. Then we will conclude the existence of a global attractor in $L^2([-L,L]^2)$. □

Global Well-Posedness for (2.2.1) in $L^2([-L,L]^2)$

In this section, we will first show the local well-posedness for (2.2.1) and then iterate the local well-posedness result to a global one by using the a priori bound for the solution. We will need some estimates throughout the proof which we collect in the following lemma.

Lemma 2.2.2. *Let* $f \in L^2([-L,L]^2)$ *then we have*

$$\|e^{-t\triangle^2} f\|_{\dot{H}^1([-L,L]^2)} \leqslant \frac{C}{t^{1/2}} \|f\|_{L^1([-L,L]^2)} \tag{2.2.6}$$

$$\|e^{-t\triangle^2} f\|_2 \leqslant C\|f\|_2 \tag{2.2.7}$$

$$\|e^{-t\triangle^2} f\|_{\dot{H}^2([-L,L]^2)} \leqslant \frac{C}{t^{1/2}} \|f\|_2 \tag{2.2.8}$$

Proof. Duality principle implies that showing the following inequality

$$\|\nabla e^{-t\triangle^2} f\|_{L^\infty([-L,L]^2)} \leqslant \frac{C}{t^{1/2}}\|f\|_2$$

is equivalent to showing (2.2.6). By (1.2.5) and (1.2.9), we have

$$\begin{aligned}
&\|\nabla e^{-t\triangle^2} f\|_{L^\infty([-L,L]^2)} \leqslant \|\nabla \widehat{e^{(-D^4)t}} f\|_{L^1([-L,L]^2)} = \frac{1}{2L}\sum_{n\in\mathbb{Z}^2}\frac{2\pi}{L} n e^{-t(\frac{2\pi}{L})^4 n^4}|a_n| \\
&\leqslant \sum_{|n|:n\lesssim t^{-1/4}L}\frac{\pi}{L^2} n e^{-t(\frac{2\pi}{L})^4 n^4}|a_n| + \sum_{m\in\mathbb{Z}}\sum_{|n|\sim 2^m t^{-1/4}L}\frac{\pi}{L^2} n e^{-2^{4m}(2\pi)^4}|a_n| \\
&\leqslant (\sum_n |a_n|^2)^{1/2}\frac{\pi}{t^{1/2}} + \sum_m \left((\pi 2^m t^{-1/4})(e^{-2^{4m}(2\pi)^4})(\sum_n |n|^2|a_n|^2)^{1/2} \right) \\
&\lesssim (\sum_n |a_n|^2)^{1/2}\frac{1}{t^{1/2}} + \sum_m(\sum_n |a_n|^2)^{1/2}(2^m t^{-1/4})(2^m t^{-1/4})(e^{-2^{4m}16\pi^4}) \\
&= \|f\|_2\frac{1}{t^{1/2}} + \|f\|_2\frac{1}{t^{1/2}}\sum_m 2^{2m}e^{-2^{4m}16\pi^4} \\
&\leqslant \frac{C}{t^{1/2}}\|f\|_2 \qquad\qquad \text{(since } \sum_m 2^{2m}e^{-2^{4m}16\pi^4} \text{ converges.)}
\end{aligned}$$

In order to prove (2.2.7), we will also use (1.2.5) and (1.2.9), that is,

$$\|e^{-t\triangle^2} f\|_2 = \|\widehat{e^{-t\triangle^2}} f\|_2 = \left(\sum_{n\in\mathbb{Z}^2} e^{-2t(\frac{2\pi}{L})^4 n^4}|a_n|^2\right)^{1/2} \leqslant C\|f\|_2$$

(2.2.8) also follows from (1.2.5) and (1.2.9):

$$\begin{aligned}
&\|e^{-t\triangle^2} f\|_{\dot{H}^2([-L,L]^2)} = \|\triangle e^{-t\triangle^2} f\|_2 = \left(\sum_{n\in\mathbb{Z}^2}(\frac{2\pi}{L})^4 n^4 e^{-2t(\frac{2\pi}{L})^4 n^4}|a_n|^2\right)^{1/2} \\
&= \frac{1}{t^{1/2}}(\sum_{n\in\mathbb{Z}^2} t(\frac{2\pi}{L})^4 n^4 e^{-2t(\frac{2\pi}{L})^4 n^4}|a_n|^2)^{1/2} \leqslant \frac{1}{t^{1/2}}\sup_m m e^{-2m^2}(\sum |a_n|^2)^{1/2} \\
&\leqslant \frac{C}{t^{1/2}}\|f\|_2 \quad (m = t(\frac{2\pi}{L})^4 n^4 \text{ and } \sup_m m e^{-2m^2} \text{ exists.)}
\end{aligned}$$

□

Local well-posedness for (2.2.1) in $L^2([-L,L]^2)$

We will show that $\Lambda : L^2([-L,L]^2) \to L^2([-L,L]^2)$ defined by

$$\Lambda u = e^{-t\triangle^2}u(0) + \int_0^t e^{-(t-s)\triangle^2}(-\triangle u - uu_x - uu_y + g)ds \tag{2.2.9}$$

has a fixed point in $X_{R,T} = \{u \in L^\infty((0,T), L^2([-L,L]^2)) : \sup \|u(t,\cdot)\|_2 \leqslant R\}$. By triangular inequality, we have

$$\|\Lambda u\|_2 \lesssim \|e^{-t\triangle^2}u(0)\|_2 + \int_0^t \|e^{-(t-s)\triangle^2}u\|_{\dot{H}^2([-L,L]^2)} + \|e^{-(t-s)\triangle^2}(u^2)\|_{\dot{H}^1([-L,L]^2)} + \|e^{-(t-s)\triangle^2}g\|_2 ds,$$

Applying Lemma (2.2.2), we get

$$\|\Lambda u\|_2 \leqslant C\|u(0)\|_2 + \int_0^t \frac{C_1}{(t-s)^{1/2}}\|u\|_2 + \frac{C_2}{(t-s)^{1/2}}\|u^2\|_{L^1([-L,L]^2)} + C_3\|g\|_2 ds$$

Since t is in $[0,T]$, we have

$$\|\Lambda u\|_2 \leqslant C\|u(0)\|_2 + 2C_1T^{1/2}\|u\|_2 + 2C_2T^{1/2}\|u\|_2^2 + C_3T\|g\|_2$$

If we choose R such that $C\|u(0)\|_2 \leqslant R/2$ and T such that

$$2C_1T^{1/2}\|u\|_2 + 2C_2T^{1/2}\|u\|_2^2 + C_3T\|g\|_2 \leqslant R/2,$$

we have $\|\Lambda u\|_2 \leqslant R$. Similarly one can show that Λ is a contraction.

Global well-posedness for (2.2.1) in $L^2([-L,L]^2)$

In order to prove global well-posedness in $L^2([-L,L]^2)$, it is enough to show that there is a time-independent bound for the solution

$$\sup_{0\leq t\leq T} \|u(t,\cdot)\|_2 \leqslant C.$$

In Section (2.2.2), we will see that C only depends on $\|g\|_2$ and L. We will show that for the local L^2 solution $u(t,.)$, there exists a Lyapunov function $\phi = \phi(x) \in H^2([-L,L]^2)$ such that one has the estimate

$$\|u(t,\cdot)\|_2 \leqslant \|\phi\|_2 + \sqrt{e^{-\frac{\lambda_0}{2}t}\|u(0)\|_2^2 + \frac{2P^2}{\lambda_0}} \tag{2.2.10}$$

for some constants $\lambda_0 > 0$ and P, and for every $0 < t < T$, where T is its life span. Assuming (2.2.10), let us prove that the solution is global. Fix $u_0 \in L^2([-L,L]^2)$, assume $\phi = \phi(x) \in \dot{H}^2([-L,L]^2)$ and define for every (sufficiently large) integer n

$$T_n = \sup\{t : L^2 \text{ solution is defined in}(0,t), \sup_{0<t_1<t} \|u(t,\cdot)_{L^2}\| < n\}$$

and define $T^* := \limsup_n T_n$. If $T^* = \infty$, there is nothing to prove, the solution is global. If $T^* < \infty$, then $\limsup_{t\to T^*} \|u(t,\cdot)\|_{L^2([-L,L]^2)} = \infty$. On the other hand, take a sequence $t_n \to T^*$, so that $\lim_{n\to\infty} \|u(t_n,\cdot)\|_2 = \infty$.
By (2.2.10), we have

$$\limsup_{n\to\infty} \|u(t_n,\cdot)\|_2 \leqslant \|\phi\|_2 + \sqrt{e^{-\frac{\lambda_0}{2}T^*}\|u(0)\|_2^2 + \frac{2P^2}{\lambda_0}}$$

where $P = C\Big(\|g\|_2, \|\phi\|_{H^2([-L,L]^2)}\Big)$ and λ_0 are positive constants. Thus, there exist positive C_1, C_2 and C_3 such that

$$\limsup_{n\to\infty} \|u(t_n,\cdot)\|_2 \leqslant C_1\|\phi\|_{H^2([-L,L]^2)} + C_2 \sup_{0\le s\le T^*} \|g(s,\cdot)\|_2 + C_3\|u_0\|_2 < \infty,$$

but this is a contradiction. Thus we can conclude that the solution is globally well-posed.

Existence of the Global Attractor

We will prove point dissipativeness and asymptotic compactness to conclude the existence of a global attractor by Theorem 1.1.21. In order to show point dissipativeness, we need to verify that for any $t_n \to \infty$, $B > 0$ and any sequence of initial data $\{u_n\} \in L^2([-L,L]^2)$ with $\sup_n \|u_n\|_2 \le B$, we have

$$\sup_{u_0\in L^2([-L,L]^2)} \limsup_{t\to\infty} \|S(t)u_0\|_2 \le C(g,L) \tag{2.2.11}$$

In order to obtain the compactness result, we will use Proposition (1.1.20), which requires to prove the following two:

$$\sup_n \|S(t_n)u_n)\|_2 \leqslant C(g,B,L) \tag{2.2.12}$$

$$\lim_N \limsup_n \|P_{>N}S(t_n)u_n\|_2 = 0 \text{ as } N\to\infty \tag{2.2.13}$$

Then we will conclude that the sequence $\{u_n(t_n,\cdot)\}$ is point dissipative by (2.2.11) and asymptotically compact by (2.2.12) and (2.2.13) in $L^2([-L,L]^2)$.

Point dissipativeness

In this section, our aim will be to prove (2.2.11). The lemmas and the theorem in this section will be based on Lyapunov approach.

Lemma 2.2.3. *Given* $u = u(t;x,y) \in L^2([-L,L]^2)$ *for all* $t \geq 0$*, and* $\phi(t;x,y) = \phi(x) \in L^2([-L,L])$ *satisfying the following inequality:*

$$\frac{d}{dt}\|u-\phi\|_2^2 \leqslant -\lambda_0\|u\|_2^2 + P^2 \tag{2.2.14}$$

for some constants $\lambda_0 > 0$ *and* P*, then* $B(O,R^{**})$*, the ball of radius* R^{**} *centered about the origin, is an attracting region, where the radius* R^{**} *is given by*

$$R^{**} = \sqrt{2\|\phi\|_2^2 + \frac{2P^2}{\lambda_0}} + \|\phi\|_2 \tag{2.2.15}$$

Proof. The proof for 1D is in [7], and it also works for 2D. By the parallelogram law $-\lambda_0\|u-\phi\|_2^2 \geqslant -2\lambda_0\|u\|_2^2 - 2\|\phi\|_2^2$, which gives

$$\frac{d}{dt}\|u-\phi\|_2^2 + \frac{\lambda_0}{2}\|u-\phi\|_2^2 \leqslant \lambda_0\|\phi\|_2^2 + P^2$$

If we multiply each side by $e^{\frac{\lambda_0}{2}t}$, we get $\frac{d}{dt}(e^{\frac{\lambda_0}{2}t}\|u-\phi\|_2^2) \leqslant e^{\frac{\lambda_0}{2}t}(\lambda_0\|\phi\|_2^2 + P^2)$. By integrating we get $\|u-\phi\|_2^2 \leqslant e^{-\frac{\lambda_0}{2}t}\|u(0)\|_2^2 + \frac{2}{\lambda_0}(\lambda_0\|\phi\|_2^2 + P^2)$. Thus we have the following result

$$\|(u(t,\cdot)\|_2 \leqslant \|\phi\|_2 + \sqrt{e^{-\frac{\lambda_0}{2}t}\|u(0)\|_2^2 + \frac{2P^2}{\lambda_0}} \tag{2.2.16}$$

It is clear that $B(\phi,R^*)$, the ball of radius R^* centered about ϕ, is exponentially attracting, with $R^{*2} = 2\|\phi\|_2^2 + (\frac{2P^2}{\lambda_0})$. The triangle inequality implies $B(\phi,R^*) \subset B(0,R^{**})$. This will guarantee the existence of an absorbing set. $\square$

Lemma 2.2.4. *For any* $\phi(t;x,y) = \phi(x) \in \bar{H}^2_{per}[-L,L]$ *and* $u(t;x,y)$ *solving (2.2.1) we have the inequality*

$$\frac{1}{4}\frac{d}{dt}\int\limits_{[-2L,2L]^2}(u-8\tilde{\phi})^2 d\tilde{x}d\tilde{y} \leqslant 4\int\limits_{[-2L,2L]^2}(\tilde{\nabla}u)^2-(\tilde{\triangle}u)^2+(\frac{1}{4}-\tilde{\phi}_{\tilde{x}})u^2 d\tilde{x}d\tilde{y} + \int\limits_{[-2L,2L]^2} 32(\tilde{\phi}_{\tilde{x}})^2+256(\tilde{\phi}_{\tilde{x}\tilde{x}})^2+16\tilde{\phi}^2+\frac{g^2}{2}d\tilde{x}d\tilde{y} \qquad (2.2.17)$$

Proof. Our proof will be similar to the one for the space dimension one given in [7]. A straightforward calculation gives

$$\frac{1}{2}\frac{d}{dt}\|u-\phi\|_2^2 = \int\limits_{[-L,L]^2} u_t(u-\phi)dxdy = \int\limits_{[-L,L]^2}(-\Delta^2 u-\Delta u-uu_x-uu_y+g)(u-\phi)dxdy.$$

After integration by parts and applying periodic boundary conditions this becomes

$$\frac{1}{2}\frac{d}{dt}\|u-\phi\|_2^2 = \int\limits_{[-L,L]^2}\left((\nabla u)^2-(\triangle u)^2-\phi_x u_x+\phi_{xx}\triangle u-\frac{1}{2}\phi_x u^2+gu-g\phi\right)dxdy.$$

Applying the Cauchy-Schwartz inequality in the form $\langle f,g\rangle \le p/2\langle f,f\rangle + 1/2p\langle g,g\rangle$ and making substitution $\phi = 8\tilde{\phi}, \tilde{x} = 2x, \tilde{y} = 2y$, we get (2.3.9). □

Note that (2.2.14) and (2.3.9) show that if we can construct $\phi \in \bar{H}^2_{per}[-L,L]^2$ such that the coercivity estimate

$$\langle u,Ku\rangle = \int\limits_{[-L,L]^2}\left((\triangle u)^2-(\nabla u)^2+(\phi_x-\frac{1}{4})u^2\right)dxdy \ge \lambda_0\|u\|_2^2 > 0 \qquad (2.2.18)$$

holds for some λ_0 independent of L, then we get an estimate of the form

$$\limsup_{t\to\infty}\|u\|_2 \leqslant R^{**} = \sqrt{c_1\|\phi\|_2^2 + c_2\|\phi_x\|_2^2 + c_3\|\phi_{xx}\|_2^2 + c_4\|g\|_2^2} + \|\phi\|_2 \qquad (2.2.19)$$

$$\leqslant C(\|\phi\|_{\bar{H}^2_{per}}, \|g\|_2) < \infty \qquad (2.2.20)$$

In order to prove (2.4.12), we will use the same potential function $\phi(x)$ as constructed in [7]. We will also use some results from [7] such as

$$\int_{-L}^{L} u_{xx}^2 - u_x^2 + (\phi_x - \frac{1}{2})u^2 dx \geqslant \frac{1}{4}\int_{-L}^{L} u_{xx}^2 + u^2 dx \qquad (2.2.21)$$

for all $u \in C^3[-L,L]$ with $u(0) = 0$. In fact (2.4.1) is not the exact inequality that is proved in [7]. However one can reconstruct the potential $\phi(x)$ so that (2.4.1) holds.

Lemma 2.2.5. *For $u(t;x,y)$ solving (2.2.1) we have the inequality*

$$\int_{[-L,L]^2} \left(u_{yy}^2 + 2u_{xx}u_{yy} - u_y^2 + \frac{1}{4}u^2\right) dxdy \geqslant 0 \qquad (2.2.22)$$

Proof. By rearranging the terms, then applying Plancherel's Theorem, integration by parts with the periodic boundary conditions we get

$$\begin{aligned}
&\int_{[-L,L]^2} \left(u_{yy}^2 + 2u_{xx}u_{yy} - u_y^2 + \frac{1}{4}u^2\right) dxdy \\
&= \int_{[-L,L]^2} \left(u_{yy}^2 + \frac{1}{4}u^2 - u_y^2\right) dxdy + \int_{[-L,L]^2} (2u_{xx}u_{yy})\, dxdy \\
&= \left(\frac{4\pi^2 n^2}{L^2} - \frac{1}{2}\right)^2 \|u\|_2^2 + 2\|u_{xy}\|_2^2 \geqslant 0
\end{aligned}$$

□

By using Lemma (2.2.5), we obtain the coercivity estimate (2.2.18):

$$\int\limits_{[-L,L]^2} \left((\triangle u)^2 - (\nabla u)^2 + (\phi_x - \frac{1}{4})u^2 \right) dxdy =$$

$$= \int\limits_{[-L,L]^2} \left(u_{xx}^2 + 2u_{xx}u_{yy} + u_{yy}^2 - u_x^2 - u_y^2 + (\phi_x - \frac{1}{4})u^2 \right) dxdy$$

$$= \int\limits_{[-L,L]^2} \left(u_{xx}^2 - u_x^2 + (\phi_x - \frac{1}{2})u^2 \right) dxdy + \int\limits_{[-L,L]^2} \left(u_{yy}^2 + 2u_{xx}u_{yy} - u_y^2 + \frac{1}{4}u^2 \right) dxdy$$

$$\geqslant \frac{1}{4} \int_{-L}^{L} \int_{-L}^{L} (u_{xx}^2 + u^2) dxdy \geqslant \frac{1}{4} \|u\|_2^2$$

Lemma 2.2.6. *The potential ϕ satisfies $\|\phi\|_{\bar{H}^2_{per}([-L,L]^2)} \leqslant CL^2$.*

Proof. From [7], since $\|\phi\|_{\bar{H}^2_{per}([-L,L])} \leqslant CL^{3/2}$, we get $\int_{-L}^{L} \|\phi\|^2_{H^2([-L,L])} dy \leqslant CL^4$ □

Remark 8. *The results claimed above are for odd initial data with the assumption of the external force to be odd. Since the theorem proved in [7] requires the assumption of u(0)=0, it is clear that it holds for any odd initial data. These results can be extended to arbitrary mean-zero initial data in the manner done by ([16]) or ([27]). So we can conclude that one can construct a potential function ϕ satisfying $\|\phi\|_{\bar{H}^2_{per}} \leqslant CL^2$ for any initial data.*

Proof of (2.2.11):

Fix the initial data u_0 with $\|u_0\|_2 \leq B$, and define $u(t,\cdot) = S(t)u_0$ we can conclude (2.2.11) because from (2.2.10) we have the following result.

$$\|S(t)u_0\|_2 \leqslant \|\phi\|_2 + \sqrt{e^{-\frac{\lambda_0}{2}t}\|u(0)\|_2^2 + \frac{2P^2}{\lambda_0}} \tag{2.2.23}$$

It follows that

$$\limsup_{t\to\infty} \|u(t,\cdot)\|_2 \leqslant R^{**} \leqslant C_1\|\phi\|_{H^2([-L,L]^2)} + C_2\|g\|_2 \leq C(g,L) \tag{2.2.24}$$

which is the point dissipativeness of $S(t)$.

