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ON THE TOPOLOGY OF GRAPH PICTURE SPACES

JEREMY L. MARTIN

Abstract. We study the space $\mathcal{X}^d(G)$ of pictures of a graph G in complex projective d-space. The main result is that the homology groups (with integer coefficients) of $\mathcal{X}^d(G)$ are completely determined by the Tutte polynomial of G. One application is a criterion in terms of the Tutte polynomial for independence in the *d-parallel matroids* studied in combinatorial rigidity theory. For certain special graphs called *orchards*, the picture space is smooth and has the structure of an iterated projective bundle. We give a Borel presentation of the cohomology ring of the picture space of an orchard, and use this presentation to develop an analogue of the classical Schubert calculus.

1. Introduction

The theory of *configuration varieties*, such as the Grassmannian, flag and Schubert varieties, is marked by an interplay between different fields of mathematics, including algebraic geometry, topology and combinatorics. In this paper, we study a class of configuration varieties called *picture spaces of graphs*, a program initiated in [10]. As we will see, there is a close connection between the combinatorial structure of a graph and the topology and geometry of its picture space.

Let G be a graph with vertices V and edges E. The *d-dimensional picture space* $\mathcal{X}^d(G)$ is defined as the projective algebraic set whose points are *pictures* of G in complex projective d-space $\mathbb{P}^d = \mathbb{P}^d_{\mathbb{C}}$. A picture $\mathbf{P}$ consists of a point $\mathbf{P}(v) \in \mathbb{P}^d$ for each vertex v of G and a line $\mathbf{P}(e)$ for each edge e, subject to the conditions $\mathbf{P}(v) \in \mathbf{P}(e)$ whenever v is an endpoint of e.

Two fundamental operations of graph theory are *deletion* and *contraction*: given a graph G and an edge e, we may delete e to form a graph $G-e$, or identify the endpoints of e to form a graph G/e. Many combinatorial invariants, such as the number of spanning forests, the chromatic polynomial, etc., satisfy a deletion-contraction recurrence; the most general and powerful of these invariants is the *Tutte polynomial* $\mathbf{T}_G(x,y)$. (For those unfamiliar with the properties of the Tutte polynomial, we give a brief sketch in Section 2.1 below; a much more comprehensive treatment may be found in [3].) In the context of the present study, the graph-theoretic operations of deletion and contraction correspond to canonical morphisms (3) and (7) between picture spaces. This suggests that there is a connection between the geometry or topology of $\mathcal{X}^d(G)$ and the Tutte polynomial of G.

The main result of this paper (Theorem 1 below) characterizes the integral homology groups of $\mathcal{X}^d(G)$ completely in terms of the Tutte polynomial. Rather than attempting to describe the homology directly, we apply topological machinery, such

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as the Mayer-Vietoris sequence, to the morphisms arising from deletion and contraction. The result is a recurrence describing the homology groups of $\mathcal{X}^d(G)$ in terms of those of $\mathcal{X}^d(G-e)$ and $\mathcal{X}^d(G/e)$, where e is any edge of G. This recurrence may in turn be phrased in terms of the Tutte polynomial, as follows.

Theorem 1. *Let G be a graph and $d \geq 2$ an integer. Then*

(1) *the picture space $\mathcal{X}^d(G)$ is path-connected and simply connected;*
(2) *the homology groups $H_i(\mathcal{X}^d(G)) = H_i(\mathcal{X}^d(G); \mathbb{Z})$ are free abelian for i even and zero for i odd; and*
(3) *the "compressed Poincaré series"*

$$P_G^d(q) \;:=\; \sum_i q^i \;\operatorname{rank}_{\mathbb{Z}}\, H_{2i}(\mathcal{X}^d(G))$$

is a specialization of the Tutte polynomial $\mathbf{T}_G(x,y)$, namely

$$P_G^d(q) \;=\; ([d]_q - 1)^{\mathbf{v}(G)-\mathbf{c}(G)}\; [d+1]_q^{\mathbf{c}(G)}\; \mathbf{T}_G\left(\frac{[2]_q\,[d]_q}{[d]_q - 1},\, [d]_q\right)$$

where $\mathbf{v}(G)$ is the number of vertices of G, $\mathbf{c}(G)$ is the number of connected components, and $[d]_q = (1-q^d)/(1-q)$.

In Section 2, we set forth some elementary facts and notation involving graphs and picture spaces. We assume some familiarity with basic graph theory, for which [11] is an excellent reference (among many others). For a more leisurely treatment of the picture spaces of graphs, the reader is referred to [10].

Section 3 contains the proof of Theorem 1. We continue by exploring some natural extensions of this main result. In Section 4, we consider the space $\mathcal{X}^M(G)$ of "pictures" of a graph G on a complex manifold M. With suitable conditions on M, we may mimic the methods of Section 3 to describe the integral homology groups of $\mathcal{X}^M(G)$ in terms of the Tutte polynomial of G and the dimension and Poincaré series of M.

For a graph G without loops or parallel edges, the *picture variety* $\mathcal{V}^d(G) \subset \mathcal{X}^d(G)$ is the (irreducible) algebraic variety defined as the closure of the pictures $\mathbf{P}$ for which the points $\mathbf{P}(v)$ are distinct. For more on this subject, seee [10]. The problem of finding a combinatorial interpretation of the Poincaré series of $\mathcal{V}^d(G)$ appears to be quite difficult. We work out a sample Poincaré series calculation in Section 5, and attempt to give some idea of the obstacles that are likely to arise.

Section 6 describes an application of the main result to the theory of *combinatorial rigidity*. This subject (for which an excellent reference is [6]) concerns questions such as the following. Suppose that we are given a physical framework in d-dimensional space, built out of "joints" and "bars" corresponding to the vertices and edges of some graph G. For our purposes, we suppose that the bars may vary in length, but meet the joints at fixed angles. How can one tell from the combinatorial structure of G whether the framework will hold its shape—whether it is "rigid" or "flexible"? We show in Section 6 that this information may be read off from the Tutte polynomial specialization of Theorem 1. In the language of rigidity theory, this says that all of the information about the *d-parallel matroid* of a graph G is contained in the Tutte polynomial $\mathbf{T}_G(x,y)$.

In Section 7, we study the multiplicative structure of the cohomology ring $H^*(\mathcal{X}^d(G); \mathbb{Z})$ in the case that $\mathcal{X}^d(G)$ is smooth. This turns out to be equivalent to the property that G is an "orchard": that is, every edge is either a loop or

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an isthmus. For such a graph, $\mathcal{X}^d(G)$ is an iterated projectivized vector bundle, so its cohomology ring has a presentation in terms of Chern classes of line bundles (see [2]); we give this presentation as Theorem 17. This in turn leads to a "Schubert calculus of orchards": that is, we can answer certain questions involving the enumerative geometry of points and lines in $\mathbb{P}^d_{\mathbb{C}}$ by means of polynomial calculations in $H^*(\mathcal{X}^d(G);\, \mathbb{Z})$.

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2. Graphs and their Picture Spaces

2.1. **Graphs.** We assume some familiarity with the basics of graph theory on the part of the reader; a good general reference is [11].

A *graph* is a pair $G = (V, E)$, where $V = V(G)$ is a finite nonempty set of *vertices* and $E = E(G)$ is a set of *edges*. Each edge e has two vertices v, w, not necessarily distinct, called its *endpoints*. If $v = w$ then e is called a *loop*. When no ambiguity can arise, we sometimes denote an edge by its endpoints, e.g., "$e = vw$". A *subgraph* G' of G is a graph with $V(G') \subset V(G)$ and $E(G') \subset E(G)$.

The set of edges with v as an endpoint is denoted $E(v)$. The set $E(v) \cap E(w)$ is called a *parallel class*. A graph is *simple* if all its of nonloop parallel classes are singletons. (In [10], there is the additional condition that simple graphs contain no loops. However, allowing loops does no harm to the results of [10] used here.) An *underlying simple graph* of a graph G is a graph $G' = (V, E')$, where E' consists of one member of each nonloop parallel class of G.

A graph G is *connected* if for every $v, w \in V(G)$, there is a sequence of vertices $v = v_0, v_1, \dots, v_r = w$ with $\{v_i, v_{i+1}\} \in E(G)$ for $0 \leq i < r$. The maximal connected subgraphs of G are called its *connected components.* We write $\mathbf{v}(G)$, $\mathbf{e}(G)$, and $\mathbf{c}(G)$ for, respectively, the number of vertices, edges and connected components of G.

Definition 2. Let $e \in E(G)$. The *deletion* $G - e$ is the graph $(V, E \setminus \{e\})$. If e is not a loop, the *contraction* G/e is obtained from $G - e$ by identifying the endpoints of e with each other.

For an example of contraction, see Example 13. In general,

$$\begin{aligned} \mathbf{e}(G-e) &= \mathbf{e}(G) - 1, & \mathbf{e}(G/e) &= \mathbf{e}(G) - 1,\\ \mathbf{v}(G-e) &= \mathbf{v}(G), & \mathbf{v}(G/e) &= \mathbf{v}(G) - 1,\\ \mathbf{c}(G-e) &= \mathbf{c}(G) \text{ or } \mathbf{c}(G) + 1, & \mathbf{c}(G/e) &= \mathbf{c}(G). \end{aligned}$$

An edge e is called an *isthmus* if $\mathbf{c}(G - e) = \mathbf{c}(G) + 1$.

Some examples of graphs are as follows:

- the *complete graph* K_n, a simple graph with vertices $\{1, \dots, n\}$, and edges $\{ij \mid 1 \leq i < j \leq n\}$.
- the *empty graph* N_n, with n vertices and no edges;
- the *loop graph* L_1, consisting of one vertex and a loop, and
- the *digon* D_2, with two vertices and two parallel nonloop edges.

K_4 $\quad$ N_4 $\quad$ L_1 $\quad$ D_2

A fundamental isomorphism invariant of a graph $G = (V, E)$ is its *Tutte polynomial* $\mathbf{T}_G(x, y)$. We describe here only a few of the many properties of the Tutte polynomial; see the excellent survey by Brylawski and Oxley [3] for more information. For our present purposes, the following recursive definition of the Tutte polynomial will be the most useful.

Definition 3. Let $G = (V, E)$ be a graph. The *Tutte polynomial* $\mathbf{T}_G(x, y)$ is defined as follows. If $\mathbf{e}(G) = 0$, then $\mathbf{T}_G(x, y) = 1$. Otherwise, $\mathbf{T}_G(x, y)$ is defined recursively as

$$\mathbf{T}_G(x,y) \;=\; \begin{cases} x \cdot \mathbf{T}_{G/e}(x,y) & \text{if } e \text{ is an isthmus,} \\ y \cdot \mathbf{T}_{G-e}(x,y) & \text{if } e \text{ is a loop,} \\ \mathbf{T}_{G-e}(x,y) + \mathbf{T}_{G/e}(x,y) & \text{otherwise.} \end{cases} \tag{1}$$

for any $e \in E(G)$. (It is a standard fact, albeit not immediate from the definition, that the choice of e does not matter.)

