The Theory of Functional Forms of the Consumer Demand System and its Application

BY

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Dedication

This dissertation is dedicated to my mother, Fumiko Usui, who passed away during my graduate study.

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I also would like to thank the many people I have encountered during my time in Lawrence, Kansas. Lawrence will be remembered as my second hometown.

Abstract

This dissertation studies the consumer demand system by focusing on its functional form. The theoretical part investigates the regularity property of the flexible consumer demand system characterized by its normalized quadratic functional form. The regularity conditions of monotonicity and curvature are two of the axioms of the consumer demand theory. While other axioms are maintained by construction, these two conditions are only attained in the limited price-income space. We display the regular regions of the model using estimated parameter values from true underlying preferences. The model is estimated using different methods of imposing curvature: global imposition, local imposition, and no imposition. We find that the model often violates the monotonicity condition regardless of the way the curvature is imposed. We find a case where local and global curvature impositions achieve a global regularity within a very large space without causing any biases in estimating the true preference when the unconstrained model produces a non-regular region reflected by the violation of curvature. We also find a case where the globally concave model makes substitute goods more substitute and complement goods more complement.

In the empirical part, functional forms of the consumer demand system which are flexible in the total expenditure are used to estimate the cost of a child using Japanese household expenditure data. The consumer demand system which can describe complicated shapes of Engel curves is necessary to model household behaviors which can vary substantially in expenditure level as well as in demographic characteristics. We estimate the equivalence scales for types of households which differ in the number of children. In doing so, we employ the identifiable expenditure-dependent equivalence scales rather than the constant-equivalence scales usually used in the household welfare literature. A large number of observations with zero expenditures on some goods are addressed by using the Amemiya-Tobit type estimation method to correct potential biases in parameter estimation. The results show that the Japanese household equivalence scales are decreasing in total expenditure as well as increasing in number of children. This suggests the intuitive policy design that the child-support benefits, if any, should depend on household income to preserve equality in welfare level. The results also suggest that the new child-support program proposed by the current Japanese government may need to be reevaluated since it does not consider limiting income level in distributing these benefits.

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Chapter 1

Introduction

The field of consumer demand models stands out in applied economics for its ability to usefully incorporate economic theory into empirical practice. It is an area where empirical investigation benefits from theoretical insight and where theoretical concepts are observed and substantiated by the discipline of empirical relevance and policy design. Earlier development of neoclassical consumer demand theory focused on the restrictions on demand functions implied by the optimizing consumer behavior under the budget constraint. Standard axioms of consumer preference lead to demand systems having the properties of homogeneity, monotonicity, adding-up, symmetry, and quasi-concavity. These restrictions give practical guidance to construct a parsimonious (parametric) statistical consumer demand model. Imposing theoretical restrictions through explicit side constraints on the parameters can permit the testing of theories using easily implementable statistical testing procedures. Furthermore, having chosen a parameterization one can seek to improve the precision of one's estimators by imposing theoretically acceptable restrictions on the parameters.

Earlier empirical work was based on estimated behavioral models obtained using aggregate data. It was limited to the extent that it imposed the conditions required to be able to infer individual behavior from aggregate data. Important contributions by Muellbauer (1975, 1976), by Jorgenson, Lau and Stoker (1980) and by Jorgenson (1990), building upon the pioneering work of Gorman (1953, 1961) established exact conditions under which it is possible to make such inferences from aggregate data.[1] These are restrictive and the increasing availability of accurate microdata in recent years allows a much more general analysis of preferences and constraints, opening up a large new set of empirically-motivated issues.[2]

One approach to the translation of the restrictions implied by the theory into empirical application is to derive the demand equations literally by specifying a direct utility function and solving the constrained maximization problem. While this approach, called the primal approach, leads to demand systems which satisfy the above axioms of consumer preference by construction, the need to derive analytical solutions to the set of first order conditions restricts its application to utility functions in the limited space of neoclassical consumer demand functions, such as the origin-translated CES form of the Klein-Rubin type utility function. Even though the implicit utility model is found to be relatively flexible, it is not possible to obtain the closed form solution to the constrained optimization problem and thus the estimable demand equations. The estimation cannot be carried out without appealing to unconventional methods.[3]

Another methodology is the differentiable demand system approach which has produced the models such as the Rotterdam model and the Constant Slutsky Elasticity (CSE) model. This approach attempts to impose theoretical restrictions on log-differential approximations to the demand equations.

[1] A series of papers in Barnett and Serletis (2004) illustrates how the theory of monetary aggregation is structured based on the micro-founded aggregation theory.

[2] The UK Family Expenditure Survey is often used in household demand studies. This dissertation uses the Japanese household expenditure survey panel data which will be examined in chapter 4.

[3] See chapter 2.1.1.1.

The most popular approach is to exploit the theory of duality among direct utility functions, indirect utility functions, the expenditure functions, and the integrability conditions on these functions which make them equivalent representations of the underlying preferences.[4] Duality theory allows systems of demand equations to be derived from these dual representations by simple differentiation according to Roy's identity or Shephard's lemma. This approach was popularized by Diewert (1974, 1982), and led to the use of flexible functional forms such as the generalized Leontief (GL) of Diewert (1971), the translog (TL) of Christensen, Jorgenson and Lau (1975) and the almost ideal demand system (AIDS) of Deaton and Muellbauer (1980).

Flexible functional forms are defined by Diewert (1971) as a class of functions that have enough free parameters to provide a local second-order approximation to any twice continuously differentiable function. The "up to" second-order approximation property is sufficient to generate preferences or technologies represented by any usual kind of elasticity relations at a point. Based on this definition, the constant elasticity of substitution (CES) form of utility function has a unit income elasticity and constant elasticities of substitutions, and therefore it does not belong to this class. Diewert also defined the "parsimonious" flexible functional form as having no more parametric freedom than needed to satisfy the above definition. The "parsimoniety" is desirable since the number of parameters to be estimated as the number of consumption goods are added in the system increases quadratically, whereas the number of effective observations increase only linearly.[5]

[4] See Hurwicz and Uzawa (1971) on the integrabiliy theory.

[5] The recent advancement of the personal computer lessens the problem to some extent.

Given the large number of available models of flexible functional form, the selection of which flexible functional forms to use for a particular empirical application can be determined merely by the individual researcher's taste or by some criteria of goodness of fit after experimenting with several models. But the pre-selection can be done by looking at the regularity properties of the model.[6]

While such flexible functional forms lead to demand equations which can attain arbitrary elasticities at a point in price-expenditure space, such systems generally satisfy globally only homogeneity, symmetry, and adding-up, and often violate monotonicity and, in particular, curvature restrictions, usually referred to as regularity conditions, either within the sample or at points close to the sample. Regularity conditions: monotonicity and concavity/convexity are two of the axioms of consumer preferences, but are often ignored in papers of characterization theorems of consumer demand models (Gorman 1981, Lewbel 1987a).[7] The regularity conditions are often ignored because it is impossible to satisfy the local flexibility property of

[6] Other criteria may include Engel curve shapes, exact aggregability, price aggregability, and rank conditions. Discussions on Engel curve and exact aggregability will be in chapter 2. See Lewbel (1986) for price aggregability.

[7] In this dissertation, we refer to a full condition of theoretically legitimate consumer demand functions as the integrability condition and refer to both monotonicity and curvature conditions as the regularity condition. Lewbel (2001) distinguishes the regularity conditions from the integrabiltiy conditions by stating that a set of demand functions is defined to be "integrable" if it satisfies adding-up, homogeneity, and Slutsky symmetry and that a set of demand functions is defined to be "rational" if it is integrable and also satisfies negative semi-definiteness of Slutsky substitution matrix. Some authors refer to a full set of consumer demand axioms as a regularity condition. It is convenient to refer to both conditions of monotonicity and curvature separately from a complete set of axioms of demand functions since conditions of homogeneity, symmetry, and adding-up in empirical models are almost always satisfied by construction. Therefore, the violation of the integrability condition is exclusively blamed upon the violations of monotonicity and curvature properties.

flexible functional form and regularity conditions in the entire price-income space simultaneously given the limited number of degrees of freedom.[8] The practical implementation of imposing the regularity conditions in the entire price-income space is possible, provided that imposition of adding-up, homogeneity, and Slutsky symmetry are maintained by construction, but not without sacrificing the flexibility property. Therefore, the satisfaction of regularity conditions is partly dependent upon a specific empirical application. One can hope that they will be attained by luck, at least at all data points, and a "better" model will be the one which has a wider domain of regularity.

In this spirit, Caves and Christensen (1980) devised the procedure to visualize the regular regions of parsimonious flexible functional forms for the translog and the generalized Leontief (GL) models. Their method only applies to "parsimonious" models (with a minimum number of parameters neccesary to be locally flexible). Since their procedure requires a unique set of parameters to be solved for, corresponding to a particular preference setting, enough parameter restrictions are necessary to solve the system of simultaneous equations. Their results show that particular models cover particular regular regions in price-income space, depending on the preference settings. Barnett, Lee and Wolfe (1985, 1987) and Barnett and Lee (1985) extended their work to newly-developed flexible functional forms. They found that the variants of minflex Laurent flexible functional models that can have extra free parameters while maintaining their parsimonious property cover wider regular regions than any other available flexible func-

[8] In this dissertation, "global" implies an entire space and "local" implies at one point, but some authors consider that the flexible functional forms are globally regular if they satisfy the regularity condition within a convex hull of the data points.

tional forms.[9]

In chapter 3, we investigate the regularity property of a newer model: the normalized quadratic model (NQ) developed by Diewert and Wales (1987). This model is particularly interesting to subject to the regularity assessment since there exist methods to impose a global curvature condition and to impose a local curvature condition (Ryan and Wales, 1998b), whose characteristics are not shared by other popular flexible functional models. Unfortunately, the global imposition of curvature destroys the flexibility, but the severity is unknown and may be moderate. On the other hand, it is known that the global imposition of curvature on translog reduces to Cobb-Douglas, the consequence of which invalidates the justification of using the model. Part of our discussion is focused on the monotonicity condition which is more often neglected than the curvature condition, as pointed out by Barnett (2002). Our displays of the regular regions differentiate the regular regions from non-regular regions, and the non-regular regions are further differentiated into three different parts: violations of monotonicity, violations of concavity, and violations of both of them simultaneously This improved visualization method reveals the full regularity property of NQ functional form. Discussions of the more detailed issues that motivated our study are relegated to the introduction of chapter 3.