Asymptotic compactness

Proof of (2.2.12): The uniform boundedness follows from (2.2.10) as well.

If $u(t_n.\cdot) = S(t_n)u_n$, where $u_n \in L^2([-L,L]^2)$ and $\|u_n\|_2 \leq B$ then from (2.2.10)

$$\|S(t_n)u_n\|_2 \leqslant \|\phi\|_2 + \sqrt{e^{-\frac{\lambda_0 t_n}{2}}\|u_n(0)\|_2^2 + \frac{2P^2}{\lambda_0}} \leqslant C(g,B,L)$$

Proof of (2.2.13): Define $u_k := P_k u$ and $g_k = P_k(g(x))$ where P_k is the Littlewood-Paley operator. If we apply P_k to (2.2.1) we get

$$P_k u_t = P_k(-\triangle^2 u) - P_k(\triangle u) - P_k(uu_x + uu_y) + P_k(g(x)).$$

We can rewrite this as

$$(u_k)_t = -\triangle^2 u_k - \triangle u_k - P_k(uu_x + uu_y) + g_k.$$

Multiplying each side by u_k, integrating over the domain $[-L,L]^2$ and applying integration by parts gives:

$$\partial_t \frac{1}{2}\|u_k(t,\cdot)\|_2^2 + \int (\triangle u_k)^2 - \int (\nabla u_k)^2 + \int P_k(uu_x + uu_y)u_k = \int g_k u_k.$$

Now from (1.2.9), we have

$$\int(\triangle u_k)^2 \geqslant C_1 2^{4k}\|u_k\|_2^2 \; and \; \int(\nabla u_k)^2 \leqslant C_2 2^{2k}\|u_k\|_2^2 \tag{2.2.25}$$

We can also find a bound for $\|g_k\|_2\|u_k\|_2$.

$$\|g_k\|_2\|u_k\|_2 \leqslant \frac{C_3}{2}2^{4k}\|u_k\|_2^2 + \frac{1}{2C_3}2^{-4k}\|g_k\|_2^2 \tag{2.2.26}$$

To find a bound for $|\int P_k(uu_x+uu_y)u_k dx|$, we define $v = \frac{1}{2}[\partial_x(u^2)+\partial_y(u^2)]$. If we apply integration by parts, we get

$$\int P_k(v)u_k dx = -\frac{1}{2}\int(u^2)_k[\partial_x u_k + \partial_y u_k]dx \leqslant C\|P_k(u^2)\|_2\|\nabla u_k\|_2. \tag{2.2.27}$$

By using Lemma (1.2.4), and (2.2.25) and (2.2.27), we can say that

$$|\int P_k(uu_x+uu_y)u_k| \leqslant C\|P_k(u^2)\|_2\|\nabla u_k\|_2 \lesssim 2^k\|u^2\|_{L_1}2^k\|u_k\|_2 \tag{2.2.28}$$

Finally using Cauchy- Schwartz inequality, we get

$$|\int P_k(uu_x+uu_y)u_k| \leqslant \varepsilon 2^{5k}\|u_k\|_2^2 + \frac{\|u\|_2^4}{\varepsilon 2^k} \tag{2.2.29}$$

Thus from (2.2.26), (2.2.27) and, (2.2.28),

$$\begin{aligned}&\partial_t \frac{1}{2}\|u_k(t,\cdot)\|_2^2 + C_1 2^{4k}\|u_k\|_2^2 \\ &\leqslant C_2 2^{2k}\|u_k\|_2^2 + \varepsilon 2^{5k}\|u_k\|_2^2 + \frac{\|u\|_2^4}{\varepsilon 2^k} + \frac{C_3}{2}2^{4k}\|u_k\|_2^2 + \frac{1}{2C_3}2^{-4k}\|g_k\|_2^2\end{aligned}$$

Now defining $I_k(t)$ by $I_k(t) = \|u_k(t,\cdot)\|_2^2$, choosing $\varepsilon = \frac{C_1}{2}2^{-k}$, we obtain the following inequality:

$$\partial_t I_k(t) + C_4 2^{4k} I_k(t) \leqslant \frac{2}{C_1}\|u\|_2^4 + \frac{1}{2C_3}2^{-4k}\|g_k\|_2^2 \tag{2.2.30}$$

Since $\limsup_{t\to\infty} \|u\|_2 = C(g,L)$ from (2.2.19) and using Gronwall inequality, we get

$$I_k(t) \leqslant I_k(0)e^{C_4 2^{-4k}t} + \frac{2^{-4k}}{C_4}\left(\frac{2}{C_1}C(g,L)^4 + \frac{1}{2C_3}2^{-4k}\|g_k\|_2^2\right)$$

Since $g \in L^2([-L,L]^2)$, we have that $(\frac{2}{C_1}C(g,L)^4 + \frac{1}{2C_3}2^{-4k}\|g_k\|_2^2)$ is bounded. Thus we get

$$\|P_{>N}u_n\|_2^2 \approxeq \sum_{k:2^kL\geq N}\|u_k\|_2^2 = \sum_{k:2^kL\geq N} I_k \leqslant \sum_{k:2^kL\geq N} I_k(0)e^{C_4 2^{-4k}t} + \frac{2^{-4k}}{C_4}\tilde{C}(g,L)$$

which tends to 0 as $N \to \infty$.

2.3 Burgers-Sivashinsky Equation in 1D and 2D

2.3.1 Formulation of the Problem

Burgers-Sivashinsky Equation,

$$\phi_t = \Delta\phi + \phi - |\nabla\phi|^2 \tag{2.3.1}$$

is often used as a model problem for fluid dynamical systems. In the first part, we will work on a bounded domain $[-L,L]$ in the case of one space dimension. When we differentiate (2.3.1) and define $u = \phi_x$, then we get

$$u_t = u_{xx} + u - 2uu_x \tag{2.3.2}$$

We will work with this equation assuming first that the initial data is odd, then we will generalize the result to general initial data.We will have the following boundary conditions and will assume that u is the mean-zero solution, i.e.,

$$\frac{d^j u}{dx^j}(L) = \frac{d^j u}{dx^j}(-L), \quad j = 0,1 \quad \int_{-L}^{L} u_0(x)dx = 0, \quad x \in (-L,L) \tag{2.3.3}$$

In the second part, we will study the long-time behavior of the radially symmetric solutions of Burger-Sivashinksy equation in a two-dimensional domain $\Omega = \{x \in \mathbb{R}^2,\ 0 \leq \|x\| < R_0\}$. Changing the rectangular coordinates to polar coordinates and defining the radially symmetric solution as $v(r)$, (2.3.1) becomes

$$v_t - \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r}\right) - v + \left(\frac{\partial v}{\partial r}\right)^2 = 0 \tag{2.3.4}$$

And we know that v is even due to its construction. Differentiating (2.3.4), and defining $u = \frac{\partial v}{\partial r}$ we have

$$u_t - \frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}u - u - 2u\frac{\partial u}{\partial r} = 0 \tag{2.3.5}$$

We will have boundary conditions with the assumption of $u(r)$ being the mean-zero solution, i.e.,

$$\frac{d^j u}{dr^j}(R) = \frac{d^j u}{dr^j}(0), \quad j = 0,1 \quad \int_{0}^{R} u_0(r)dr = 0, \quad r \in (0,R) \tag{2.3.6}$$

We will work on the following equation, and the space will be $L^2[0,R]$ and we know that u is an odd function, due to its construction.

2.3.2 Results

Theorem 2.3.1. *For the solution $u(t;x)$ satisfying (2.3.2) and (2.3.3), we have*

$$\limsup_{t\to\infty} \|u\|_{L^2[-L,L]} \leqslant C_1 L^{3/2} \tag{2.3.7}$$

Theorem 2.3.2. *For the radially symmetric solution $u(t;r)$ satisfying (2.3.5)-(2.3.6), we have*

$$\limsup_{t\to\infty} \|u\|_{L^2[0,R]} \leqslant C_2 R^{3/2} \tag{2.3.8}$$

where C_1 and C_2 are independent of L and R.

Proof of Theorem 2.3.1:

Our results for BS equations are also based on Lyapunov Function approach. We will present some lemmas which will help us to construct a potential function ϕ as in Lemma 2.2.3.

Lemma 2.3.3. *For any $\phi \in \dot{H}^1[-L,L]$ and $u(t;x)$ a solution of (2.3.2) and satisfying (2.3.3), we have the inequality*

$$\frac{d}{dt}\int_{-L}^{L}(u-\phi)^2 dx \leqslant 2\int_{-L}^{L}\left((2-\phi_x)u^2 - \frac{3}{4}u_x^2\right)dx + \int_{-L}^{L}\left(\frac{1}{2}\phi^2 + 2\phi_x^2\right)dx \tag{2.3.9}$$

Proof. A straightforward calculation gives

$$\frac{1}{2}\frac{d}{dt}\|u-\phi\|^2 = \int_{-L}^{L} u_t(u-\phi)dx = \int_{-L}^{L}(u_{xx}+u-2uu_x)(u-\phi)dx \tag{2.3.10}$$

After integrating by parts and applying periodic boundary conditions, we have

$$\frac{1}{2}\frac{d}{dt}\|u-\phi\|^2 = \int_{-L}^{L}\left(-u_x^2 + u^2 + u_x\phi_x - u\phi - u^2\phi_x\right)dx \tag{2.3.11}$$

Applying the Cauchy-Schwartz inequality: $\langle f,g\rangle \leq p/2\langle f,f\rangle + 1/2p\langle g,g\rangle$ gives

$$\frac{1}{2}\frac{d}{dt}\|u-\phi\|^2 \leqslant \int_{-L}^{L}\left(\left(1+\frac{1}{2p}-\phi_x\right)u^2+\left(-1+\frac{1}{2q}\right)u_x^2+\frac{p}{2}\phi^2+\frac{q}{2}\phi_x^2\right)dx \tag{2.3.12}$$

Taking $p=\frac{1}{2}$ and $q=2$ we get

$$\frac{1}{2}\frac{d}{dt}\|u-\phi\|^2 \leqslant \int_{-L}^{L}\left((2-\phi_x)u^2-\frac{3}{4}u_x^2\right)dx+\int_{-L}^{L}\left(\frac{1}{4}\phi^2+\phi_x^2\right)dx \tag{2.3.13}$$

□

Remark 9. *Our aim is to construct $\phi \in \dot{H}^1[-L,L]$ such that the coercivity estimate*

$$\langle u,Ku\rangle = 2\int_{-L}^{L}\left(\frac{3}{4}u_x^2+(\phi_x-2)u^2\right)dx \geq \lambda_0\|u\|_2^2 > 0 \tag{2.3.14}$$

holds for some λ_0 independent of L, then one gets an estimate of the form

$$\limsup_{t\to\infty}\|u\|_2 \leqslant R^{**} = \sqrt{c_1\|\phi\|_2^2+c_2\|\phi_x\|_2^2}+\|\phi\|_2 \leqslant C\|\phi\|_{\dot{H}^1} < \infty \tag{2.3.15}$$

In the following section, defining $\lambda = \frac{\lambda_0}{2}$, we will construct $\phi \in \dot{H}^1[-L,L]$ such that

$$\int_{-L}^{L}\left(\frac{3}{4}u_x^2+(\phi_x-2)u^2\right)dx \geqslant \frac{\lambda_0}{2}\|u\|^2 = \lambda\|u\|^2 \tag{2.3.16}$$

Construction of ϕ

We construct the mean zero function in this form: $\phi_x := q(x)-\langle q\rangle$. Here $\langle.\rangle$ denotes the mean value on $[-L,L]$:

$$\langle q\rangle = \frac{1}{2L}\int_{-L}^{L}q(x)dx \tag{2.3.17}$$

We define an even function $\varphi \in C_0^\infty$ such that supp $\varphi \in [-4,-\frac{1}{4}]\bigcup[\frac{1}{4},4]$ satisfying $\varphi(x)=1$ if $|x| \in [\frac{1}{2},2]$ and $\int_{-L}^{L}\varphi(x)dx \leq 1$. We define the following potential function q_M as

$$q_M = -M^2\varphi(Mx) \tag{2.3.18}$$

Note that that $\langle q_M(x)\rangle \leqslant -\dfrac{M}{2L}$ since

$$\int_{-L}^{L} q_M(x)dx = \int_{-L}^{L} -M^2\varphi(Mx)dx = -M\int_{-L}^{L}\varphi(y)dy \tag{2.3.19}$$

Our aim is to choose q such that it satisfies

$$\checkmark\, \langle q\rangle \leqslant -2-\lambda \tag{2.3.20}$$

$$\checkmark \int_{-L}^{L}\frac{3}{4}u_x^2 + q(x)u^2dx \geqslant 0 \tag{2.3.21}$$

If we choose $q=\dfrac{q_M}{100}$, then (2.3.20) implies that there exists M such that

$$M \geqslant 200(2+\lambda)L$$

So for such M, we have to show (2.4.12). By Hardy's Inequality with the assumption of $u(0)=0$, we get

$$\int_{-L}^{L}u_x^2dx \geqslant \frac{1}{4}\int_{-L}^{L}\frac{u^2}{x^2}dx \tag{2.3.22}$$

Using (2.3.22), we get

$$\begin{aligned}\int_{-L}^{L}\left(\frac{3}{4}u_x^2+q(x)u^2\right)dx &\geqslant \int_{-L}^{L}\left(\frac{3}{16}\frac{u^2}{x^2}+q(x)u^2\right)dx\\ &= \int_{-L}^{L}\left(\frac{3}{16x^2}-\frac{M^2}{100}\varphi(Mx)\right)u^2dx\end{aligned}$$

Observe that if $|x|$ is not in $[\frac{1}{4M}, 4M]$, then (2.4.12) automatically satisfies. Otherwise

$$\int_{-L}^{L}\left(\frac{3}{4}u_x^2+q(x)u^2\right)dx \geqslant \int_{-L}^{L} M^2\left(\frac{3}{256}-\frac{1}{100}\varphi(Mx)\right)u^2dx \geqslant 0 \qquad (2.3.23)$$

Thus we can conclude that

$$\int_{-L}^{L}\left(\frac{3}{4}u_x^2+(\phi_x-2-\lambda_0)u^2\right)dx \geqslant \int_{-L}^{L}\left(\frac{3}{4}u_x^2+q(x)u^2\right)dx \geqslant 0 \qquad (2.3.24)$$

Note that in order to minimize $\|\phi\|_{H^1}$, we will choose $M = O(L)$ since $M \geqslant 200(2+\lambda)L$. Thus an optimal potential function is in the form $q(x) = -L^2\varphi(Lx)$.

Lemma 2.3.4. *The potential ϕ satisfies $\|\phi\|_{H^1} \leqslant CL^{3/2}$.*

Proof. From the definition $\phi_x = q(x) - \langle q\rangle = -L^2\varphi(Lx) - \langle q\rangle$

$$\|\phi_x\|_{L^2}^2 = \int_{-L}^{L} q^2(x)dx - \int_{-L}^{L}\langle q\rangle^2 dx = \int_{-L}^{L} L^4\varphi^2(Lx)dx - \int_{-L}^{L}\langle q\rangle^2 dx$$

After rescaling we get $\|\phi_x\|_2 = O(L^{3/2})$. From the definition of ϕ_x, we have

$$\phi(x) = \int_0^x \phi_s ds = \int_0^x (q(s)-\langle q\rangle)\,ds = \int_0^x \left(-L^2\varphi(Ls)-\langle q\rangle\right)ds \qquad (2.3.25)$$

After the substitution $y = Ls$, we get $\phi(x) = -L\int_0^{Lx}\varphi dy - \langle q\rangle x$
Since φ is bounded and supported on $[-4,-\frac{1}{4}]\bigcup[\frac{1}{4},4]$, we conclude that $|\phi(x)| \leqslant CL + \langle q\rangle L = O(L)$, which implies $\|\phi\|_2^2 \leqslant (2L)\|\phi\|_{L^\infty}^2 = O(L^3)$. □

Extension to arbitrary initial data

In order to extend our claims to any arbitrary initial data, we will use the idea introduced in [16]. We will define the potential function as $\phi_b(x) = \phi(x+b)$ where

$b = b(t)$. The point b will be chosen in such a way that the gradient of the distance function is parallel to a line connecting u to the closest point on the curve, union of such comparison functions ϕ_b. We will also define $b(t)$ such that $b(0) = 0$ and satisfying $b'(t) = \frac{1}{2L(\lambda+2)} \int_{-L}^{L} u\phi_b' dx$. Note that we have

$$-\int_{-L}^{L} u\phi_b' dx = \frac{1}{2}\partial_b \int_{-L}^{L} (u-\phi_b)^2 dx \quad \text{and} \quad \int_{-L}^{L} u\phi_b' dx = \int_{-L}^{L} (u-\phi)\phi_b' dx \tag{2.3.26}$$

Lemma 2.3.5.

$$\frac{1}{2}\frac{d}{dt}\|u-\phi_b\|_2^2 + \frac{1}{2L(\lambda+2)}\left(\int_{-L}^{L} u\phi_b' dx\right)^2 \leqslant \int_{-L}^{L}\left((2-\phi_b')u^2 - \frac{3}{4}u_x^2 + \phi^2 + \frac{1}{4}\phi_b^2\right) dx \tag{2.3.27}$$

Proof.

