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FACTORIZATIONS OF SOME WEIGHTED SPANNING TREE ENUMERATORS

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ABSTRACT. We give factorizations for weighted spanning tree enumerators of Cartesian products of complete graphs, keeping track of fine weights related to degree sequences and edge directions. Our methods combine Kirchhoff's Matrix-Tree Theorem with the technique of identification of factors.

1. Introduction

Cayley's celebrated formula n^{n-2} for the number of spanning trees in the complete graph K_n has many generalizations (see [5]). Among them is the following well-known factorization for the enumerator of the spanning trees according to their degree sequence, which is a model for our results.

Cayley-Prüfer Theorem.

$$\sum_{T \in \mathrm{Tree}(K_n)} x^{\deg(T)} = x_1 x_2 \cdots x_n (x_1 + \cdots + x_n)^{n-2}$$

where $\mathrm{Tree}(K_n)$ *is the set of all spanning trees and* $x^{\deg(T)} := \prod_{i=1}^n x_i^{\deg_T(i)}$.

Although this is most often deduced from the bijective proof of Cayley's formula that uses Prüfer coding (see, e.g., [5, pp. 4-6]), we do not know of such a bijective proof for most of our later results. Section 2 gives a quick proof (modelling those that will follow) using a standard weighted version of *Kirchhoff's Matrix-Tree Theorem*, along with the method of *identification of factors*.

We generalize the Cayley-Prüfer Theorem to Cartesian products of complete graphs $G = K_{n_1} \times \cdots \times K_{n_r}$. The number of spanning trees for such product graphs can be computed using Laplacian eigenvalues. Section 3 generalizes this calculation to keep track of the directions of edges in the tree, as we now explain. Note that vertices in G are r-tuples $(j_1, \ldots, j_r) \in [n_1] \times \cdots \times [n_r]$, and each edge connects two such r-tuples that differ in only one coordinate. Say that such an edge lies *in direction* i if its two endpoints differ in their i^{th} coordinate. Given a spanning tree T in G, define the direction monomial

$$q^{\mathrm{dir}(T)} := \prod_{i=1}^{r} q_i^{|\{\text{edges in } T \text{ in direction } i\}|}.$$

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Theorem 1.

$$\sum_{T\in \mathrm{Tree}(K_{n_1}\times\cdots\times K_{n_r})} q^{\mathrm{dir}(T)} = \frac{1}{n_1\cdots n_r}\prod_{\emptyset\neq A\subset[r]}\left(\sum_{i\in A} q_i n_i\right)^{\prod_{i\in A}(n_i-1)}$$
$$= \prod_{i=1}^{r} q_i^{n_i-1} n_i^{n_i-2} \prod_{\substack{A\subset[r]\\ |A|\geq 2}}\left(\sum_{i\in A} q_i n_i\right)^{\prod_{i\in A}(n_i-1)}.$$

One might hope to generalize the previous result by keeping track of edge directions and vertex degrees simultaneously. Empirically, however, such generating functions do not appear to factor nicely. Nevertheless, if one "decouples" the vertex degrees in a certain way that we now explain, nice factorizations occur. Create a variable $x_j^{(i)}$ for each pair (i,j) in which $i\in\{1,2,\ldots,r\}$ is a direction and j is in the range $1,2,\ldots,n_i$. In other words, there are r sets of variables, with n_i variables $x_1^{(i)},x_2^{(i)},\ldots,x_{n_i}^{(i)}$ in the i^{th} set. Given a spanning tree T of G, define the decoupled degree monomial

$$x^{\mathrm{dd}(T)} := \prod_{v=(j_1,\ldots,j_r)\,\in\,[n_1]\times\cdots\times[n_r]} \left(x_{j_1}^{(1)}\cdots x_{j_r}^{(r)}\right)^{\deg_T(v)}. \tag{1}$$

In Section 3, we prove the following generalization of the Cayley-Prüfer Theorem.

Theorem 2. *The spanning tree enumerator*

$$f_{n_1,\ldots,n_r}(q,x) := \sum_{T\in\mathrm{Tree}(K_{n_1}\times\cdots\times K_{n_r})} q^{\mathrm{dir}(T)} x^{\mathrm{dd}(T)}$$

is divisible by

$$q_i^{n_i-1},$$

by

$$\left(x_j^{(i)}\right)^{n_1\cdots n_{i-1}n_{i+1}\cdots n_r}$$

and by

$$\left(x_1^{(i)}+\cdots+x_{n_i}^{(i)}\right)^{n_i-2}$$

for each $i\in[r]$ and $j\in[n_i]$.

Conjecture. *The quotient polynomial*

$$\frac{f_{n_1,\ldots,n_r}(q,x)}{\prod_{i=1}^{r} q_i^{n_i-1}\left(x_1^{(i)}\cdots x_{n_i}^{(i)}\right)^{n_1\cdots n_{i-1}n_{i+1}\cdots n_r}\left(x_1^{(i)}+\cdots+x_{n_i}^{(i)}\right)^{n_i-2}}$$

in $\mathbf{Z}[q_i,x_j^{(i)}]$ has non-negative coefficients.