Chapter 4 serves as an illustration of how the theory of consumer demand behavior is used to conduct welfare analysis of household units. Households with different demographic characteristics are classified by the number of

[9] Specifically, the regular region expands as the real income levels grow. This class of flexible functional forms is defined as "effectively regular" flexible functional forms by Cooper and McLaren (1996).

additional household members, in addition to the husband and wife, and the procedure to incorporate the demographic characteristics into the individual consumer demand systems to create the household demand systems is introduced (Lewbel, 1985). The particular models used in this chapter are the quadratic AIDS model (QUAIDS) of Banks, Blundell and Lewbel (1997) and the translated QUAIDS of Lewbel (2003). These models are characterized as being more flexible in terms of total expenditure variation. The expenditure levels of household data varies substantially, and it is expected that the consumption behaviors of rich households (usually with high expenditure levels) and poor households (usually with low expenditure levels) can differ. Unfortunately, local flexible functional forms only cannot capture the higher degree of total expenditure variation, and a more appropriate model is necessary unless the analysis is only applied to a subpopulation of a group of households of similar income levels. The flexibility property in terms of expenditure level is summarized as the shape of Engel curve of demand functions. The shape of Engel curve in most of the parametric consumer demand models is described as a linear combination of linearly independent functions of total expenditures. The space spanned by the functions of income is defined as the "rank" of the demand functions (Gorman 1981, Lewbel 1987a, 1989b, 1991). Its definition and implications are presented in chapter 2 as background theory. Frankly, demand functions with higher rank can account for more variation in total expenditure. The QUAIDS is rank three, the translated QUAIDS is rank four, and the NQ model of expenditure function is rank two. On the other hand, models based on the Laurent series and two globally flexible functional forms: the

Fourier flexible functional form of Gallant (1981) and the Asymptotically Ideal Model of Barnett and Jonas (1983) and Barnett and Yue (1988) have complicated nonlinear Engel curves and cannot be defined in terms of the rank of the demand functions.

Using these tools, chapter 4 estimates the cost of a child using Japanese household expenditure data. The current Japanese public policies and ongoing discussions on the Japanese child-subsidy program and low fertility rate, the issues motivating our study, are discussed in the early part of the chapter. The concept of equivalence scales is introduced as a measure of the welfare of different household types. We use the demographically modified consumer demand models by the expenditure-dependent equivalence scales developed by Donaldson and Pendakur (2004) in order to capture the effect of additional family members on welfare differences between rich and poor households. The presence of a number of zero-expenditure observations on some goods called for the use of Tobit-type estimation procedure to correct the bias in the parameter estimation. The results of parameter estimation produced equivalence scales that are increasing (decreasing) in total expenditure. It indicates that poor households require more compensation than rich households to keep the same level of utility when more household members are added. This welfare effect on Japanese households has straightforward implications for public policy design. The recommended policy implementation based on the current discussion on the child-subsidy programs is stated.

The next chapter describes most of the relevant theoretical results on consumer demand literature to make the materials in chapters 3 and 4 more comprehensible. We have tried to make it rather self-contained, but in case we have missed certain details, we have tried to provide the references to consult as much as possible. In writing much of the material in chapter 2, we have referenced the recent excellent survey papers on consumer demand systems by Barnett and Serletis (2008, 2009). Other excellent sources of consumer demand system literature include: Deaton and Muellbauer (1980), Blundell (1988), Lewbel (1997), Pollak and Wales (1992), Deaton (1986). Deaton (1997) extensively illustrates practical procedures on how to conduct household behavior analysis. The final chapter concludes with a summary of our contributions and ideas for possible directions for future research.

Chapter 2

This chapter presents relevant theories of consumer demand models to facilitate understanding of the subsequent materials. The first part illustrates the basic theory of consumer optimization behavior subject to the budget constraint and the properties of the resulting demand functions and discusses the duality theory. The second part describes the various consumer demand models and characterizes them in terms of the "rank" of consumer demand systems. In fact, this chapter spells out all of the consumer demand systems used in chapters 3 and 4.

We try to keep our notational convention consistent throughout this dissertation. The scalar variables are written in lower-case and un-emphasized letters. The vectors and matrices are written in boldface. Specifically, the vectors are written in bold symbolic and the matrices are in block-looking boldface. Any deviation from this convention should be obvious in its context; when it comes to a conflict with our notational convention, we follow the traditional notation in the literature.

2.1 Neoclassical Demand Theory

Neoclassical demand theory can begin with assuming the existence of a utility function of individual, $U(\boldsymbol{q})$ where $\boldsymbol{q}$ is a $n \times 1$ nonnegative vector of consuming goods (sometimes called items or commodities). The individual's consumption decision reduces to the standard utility maximization problem:

$$\max U(\boldsymbol{q}) \qquad \text{subject to } \boldsymbol{p}'\boldsymbol{q} = x, \tag{1}$$

where $\boldsymbol{p}$ is a $n \times 1$ nonnegative vector of prices corresponding to $\boldsymbol{q}$, and x is the total expenditure on these goods (sometimes called outlay or nominal income).

This opimization problem is solved by the following steps:

1) constructing the Lagrangian

2) deriving the first order conditions with respect to choice variables $\boldsymbol{q}$

3) solving for $\boldsymbol{q}$ in terms of exogneous variables $\boldsymbol{p}$ and x.

2.1.1 Marshallian Demands

The empirical analysis requires observable variables. The Marshallian ordinary functions describe demands as functions of prices and expenditures, and these are all observable as data. It takes the form of

$$\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{p}, x), \tag{2}$$

which is derived as the solution to the first-order conditions of the utility maximization.

Demand systems are often expressed in budget share form, $\boldsymbol{w}$, where $w_i = p_i q_i(\boldsymbol{p}, x)/x$ is the expenditure share of good i, and $\boldsymbol{w} = (w_1, ..., w_n)'$.

As a textbook example, consider the Cobb-Douglas utility function

$$U(\boldsymbol{q}) = \prod_{k=1}^{n} q_k^{\alpha_k} = q_1^{\alpha_1} q_2^{\alpha 2} q_3^{\alpha_3} \cdots q_n^{\alpha_n}, \tag{3}$$

with $\alpha_k > 0$ and $\sum_{k=1}^{n} \alpha_k = 1$. Setting up the Lagrangian for this optimization problem

$$\begin{aligned}\mathcal{L} &= U(\boldsymbol{q}) + \lambda\left(x - \sum_{k=1}^{n} p_k q_k\right) \\ &= \prod_{k=1}^{n} q_k^{\alpha_k} + \lambda\left(x - \sum_{k=1}^{n} p_k q_k\right),\end{aligned} \tag{4}$$

and by taking first derivatives for each q_i, we get the following set of first order conditions:

$$\begin{aligned}\frac{\alpha_i}{q_i}\prod_{k=1}^{n} q_k^{\alpha_k} - \lambda p_i &= 0, \qquad i = 1, ..., n; \\ x - \sum_{k=1}^{n} p_k q_k &= 0,\end{aligned} \tag{5}$$

which, after solving the system of simultaneous equations for n values of $\boldsymbol{q}$, yields the Marshallian demand functions

$$q_i = \alpha_i \frac{x}{p_i}, \qquad i = 1, ..., n, \tag{6}$$

after using the fact that $\sum_{k=1}^{n} \alpha_k = 1$.

Marshallian demands satisfy the following properties: positivity, adding-up ($\boldsymbol{p}'\boldsymbol{q}(\boldsymbol{p}, x) = x$), homogeneity of degree zero in $(\boldsymbol{p}, x)$ (the absence of money illusion), and the matrix of Slutsky substitution effects

$$\left[\mathbf{S} = \partial\boldsymbol{q}(\boldsymbol{p}, x)/\partial\boldsymbol{p}' + \left(\partial\boldsymbol{q}(\boldsymbol{p}, x)/\partial x\right)\boldsymbol{q}(\boldsymbol{p}, x)'\right] \tag{7}$$

is symmetric and negative semidefinite implying that the substitution effect of each good with respect to its own price is always nonpositive.

These properties of the demand functions are referred to as the "integrability conditions" since the fullfillment of all of these properties permits the reconstruction of the preference ordering from the demand functions (see, Hurwicz and Uzawa, 1971). If the properties are tested empirically and cannot be rejected, then we can infer that there exists a utility function that generates the demand functions.

The Lagrange multipier λ in (4) is sometimes interpreted as marginal utility of income. Differentiating the Lagrangian (4) with respect to total expenditure gives

$$\frac{\partial \mathcal{L}}{\partial x} = \lambda.$$

Hence, the optimal Lagrange multiplier λ tells how much utility increases if an extra unit of income is available.

2.1.2 Indirect utility

The maximum level of utility at given prices and income is obtained by substituting the Marshallian demand functions in equation (2) to the utility function $U(\boldsymbol{q})$, denoted by $V(\boldsymbol{p}, x) = U[\boldsymbol{q}(\boldsymbol{p}, x)]$ which is called the indirect utility function. Hence, it traces the maximum level of utility achievable given particular prices and income. Using the example of the Cobb-Douglas preferences, the indirect utility function is obtained by substituting the demand system in equation (6) into the direct utility function in equation (3) to get

$$V(p,x) = \prod_{k=1}^{n} q_k^{\alpha_k} = \prod_{k=1}^{n} \left(\frac{\alpha_k}{\sum_{j=1}^{n} \alpha_j} \frac{x}{p_k} \right)^{\alpha_k} = x \prod_{k=1}^{n} \left(\frac{\alpha_k}{p_k} \right)^{\alpha_k}, \tag{8}$$

again using $\sum_{k=1}^{n} \alpha_k = 1$.