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u-\phi_b\|_2^2 &= \frac{1}{2}\frac{d}{dt}\int_{-L}^{L}(u-\phi_b)^2 dx = \int_{-L}^{L}(u-\phi_b)u_t dx - \int_{-L}^{L}(u-\phi_b)\frac{\partial}{\partial b}\phi_b b'(t) dx \\ &= \int_{-L}^{L}(u-\phi_b)u_t dx - b'(t)\int_{-L}^{L} u\phi_b' dx\end{aligned}$$

Thus we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u-\phi_b\|_2^2 + \frac{1}{2L(\lambda+2)}\left(\int_{-L}^{L} u\phi_b' dx\right)^2 &= \int_{-L}^{L}(u-\phi_b)(u_{xx}+u-2uu_x) dx \\ &\quad -b'(t)\int_{-L}^{L} u\phi_b' dx + \frac{1}{2L(\lambda+2)}\left(\int_{-L}^{L} u\phi_b' dx\right)^2\end{aligned}$$

Since we choose $b'(t) = \frac{1}{2L(\lambda+2)}\left(\int_{-L}^{L} u\phi_b' dx\right)$, then the two last terms cancel and we get

$$\frac{1}{2}\frac{d}{dt}\|u-\phi_b\|_2^2 + \frac{1}{2L(\lambda+2)}\left(\int_{-L}^{L} u\phi_b' dx\right)^2 = \int_{-L}^{L}(u-\phi_b)(u_{xx}+u-2uu_x) dx$$

As in the case of odd initial data, because of (2.3.9), the claim follows. □

Construction of ϕ_b

Similar to the idea in [16], our aim will be to construct the potential function ϕ_b such that we have

$$\int_{-L}^{L}\left(\frac{3}{4}u_x^2+(\phi_b'-2)u^2\right)dx+\frac{1}{2L(\lambda+2)}\left(\int_{-L}^{L}u\phi_b'dx\right)^2 \geqslant \frac{\lambda_0}{2}\|u\|_2^2=\lambda\|u\|_2^2 \quad (2.3.28)$$

which is equivalent to

$$\int_{-L}^{L}\left(\frac{3}{4}u_x^2+(\phi_b'-\lambda-2)u^2\right)dx+\frac{1}{2L(\lambda+2)}\left(\int_{-L}^{L}u\phi_b'dx\right)^2 \geqslant 0 \qquad (2.3.29)$$

It suffices to prove the claim for $b=0$ since (2.3.29) is invariant under translation. So we will write $\phi_{b=0}=\phi$ and $u=u(0)+u_a+u_s$ where u_a is the antisymmetric part and u_s is the symmetric part of u.
First, observe that since $\int_{-L}^{L}u_a'u_s'dx=0$, then $\int_{-L}^{L}u_x^2dx=\int_{-L}^{L}u_a'^2+u_s'^2dx$. Second, we can partition the integral as in the following form:

$$\begin{aligned}\int_{-L}^{L}(\phi_x-\lambda-2)u^2dx &= \int_{-L}^{L}(\phi_x-\lambda-2)(u(0)+u_s+u_a)^2dx\\ &= \int_{-L}^{L}(\phi_x-\lambda-2)(u_s^2+u_a^2)dx+\int_{-L}^{L}(\phi_x-\lambda-2)u(0)^2dx\\ &\quad +2\int_{-L}^{L}(\phi_x-\lambda-2)u(0)u_sdx+2\int_{-L}^{L}(\phi_x-\lambda-2)u(0)u_adx\\ &\quad +2\int_{-L}^{L}(\phi_x-\lambda-2)u_au_sdx\end{aligned}$$

Using the fact that ϕ_x is even and $\int_{-L}^{L} u_s dx = -2Lu(0)$, we get $\int_{-L}^{L} (\phi_x - \lambda - 2)u^2 dx =$

$$= \int_{-L}^{L} (\phi_x - \lambda - 2)(u_s^2 + u_a^2)dx + 2L(\lambda+2)u(0)^2 + 2u(0)\int_{-L}^{L} \phi_x u dx$$

Thus we get $\int_{-L}^{L} \left(\frac{3}{4}u_x^2 + (\phi_x - \lambda - 2)u^2\right) dx + \frac{1}{2L(\lambda+2)}\left(\int_{-L}^{L} u\phi_x dx\right)^2 =$

$$\begin{aligned} &= \int_{-L}^{L} \left(\frac{3}{4}u_a'^2 + (\phi_x - \lambda - 2)u_a^2\right) dx + \int_{-L}^{L} \left(\frac{3}{4}u_s'^2 + (\phi_x - \lambda - 2)u_s^2\right) dx \\ &+ 2L(\lambda+2)u(0)^2 + 2u(0)\int_{-L}^{L} \phi_x u dx + \frac{1}{2L(\lambda+2)}\left(\int_{-L}^{L} u\phi_x dx\right)^2 \end{aligned}$$

Since $u_s(0) = 0$ and $u_a(0) = 0$, we can construct ϕ such that the coercivity estimates for u_s and u_a hold. Thus we can have the following inequality which proves the coercivity estimate.

$$\begin{aligned} &\int_{-L}^{L} \frac{3}{4}u_x^2(\phi_x - \lambda - 2)u^2 dx + \frac{1}{2L(\lambda+2)}\left(\int_{-L}^{L} u\phi_x dx\right)^2 \\ &\quad = \int_{-L}^{L} \left(\frac{3}{4}u_a'^2 + (\phi_x - \lambda - 2)u_a^2\right) dx + \int_{-L}^{L} \left(\frac{3}{4}u_s'^2 + (\phi_x - \lambda - 2)u_s^2\right) dx \\ &\quad + \left(\frac{1}{\sqrt{2L(\lambda+2)}}\int_{-L}^{L} u\phi_x dx + \sqrt{2L(\lambda+2)}u(0)\right)^2 \geqslant 0 \end{aligned}$$

This completes the proof of Theorem 2.3.1, because Lemma 2.3.4 and 2.3.15 imply that

$$\limsup_{t\to\infty} \|u\|_2 \leqslant C_1 L^{3/2}$$

Proof of Theorem 2.3.2:

Lemma 2.3.6. *For any $\phi \in \dot{H}^1[0,R]$ and $u(r,t)$ solving (2.3.5),we have the inequality*

$$\frac{1}{2}\frac{d}{dt}\|u-\phi\|^2 \leqslant \int_0^R \left(-\frac{3}{4}u_r^2 + (2-\phi_r)u^2\right)dr + \int_0^R \left(126\phi_r^2 + \frac{1}{4}\phi^2\right)dr \quad (2.3.30)$$

Proof. A straightforward calculation gives that

$$\frac{1}{2}\frac{d}{dt}\|u-\phi\|^2 = \int_0^R u_t(u-\phi)dr = \int_0^R (u_{rr} + \frac{1}{r}u_r - \frac{1}{r^2}u + u - 2uu_r)(u-\phi)dr \quad (2.3.31)$$

After integrating by parts and applying periodic boundary conditions, we have

$$\frac{1}{2}\frac{d}{dt}\|u-\phi\|^2 = \int_0^R \left(-u_r^2 + \frac{1}{r}uu_r - \frac{1}{r^2}u^2 + u^2 + u_r\phi_r - \frac{1}{r}u_r\phi + \frac{1}{r^2}u\phi - u\phi - \phi_r u^2\right)dr \quad (2.3.32)$$

We will estimate the terms on the right hand side. Applying the Cauchy-Schwartz inequality, we have

$$\int_0^R \frac{1}{r}uu_r dr \leqslant \int_0^R \frac{2u^2}{r^2}dr + \int_0^R \frac{1}{8}u_r^2 dr \quad \text{and} \quad \int_0^R u_r\phi_r dr \leqslant \int_0^R \left(\frac{1}{2p}u_r^2 + \frac{p}{2}\phi_r^2\right)dr \quad (2.3.33)$$

and

$$\int_0^R u\phi dr \leqslant \int_0^R \left(\frac{1}{2s}u^2 + \frac{s}{2}\phi^2\right)dr \quad (2.3.34)$$

Using Hardy's inequality, and assuming that $\phi(0) = 0$. (We will construct ϕ so that it will satisfiy $\phi(0) = 0$.) We have

$$\int_0^R \frac{1}{r}\phi u_r dr \leqslant \left(\int_0^R (\frac{\phi}{r})^2 dr\right)^{1/2} \left(\int_0^R u_r^2 dr\right)^{1/2} \leqslant 2\|\phi_r\|_2\|u_r\|_2 \leqslant \int_0^R \left(q\phi_r^2 + \frac{1}{q}u_r^2\right)dr \quad (2.3.35)$$

In order to find an estimate for $\int_0^R \frac{1}{r^2} u\phi dr$, we will first use Cauchy-Schwartz inequality then Hardy's Inequality. (since u(0)=0)

$$\int_0^R \frac{1}{r^2} u\phi dr \leqslant \int_0^R \left(\frac{1}{2m}\left(\frac{u}{r}\right)^2 + \frac{m}{2}\left(\frac{\phi}{r}\right)^2 \right) dr \leqslant \int_0^R \left(\frac{2}{m}u_r^2 + 2m\phi_r^2 \right) dr \quad (2.3.36)$$

When we add all, the right hand side of (2.3.32) becomes

$$\leqslant \int_0^R \left(\left(-\frac{7}{8} + \frac{1}{2p} + \frac{1}{q} + \frac{2}{m} \right) u_r^2 + \left(1 + \frac{1}{2s} - \phi_r \right) u^2 + \left(\frac{p}{2} + q + 2m \right) \phi_r^2 + \frac{s}{2}\phi^2 \right) dr \quad (2.3.37)$$

If we take $p = 12$, $q = 24$, $m = 48$, $s = 1/2$, then we get (2.3.30). □

Remark 10. *The preceding lemmas show that if we can construct $\phi \in \dot{H}^1[0,R]$ such that the coercivity estimate*

$$\langle u, Ku \rangle = 2\int_0^R \left(\frac{3}{4}u_r^2 + (\phi_r - 2)u^2 \right) dr \geqslant \lambda_0 \|u\|^2 \quad (2.3.38)$$

holds for some λ_0 independent of L, then we get

$$\limsup_{t\to\infty} \|u\|_2 \leqslant c\|\phi\|_{H^1[0,R]} \quad (2.3.39)$$

One can observe that the coercivity equation for radial case (2.3.38) is same as the coercivity equation (2.3.16) for one dimensional case. Thus the function ϕ_r exists.

Also note that for dimension n, the only difference as compared to dimension 2, will be

$$\Delta\varphi = \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} \right) \varphi. \quad (2.3.40)$$

The steps in the proof for dimension n will also be similar. The coefficients of the (2.3.6) will change, however these changes will not affect the exponent of the bound.

2.4 Radially symmetric Kuramoto-Sivashinsky Equation in 3D

2.4.1 Formulation of the Problem

Inspired by the paper [6], our goal is to show that $\limsup_{t\to\infty} \|u\|_2 \leq C_{r_0}(R_0 - r_0)^{3/2}$ where u is the radially symmetric solution of the differentiated Kuramoto-Sivashinsky equation (2.1.1) in a shell domain $\Omega = \{x \in \mathbb{R}^n \text{ such that } 0 < r_0 < \|x\| < R_0\}$. We work with the differentiated Kuramoto-Sivashinsky equation in Ω with boundary conditions similar to [6]

$$u = \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r} + \frac{2}{r}u\right) = 0 \text{ at } r = r_0, \text{ and } r = R_0. \tag{2.4.1}$$

We assume that the initial condition u_0 is a radial function $u_0(x) = u_0(r)$, differentiate (2.1.1) and introduce a new variable $u = \dfrac{d\varphi}{dr}$. Thus we get the reduced radial system:

$$u_t + u_{rrrr} + \frac{4}{r}u_{rrr} + \left(1 - \frac{4}{r^2}\right)u_{rr} + \frac{2}{r}u_r - \frac{2}{r^2}u + uu_r = 0 \tag{2.4.2}$$

$$u = u_{rr} + \frac{2}{r}u_r = 0 \qquad \text{for} \quad r = r_0, r = R_0 \tag{2.4.3}$$

$$u(x,0) = u_0(x) = u_0(|x|) \qquad \text{in } \Omega \tag{2.4.4}$$

We will use the following notations. If Ω is a smooth, bounded domain in $\mathbb{R}^n$, then $\mathcal{Q}_t = \Omega \times (0,t)$, $\Gamma_t = \partial\Omega \times (0,t)$, $\Omega_t = \Omega \times \{t\}$, $|\nabla\varphi| = (\nabla\varphi, \nabla\varphi)^{1/2}$ and $(\cdot,\cdot)$ is the usual Euclidian dot product in $\mathbb{R}^n$. Changing the coordinates from rectangular to polar and assuming that φ is radially symmetric, we get the usual formulas

$$|\nabla\varphi|^2 = \left(\frac{\partial\varphi}{\partial r}\right)^2, \qquad \Delta\varphi = \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right)\varphi, \tag{2.4.5}$$

$$\Delta^2\varphi = \left(\frac{\partial^4}{\partial r^4} + \frac{2(n-1)}{r}\frac{\partial^3}{\partial r^3} + \frac{(n-1)(n-3)}{r^2}\frac{\partial^2}{\partial r^2} - \frac{(n-1)(n-3)}{r^3}\frac{\partial}{\partial r}\right)\varphi \quad (2.4.6)$$

2.4.2 Results

Theorem 2.4.1. *Consider the Kuramoto-Sivashinsky equation (2.4.2) with* $0 < r_0 < R_0 < \infty$*, subject to the boundary and initial conditions given by (2.4.3), (2.4.4). Assume also* $(R_0 - r_0) \geq \alpha(1+\frac{1}{r_0^2})^{-1/2}$ *for some* $\alpha > 0$*. Then, there is constant* $C = C_\alpha$*, so that*

$$\limsup_{t\to\infty} \|u(t)\|_{L^2[r_0,R_0]} \leqslant C_\alpha (R_0 - r_0)^{3/2}\left(1+\frac{1}{r_0^2}\right)^3. \quad (2.4.7)$$

For the related problem (2.1.6) *with* $(R_0 - r_0) \geq \alpha(1+\frac{1}{r_0^2})^{-1/2}$ *subject to the radial initial conditions and the boundary conditions [6], we also have*

$$\limsup_{t\to\infty} \|\partial_r\varphi(t)\|_{L^2[r_0,R_0]} \leqslant C_\alpha (R_0 - r_0)^{3/2}\left(1+\frac{1}{r_0^2}\right)^3. \quad (2.4.8)$$

If $(R_0 - r_0) \leq (1+\frac{1}{r_0^2})^{-1/2}$*, then*

$$\limsup_{t\to\infty} \|u(t)\|_{L^2[r_0,R_0]} \leq C\frac{(1+\frac{1}{r_0^2})^2}{\sqrt{R_0 - r_0}}.$$

and similar estimate holds for the derivative of the solution φ_r *of* (2.1.6).

Proof of Theorem 2.4.1:

The proof of the theorem is based on the Lyapunov function approach, which is mainly the Lemma (2.2.3). As in the previous sections, we will construct a potential function $\phi \in L^2([r_0,R_0])$ such that if we can get the following inequality

$$\frac{d}{dt}\|u-\phi\|_2^2 \leqslant -\lambda_0\|u\|_2^2 + P^2 \quad (2.4.9)$$

for some constants $\lambda_0 > 0$ and P, then we will conclude the existence of an attracting region, ball of radius R^{**} centered about the origin,

$$R^{**} = \sqrt{2\|\phi\|_2^2 + \frac{2P^2}{\lambda_0}} + \|\phi\|_2. \tag{2.4.10}$$

An energy estimate

Next lemma will be our main energy estimate, which we will use in conjunction with Lemma 2.2.14.

Lemma 2.4.2. *For any $\phi \in \dot{\bar{H}}^2[r_0, R_0]$ and $u(t;r)$ solving (2.4.2) we have the inequality*

$$\begin{aligned} \tfrac{d}{dt}\int_{r_0}^{R_0}(u-\phi)^2 dr \quad &\leqslant \int_{r_0}^{R_0}\left(-u_{rr}^2 + \left(4+\tfrac{16}{r_0^2}\right)u_r^2 + (1-\phi_r)u^2\right) dr \\ &\quad + \int_{r_0}^{R_0}\left(4\phi_{rr}^2 + \left(\tfrac{1}{2}+\tfrac{18}{r_0^2}\right)\phi_r^2\right) dr \end{aligned} \tag{2.4.11}$$

Note that (2.4.9), (2.4.10) and (2.4.11) show that if one can construct $\phi \in \dot{\bar{H}}^2[r_0, R_0]$ such that the coercivity estimate

$$\langle u, Ku\rangle = \int_{r_0}^{R_0}\left(u_{rr}^2 - B_{r_0}u_r^2 + (\phi_r - 1)u^2\right) dr \geq \lambda_0\|u\|_2^2 > 0 \tag{2.4.12}$$

holds for some λ_0 independent of r_0 and R_0, where $B_{r_0} = 4 + \dfrac{16}{r_0^2}$, then one gets an estimate of the form

$$\limsup_{t\to\infty}\|u\|_2 \leqslant R^{**} = \sqrt{c_1\|\phi\|_2^2 + c_2\|\phi_r\|_2^2 + c_3\|\phi_{rr}\|_2^2} + \|\phi\|_2 \leqslant C\|\phi\|_{\bar{H}^2} < \infty.$$

Next, we prove the lemma.

Proof. A straightforward calculation gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u-\phi\|_2^2 &= \int_{r_0}^{R_0} u_t(u-\phi)dr \\ &= \int_{r_0}^{R_0}\left(-u_{rrrr}-\frac{4}{r}u_{rrr}-\left(1-\frac{4}{r^2}\right)u_{rr}-\frac{2}{r}u_r+\frac{2}{r^2}u-uu_r\right)(u-\phi)dr\end{aligned}$$

After integration by parts, applying periodic boundary conditions and simplifying, one gets

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u-\phi\|_2^2 = u_{rr}(R_0)u_r(R_0)-u_{rr}(r_0)u_r(r_0)\\ &+\int_{r_0}^{R_0}\left(-u_{rr}^2+\frac{4}{r}u_{rr}u_r+u_r^2+\frac{2}{r}u_ru\right)dr-u_{rr}(R_0)\phi_r(R_0)+u_{rr}(r_0)\phi_r(r_0)\\ &+\int_{r_0}^{R_0}\left(u_{rr}\phi_{rr}-\frac{4}{r}u_{rr}\phi_r-u_r\phi_r-\frac{2}{r}u\phi_r-\frac{1}{2}u^2\phi_r\right)dr\end{aligned}$$

Using the boundary conditions, one can find estimate for $u_{rr}(R_0)u_r(R_0)-u_{rr}(r_0)u_r(r_0)$ as follows:

$$\begin{aligned}u_{rr}(R_0)u_r(R_0)-u_{rr}(r_0)u_r(r_0) &= -\frac{2}{R_0}u_r^2(R_0)+\frac{2}{r_0}u_r^2(r_0) = -2\int_{r_0}^{R_0}\left(\frac{u_r^2}{r}\right)'dr \\ &= 2\int_{r_0}^{R_0}\frac{u_r^2}{r^2}dr-4\int_{r_0}^{R_0}\frac{u_ru_{rr}}{r}dr\end{aligned}$$

Similarly

$$-u_{rr}(R_0)\phi_r(R_0)+u_{rr}(r_0)\phi_r(r_0) = -2\int_{r_0}^{R_0}\frac{u_r\phi_r}{r^2}dr+2\int_{r_0}^{R_0}\frac{u_{rr}\phi_r+u_r\phi_{rr}}{r}dr$$

Next combine these terms and rewrite again to get

$$\frac{1}{2}\frac{d}{dt}\|u-\phi\|_2^2 = \int_{r_0}^{R_0}\left(-u_{rr}^2+u_r^2+\frac{2}{r}uu_r+\frac{2}{r^2}u_r^2+u_{rr}\phi_{rr}-\frac{2}{r}u_{rr}\phi_r\right)dr$$
$$+\int_{r_0}^{R_0}\left(\frac{2}{r}u_r\phi_{rr}-u_r\phi_r-\frac{2}{r}u\phi_r-\frac{2}{r^2}u_r\phi_r-\frac{1}{2}u^2\phi_r\right)dr$$

Applying the Cauchy-Schwartz inequality $\langle f,g\rangle \le p/2\langle f,f\rangle + 1/2p\langle g,g\rangle$ gives

$$\frac{1}{2}\frac{d}{dt}\|u-\phi\|_2^2 \leqslant \int_{r_0}^{R_0}\left(-1+\frac{m}{2}+p\right)u_{rr}^2+\left(1+\frac{q}{r_0^2}+\frac{2}{r_0^2}+\frac{d}{2}+\frac{k}{r_0^2}+\frac{c}{r_0^2}\right)u_r^2 dr$$
$$+\int_{r_0}^{R_0}\left(\frac{1}{q}+s-\frac{1}{2}\phi_r\right)u^2+\left(\frac{1}{2m}+\frac{1}{c}\right)\phi_{rr}^2+\left(\frac{1}{pr_0^2}+\frac{1}{2d}+\frac{1}{sr_0^2}+\frac{1}{kr_0^2}\right)\phi_r^2 dr$$

The choice $m=1/2$, $p=1/4$, $q=4$, $s=1/4$, $k=1$, $d=2$ and $c=1$ gives (2.3.9). □

This shows that for the proof of Theorem 2.4.1, it remains to establish the coercivity estimate (2.4.12).

Constructing the function ϕ

In order to prove the coercivity estimate (2.4.12) for $L \ge 1$, we will use the following result, which is in essence what was proved in [7].