Empirically, this quotient polynomial seems not to factor further in general, although when one examines the coefficient of particular "extreme" monomials in the q_i, the resulting polynomial in the $x_j^{(i)}$ factors nicely. Such nice factorizations seem to fail for the coefficients of non-extreme monomials in q_i when there are at least two $n_i\geq 3$.

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Section 5 shows that when all $n_i = 2$, the spanning tree enumerator $f_{n_1,\dots,n_r}(q,x)$ factors beautifully. Here we consider the Cartesian product

$$Q_n := \underbrace{K_2 \times \cdots \times K_2}_{n \text{ times}},$$

which is the 1-skeleton of the n-dimensional cube. For the sake of a cleaner statement, we make the following substitution of the $2n$ variables $\{x_1^{(i)}, x_2^{(i)}\}_{i=1}^n$:

$$\begin{aligned} x_1^{(i)} &= x_i^{-\frac{1}{2}}, \\ x_2^{(i)} &= x_i^{\frac{1}{2}}. \end{aligned} \tag{2}$$

The substitution (2) is harmless, because it is immediate from (1) that the polynomial $f_{2,\dots,2}(q,x)$ is homogeneous of total degree $2(2^n-1)$ in each of the sets of two variables $\{x_1^{(i)}, x_2^{(i)}\}$. Our result may now be stated as follows:

Theorem 3.

$$\left[\sum_{T\in\mathrm{Tree}(Q_n)} q^{\mathrm{dir}(T)} x^{\mathrm{dd}(T)}\right]_{x_1^{(i)}=x_i^{-\frac{1}{2}},\, x_2^{(i)}=x_i^{\frac{1}{2}}} = q_1\cdots q_n \prod_{\substack{A\subset[n]\\|A|\geq 2}} \sum_{i\in A} q_i\left(x_i^{-1}+x_i\right).$$

This result may shed light on the problem of finding a bijective proof for the known number of spanning trees in the n-cube (see [7, pp. 61-62]).

Section 6 proves a result (Theorem 4 below), generalizing the Cayley-Prüfer Theorem in two somewhat different directions. In one direction, it deals with threshold graphs, a well-behaved generalization of complete graphs. Threshold graphs have many equivalent definitions (see, e.g., [4, Chapters 7-8]), but one that is convenient for our purpose is the following. A graph G is *threshold* if, after labelling its vertices by $[n] := \{1,2,\dots,n\}$ in weakly decreasing order of their degrees, the degree sequence $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ determines the graph completely by the rule that the neighbors of vertex i are the λ_i smallest members of $[n]$ other than i itself. A result of Merris [3] implies the following generalization of Cayley's formula to all threshold graphs. It uses the notion of the *conjugate partition* λ' to the degree sequence λ, whose Ferrers diagram is obtained from that of λ by flipping across the diagonal.

Merris' Theorem. *Let G be a threshold graph with vertices $[n]$ and degree sequence λ. Then the number of spanning trees in G is $\prod_{r=2}^{n-1}\lambda'_r$.*

The natural vertex-ordering by degree for a threshold graph G induces a canonical edge orientation in any spanning tree T of G, by orienting the edge $\{i,j\}$ from j to i if $j > i$. Thus given a spanning tree T and a vertex i, one can speak of its *indegree* $\mathrm{indeg}_T(i)$ and *outdegree* $\mathrm{outdeg}_T(i)$.

Theorem 4. *Let G be a connected threshold graph with vertices $[n]$ and degree sequence λ. Then*

$$\sum_{T\in\mathrm{Tree}(G)} \prod_{i=1}^{n} x_i^{\mathrm{indeg}_T(i)} y_i^{\mathrm{outdeg}_T(i)} = x_1 y_n \prod_{r=2}^{n-1}\left(\sum_{i=1}^{\lambda'_r} x_{\min\{i,r\}} y_{\max\{i,r\}}\right).$$

In particular, setting $y_i = x_i$ *gives*

$$\sum_{T \in \mathrm{Tree}(G)} x^{\deg(T)} = x_1 x_2 \cdots x_n \prod_{r=2}^{n-1} \left(\sum_{i=1}^{\lambda'_r} x_i \right).$$

The proof, sketched in Section 6, proceeds by identification of factors. The authors thank M. Rubey and an anonymous referee for pointing out that it can also be deduced bijectively from a very special case of a recent encoding theorem of Remmel and Williamson [6].

2. Proof of Cayley-Prüfer Theorem: the model

The goal of this section is to review Kirchhoff's Matrix-Tree Theorem, and use it to give a proof of the Cayley-Prüfer Theorem. Although this proof is surely known, we included it both because we were unable to find it in the literature, and because it will serve as a model for our other proofs.