Many important graph isomorphism invariants satisfy deletion-contraction recurrences of this form, and consequently may be obtained as specializations of the Tutte polynomial. For instance, $\mathbf{T}_G(1, 1)$ equals the number of spanning forests of G, while $\mathbf{T}_G(2, 2) = 2^{\mathbf{e}(G)}$. In addition, one can obtain more refined combinatorial data, such as the chromatic and flow polynomials of G, by specializing the arguments x and y appropriately. Again, the reader is referred to [3] for the full story.

There is an equivalent definition of the Tutte polynomial as a certain generating function for the edge subsets $F \subset E$. Define the *rank* of F, denoted $r(F)$, as the cardinality of a maximal acyclic subset of F; equivalently, $r(F) = \mathbf{v}(G|_F) - \mathbf{c}(G|_F)$, where $G|_F$ is the edge-induced subgraph of G. Then the Tutte polynomial may be defined in closed form as the generating function

$$\mathbf{T}_G(x,y) \;=\; \sum_{F \subset E} (x-1)^{r(E)-r(F)}(y-1)^{|F|-r(F)} \tag{2}$$

[3, eq. 6.13]; this formula will be useful when we study the d-parallel matroid in Section 6.

2.2. **Picture Spaces.** The main objects of our study are *picture spaces*, projective algebraic sets which parameterize "pictures" of a graph in projective d-space $\mathbb{P}^d$ over a field $\mathbf{k}$. In this paper, we shall be concerned exclusively with the case $\mathbf{k} = \mathbb{C}$; however, the picture space may be defined over an arbitrary field. The reader is referred to [10], especially Section 3, for a more thorough discussion of the basic theory of picture spaces.

Definition 4. Let $G = (V, E)$ be a graph and $d \geq 2$ a positive integer. A *picture* of G in $\mathbb{P}^d$ is a tuple $\mathbf{P}$, consisting of a point $\mathbf{P}(v) \in \mathbb{P}^d$ for each $v \in V$ and a line $\mathbf{P}(e)$ in $\mathbb{P}^d$ for each $e \in E$, such that $\mathbf{P}(v) \in \mathbf{P}(e)$ whenever $e \in E(v)$. The set of all d-dimensional pictures is called the *d-dimensional picture space* of G, denoted $\mathcal{X}^d(G)$.

Example 5. A picture of N_1 is a point, so $\mathcal{X}^d(N_n) \cong \mathbb{P}^d$. More generally, a picture of the empty graph N_n consists of an ordered n-tuple of points in $\mathbb{P}^d$, so $\mathcal{X}^d(N_n) \cong (\mathbb{P}^d)^n$.

The data for a picture of the complete graph K_2 consists of two points $\mathbf{P}(1), \mathbf{P}(2)$ and a line $\mathbf{P}(12)$ containing both points. If $\mathbf{P}(1) \neq \mathbf{P}(2)$, then $\mathbf{P}(e)$ is determined uniquely, but if $\mathbf{P}(1) = \mathbf{P}(2)$, then the set of lines containing that point is isomorphic to $\mathbb{P}^{d-1}$. In fact, $\mathcal{X}^d(K_2)$ is the blowup of $\mathbb{P}^d \times \mathbb{P}^d$ along the diagonal $\{(p,p) \mid p \in \mathbb{P}^d\}$ (see also [10, Example 3.6]). This is a smooth, irreducible variety.

An easy consequence of the definition is that if $G_1, \dots, G_r$ are the connected components of G, then $\mathcal{X}^d(G) \cong \mathcal{X}^d(G_1) \times \dots \times \mathcal{X}^d(G_r)$. In particular, if $\mathbf{e}(G) = 0$, then $\mathcal{X}^d(G) \cong (\mathbb{P}^d)^{\mathbf{v}(G)}$.

Another elementary consequence is that if e is a loop, then $\mathcal{X}^d(G)$ is a $\mathbb{P}^{d-1}$-bundle over $\mathcal{X}^d(G-e)$. Indeed, a picture of G may be regarded as a picture of $G-e$, together with a line $\mathbf{P}(e)$ containing the point $\mathbf{P}(v)$ (where v is the single endpoint of e). The fiber $\mathbb{P}^{d-1}$ corresponds to the space of lines through $\mathbf{P}(v)$ in $\mathbb{P}^d$.

This last observation may be generalized as follows. Let $G = (V, E)$ be a graph and $G' = (V', E')$ a subgraph of G. There is a natural epimorphism

$$\mathcal{X}^d(G) \twoheadrightarrow \mathcal{X}^d(G') \tag{3}$$

given by forgetting the picture data for vertices and edges not in G'. Moreover, if $G'' = (V'', E'')$ is another subgraph of G, then the commutative diagram

$$\begin{tikzcd} & \mathcal{X}^d(G' \cup G'') \arrow[dl] \arrow[dr] & \\ \mathcal{X}^d(G') \arrow[dr] & & \mathcal{X}^d(G'') \arrow[dl] \\ & \mathcal{X}^d(G' \cap G'') & \end{tikzcd} \tag{4}$$

is easily seen to be a fiber product square. Here $G' \cup G''$ is the graph with vertices $V(G') \cup V(G'')$ and edges $E(G') \cup E(G'')$, and $G' \cap G''$ is defined similarly.

Consider the Boolean algebra on E, where each subset $E' \subset E$ is associated with the space $\mathcal{X}^d(V, E')$. By (3), there is an epimorphism $\mathcal{X}^d(V, E') \to \mathcal{X}^d(V, E'')$ whenever $E'' \subset E'$. Moreover, (4) may be interpreted as saying that the join of two spaces is the fiber product over their meet. Accordingly, $\mathcal{X}^d(G)$ is a fiber product of picture spaces of simple graphs—indeed, of graphs with one edge each, which correspond to the atoms of the Boolean algebra.

Example 6. Consider the digon D_2, with vertices $1, 2$ and parallel edges e_1, e_2. A picture of D_2 consists of two points $\mathbf{P}(1), \mathbf{P}(2)$ and two lines $\mathbf{P}(e_1), \mathbf{P}(e_2)$, such that $\mathbf{P}(i) \in \mathbf{P}(e_j)$ for $i, j \in \{1, 2\}$. By the previous remarks, we may describe $\mathcal{X}^d(D_2)$ as a fiber product:

$$\mathcal{X}^d(D_2) \;=\; \mathcal{X}^d(K_2) \underset{\mathcal{X}^d(N_2)}{\times} \mathcal{X}^d(K_2).$$

This space is neither smooth nor irreducible. Its irreducible components are

$$X_1 \;=\; \left\{\mathbf{P} \in \mathcal{X}^d(D_2) \;\mid\; \mathbf{P}(e_1) = \mathbf{P}(e_2)\right\},$$

which is isomorphic to $\mathcal{X}^d(K_2)$, and

$$X_2 \;=\; \left\{\mathbf{P} \in \mathcal{X}^d(D_2) \;\mid\; \mathbf{P}(1) = \mathbf{P}(2)\right\},$$

which is a bundle over $\mathbb{P}^d$ with fiber $\mathbb{P}^{d-1} \times \mathbb{P}^{d-1}$. The singular locus of $\mathcal{X}^d(D_2)$ is $X_1 \cap X_2$, which is isomorphic to a $\mathbb{P}^{d-1}$-bundle over $\mathbb{P}^d$.

It will be useful to classify the pictures $\mathbf{P}$ of a graph $G = (V, E)$ according to which points $\mathbf{P}(v)$ coincide. Thus we are led to the notion of a *cellule*.

Definition 7. Let $\sim_{\mathcal{A}}$ be an equivalence relation on $V(G)$ with equivalence classes $\mathcal{A} = \{A_1, \dots, A_s\}$. The corresponding *cellule* in $\mathcal{X}^d(G)$ is defined as

$$\mathcal{X}^d_{\mathcal{A}}(G) \;=\; \left\{\mathbf{P} \in \mathcal{X}^d(G) \;\mid\; \mathbf{P}(v) = \mathbf{P}(w) \iff v \sim_{\mathcal{A}} w\right\}.$$

A picture $\mathbf{P}$ is called *generic* if $\mathbf{P}(v) \neq \mathbf{P}(w)$ whenever $v \neq w$. Equivalently, $\mathbf{P}$ belongs to the *discrete cellule* $\mathcal{X}^d_{\mathcal{D}}(G)$, where $\mathcal{D}$ is the equivalence relation whose equivalence classes are all singletons.

Note that the cellules are pairwise disjoint, and their union is $\mathcal{X}^d(G)$. Furthermore, if e is an edge whose endpoints lie in different blocks of $\mathcal{A}$, then the points $\mathbf{P}(v)$ determine the line $\mathbf{P}(e)$ uniquely for each $\mathbf{P} \in \mathcal{X}^d_{\mathcal{A}}(G)$. In this case we say that e is *constrained* with respect to $\mathcal{A}$. Otherwise, varying $\mathbf{P}(e)$ while keeping the other data of $\mathbf{P}$ fixed gives a family of pictures in $\mathcal{X}^d_{\mathcal{A}}(G)$; this family is isomorphic to $\mathbb{P}^{d-1}$. Therefore $\mathcal{X}^d_{\mathcal{A}}(G)$ has the structure of a fiber bundle, whose base is $(\mathbb{P}^d)^{|\mathcal{A}|}$ with diagonals deleted (that is, the discrete cellule of the empty graph with vertices $V(G)$), and whose fiber is $(\mathbb{P}^{d-1})^{u(\mathcal{A})}$, where $u(\mathcal{A})$ is the number of unconstrained edges. In particular

$$\dim \mathcal{X}^d_{\mathcal{A}}(G) \;=\; d|\mathcal{A}| \;+\; (d-1) \cdot u(\mathcal{A}) \tag{5}$$

In addition to (3), there is a second canonical morphism between picture spaces, associated with any nonloop edge $e = vw$. First, we define the *coincidence locus* $Z_{vw}(G) = Z_e(G)$ as

$$Z_{vw}(G) \;:=\; \left\{\mathbf{P} \in \mathcal{X}^d(G) \;\mid\; \mathbf{P}(v) = \mathbf{P}(w)\right\} \;=\; \bigcup_{\mathcal{A}\,:\, v \sim_{\mathcal{A}} w} \mathcal{X}^d_{\mathcal{A}}(G). \tag{6}$$

Then there is a natural monomorphism

$$\mathcal{X}^d(G/e) \hookrightarrow \mathcal{X}^d(G - e) \tag{7}$$

whose image is the *coincidence locus* $Z_{vw}(G - e)$.