Lemma 2.4.3. *(see Theorem 1, [7]) Let $z \in C^3[0,L], L \ge 1$ with $z(0)=0$. Then there exists a function $\psi \in C^\infty[0,L]$, so that one has the estimate*

$$\int_0^L (z_{xx}^2+\psi' z^2)dx \ge 10\int_0^L z^2 dx.$$

In addition, ψ is in the form $\psi'(x) = L^{4/3}\chi(L^{1/3}x) - \int_0^L \chi(y)dy$, where $\chi \in C^\infty$, supported on a set with diameter $O(1)$ and so that $\sup_x |\chi^{(\alpha)}(x)| \le C_\alpha, \alpha = 0,1,\ldots$.

Our next result will address the question for the coercivity estimates when $L < 1$. We shall need this result to finish the proof of the theorem in one of the two cases considered. Although it's proof reduces in a simple fashion to Lemma 2.4.3, we include it for completeness.

Lemma 2.4.4. *Let $z \in C^3[0,\varepsilon], \varepsilon \le 1$ with $z(0) = 0$. Then there exists a function $\psi \in C_0^\infty[0,\varepsilon]$, so that*

$$\int_0^\varepsilon (z_{xx}^2 + \psi' z^2) dx \ge 10 \int_0^\varepsilon z^2 dx.$$

In addition, ψ is in the form $\psi'(x) = \chi(x/\varepsilon) - \int_0^1 \chi(y) dy$, where $\chi \in C^\infty$, supported on $(0,1)$ and so that $\sup_{0 \le x < 1} |\chi^{(\alpha)}(x)| \le C_\alpha, \alpha = 0, 1, \ldots$.

Proof. Introduce v, so that $z(x) = v(x/\varepsilon)$. Clearly $v \in C^3[0,1] : v(0) = 0$ and we need to show

$$\int_0^1 (v_{yy}^2(y) + \varepsilon^4 \psi'(\varepsilon y) v^2(y)) dy \ge 10\varepsilon^4 \int_0^1 v^2(y) dy.$$

Clearly, that puts us in the situation of Lemma 2.4.3 with $L = 1$ and thus, it will suffice to take $\psi : \psi'(\varepsilon y) = \chi(y) - \int_0^1 \chi(x) dx$. Indeed,

$$\int_0^1 (v_{yy}^2(y) + \varepsilon^4 \psi'(\varepsilon y) v^2(y)) dy \ge \varepsilon^4 \int_0^1 (v_{yy}^2(y) + \psi'(\varepsilon y) v^2(y)) dy \ge 10\varepsilon^4 \int_0^1 v^2(y) dy,$$

where we have used the construction of Lemma 2.4.3 in the last inequality. Thus,

$$\psi'(x) = \chi(x/\varepsilon) - \int_0^1 \chi(y) dy.$$

and the proof of Lemma 2.4.4 is complete. □

Completion of the proof of Theorem 2.4.1

We will do a rescaling argument, which will show how to obtain (2.4.12) from Lemma 2.4.3 or Lemma 2.4.4. To prove the theorem, we need to construct ϕ_r such that

$$\int_{r_0}^{R_0} (u_{rr}^2 - B_{r_0} u_r^2 + (\phi_r - 1 - \lambda_0) u^2) dr \geqslant 0 \tag{2.4.13}$$

After applying the Cauchy-Schwartz inequality to estimate

$$-B_{r_0} \int_{r_0}^{R_0} u_r^2 dr \geqslant -\frac{1}{2} \int_{r_0}^{R_0} u_{rr}^2 dr - \frac{B_{r_0}^2}{2} \int_{r_0}^{R_0} u^2 dr,$$

we see that it will be enough to show that for all $u \in C^3[r_0, R_0] : u(r_0) = 0$,

$$\int_{r_0}^{R_0} (u_{rr}^2 + \phi_r u^2) dr \geq K \int_{r_0}^{R_0} u^2 dr \tag{2.4.14}$$

where $K = 10 + B_{r_0}^2$. Let $L = R_0 - r_0$. Introduce $v \in C^3[0, L] : v(r) = u(r + r_0)$. Clearly $v(0) = 0$ and we need to show (for appropriate ϕ)

$$\int_0^L (v_{rr}^2 + \phi'(r + r_0) v^2) dr \geq K \int_0^L v^2 dr$$

Next, introduce $w : v(r) = w(K^{1/4} r)$. Again $w(0) = 0$ and we need

$$\int_0^{LK^{1/4}} (w_{rr}^2 + \frac{1}{K} \phi'(K^{-1/4} r + r_0) w^2) dr \geq \int_0^{LK^{1/4}} w^2 dr, \tag{2.4.15}$$

At this point, we will have to consider two separate cases, depending on the relative size of $LK^{1/4}$. These will be handled either by Lemma 2.4.3 or by Lemma 2.4.4. We will be mainly interested in the first case which holds always when r_0 is small and we are tracking the dependence of the constant on $\frac{1}{r_0}$ in this case.

Case I: $LK^{1/4} \geq 1$

By Lemma 2.4.3, the following choice of ϕ (recall $LK^{1/4} \geq 1$)

$$\frac{1}{K}\phi'(K^{-1/4}r + r_0) = \psi'(r) = (LK^{1/4})^{4/3}\chi((LK^{1/4})^{1/3}x) - c_0,$$

will guarantee (2.4.12). Note that $c_0 = \int_0^1 \chi(y)dy = O(1)$, according to Lemma 2.4.3. We get

$$\phi_r(r) = (LK)^{4/3}\chi(L^{1/3}K^{1/3}(r - r_0)) - c_0 K : [r_0, R_0] \to \mathbf{R}^1.$$

Clearly now $\|\phi_{rr}\|_{L^2} \leq C(LK)^{3/2}$, $\|\phi_r\|_{L^2} \leq C(LK)^{7/6}$, while since $\|\phi\|_{L^\infty} \leq C(LK)$, we get $\|\phi\|_{L^2} \leq CL^{3/2}K$.

From Lemma 2.3.9 it follows that

$$\limsup_{t\to\infty} \|u\|_2 \leqslant R^{**} \leq \sqrt{2\|\phi\|_2^2 + c_1(1 + \frac{1}{r_0^2})\|\phi_r\|_2^2 + c_2\|\phi_{rr}\|_2^2} + \|\phi\|_2,$$

where the constants are independent of r_0. Thus, we get the estimate

$$\limsup_{t\to\infty} \|u\|_2 \leq C(R_0 - r_0)^{3/2}\left(1 + \frac{1}{r_0^2}\right)^3,$$

whenever $R_0 - r_0 = L \geq K^{-1/4}$.

Case II: $LK^{1/4} < 1$

Going back to the proof of of (2.4.15), we now have $LK^{1/4} < 1$ and hence, we use Lemma 2.4.4 with $\varepsilon = LK^{1/4} < 1$. Thus,

$$\frac{1}{K}\phi'(r_0 + K^{-1/4}r) = \psi'(r) = \chi(r/\varepsilon) - c_0.$$

or written otherwise

$$\phi_r(r) = K\chi\left(\frac{r-r_0}{L}\right) - Kc_0 : [r_0, R_0] \to \mathbf{R}^1.$$

Clearly, $\|\phi_{rr}\|_{L^2[r_0,R_0]} \leq CKL^{-1/2}$, $\|\phi_r\|_{L^2[r_0,R_0]} \leq CKL^{1/2}$ and $\|\phi\|_{L^\infty} < C(KL)$, which implies $\|\phi\|_{L^2} \leq CL^{3/2}K$. Hence

$$\limsup_{t\to\infty} \|u\|_2 \leq C\frac{(1+\frac{1}{r_0^2})^2}{\sqrt{R_0 - r_0}}.$$

whenever $R_0 - r_0 = L \leq K^{-1/4}$.

n-dimensional case

In this section we will describe similar results for the general n dimensional case. The statement of the theorem remains the same as in the three-dimensional case, even though after the tedious computations some additional terms appear. In what follows we will show that the same lemmas can be applied and the coefficients remain similar and produce same result for the dependence of the limit on $\frac{1}{r_0}$. As before, we differentiate (2.1.1), define $u = \frac{d\varphi}{dr}$ and use the same boundary conditions as in (2.4.1). Thus we get the following reduced radial system, where n is the dimension.

$$\begin{aligned} u_t + u_{rrrr} + \frac{2(n-1)}{r}u_{rrr} &+ \left(\tfrac{n^2-6n+5}{r^2}+1\right)u_{rr} + \left(\tfrac{n-1}{r} - \tfrac{3(n^2-4n+3)}{r^3}\right)u_r \\ &+ \left(\tfrac{3(n^2-4n+3)}{r^4} - \tfrac{n-1}{r^2}\right)u + uu_r = 0 \end{aligned} \tag{2.4.16}$$

$$u = u_{rr} + \left(\frac{n-1}{r}\right)u_r = 0 \qquad \text{for} \quad r = r_0,\, r = R_0 \tag{2.4.17}$$

$$u(x,0) = u_0(x) = u_0(|x|) \qquad \text{in } \Omega \tag{2.4.18}$$

An energy estimate

Similar to what we did in Lemma 2.3.9, we will find the energy estimate for the equation (2.4.16), which we will use in conjunction with the coercivity to prove that Theorem 2.4.1 holds in this case as well.

Lemma 2.4.5. *For any $\phi \in \dot{H}^2[r_0,R_0]$ and $u(t;r)$ solving (2.4.16) we have the inequality*

$$\begin{aligned}\tfrac{d}{dt}\|u-\phi\|_2^2 \leqslant \textstyle\int_{r_0}^{R_0} -u_{rr}^2 + \left(\tfrac{12(n-1)}{r_0^2}+3\right)u_r^2 + \left((n-1)\tfrac{(n-3)^2(4n-1)+9|n-3|+1}{r_0^4}\right)u^2 dr \\ + \textstyle\int_{r_0}^{R_0}\left(\tfrac{2(n-1)}{r_0^2}+(n-1-\phi_r)\right)u^2+(n+3)\phi_{rr}^2+\tfrac{(n-1)(4n-3)+1}{r_0^2}\phi_r^2 dr \\ + \textstyle\int_{r_0}^{R_0}\left(\tfrac{|n^2-4n+3|((n-3)(4n-1)+3)}{r_0^4}+2(n-1)\right)\phi^2 dr\end{aligned}$$

Proof. A straightforward calculation gives

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}\|u-\phi\|_2^2 = \textstyle\int_{r_0}^{R_0}\left(-u_{rrrr}-\tfrac{2(n-1)}{r}u_{rrr}-\left(\tfrac{n^2-6n+5}{r^2}+1\right)u_{rr}\right)(u-\phi)dr \\ -\textstyle\int_{r_0}^{R_0}\left(\left(\tfrac{n-1}{r}-\tfrac{3(n^2-4n+3)}{r^3}\right)u_r-\left(\tfrac{3(n^2-4n+3)}{r^4}-\tfrac{n-1}{r^2}\right)u-uu_r\right)(u-\phi)dr\end{aligned} \tag{2.4.19}$$

After integration by parts, applying periodic boundary conditions and simplifying, one gets

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u-\phi\|_2^2 = u_{rr}(R_0)u_r(R_0)-u_{rr}(r_0)u_r(r_0)-u_{rr}(R_0)\phi_r(R_0)+u_{rr}(r_0)\phi_r(r_0)\\ &+\int_{r_0}^{R_0}\left(-u_{rr}^2+2(n-1)\left(\frac{u_{rr}u_r}{r}-\frac{u_{rr}u}{r^2}\right)-\frac{(n^2-6n+5)}{r^2}u_{rr}u+u_r^2\right)dr\\ &+\int_{r_0}^{R_0}\left(\left(\frac{3(n^2-4n+3)}{r^3}-\frac{n-1}{r}\right)u_ru+\left(\frac{n-1}{r^2}-\frac{3(n^2-4n+3)}{r^4}\right)u^2\right)dr\\ &+\int_{r_0}^{R_0}\left(u_{rr}\phi_{rr}-2(n-1)\left(\frac{u_{rr}\phi_r}{r}-\frac{u_{rr}\phi}{r^2}\right)+(n^2-6n+5)\frac{u_{rr}\phi}{r^2}-u_r\phi_r\right)dr\\ &+\int_{r_0}^{R_0}\left((n-1)\frac{u_r\phi}{r}-3(n^2-4n+3)\frac{u_r\phi}{r^3}+3(n^2-4n+3)\frac{u\phi}{r^4}-(n-1)\frac{u\phi}{r^2}-\frac{u^2\phi_r}{2}\right)dr\end{aligned}$$

Using the boundary conditions, one can find estimate for

$u_{rr}(R_0)u_r(R_0) - u_{rr}(r_0)u_r(r_0)$ as follows:

$$u_{rr}(R_0)u_r(R_0) - u_{rr}(r_0)u_r(r_0) = -\frac{(n-1)}{R_0}u_r^2(R_0) + \frac{(n-1)}{r_0}u_r^2(r_0) =$$

$$= -(n-1)\int_{r_0}^{R_0}\left(\frac{u_r^2}{r}\right)' dr = (n-1)\int_{r_0}^{R_0}\frac{u_r^2}{r^2}dr - 2(n-1)\int_{r_0}^{R_0}\frac{u_r u_{rr}}{r}dr$$

Similarly

$$\begin{aligned}&-u_{rr}(R_0)\phi_r(R_0) + u_{rr}(r_0)\phi_r(r_0)\\ &\qquad = -(n-1)\int_{r_0}^{R_0}\frac{u_r\phi_r}{r^2}dr + (n-1)\int_{r_0}^{R_0}\frac{u_{rr}\phi_r + u_r\phi_{rr}}{r}dr\end{aligned}$$

Next combine these terms and rewrite again to get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u-\phi\|_2^2 = \int_{r_0}^{R_0}\left((n-1)\frac{u_r^2}{r^2} - u_{rr}^2 - \frac{(n^2-4n+3)}{r^2}u_{rr}u + u_r^2\right)dr\\ &+\int_{r_0}^{R_0}\left(\left(\frac{3(n^2-4n+3)}{r^3} - \frac{n-1}{r}\right)u_r u + \left(\frac{n-1}{r^2} - \frac{3(n^2-4n+3)}{r^4}\right)u^2\right)dr\\ &+\int_{r_0}^{R_0}\left(-(n-1)\frac{u_{rr}\phi_r}{r} + (n-1)\frac{\phi_{rr}u_r}{r} - (n-1)\frac{u_r\phi_r}{r^2} + u_{rr}\phi_{rr}\right)dr\\ &+\int_{r_0}^{R_0}\left((n^2-4n+3)\frac{u_{rr}\phi}{r^2} - u_r\phi_r + (n-1)\frac{u_r\phi}{r}\right)dr\\ &+\int_{r_0}^{R_0}\left(-3(n^2-4n+3)\frac{u_r\phi}{r^3} + 3(n^2-4n+3)\frac{u\phi}{r^4} - (n-1)\frac{u\phi}{r^2} - \frac{u^2\phi_r}{2}\right)dr\end{aligned}$$

Applying the Cauchy-Schwartz inequality $\langle f,g\rangle \le p/2\langle f,f\rangle + 1/2p\langle g,g\rangle$ gives

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u-\phi\|_2^2 \leqslant \\
&\int_{r_0}^{R_0}\left(-1+|n^2-4n+3|\frac{p}{2}+(n-1)\frac{m}{2}+\frac{f}{2}+|n^2-4n+3|\frac{h}{2}\right)u_{rr}^2dr \\
&+\int_{r_0}^{R_0}\left(\frac{(n-1)}{r_0^2}+1+\frac{3|n^2-4n+3|}{r_0^2}\frac{q}{2}+\frac{(n-1)}{r_0^2}\frac{z}{2}+\frac{(n-1)}{r_0^2}\frac{c}{2}+\frac{(n-1)}{r_0^2}\frac{d}{2}\right)u_r^2dr \\
&+\int_{r_0}^{R_0}\left(\frac{y}{2}+\frac{(n-1)}{r_0^2}\frac{w}{2}+\frac{3|n^2-4n+3|}{r_0^2}\frac{j}{2}\right)u_r^2dr \\
&+\int_{r_0}^{R_0}\left(\frac{|n^2-4n+3|}{r_0^4}\frac{1}{2p}+\frac{3|n^2-4n+3|}{r_0^4}\frac{1}{2q}+\frac{3|n^2-4n+3|}{r_0^4}\frac{t}{2}\right)u^2dr \\
&+\int_{r_0}^{R_0}\left(\frac{(n-1)}{2z}+\frac{(n-1)}{r_0^2}+\frac{3|n^2-4n+3|}{r_0^4}+\frac{(n-1)}{r_0^4}\frac{\tilde{w}}{2}-\frac{\phi_r}{2}\right)u^2dr \\
&+\int_{r_0}^{R_0}\left(\frac{(n-1)}{2c}+\frac{1}{2f}\right)\phi_{rr}^2+\left(\frac{(n-1)}{r_0^2}\frac{1}{2d}+\frac{1}{2y}+\frac{(n-1)}{r_0^2}\frac{1}{2m}\right)\phi_r^2dr+ \\
&\int_{r_0}^{R_0}\left(\frac{|n^2-4n+3|}{2hr_0^4}+\frac{3|n^2-4n+3|}{2jr_0^4}+\frac{3|n^2-4n+3|}{2tr_0^4}+\frac{n-1}{2w}+\frac{n-1}{2\tilde{w}}\right)\phi^2dr
\end{aligned}
$$

Choosing $p=\frac{1}{4|n^2-4n+3|}$, $m=\frac{1}{4(n-1)}$, $f=\frac{1}{4}$, $h=\frac{1}{4|n^2-4n+3|}$, $q=\frac{1}{|n-3|}$, $c=1$, $d=1$, $j=\frac{1}{|n-3|}$, $z=1$, $y=1$, $w=1$, $t=1$, $z=1$, $\tilde{w}=1$, we get

$$
\begin{aligned}
&\frac{d}{dt}\|u-\phi\|_2^2 \leqslant \int_{r_0}^{R_0}-u_{rr}^2+\left(\frac{12(n-1)}{r_0^2}+3\right)u_r^2dr \\
&+\int_{r_0}^{R_0}\left((n-1)\frac{(n-3)^2(4n-1)+9|n-3|+1}{r_0^4}\right)u^2dr \\
&+\int_{r_0}^{R_0}\left(\left(\frac{2(n-1)}{r_0^2}+(n-1-\phi_r)\right)u^2+\left(\frac{(n-1)(4n-3)}{r_0^2}+1\right)\phi_r^2\right)dr \\
&+\int_{r_0}^{R_0}\left(\left(\frac{|n^2-4n+3|((n-3)(4n-1)+3)}{r_0^4}+2(n-1)\right)\phi^2+(n+3)\phi_{rr}^2\right)dr
\end{aligned}
$$

□

We will prove the coercivity in the n-dimensional case using the lemmas from the previous section. For the modified coefficients $B_{r_0}=\frac{12(n-1)}{r_0^2}+3$, and

$C_{r_0} = (n-1)\frac{(n-3)^2(4n-1)+9|n-3|+1}{r_0^4} + \frac{2(n-1)}{r_0^2} + n - 1$ we have to show that there exists potential function ϕ_r, such that we have the following coercivity estimate.

$$\int_{r_0}^{R_0} (u_{rr}^2 - B_{r_0} u_r^2 + (\phi_r - C_{r_0})u^2)dr \geqslant \lambda_0 \int_{r_0}^{R_0} u^2 dr \tag{2.4.20}$$

Applying Cauchy-Schwartz inequality once again,

$$-B_{r_0} \int_{r_0}^{R_0} u_r^2 dr \geqslant -\frac{1}{2} \int_{r_0}^{R_0} u_{rr}^2 dr - \frac{B_{r_0}^2}{2} \int_{r_0}^{R_0} u^2 dr,$$

(2.4.20) will be equivalent to

$$\int_{r_0}^{R_0} \left(\frac{1}{2} u_{rr}^2 + (\phi_r - D_{r_0})u^2 \right) dr \geqslant \lambda_0 \int_{r_0}^{R_0} u^2 dr \tag{2.4.21}$$

where $D_{r_0} = (n-1)\frac{(n-3)^2(4n-1)+72(n-1)+9|n-3|+1}{r_0^4} + \frac{38(n-1)}{r_0^2} + n + \frac{7}{2}$. Once again we have to prove that

$$\int_{r_0}^{R_0} (u_{rr}^2 + \phi_r u^2)dr \geq K \int_{r_0}^{R_0} u^2 dr$$

where $K = 10 + D_{r_0} \sim (1 + \frac{1}{r_0^2})^2$, which follows from Lemma 2.4.3 or Lemma 2.4.4. To finish the proof of the theorem, notice that now by Lemma 2.4.5 one has

$$\limsup_{t\to\infty} \|u\|_2 \leqslant R^{**} \leq \sqrt{c_1(1+\frac{1}{r_0^2})^2\|\phi\|_2^2 + c_2(1+\frac{1}{r_0^2})\|\phi_r\|_2^2 + c_3\|\phi_{rr}\|_2^2} + \|\phi\|_2,$$

where the constants are independent of r_0. Using the estimates for $\|\phi\|_2^2, \|\phi_r\|_2^2, \|\phi_{rr}\|_2^2$ in this inequality gives the same results as in the three-dimensional case and proves the theorem.