Introduce a variable e_{ij} for each edge $\{i, j\}$ in the complete graph K_n, with the conventions that $e_{ij} = e_{ji}$ and $e_{ii} = 0$. Let L be the $n \times n$ weighted Laplacian matrix defined by

$$L_{ij} := \begin{cases} \sum_{k=1}^{n} e_{ik} & \text{for } i = j \\ -e_{ij} & \text{for } i \neq j. \end{cases} \tag{3}$$

Kirchhoff's Matrix-Tree Theorem [5, §5.3] *For any* $r, s \in [n]$,

$$\sum_{T \in \mathrm{Tree}(K_n)} \prod_{\{i,j\} \in T} e_{ij} = (-1)^{r+s} \det \hat{L}$$

where $\hat{L}$ *is the reduced Laplacian matrix obtained from* L *by removing row* r *and column* s.

We now restate and prove the Cayley-Prüfer Theorem.

Cayley-Prüfer Theorem.

$$\sum_{T \in \mathrm{Tree}(K_n)} x^{\deg(T)} \;=\; x_1 x_2 \cdots x_n (x_1 + \cdots + x_n)^{n-2}$$

where $x^{\deg(T)} := \prod_{i=1}^{n} x_i^{\deg_T(i)}$.

Proof. Apply the substitution $e_{ij} = x_i x_j$ to the weighted Laplacian matrix L in Kirchhoff's Theorem. Setting $f := x_1 + \cdots + x_n$, one has from (3)

$$L_{ij} = \begin{cases} (f - x_i) x_j & \text{for } i = j \\ -x_i x_j & \text{for } i \neq j. \end{cases}$$

By Kirchhoff's Theorem, the left-hand side of the Cayley-Prüfer Theorem coincides with the determinant $\det \hat{L}$, where $\hat{L}$ is the reduced Laplacian $\hat{L}$ obtained from this substituted L by removing the last row and column. We wish to show that this determinant coincides with the right-hand-side of the Cayley-Prüfer Theorem. Note that both sides are polynomials in the x_i of degree $2n-2$, and both have coefficient 1 in the monomial $x_1^{n-1} x_2 x_3 \cdots x_n$. Therefore it suffices to show that the determinant is divisible by each of the variables x_j, and also by f^{n-2}. Divisibility by x_j is clear since x_j divides every entry in the j^{th} column of L (and hence also $\hat{L}$). Divisibility by f follows from Lemma 5 below, once one notices that in the quotient ring $\mathbf{Q}[x_1, \ldots, x_n]/(f)$, this weighted Laplacian L (and hence $\hat{L}$) reduces to a rank one matrix of the form $L = -v^T \cdot v$, where v is the row vector $\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}$. □

The following lemma, used in the preceding proof, is one of our main tools. It generalizes from one to several variables the usual statement on identification of factors in determinants over polynomial rings (see [2, §2.4]).

Lemma 5. *(Identification of factors) Let R be a Noetherian integral domain (e.g., a polynomial or Laurent ring in finitely many variables over a field). Let $f \in R$ be a prime element, so that the quotient ring $R/(f)$ is an integral domain, and let K denote the field of fractions of $R/(f)$. Let $A \in R^{n \times n}$ be a square matrix. If the reduction $\bar{A} \in (R/(f))^{n \times n}$ has K-nullspace of dimension at least d, then f^d divides* $\det A$ *in R.*

Proof. Let $\{\bar{v}_i\}_{i=1}^d$ be d linearly independent vectors in K^n lying in the nullspace of $\bar{A}$. Extend them to a basis $\{\bar{v}_i\}_{i=1}^n$ of K^n. By clearing denominators, one may assume that $\{\bar{v}_i\}_{i=1}^n$ lie in $R/(f)^n$, and then choose pre-images $\{v_i\}_{i=1}^n$ in R.

Letting F denote the fraction field of R, we claim that $\{v_i\}_{i=1}^n$ is a basis for F^n. To see this, assume not, so that there are scalars $c_i \in F$ which are not all zero satisfying

$$\sum_{i=1}^n c_i v_i = 0. \tag{4}$$

Clearing denominators, one may assume that $c_i \in R$ for all i. If every c_i is divisible by f, one may divide the equation (4) through by f, and repeat this division until at least one of the c_i is not divisible by f. (This will happen after finitely many divisions because R is Noetherian.) But then reducing (4) modulo (f) leads to a nontrivial K-linear dependence among the vectors $\{\bar{v}_i\}_{i=1}^n$, a contradiction.

Let $P \in F^{n \times n}$ be the matrix whose columns are the vectors v_i. Note that $\det P$ is not divisible by f, or else the reductions $\{\bar{v}_i\}_{i=1}^n$ would not form a K-basis in K^n. Therefore, by Cramer's Rule, every entry of P^{-1} belongs to the localization $R_{(f)}$ at the prime ideal (f). Note the following commutative diagram in which horizontal maps are inclusions and vertical maps are reductions modulo (f):

$$\begin{array}{ccccc} R & \longrightarrow & R_{(f)} & \longrightarrow & F \\ \downarrow & & \downarrow & & \\ R/(f) & \longrightarrow & K & & \end{array}$$

Since P^{-1} has entries in $R_{(f)}$, so does $P^{-1}AP$. For each $i \in [d]$, the reduction of Av_i vanishes in K, so every entry in the first d columns of $P^{-1}AP$ lies in the ideal (f). Hence $\det P^{-1}AP = \det A$ is divisible by f^d in $R_{(f)}$, thus also in R. □

The authors thank W. Messing for pointing out a more general result, deducible by a variation of the above proof that uses Nakayama's Lemma:

Lemma 6. *Let S be a (not necessarily Noetherian) local ring, with maximal ideal $\mathfrak{m}$ and residue field $K := S/\mathfrak{m}$. Let $A \in S^{n \times n}$ be a square matrix such that the reduction $\bar{A}$ has K-nullspace of dimension at least d. Then* $\det A \in \mathfrak{m}^d$.