Remark 8. In light of (3) and (7), one may regard $\mathcal{X}^d$ as a contravariant functor from the category of graphs to that of projective algebraic sets. Here a morphism $\phi : G \to G'$ of graphs is a pair of maps $V(G) \to V(G')$ and $E(G) \to E(G')$ such that if v is an endpoint of e, then $\phi(v)$ is an endpoint of $\phi(e)$. Thus $\mathcal{X}^d$ sends ϕ to a morphism $\phi^{\#} :\ \mathcal{X}^d(G') \to \mathcal{X}^d(G)$ defined by

$$(\phi^{\#}\mathbf{P})(a) \;=\; \mathbf{P}(\phi(a)),$$

where $\mathbf{P} \in \mathcal{X}^d(G')$ and a is a vertex or an edge of G. Furthermore, if G is a subgraph of G', then $\phi^{\#}$ is an epimorphism, while if G' is a quotient of G (that is, it is obtained by a sequence of contractions), then $\phi^{\#}$ is a monomorphism.

3. Proof of the Main Theorem

For the rest of the paper, we work over the ground field $\mathbf{k} = \mathbb{C}$. In this section, we show that the homology groups (with integer coefficients) of $\mathcal{X}^d(G)$ are completely determined by the Tutte polynomial $\mathbf{T}_G(x, y)$. We have observed how the operations of deletion and contraction correspond to the morphisms (3) and (7) of picture spaces. In fact, these morphisms may be extended to a homotopy pushout square (10), which induces a Mayer-Vietoris long exact sequence of homology groups. The Mayer-Vietoris sequence permits us to write down a recursive formula for the Poincaré series of $\mathcal{X}^d(G)$; this formula may in turn be expressed as a specialization of the Tutte polynomial. For the tools of topology that we use here, an excellent reference is [7].

We will work with the *q-analogues* $[q]_d$ of integers d, defined as

$$[q]_d \;:=\; (1 - q^d)/(1 - q) \;=\; 1 + q + \cdots + q^{d-1}.$$

We shall see that for every graph G, the picture space $\mathcal{X}^d(G)$ has only free abelian even-dimensional homology. That is, the groups $H_i(\mathcal{X}^d(G)) = H_i(\mathcal{X}^d(G);\, \mathbb{Z})$ are free abelian for all i, and zero when i is odd. Accordingly, the structure of the homology may be encoded conveniently by the *compressed Poincaré series* whose coefficients are the even Betti numbers:

$$P^d_G(q) \;:=\; \sum_i q^i \;\operatorname{rank}_{\mathbb{Z}}\; H_{2i}(\mathcal{X}^d(G)).$$

We first consider two simple cases. If $\mathbf{e}(G) = 0$, then $\mathcal{X}^d(G) \cong (\mathbb{P}^d)^{\mathbf{v}(G)}$, while if $\mathbf{v}(G) = 1$, then $\mathcal{X}^d(G)$ is a $(\mathbb{P}^{d-1})^{\mathbf{e}(G)}$-bundle over $\mathbb{P}^d$. In both cases, $\mathcal{X}^d(G)$ is a simply connected complex manifold with only free abelian even-dimensional homology. Since the compressed Poincaré series of $\mathbb{P}^d_{\mathbb{C}}$ is $[d+1]_q$, we have

$$P^d_G(q) \;=\; \begin{cases} [d+1]_q^{\mathbf{v}(G)} & \text{if } \mathbf{e}(G) = 0, \\ [d]_q^{\mathbf{e}(G)}\,[d+1]_q & \text{if } \mathbf{v}(G) = 1. \end{cases} \tag{8}$$

In the case $\mathbf{v}(G) = 1$, we are using the fact that the Poincaré series of $\mathcal{X}^d(G)$ is the same as that of $\mathbb{P}^d \times (\mathbb{P}^{d-1})^{\mathbf{e}(G)}$ (see, e.g., Proposition 2.3 of [5]).

Now let G be an arbitrary graph and $e \in E(G)$ a nonloop edge. As in (3) and (7), we have maps

$$\begin{tikzcd} & \mathcal{X}^d(G) \arrow[d, "\pi"] \\ \mathcal{X}^d(G/e) \arrow[r, hook, "\iota"] & \mathcal{X}^d(G-e) \end{tikzcd} \tag{9}$$

Since $\iota(\mathcal{X}^d(G/e)) \subset Z_e(G-e)$ and $\pi^{-1}(\mathcal{X}^d(G/e)) = Z_e(G)$, we may "complete the square" to a commutative diagram

$$\begin{tikzcd} Z_e(G) \arrow[r, hook] \arrow[d] & \mathcal{X}^d(G) \arrow[d, "\pi"] \\ \mathcal{X}^d(G/e) \arrow[r, hook, "\iota"] & \mathcal{X}^d(G-e) \end{tikzcd} \tag{10}$$

Lemma 9. *The map $\pi : Z_e(G) \to \mathcal{X}^d(G/e)$ is a $\mathbb{P}^{d-1}$-fibration, and the diagram (10) is a homotopy pushout square.*

Proof. Let x, y be the endpoints of e. A picture in $Z_e(G)$ may be described by a picture of G/e, together with a line in $\mathbb{P}^d$ through the point $\mathbf{P}(x) = \mathbf{P}(y)$. Hence $\pi^{-1}(\mathbf{P}) \cong \mathbb{P}^{d-1}$. In particular, π induces an isomorphism between the fundamental groups of $Z_e(G)$ and $\mathcal{X}^d(G/e)$ (since complex projective d-space is simply connected). Since the map ι is a monomorphism, the diagram is a homotopy pushout. $\square$

It follows from Lemma 9 that (10) induces a Mayer-Vietoris long exact sequence

$$\begin{aligned} \ldots \quad \to \quad H_i(Z_e(G)) \quad \to \quad H_i(\mathcal{X}^d(G/e)) \oplus H_i(\mathcal{X}^d(G)) \quad \to \quad H_i(\mathcal{X}^d(G-e)) \\ \to \quad H_{i-1}(Z_e(G)) \quad \to \quad \ldots . \end{aligned} \tag{11}$$

This exact sequence allows us to compute the homology groups recursively, in the same manner as the definition (1) of the Tutte polynomial. We can now prove the main theorem, which we restate for convenience.

Theorem 1. Let G be a graph and $d \geq 2$ an integer. Then

(1) the picture space $\mathcal{X}^d(G)$ is path-connected and simply connected;
(2) the homology groups $H_i(\mathcal{X}^d(G))$ are free abelian for i even and zero for i odd; and
(3) the compressed Poincaré series $P_G^d(q)$ may be obtained from the Tutte polynomial $\mathbf{T}_G(x, y)$ by the formula

$$P_G^d(q) \;=\; ([d]_q - 1)^{\mathbf{v}(G)-\mathbf{c}(G)}\; [d+1]_q^{\mathbf{c}(G)}\; \mathbf{T}_G\left(\frac{[2]_q\,[d]_q}{[d]_q - 1},\, [d]_q \right). \tag{12}$$

Proof. We first show that $\mathcal{X}^d(G)$ is path-connected. Let $\mathcal{I}$ be the equivalence relation on V in which all vertices are equivalent. The corresponding *indiscrete cellule* $\mathcal{X}^d_{\mathcal{I}}(G)$ consists of the pictures $\mathbf{P}$ with $\mathbf{P}(v) = \mathbf{P}(w)$ for all $v, w \in V$. Thus $\mathcal{X}^d_{\mathcal{I}}(G)$ is a $(\mathbb{P}^{d-1})^{\mathbf{e}(G)}$-bundle over $\mathbb{P}^d$; in particular it is path-connected. On the other hand, an arbitrary picture $\mathbf{P}$ can be deformed continuously into a picture in the indiscrete cellule $\mathcal{X}^d_{\mathcal{I}}(G)$ as follows. Choose a system of local affine coordinates for $\mathbb{P}^d$ such that no $\mathbf{P}(v)$ lies on the hyperplane at infinity, rescale the coordinates of all $\mathbf{P}(v)$ uniformly by a constant λ, and let λ tend to zero. Since $\mathcal{X}^d_{\mathcal{I}}(G)$ is path-connected, so is $\mathcal{X}^d(G)$.

When $\mathbf{v}(G) = 1$ or $\mathbf{e}(G) = 0$, the formula for $P_G^d(q)$ follows from (8). For the general case, we induct on $\mathbf{v}(G)$ and $\mathbf{e}(G)$. In particular, we may assume that the theorem holds for $\mathcal{X}^d(G-e)$ and $\mathcal{X}^d(G/e)$. Since $Z_e(G)$ is a $\mathbb{P}^{d-1}$-bundle over $\mathcal{X}^d(G/e)$, it follows from Proposition 2.3 of [5] that $Z_e(G)$ has only free abelian even-dimensional homology, and its compressed Poincaré series is $[d]_q\, P^d_{G/e}(q)$. Since (10) is a homotopy pushout and the map $Z_e(G) \to \mathcal{X}^d(G/e)$ induces an isomorphism of fundamental groups, the map π also induces an isomorphism. By induction, $\mathcal{X}^d(G)$ is simply connected. Furthermore, the Mayer-Vietoris sequence (11) gives $H_i(\mathcal{X}^d(G)) = 0$ for i odd, and splits into short exact sequences

$$0 \;\to\; H_i(Z) \;\to\; H_i(\mathcal{X}^d(G/e)) \oplus H_i(\mathcal{X}^d(G)) \;\to\; H_i(\mathcal{X}^d(G-e)) \;\to\; 0 \tag{13}$$

for i even. In particular, $H_i(\mathcal{X}^d(G))$ is free abelian for all i, Recall the definition (1) of the Tutte polynomial. In order to establish (12), it suffices to show that

$$P^d_G(q) = [d+1]_q^{\mathbf{v}(G)} \qquad \text{if } \mathbf{e}(G) = 0, \tag{14a}$$

$$P^d_G(q) = [d]_q \; P^d_{G-e}(q) \qquad \text{if } e \in E \text{ is a loop,} \tag{14b}$$

$$P^d_G(q) = [2]_q \; [d]_q \; P^d_{G/e}(q) \qquad \text{if } e \text{ is an isthmus,} \tag{14c}$$

$$P^d_G(q) = P^d_{G-e}(q) \; + \; ([d]_q - 1) P^d_{G/e}(q) \qquad \text{otherwise.} \tag{14d}$$