2.5 Summary, remarks and open questions

Kuramoto-Sivashinsky equation arises when studying the propagation of instabilities in combustion theory and hydrodynamics and is well studied in dimension one. A major characteristic of the periodic case in dimension one is the existence of globally invariant, exponentially attracting inertial manifold, which is finite-dimensional. Thus the long-term dynamics is well-known in this case. For the higher-dimensional Kuramoto-Sivashinsky equation the question of long-term dynamics is still open for any general solution, but some results are available when the equation is considered on a thin domain or restricted to a periodic solution on a shell domain that excludes zero. In this paper, we worked with the radially symmetric solutions of Kuramoto-Sivashinsky equation in a shell domain $\Omega = \{x \in \mathbb{R}^n \text{ such that } 0 < r_0 < \|x\| < R_0\}$ and established a time-independent bound for the L^2 norm of the radially symmetric solutions. In particular, we proved that $\limsup_{t\to\infty} \left(\int_{r_0}^{R_0} |u(t,r)|^2 dr \right)^{1/2} \leq C_{r_0}(R_0 - r_0)^{3/2}$ and we explicitly calculate the dependence of the constant C_{r_0} on $\frac{1}{r_0}$. This is important when r_0 tends to 0 since it might shed some light on the potential formation of singularity at the origin and is subject of future research. Thus we were not able to prove similar bounds for the whole disk/ball, but our results can be interpreted as showing that if the dimension is high enough there is no singularity at the origin. In particular if one considers the standard L^2-norm in polar coordinates on R^n as $\int_{r_0}^{R_0} |u(t,r)|^2 r^{n-1} dr$ instead of the norm that we have used one gets no singularity at zero in dimension seven and above immediately. This result does not seem optimal and we are currently working on the regularity and long time behavior for axisymmetric solutions of the same equation of the form $r^s u(r)$ for an appropriate power s in the standard norm. We have written the paper using the same norm and Neumann boundary conditions as in [6].

Although these boundary conditions are quite standard when dealing with radial and axisymmetric solutions, it might be of interest to consider similar problem with different boundary conditions. Our initial calculations show that one can get analogous results in many different situations and the question becomes which boundary conditions are most interesting for the applications.

Finally, it might be feasible to reconsider the general solutions of the Kuramoto-Sivashinsky equation in higher dimensions. We have proved the result using Lyapunov function methods that work fairly well in the case of one variable only. If one considers general solution in dimension two and higher the resulting equations contain mixed nonlinear terms that are very hard to treat using coercivity. It might be possible to drop the radial symmetry assumptions, but still use polar coordinates to respect the geometry of the circle to prove similar results by extending the methods used in this paper. This is going to require additional estimates beyond the scope of this work. These questions will be the subject of a future investigation.

Chapter 3

Conditional Stability results for Klein-Gordon equation

3.1 Introduction and Previous Results

In this chapter, our interest will be the conditional stability of the steady state solutions of one-dimensional Klein-Gordon equation:

$$u_{tt} - \Delta u + u - \mathcal{N}(u) = 0 \qquad (t,x) \in \mathbf{R}^+ \times \mathbf{R^d} \tag{3.1.1}$$

where $\mathcal{N}(u) = |u|^{p-1}u$ and $p \geq 5$. With some assumptions on the nonlinear term $\mathcal{N}(u)$, it has been proved by the authors of [32] that these solutions are in fact linearly and nonlinearly unstable. Our interest is the conditional stability of such steady state solutions. This kind of stability has been extensively studied recently. For example for the equation $u_{tt} - \Delta u = u^5$, in [32], the existence of steady state solutions, the linear and the nonlinear instability of such solutions have been proved. However it has been also proved in [33] that for the special perturbation to the steady state solution of $u_{tt} - \Delta u = u^5$, the solution exists globally and remains near the steady state. Thus, a center-stable manifold for the steady state in the sense of Bates and Jones [2] is described. In 1989, Bates and Jones [2], [3] proved that for a large class of semilinear equations, including the Klein-Gordon equation, the space of solutions decomposes into an unstable and center-stable manifold. Similar result was proved in [26] for the semilinear Schrödinger equation in any dimension. Both are abstract results and do not deal with the global in time behavior of the solutions, e.g. existence and asymptotic behavior. The first asymptotic stability result was obtained by Soffer and Weinstein, [52], [53] (see also [54]), followed by works of Pillet and Wayne [42], Buslaev, Perelman, Sulem [9], [10], [11], Rodnianski-Schlag-Soffer [45], [46] etc. In this context we would like to mention some recent work of Schlag [47] and Beceanu [4],[5] on the existence of center-stable manifold for the pulse solutions of the focusing cubic nonlinear Schrödinger equation in dimension three. It identifies a center-stable manifold in the critical for the equation space $H^{1/2}$ and shows that solutions starting on the manifold exist globally in time and remain on the manifold for all time answering an open question in [26]. Recently the authors of [57] proved a conditional stability of the steady state solutions of (3.1.1) with $\mathcal{N}(u) = |u|^{p-1}u$ for the dimension $d = 2,3$ and 4 where $p \geq 1+4/d$. In terms of center-stable manifold for the solution, their result shows the global in time behavior of the solutions and a precise description of the manifold which includes its co-dimension and decay rates.

In this chapter, we will consider the steady state solutions for the equation

$$u_{tt} - u_{xx} + u - |u|^{p-1}u = 0 \qquad (t,x) \in \mathbf{R}^+ \times \mathbf{R^1} \tag{3.1.2}$$

for $p \geq 5$ and explicitly construct the center-stable manifold for such solutions. In these problems, since Strichartz estimates are key, the lower the dimension, the harder it is to close the argument. The main difficulty in the one-dimensional case is that the required decay of the Klein-Gordon semigroup does not follow from Strichartz estimates alone.

One needs to further refine the function spaces and use additional decay estimates to resolve this issue. The techniques we use are similar to the ones used in [37] and [57].

Note that throughout this chapter, we will use the following norms that are defined on the weighted spaces

$$\|f\|_{L_t^r L_x^s(\mathbb{R};\langle x\rangle^p dx)} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \langle x\rangle^p |f(t,x)|^s dx\right)^{r/s} dt\right)^{1/r}$$
$$\|f\|_{L_x^s(\mathbb{R};\langle x\rangle^p dx) L_t^r} = \left(\int_{\mathbb{R}} \langle x\rangle^p \left(\int_{\mathbb{R}} |f(t,x)|^r dt\right)^{s/r} dx\right)^{1/s}$$

and $\langle x\rangle = \sqrt{1+x^2}$

3.2 Preliminary Lemmas

The existence and uniqueness of steady state solutions of (3.1.1) are shown in [34] for $p < \frac{d+2}{d-2}$ when $d \geq 3$ and for any p when $d = 1,2$. These solutions are positive, radial and exponentially decaying. Next lemma in [13] shows the explicit form of such solutions for (3.1.2) in one-dimensional space, see e.g. [13].

Lemma 3.2.1. *For all $p \in (1,\infty)$ the steady state solution $\phi(x)$ of (3.1.2) has the explicit form*

$$\phi(x) = c_p \cosh^{-\beta}\left(\frac{x}{\beta}\right), \quad c_p = \left(\frac{p+1}{2}\right)^{\frac{1}{p-1}}, \quad \beta := \frac{2}{p-1} \tag{3.2.1}$$

$\phi(x)$ satisfies (3.1.2) and is the unique $H^1(\mathbb{R})$-solution up to translation.

The linearization of (3.1.2) around ϕ:

Define $v(x,t) := u(x,t) - \phi(x)$. If we plug $u = v + \phi$ into (3.1.2), we get

$$v_{tt} - v'' + v - ((\phi + v)^p - \phi^p) = 0 \tag{3.2.2}$$

As in the Section (1.1.2), we can find the linearized equation of (3.2.2):

$$v_{tt} + \mathscr{H} v = 0 \qquad \text{where } \mathscr{H} := -\partial_x^2 + 1 - p\phi^{p-1} \tag{3.2.3}$$

We can write the system (3.2.3) as a first order system by introducing a new variable $w := v_t$, which can be described as

$$X_t + \tilde{\mathscr{H}} X = 0, \quad \text{where } \tilde{\mathscr{H}} = \begin{pmatrix} 0 & -1 \\ \mathscr{H} & 0 \end{pmatrix}, \; X = \begin{pmatrix} v \\ w \end{pmatrix}. \tag{3.2.4}$$

The spectral stability of the steady state solutions, that is the spectrum of $\tilde{\mathscr{H}}$ is determined by the spectrum of the operator $\mathscr{H}$. Next lemma gives the spectrum and the corresponding eigenfunctions.

Lemma 3.2.2. *(See Theorem 3.1 in [14]) For the equation (3.1.2), assume* $3 \leq p < \infty$*. Then there exists* $\sigma = \sigma(p) > 0$*, such that the spectrum of* $\mathscr{H}$ *is given by*

$$\sigma(\mathscr{H}) = \{-\sigma^2\} \cup \{0\} \cup [1, \infty) \tag{3.2.5}$$

with $\mathscr{H}\psi = -\sigma^2\psi$*. The eigenfunctions* $\{\psi\}$ *and* $\{\phi'\}$ *(corresponding to the eigenvalue at* $-\sigma^2$ *and* 0 *respectively) are decaying at infinity and mutually orthogonal.*

In particular, in the one-dimensional case the so called "gap lemma" for the spectrum is satisfied if $p \geq 3$, namely there are no eigenvalues in $(0, 1]$ and the point 1 is not a resonance. Regarding the eigenvalue at zero, it is well-known that at least some of its eigenvectors arise out of symmetries for the problem. Thus, 0 is an eigenvalue, due to translation invariance and it might be of multiplicity 1 or 2, see [61].

Remark 11. *One can easily observe that if $-\sigma^2$ is an eigenvalue of $\mathscr{H}$ with the eigenfunction ψ, then $\lambda_1 = -\sigma$ and $\lambda_2 = \sigma$ are the eigenvalues for $\tilde{\mathscr{H}}$ with the corresponding eigenfunctions $\vec{x_1} = \begin{pmatrix} \psi \\ \sigma\psi \end{pmatrix}$ and $\vec{x_2} = \begin{pmatrix} \psi \\ -\sigma\psi \end{pmatrix}$ respectively. Then the solution X to the linearized equation (3.2.4) is in the following form:*

$$X(t) = c_1 e^{\sigma t} \begin{pmatrix} \psi \\ \sigma\psi \end{pmatrix} + c_2 e^{-\sigma t} \begin{pmatrix} \psi \\ -\sigma\psi \end{pmatrix} + Z(t) \tag{3.2.6}$$

where $Z(t) = P_{a.c.}(\tilde{H})$ and $P_{a.c.}$ is the spectral projection associated to the continuous spectrum of $\tilde{\mathscr{H}}$.

In order to have linear stability, we should have $c_1 = 0$. Thus if the initial value to the linearized problem (3.2.4) is chosen as $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, then $c_1 = 0$ holds if and only if we have $\langle \sigma f_1 + f_2, \psi \rangle = 0$.

3.3 Results

We present a theorem that describes an explicit construction of the center-stable manifold, which is our main result.

Theorem 3.3.1. *For (3.1.2) with $5 \le p < \infty$, and $\mathscr{H}\psi = -\sigma^2\psi$ where $\sigma = \sigma(p)$, there exists $0 < \varepsilon = \varepsilon(p) << 1$ and $0 < \delta = \delta(p) << 1$, and a function*

$$h : B_{H^1}(\delta\varepsilon) \times B_{L^2}(\delta\varepsilon) \cap \{(f,g) : \langle \sigma f + g, \psi \rangle = 0\} \to \mathbf{R}^1$$

so that whenever the <u>*real-valued initial data is even*</u> *and*

$$\begin{aligned} u(0) = &\quad \phi + f_1 + h(f_1, f_2)\psi \\ u_t(0) = &\quad\quad\quad\quad f_2 \end{aligned}$$

$$\langle \sigma f_1 + f_2, \psi \rangle = 0; \|(f_1, f_2)\|_{H^1 \times L^2} < \delta\varepsilon,$$

then

$$u(t,x) = \phi(x) + a(t)\psi + \mathbf{z}(t,x) \quad \textit{where } \mathbf{z} = P_{a.c.}(\mathscr{H})\mathbf{z} \tag{3.3.1}$$

and

$$\|\mathbf{z}\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1 \cap L_x^\infty(\mathbb{R}; \langle x \rangle^{-3/2} dx) L_t^2} \leq \varepsilon, \qquad \|a\|_{L_t^3[0,\infty) \cap L_t^\infty[0,\infty)} \leq \varepsilon.$$

This theorem states that if the initial data u_0 satisfies $u_0 - \phi \in \Sigma$, where Σ is the center-stable manifold we construct, then the solution will approach in an exponential way or slower the steady state ϕ. In this theorem, we assume the initial data to be even. This destroys the eigenvalue at 0. Since the evolution preserves even solutions and the zero eigenvalue has only odd eigenfunctions, the whole evolution proceeds perpendicularly to that marginally stable direction. Thus we will be looking for a solution u in the form (3.3.1). More precisely, we write differential equations for the unknown functions $a(t)$ and $z(t)$, which we solve using fixed points for certain maps. We show that these maps do indeed have fixed points, in view of the linear estimates that they satisfy. These are in turn a consequence of the spectral assumptions and the decay of the bound state.

Remark 12. *As we discussed in Remark 11, we needed the orthogonality condition* $\langle \sigma f_1 + f_2, \psi \rangle = 0$ *in order to get linear stability. However our interest is prove nonlinear stability. Thus besides having the same orthogonal condition on the initials, we will* *be allowed to have some amount in the direction of unstable direction* $\begin{pmatrix} \psi \\ 0 \end{pmatrix}$ *because of the nonlinear term.*

Proof of the Theorem 3.3.1:

3.3.1 Main linear estimates

The proof of the conditional stability theorem is based on a spectral decomposition or modulation argument and a contraction mapping argument in the appropriate spaces. The key is to define the spaces and the norm in such a way that one is able to close the argument, and infers the decay rates. In this section we will explain how to prove Lemma 3.3.2 and Lemma 3.3.3 which are the main tools needed to show the conditional stability result. The lemmas in this section will also help to understand the reason why we are choosing these particular spaces.

Let $P_{a.c.}$ be a spectral projection associated to the continuous spectrum of $\mathscr{H} = -\partial_x^2 + 1 - p\phi^{p-1}$.

Lemma 3.3.2. *There exists a positive constant C such that for any* $g(t,x) \in \mathscr{S}(\mathbf{R^2})$ *and* $t \in \mathbb{R}$*,*

$$\left\| \int_0^t \frac{e^{-i(t-s)\sqrt{\mathscr{H}}}}{\sqrt{\mathscr{H}}} P_{a.c.} g(s,\cdot) ds \right\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} \leq C \|g\|_{L_t^2 L_x^2(\mathbb{R}, \langle x \rangle^5 dx)} \tag{3.3.2}$$

Lemma 3.3.3. *There exists a positive constant C such that for any* $g(t,x) \in \mathscr{S}(\mathbf{R^2})$ *and* $t \in \mathbf{R}$*,*

$$\left\| \langle x \rangle^{-3/2} \int_0^t e^{-i(t-s)\sqrt{\mathscr{H}}} P_{a.c.} g(s,\cdot) ds \right\|_{L_x^\infty L_t^2} \leq C \| \langle x \rangle^{3/2} g \|_{L_x^1 L_t^2} \tag{3.3.3}$$

Remark 13. *In order to prove Lemma 3.3.2 and Lemma 3.3.3, we will prove Lemma 3.3.4 and Lemma 3.3.5 first.*

Lemma 3.3.4. *There exists a positive constant C such that for any $f \in \mathscr{S}(\mathbb{R})$*

$$\|\langle x\rangle^{-3/2} e^{-it\sqrt{\mathscr{H}}} P_{a.c.} f\|_{L_x^\infty L_t^2} \le C\|f\|_{L^2} \tag{3.3.4}$$

where $P_{a.c.}(\mathscr{H})$ is the spectral projection associated to the continuous spectrum of $\mathscr{H} = -\partial_x^2 + 1 - p\phi^{p-1}$.

Lemma 3.3.5. *There exists a positive constant C such that for any $g(t,x) \in \mathscr{S}(\mathbf{R}^2)$*

$$\left\|\int_R e^{is\sqrt{\mathscr{H}}} P_{a.c.} g(s,\cdot)ds\right\|_{L_x^2} \le C\|\langle x\rangle^{3/2} g\|_{L_x^1 L_t^2} \tag{3.3.5}$$

In order to explain the difficulties involved and why we need to resort to the weighted estimates above , let us consider a very simple and naive model,which is nevertheless instructive. Consider a Schrödinger equation.

$$w_t + i\partial_{xx} w = w^2\eta + w^p, \ \ (t,x) \in \mathbf{R}^{1+1}.$$

with small data, where η is a rapidly decaying function and $p \ge 5$. It is not hard to check that for the equation $w_t + i\partial_{xx} w = w^p$, one may apply the standard Strichartz estimates for $e^{it\partial_{xx}}$ and be done with it very quickly. The addition of the highly-localized in x (but not rapidly decaying in time) term $w^2\eta$ presents a new challenge in one spatial dimension in particular. This necessitates the introduction of the weighted estimates in Lemmas 3.3.2-3.3.5, which in essence make way to exchange this extra spatial decay for some extra time decay, just enough to close the fixed point arguments. Before we embark on the proofs of these lemmas which are, as we saw, necessary ingredients in the proof of our main results, let us comment on the strategy and previous results in this direction. We follow mostly the methodology of Mizumachi, [37], which we consider a breakthrough in the area. As is well-understood by now, one splits the estimates in high and low frequency regimes. In the high frequency regimes, one basically uses integration by parts (although this is accomplished by a non-trivial Born series expansion of the resolvents, together with a precise knowledge of the free resolvents). In low frequency, we have to heavily utilize known properties of the Jost solutions, which generate the perturbed resolvents directly. In all of this, we use what has become a standard way of approaching these weighted dispersive estimates. On the other hand, our arguments are being applied to study the Klein-Gordon's equation and as such, it is new and it has subtleties, which are not present in the work of Mizumachi.