Lemma 5 follows from this by taking S to be the localization $R_{(f)}$.

3. Proof of Theorem 1

We recall the statement of Theorem 1.

Theorem 1.

$$\sum_{T\in\mathrm{Tree}(K_{n_1}\times\cdots\times K_{n_r})} q^{\mathrm{dir}(T)} = \frac{1}{n_1\cdots n_r}\prod_{\emptyset\neq A\subset[r]}\left(\sum_{i\in A} q_i n_i\right)^{\prod_{i\in A}(n_i-1)}.$$

As a prelude to the proof, we discuss some generalities about Laplacians and eigenvalues of Cartesian products of graphs. We should emphasize that all results in this section refer only to *unweighted* Laplacians, that is, one substitutes $e_{ij}=1$ for $i\neq j$ in the usual weighted Laplacian $L(G)$ defined in (3).

The *Cartesian product* $G_1\times\cdots\times G_r$ of graphs G_i with vertex sets $V(G_i)$ and edge sets $E(G_i)$ is defined as the graph with vertex set

$$V(G_1\times\cdots\times G_r) \;=\; V(G_1)\times\cdots\times V(G_r)$$

and edge set

$$\begin{aligned}&E(G_1\times\cdots\times G_r)\\ &\qquad=\bigsqcup_{i=1}^{r} V(G_1)\times\cdots\times V(G_{i-1})\times E(G_i)\times V(G_{i+1})\times\cdots\times V(G_r).\end{aligned}$$

where $\bigsqcup$ denotes a disjoint union. The following proposition follows easily from this description; we omit the proof.

Proposition 7. *If $G_1,\ldots,G_r$ are graphs with (unweighted) Laplacian matrices $L(G_i)$, then*

$$L(G_1\times\cdots\times G_r)=\sum_{i=1}^{r}\mathrm{id}\otimes\cdots\mathrm{id}\otimes L(G_i)\otimes\mathrm{id}\otimes\cdots\mathrm{id}$$

where id *denotes the identity, and $L(G_i)$ appears in the i^{th} tensor position.*

As a consequence, a complete set of eigenvectors for $L(G_1\times\cdots\times G_r)$ can be chosen of the form $v_1\otimes\cdots\otimes v_r$, where v_i is an eigenvector for $L(G_i)$. Furthermore, this eigenvector will have eigenvalue $\lambda_1+\cdots+\lambda_r$ if v_i has eigenvalue λ_i for $L(G_i)$.

We also will make use of the following variation of the Matrix-Tree Theorem; see, e.g., [7, Theorem 5.6.8].

Theorem 8. *If the (unweighted) Laplacian matrix $L(G)$ has eigenvalues $\lambda_1,\ldots,\lambda_n$, indexed so that $\lambda_n=0$, then the number of spanning trees in G is*

$$\frac{1}{n}\lambda_1\cdots\lambda_{n-1}.$$

Proof of Theorem 1. Both sides in the theorem are polynomials in the q_i, and hence it suffices to show that they coincide whenever the q_i are all positive integers. In that case, the left-hand side of the theorem has the following interpretation. Let $K_n^{(q)}$ denote the multigraph on vertex set $[n]$ having q parallel copies of the edge $\{i,j\}$ for every pair of vertices i,j. Then the left-hand side of Theorem 1 counts the number of spanning trees in the Cartesian product

$$K_{n_1}^{(q_1)}\times\cdots\times K_{n_r}^{(q_r)},$$

as each spanning tree T in $K_{n_1}\times\cdots\times K_{n_r}$ gives rise in an obvious way to exactly $q^{\mathrm{dir}(T)}$ spanning trees in $K_{n_1}^{(q_1)}\times\cdots\times K_{n_r}^{(q_r)}$. It is well-known that the (unweighted)

Laplacian $L(K_n)$ has eigenvalues $n, 0$ with multiplicities $n-1, 1$, respectively [7, Example 5.6.9]. Hence $L(K_n^{(q)}) = qL(K_n)$ has eigenvalues $qn, 0$ with multiplicities $n-1, 1$, respectively. By Proposition 7, $L(K_{n_1}^{(q_1)} \times \cdots \times K_{n_r}^{(q_r)})$ has an eigenvalue $\sum_{i\in A} q_i n_i$ for each subset $A \subset [r]$, and this eigenvalue occurs with multiplicity $\prod_{i\in A}(n_i - 1)$. As the zero eigenvalue arises (with multiplicity 1) only by taking $A = \emptyset$, Theorem 8 implies that the number of spanning trees in $K_{n_1}^{(q_1)} \times \cdots \times K_{n_r}^{(q_r)}$ is

$$\frac{1}{n_1 \cdots n_r} \prod_{\emptyset \neq A \subset [r]} \left(\sum_{i\in A} q_i n_i \right)^{\prod_{i\in A}(n_i-1)}. \qquad \square$$

4. Proof of Theorem 2

We recall here the statement of Theorem 2.