Indeed, (14a) is precisely (8), and (14d) follows from (13). If e is a loop, then $P^d_G(q)$ is a $\mathbb{P}^{d-1}$-bundle over $P^d_{G/e}(q)$, which implies (14b). Now suppose that e is an isthmus with endpoints v_1, v_2. It suffices to consider the case that G is connected and that $G - e$ has two connected components G_1, G_2, with $v_i \in V(G_i)$. Then

$$\mathcal{X}^d(G-e) \;\cong\; \mathcal{X}^d(G_1) \;\times\; \mathcal{X}^d(G_2)$$

and

$$\mathcal{X}^d(G/e) \;\cong\; \mathcal{X}^d(G_1) \underset{\mathbb{P}^d}{\times} \mathcal{X}^d(G_2)$$

where the maps $\mathcal{X}^d(G_i) \to \mathbb{P}^d$ are given by $\mathbf{P} \mapsto \mathbf{P}(v_i)$ for $i = 1, 2$. It follows that

$$P^d_{G-e}(q) \;=\; [d+1]_q \; P^d_{G/e}(q).$$

Substituting this into the recurrence defining $P^d_G(q)$, we obtain

$$P^d_G(q) \;=\; ([d]_q - 1) \; P^d_{G/e}(q) \;+\; [d+1]_q \; P^d_{G/e}(q) \;=\; [2]_q \; [d]_q \; P^d_{G/e}(q)$$

yielding (14c) as desired. □

4. Pictures on a Complex Manifold

We may generalize the preceding results by replacing $\mathbb{P}^d_{\mathbb{C}}$ with a complex manifold M of dimension d, provided that M has only free abelian even-dimensional homology. That is, we study the space $\mathcal{X}^M(G)$ of "M-pictures" of G. In this setting, we are faced immediately with the problem of how to describe "lines" in M corresponding to edges of G. To resolve this question, we rely on the observations that for $e = vw \in E$, the data $\mathbf{P}(e)$ is redundant if $\mathbf{P}(v) \neq \mathbf{P}(w)$, and if $\mathbf{P}(v) = \mathbf{P}(w)$, then $\mathbf{P}(e)$ may be described as a tangent direction at $\mathbf{P}(v)$. With this in mind, we can formulate a definition of $\mathcal{X}^M(G)$ which specializes to $\mathcal{X}^d(G)$ in the case that $M = \mathbb{P}^d$, and whose Poincaré series obeys a recurrence akin to (12). We omit the proofs, which are analogous to those of Theorem 1.

Denote by TM the tangent bundle of M, and by T_pM the tangent space of M at a point p. Note that TM is a rank-d complex vector bundle over M, and its projectivization $\mathbb{P}(TM)$ (see Section 7.1) is a bundle over M with fiber $\mathbb{P}^{d-1}$.

Definition 10. Let $G = (V, E)$ be a graph and M a simply connected, compact complex manifold of dimension d which has only free abelian even-dimensional homology. The *space of M-pictures of G*, denoted $\mathcal{X}^M(G)$, is defined recursively as follows:

- If $E = \emptyset$, then $\mathcal{X}^M(G)$ is the direct product of $\mathbf{v}(G)$ copies of M.
- If $G = L_1$, then $\mathcal{X}^M(G) = \mathbb{P}(TM)$.
- If $G = K_2$, then $\mathcal{X}^M(G)$ is the blowup of $M \times M$ along the diagonal. (Note that blowups exist in the category of manifolds.)

- Suppose that E is the disjoint union of nonempty sets E_1 and E_2, and let $G_i = (V, E_i)$ for $i = 1, 2$. Also, let $G_0 = (V, \emptyset)$. In this case, we define $\mathcal{X}^M(G)$ as a fiber product:

$$\mathcal{X}^M(G) \;=\; \mathcal{X}^M(G_1) \underset{\mathcal{X}^M(G_0)}{\times} \mathcal{X}^M(G_2).$$

Let $e = vw \in E$. As before, we define the coincidence locus $Z_e(\mathcal{X}^M(G))$ to be the set of pictures $\mathbf{P} \in \mathcal{X}^M(G)$ for which $\mathbf{P}(v) = \mathbf{P}(w)$. Then the analogue of (10) is the commutative diagram

$$\begin{tikzcd} & Z_e(\mathcal{X}^M(G)) \arrow[r, hook] \arrow[d, "\pi"] & \mathcal{X}^M(G) \arrow[d] \\ \mathcal{X}^M(G/e) \arrow[r, phantom, "\cong"] & Z_e(\mathcal{X}^M(G-e)) \arrow[r, hook] & \mathcal{X}^M(G-e) \end{tikzcd} \tag{15}$$

which is a homotopy pushout square, with the map π a $\mathbb{P}^{d-1}$-fibration. As in the proof of Theorem 1, the Mayer-Vietoris sequence associated with (15) implies (by induction) that $\mathcal{X}^M(G)$ has has only free abelian even-dimensional homology, and leads to a recurrence for its Poincaré series. This recurrence may in turn be translated into a formula in terms of the Tutte polynomial of G:

$$\sum_i q^i \;\operatorname{rank}_{\mathbb{Z}}\; H_{2i}(\mathcal{X}^M(G)) \;=\; ([d]_q - 1)^{\mathbf{v}(G)-\mathbf{c}(G)} \;\; P(M)^{\mathbf{c}(G)} \;\; \mathbf{T}_G\left(\frac{P(M) + [d]_q - 1}{[d]_q - 1},\, [d]_q\right) \tag{16}$$

where $P(M) = \operatorname{Poin}(M;\; q^{1/2})$. One can easily check that this formula specializes to that of Theorem 1 in the case $M = \mathbb{P}^d$, $P(M) = [d+1]_q$.

5. The Picture Variety

It is natural to ask if one can inductively calculate the Poincaré series of the picture variety $\mathcal{V}^d(G)$, as we did above for the picture space $\mathcal{X}^d(G)$. However, it seems that the only part of Theorem 1 that can be extended to $\mathcal{V}^d(G)$ in general is the simple argument for path-connectivity. The problem is that the coincidence locus of an edge in a picture variety is difficult to describe in general; it is not always isomorphic to $\mathcal{V}^d(G/e)$. The analogue of (10) for picture varieties is the blowup diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"] & \mathcal{V}^d(G) \arrow[d] \\ Z_e(G-e) \arrow[r] & \mathcal{V}^d(G-e) \end{tikzcd} \tag{17}$$

where $Z_e(G-e)$ now denotes the coincidence locus of e in $\mathcal{V}^d(G-e)$ and E is the exceptional divisor of the blowup (see Remark 3.5 of [10]). While (17) is a homotopy pushout square, the coincidence locus $Z_e(G-e)$ may not be irreducible or even equidimensional. So the map p in (17) may not be a $\mathbb{P}^{d-1}$-fibration, and there appears to be no general way to calculate the Poincaré series of $\mathcal{V}^d(G)$ using a Mayer-Vietoris argument.

Example 11. To give a sense of the difficulties involved, we sketch the calculation of the Poincaré series of $\mathcal{V}^d(G)$ in the case $G = K_4$, $d = 2$. In a sense, this is the simplest case not covered by Theorem 1, since K_4 is the only simple graph with 4 or fewer vertices for which $\mathcal{V}^2(G) \neq \mathcal{X}^2(G)$.

Let e be the edge joining vertices 1 and 2: then $\mathcal{V}^2(G)$ is the blowup of $\mathcal{V}^2(G-e)$ along the coincidence locus of e. This is the disjoint union of five cellules, corresponding to the five equivalence relations $\mathcal{A}$ on $V(G) = \{1,2,3,4\}$ with $1,2$ equivalent. Using the cellule dimension formula (5), one finds that one of these five cellules (the indiscrete cellule) has codimension 1, three have codimension 2, and one has codimension 3. Every component of Z must have codimension at most 2 (since Z is defined locally by two equations; see [10]), so we can ignore the codimension-3 cellule. On the other hand, blowing up a codimension-1 subvariety is a trivial operation, so we can also ignore the indiscrete cellule. Let Z' be the union of the three remaining cellules; then $\mathcal{V}^2(G)$ is the blowup of $\mathcal{V}^2(G-e)$ along Z', and the exceptional divisor E' of the blowup is a $\mathbb{P}^1$-bundle over Z'. Moreover, replacing Z, E with Z', E' in (17), we obtain a homotopy pushout square, and a Mayer-Vietoris sequence akin to (11).

We can now verify that Z' is connected, simply connected, and has only free abelian even-dimensional homology. Moreover, the Poincaré series of Z' may be calculated by applying the Mayer-Vietoris sequence to its decomposition into three irreducible components. By Proposition 2.3 of [5], E' shares the first three properties, and its compressed Poincaré series is $(1+q)$ times that of Z'. Finally, $\mathcal{V}^2(G-e) = \mathcal{X}^2(G-e)$ because $G-e$ is rigidity-independent (see [10, Theorem 4.5]), so the Poincaré series of that space is given by Theorem 1. By an argument similar to the proof of Theorem 1, $\mathcal{V}^2(G)$ has only free abelian even-dimensional homology, and its compressed Poincaré series is

$$(1+q)(1+q+q^2)(q^5+13q^4+32q^3+24q^2+8q+1).$$

It is not clear what combinatorial significance this polynomial has, if any.

The technique of ignoring the codimension-1 components from the coincidence locus does allow us to prove a general fact about picture varieties in the projective plane.

Proposition 12. *Let G be a simple graph. Then $\mathcal{V}^2(G)$ is simply connected.*

Proof. Let $e \in E(G)$. We have seen that in the blowup diagram (17), every component of the coincidence locus $Z_e(G-e)$ has codimension 1 or 2. As in the preceding example, the codimension-1 components may safely be ignored, and the map p is a fibration with simply connected fiber $\mathbb{P}^1$, hence induces an isomorphism of fundamental groups. Since (17) is a homotopy pushout, we also have an isomorphism between the fundamental groups of $\mathcal{V}^2(G)$ and $\mathcal{V}^2(G-e)$, so we are done by induction on the number of edges. $\square$

If $d > 2$, then this argument does not go through because Z' may not be equidimensional (the codimensions of its components may vary between 2 and d), in which case p is not a fibration.

A fundamental difficulty in extending our results to picture varieties is that it is unclear how to define $\mathcal{V}^d(G)$ in the case that G is not simple. If we "naively" take $\mathcal{V}^d(G)$ to be the closure of the locus of generic pictures in $\mathcal{X}^d(G)$, then the structure of parallel edges is lost. Indeed, if e, e' are parallel, then $\mathbf{P}(e) = \mathbf{P}(e')$

for all generic pictures, hence for all pictures in $\mathcal{V}^d(G)$, which implies that $\mathcal{V}^d(G)$ is isomorphic to the picture variety of its underlying simple graph.

A less trivial approach is to define $\mathcal{V}^d(G)$ as the locus of all pictures $\mathbf{P} \in \mathcal{X}^d(G)$ such that each "underlying simple picture" of $\mathbf{P}$ belongs to the picture variety of the corresponding underlying simple graph. This may be described as a fiber product, in the spirit of (4). Specifically, we define $\mathcal{V}^d(G)$ as the fiber product of the picture varieties $\mathcal{V}^d(G_i)$, where $G_1, G_2, \dots$ are the underlying simple graphs of G. This definition preserves the structure of parallel edges; however, there still exist coincidence loci which are hard to describe.