Proof of Lemma 3.3.4 and Lemma 3.3.5

Define $\varphi(x)$ to be a smooth function satisfying $0 \leq \varphi(x) \leq 1$ for $x \in \mathbf{R}$ and

$$\varphi(x) = \begin{cases} 1 & \text{if } x \geq 2 \\ 0 & \text{if } x \leq 1 \end{cases} \tag{3.3.6}$$

and let $\varphi_M(x)$ be an even function satisfying $\varphi_M(x) = \varphi(x - M)$ for $x \geq 0$ and let $\tilde{\varphi}_M(x) = 1 - \varphi_M(x)$. Then define $L := \mathscr{H} - 1 = -\partial_x^2 - p\phi^{p-1}$

$$P_{a.c.} e^{-it\sqrt{\mathscr{H}}} f = P_{a.c.} e^{-it\sqrt{L+1}} f = e^{-it\sqrt{L+1}} \varphi_M(\sqrt{L+1}) f + P_{a.c.} e^{-it\sqrt{L+1}} \tilde{\varphi}_M(\sqrt{L+1}) f \tag{3.3.7}$$

Let $R(\lambda) = (\lambda - L)^{-1}$, from Spectral Decomposition Theorem and Complex Analysis since

$$f(L) = \frac{1}{2\pi i} \int_{\wp} f(\lambda)(\lambda - L)^{-1} d\lambda \tag{3.3.8}$$

where $\wp$ is the closed curve containing the absolute continuous spectrum of L, we have

$$\varphi_M(\sqrt{L+1})e^{-it\sqrt{L+1}}f = \frac{1}{2\pi i}\int_0^\infty e^{-it\sqrt{\lambda+1}}\varphi_M(\sqrt{\lambda+1})(R(\lambda-i0)-R(\lambda+i0))fd\lambda \tag{3.3.9}$$

and

$$P_{a.c.}e^{-it\sqrt{L+1}}\tilde{\varphi}_M(\sqrt{L+1})f = \frac{1}{2\pi i}\int_0^\infty e^{-it\sqrt{\lambda+1}}\tilde{\varphi}_M(\sqrt{\lambda+1})P_{a.c.}(R(\lambda-i0)-R(\lambda+i0))fd\lambda \tag{3.3.10}$$

By change of variables $\mu := \sqrt{\lambda+1}$, (3.3.9) becomes

$$= \frac{1}{\pi i}\int_{-\infty}^\infty \chi_{[1,\infty]}e^{-it\mu}\varphi_M(\mu)(R(\mu^2-1-i0)-R(\mu^2-1+i0))\mu fd\mu \tag{3.3.11}$$

Applying integration by parts for j times, we get

$$\varphi_M(\sqrt{L+1})e^{-it\sqrt{L+1}}f = \frac{(it)^{-j}}{\pi i}\int_{-\infty}^\infty e^{-it\mu}\partial_\mu^j\{\chi_{[1,\infty]}\varphi_M(\mu)(R(\mu^2-1-i0)-R(\mu^2-1+i0))\mu\}fd\mu \tag{3.3.12}$$

in $\mathscr{S}_x'(\mathbf{R})$ for any $t \neq 0$ and $f \in \mathscr{S}_x(\mathbf{R^2})$. Since

$$\|\partial_\lambda^j P_{a.c.}R(\lambda \pm i0)\|_{B(L^{2,(j+1)/(2+0)},L^{2,-(j+1)/(2-0)})} \lesssim \langle\lambda\rangle^{-(j+1)/2} \tag{3.3.13}$$

the integral is absolutely convergent in $L_x^{2,-(j+1)/2}$ for $j \geq 2$. Suppose $g(t,x) = g_1(t)g_2(x)$ where $g_1 \in C_0^\infty(\mathbf{R}-\{0\}), g_2 \in \mathscr{S}(\mathbf{R})$. Define

$$\langle u_1,u_2\rangle_x := \int_{-\infty}^\infty u_1(x)u_2(x)dx \tag{3.3.14}$$

$$\langle v_1,v_2\rangle_{t,x} := \int_{-\infty}^\infty\int_{-\infty}^\infty v_1(t,x)v_2(t,x)dxdt \tag{3.3.15}$$

Thus

$$\langle \varphi_M(\sqrt{L+1})e^{-it\sqrt{L+1}}f,g\rangle_{t,x} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi_M(\sqrt{L+1})e^{-it\sqrt{L+1}}fg_1(t)g_2(x)dxdt \quad (3.3.16)$$

Using (3.3.12), we get

$$\langle \varphi_M(\sqrt{L+1})e^{-it\sqrt{L+1}}f,g\rangle_{t,x}$$
$$=\frac{1}{\pi i}\int_{-\infty}^{\infty} dt(it)^{-j}g_1(t)\int_{-\infty}^{\infty} d\mu e^{-it\mu}\partial_\mu^j\langle\chi_{[1,\infty]}(R(\mu^2-1-i0)-R(\mu^2-1+i0))\varphi_M(\mu)\mu f,g_2\rangle_x$$

By Fubini's Theorem

$$=\frac{1}{\pi i}\int_{-\infty}^{\infty} d\mu\partial_\mu^j\langle\chi_{[1,\infty]}(R(\mu^2-1-i0)-R(\mu^2-1+i0))\varphi_M(\mu)\mu f,g_2\rangle_x\int_{-\infty}^{\infty} dt(it)^{-j}e^{-it\mu}g_1(t)$$

Doing integration by parts for j times

$$=\frac{\sqrt{2}}{\sqrt{\pi i}}\int_{-\infty}^{\infty} d\mu(\mathscr{F}_t g_1)(\mu)\langle\chi_{[1,\infty]}(R(\mu^2-1-i0)-R(\mu^2-1+i0))\varphi_M(\mu)\mu f,g\rangle_x$$

From Fubini's Theorem

$$=\frac{\sqrt{2}}{\sqrt{\pi i}}\int_{-\infty}^{\infty} dx\int_{-\infty}^{\infty} d\mu\langle\chi_{[1,\infty]}(R(\mu^2-1-i0)-R(\mu^2-1+i0))\varphi_M(\mu)\mu f\mathscr{F}_t g(\mu,x))$$

Using Plancherel's Theorem and Cauchy Schwartz Inequality

$$|\langle \varphi_M(\sqrt{L+1})e^{-it\sqrt{L+1}}f,g\rangle_{t,x}|$$
$$\leqslant\frac{\sqrt{2}}{\sqrt{\pi i}}\|\varphi_M(\mu)\mu(R(\mu^2-1-i0)-R(\mu^2-1+i0))f\|_{L_x^\infty L_\mu^2}\|g(\mu,x))\|_{L_x^1L_\mu^2}$$

Similarly

$$|\langle P_{a.c.}e^{-it\sqrt{L+1}}\tilde{\varphi}_M(\sqrt{L+1})f,g\rangle_{t,x}|$$
$$\leqslant \frac{\sqrt{2}}{\sqrt{\pi i}}\|\langle x\rangle^{-3/2}\tilde{\varphi}_M(\mu)P_{a.c.}(R(\mu^2-1-i0)-R(\mu^2-1+i0))\mu f\|_{L_x^\infty L_\mu^2}\|\langle x\rangle^{3/2}g(\mu,x))\|_{L_x^1 L_\mu^2}$$

If we combine these two and assuming the next two inequalities (3.3.17) and (3.3.18) hold

$$\|\varphi_M(\mu)(R(\mu^2-1-i0)-R(\mu^2-1+i0))\mu f\|_{L_x^\infty L_\mu^2} \leqslant C\|f\|_{L^2} \tag{3.3.17}$$

$$\|\langle x\rangle^{-3/2}\tilde{\varphi}_M(\mu)P_{a.c.}(R(\mu^2-1-i0)-R(\mu^2-1+i0))\mu f\|_{L_x^\infty L_\mu^2} \leqslant C\|f\|_{L^2} \tag{3.3.18}$$

we get

$$|\langle x\rangle^{-3/2}\langle e^{-it\sqrt{L+1}}P_{a.c.}f,g\rangle_{t,x}| \leqslant C\|f\|_{L^2}\|g\|_{L_x^1 L_\mu^2} \tag{3.3.19}$$

Since $\mathscr{C}_0^\infty(\mathbb{R}_t-\{0\})\otimes\mathscr{S}(\mathbb{R}_x)$ is dense in $L_x^1L_t^2$ and by duality principle

$$\|\langle x\rangle^{-3/2}e^{-it\sqrt{L+1}}P_{a.c.}f\|_{L_x^\infty L_\mu^2} \leqslant C\|f\|_{L^2} \tag{3.3.20}$$

Now we will prove (3.3.17) then (3.3.18) in order to complete the proof of the lemma. We will use Green's functions to show (3.3.17), Scattering Theory and Jost functions to prove (3.3.18).

Proof of (3.3.17)**: High Energy Estimate**

Let $R_0(\lambda) = (\lambda + \partial_x^2)^{-1}$ and $G_1(x,k) = \frac{e^{ik|x|}}{2ik}$, and $\lambda = k^2$ with $k \geq 0$ and $V := -p\phi^{p-1}$. Then $R_0(\lambda \pm i0)\delta = G_1(x, \mp k)$. If M is sufficiently large enough, we have

$$R(\lambda \pm i0) = \sum_{j=0}^{\infty} R_0(\lambda \pm i0)(VR_0(\lambda \pm i0))^j u \tag{3.3.21}$$

for $\lambda \in \mathbb{R}$ with $|\lambda| > M$ and $u \in \mathscr{S}(\mathbb{R})$ since

$$\|\langle x\rangle^{-1} R_0(\lambda \pm i0)\langle x\rangle^{-1}\|_{B(L^2(\mathbb{R}))} \lesssim \langle \lambda \rangle^{-1/2} \tag{3.3.22}$$

The sum is absolutely convergent because

$$\begin{aligned} R(\lambda \pm i0)u =& R_0(\lambda \pm i0)u + R_0(\lambda \pm i0)VR_0(\lambda \pm i0)u + \ldots \\ =& \langle x\rangle\langle x\rangle^{-1}R_0(\lambda \pm i0)\langle x\rangle^{-1}\langle x\rangle u \\ &+ \langle x\rangle\langle x\rangle^{-1}R_0(\lambda \pm i0)\langle x\rangle^{-1}\langle x\rangle V\langle x\rangle\langle x\rangle^{-1}R_0(\lambda \pm i0)\langle x\rangle^{-1}\langle x\rangle u + \ldots \end{aligned}$$

Since V is exponentially decreasing and $u \in \mathscr{S}(\mathbb{R})$, the absolute sum in L^2 is bounded by $C\sum_{j=1}^{j=\infty}\langle\lambda\rangle^{-j/2}$. Since $|\lambda| > M$ and M is large enough, the geometric series converges. Now if we assign $\lambda = \mu^2 - 1$, then we can write

$$\begin{aligned} \|\varphi_M(\mu)R(\mu^2 - 1 \pm i0)\mu u\|_{L_x^\infty L_\mu^2} \leqslant& \|\varphi_M(\mu)R_0(\mu^2 - 1 \pm i0)\mu u\|_{L_x^\infty L_\mu^2} \\ &+ \sum_{n=1}^{\infty}\|\varphi_M(\mu)F_{1,n}(x, \mp k)\|_{L_x^\infty L_\mu^2} \end{aligned}$$

where

$$F_{1,n}(x, \mp k) := R_0(\mu^2 - 1 \pm i0)(VR_0(\mu^2 - 1 \pm i0))^n \mu u(x) \tag{3.3.23}$$

$$\begin{aligned}
\|\varphi_M(\mu)R_0(\mu^2-1\pm i0)\mu u\|^2_{L^\infty_x L^2_\mu} &= \sup_x \int_{\mathbf{R}} |\varphi_M(\mu)R_0(\mu^2-1\pm i0)\mu u|^2 d\mu \\
&= \sup_x \int_{\mathbf{R}} \frac{k}{\sqrt{k^2+1}} |\varphi_M(\sqrt{k^2+1})R_0(k^2\pm i0)\sqrt{k^2+1}u|^2 dk \\
&= \sup_x \int_{\mathbf{R}} k\sqrt{k^2+1} |\varphi_M(\sqrt{k^2+1})(G_1(\cdot,\mp k)*u)(x)|^2 dk \\
&\lesssim \sup_x \int_{\mathbf{R}} \left(\left| \int_x^\infty u(y)e^{\pm iky} dy \right|^2 + \left| \int_{-\infty}^x u(y)e^{\mp iky} dy \right|^2 \right) dk \\
&\lesssim \|u\|^2_{L^2_x}
\end{aligned}$$

Similarly one can write

$$F_{1,n}(x,\pm k) = \int_{\mathbf{R}^{n+1}} G_1(x-x_1,\pm k) \prod_{j=1}^n (V(x_j)G_1(x_j-x_{j+1},\pm k))\sqrt{k^2+1}u(x_{n+1})dx_1...dx_{n+1} \tag{3.3.24}$$

Since

$$\int_{\mathbb{R}} G_1(x_n-x_{n+1})u(x_{n+1})dx_{n+1} = G_1(x_n)*u(x_n) \tag{3.3.25}$$

by Minkowski's Inequality, we get

$$\begin{aligned}
&\|\varphi_M(\mu)F_{1,n}(x,\pm k)\|_{L^2_\mu} = \left(\int_{\mathbf{R}} |\varphi_M(\mu)F_{1,n}(x,\pm k)|^2 d\mu \right)^{1/2} \lesssim \int_{\mathbf{R}^n} \prod_{j=1}^n V(x_j)dx_1...dx_n \\
&\times \left(\int_{\mathbf{R}} k(k^2+1)|\varphi_M(\sqrt{k^2+1})G_1(x-x_1)...G_1(x-x_n)|^2 |(G_1*u)(x_n)|^2 dk \right)^{1/2} \\
&\lesssim \|V\|^n_{L^1} \sup_{x_n} \left(\int_{\mathbf{R}} k^{-2n}k(k^2+1)|\varphi_M(\sqrt{k^2+1})(G_1*u)(x_n)|^2 dk \right)^{1/2} \\
&\lesssim \|V\|^n_{L^1} M^{(-2n+1)/2} \|u\|_{L^2} \qquad \text{for } n \geq 1
\end{aligned}$$

Since $V \in L^1(\mathbb{R})$, $u \in \mathscr{S}(\mathbb{R})$ and M is sufficiently large, we have

$$\begin{aligned}\|\varphi_M(\mu)R(\mu^2-1\mp i0)\mu u\|_{L^\infty_x L^2_\mu} &\lesssim \|u\|_{L^2} + \sum_{n=1}^{\infty} \|V\|_{L^1}^n M^{-n+1/2}\|u\|_{L^2} \\ &\lesssim \|u\|_{L^2}\end{aligned}$$

Proof of (3.3.18)**: Low Energy Estimate**

This section is based on Jost functions and Scattering Theory. Let $f_1(x,k)$ and $f_2(x,k)$ be the solutions to $Lu = k^2 u$ satisfying

$$\lim_{x\to\infty} |e^{-ikx} f_1(x,k) - 1| = 0, \qquad \lim_{x\to-\infty} |e^{ikx} f_2(x,k) - 1| = 0 \tag{3.3.26}$$

Define

$$m_1(x,k) := e^{-ikx} f_1(x,k), \qquad m_2(x,k) := e^{ikx} f_2(x,k)$$

Then

$$\begin{aligned} m_1(x,k) &= 1 + \int_x^\infty \frac{e^{2ik(y-x)}}{2ik} V(y) m_1(y,k)dy \\ m_2(x,k) &= 1 + \int_{-\infty}^x \frac{e^{2ik(x-y)}}{2ik} V(y) m_2(y,k)dy \end{aligned}$$

By the results of [17], for $x \in \mathbb{R}$ and $k \in \mathbb{C}$ with nonnegative imaginary part, we have

$$|m_1(x,k) - 1| \lesssim \langle k\rangle^{-1}(1+\max(-x,0)) \int_x^\infty \langle y\rangle |V(y)|dy \tag{3.3.27}$$

$$|m_2(x,k) - 1| \lesssim \langle k\rangle^{-1}(1+\max(x,0)) \int_{-\infty}^x \langle y\rangle |V(y)|dy \tag{3.3.28}$$

For every $\delta > 0$, there exists $C_\delta > 0$ such that for every $x \in \mathbb{R}$ and $k \in \mathbb{C}$ with nonnegative imaginary part and $|k| \geq \delta$

$$|m_1(x,k) - 1| \leqslant C_\delta \int_x^\infty |V(y)|dy \tag{3.3.29}$$

$$|m_2(x,k) - 1| \leqslant C_\delta \int_{-\infty}^x |V(y)|dy \tag{3.3.30}$$

The resolvent operator $R(\lambda \pm i0)$ with $\lambda = k^2$ has the kernel

$$K_\pm(x,y,k) = \begin{cases} -\frac{f_1(x,\pm k) f_2(y,\pm k)}{W(\pm k)} & \text{if } x > y \\ -\frac{f_2(x,\pm k) f_1(y,\pm k)}{W(\pm k)} & \text{if } x < y \end{cases} \tag{3.3.31}$$

where $W(k) = f_1'(x,k) f_2(x,k) - f_1(x,k) f_2'(x,k) \neq 0$ where the Wronskian $W(k)$ is independent of x.

$$R(\lambda \pm i0)u = -\frac{f_1(x,\pm k)}{W(\pm k)}(I_1 + I_2 + I_3) - \frac{f_2(x,\pm k)}{W(\pm k)}(II_1 + II_2)$$

where $I_1(k) = \int_{-\infty}^0 e^{-iky} u(y)dy$, $I_2(k) = \int_\infty^0 e^{-iky}(m_2(y,k) - 1)u(y)dy$, $I_3(k) = \int_0^x f_2(y,k)u(y)dy$ and $II_1(k) = \int_x^\infty e^{iky} u(y)dy$, $II_2(k) = \int_x^\infty e^{iky}(m_2(y,k) - 1)u(y)dy$.

Bound for I_1: Assuming $x > 0$, (3.3.27) (3.3.28) imply that

$$\sup_{x>0} (|f_1(x,k)| + \langle x \rangle^{-1} |f_2(x,k)|) < \infty \tag{3.3.32}$$

Then

$$|I_1| = \left| \int_0^x f_2(y,k)u(y)dy \right| \lesssim \int_0^x \langle y \rangle |u(y)|dy \lesssim \left(\int_0^x \langle y \rangle^2 dy \right)^{1/2} \left(\int_0^x |u(y)|^2 dy \right)^{1/2}$$
$$\lesssim \langle x \rangle^{3/2} \|u\|_{L^2}$$

By using (3.3.27), (3.3.28), Cauchy-Schwartz Inequality and the properties of Fourier Transform, one can also bound I_2,I_3,II_1 and II_2 by $C\|u\|_{L^2}$. Then since $W(k) \neq 0$ for every $k \in \mathbb{R}$ and $\tilde{\varphi}_M(k)$ is compactly supported, it follows that

$$\begin{aligned}
&\|\tilde{\varphi}_M(\mu)P_{a.c.}R(\mu^2 - 1 \pm i0)\mu f\|_{L^\infty_x L^2_\mu} \\
&\qquad = \sup_x \left(\int_{\mathbb{R}} |k|(k^2+1) \mid \tilde{\varphi}_M(k) \int_{\mathbb{R}} K_\pm(x,y,k)u(y)dy \mid^2 dk \right)^{1/2} \\
&\qquad \lesssim \langle x \rangle^{3/2} \|u\|_{L^2}
\end{aligned}$$

This finishes the proof of Lemma 3.3.4.