Theorem 2. *The spanning tree enumerator*

$$f_{n_1,\ldots,n_r}(q, x) := \sum_{T \in \mathrm{Tree}(K_{n_1} \times \cdots \times K_{n_r})} q^{\mathrm{dir}(T)} x^{\mathrm{dd}(T)}$$

is divisible by

$$q_i^{n_i - 1},$$

by

$$\left(x_j^{(i)}\right)^{n_1 \cdots n_{i-1} n_{i+1} \cdots n_r}$$

and by

$$\left(x_1^{(i)} + \cdots + x_{n_i}^{(i)}\right)^{n_i - 2}$$

for each $i \in [r]$ and $j \in [n_i]$.

Proof. To see divisibility by $q_i^{n_i-1}$, note that every spanning tree in $K_{n_1} \times \cdots \times K_{n_r}$ is connected, and hence gives rise to a connected subgraph of K_{n_i} when one contracts out all edges not lying in direction i. This requires at least $n_i - 1$ edges in direction i in the original tree.

To see divisibility by $(x_j^{(i)})^{n_1 \cdots n_{i-1} n_{i+1} \cdots n_r}$, note that every spanning tree has an edge incident to each vertex, and therefore to each of the $n_1 \cdots n_{i-1} n_{i+1} \cdots n_r$ different vertices which have i^{th} coordinate equal to some fixed value $j \in [n_i]$.

Lastly, we check divisibility by $(x_1^{(i)} + \cdots + x_{n_i}^{(i)})^{n_i-2}$. Starting with the weighted Laplacian matrix (3) for $K_{n_1} \times \cdots \times K_{n_r}$ (regarded as a subgraph of $K_{n_1 \cdots n_r}$), let L be the matrix obtained by the following substitution: if $\{k, l\}$ represents an edge of $K_{n_1} \times \cdots \times K_{n_r}$ in direction i between the two vertices $k = (k_1, \ldots, k_r)$ and $l = (l_1, \ldots, l_r)$, then we set

$$e_{kl} = q_i \cdot x_{k_1}^{(1)} \cdots x_{k_r}^{(r)} \cdot x_{l_1}^{(1)} \cdots x_{l_r}^{(r)} ; \qquad (5)$$

otherwise we set $e_{kl} = 0$. Then Kirchhoff's Theorem says that $f_{n_1,\ldots,n_r}(q, x) = \pm \det \hat{L}$ for any reduced matrix $\hat{L}$ obtained from L by removing a row and column. Thus by Lemma 5, it suffices for us to show that $\hat{L}$ has nullspace of dimension at least $n_i - 2$ modulo $f^{(i)} := x_1^{(i)} + \cdots + x_{n_i}^{(i)}$. In fact, we will show that L itself has nullspace of dimension at least $n_i - 1$ in this quotient. To see this, one can check

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that, as in Proposition 7, the matrix L has the following simpler description, due to our "decoupling" substitution of variables:

$$L = \sum_{i=1}^{r} X^{(1)} \otimes \cdots \otimes X^{(i-1)} \otimes q_i L^{(i)} \otimes X^{(i+1)} \otimes \cdots \otimes X^{(r)},$$

where $X^{(i)}$ is the diagonal matrix with entries $(x_1^{(i)})^2, \ldots, (x_{n_i}^{(i)})^2$, and $L^{(i)}$ is obtained by making the substitution $e_{kl} = x_k^{(i)} x_l^{(i)}$ in the weighted Laplacian matrix for K_{n_i}. In the proof of the Cayley-Prüfer Theorem, we saw that $L^{(i)}$ has rank 1 modulo $(f^{(i)})$, and thus a nullspace of dimension $n_i - 1$. If v is any nullvector for $L^{(i)}$ modulo $(f^{(i)})$, then the following vector is a nullvector for L modulo $(f^{(i)})$:

$$\mathbf{1}_{n_1} \otimes \cdots \otimes \mathbf{1}_{n_{i-1}} \otimes v \otimes \mathbf{1}_{n_{i+1}} \otimes \cdots \otimes \mathbf{1}_{n_r}$$

where $\mathbf{1}_m$ represents a vector of length m with all entries equal to 1. Since varying v leads to $n_i - 1$ linearly independent such nullvectors, the proof is complete. $\square$