The direct utility function and the indirect utility function are equivalent representations of the underlying preference pre-ordering. In fact, there is a duality relationship between the direct utility function and the indirect utility function, in the sense that maximization of $U(\boldsymbol{q})$ with respect to $\boldsymbol{q}$ given $(\boldsymbol{p}, x)$ fixed, and the minimization of $V(\boldsymbol{p}, x)$ with respect to $(\boldsymbol{p}, x)$, given $\boldsymbol{q}$ fixed, leads to the same demand functions.

Being able to represent preferences by the indirect utility function has its advantages. As a statistical convenience, the indirect utility function has prices as exogenous in explaining consumer behavior.[10] Moreover, using V, we can easily derive the demand system by straightforward differentiation, without having to solve a system of simultaneous equations, as would be the case with the direct utility function's first-order conditions. In particular, Roy's identity,

$$\boldsymbol{q}(\boldsymbol{p}, x) = -\frac{\partial V(\boldsymbol{p}, x)/\partial \boldsymbol{p}}{\partial V(\boldsymbol{p}, x)/\partial x}, \tag{9}$$

allows us to derive the demand system, provided that there is an interior

[10] Inverse demand systems assume that the prices are endogenous and depend on demand goods as exogenous variables.

solution and that $\boldsymbol{p} > 0$ and $x > 0$. Alternatively, the logarithmic form of Roy's identity,

$$\boldsymbol{w}(\boldsymbol{p}, x) = \frac{-\partial \ln V(\boldsymbol{p}, x)/\partial \ln \boldsymbol{p}}{\partial \ln V(\boldsymbol{p}, x)/\partial \ln x}, \tag{10}$$

or Diewert's (1974, p.126) modified version of Roy's identity,

$$w_i(\boldsymbol{\nu}) = \frac{\nu_i \nabla V(\boldsymbol{\nu})}{\boldsymbol{\nu}' \nabla V(\boldsymbol{\nu})} \tag{11}$$

or

$$w_i(\boldsymbol{p}, x) = -\frac{\partial V\left(\boldsymbol{p}, x\right)/\partial \ln p_i}{\partial V\left(\boldsymbol{p}, x\right)/\partial \ln x},$$

can be used to derive the budget share equations, where $\boldsymbol{\nu} = [\nu_1, ..., \nu_n]'$ is a vector of expenditure normalized prices, with ith element being $\nu_i = p_i/x$, and $\nabla V(\boldsymbol{\nu}) = \partial V(\boldsymbol{\nu})/\partial \boldsymbol{\nu}$. The indirect utility function is continuous in $(\boldsymbol{p}, x)$ and has the following properties: homogeneity of degree zero in $(\boldsymbol{p}, x)$, decreasing in $\boldsymbol{p}$ and increasing in x, strictly quasiconvex in $\boldsymbol{p}$, and it satisfies Roy's identity in equation (9). The equation (11) tells that the budget shares (or quantities demanded) will never become negative if V is merely non-decreasing in $\boldsymbol{\nu}$ while they could be still positive, for example, if $\partial V/\partial \nu_i < 0$ for all $i = 1, ..., n$. Therefore, positivity of the quantity variables does not imply the monotonicity of indirect utility function.

In the terminology of Caves and Christensen (1980), indirect utility function is regular at a given $(\boldsymbol{p}, x)$, at which it satisfies quasiconcavity and monotonicity, provided that other axioms of consumer demand functions are satisfied. Similarly, the "regular region" is the set of prices and income at which an indirect utility function satisfies the regularity conditions.

2.1.3 Hicksian demands

The utility maximization problem is dual to the problem of minimizing the cost or expenditure necessary to attain a fixed level of utility u, given market prices $\boldsymbol{p}$:[11]

$$E(u, \boldsymbol{p}) = \min_{\boldsymbol{q}} \boldsymbol{p}'\boldsymbol{q} \qquad \text{subject to } U(\boldsymbol{q}) \geq u. \tag{12}$$

Given Cobb-Douglas preferences, the Lagrangian for this minimization problem is

$$\mathcal{L} = \sum_{k=1}^{n} p_k q_k + \lambda \left(u - \prod_{k=1}^{n} q_k^{\alpha_k} \right),$$

with the following set of first order conditions:

$$\begin{aligned} p_i - \lambda \frac{\alpha_i}{q_i} \prod_{k=1}^{n} q_k^{\alpha_k} &= 0, \qquad i = 1, ..., n; \\ u - \prod_{k=1}^{n} q_k^{\alpha_k} &= 0, \end{aligned}$$

which, solving the system of $n+1$ simultaneous equations for $\boldsymbol{q}$ and λ, yields the expenditure minimizing demands

$$q_i^c(u, \boldsymbol{p}) = u \frac{\alpha_i}{p_i} \prod_{k=1}^{n} \left(\frac{p_k}{\alpha_k} \right)^{\alpha_k}, \qquad i = 1, ..., n. \tag{13}$$

The expenditure minimizing demands are also known as Hicksian or compensated demands. They tell us how $\boldsymbol{q}$ is affected by prices with u held

[11] We use the term "expenditure" function for consumer demand context and "cost" function for producer input demand context and use notations E and C respectively to maintain the consistent use of terminology.

constant. Finally, substituting the Hicksian demands into the expenditure function in (12) yields

$$
\begin{aligned}
E(u,\boldsymbol{p}) &= \sum_{k=1}^{n} p_k q_k^c \\
&= \sum_{j=1}^{n} p_j \left[u \frac{\alpha_j}{p_j} \prod_{k=1}^{n} \left(\frac{p_k}{\alpha_k} \right)^{\alpha_k} \right] \\
&= u \prod_{k=1}^{n} \left(\frac{p_k}{\alpha_k} \right)^{\alpha_k}, \qquad (14)
\end{aligned}
$$

using $\sum_{k=1}^{n} \alpha_k = 1$.

In general, assuming that the expenditure function is differentiable with respect to $\boldsymbol{p}$, Shephard's (1953) lemma,

$$
\boldsymbol{q}^c(u,\boldsymbol{p}) = \frac{\partial E(u,\boldsymbol{p})}{\partial \boldsymbol{p}}, \qquad (15)
$$

can be applied to obtain the expenditure minimizing demands $\boldsymbol{q}^c(u,\boldsymbol{p})$. Alternatively, the logarithmic form of Shephard's lemma obtains the budget share equations,

$$
\boldsymbol{w}^c(u,\boldsymbol{p}) = \frac{\partial \ln E(u,\boldsymbol{p})}{\partial \ln \boldsymbol{p}}. \qquad (16)
$$

For example, applying Shephard's lemma (15) to the expenditure function in equation (14), it is easy to see that Hicksian demands in equation (13) are obtained.

Hicksian demands are positive valued and have the following properties: homogeneity of degree zero in $\boldsymbol{p}$, and the Slutsky matrix, $[\partial \boldsymbol{q}^c(u,\boldsymbol{p})/\partial \boldsymbol{p}']$, is symmetric and negative semidefinite.

Finally, the expenditure function, $E(u, \boldsymbol{p}) = \boldsymbol{p}'\boldsymbol{q}^c(u, \boldsymbol{p})$, has the following properties: continuous in $(u, \boldsymbol{p})$, homogeneous of degree one in $\boldsymbol{p}$, increasing in $\boldsymbol{p}$ and u, concave in $\boldsymbol{p}$, and satisfies Shephard's lemma in equation (15).

2.1.4 Elasticity Relations

A demand system provides a complete characterization of consumer preferences and can be used to estimate the income elasticities, the own-and cross-price elasticities, as well as the elasticities of substitution. These elasticities are particularly useful in judging the validity of the parameter estimates which are sometimes difficult to interpret due to the complexity of the demand system specifications, unlike regressions of linear specifications where each coefficient (parameter) often has a particular interpretation.

The elasticity measures can be calculated from the Marshallian demand functions, $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{p}, x)$. In particular, the income elasticity of demand for $i = 1, .., n$, is calculated as

$$\eta_{ix}(\boldsymbol{p}, x) = \frac{\partial q_i(\boldsymbol{p}, x)}{\partial x} \frac{x}{q_i(\boldsymbol{p}, x)} = \frac{\partial \ln q_i(\boldsymbol{p}, x)}{\partial \ln x}. \qquad (17)$$

If $\eta_{ix}(\boldsymbol{p}, x) > 0$, the ith good is classified as normal at $(\boldsymbol{p}, x)$, and if $\eta_{ix}(\boldsymbol{p}, x) < 0$, it is classified as inferior. Another dividing line in classifying goods according to their income elasticities is the number one. If $\eta_{ix}(\boldsymbol{p}, x) > 1$, the ith good is classified as a necessity. For example, with Cobb-Douglas preferences represented by direct utility function of (3) and with constant elasticity of substitution (CES) preferences also used in chapter 3, $\eta_{ix}(\boldsymbol{p}, x) = 1$ (for all i) since Marshallian demands in this case are linear in income. One of the properties of income elasticity is derived by differentiating the budget constraint with respect to x :

$$\sum_{k=1}^{n} w_k \eta_{kx} = 1.$$

For $i, j = 1, ..., n$, the uncompensated (Cournot) price elasticities, $\eta_{ij}(\boldsymbol{p}, x)$, can be calculated as

$$\eta_{ij}(\boldsymbol{p}, x) = \frac{\partial q_i(\boldsymbol{p}, x)}{\partial p_j} \frac{p_j}{q_i(\boldsymbol{p}, x)} = \frac{\partial \ln q_i(\boldsymbol{p}, x)}{\partial \ln p_j}. \tag{18}$$

If $\eta_{ij}(\boldsymbol{p}, x) > 0$, the goods are gross substitutes meaning that when q_j becomes more expensive, the consumer increases consumption of good q_i and decreases consumption of good q_j.[12] If $\eta_{ij}(\boldsymbol{p}, x) < 0$, they are gross complements meaning that when q_j becomes more expensive, the consumer reduces the consumption of q_j and also of q_i. If $\eta_{ij}(\boldsymbol{p}, x) = 0$, they are independent. Differentiating the budget constraint with respect to p_i yields one of the identity properties relating the uncompensated price elasticities and budget shares of all goods,

$$w_i + \sum_{k=1}^{n} w_k \eta_{ki} = 0.$$

With Cobb-Douglas preferences represented by the direct utility function of (3), using the Marshallian demands in equation (6), the own-price elasticities are $\eta_{ii} = -\alpha_i \left(x / p_i q_i \right)$, and the cross-price elasticities are $\eta_{ij} = 0$, since the demands for the ith good depends only on the ith price.