In fact, the proof of Lemma 3.3.5 relies on a simple duality argument, based on Lemma 3.3.4. Define $Tf := \langle x \rangle^{-3/2} e^{-it\sqrt{\mathscr{H}}} P_{a.c.} f$. From Lemma 3.3.4, we have

$$\|Tf\|_{L^\infty_x L^2_t} \leq C\|f\|_{L^2}.$$

Then using Fubini's Theorem and Duality Principle we get

$$| \int_{\mathbb{R}} \int_{\mathbb{R}} \langle x \rangle^{-3/2} e^{-it\sqrt{\mathscr{H}}} P_{a.c.} f h dx dt | = | \langle f, \int_{\mathbb{R}} dt e^{it\sqrt{\mathscr{H}}} P_{a.c.} \langle x \rangle^{-3/2} h \rangle_x | \leq C\|f\|_{L^2} \|h\|_{L^1_x L^2_t} \tag{3.3.33}$$

If we define $g := \langle x \rangle^{-3/2} h$, then (3.3.5) follows by duality principle.

Proof of Lemma 3.3.2 and Lemma 3.3.3

First, we need the following

Proof of Lemma 3.3.2. From Strichartz estimates for the Klein Gordon equation, we have

$$\|e^{-it\sqrt{\mathscr{H}}} P_{a.c.} f\|_{L^5_t L^{10}_x \cap L^\infty_t H^1_x} \leq C\|f\|_{H^1} \tag{3.3.34}$$

Similarly, we get

$$\left\| \frac{e^{-it\sqrt{\mathscr{H}}}}{\sqrt{\mathscr{H}}} P_{a.c.} f \right\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} \leq C \|f\|_{L^2} \tag{3.3.35}$$

and from Lemma 3.3.5, we know that

$$\left\| \int_{\mathbb{R}} e^{is\sqrt{\mathscr{H}}} P_{a.c.} g(s,\cdot) ds \right\|_{L_x^2} \leq C \|\langle x \rangle^{3/2} g\|_{L_x^1 L_t^2} \tag{3.3.36}$$

Let

$$Tg(t) = \int_{\mathbb{R}} \frac{e^{-i(t-s)\sqrt{\mathscr{H}}}}{\sqrt{\mathscr{H}}} P_{a.c.} g(s) ds \tag{3.3.37}$$

Choose

$$f := \int_{\mathbb{R}} e^{is\sqrt{\mathscr{H}}} P_{a.c} g(s) ds \in L^2(\mathbb{R}) \tag{3.3.38}$$

Then using (3.3.35),(3.3.36) and Cauchy-Schwartz inequality, we get

$$\begin{aligned} \|Tg\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} &= \left\| \int_{\mathbb{R}} \frac{e^{-i(t-s)\sqrt{\mathscr{H}}}}{\sqrt{\mathscr{H}}} P_{a.c.} g(s,\cdot) ds \right\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} \\ &\leqslant C \|f\|_{L^2} \lesssim C \|\langle x \rangle^{3/2} g\|_{L_x^1 L_t^2} \leqslant \|\langle x \rangle^{5/2} g\|_{L_x^2 L_t^2} \|\langle x \rangle^{-1}\|_{L_x^2} \\ &\leqslant C \|g\|_{L_t^2 L_x^2(\mathbb{R}; \langle x \rangle^5 dx)} \end{aligned}$$

Using the Christ-Kiselev lemma ([15]), it follows that

$$\left\| \int_{s<t} e^{-i(t-s)\sqrt{\mathscr{H}}} P_{a.c.} g(s) ds \right\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} \lesssim \|g\|_{L_t^2 L_x^2(\mathbb{R}; \langle x \rangle^5 dx)} \tag{3.3.39}$$

Thus we complete the proof of Lemma 3.3.2. □

Proof of Lemma 3.3.3. In order to show Lemma 3.3.3, we shall need two modifications of results appearing in [37]. These will be needed to control various terms, arising in the analysis of the estimate (3.3.3).

The first result is stated in [37] for self-adjoint operators $H = -\partial_x^2 + V$, but in fact, it is applicable for any self-adjoint operator acting on L^2.

Proposition 3.3.6. *(Lemma 11, [37]) Let H be a self-adjoint operator and $g(t,x) = g_1(t)g_2(x)$. Define the function*

$$U(t,x) = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-it\lambda} \check{g}_1(\lambda)\{R(\lambda - i0) + R(\lambda + i0)\}[P_{a.c.}(H)g_2]d\lambda,$$

where $\check{g}_1$ is the inverse Fourier transform of g_1. Then

$$\begin{aligned} U(t,x) &= 2\int_0^t e^{-i(t-s)H} P_{a.c.}(H)g(s,\cdot)ds + \int_{-\infty}^0 e^{-i(t-s)H} P_{a.c.}(H)g(s,\cdot)ds \\ &- \int_0^{\infty} e^{-i(t-s)H} P_{a.c.}(H)g(s,\cdot)ds \end{aligned}$$

One can obtain similar results for expressions in the form $\int_0^t e^{-i(t-s)\sqrt{\mathscr{H}}} P_{a.c.}(\mathscr{H})g(s,\cdot)ds$. Namely, based on the argument in Proposition 3.3.6, we get the following formula for the Duhamel's operator, associated with our evolution

$$\begin{aligned} &\int_0^t e^{-i(t-s)\sqrt{\mathscr{H}}} P_{a.c.}(\mathscr{H})g(s,\cdot)ds = \\ &= \frac{i}{2\sqrt{2\pi}} \int_1^{\infty} e^{-it\sqrt{\lambda}} \check{g}_1(\lambda)\{R(\lambda - i0) + R(\lambda + i0)\}[P_{a.c.}(\mathscr{H})g_2]d\lambda + \\ &+ \int_0^{\infty} e^{-i(t-s)\sqrt{\mathscr{H}}} P_{a.c.}(\mathscr{H})g(s,\cdot)ds - \int_{-\infty}^0 e^{-i(t-s)\sqrt{\mathscr{H}}} P_{a.c.}(\mathscr{H})g(s,\cdot)ds \end{aligned}$$

Combining Lemma 8 and Lemma 10 from [37] yields the following.

Proposition 3.3.7. *Let $H = -\partial_x^2 + V(x)$, where $V(x)$ is a real valued potential, which decays sufficiently fast. Then*

$$\sup_{\lambda} \| < x >^{-1} R_H(\lambda \pm i0) P_{a.c.}(H)u\|_{L_x^\infty} \leq \frac{C}{< \lambda >^{1/2}} \| < x > u\|_{L_x^1}. \tag{3.3.40}$$

Note: The constant $<\lambda>^{-1/2}$ in (3.3.40) is not stated in Lemma 8, [37] (which is the high-frequency version regime, i.e. $\lambda >> 1$), but it is very explicit in the estimates there.

We are now ready to proceed with the proof of Lemma 3.3.3. First, it is standard that in order to establish (3.3.3), it suffices to consider functions $g(t,x) = g_1(t)g_2(x)$. Therefore, in view of our formula for $\int_0^t e^{-i(t-s)\sqrt{\mathscr{H}}} P_{a.c.}(H)g(s,\cdot)ds$, it remains to establish

$$\| <x>^{-1} \int_1^\infty e^{-it\sqrt{\lambda}} \check{g}_1(\lambda) R(\lambda \pm i0)[P_{a.c.}(\mathscr{H})g_2] d\lambda \|_{L^\infty_x L^2_t} \leq C \|g_1\|_{L^2_t} \| <x> g_2\|_{L^1_x} \tag{3.3.41}$$

$$\| <x>^{-3/2} \int_0^\infty e^{-i(t-s)\sqrt{\mathscr{H}}} P_{a.c.}(\mathscr{H}) g(s,\cdot) ds\|_{L^\infty_x L^2_t} \leq C \| <x>^{3/2} g\|_{L^1_x L^2_t} \tag{3.3.42}$$

$$\| <x>^{-3/2} \int_{-\infty}^0 e^{-i(t-s)\sqrt{\mathscr{H}}} P_{a.c.}(\mathscr{H}) g(s,\cdot) ds\|_{L^\infty_x L^2_t} \leq C \| <x>^{3/2} g\|_{L^1_x L^2_t} \tag{3.3.43}$$

The proof of (3.3.42) and (3.3.43) are similar, so we concentrate on (3.3.42). We have from (3.3.4) and (3.3.5)

$$\begin{aligned} & \| <x>^{-3/2} \int_0^\infty e^{-i(t-s)\sqrt{\mathscr{H}}} P_{a.c.}(\mathscr{H}) g(s,\cdot) ds\|_{L^\infty_x L^2_t} = \\ = \ & \| <x>^{-3/2} e^{-it\sqrt{\mathscr{H}}} P_{a.c.}(\mathscr{H}) \int_0^\infty e^{is\sqrt{\mathscr{H}}} P_{a.c.}(H) g(s,\cdot) ds\|_{L^\infty_x L^2_t} \leq \\ \leq \ & C \| \int_0^\infty e^{is\sqrt{\mathscr{H}}} P_{a.c.}(\mathscr{H}) g(s,\cdot) ds\|_{L^2_x} \leq C \| <x>^{3/2} g\|_{L^1_x L^2_t}. \end{aligned}$$

Regarding (3.3.41), we have by Plancherel's theorem in the time variable and Cauchy-Schwartz inequality that

$$\begin{aligned}
&\| <x>^{-1} \int_1^\infty e^{-it\sqrt{\lambda}} \check{g}_1(\lambda) R(\lambda \pm i0)[P_{a.c.}(\mathscr{H}) g_2] d\lambda \|_{L_x^\infty L_t^2} \leq \\
&\leq 2 \sup_x | <x>^{-1} \| \int_1^\infty e^{-it\mu} \mu \check{g}_1(\mu^2) R(\mu^2 \pm i0)[P_{a.c.}(\mathscr{H}) g_2] d\mu \|_{L_t^2} \leq \\
&\leq C (\int_{-\infty}^\infty |\mu| |\check{g}_1(\mu^2)|^2 d\mu)^{1/2} \sup_\mu |\mu|^{1/2} \sup_x | <x>^{-1} R(\mu^2 \pm i0)(P_{a.c.}(\mathscr{H}) g_2)(x)|
\end{aligned}$$

From (3.3.40), we have $\| <x>^{-1} R(\mu^2 \pm i0) P_{a.c.}(\mathscr{H}) <x>^{-1} \|_{L_x^1 \to L_x^\infty} \leq C <\mu>^{-1}$, whence

$$\sup_x | <x>^{-1} R(\mu^2 \pm i0)(P_{a.c.}(\mathscr{H}) g_2)(x)| \leq C <\mu>^{-1} \| <x> g_2 \|_{L_x^1}.$$

Overall, observing that

$$(\int_{-\infty}^\infty |\mu| |\check{g}_1(\mu^2)|^2 d\mu)^{1/2} \leq \|\check{g}_1\|_{L^2} = \|g_1\|_{L_t^2} \text{ and } |\mu|^{1/2} <\mu>^{-1} < 1,$$

we conclude

$$\| <x>^{-1} \int_1^\infty e^{-it\sqrt{\lambda}} \check{g}_1(\lambda) R(\lambda \pm i0)[P_{a.c.}(\mathscr{H}) g_2] d\lambda \|_{L_x^\infty L_t^2} \leq C \|g_1\|_{L_t^2} \| <x> g_2 \|_{L_x^1}$$

which is (3.3.41). □

3.3.2 Analysis of $a(t)$ and $\mathbf{z}(t)$ equations

In this section we will prove the conditional stability result by applying the fixed point theorem. We will set the contraction map and the function spaces. In order to prove the contraction mapping theorem, for the decay estimates, we will apply Lemma 3.3.2 and Lemma 3.3.3 and for the Strichartz estimates, we will use Lemma 1.2.4.

Taking the ansatz (3.3.1) into (3.1.2), we get

$$\mathbf{z}_{tt} + \mathscr{H}\mathbf{z} + \psi(a''(t) - \sigma^2 a(t)) - F(t,x) = 0 \tag{3.3.44}$$

where

$$F(t,x) = |\phi + a(t)\psi + \mathbf{z}|^{p-1}(\phi + a(t)\psi + \mathbf{z}) - \phi^p - p\phi^{p-1}(a(t)\psi + \mathbf{z}(t)) \tag{3.3.45}$$

Taking the spectral projections, we derive the equations

$$a''(t) - \sigma^2 a(t) - \langle F(t,\cdot), \psi \rangle = 0 \tag{3.3.46}$$

$$\mathbf{z}_{tt} + \mathscr{H}\mathbf{z} - P_{a.c.}[F] = 0 \tag{3.3.47}$$

The explicit solution of (3.3.46) is in the form

$$a(t) = \cosh(\sigma t)a(0) + \frac{1}{\sigma}\sinh(\sigma t)a'(0) + \frac{1}{\sigma}\int_0^t \sinh(\sigma(t-s))\langle F(s,\cdot), \psi \rangle ds \tag{3.3.48}$$

Note that, if we separate the exponentially growing terms from the exponentially decaying ones, we come up with

$$a(t) = \frac{e^{\sigma t}}{2}\left[a(0) + \frac{1}{\sigma}a'(0) + \frac{1}{\sigma}\int_0^t e^{-\sigma s}\langle F(s,\cdot), \psi \rangle ds\right] + \text{exponentially decaying term.}$$

In order to have a vanishing solution, we must have $a(t) \to 0$, and so, at the very least, we must ensure (by taking appropriate initial data)

$$a(0) + \frac{1}{\sigma} a'(0) + \frac{1}{\sigma} \int_0^\infty e^{-\sigma s} \langle F(s,\cdot), \psi \rangle ds = 0. \tag{3.3.49}$$

The non-explicit non-linear equation (3.3.49) defines the center stable manifold as we shall show below and in that sense, it is useful in its own right. It also shows (modulo the successful completion of our argument) that it is co-dimension one. This, despite heuristically expected (due to the presence of a single unstable direction of the linearized operator), is not at all obvious statement.

According to our definitions $a(0) = \langle (f_1 + h\psi), \psi \rangle = h + \langle f_1, \psi \rangle$. Similarly, $a'(0) = \langle f_2, \psi \rangle$. Taking into account $\langle f_1 + \frac{1}{\sigma} f_2, \psi \rangle = 0$, we have no choice, but to set (as in [57])

$$h(f_1, f_2) = -\frac{1}{\sigma} \int_0^\infty e^{-\sigma s} \langle F(m(s)), \psi \rangle ds \tag{3.3.50}$$

Thus, (3.3.48) becomes equivalent to

$$a(t) = \frac{e^{-t\sigma}}{2} [a(0) - \frac{1}{\sigma} a'(0)] - \frac{1}{2\sigma} \int_0^t e^{-\sigma(t-s)} \langle F(s,\cdot), \psi \rangle ds - \frac{1}{2\sigma} \int_t^\infty e^{\sigma(t-s)} \langle F(s,\cdot), \psi \rangle ds \tag{3.3.51}$$

Taking into account $P_{a.c.}(\mathscr{H})\psi = 0$, the explicit solution of (3.3.47) is in the form

$$\mathbf{z}(t) = \cos(t\sqrt{\mathscr{H}}) P_{a.c.} f_1 + \frac{\sin(t\sqrt{\mathscr{H}})}{\sqrt{\mathscr{H}}} P_{a.c.} f_2 + \int_0^t \frac{\sin((t-s)\sqrt{\mathscr{H}})}{\sqrt{\mathscr{H}}} P_{a.c.} [F(s,\cdot)] ds \tag{3.3.52}$$

3.3.3 Setting the contraction map and the function spaces

Let Λ be a contraction map defined as $\Lambda : X \to X$ such that $\Lambda(m) = \tilde{m}$ where $m := (h, a(t), z(t))$ defined as (3.3.50),(3.3.51),(3.3.52) and $\tilde{m} = (\tilde{h}, \tilde{a(t)}, \tilde{z(t)})$

$$\tilde{h} := -\frac{1}{\sigma}\int_0^\infty e^{-\sigma s}\langle F(m(s)), \psi\rangle ds,$$
$$\tilde{a}(t) := \frac{e^{-t\sigma}}{2}[a(0) - \frac{1}{\sigma}a'(0)] - \frac{1}{2\sigma}\int_0^t e^{-\sigma(t-s)}\langle F(s,\cdot), \psi\rangle ds - \frac{1}{2\sigma}\int_t^\infty e^{\sigma(t-s)}\langle F(s,\cdot), \psi\rangle ds,$$
$$\tilde{\mathbf{z}}(t) := \cos(t\sqrt{\mathscr{H}})P_{a.c.}f_1 + \frac{\sin(t\sqrt{\mathscr{H}})}{\sqrt{\mathscr{H}}}P_{a.c.}f_2 + \int_0^t \frac{\sin((t-s)\sqrt{\mathscr{H}})}{\sqrt{\mathscr{H}}}P_{a.c.}[F(s,\cdot)]ds.$$

Let the norm on X be defined as $\|m\|_X := \max(M_0(m), M_1(m), M_2(m))$ such that

$$M_0(m) := |h|$$
$$M_1(m) := \|a\|_{L_t^3([0,\infty)) \cap L_t^\infty([0,\infty))}$$
$$M_2(m) = \|\mathbf{z}\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1 \cap L_x^\infty(\mathbb{R}; \langle x\rangle^{-3/2}dx)L_t^2}$$

Our goal is to show that Λ is a contraction map defined on the Banach Space X, whose fixed point will be the desired solution.