5. Proof of Theorem 3

We recall here the statement of Theorem 3, using slightly different notation. Regard the vertex set of Q_n as the power set $2^{[n]}$, so that vertices correspond to subsets of $S \subset [n]$. For any subset $S \subset [n]$, let $x_S := \prod_{i \in S} x_i$. Write $x^{\mathrm{wt}(T)}$ for the decoupled degree monomial corresponding to a tree T under the substitution (2), that is,

$$\begin{aligned} x^{\mathrm{wt}(T)} &= \prod_{S \subset [n]} \left(\frac{x_S}{x_{[n] \setminus S}} \right)^{\frac{1}{2} \deg_T(S)} \\ &= \prod_{\text{edges } \{S,R\} \text{ in } T} \frac{x_S x_R}{x_{[n]}} \end{aligned} \tag{6}$$

Theorem 3.

$$\sum_{T \in \mathrm{Tree}(Q_n)} q^{\mathrm{dir}(T)} x^{\mathrm{wt}(T)} = q_1 \cdots q_n \prod_{\substack{A \subset [n] \\ |A| \geq 2}} \sum_{i \in A} q_i \left(x_i^{-1} + x_i \right).$$

Proof. As before, regard the vertex set of Q_n as the power set $2^{[n]}$. Denote the symmetric difference of two sets S and R by $S \triangle R$, and abbreviate $S \triangle \{i\}$ by $S \triangle i$. Thus two vertices S, R form an edge in Q_n exactly when $S \triangle R$ is a singleton set $\{i\}$; in this case the *direction* of this edge is $\mathrm{dir}(e) := i$. It is useful to note that the neighbors of S are

$$N(S) = \{S \triangle i \mid i \in [n]\}.$$

Our goal is to show that the two sides of the theorem coincide as elements of $\mathbf{Z}[q_i, x_i, x_i^{-1}]$. Note that the two sides coincide as polynomials in q_i after setting $x_i = 1$ for all i, using the special case of Theorem 1 in which all $n_i = 2$.

We next show that both sides have the same maximum and minimum total degrees as Laurent polynomials in the x_i. Each side is easily seen to be invariant under the substitution $x_i \mapsto x_i^{-1}$ (this follows from the antipodal symmetry of the n-cube for the left-hand side), so it suffices to show that both sides have the same maximum total degree. For the right-hand side, the maximum total degree in the x_i is simply the number of subsets $S \subset [n]$ with $|S| \geq 2$, that is, $2^n - n - 1$.

For the left-hand side, we argue as follows. Denote by V' the set of vertices of Q_n other than $[n]$. For any spanning tree T and vertex $S \in V'$, define $\phi(S)$ to be

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the parent vertex of the vertex S when the tree T is rooted at the vertex $[n]$ (that is, $\phi(S)$ is the first vertex on the unique path in T from S to $[n]$); then the edges of T are precisely

$$E(T) = \{\{S, \phi(S)\} \mid S \in V'\}.$$

One has $|\phi(S)| = |S| \pm 1$ (because $\phi(S) = S \triangle i$ for some i). Therefore the total degree of the monomial $x^{\mathrm{wt}(T)}$ will be maximized when $|\phi(S)| = |S| + 1$ for all $S \in V'$; for instance, when $\phi(S) = S \cup \{\max([n] \setminus S)\}$. In this case, that total degree is

$$\sum_{\{S,\phi(S)\}\in T} (|S| + |\phi(S)| - n) \;=\; \sum_{S\in V'} (2|S| + 1 - n) \;=\; 2^n - n - 1.$$

Having shown that both sides have the same total degree in the q_i, the same maximum and minimum total degrees in the x_i, and that they coincide when all $x_i = 1$, it suffices by unique factorization to show that the left-hand side is divisible by each factor on the right-hand side, that is, by

$$f_A \;:=\; \sum_{i\in A} q_i \left(x_i^{-1} + x_i\right).$$

Henceforth, fix $A \subset [n]$ of cardinality ≥ 2. It is not hard to check that f_A is irreducible in $\mathbf{Z}[q_i, x_i, x_i^{-1}]$, using the fact that it is a linear form in the q_i.

Starting with the weighted Laplacian matrix (3) for Q_n, whose rows and columns are indexed by subsets $S \subseteq [n]$, let L be the matrix obtained by making the substitutions

$$e_{S,R} = \begin{cases} \dfrac{q_i x_S x_{S\triangle i}}{x_{[n]}} & \text{for } S \triangle R = \{i\}, \\ 0 & \text{for } |S \triangle R| > 1. \end{cases}$$

By Kirchhoff's Theorem, the left-hand side in Theorem 3 is the determinant of the reduced Laplacian matrix $\hat{L}$ obtained from L by removing the row and column indexed by $S = \emptyset$. It therefore suffices to show that the reduction of $\hat{L}$ modulo (f_A) has nontrivial nullspace. We will show that

$$v := \sum_{\emptyset \neq S \subset [n]} \left(x_A^2 - (-1)^{|A\cap S|}(x_{A\setminus S})^2\right) \epsilon_S \tag{7}$$

is a nullvector[1], where ϵ_S is the standard basis vector corresponding to S. Note that the entries of v are not all zero modulo (f_A); it remains to check that every entry $(\hat{L}v)_R$ of $\hat{L}v$ is a multiple of f_A. Since $\hat{L}_{R,S} = 0$ unless $S = R$ or $S = R\triangle i$