[12] Two goods are said to be net substitutes if the notion is applied to compensated (Hicksian) demands.

The definitions given above are in gross terms, because they ignore the income effect, that is, the change in demand of good q_i due to the change in purchasing power resulting from the change in the price of good q_j. The Slutsky equation, however, decomposes the total effect of a price change on demand into a substitution effect and an income effect. In particular, differentiating the second identity in

$$q_i(\boldsymbol{p}, x) = q_i\left(\boldsymbol{p}, E(u, \boldsymbol{p})\right) = q_i^c(u, \boldsymbol{p}),$$

with respect to p_j using the chain rule, noting that $x = E$, and rearranging, we acquire the Slutsky equation,

$$\frac{\partial q_i(\boldsymbol{p}, x)}{\partial p_j} = \frac{\partial q_i^c(u, \boldsymbol{p})}{\partial p_j} - q_j(\boldsymbol{p}, x)\frac{\partial q_i(\boldsymbol{p}, x)}{\partial x}, \tag{19}$$

for all $(\boldsymbol{p}, x)$, $u = V(\boldsymbol{p}, x)$, and $i, j = 1, ..., n$.[13] The derivative $\partial q_i(\boldsymbol{p}, x)/\partial p_j$ is the total effect of a price change on demand, while the first term, $\partial q_i^c(u, \boldsymbol{p})/\partial p_j$ is the substitution effect of a compensated price change on demand, and $-q_j(\boldsymbol{p}, x)\partial q_i(\boldsymbol{p}, x)/\partial x$ in the second term is the income effect, resulting from a change in price. Hicks (1936) suggested using the sign of the cross-substitution effect to classify goods as substitutes, whenever $\partial q_i^c(u, \boldsymbol{p})/\partial p_j$ is positive. In fact, according to Hicks (1936), $\partial q_i^c(u, \boldsymbol{p})/\partial p_j > 0$ indicates substitutability, $\partial q_i^c(u, \boldsymbol{p})/\partial p_j < 0$ indicates complementarity, and $\partial q_i^c(u, \boldsymbol{p})/\partial p_j = 0$ indicates independence.

One important property of the Slutsky equation is that the cross-substitution effects are symmetric expressed as $\partial q_i^c(u, \boldsymbol{p})/\partial p_j = \partial q_j^c(u, \boldsymbol{p})/\partial p_i$. This sym-

[13] This derivation of Slutsky decomposition was due to Cook (1972).

metry restriction may also be written in elasticity terms.[14] It is easy to verify that the Slutsky substitution matrix can be rewritten in terms of elasticities as

$$w_i \eta_{ij} + w_i w_j \eta_{ix} = w_j \eta_{ji} + w_i w_j \eta_{jx},$$

and therefore the symmetry restriction may be written as

$$\eta_{ix} + \frac{\eta_{ij}}{w_j} = \eta_{jx} + \frac{\eta_{ji}}{w_i}. \tag{20}$$

The notion of elasticity of substitution was developed mainly in the producer context to study the evolution of relative factor shares in a growing economy. A logarithmic derivative of a quantity ratio with respect to a marginal rate of technical substitution is an intuitive measure of curvature of an isoquant and provides information about the comparative statics of factor shares. Out of two generalizations for more than two inputs suggested by Allen and Hicks (1934), only one is currently used. That notion became known as the Allen-Uzawa elasticity of substitution after Uzawa (1962) provided a more general formulation in the dual in terms of derivatives of the cost function. He expressed the elasticity of substitution between goods i and j as

$$\sigma_{ij}^{AU} = \frac{EE_{ij}}{E_i E_j},$$

[14] Other useful results include:

$$\begin{aligned} \frac{\partial w_i}{\partial \ln p_j} &= w_i \delta_{ij} + w_i \eta_{ij}, \\ \frac{\partial w_i}{\partial \ln x} &= w_i \eta_{ix} - w_i, \end{aligned}$$

where $\delta_{ij} = 1$ when $i = j$, or $\delta_{ij} = 0$ otherwise.

where $E_{ij} = \partial^2 E/\partial p_i \partial p_j$, $E_i = \partial E/\partial p_i$, and $E_j = \partial E/\partial p_j$. It is symmetry by construction. It also can be written in terms of Hicksian demand elasticities as

$$\sigma_{ij}^{AU} = \frac{\ln q_i^c(u, \boldsymbol{p})/\partial \ln p_j}{w_j} = \sigma_{ji}^{AU},$$

where $\partial \ln q_i^c(u, \boldsymbol{p})/\partial \ln p_j$ is the Hicksian elasticity of demand. Hence, the Allen-Uzawa elasticity of substitution is the Hicksian demand elasticity divided by the budget share. For this reason, reporting both the Hicksian demand elasticity and the Allen-Uzawa elasticity of substitution is redundant. Alternatively, since the Hicksian demand elasticity is related to the Marshallian demand elasticity through the elasticity form of the Slutsky equation in (19), the Allen-Uzawa elasticities of substitution can be written in terms of Marshallian demand elasticities as

$$\sigma_{ij}^{AU} = \eta_{ix}(\boldsymbol{p}, x) + \frac{\eta_{ij}(\boldsymbol{p}, x)}{w_j} = \eta_{jx}(\boldsymbol{p}, x) + \frac{\eta_{ji}(\boldsymbol{p}, x)}{w_i} = \sigma_{ji}^{AU}.$$

If $\sigma_{ij}^{AU} > 0$, goods i and j are said to be Allen substitutes, in the sense that an increase in the price of good j causes an increased consumption of good i. If, however, $\sigma_{ij}^{AU} < 0$, then the goods are said to be Allen complements, in the sense that an increase in the price of good j causes a decreased consumption of good i.

As another textbook example, consider a constant elasticity of substitution (CES) utility function,

$$U(\boldsymbol{q}) = \sum_{k=1}^{n} (a_k q_k^r)^{1/r}, \tag{21}$$

where $0 < \alpha_k < 1$, $-\infty < r < 1$, but $r \neq 0$. The limiting case $r = 1$ corresponds to the Cobb-Douglas form and the limiting case $r = -\infty$ corresponds to the Leontief utility functional form,

$$U(\boldsymbol{q}) = \min\{\alpha_1 q_1, \alpha_2 q_2, ..., \alpha_n q_n\}.$$

Following the procedure described in section 2.1.1 for the case of the Cobb-Douglas utility function, we obtain the Marshallian demand functions,

$$q_i(\boldsymbol{p}, x) = \frac{\alpha_i^\sigma p_i^{-\sigma}}{\sum_{k=1}^n \alpha_k^\sigma p_k^{1-\sigma}} x, \qquad i = 1, .., n, \tag{22}$$

where $\sigma = 1/(1 - r)$. The limiting case of $r = -\infty$ (Leontief) of the Marshallian demands are

$$q_i(\boldsymbol{p}, x) = \frac{\alpha_i}{\sum_{k=1}^n \alpha_k p_k} x \qquad i = 1, ..., n. \tag{23}$$

Using Roy's identity, the indirect utility function is obtained as

$$V(\boldsymbol{p}, x) = x \left[\sum_{k=1}^n \alpha_k^\sigma p_k^{1-\sigma}\right]^{1/(\sigma-1)},$$

and the Leontief case as

$$V(\boldsymbol{p}, x) = \frac{x}{\sum_{k=1}^n \alpha_k p_k}. \tag{24}$$

As mentioned, $\eta_{ix} = 1$, and using (18), we obtain

$$\eta_{ij} = \frac{-(1-\sigma)\alpha_j^\sigma p_j^{1-\sigma}}{\sum_{k=1}^n \alpha_k^\sigma p_k^{1-\sigma}}, \qquad i, j = 1, ..., n. \tag{25}$$

Constructing $w_j = p_j q_j / x$ using the equation (22) and substituting the w_j and the equation (25) into (20), we get

$$\begin{aligned}
\sigma_{ij}^{AU} &= \eta_{ix}(\boldsymbol{p}, x) + \frac{\eta_{ij}(\boldsymbol{p}, x)}{w_j} \\
&= 1 + \frac{-(1-\sigma)\alpha_j^\sigma p_j^{1-\sigma} / \sum_{k=1}^n \alpha_k^\sigma p_k^{1-\sigma}}{\alpha_i^\sigma p_i^{1-\sigma} / \sum_{k=1}^n \alpha_k^\sigma p_k^{1-\sigma}} \\
&= 1 - (1-\sigma) \\
&= \sigma,
\end{aligned}$$

just as we expect.

Therefore, this functional form relaxes the unitary elasticity of substitution restrictions imposed by the Cobb-Douglas, but imposes the restriction that the Allen-Uzawa elasticity of substitution (and the Hicksian elasticity of substitution) between any pair of goods is always constant, $1/(1-r)$.

Thousands of Allen-Uzawa elasticities have been estimated to analyze substitutability and complementarity relationships among inputs and among consumption goods and to measure structural instability in a variety of contexts. There are, however, other elasticities that can be used to assess the substitutability and complementarity relationships between goods. Blackorby and Russell (1981, 1989) show that the Allen-Uzawa elasticity of substitution preserves none of the salient properties of the original Hicksian notion and proposed an alternative elasticity, first formulated by Morishima

(1967). The Morishima elasticity of substitution is shown to be the natural generalization of the original notion of Hicks when there are more than two inputs (goods). They suggest that the Morishima elasticity of substitution is the natural generalization of the original Hicksian concept. The Morishima elasticity of substitution is gradually making its way into the empirical studies on substitutability and complementarity. Davis and Gauger (1996) show how three different elasticity measures reach different conclusions on substitutability/complementarity relationships in monetary assets.