Example 13. Consider the graphs G and G/e given by the following figure.

G G/e

Note that the nonparallel edges f, g become parallel in G/e. Let G_1 (resp. G_2) be the underlying simple graph of G/e containing f (resp. g), and let $\mathcal{A}$ be the partition of $V(G)$ in which $\{y, z\}$ is the only nonsingleton block. Since each G_i is isomorphic to K_4, Theorem 5.5 of [10]) and our fiber product construction imply that the picture variety $\mathcal{V}^d(G/e)$ is the vanishing locus in $\mathcal{X}^d(G/e)$ of two polynomials $\tau(G_1), \tau(G_2)$ (called in [10] the *tree polynomials* of G_1 and G_2 respectively). Then $\tau(G_1)$ vanishes on $\mathcal{V}^d_{\mathcal{A}}(G - e)$ but $\tau(G_2)$ does not, because $E(G) \setminus \{e, g\}$ is a rigidity circuit (see [10]) while $E(G) \setminus \{e, f\}$ is independent. Accordingly, $\mathcal{V}^d_{\mathcal{A}}(G - e)$ is isomorphic neither to $\mathcal{X}^d(G/e)$ nor $\mathcal{V}^d(G/e)$.

We suspect that $\mathcal{V}^d(G)$ has in general only free abelian even-dimensional homology; it may be possible to describe the centers of the iterated blowings-up sufficiently well to apply a result such as Proposition 2.3 of [5].

6. Applications to Parallel Independence

Let $\mathbf{P}$ be a d-dimensional picture of a graph $G = (V, E)$. Consider a physical model of $\mathbf{P}$ consisting of a "bar" for each edge e and a "joint" for each vertex v. If e has v as an endpoint, then the corresponding bar is attached to the corresponding joint. The bars may cross, and their lengths are allowed to vary, but we fix the angles at which the bars are attached to the joints. Thus, for example, a square framework may be deformed to produce an arbitrary rectangle, but not any other rhombus. Under what conditions on G is such a model rigid? That is, when is the model determined up to scaling by specifying the attaching angles? These and similar questions are the focus of *combinatorial rigidity theory*; for more details, see, e.g., [6] and [12].

The graph G (or, more properly, its edge set) is said to be *d-parallel independent* if for a generic picture in $\mathcal{X}^d(G)$, the directions of the lines representing edges are mutually unconstrained. For instance, K_3 is 2-parallel independent, because the slopes of two lines of a triangle in the plane do not determine that of the third.

However, K_3 is d-parallel dependent for $d \geq 3$, because the three sides of a triangle must be coplanar. (The smallest simple graph which is 2-parallel dependent is K_4.)

The condition of d-parallel independence is in fact a *matroid* independence condition; for the reader not familiar with matroids, we remark here only that it satisfies certain axioms which abstract the idea of linear independence in a vector space. Returning to rigidity for a moment, a generic d-dimensional model of G, as described above, will hold its shape if and only if $E(G)$ contains a d-parallel independent set of cardinality $d \cdot \mathbf{v}(G) - (d+1)$ [12, Theorem 8.2.2].

The Poincaré series formula of Theorem 1 may be applied to give a criterion for d-parallel independence in terms of the Tutte polynomial. The central idea is that parallel independence can be determined from the dimension and number of maximal-dimensional components of $\mathcal{X}^d(G)$, which can in turn be read off from its Poincaré series. Specifically, let X be a connected algebraic subset of $\mathbb{P}^n_{\mathbb{C}}$ of (complex) dimension d, and let c be the number of irreducible components of X of dimension d. By [4, Appendix A, Lemmas 2 and 4], the leading term of the Poincaré series $\mathrm{Poin}(X;q)$ is cq^{2d}. We apply this fact in the case that $X = \mathcal{X}^d(G)$. (Note that the picture space is defined as an algebraic subset of a product of Grassmannians, which may be identified in some complex projective space by means of the Plücker and Segre embeddings (see, e.g., [4, §9.1].)

Theorem 14. *Let $G = (V,E)$ be a graph and $d \geq 2$ an integer. Then E is independent in the generic d-parallel matroid if and only if the polynomial*

$$([d]_q - 1)^{\mathbf{v}(G)-\mathbf{c}(G)} \; \mathbf{T}_G\left(\frac{[2]_q\,[d]_q}{[d]_q - 1},\, [d]_q\right)$$

is monic of degree $d(\mathbf{v}(G) - \mathbf{c}(G))$.

Proof. It suffices to consider the case that G is simple, since L_1 and D_2 are circuits in the d-parallel matroid for every d (this follows from Theorem 8.2.2 of [12]). In this case, Theorem 4.5 of [10] tells us that E is independent with respect to the 2-dimensional generic rigidity matroid if and only if $\mathcal{V}^2(G) = \mathcal{X}^2(G)$. The argument of [10] can be generalized as follows: for all d, E is d-parallel independent if and only if $\mathcal{V}^d(G) = \mathcal{X}^d(G)$. Recall that $\mathcal{V}^d(G)$ is an irreducible component of $\mathcal{X}^d(G)$ of dimension $d \cdot \mathbf{v}(G)$ and all other components have equal or greater dimension. By the preceding remarks, G is d-parallel independent if and only if the compressed Poincaré series of $\mathcal{X}^d(G)$, as given by Theorem 1, is monic of degree $d \cdot \mathbf{v}(G)$. By (2), we have

$$\mathbf{T}_G\left(\frac{[2]_q\,[d]_q}{[d]_q - 1},\, [d]_q\right) \;=\; \frac{f(q)}{([d]_q - 1)^{r(E)}} \;=\; \frac{f(q)}{([d]_q - 1)^{\mathbf{v}(G)-\mathbf{c}(G)}}$$

where $f(q)$ is a polynomial in q, and r is the rank function on subsets of E. Therefore, dividing the compressed Poincaré series by $[d+1]_q^{\mathbf{c}(G)}$, which itself is monic of degree $d \cdot \mathbf{c}(G)$, produces a polynomial, which is monic of degree $d(\mathbf{v}(G) - \mathbf{c}(G))$ if and only if G is d-parallel independent. □

One should note that this does not lead to an efficient algorithm for computing the d-parallel behavior of an edge set. The Tutte polynomial is exponentially hard to compute (see [8]), and determining whether an edge set F is d-parallel independent appears to be polynomial-time in $|F|$ (see, e.g., [6, §4.10].

7. Orchards

A graph $G = (V, E)$ is an *orchard* if every edge of G is either a loop or an isthmus.[1] The orchards are precisely the graphs whose picture spaces are smooth; indeed, the picture space of an orchard G is an iterated bundle in which each fiber is a complex projective space. Thus we may write down an explicit "Borel" presentation for the integral cohomology ring $H^*(\mathcal{X}^d(G)) = H^*(\mathcal{X}^d(G); \mathbb{Z})$ in terms of the Chern classes of natural line bundles on $\mathcal{X}^d(G)$. The Borel presentation allows us to formulate a "Schubert calculus of orchards", that is, a method of answering enumerative geometry questions about $\mathcal{X}^d(G)$ in terms of Schubert polynomial calculations.

7.1. **The Cohomology Ring.** Denote the number of isthmuses and loops of a graph G by $\mathbf{i}(G)$ and $\mathbf{l}(G)$ respectively. If G is an orchard, then by (2) its Tutte polynomial is $\mathbf{T}_G(x, y) \;=\; x^{\mathbf{i}(G)} y^{\mathbf{l}(G)}$, so by Theorem 1 the compressed Poincaré series of $\mathcal{X}^d(G)$ is

$$P^d_G(q) \;=\; [d+1]_q^{\mathbf{c}(G)}\ [2]_q^{\mathbf{i}(G)}\ [d]_q^{\mathbf{e}(G)}. \tag{18}$$

This polynomial is palindromic, suggesting that the picture space of an orchard is smooth (by Poincaré duality). In fact, more is true.

Proposition 15. *Let $G = (V, E)$ be a graph and $d \geq 2$. The picture space $\mathcal{X}^d(G)$ is smooth if and only if G is an orchard.*

Proof. Suppose that G is an orchard. If $\mathbf{e}(G) = 0$, then $\mathcal{X}^d(G) = (\mathbb{P}^d)^{\mathbf{v}(G)}$ is smooth. Otherwise, let $e \in E$. We shall shortly prove in Theorem 17 that $\mathcal{X}^d(G)$ is the projectivization of a complex vector bundle with base $\mathcal{X}^d(G - e)$ (if e is a loop) or $\mathcal{X}^d(G/e)$ (if e is an isthmus). It follows by induction on $\mathbf{e}(G)$ that $\mathcal{X}^d(G)$ is smooth.

Now suppose that G is not an orchard. It suffices to consider the case that G is connected and has no loops (since $\mathcal{X}^d(G)$ is the Cartesian product of the picture spaces of the connected components of G, and loops correspond to $\mathbb{P}^{d-1}$-bundles). In this case $\mathbf{e}(G) \geq \mathbf{v}(G)$. Let $\mathbf{P}$ be a generic picture, so that $\mathcal{X}^d(G)$ looks locally like $(\mathbb{P}^d)^{\mathbf{v}(G)}$ near $\mathbf{P}$. Therefore

$$\dim T_{\mathbf{P}}(\mathcal{X}^d(G)) \;=\; d \cdot \mathbf{v}(G). \tag{19}$$

At the other extreme, let $\mathbf{Q}$ be a picture such that $\mathbf{Q}(v) = \mathbf{Q}(w) = p$ for all $v, w \in V$ and $\mathbf{Q}(e) = \mathbf{Q}(f) = \ell$ for all $e, f \in E$. Then we can deform $\mathbf{Q}$ into another picture of G in the following ways:

- for any $v \in V$, move the point $\mathbf{Q}(v)$ along the line ℓ;
- for any $e \in E$, rotate $\mathbf{Q}(e)$ through the space of lines containing p; or
- move the line in a direction "orthogonal" to itself (producing another totally degenerate picture).

(Here "orthogonal" really means "orthogonal with respect to local affine coordinates in which the hyperplane at infinity does not contain p".) It follows that

$$\begin{aligned} \dim T_{\mathbf{Q}}(\mathcal{X}^d(G)) \;&\geq\; \mathbf{v}(G) \;+\; (d-1)\cdot\mathbf{e}(G) \;+\; (d-1) \\ &\geq\; d\cdot\mathbf{v}(G) + d - 1 \\ &>\; d\cdot\mathbf{v}(G), \end{aligned} \tag{20}$$

[1] To justify this terminology, G is a forest with possibly some added loops, which resemble fruit hanging from the trees.

which together with (19) implies that $\mathcal{X}^d(G)$ is not smooth. □

Remark 16. Let G be the "acetylene" graph . This is not an orchard, so $\mathcal{X}^2(G)$ is not smooth. However, its Poincaré series is palindromic: the formula of Theorem 1 yields $P^2_G(q) = 1 + 5q + 9q^2 + 9q^3 + 5q^4 + q^5$.