[1] The form of the nullvector (7) was suggested by experimentation using the computational commutative algebra package `Macaulay` [1] to compute the nullspace of $\hat{L}$ in the quotient ring modulo (f_A).

for some i, one has

$$\begin{aligned}
(\hat{L}v)_R &= \hat{L}_{R,R}\, v_R \;+\; \sum_{i=1}^{n} \hat{L}_{R,R\triangle i}\, v_{R\triangle i} \\
&= \sum_{i=1}^{n} \frac{q_i x_R x_{R\triangle i}}{x_{[n]}} \left(x_A^2 - (-1)^{|A\cap R|}(x_{A\setminus R})^2\right) \\
&\qquad - \sum_{i=1}^{n} \frac{q_i x_R x_{R\triangle i}}{x_{[n]}} \left(x_A^2 - (-1)^{|A\cap (R\triangle i)|}(x_{A\setminus (R\triangle i)})^2\right) \\
&= \frac{x_R}{x_{[n]}} \sum_{i=1}^{n} q_i x_{R\triangle i} \left((-1)^{|A\cap (R\triangle i)|}(x_{A\setminus (R\triangle i)})^2 - (-1)^{|A\cap R|}(x_{A\setminus R})^2\right) \;. \qquad (8)
\end{aligned}$$

If $i \notin A$, then $A \cap R = A \cap (R\triangle i)$ and $A \setminus R = A \setminus (R\triangle i)$, so the summand in (8) is zero. If $i \in A$, then $|A\cap R| = |A \cap (R\triangle i)| \pm 1$, so one may rewrite (8) as follows:

$$(\hat{L}v)_R \;=\; -(-1)^{|A\cap R|}\, \frac{x_R}{x_{[n]}} \sum_{i\in A} q_i x_{R\triangle i} \left((x_{A\setminus (R\triangle i)})^2 + (x_{A\setminus R})^2\right). \qquad (9)$$

Note also that when $i \in A$,

$$x_{R\triangle i} \;=\; \begin{cases} x_R\, x_i^{-1} & \text{for } i \in R \\ x_R\, x_i & \text{for } i \notin R \end{cases}$$

and

$$x_{A\setminus (R\triangle i)} \;=\; \begin{cases} x_{A\setminus R}\, x_i & \text{for } i \in R \\ x_{A\setminus R}\, x_i^{-1} & \text{for } i \notin R. \end{cases}$$

Therefore one may rewrite (9) as follows:

$$\begin{aligned}
(\hat{L}v)_R &= \pm\frac{x_R}{x_{[n]}} \left(\sum_{i\in A\cap R} q_i x_R x_i^{-1} \left((x_{A\setminus R}\, x_i)^2 + (x_{A\setminus R})^2\right) \right. \\
&\qquad \left. + \sum_{i \in A\setminus R} q_i x_R x_i \left((x_{A\setminus R}\, x_i^{-1})^2 + (x_{A\setminus R})^2\right) \right) \\
&= \pm\frac{(x_R x_{A\setminus R})^2}{x_{[n]}} \left(\sum_{i\in A\cap R} q_i x_i^{-1}(x_i^2+1) \;+\; \sum_{i\in A\setminus R} q_i x_i (x_i^{-2}+1) \right) \\
&= \pm\frac{(x_R x_{A\setminus R})^2}{x_{[n]}} \sum_{i\in A} q_i (x_i + x_i^{-1}) \\
&= \pm\frac{(x_R x_{A\setminus R})^2}{x_{[n]}} f_A,
\end{aligned}$$

which shows that $(\hat{L}v)_R$ is zero modulo (f_A) as desired. □

6. Proof of Theorem 4

We recall the statement of Theorem 4.

Theorem 4. *Let G be a connected threshold graph with vertices $[n]$, edges E, and degree sequence λ. Then*

$$\sum_{T\in\mathrm{Tree}(G)} \prod_{i=1}^{n} x_i^{\mathrm{indeg}_T(i)} y_i^{\mathrm{outdeg}_T(i)} = x_1 y_n \prod_{r=2}^{n-1}\left(\sum_{i=1}^{\lambda'_r} x_{\min\{i,r\}} y_{\max\{i,r\}}\right).$$

As noted in the Introduction, this result is a special case of Theorem 2.4 of [6]. For this reason, and because the ideas of the proof are quite similar to those of Theorem 3, we omit most of the technical details. We write $N(v)$ for the neighbors of a vertex v, and denote the set $\{i, i+1, \ldots, j\}$ by $[i, j]$.