Morishima net elasticity of substitution can be used to measure the percentage change in relative demands (quantity ratios) with respect to a percentage change in one price. In particular, under the assumption that a change in p_j/p_i is due solely to a change in p_j, the Morishima elasticity of substitution for q_i/q_j is given by

$$\begin{aligned}
\sigma_{ij}^{M} &= \frac{\partial \ln \left(q_i^c(u, \boldsymbol{p}) / q_j^c(u, \boldsymbol{p}) \right)}{\partial \ln \left(p_j / p_i \right)} \\
&= \frac{p_i E_{ij}}{E_j} - \frac{p_i E_{ii}}{E_i} \\
&= \frac{\partial \ln q_i^c(u, \boldsymbol{p})}{\partial \ln p_j} - \frac{\partial \ln q_j^c(u, \boldsymbol{p})}{\partial \ln p_j} \\
&= w_j \left(\sigma_{ij}^{AU} - \sigma_{jj}^{AU} \right),
\end{aligned}$$

and measures the net change in the compensated demand for good i, when the price of good j changes. A change in p_j, holding p_i constant, has two effects on the quantity ratio q_i/q_j: one on q_i captured by $\partial \ln q_i^c(u, \boldsymbol{p}) / \partial \ln p_j$ and one on q_j captured by $\partial \ln q_j^c(u, \boldsymbol{p}) / \partial \ln p_j$. Two goods will be Morishima substitutes (complements), if an increase in the price of j causes q_i/q_j to decrease (increase). Comparing the Allen-Uzawa and Morishima elasticities of substitution, we see that if two goods are Allen-Uzawa substitutes, $\sigma_{ij}^{AU} > 0$, they must also be Morishima substitutes, $\sigma_{ij}^{M} > 0$. However, two goods may be Allen-Uzawa complements, $\sigma_{ij}^{AU} < 0$, but Morishima substitutes if $\left|\sigma_{jj}^{AU}\right| > \left|\sigma_{ij}^{AU}\right|$. It suggestes that the Allen-Uzawa elasticity of substitution matrix is symmetric, $\sigma_{ij}^{AU} = \sigma_{ji}^{AU}$, but the Morishima elasticity of substitution matrix is not. The Morishima elasticity of substitution matrix is symmetric only when the aggregator function is a member of the constant elasticity of substitution family.

The Morishima elasticity of substitution is a "two-good one-price" elasticity of substitution, unlike the Allen-Uzawa elasticity of substitution, which is a "one-good one-price" elasticity of substitution. They differ in that the former measures elasticities of ratios of variables rather than those of variables themselves. Another "two-good one-price" elasticity of substitution that can be used to assess the substitutability/complementarity relationship between goods is the Mundlak elasticity of substitution (Mundlak, 1968),

$$
\begin{aligned}
\sigma_{ij}^{MU} &= \frac{\partial \ln \left(q_i(\boldsymbol{p},x)/q_j(\boldsymbol{p},x)\right)}{\partial \ln \left(p_i/p_j\right)} \\
&= \eta_{ij}(\boldsymbol{p},x) - \eta_{jj}(\boldsymbol{p},x) \\
&= \sigma_{ij}^{M} + w_j \left(\eta_{jx}(\boldsymbol{p},x) + \eta_{ix}(\boldsymbol{p},x)\right).
\end{aligned}
$$

The Mundlak elasticity of substitution, like the Marshallian demand elasticity, is a measure of gross substitution (with income held constant). Two goods will be Mundlak substitutes (complements) if an increase in the price of j causes q_i/q_j to decrease (increase).

Another kind of elasticity of substitution which is "two-good two-price" elasticity of substitution is Shadow elasticity of substitution (McFadden, 1963). It is defined as the negative of the elasticity of the demand ratio $q_i(\boldsymbol{p}, x)/q_j(\boldsymbol{p}, x)$ with respect to a change in the price ratio p_i/p_j holding utility level, all other prices, and total expenditure constant. It takes the form,

$$
\begin{aligned}
\sigma_{ij}^{S} &= -\frac{\partial \ln \left(q_i\left(\boldsymbol{p}, x\right)/q_j\left(\boldsymbol{p}, x\right)\right)}{\partial \ln \left(p_i/p_j\right)}\Bigg|_{u, E \text{ and } p_k, k \neq i, j, \text{ constant}} \\
&= \frac{-E_{ii}/E_i^2 + 2\left(E_{ij}/E_i E_j\right) - E_{jj}/E_j^2}{1/p_i E_i + 1/p_j E_j}, \qquad (26) \\
&= \frac{w_j \sigma_{ij}^{M} + w_j \sigma_{ji}^{M}}{w_i + w_j}, \qquad (27)
\end{aligned}
$$

for $i, j = 1, ..., n$. The Shadow elasticity of substitution measures the curvature of a level surface of the expenditure function in a particular direction $\boldsymbol{\xi}$ such that $\xi_i q_i + \xi_j q_j = 0$, and $\xi_k = 0$ for $k \neq i, j$ in two-dimensional subspace of the price space spanned by the ith and jth basis vectors. It follows that $\sigma_{ij}^{S} \geqq 0$ for all $i, j = 1, ..., n, i \neq j$ directly from the concavity of the expenditure function.

The further generalization of Shadow elasticity of substitution was formulated by Frenger (1978, 1985a, 1985b), who defined the directional shadow elasticity of substitution. This elasticity measures the curvature of the level surface of the expenditure function at a point $\boldsymbol{p}$ for an arbitrary price change $\boldsymbol{\xi}$ which leaves total expenditure and utility constant. One definition based on this idea is given by

$$\sigma_{ij}^{D}(\boldsymbol{\xi}) = -\frac{\sum_{i=1}^{n}\sum_{j=1}^{n} E_{ij}\xi_i\xi_j}{\sum_{i=1}^{n} q_i\xi_i\left(\xi_i/p_i\right)}, \qquad \boldsymbol{\xi} \in T\left(\boldsymbol{p}\right), \quad \boldsymbol{\xi} \neq \mathbf{0}.$$

For each $\boldsymbol{p}$ this elasticity is a function defined on the tangent plane $T(\boldsymbol{p}) = \left\{\boldsymbol{\xi}|\boldsymbol{\xi}'\boldsymbol{q} = 0\right\}$ to the level surface of the expenditure function at $\boldsymbol{p}$. The domain of $\sigma_{ij}^{D}(\boldsymbol{\xi})$ is all of $T\left(\boldsymbol{p}\right)$ except for the point $\boldsymbol{\xi} = \mathbf{0}$, and $\sigma_{ij}^{D}(\boldsymbol{\xi})$ is a homogeneous function of degree zero in $\boldsymbol{\xi}$. It is shown that $\sigma_{ij}^{D}(\boldsymbol{\xi}) \geqq 0$ for every $\boldsymbol{\xi} \in T\left(\boldsymbol{p}\right)$, and that it reduces to σ_{ij}^{S} when $\xi_i q_i + \xi_j q_j = 0$ and $\xi_k \neq 0, k \neq i, j$.

2.1.5 Curvature

Often time, we would like to check if the functional form of demand models has a correct curvature property. The condition can be translated to the definiteness of the Slutsky substitution matrix or the Allen-Uzawa elasticity of substitution matrix. Since the definition of the definiteness of matrix is not applicable in practice, the popular method to check the definiteness is to apply the Cholesky factorization and check the signs of the resulting Cholesky values. For example, the Hermitian (symmetric with real entries) matrix is negative semi-definite if the all Cholesky values are nonpositive.

Lau (1978b, p. 427) shows that every positive semi-definite matrix $\mathbf{A}$ has the following representation (Cholesky factorization):

$$\mathbf{A} = \mathbf{LDL}', \tag{28}$$

where $\mathbf{L}$ is a unit lower triangular matrix with all diagonal entries unity and $\mathbf{D}$ is a diagonal matrix with all the main diagonal entries nonnegative. The diagonal entries in $\mathbf{D}$ are called Cholesky values.

Because rank of the Slutsky matrix and the Allen-Uzawa elasticity matrix is $n-1$ due to the homogeneity condition, the condition is applied to the $(n-1)$ by $(n-1)$ matrix resulting from removing jth row and jth column from the original substitution matrix.[15] Noting that the matrix $\mathbf{D}$ can be always written as $\mathbf{D}^{1/2}\mathbf{D}^{1/2}$ because of the non-negative diagonal entries, the equation (28) can be written as,

$$\mathbf{A} = \mathbf{LDL}' = \mathbf{LD^{1/2}D^{1/2}L}' = \left(\mathbf{LD^{1/2}}\right)\left(\mathbf{LD^{1/2}}\right)' = \mathbf{BB}', \qquad (29)$$

where $\mathbf{B}$ is some lower triangular matrix (See also Theorem 9 in Diewert and Wales, 1987).

Alternatively, the eigenvalues of Slutsky matrix can be used to check its negative semi-definiteness: the eigenvalues of negative semi-definite matrix are all nonpositive. The representation in (29) is more convenient than (28) if one wants to impose the semi-definiteness on the matrix $\mathbf{A}$. It simply requires the specification of lower triangular matrix $\mathbf{B}$, which is easier than specifying $\mathbf{L}$ and $\mathbf{D}$ in (28).

In the case of three goods, given the Allen-Uzawa substitution matrix σ^{AU}, after deleting any one row and the corresponding column, say third,

[15] See Moschini's (1997) Lemma that S is negative semi-definite matrix if and only if $\widetilde{S}$ whose jth row and jth column are removed from S is negative semi-definite matrix.

Cholesky factorization yields

$$\widetilde{\sigma}^{AU} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \sigma_{12}/\sigma_{11} & 1 \end{bmatrix} \begin{bmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} - \sigma_{12}/\sigma_{11} \end{bmatrix} \begin{bmatrix} 1 & \sigma_{12}/\sigma_{11} \\ 0 & 1 \end{bmatrix},$$

where $\widetilde{\sigma}^{AU}$ is a matrix after the third row and column are removed from σ^{AU}. Arranging the statement for the negative semi-definite matrix, the required conditions are

$$\sigma_{11} \leq 0 \qquad \text{and} \quad \sigma_{22} - \sigma_{12}/\sigma_{11} \leq 0.$$

This condition was used to evaluate the negative semi-definiteness in Caves and Christen (1980) and Barnett and Seck (2007).