In order to give a presentation of the cohomology ring $H^*(\mathcal{X}^d(G))$, we will need several facts about vector bundles over complex manifolds. We summarize these briefly. (For more details, see chapter IV of [2], especially pp. 269–271.)

Let M be a complex manifold and $\mathcal{E}$ a complex vector bundle of rank d over M. The *projectivization* of $\mathcal{E}$ is the fiber bundle $\mathbb{P}(\mathcal{E}) \xrightarrow{\pi} M$ whose fiber at a point $m \in M$ is $\mathbb{P}(\mathcal{E})_m = \mathbb{P}(\mathcal{E}_m)$, that is, the space of lines through the origin in the fiber of $\mathcal{E}$ at m. Thus $\pi^{-1}\mathcal{E}$ is a rank-d vector bundle over $\mathbb{P}(\mathcal{E})$. The *tautological subbundle* $\mathcal{L}$ is the line bundle on $\mathbb{P}(\mathcal{E})$ defined fiberwise by $\mathcal{L}_p = p$ (regarding p as a line in $\mathcal{E}_{\pi(p)}$). Thus we have a commutative diagram

$$\begin{tikzcd} \mathcal{L} \arrow[r] \arrow[dr] & \pi^{-1}\mathcal{E} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ & \mathbb{P}(\mathcal{E}) \arrow[r, "\pi"] & M \end{tikzcd} \tag{21}$$

and with this setup, one has

$$H^*(\mathbb{P}(\mathcal{E})) \;\cong\; H^*(M)[x] \,/\, \left\langle x^d + c_1(\mathcal{E})x^{d-1} + \cdots + c_d(\mathcal{E}) \right\rangle \tag{22}$$

where $c_i(\mathcal{E})$ denotes the ith Chern class of $\mathcal{E}$, and $x = c_1(\mathcal{L}^*)$. The Chern classes satisfy the following properties (see [2, p. 271]). First, if $\mathcal{E}$ is a trivial bundle then $c_i(\mathcal{E}) = 0$ for $i > 0$. Second, if $\pi : X \to Y$ is a map of spaces and $\mathcal{E}$ is a vector bundle on Y, then

$$c_i(\pi^{-1}\mathcal{E}) = \pi^*(c_i(\mathcal{E})). \tag{23}$$

Third, there is the *Whitney product formula*: if $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ is an exact sequence of vector bundles on X, then

$$c(\mathcal{E}) \;=\; c(\mathcal{E}')\, c(\mathcal{E}'') \tag{24}$$

where $c(\mathcal{E}) = 1 + c_1(\mathcal{E}) + c_2(\mathcal{E}) + \dots$ is the *total Chern class*. In particular, if $\mathcal{L}$ is a line bundle with dual bundle $\mathcal{L}^*$, then $\mathcal{L} \otimes \mathcal{L}^*$ is a trivial bundle, so

$$c_i(\mathcal{L}^*) = -c_i(\mathcal{L}). \tag{25}$$

Given an orchard G, we may "prune" G to obtain a smaller orchard G' such that $\mathcal{X}^d(G)$ is the projectivization of a vector bundle $\mathcal{E}$ on $\mathcal{X}^d(G')$. The tautological line bundle of $\mathcal{E}$ may be expressed fiberwise in terms of the data $\mathbf{P}(v)$ and $\mathbf{P}(e)$, allowing us to describe the cohomology ring of $\mathcal{X}^d(G)$ inductively by means of (22).

Denote by $Gr(r, \mathbb{C}^n)$ the Grassmannian variety of r-dimensional vector subspaces of $\mathbb{C}^n$. Thus $\mathbf{P}(v)$ and $\mathbf{P}(e)$ may be regarded as elements of $Gr(1, \mathbb{C}^{d+1})$ and $Gr(2, \mathbb{C}^{d+1})$, respectively. All the vector bundles we shall consider on $\mathcal{X}^d(G)$ are subbundles of the trivial bundle $\mathcal{W} = \mathbb{C}^{d+1} \times \mathcal{X}^d(G)$. For each $v \in V$, there is a line bundle $\mathcal{L}_v \subset \mathcal{W}$ with fiber

$$(\mathcal{L}_v)_{\mathbf{P}} \;=\; \mathbf{P}(v) \tag{26}$$

and for each edge $e \in E(v)$, there is a line bundle $\mathcal{K}_{e,v}$ with fiber

$$(\mathcal{K}_{e,v})_{\mathbf{P}} \;=\; \mathbf{P}(e)/\mathbf{P}(v). \tag{27}$$

The Chern classes of these bundles generate the cohomology ring of $\mathcal{X}^d(G)$, as we now prove.

Theorem 17. *Let $G = (V, E)$ be an orchard. Then $H^*(\mathcal{X}^d(G); \mathbb{Z}) \cong R/I$, where*

$$R \;=\; \mathbb{Z}\big[x_v, y_{e,v} \;:\; v \in V,\; e \in E(v)\big]$$

and I is the ideal

$$\left\langle \begin{array}{ll} x_v^{d+1} & \text{for } v \in V, \\ h_d(x_v,\, y_{e,v}) & \text{for } v \in V,\ e \in E(v), \\ x_v - x_w + y_{e,v} - y_{e,w},\ \ x_v y_{e,v} - x_w y_{e,w} & \text{for } e = vw \end{array} \right\rangle$$

where $h_d(x, y) = x^d + x^{d-1}y + \cdots + xy^{d-1} + y^d$.

Proof. It suffices to consider the case that G is connected, because the picture space of a graph is the product of the picture spaces of its connected components.

We induct on $\mathbf{e}(G)$. If $E = \emptyset$, then

$$\begin{aligned} H^*(\mathcal{X}^d(G)) \;&\cong\; \mathbb{Z}[x_v \;:\; v \in V] \;/\; \big\langle x_v^{d+1} \;:\; v \in V \big\rangle \\ &\cong\; \bigotimes_{v \in V} \mathbb{Z}[x_v] \;/\; \big\langle x_v^{d+1} \big\rangle \end{aligned}$$

(the tensor product over $\mathbb{Z}$), because $\mathcal{X}^d(G) \cong (\mathbb{P}^d)^{\mathbf{v}(G)}$. Furthermore, x_v is the dual Chern class of the line bundle with fiber $\mathbf{P}(v)$ (that is, the pullback to $\mathcal{X}^d(G)$ of the tautological line bundle on the copy of $\mathbb{P}^d$ indexed by v).

Now, suppose that $\mathbf{e}(G) > 0$ and that the theorem holds for all orchards with fewer edges than G. First, consider the case that there is at least one loop e, incident to a vertex v. Let $G' = G - e$. Consider the vector bundles $\mathcal{L}_v$ and $\mathcal{W}$ on $\mathcal{X}^d(G')$, of ranks 1 and $d+1$ respectively. A picture of G may be specified by giving a picture $\mathbf{P}$ of G', together with a plane $\mathbf{P}(e) \in Gr(2, \mathbb{C}^{d+1})$ such that $\mathbf{P}(v) \subset \mathbf{P}(e)$. Equivalently, $\mathcal{X}^d(G) = \mathbb{P}(\mathcal{Q})$, where $\mathcal{Q}$ is the quotient bundle $\mathcal{W}/\mathcal{L}_v$. Let $\mathcal{K}_{e,v}$ be the tautological line bundle associated to $\pi^{-1}\mathcal{Q}$, so we have a diagram

$$\begin{tikzcd}
\mathcal{K}_{e,v} \arrow[r] \arrow[dr] & \pi^{-1}\mathcal{Q} \arrow[d] & & \mathcal{L}_v \arrow[r] \arrow[dr] & \mathcal{W} \arrow[r] \arrow[d] & \mathcal{Q} \arrow[dl] \\
& \mathcal{X}^d(G) = \mathbb{P}(\mathcal{Q}) \arrow[rrr, "\pi"] & & & \mathcal{X}^d(G') &
\end{tikzcd}$$

Note that the map π is a $\mathbb{P}^{d-1}$-fibration. Let $y_{e,v} = c_1(\mathcal{K}^*_{e,v})$. By (22),

$$H^*(\mathcal{X}^d(G)) \;=\; H^*(\mathcal{X}^d(G'))[y_{e,v}] \;/\; \big\langle y_{e,v}^d + c_1(\mathcal{Q})y_{e,v}^{d-1} + \cdots + c_d(\mathcal{Q}) \big\rangle .$$

By the Whitney formula, $c(\mathcal{L}_v)c(\mathcal{Q}) = c(\mathcal{W}) = 1$. Also, $c(\mathcal{L}_v) = 1 - x_v$, so

$$H^*(\mathcal{X}^d(G)) \;\cong\; H^*(\mathcal{X}^d(G'))[y_{e,v}] \;/\; \big\langle x_v^{d+1},\, x_v^d + x_v^{d-1}y_{e,v} + \cdots + y_{e,v}^d \big\rangle$$

as desired.

Now, suppose that G has a vertex v with $E(v) = \{e\}$. Since G is connected, e is not a loop. Let w be the other endpoint of e, and let G' be the orchard obtained from G by deleting v and e and adding a loop $\tilde{e}$ incident to w. Forgetting the coordinates of $\mathbf{P}(v)$ and setting $\mathbf{P}(\tilde{e}) = \mathbf{P}(e)$ gives an epimorphism $\mathcal{X}^d(G) \to \mathcal{X}^d(G')$. Moreover, a picture of G may be specified by giving a picture $\mathbf{P}$ of G', together with a line $\mathbf{P}(v) \subset \mathbf{P}(e)$. Hence $\mathcal{X}^d(G)$ is the projectivization of the rank-2 bundle $\mathcal{F} \to \mathcal{X}^d(G')$ whose fiber is $\mathcal{F}_{\mathbf{P}} = \mathbf{P}(e)$, and whose tautological subbundle is $\mathcal{L}_v$. We thus have a diagram analogous to (21):

$$\begin{tikzcd} \mathcal{L}_v \arrow[r] \arrow[dr] & \pi^{-1}\mathcal{F} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ & \mathcal{X}^d(G) = \mathbb{P}(\mathcal{F}) \arrow[r, "\pi"] & \mathcal{X}^d(G') \end{tikzcd}$$