Sketch of proof. The partitions λ which arise as degree sequences of threshold graphs have been completely characterized (see, e.g., [4, Theorem 8.5]). In particular, suppose that Durfee square of λ (the largest square which is a subshape of λ) has side length s. Then for all $r \in [n]$,

$$\begin{array}{rl} \text{either} & r \le s < \lambda'_r = 1 + \lambda_r \\ \text{or} & r > s \ge \lambda'_r = \lambda_{r+1}. \end{array} \tag{10}$$

Using these identities, one may rewrite the desired equality as

$$\sum_{T\in\mathrm{Tree}(G)} \prod_{i=1}^{n} x_i^{\mathrm{indeg}_T(i)} y_i^{\mathrm{outdeg}_T(i)} = x_1 \cdot \prod_{r=2}^{s} f_r \cdot \prod_{r=s+1}^{n-1} g_r \cdot \prod_{r=s+1}^{n} y_r. \tag{11}$$

where

$$f_r := y_r \sum_{i=1}^{r} x_i + x_r \sum_{i=r+1}^{1+\lambda_r} y_i \qquad \text{and} \qquad g_r := \sum_{i=1}^{\lambda_{r+1}} x_i.$$

Both the left-hand and right-hand sides of (11) are polynomials in the x_i, y_i of total degree $2n-2$, and both have coefficient of $x_1^{n-1} y_2 y_3 \cdots y_n$ equal to 1 (because $N(1) = [2, n]$). Thus it suffices to prove that the left-hand side is divisible by each of the factors on the right-hand side. By Kirchhoff's Theorem, this left-hand side is the determinant of the matrix $\hat{L}$ obtained from the usual weighted Laplacian matrix by removing the first row and column and making the substitution

$$e_{ij} = \begin{cases} x_{\min\{i,j\}} y_{\max\{i,j\}} & \text{for } \{i,j\} \in E \\ 0 & \text{for } \{i,j\} \notin E. \end{cases}$$

First, one must show that the left-hand side of the theorem is divisible by the monomial factor $x_1 \prod_{r=s+1}^{n} y_r$. Every spanning tree T of G contains an edge of the form $\{1, j\}$, which contributes a factor of $x_1 y_j$ to the monomial corresponding to T. In particular, x_1 divides the left-hand side of (11). Furthermore, if $r > s$, then $q < r$ whenever $q \in N(r)$. In particular, y_r divides every entry in the r^{th} row of $\hat{L}$.

Second, one must show that f_r divides $\det \hat{L}$ for $r \in [2, s]$. Clearly f_r is irreducible, since neither sum in the definition of f_r is empty. Define a column vector[2]

$$v = \sum_{i=1}^{r} x_i \epsilon_r + \sum_{i=r+1}^{\lambda'_r} x_r \epsilon_i,$$

where ϵ_i denotes the i^{th} standard basis vector. Note that the entries of v are not all divisible by f_r, so that v is a non-zero vector modulo (f_r). By Lemma 5, it is

[2] Found using `Macaulay`; see the earlier footnote.

now sufficient to show that for each j, the entry $(\hat{L}v)_j$ of $\hat{L}v$ is divisible by f_r. One must consider four cases depending on the value of j: (i) $j < r$, (ii) $j = r$, (iii) $j > r$ and $\{j, r\} \in E$, (iv) $j > r$ and $\{j, r\} \notin E$. We omit the routine calculations, which are similar to the proof of Theorem 3.

Third, one must show that g_r divides $\det \hat{L}$ for all $r \in [s+1, n-1]$. In fact, some higher power of g_r may divide $\det \hat{L}$, as we now explain. If λ has exactly b columns of height λ'_r, i.e.,

$$\lambda'_{a-1} > \lambda'_a = \cdots = \lambda'_r = \cdots = \lambda'_{a+b-1} > \lambda'_{a+b}$$

for some $a > s$, then $N(i) = [\lambda_r]$ for all vertices $i \in [a+1, a+b]$, so

$$g_a = g_{a+1} = \cdots = g_r = \cdots = g_{a+b-1}.$$

Accordingly, one must show that g_r^b divides $\det \hat{L}$. Restricting the Laplacian matrix L to the columns $[a+1, a+b]$ yields a rank-1 matrix of the form

$$-\begin{bmatrix} x_1 & x_2 & \cdots & x_{\lambda_r} & 0 & \cdots & 0 \end{bmatrix}^T \cdot \begin{bmatrix} y_{a+1} & y_{a+2} & \cdots & y_{a+b} \end{bmatrix}.$$

Consequently, both L and $\hat{L}$ have $b-1$ linearly independent nullvectors modulo (g_r) supported in coordinates $[a+1, a+b]$. It remains only to exhibit one further nullvector for $\hat{L}$ which is supported in at least one coordinate outside that range. We claim that such a vector[3] is

$$v = \sum_{i=1+\lambda_r}^{a} (y_{a+b}\epsilon_i - y_i\epsilon_{a+b}).$$

One must verify that for each k, the k^{th} coordinate $(\hat{L}v)_k$ vanishes modulo (g_r). This calculation splits into four cases: (i) $k \in [2, \lambda_r]$, (ii) $k \in [1+\lambda_r, a]$, (iii) $k \in [a+1, n-1] \setminus \{a+b\}$, (iv) $k = a+b$. Once again, we omit the routine verification. $\square$

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[3]Found using `Macaulay`; see the earlier footnote.