It is also possible to check the quasiconcavity of the function by the inspection of bordered Hessian. Let $f(\boldsymbol{x})$ be a real-valued function defined on $\mathbb{R}^n_+$, and the bordered Hessian is formed as

$$\mathbf{H}^* = \begin{bmatrix} f_{11} & \cdots & f_{1n} & f_1 \\ \vdots & \ddots & \vdots & \vdots \\ f_{n1} & \cdots & f_{nn} & f_n \\ f_1 & \cdots & f_{1n} & 0 \end{bmatrix}. \tag{30}$$

In (30), the matrix of second-order partial derivatives is bordered by the first-order partials and a zero to complete the square matrix. The principal minors of this matrix are the determinants

$$D_1 = \begin{vmatrix} f_{11} & f_1 \\ f_1 & 0 \end{vmatrix}, \, D_2 = \begin{vmatrix} f_{11} & f_{12} & f_1 \\ f_{21} & f_{22} & f_2 \\ f_1 & f_2 & 0 \end{vmatrix}, \, \cdots, \, D_n = |\mathbf{H}^*| \, .$$

Arrow and Enthoven (1961) use the sign pattern of these principal minors to establish the following useful results, which are used to evaluate the quasiconcavity of the Normalized Quadratic reciprocal indirect utility function in chapter 3:

1 If $f(\boldsymbol{x})$ is quasiconcave, these principal minors take an alternating sign as follows: $D_1 \leq 0$, $D_2 \geq 0$, ...

2 If for all $\boldsymbol{x} \gg 0$, these principal minors alternate in sign beginning with strictly negative: $D_1 < 0$, $D_2 > 0$,..., then f is strictly quasiconcave on the positive orthant.

Galland and Golub (1984), following Diewert, Avriel and Zang (1977), argue that a necessary and sufficient condition for quasiconvexity of $V(\nu)$ is

$$g(\boldsymbol{\nu}) = \min_{\boldsymbol{Z}} \left\{ \boldsymbol{z}' \nabla^2 V\left(\boldsymbol{\nu}\right) \boldsymbol{z} : \boldsymbol{z}' \nabla V\left(\boldsymbol{\nu}\right) = 0, \boldsymbol{z}'\boldsymbol{z} = 1 \right\} \tag{31}$$

where $\nabla^2 V = \partial^2 V / \partial \nu \partial \nu'$ and $\nabla V = \partial V / \partial \nu$, and g is nonnegative when quasiconvexty (curvature) constraint is satisfied and negative when it is violated. They devised the numerical procedure to impose quasiconvexity during estimation on demand functions based on the idea in (31), and Serletis and Shahmoradi (2008) succeessfully applied it to the estimation of demands for monetary assets with globally flexible demand systems.

When characterizing a concave function, for example, expenditure function, Lewbel (1985) used the theorem (Hardy, Littlewood and Polya, 1952) that the function $E(u, \boldsymbol{p})$ is concave in the vector $\boldsymbol{p}$ if at $t = 0$, $\partial^2 E(u, \boldsymbol{p} + \boldsymbol{\xi} t)/\partial t^2 \leq 0$ for all n-vectors $\boldsymbol{\xi}$.

For example, given the expenditure function for Cobb-Douglas preference given in equation (14),

$$E(u, \boldsymbol{p}) = u \prod_{j=1}^{n} \left(\frac{p_j}{\alpha_j} \right)^{\alpha_j},$$

where $\sum_{k=1}^{n} \alpha_k = 1$, the straightforward differentiation of equation (14) with respect to t shows that

$$\left. \frac{\partial^2 E(u, \boldsymbol{p} + \boldsymbol{\xi} t)}{\partial t^2} \right|_{t=0} = u \sum_{j=1}^{n} \left(\frac{\alpha_j^2 - \alpha_j}{p_j^2} \right) \left(\frac{p_j}{\alpha_j} \right)_j^{\alpha_j} \xi_j^2 \leq 0.$$

This inequality requires that $\alpha_j^2 - \alpha_j \leq 0$ or $0 \leq \alpha_j \leq 1$ for all $j = 1, ..., n$, which is essentially the condition that is known to have the globally regular Cobb-Douglas functional form.

2.2 Demand System Specification

This part of the chapter illustrates the parametric approach of demand systems. Although the differentiable approach of demand models is not considered since it is not used in this dissertation, one important model that has been frequently used to test the consumer demand theory and to estimate many kinds of elasticity values is the Rotterdam model, introduced by

Theil (1965) and Barten (1966). The Rotterdam model is a member of the class of the differential approach of demand systems. Those who are interested in this class of model can refer to Barnett and Serletis (2008, 2009a, 2000b). Barnett and Seck (2007) recently compared the performance of the Rotterdam model to AIDS model of Deaton and Muellbauer (1980).

2.2.1 Models with separable preferences

In this section, we discuss demand systems generated by separable preferences. Types of separability assumptions such as strong separability or weak separability restrict preferences by precluding certain types of specific interactions among goods. A utility function is weakly separable if and only if the goods can be partitioned into subsets in such a way that every marginal rate of substitution involving two goods from the same subset depends only on the goods in that subset. A utility function is strongly separable if and only if the goods can be partitioned into subsets in such a way that every marginal rate of substitution involving goods from different subsets depends only on the goods in those two subsets. Additive separability is a form of strong separability. As expected, a utility function that is strongly separable with m subsets is also weakly separable with m subsets. We illustrate the consumer demand systems based on separable preferences here since those models have been developed under the premise that the models are relatively flexible (thus not satisfying the local flexibility definition) and globally integrable, and therefore are convenient to generate artificial data that is consistent with the optimizing consumer behaviors.[16] Since this dissertation is not concerned with any of the issues on separability, we give only a brief note on the literature before illustrating the several models with separable preference. Leontief (1947a, 1947b) investigated the underlying mathematical structure of separability in the producer context and Sono (1945) in the consumer context. Debreu (1960) provided an important characterization of additivity. The notions of weak and strong separability were developed in Strotz (1957, 1959) and Gorman (1959). Goldman and Uzawa (1964) developed a characterization of separability in terms of the partial derivatives of the demand functions. Gorman (1968) provides a definitive discussion of separability concepts. One may enjoy reading the subsequent exchange between Vind (1971a, 1971b) and Gorman (1971a, 1971b). Blackorby, Primont and Russel (1978) provide a thorough discussion and rigorous analysis of separability. There is a large amount of literature on the empirical examination of separable preference structures. Most of the nonparametric tests that have been developed are based on Varian (1982, 1983, 1985). Works on tests for weak separability in the producer model include Berndt and Christensen (1973, 1974), Berndt and Wood (1975), Denny and Fuss (1977), Woodland (1987), Blackorby, Schworm and Fisher (1986), Diewert and Wales (1995). In the consumer context, works on tests of separability include Jorgenson and Lau (1975). Large numbers of studies are interested in the separability of the monetary aggregates from other aggregate variables.[17] Monte Carlo studies that assess

[16] Blackorby, Primont and Russell (1977, 1978) showed that if we start with any flexible functional form, then under the hypothesis of weak separability, the resulting function would be necessarily inflexible.

[17] See Fisher and Fleissig (1997), Fleissig and Swofford (1996), Fleissig and Whitney (2003), Jones and Stracca (2006), Swofford and Whitney (1987, 1988, 1994).

the performance of the various available tests of separability include Barnett and Choi (1983), Elger, Jones and Binner (2006). O (2008) investigates the performances of several non-parametric tests including the newly developed de Peretti's (2005) test.

2.2.1.1 Explicit and Implicit Additivity utility models

We say that a preference ordering is directly additive if it can be represented by a direct utility function of the form

$$U(\boldsymbol{q}) = F\left[\sum_{k=1}^{n} u^k(q_k)\right], \tag{32}$$

where $F(\cdot) > 0$. Additive separability is a member of strong separable preference.

The direct utility function is implicitly additive if it may be defined as the identity of the form

$$\sum_{k=1}^{n} F^k(q_k, u) \equiv 1,$$

where F^k's are n functions of two variables. Under appropriate conditions on the function F^k, a unique solution $u = U(\boldsymbol{q})$ to the above equation exists for u as a function of $\boldsymbol{q}$ such that the resulting function U is monotonically increasing and strictly quasiconcave. But those conditions (although sufficient for the existence of a regular neoclassical utility function) are not themselves sufficient for the existence of a closed form explicit representation of U.

The explicit and implicit indirect additive utility functions are similarly defined as

$$V(\boldsymbol{p}, x) = G\left[\sum_{k=1}^{n} v^k(p_k/x)\right]$$

and

$$\sum_{k=1}^{n} G^k((p_k/x), v) \equiv 1,$$

where $G(\cdot) > 0$ and G^k's are n functions of two variables.

Hanoch (1975) showed that implicit additive utility functions are less restricted than their explicit counterparts. The restrictive nature of additivity implies that the ratios of the Allen-Uzawa elasticities of substitution for implicit additive direct utility function for ith and jth goods are functions of q_i, q_j and $(\boldsymbol{p}/x)$ only, and are independent of any $k(\neq i, j)$th variables. But, for the explicit additive case, the ratio of a pair of Allen-Uzawa elasticities of substitution with respect to some third variable is equal to the ratio of the income elasticities. For implicit additive indirect utility function, the difference of any pair of Allen-Uzawa elasticities of substitution with respect to some third variable depends only on their own variables. But in the explicit case, the differences are equal to income elasticity differences.