Let $x_v = c_1(\mathcal{L}_v^*)$. Then $H^*(\mathcal{X}^d(G))$ is generated by x_v as an algebra over $H^*(\mathcal{X}^d(G'))$. Let $\mathcal{K}_{e,v} = \mathcal{F}/\mathcal{L}_v$ and $y_{e,v} = c_1(\mathcal{K}_{e,v}^*)$. By the Whitney formula, $c(\mathcal{F}) = c(\mathcal{L}_w)c(\mathcal{K}_{e,w}) = c(\mathcal{L}_v)c(\mathcal{K}_{e,v})$, that is,

$$(1 - x_v)(1 - y_{e,w}) \;=\; (1 - x_w)(1 - y_{e,v}).$$

Extracting the homogeneous parts of this equation, we find that $x_v - x_w + y_{e,v} - y_{e,w}$ and $x_v y_{e,v} - x_w y_{e,w}$ are zero in $H^*(\mathcal{X}^d(G))$. Eliminating $y_{e,v}$ from these equations recovers the presentation (22) of the cohomology ring, so it remains only to check the equations

$$x_v^{d+1} = 0, \qquad h_d(x_v, y_{e,v}) = 0 \tag{28}$$

in $H^*(\mathcal{X}^d(G))$. Consider the loop graph L_1 with vertex v' and edge e'. Setting $\mathbf{P}(v') = \mathbf{P}(v)$ and $\mathbf{P}(e') = \mathbf{P}(e)$ gives an epimorphism $\pi : \mathcal{X}^d(G) \to \mathcal{X}^d(L_1)$. From the first part of the proof, we have a presentation

$$H^*(\mathcal{X}^d(L_1)) \;\cong\; \mathbb{Z}[x,y] \,/\, \left\langle x^{d+1},\, h_d(x,y) \right\rangle.$$

Here $x = c_1(\mathcal{L}^*)$, where $\mathcal{L}$ is the line bundle with fiber $\mathbf{P}(\tilde{v})$, and $y = c_1(\mathcal{K}^*)$, where $\mathcal{K}$ is the line bundle with fiber $\mathbf{P}(\tilde{e})/\mathbf{P}(\tilde{v})$. Then $\mathcal{L}_v = \pi^{-1}\mathcal{L}$ and $\mathcal{K}_{e,v} = \pi^{-1}\mathcal{K}$, so the desired equations (28) follow from (23). $\square$

A more concise presentation of the cohomology ring can be obtained by using the linear relations $x_v - x_w + y_{e,v} - y_{e,w}$ to eliminate variables. The most symmetric way to do this is to introduce a new variable $z_e = x_v + y_{e,v} = x_w + y_{e,w}$ and then eliminate $y_{e,v}$ and $y_{e,w}$. The resulting presentation is as follows:

$$H^*(\mathcal{X}^d(G)) \;\cong\; \mathbb{Z}\,[x_v, z_e \;:\; v \in V, e \in E] \;\Big/ \left\langle \begin{array}{rcl} x_v^{d+1} & : & v \in V, \\ h_d(x_v,\, z_e - x_v) & : & e \in E(v), \\ (x_v - x_w)(z_e - x_v - x_w) & : & e = vw \end{array} \right\rangle. \tag{29}$$

7.2. **Orchard Schubert Calculus.** In this section, we give some examples of how Theorem 17 may be used to answer enumerative geometry questions, in the spirit of the classical Schubert calculus on Grassmannian and flag varieties. Briefly, we can find the number of pictures of a given orchard G in $\mathbb{P}^d_{\mathbb{C}}$ satisfying certain incidence conditions, by means of polynomial calculations in $H^*(\mathcal{X}^d(G))$.

We begin with a brief summary of the classical theory; for more details, see [4] or [9]. Let S_n be the symmetric group on n letters, and let $F\ell(n)$ be the complete flag variety

$$F\ell(n) \;=\; \{F_\bullet = F_0 \subset \cdots \subset F_n \;\mid\; F_i \in Gr(i, \mathbb{C}^n)\}\,,$$

a complex manifold of dimension $n(n-1)/2$. Fix a flag $F_\bullet \in F\ell(n)$. For each $w \in S_n$, there is a corresponding *Schubert cell*, consisting of flags "in position w

with respect to $F_\bullet$", that is,

$$\Omega^\circ_\sigma \;=\; \{E_\bullet \in F\ell(n) \;\mid\; \dim(E_p \cap F_q) = \#\{i \leq p \;\mid\; w(i) \geq n+1-q\}\}\,. \qquad (30)$$

One has $\Omega^\circ_\sigma \cong \mathbb{C}^{n(n-1)/2-\ell(w)}$, where $\ell(w) = |\{i<j \;\mid\; w(i) > w(j)\}|$. The flag variety is the disjoint union of the Schubert cells. The *Schubert variety* Ω_σ, defined as the closure of Ω°_σ, is a union of Schubert cells:

$$\Omega_\sigma \;=\; \overline{\Omega^\circ_\sigma} \;=\; \bigcup_{\rho\geq\sigma} X^\circ_\rho \qquad (31)$$

where $\geq$ is a certain partial order on S_n, the *strong Bruhat order*. The cohomology classes $[X_\sigma] \in H^{2\ell(w)}(F\ell(d);\,\mathbb{Z})$ are a $\mathbb{Z}$-basis for $R_n = H^*(F\ell(d);\,\mathbb{Z})$.

Alternatively, R_n may be described in terms of Chern classes. Let U_i be the rank-i vector bundle on $F\ell(n)$ whose fiber at a flag $E_\bullet$ is E_i, and let $\xi_i = -c_1(U_i/U_{i-1})$. Then R_n is the quotient of $\mathbb{Z}[\xi_1,\dots,\xi_n]$ by the ideal generated by the elementary symmetric functions in the ξ_i. The *Schubert polynomials* $\mathfrak{S}_w$ express the cohomology classes $[\Omega_w]$ of the Schubert varieties as polynomials in the ξ_i. Since $F\ell(n)$ is a smooth variety, its cohomology ring is the same as its Chow or intersection ring, allowing one to solve problems in enumerative geometry by means of computations in R_n involving Schubert polynomials. This is the classical Schubert calculus.

The Schubert polynomials may be calculated using Demazure's divided difference operators; see [4, pp. 170–173]. We note here some special cases for later use, omitting the calculations. We write a permutation $w \in S_n$ in one-line notation as a sequence $(w(1), w(2), \dots, w(n))$ (sometimes omitting the commas for brevity):

- If w is the identity permutation, then $\mathfrak{S}_w = 1$ (the fundamental class, since Ω_w is the whole flag variety).
- If $w = (n, n-1, \dots, 2, 1)$ is the unique permutation of maximal length, then $\mathfrak{S}_w = \xi_1^{n-1}\xi_2^{n-2}\dots\xi_{n-1}$. This is the cohomology class of a point in $F\ell(n)$.
- If w is the transposition of i with $i+1$, then $\mathfrak{S}_w = \xi_1 + \dots + \xi_i$.
- If $w = (n, 1, 2, \dots, n-1)$, then $\mathfrak{S}_w = \xi_1^d$.

The Schubert calculus may be extended to the partial flag manifold

$$F\ell^{1,2}(d+1) \;=\; \left\{F_1 \subset F_2 \subset \mathbb{C}^{d+1} \;\mid\; \dim F_i = i\right\},$$

which is isomorphic to the picture space $\mathcal{X}^d(L_1)$. The natural surjection $F\ell(d{+}1) \to F\ell^{1,2}(d+1)$, forgetting the data $F_3,\dots,F_d$, induces a decomposition of $F\ell^{1,2}(d)$ into Schubert cells Ω°_w of a form analogous to (30), where $w \in S_{d+1}$ is a permutation such that

$$w_3 > w_4 > \dots > w_{d+1}. \qquad (32)$$

For such permutations, the Schubert polynomial $\mathfrak{S}_w = [\Omega_w]$ involves only the variables ξ_1 and ξ_2.

More generally, if $G = (V, E)$ is an orchard, $v \in V$, and $e \in E(v)$, then there is a fibration

$$\pi = \pi_{v,e} :\; \mathcal{X}^d(G) \to \mathcal{X}^d(L_1) \cong F\ell^{1,2}(d+1)$$

sending a picture $\mathbf{P}$ to the partial flag $\mathbf{P}(v) \subset \mathbf{P}(e)$. Thus $\mathcal{X}^d(G)$ is a disjoint union of "Schubert cells" of the form

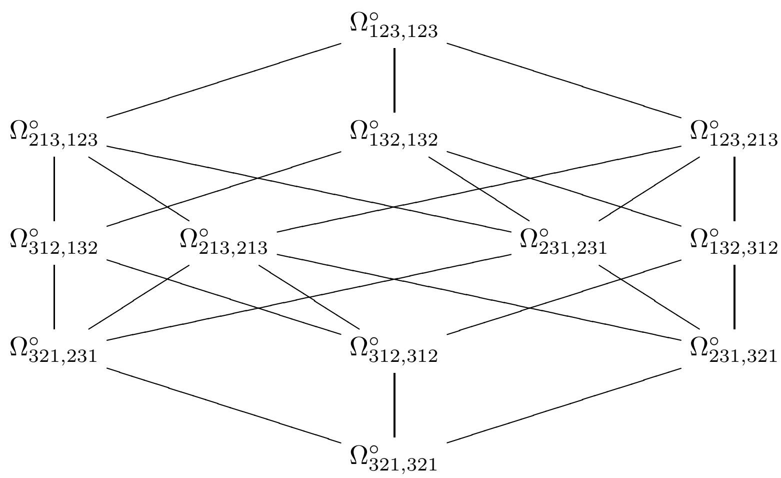
$$\Omega^\circ_{\mathbf{w}} \;=\; \bigcap_{e\in E(v)} \pi^{-1}(\Omega^\circ_{w_{v,e}})$$

indexed by tuples $\mathbf{w}$ of permutations $w_{v,e} \in S_{d+1}$. It is easy to prove (by induction on $|E|$) that each $\Omega^\circ_{\mathbf{w}}$ is isomorphic to an affine space, and to obtain conditions on the permutations $w_{v,e}$ for $\Omega^\circ_{\mathbf{w}}$ to be nonempty. We expect that in general the "orchard Schubert variety" $\Omega_{\mathbf{w}} = \overline{\Omega^\circ_{\mathbf{w}}}$ should be a union of cells; however, it is not clear how to describe the partial order analogous to (31).

Example 18. Let $G = K_2$, with vertices v_1, v_2 and edge e. For $i \in \{1, 2\}$, there is a surjection $\pi_i : \mathcal{X}^2(K_2) \to F\ell(3)$, sending a picture $\mathbf{P}$ to the complete flag $0 \subset \mathbf{P}(v_i) \subset \mathbf{P}(e) \subset \mathbb{C}^3$. Thus $\mathcal{X}^2(K_2)$ decomposes into Schubert cells

$$\Omega^\circ_\sigma \ = \ \pi_1^{-1}(\Omega_{\sigma_1}) \ \cap \ \pi_2^{-1}(\Omega_{\sigma_2})$$

where $\sigma = (\sigma_1, \sigma_2) \in S_3 \times S_3$. In fact, Ω°_σ is nonempty if and only if $\sigma_1(3) = \sigma_2(3)$. One may verify that the closure of a cell is indeed a union of cells, and that the closure order analogous to (31) is given by the following diagram:



Note that this is strictly weaker than the product of two copies of the strong Bruhat order. For instance, $231 > 213$ in Bruhat order, but the cells $\Omega^\circ_{231,231}$ and $\Omega^\circ_{213,213}$ both have complex dimension 2, hence are incomparable.