Estimating $M_0(\tilde{m})$

$$M_0(\tilde{m}) = |\tilde{h}| \le \frac{1}{\sigma}\int_0^\infty e^{-\sigma s}|\langle F(m(s)), \psi\rangle| ds \tag{3.3.53}$$

From Proposition 3 in [57], we have

$$|F(t,x)| \le C_p(\phi^{p-2}(|a(t)|^2\psi^2 + |\mathbf{z}(t)|^2) + |a(t)|^p\psi^p + |\mathbf{z}(t)|^p) \tag{3.3.54}$$

Then it follows

$$|\langle F(m(s)),\psi\rangle| \leq C(|a(s)|^2 + \|\mathbf{z}(s,\cdot)\|_{L_x^2}^2 + |a(t)|^p + \|\mathbf{z}(s,\cdot)\|_{L_x^2}^p) \tag{3.3.55}$$

where C depends on various L^w norms of the decaying functions ϕ, ψ. It follows that

$$\begin{aligned}
M_0(\tilde{m}) &\leq \frac{C}{\sigma}\int_0^\infty e^{-\sigma s}(|a(s)|^2 + \|\mathbf{z}(s,\cdot)\|_{L_x^2}^2 + |a(t)|^p + \|\mathbf{z}(s,\cdot)\|_{L_x^2}^p)ds \\
&\leq \frac{C}{\sigma^2}(\|a\|_{L^\infty}^2 + \|\mathbf{z}\|_{L_t^\infty L_x^2}^2 + \|a\|_{L^\infty}^p + \|\mathbf{z}\|_{L_t^\infty L_x^2}^p) \\
&\leq \frac{C}{\sigma^2}(M_1(m)^2 + M_2(m)^2 + M_1(m)^p + M_2(m)^p) \leq \frac{2C}{\sigma^2}(\varepsilon^2 + \varepsilon^p) \leq \min(1,\sigma)\frac{\varepsilon}{10}
\end{aligned}$$

provided $C(\varepsilon + \varepsilon^{p-1}) \leq \frac{1}{20}\sigma^2 \min(1,\sigma)$. Note that we used Sobolev embedding and Gagaliardo-Nirenberg's inequality to estimate $\|\mathbf{z}\|_{L_t^\infty L_x^2}$ which states that for any KG admissible pair (q,r), one has the following estimate:

$$\|z\|_{L_t^q W_x^{1-(d/2-2/q-d/r)-1/q-2/(dq),r}} \leq M_2(m) \tag{3.3.56}$$

Estimating $M_1(\tilde{m})$

In order to estimate M_1, we will use the fact that if $h = \tilde{h}$ and $\langle \sigma f_1 + f_2, \psi\rangle = 0$, then $2\langle f_1,\psi\rangle + \tilde{h} = a(0) - \frac{a'(0)}{\sigma}$. $M_1(\tilde{m})$ has two components. First, we estimate

$$\begin{aligned}
\sup_t |\tilde{a}(t)| \leq \frac{1}{2}(2|\langle f_1,\psi\rangle| + |\tilde{h}|) + \frac{1}{2\sigma}\sup_t \int_0^t e^{-\sigma(t-s)}|\langle F(m(s)),\psi\rangle| ds \\
+ \frac{1}{2\sigma}\sup_t \int_t^\infty e^{\sigma(t-s)}|\langle F(m(s)),\psi\rangle| ds
\end{aligned}$$

From (3.3.55) and the estimates for $M_0(\tilde{m})$, it follows

$$\begin{aligned}\sup_t |\tilde{a}(t)| &\leq \delta\varepsilon + \frac{\varepsilon}{10} + \frac{1}{\sigma^2}\sup_s |\langle F(m(s)),\psi\rangle| \\ &\leq \delta\varepsilon + \frac{\varepsilon}{10} + \frac{C}{\sigma^2}(M_1(m)^2 + M_2(m)^2 + M_1(m)^p + M_2(m)^p) \leq \varepsilon\end{aligned}$$

provided $\delta < 1/2$ and $2C(\varepsilon + \varepsilon^{p-1}) \leq \sigma^2/4$. For the second component, we use Hausdorf-Young's inequality

$$\begin{aligned}\|\tilde{a}\|_{L^3_t} &\leq (\|f_1\|_{L^2} + |\tilde{h}|)\left(\int_0^\infty e^{-3\sigma t}dt\right)^{1/3} + \frac{1}{2\sigma}\|e^{-\sigma|\cdot|}\|_{L^1}\|\langle F(m(s)),\psi\|_{L^3_s} \\ &\leq (\delta\varepsilon + \frac{\varepsilon}{10}\min(1,\sigma))\frac{1}{\min(1,\sigma)} + \frac{1}{2\sigma^2}\left(\int_0^\infty |\langle F(m(s)),\psi\rangle|^3 ds\right)^{1/3}\end{aligned}$$

From Proposition 3 in [57], we have

$$|\langle F(m(s)),\psi\rangle| \leq C(|a(s)|^2 + \|\mathbf{z}(s,\cdot)\|^2_{L^r_x} + |a(t)|^p + \|\mathbf{z}(s,\cdot)\|^p_{L^r_x}) \tag{3.3.57}$$

It follows that

$$\left(\int_0^\infty |\langle F(m(s)),\psi\rangle|^3 ds\right)^{1/3} \leq C\left(\|a\|^2_{L^6_t} + \|a\|^p_{L^{3p}_t} + \|\mathbf{z}\|^2_{L^6_t L^6_x} + \|\mathbf{z}\|^p_{L^{3p}_t L^6_x}\right) \tag{3.3.58}$$

Since $p \geq 5$, we estimate $\|a\|_{L^6}$, and $\|a\|_{L^{3p}}$. By Gagliardo-Nirenberg's inequality (or log-convexity of L^p norms), for $w \geq 3$,

$$\|a\|_{L^w(0,\infty)} \leq M_1(m). \tag{3.3.59}$$

This follows from

$$\|a\|_{L^w(0,\infty)} \leq \|a\|^{3/w}_{L^3(0,\infty)}\|a\|^{1-3/w}_{L^\infty(0,\infty)} \leq M_1(m). \tag{3.3.60}$$

Thus we have

$$\|a\|_{L_t^6}, \|a\|_{L_t^{3p}} \leq M_1(m) \leq \varepsilon \tag{3.3.61}$$

and because $(6,6), (3p,6)$ are KG admissible, it follows that

$$\|\mathbf{z}\|_{L_t^6 L_x^6}, \|\mathbf{z}\|_{L_t^{3p} L_x^6} \leq M_2(m) \leq \varepsilon \tag{3.3.62}$$

Thus we have

$$\|\tilde{a}\|_{L_t^3} \leq \frac{1}{\min(1,\sigma)}(\delta\varepsilon + \min(1,\sigma)\frac{\varepsilon}{10}) + C_\sigma(2\varepsilon^2 + 2\varepsilon^p) \tag{3.3.63}$$

and it suffices to require that $\delta < \min(1,\sigma)/2$ and $2C_\sigma(\varepsilon + \varepsilon^{p-1}) \leq 1/4$ in order to conclude that

$$M_1(\tilde{m}) = \max(\|\tilde{a}\|_{L_t^\infty}, \|\tilde{a}\|_{L_t^3}) \leq \varepsilon \tag{3.3.64}$$

Estimating $M_2(\tilde{m})$

M_2 has two components. Firstly, we will estimate

$$\|\tilde{z}\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} \leqslant C\|\cos(t\sqrt{\mathscr{H}})P_{a.c.}f_1\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} + \left\|\frac{\sin(t\sqrt{\mathscr{H}})}{\sqrt{\mathscr{H}}}P_{a.c.}f_2\right\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1}$$
$$+ \left\|\int_0^t \frac{\sin((t-s)\sqrt{\mathscr{H}})}{\sqrt{\mathscr{H}}}P_{a.c.}F(m(s))\right\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1}$$

Using Strichartz Estimates and Sobolev Embedding,

$$\|\cos(t\sqrt{\mathscr{H}})P_{a.c.}f_1\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} \leq C\|f_1\|_{H^1} \tag{3.3.65}$$

Similarly we get

$$\left\| \frac{\sin(t\sqrt{\mathscr{H}})}{\sqrt{\mathscr{H}}} P_{a.c.} f_2 \right\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} \leq C\|f_2\|_{L^2} \tag{3.3.66}$$

Using (3.3.54), we get

$$\begin{aligned}
&\left\| \int_0^t \frac{\sin((t-s)\sqrt{\mathscr{H}})}{\sqrt{\mathscr{H}}} P_{a.c.} F(m(s)) \right\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} \\
&\lesssim \left\| \int_0^t \frac{\sin((t-s)\sqrt{\mathscr{H}})}{\sqrt{\mathscr{H}}} P_{a.c.} \phi^{p-2}(|a(s)|^2 \psi^2 + |\mathbf{z}(s,\cdot)|^2) + |a(s)|^p \psi^p + |\mathbf{z}(s,\cdot)|^p \right\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} \\
&\lesssim \|a\|_{L_t^4}^2 + \|a\|_{L_t^p}^2 + \|\mathbf{z}\|^2_{L_x^\infty(\langle x \rangle^{-3/2}) L_t^2} + \|\mathbf{z}\|^p_{L_t^p L_x^{2p}}
\end{aligned}$$

We use Lemma 3.3.2 and Cauchy-Schwartz Inequality to get $\|a\|_{L_t^4}$ and $\|\mathbf{z}\|_{L_x^\infty(\langle x \rangle^{-3/2}) L_t^2}$. We have

$$\begin{aligned}
\left\| \int_0^t \frac{\sin((t-s)\sqrt{\mathscr{H}})}{\sqrt{\mathscr{H}}} P_{a.c.} \phi^{p-2} |a(s)|^2 \psi^2 \right\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} &\lesssim \|\phi^{p-2}|a(t)|^2 \psi^2\|_{L_t^2 L_x^2(\mathbb{R};\langle x \rangle^5)} \\
&\lesssim \|a\|_{L_t^4}^2
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\left\| \int_0^t \frac{\sin((t-s)\sqrt{\mathscr{H}})}{\sqrt{\mathscr{H}}} P_{a.c.} \phi^{p-2} |\mathbf{z}(s,\cdot)|^2 \right\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} &\lesssim \|\phi^{p-2}|\mathbf{z}(t,x)|^2\|_{L_t^2 L_x^2(\mathbb{R};\langle x \rangle^5)} \\
&\lesssim \|\mathbf{z}\|^2_{L_x^\infty(\langle x \rangle^{-3/2}) L_t^2}
\end{aligned}$$

We apply Lemma 1.2.4 in order to get $\|a\|_{L_t^p}^2$ and $\|\mathbf{z}\|^p_{L_t^p L_x^{2p}}$. We take $q_1' = 1$ and $r_1' = 2$.

$$\left\| \int_0^t \frac{\sin((t-s)\sqrt{\mathscr{H}})}{\sqrt{\mathscr{H}}} P_{a.c.} |a(s)|^p \psi^p + |z(s,\cdot)|^p ds \right\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} \lesssim \|a\|_{L_t^p}^2 + \|\mathbf{z}\|^p_{L_t^p L_x^{2p}}$$

Thus we have

$$\|\tilde{z}\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} \leq C(\|(f_1, f_2)\|_{H^1(R)\times L^2(R)} + \|a\|_{L_t^4}^2 + \|a\|_{L_t^p}^2 + \|\mathbf{z}\|_{L_x^\infty(\mathbb{R};\langle x\rangle^{-3/2}dx)L_t^2}^2 + \|\mathbf{z}\|_{L_t^p L_x^{2p}}^p)$$

Since $p \geq 5$, we have

$$\|a\|_{L_t^4}, \|a\|_{L_t^p} \leq M_1(m) \leq \varepsilon \tag{3.3.67}$$

From Strichartz Estimates, $p \geq 5$ implies that $\dfrac{2}{p} + \dfrac{1}{2p} \leq \dfrac{1}{2}$, thus we get

$$\|\mathbf{z}\|_{L_t^p L_x^{2p}} \leq M_2(m) \tag{3.3.68}$$

Also it is clear that $\|\mathbf{z}\|_{L_x^\infty(\mathbb{R};\langle x\rangle^{-3/2}dx)L_t^2} \leq M_2(m) \leq \varepsilon$.

It follows that

$$\|\tilde{z}\|_{L_t^5 L_x^{10} \cap L_t^\infty H_x^1} \leq C_1(\delta\varepsilon + 2\varepsilon^2 + 2\varepsilon^p) \leq \varepsilon \tag{3.3.69}$$

if $C_1\delta \leq 1/4$, $C_1(\varepsilon + \varepsilon^{p-1}) \leq 1/4$.

For the second component

$$\begin{aligned}\|\tilde{z}\|_{L_x^\infty(\mathbb{R};\langle x\rangle^{-3/2}dx)L_t^2} \leqslant C\|\cos(t\sqrt{\mathscr{H}})P_{a.c.}f_1\|_{L_x^\infty(\mathbb{R};\langle x\rangle^{-3/2}dx)L_t^2} \\ + \left\|\frac{\sin(t\sqrt{\mathscr{H}})}{\sqrt{\mathscr{H}}}P_{a.c.}f_2\right\|_{L_x^\infty(\mathbb{R};\langle x\rangle^{-3/2}dx)L_t^2} \\ + \left\|\int_0^t \frac{\sin((t-s)\sqrt{\mathscr{H}})}{\sqrt{\mathscr{H}}}P_{a.c.}F(m(s))ds\right\|_{L_x^\infty(\mathbb{R};\langle x\rangle^{-3/2}dx)L_t^2}\end{aligned}$$

By Lemma (3.3.4), we have

$$\|\cos(t\sqrt{\mathscr{H}})P_{a.c.}f_1\|_{L_x^\infty(\mathbb{R};\langle x\rangle^{-3/2}dx)L_t^2} \leq C\|f_1\|_{L^2} \tag{3.3.70}$$

and

$$\left\| \frac{\sin(t\sqrt{\mathscr{H}})}{\sqrt{\mathscr{H}}} P_{a.c.} f_2 \right\|_{L_x^\infty(\mathbb{R};\langle x\rangle^{-3/2}dx)L_t^2} \leq C\|f_2\|_{H^1} \tag{3.3.71}$$

In order to estimate the last term, we will use

$$\left\| \int_0^t \frac{\sin((t-s)\sqrt{\mathscr{H}})}{\sqrt{\mathscr{H}}} P_{a.c.} F(m(s))ds \right\|_{L_x^\infty(\mathbb{R};\langle x\rangle^{-3/2}dx)L_t^2} \lesssim \begin{cases} \|F\|_{L_t^1 L_x^2} \\ \|F\|_{L_t^2 L_x^2(\mathbb{R};\langle x\rangle^3 dx)} \end{cases} \tag{3.3.72}$$

The first inequality follows from Lemma 3.3.4.

$$\begin{aligned} &\left\| \int_0^t \frac{\sin((t-s)\sqrt{\mathscr{H}})}{\sqrt{\mathscr{H}}} P_{a.c.} F(m(s))ds \right\|_{L_x^\infty(\mathbb{R};\langle x\rangle^{-3/2}dx)L_t^2} \\ &\qquad \leq \int_0^t dt \left\| \langle x\rangle^{-3/2} e^{it\sqrt{\mathscr{H}}} \left(\frac{e^{-is\sqrt{\mathscr{H}}}}{\sqrt{\mathscr{H}}} P_{a.c.} F(m(s)) \right) \right\|_{L_x^\infty L_t^2} \\ &\qquad \lesssim \int_0^\infty dt \left\| \frac{e^{-is\sqrt{\mathscr{H}}}}{\sqrt{\mathscr{H}}} P_{a.c.} F(m(s)) \right\|_{L_x^2} \lesssim \|F\|_{L_t^1 L_x^2} \end{aligned}$$

The second inequality follows from Lemma 3.3.3 and Cauchy-Schwartz Inequality.

$$\begin{aligned} \left\| \int_0^t \frac{\sin((t-s)\sqrt{\mathscr{H}})}{\sqrt{\mathscr{H}}} P_{a.c.} F(m(s))ds \right\|_{L_x^\infty(\mathbb{R};\langle x\rangle^{-3/2}dx)L_t^2} &\lesssim \|\langle x\rangle^{3/2} F\|_{L_x^1 L_t^2} \\ &\lesssim \|F\|_{L_t^2 L_x^2(\mathbb{R};\langle x\rangle^3 dx)} \end{aligned}$$

Using (3.3.54) and (3.3.72), we get

$$\begin{aligned}\left\|\int_0^t \frac{\sin((t-s)\sqrt{\mathscr{H}})}{\sqrt{\mathscr{H}}} P_{a.c.}F(m(s))ds\right\|_{L_x^\infty(\mathbb{R};\langle x\rangle^{-3/2}dx)L_t^2} \\ \lesssim \|\phi^{p-2}|a(t)|^2\psi^2\|_{L_t^2L_x^2(\mathbb{R};\langle x\rangle^3)} + \|\phi^{p-2}|\mathbf{z}(t,x)|^2\|_{L_t^2L_x^2(\mathbb{R};\langle x\rangle^3)} \\ + \||a(t)|^p\psi^p\|_{L_t^2L_x^2(\mathbb{R};\langle x\rangle^3)} + \||z(t,x)|^p\|_{L_t^1L_x^2} \\ \lesssim \|a\|_{L_t^4}^2 + \|\mathbf{z}\|_{L_x^\infty(\langle x\rangle^{-3/2})L_t^2}^2 + \|a\|_{L_t^{2p}}^p + \|z\|_{L_t^pL_x^{2p}}^p\end{aligned}$$

Since $p \geq 5$, we can control $\|a\|_{L_t^4}$,$\|\mathbf{z}\|_{L_x^\infty(\mathbb{R};\langle x\rangle^{-3/2}dx)L_t^2}$, $\|a\|_{L_t^{2p}}$,$\|z\|_{L_t^pL_x^{2p}}$. It follows that

$$\|\mathbf{z}\|_{L_x^\infty(\mathbb{R};\langle x\rangle^{-3/2}dx)L_t^2} \leq C_2(\delta\varepsilon + 2\varepsilon^2 + 2\varepsilon^p) \leq \varepsilon \tag{3.3.73}$$

if $C_2\delta \leq 1/4$, $C_2(\varepsilon+\varepsilon^{p-1}) \leq 1/4$. Thus we can conclude

$$M_2(\tilde{m}) = \max(\|\mathbf{z}\|_{L_t^5L_x^{10}\cap L_t^\infty H_x^1}, \|\mathbf{z}\|_{L_x^\infty(\mathbb{R};\langle x\rangle^{-3/2}dx)L_t^2}) \leq \varepsilon \tag{3.3.74}$$

Thus we can say that for appropriately chosen ε and δ, so that $\|(f_1,f_2)\|_{H^1\times L^2}$, we can establish $\Lambda : B_x(\varepsilon) \to B_x(\varepsilon)$. Note that all the estimates leading to that conclusion were in the form

$$\|\Lambda(m)\|_X \leq C\|m\|_X(1+\|m_1\|_X+\|m_2\|_X)^{p-1}. \tag{3.3.75}$$

In order to finish the proof of the contraction mapping theorem, we have to prove that Λ is a contraction, i.e. $\|\Lambda(m_1)-\Lambda(m_2)\| \leq C\|m_2-m_1\|$ for some $C<1$. It is standard in this line of reasoning that if one has (3.3.75) and the non-linearity F has some "multilinear" feature, then the proof of (3.3.75) can be used to show the contraction of the same map. Indeed, all we have to observe that, similar to (3.3.54), we have

$$\begin{aligned} |F(a,\mathbf{z}) - F(b,\mathbf{w})| &\leq C_{p,\phi,\psi}(\phi^{p-2}[(|a-b|)(|a|+|b|) + |\mathbf{z}-\mathbf{w}|(|\mathbf{z}|+|\mathbf{w}|)] \\ &\quad + \psi^p|a-b|(|a|+|b|)^{p-1} + |\mathbf{z}-\mathbf{w}|(|\mathbf{z}|+|\mathbf{w}|)^{p-1}). \end{aligned} \tag{3.3.76}$$

This last estimate will allow us to do the same estimates as before, except one of the entries will be the difference term $m_1 - m_2$. This way, we show the following analogue of (3.3.75)

$$\|\Lambda(m_1) - \Lambda(m_2)\|_X \leq C\|m_1 - m_2\|_X(\|m_1\|_X + \|m_2\|_X)(1 + \|m_1\|_X + \|m_2\|_X)^{p-2},$$

which implies the desired contractivity of the map Λ for small $\|m_1\|_X, \|m_2\|_X$. This finishes the proof of the theorem.

3.4 Summary, remarks and open questions

Our result constructs the co-dimension one center-stable manifold of initial data, for which the solutions of (3.1.1) close to the steady states stay close to the steady states. The results are important in several different regards - first, it shows that the center-stable manifold is indeed a co-dimension one object, which is not *a priori* clear. Secondly, the actual construction, relies on an implicit constraint (3.3.49), which is of independent interest. Thirdly, the paper develops new spectral and functional analytic tools for proving dispersive estimates for the perturbed Klein-Gordon evolution, which might prove useful in other related situations. However there are still questions which will be the subject of our future investigation.

First of all, our conditional stability result (3.3.1) for the one dimensional Klein-Gordon equation was achieved for the even initial data. By restricting our initial data to even case, we destroyed the eigenvalue at 0. If we intend to work on the same problem to get a similar result for any arbitrary initial data, then we have to work on the solution which will be in the following form:

$$u(t,x) = \phi(x+y(t)) + a(t)\psi + b(t)\phi' + \mathbf{z}(t,x) \tag{3.4.1}$$

Because of the 0 eigenvalue, the solution $u(t)$ contains $y(t)$ which is the asymptotic phase function. This makes the problem much harder because we also need to find estimates for $y(t)$ as in the reaction-diffusion equation case in [57].

Secondly, our conditional stability result (3.3.1) was achieved for $p \geq 5$. We obtained this condition as a result of Strichartz estimates. However Strichartz estimates do not suffice in the case of $p < 5$. This will certainly require additional estimates.

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