We illustrate the implication of implicit utility models by using the specific implicit indirect utility function. One type of Hanoch's model is constant differences of elasticities of substitution function denoted by the CDE model. The CDE indirect utility function is implicitly defined as

$$\sum_{k=1}^{n} G^k((p_k/x), v) = \sum_{k=1}^{n} B_k v^{e_k b_k} (p_k/x)^{b_k} \equiv 1. \tag{33}$$

The parameter restrictions required for (33) to be a globally integrable model are: B_k, $e_k > 0$, $b_k < 1$, and either,

$$b_k \leq 0 \qquad \text{for all } k, \tag{34}$$

or

$$0 \leq b_k \leq 1 \qquad \text{for all } k. \tag{35}$$

Applying Roy's identity, we obtain the Marshallian demands,

$$q_i(\boldsymbol{p}/x) = \frac{B_i b_i v^{e_i b_i} (p_i/x)^{b_i - 1}}{\sum_{k=1}^{n} B_k b_k v^{e_k b_k} (p_k/x)^{b_k}}. \tag{36}$$

It is easy to see that the appropriate restrictions on (36) reduce to the Marshallian demands for CES form given in (23). The Allen-Uzawa elasticities of substitution is written as

$$\begin{aligned} \sigma_{ij}^{AU} &= \alpha_i((p_i/x), v) + \alpha_j((p_j/x), v) \\ &\quad - \textstyle\sum_{k=1}^{n} w_k \alpha_k((p_k/x), v) - \delta_{ij} \dfrac{\alpha_i((p_i/x), v)}{w_i}, \end{aligned} \tag{37}$$

where

$$\alpha_i((p_i/x), v) = -\left(\frac{p_i}{x}\right) \frac{\frac{\partial^2 G^i}{\partial (p_i/x)^2}}{\frac{\partial G^i}{\partial (p_i/x)}} = 1 - b_i,$$

where δ_{ij} is the Kronecker delta and is equal to 1 when $i = j$ and 0 otherwise.

The CDE model allows cases of complements since the Allen-Uzawa elasticities of substitution in (37) can be negative with appropriate values for b_i and b_j. As the name CDE implies, the difference between the two elasticities of substitution with respect to some third good is constant, $\sigma_{ik} - \sigma_{jk} = -b_i + b_j$, regardless of the price argument. Because of its ability to be globally regular and to represent complementary relationship of goods, the linear homogeneous CDE when $e_i = e$ for all i, is used to generate the artificial demand data that is consistent with rational consumer behavior in Jensen (1997) and Barnett and Usui (2007) as well as in chapter 3.

Despite the fact that greater theoretical flexibility per free parameter is sometimes possible by specifying and restricting implicit utility functions rather than explicit utility functions, the attempt to use the model in practice faces an immediate obstacle. After observing that the construction of a unique and estimable structural model from an underlying implicit utility function is troublesome since it does not have an explicit closed form solution for the implied reduced form of the demand system, one notices that the conventional maximum likelihood estimation with additive error structure cannot be used. Barnett, Kopecky and Sato (1981) managed to estimate the direct implicit addilog model (DIA). See Barnett (1981, chapter 9) for detailed discussion of their highly promising, but not yet successfully implemented approach.

2.2.1.2 WS-Branch Utility Demand System

The Barnett's (1977) nonhomothetic WS-branch utility tree is the generalization to blockwise strong separability of the S-branch utility tree (Brown and Heien, 1972). In view of (32), the block utility function u^k's are of the generalized CES form and the aggregator utility function F is CES. The WS-branch model is the only blockwise weakly separable utility function and can be shown to be a flexible form when there are no more than two goods in each block and a total of no more than two blocks. It is homothetic in supernumerary quantities, but not homothetic in the elementary quantities. Hence, a homothetic utility function can be converted into a nonhomothetic utility function by translating the quantities into supernumerary quantities. In this model, the individual aggregator (or category) functions within the tree are in the form of the generalized quadratic mean of order ρ, as in the macro-utility function defined over those aggregates. Therefore, this functional form can represent a wide range of preferences when the number of goods in the system is no more than four. Barnett and Choi (1989), Barnett and Seck (2007) and O (2008) used the WS-branch tree to generate the artificial data for their Monte Carlo studies.

The generalized quadratic mean of order ρ is of the form

$$U(u_1, .., u_m) = A \left(\sum_{k=1}^{m} \sum_{l=1}^{m} B_{kl} u_k^{\rho} u_l^{\rho} \right)^{1/2\rho}, \tag{38}$$

where $\rho < 1/2$, $B_{kl} > 0$ for all k and l, $\sum_k \sum_l B_{kl} = 1$, $B_{kl} = B_{lk}$ for $k \neq l$, and $A > 0$. These inequalities ensure the monotonicity and quasiconcavity of the function. To introduce a weakly separable structure, each u_r is itself treated as an aggregator rather than as an elementary good so that (38) becomes the macro-function defined over the aggregates. The aggregator functions producing the aggregates are of the form $u_r = u_r(\boldsymbol{q}_r - \boldsymbol{a}_r)$, where $\boldsymbol{q}_r$ is a sub-vector of $\boldsymbol{q}$, and $\boldsymbol{a}_r$ is a conformable "committed quantities" vector. Assuming these aggregator functions also take the same form as (38), the resulting nested two stage structure of means of order ρ produces the WS-branch utility tree.

2.2.2 Locally Flexible Functional Forms

A locally flexible functional form is a second-order approximation to an arbitrary function. In the demand system literature there are two different definitions of second-order approximations, one by Diewert (1971) and another by Lau (1974). Barnett (1983b) has identified the relationship of each of those definitions to existing definitions in the mathematics of local approximation orders and has shown that a second-order Taylor series approximation is sufficient but not necessary for both Diewert's and Lau's definitions of second-order approximation.

Consider a n-argument, twice continuously differentiable aggregator function, $V(\boldsymbol{\nu})$. According to Diewert (1971), $V(\boldsymbol{\nu})$ is a flexible functional form if it contains enough parameters so that it can approximate an arbitrary twice continuously differentiable function V^* to the second order at an arbitrary point $\boldsymbol{\nu}^*$ in the domain of definition of V and V^*. Thus, V must have enough free parameters to satisfy the following set of $1+n+n^2$ equations:

$$V(\boldsymbol{\nu}^*) = V^*(\boldsymbol{\nu}^*), \tag{39}$$

$$\nabla V(\boldsymbol{\nu}^*) = \nabla V^*(\boldsymbol{\nu}^*), \tag{40}$$

$$\nabla^2 V(\boldsymbol{\nu}^*) = \nabla^2 V^*(\boldsymbol{\nu}^*), \tag{41}$$

where $\nabla V(\boldsymbol{\nu}) = \partial V(\boldsymbol{\nu})/\partial \boldsymbol{\nu}$ and $\nabla^2 V(\boldsymbol{\nu}) = \partial^2 V(\boldsymbol{\nu})/\partial \nu_i \nu_j$ denotes the $n \times n$ symmetric matrix of second-order partial derivatives of $V(\boldsymbol{\nu})$ evaluated at $\boldsymbol{\nu}$. The symmetry property follows from the assumption that $V(\boldsymbol{\nu})$ is twice continuously differentiable. Under this assumption, the function does not have to satisfy all n^2 equations in (41) independently since the symmetry of second derivatives (sometimes known as Young's theorem) implies that $\partial^2 V(\boldsymbol{\nu}^*)/\partial\nu_i\partial\nu_j = \partial^2 V(\boldsymbol{\nu}^*)/\partial\nu_j\partial\nu_i$ and $\partial^2 V(\boldsymbol{\nu})/\partial\nu_i\partial\nu_j = \partial^2 V(\boldsymbol{\nu})/\partial\nu_j\partial\nu_i$ for all i and j. Thus the matrices of second order partial derivatives $\nabla^2 V(\boldsymbol{\nu}^*) = \nabla^2 V^*(\boldsymbol{\nu}^*)$ are both symmetric matrices. Hence, there are only $n(n+1)/2$ independent equations to be satisfied in the restrictions (41), so that a general locally flexible functional form must have at least $1 + n + n(n+1)/2$ free parameters.

To illustrate Diewert's flexibility concept, let us consider the basic translog indirect utility function, introduced by Christensen, Jorgenson and Lau (1975),

$$\ln V(\boldsymbol{\nu}) = \alpha_0 + \sum_{k=1}^{n} \alpha_k \ln \nu_k + \frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} \beta_{kl} \ln \nu_k \ln \nu_l, \qquad (42)$$

where $\nu_k = p_k/x$, α_0 is a scalar, $\boldsymbol{\alpha}' = [\alpha_1, ..., \alpha_n]$ is a vector of parameters, and $\mathbf{B} = [\beta_{kl}]$ is a $n \times n$ symmetric matrix of parameters for a total of $1+n+(n+1)/2$ parameters. To show that (42) is a flexible functional form, we need to show that α_0, $\boldsymbol{\alpha}'$ and $\mathbf{B}$ in (42) satisfy conditions (39)-(41) at an arbitrary point $\boldsymbol{\nu}^*$.[18] Without loss of generality, we choose the arbitrary point $\boldsymbol{\nu}^* = \mathbf{1}$. This choice is harmless at least in practice since the prices and income data can always be normalized to unities at which the second-order approximation is sought. Evaluated at $\boldsymbol{\nu}^* = \mathbf{1}$, the level, first, and second

[18] It is convenient to remember the following mathematical fact: if $\beta_{kl} = \beta_{lk}$, then

$$\frac{\partial}{\partial \log p_i} \left[\frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} \beta_{kl} \ln p_k \ln p_l \right] = \sum_{l=1}^{n} \beta_{il} \ln p_l.$$

derivatives of (42) are

$$
\begin{aligned}
\ln V(\nu)\Big|_{\nu^*=1} &= \alpha_0, \\
\nabla \ln V(\nu)\Big|_{\nu^*=1} &= \boldsymbol{\alpha}, \\
\nabla^2 \ln V(\nu)\Big|_{\nu^*=1} &= \mathbf{B}.
\end{aligned}
$$

Hence, the level and all of the first and second derivative terms of any twice-differentiable functions are freely chosen by the unrestricted parameters α_0, $\boldsymbol{\alpha}$, and $\mathbf{B}$ of the translog model.

Another locally flexible functional form is the generalized Leontief (GL), introduced by Diewert (1973) in the context of cost and profit functions. Diewert (1974) also introduced the GL reciprocal indirect utility function,

$$
V(\boldsymbol{\nu}) = \alpha_0 + \sum_{k=1}^{n} \alpha_k \nu_k^{1/2} + \frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} \beta_{kl} \nu_k^{1/2} \nu_l^{1/2}, \tag{43}
$$

where $\mathbf{B} = [\beta_{kl}]$ is an $n \times n$ symmetric matrix of parameters and α_0 and α_k's are other parameters for a total of $(n^2+3n+2)/2$ parameters. The reciprocal indirect utility function is simply an inverse of the indirect utility function and possesses properties identical to the indirect utility function except that it is quasiconcave instead of quasiconvex. This model can be viewed as the generalization of the Leontief preferences since with $\alpha_0 = \alpha_k = 0$ and $\beta_{kl} = 0$ for $k \neq l$, the model reduces to the indirect utility function corresponding to the Leontief preferences in (24) after it is inverted.