We can use the cell decomposition of $\mathcal{X}^d(G)$ to extend the methods of Schubert calculus to orchards, using the presentation of the cohomology ring given in Theorem 17. That is, we can count the number of pictures of an orchard $G = (V, E)$ meeting a given list of hyperplanes in $\mathbb{P}^d$ in certain ways. For $v \in V$ and $e \in E(v)$, the map $\pi = \pi_{v,e} : \mathcal{X}^d(G) \to F\ell^{1,2}(d+1)$ induces a graded ring homomorphism

$$\pi^* : \ H^*(F\ell^{1,2}(d+1)) \to H^*(\mathcal{X}^d(G))$$

with the property that

$$\pi^*[Z] \ = \ [\pi^{-1}Z] \tag{33}$$

for all $Z \subset F\ell^{1,2}(d+1)$. In particular, suppose that $w \in S_{d+1}$ is a permutation satisfying (32). Then we can calculate the cohomology class of $Y = \pi^{-1}(\Omega_w)$ by evaluating the Schubert polynomial $\mathfrak{S}_w = \mathfrak{S}_w(\xi_1, \xi_2)$ at $\xi_1 = x_v,\ \xi_2 = z_e - x_v$. For example, nine of the twelve varieties $\Omega_{\sigma_1,\sigma_2} \subset \mathcal{X}^2(K_2)$ are pullbacks of Schubert varieties in $F\ell(3)$ under π_1 or π_2 (the exceptions are $\Omega_{123,123}$, $\Omega_{132,132}$, and $\Omega_{231,231}$).

Before giving an example of how this theory may be applied to solve a problem in enumerative geometry, we need one final ingredient—the cohomology class of a point.

Proposition 19. *Let $G = (V, E)$ be an orchard. With respect to the presentation (29), the cohomology class of a point in $\mathcal{X}^d(G, E)$ is*

$$\prod_{v \in V} \left(x_v^d \prod_{loops\ e \in E(v)} h_{d-1}(x_v, y_e - x_v) \right).$$

Proof. Suppose first that G is a forest (that is, it has no loops). Let $v \in V$ and $p \in \mathbb{P}^d$. Define

$$Z_v \;=\; \{\mathbf{P} \in \mathcal{X}^d(G) \;\mid\; \mathbf{P}(v) = p\}.$$

Let $\pi : \mathcal{X}^d(G) \to \mathbb{P}^d$ be the map sending $\mathbf{P}$ to $\mathbf{P}(v)$. Then

$$[Z_v] \;=\; [\pi^{-1}(p_0)] \;=\; \pi^*[p_0].$$

The cohomology ring of $\mathbb{P}^d$ is $\mathbb{Z}[\xi]/\langle \xi^d \rangle$, where $\xi = c_1(\mathcal{E}^*)$ and $\mathcal{E}$ is the tautological line bundle on $\mathbb{P}^d$. Then $\pi^{-1}\mathcal{E} = \mathcal{L}_v$, so by (23) and the previous equation we have $[Z_v] = x_v^d$. The discrete cellule is dense in $\mathcal{X}^d(G)$, because trees are d-parallel independent for all d by [12, Theorem 8.2.2]. This amounts to saying that the subvarieties Z_v of $\mathcal{X}^d(G)$ intersect transversely in a point. Hence the cohomology class of a point is $\prod_{v \in V} x_v^d$.

Now, suppose that G contains one or more loops. Let $q \in \mathbb{P}^d \setminus \{p\}$. For each loop $e \in E(v)$, define

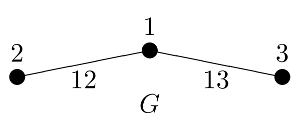
$$Y_e \;=\; \{\mathbf{P} \in \mathcal{X}^d(G) \;\mid\; q \in \mathbf{P}(e)\}.$$

Then $Y_e = {\pi_{v,e}}^{-1}(\Omega_w)$, where $w = (d+1, 1, d, d-1, \dots, 2) \in S_{d+1}$. Using the special cases of Schubert polynomials mentioned previously and the Demazure divided difference operators (we omit the details), one may show that $[Y_e] = \mathfrak{S}_w = h_{d-1}(x_v, y_e - x_v)$. Then the collection of subvarieties

$$\{Z_v \;\mid\; v \in V\} \;\cup\; \{Y_e \;\mid\; e \in E \text{ is a loop}\}$$

intersects transversely in a point, which implies the desired result. $\square$

Example 20. Let G be the tree with vertices $V = \{1, 2, 3\}$ and edges $E = \{12, 13\}$:



Let $A_1, A_2, A_3 \subset \mathbb{P}^3$ be planes, and let $A_4, \dots, A_9 \subset \mathbb{P}^3$ be lines, with the collection $\{A_i\}$ in general position. We will calculate the number of pictures of G in $\mathbb{P}^3$ satisfying the conditions

$$\begin{aligned} \mathbf{P}(i) &\in A_i && \text{for } i = 1,2,3, \\ \mathbf{P}(12) \cap A_i &\neq \emptyset && \text{for } i = 4,5,6, \\ \mathbf{P}(13) \cap A_i &\neq \emptyset && \text{for } i = 7,8,9. \end{aligned} \tag{34}$$

For $i = 1, \dots, 9$, let Y_i be the subvariety of $\mathcal{X}^3(G)$ consisting of pictures $\mathbf{P}$ for which the condition involving A_i is satisfied. Then the problem is to determine the cardinality of $Y = \bigcap_i Y_i$. Each Y_i is the pullback of some $\Omega_w \subset F\ell^{1,2}(\mathbb{C}^4)$, so its cohomology class is a Schubert polynomial in the variables $x_1, x_2, x_3, z_{12}, z_{13}$. For instance,

$$\begin{aligned} [Y_1] &= \left[{\pi_{1,12}}^{-1}(\Omega_{2134})\right] = \mathfrak{S}_{2134}(x_1, z_{12}-x_1) = x_1 \qquad \text{and} \\ [Y_4] &= \left[{\pi_{1,12}}^{-1}(\Omega_{1324})\right] = \mathfrak{S}_{1324}(x_1, z_{12}-x_1) = z_{12}. \end{aligned}$$

The other classes $[Y_i]$ may be calculated similarly. In summary,

$$\begin{aligned} &[Y_1] = x_1, && [Y_4] = [Y_5] = [Y_6] = z_{12}, \\ &[Y_2] = x_2, && [Y_7] = [Y_8] = [Y_9] = z_{13}, \\ &[Y_3] = x_3. \end{aligned}$$

Therefore $[Y] = x_1x_2x_3z_{12}^3z_{13}^3$. By Proposition 19, the cohomology class of a point in $\mathcal{X}^3(G)$ is $(x_1x_2x_3)^3$. Since $x_1x_2x_3z_{12}^3z_{13}^3 = 4(x_1x_2x_3)^3$ in $H^*(\mathcal{X}^3(G))$, we conclude that $|Y| = 4$. That is, there are four pictures of G satisfying the conditions (34). (For this and many similar computations, we used the computer algebra system *Macaulay* [1]).

This calculation can be explained purely geometrically, in the spirit of the classical Schubert calculus. First, we specialize to the case that the lines A_4 and A_5 meet in a point, as do A_7 and A_8. It is a fact that if there are four solutions to the constraints (34) under this specialization, then there are four solutions without it. (In Schubert's terminology, this is the "principle of conservation of number"; see [9].)

We will describe the set J_{12} of locations for $\mathbf{P}(1)$ for which the conditions on $\mathbf{P}(12)$ can be satisfied. That is, J_{12} consists of all points $p \in A_1$ such that there exists a line ℓ through p meeting each of A_4, A_5, A_6 nontrivially. There are two possibilities:

Case 1: $p \in A_1 \cap Q$, where Q is the plane containing A_4 and A_5. Then there is precisely one possibility for the line ℓ: it must be the unique line determined by p and the point $A_6 \cap Q$. Note that $\ell \subset Q$, so both $\ell \cap A_4$ and $\ell \cap A_5$ are nonempty.

Case 2: $p \in A_1 \cap R$, where R is the plane determined by the point $a = A_4 \cap A_5$ and the line A_6. Again, this choice of p determines ℓ uniquely: it is the line determined by p and a. Note that $\ell \subset R$, so $\ell \cap A_6 \neq \emptyset$.

Thus J_{12} is the union of two lines in A_1. By an identical argument, the set J_{13} of points for which the conditions on $\mathbf{P}(13)$ can be satisfied is also the union of two lines in A_1. Therefore $J = J_{12} \cap J_{13}$ consists of four points in A_1. For each point $p \in J$, there is a unique picture of G satisfying all the conditions of (34), with $\mathbf{P}(1) = p$. The case analysis above tells us how to choose $\mathbf{P}(12)$ and $\mathbf{P}(13)$, and $\mathbf{P}(2)$ and $\mathbf{P}(3)$ must be respectively $\mathbf{P}(12) \cap A_2$ and $\mathbf{P}(13) \cap A_3$.

This geometric result verifies *ex post facto* that the subvarieties Y_i of Example 20 intersect transversely, so that the above cohomological calculation is valid. It would be interesting to discover whether such transversality holds for all orchard Schubert varieties.

8. Some Open Problems

1. What aspects of the graph- or matroid-theoretic structure of G (other than its d-parallel behavior) can be read off from the Poincaré series of $\mathcal{X}^d(G)$? For instance, can the Tutte polynomial itself be recovered from the Poincaré series? This

seems intuitively unlikely because $\mathcal{X}^d(G)$ involves one fewer variable than $\mathbf{T}_G(x,y)$. However, the author's experimentation has thus far produced no counterexample.

2. Theorem 14 may be read as saying that the d-parallel behavior of a graph is encapsulated in the structure of the corresponding graphic matroid (which is less information than the structure of the graph itself!) Accordingly, given an arbitrary matroid M, can one define the "d-parallel matroid" of M, with the same ground set, purely in terms of the Tutte polynomial? (This idea was suggested to the author independently by M. Haiman and V. Reiner.) If so, what is the geometric meaning of such a combinatorial object?

3. Some of the material of Section 7 may warrant further investigation. Most glaring is the lack of a Schubert calculus for graphs other than orchards, for which the picture space is singular. In addition, one might study the "quasi-Bruhat" poset associated with an orchard, as in Example 18 above.

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School of Mathematics, University of Minnesota, Minneapolis, MN 55455
E-mail address: `martin@math.umn.edu`