Applying Diewert's modified version of Roy's identity (11) to (43), for

$i = 1, ..., n$, the following budget share equations result:

$$w_i = \frac{\alpha_i \nu_i^{1/2} + \sum_{k=1}^{n} \beta_{ik} \nu_k^{1/2}}{\sum_{k=1}^{n} \alpha_k \nu_k^{1/2} + \sum_{k=1}^{n} \sum_{l=1}^{n} \beta_{kl} \nu_k^{1/2} \nu_l^{1/2}}. \tag{44}$$

Since the budget share equations are homogeneous of degree zero in the parameters, the model requires a parameter normalization. Barnett and Lee (1985) use the following normalization:

$$2 \sum_{k=1}^{n} \alpha_k + \sum_{k=1}^{n} \sum_{l=1}^{n} \beta_{kl} = 1.$$

Caves and Christensen (1980) have shown that the GL form has satisfactory local properties when preferences are nearly homothetic and substitution is low. The result is intuitive since the Leontief preferences as its special case are characterized with zero elasticities of substitution. However, when preferences are not homothetic or substitution is high, the generalized Leontief has a small regular region.

Deaton and Muellbauer (1980) introduced another locally flexible demand system, the Almost Ideal Demand System (AIDS). The demand functions are written as

$$w_i = \alpha_i + \sum_{k=1}^{n} \gamma_{ik} \ln p_k + \frac{1}{2} \beta_i \ln(x/P), \tag{45}$$

for $i = 1, ..., n$, where the price deflator of the logarithm of income is

$$\ln P = \alpha_0 + \sum_{k=1}^{n} \alpha_k \ln p_k + \frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} \gamma_{kl} \ln p_k \ln p_l.$$

For more details regarding the AIDS, see Deaton and Muellbauer (1980), Barnett and Serletis (2008), and Barnett and Seck (2008).

2.2.3 Engel Curves and the Rank of Demand Systems

Applied demand analysis uses two types of data: time series and cross sectional data. Time series data offer substantial variation in relative prices and less variation in income, whereas cross-sectional data offer limited variation in relative prices and substantial variation in income levels. In time series data, prices and income vary simultaneously, whereas, in household budget data prices are almost constant. Household budget data give rise to the Engel curves (income expansion paths), which are functions describing how a consumer's purchase of goods vary as the consumer's income varies. Engel curves are Marshallian demand functions, with prices of all goods held constant. Like Marshallian demand functions, Engel curves may also depend on demographic or other non-income consumer characteristics (such as, for example, age and household composition), which we have chosen to ignore in this section.

Engel curves can be used to calculate the income elasticity of a good and hence whether a good is an inferior, normal, or luxury good, depending on whether income elasticity is less than zero, between zero and one, or greater than one, respectively. They are also used for equivalence scale calculations (welfare comparisons across households) and for determining properties of demand systems, such as aggregability and rank. For many commodities, standard empirical demand systems do not provide an accurate picture of observed behavior across income groups.

The frequently used Engel curve specification includes Working's (1943) linear budget share specification,

$$w_i = a_i + b_i \ln x, \tag{46}$$

which is known as the Working-Leser model, since Leser (1963) found that this functional form fit better than some alternatives. However, Leser obtained still better fits with what would now be called a rank three model (with "three" terms of function of prices: 1, $\ln x$, and $1/x$), specifically:

$$w_i = a_i + b_i \ln x + c_i x^{-1},$$

and in a similar, earlier comparative statistical analysis, Prais and Houthakker (1955) found that

$$q_i = a_i + b_i \ln x$$

fits best.

The Prais-Houthankker methodology is simply empirical, choosing functional forms on the ground of fit, with an attempt to classify particular forms as typically suitable for particular types of goods.[19] Much of this work is not very edifying by modern standards. The functional forms were

[19] See also Tornqvist (1941), Aitchison and Brown (1954-5), and the survey by Brown and Deaton (1972) for similar attempt.

rarely chosen with any theoretical model in mind. Indeed, all but one of Prais and Houthakker's Engel curves are incapable of satisfying the adding-up requirement, while on the econometric side, satisfactory methods for comparing different (non-nested) functional forms were not well-developed. Even the apparently straightforward comparison between a double-log and a linear specification led to considerable difficulties; see the simple statistic proposed by Sargan (1964) and the theoretically more satisfactory (but extremely complicated) solution in Aneuryn-Evans and Deaton (1980).

However, these empirical results give a motivation to construct a plausible functional form of demand functions that also satisfies the axioms of consumer demand behavior. Using (16) in which the budget shares are the logarithmic derivatives of the expenditure function, the Engel curve of the form in equation (46) corresponds to the differential equations of the form

$$\frac{\partial \ln E(u, \boldsymbol{p})}{\partial \ln p_i} = \alpha_i(\boldsymbol{p}) + \beta_i(\boldsymbol{p}) \ln E(u, \boldsymbol{p}), \tag{47}$$

which gives a solution of the general form

$$\ln E(u, \boldsymbol{p}) = u \ln b(\boldsymbol{p}) + (1-u) \ln a(\boldsymbol{p}), \tag{48}$$

where $\alpha_i(\boldsymbol{p}) = (a_i \ln b - b_i \ln a)/(\ln b - \ln a)$ and $\beta_i(\boldsymbol{p}) = b_i/(\ln b - \ln a)$ for $a_i = \partial \ln a / \partial \ln p_i$ and $b_i = \partial \ln b / \partial \ln p_i$. The form in (48) gives the expenditure function as a utility-weighted geometric mean of the linear homogeneous functions $a(\boldsymbol{p})$ and $b(\boldsymbol{p})$ representing the expenditure functions of the very poor ($u = 0$) and the very rich ($u = 1$), respectively. Such preferences have been called the PIGLOG (price independent generalized logarithmic) class by Muellbauer (1975), (1976a), (1976b). A full system of demand equations within the class of Working-Leser form can be generated by a suitable choice of the functions $a(\boldsymbol{p})$ and $b(\boldsymbol{p})$. Functional forms of the log translog (See Pollak and Wales, 1992) and the AIDS model in (45) are members of the PIGLOG class with particular choices of the two homogeneous functions. For example, if

$$\ln a(\boldsymbol{p}) = a_0 + \sum_{k=1}^{n} \alpha_k \ln p_k + \frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} \gamma_{kl} \ln p_k \ln p_l,$$

and

$$\ln b(\boldsymbol{p}) = \ln a(\boldsymbol{p}) + \beta_0 \prod_{k=1}^{n} p_k^{\beta_k},$$

we obtain the AIDS model of Deaton and Muellbauer (1980) in (45).

The log translog (log TL) indirect utility function is given by

$$V(\boldsymbol{p}, x) = -\sum_{k=1}^{n} \alpha_k \ln (p_k/x) - \frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} \beta_{kl} \ln (p_k/x) \ln (p_l/x), \qquad (49)$$

for all $k, l = 1, ..., n$, where

$$\beta_{kl} = \beta_{lk}, \qquad \sum_{k=1}^{n} \sum_{l=1}^{n} \beta_{kl} = 0, \qquad \sum_{k=1}^{n} \alpha_k = 1,$$

The corresponding share equations are given by

$$w_i(\boldsymbol{p}, x) = \frac{\alpha_i + \sum_{k=1}^{n} \beta_{ik} \ln p_k - \ln x \sum_{k=1}^{n} \beta_{ik}}{1 + \sum_{k=1}^{n} \sum_{l=1}^{n} \beta_{kl} \ln p_k \ln p_l} \tag{50}$$

for $i = 1, ..., n$. Since the log TL share equations in (50) are linear in the logarithm of expenditure, the log TL belongs to the PIGLOG class.

2.2.4 Exact Aggregation

We begin our discussion of the rank of demand systems with the definition of exactly aggregable demand systems. A demand system is "exactly aggregable" if demands can be summed across consumers to yield closed form expressions for aggregate demand. Exactly aggregable demand systems are demand systems that are linear in functions of income, as follows:

$$w_i(\boldsymbol{p}, x) = \sum_{r=1}^{R} c_{ir}(\boldsymbol{p}) \varphi_r(x), \tag{51}$$

where $c_{ir}(\boldsymbol{p})$'s are the coefficients on $\varphi_r(x)$, which is a scalar valued function independent of $\boldsymbol{p}$, and R is a positive integer. Gorman (1981), extending earlier results by Muellbauer (1975, 1976a), proved in the context of exactly aggregable demand systems that integrability forces the matrix of Engel curve coefficients to have rank three or less. The rank of a matrix is defined as the maximum number of linearly independent columns. Other related exact aggregation theorems can be found in Banks, Blundell and Lewbel (1997).

2.2.5 The Rank of Demand Systems

Lewbel (1991) extended Gorman's rank idea to all demand systems, not just exactly aggregable demand systems by defining the rank of a demand system to be the dimension of the space spanned by its Engel curves, holding demographic or other non-income consumer characteristics fixed. He showed that demands that are not exactly aggregable can have rank higher than three and still be consistent with utility maximization.

Formally, the rank of any given demand system is the smallest value of R such that each w_i can be written as

$$w_i(\boldsymbol{p}, x) = \sum_{r=1}^{R} \phi_{ir}(\boldsymbol{p}) f_r(\boldsymbol{p}, x), \tag{52}$$

for some $R \leq n$, where for each $r = 1, ..., R$, ϕ_{ir} is a function of prices and f_r is a scalar valued function of prices and income. That is, the rank of the system is the number of linearly independent vectors of price functions. All demand systems have rank $R \leq n$, where n is the number of goods. Clearly, demands that are not exactly aggregable can have rank greater than three (i.e., $R > 3$). Equation (52) is a generalization of the concept of rank. That generalization, defined by Gorman (1981), only applies to exactly aggregable demands. Notice that φ_r in equation (51) is not a function of $\boldsymbol{p}$.

Hence, any demand system has rank R, if there exist R goods such that the Engel curve of any good equals a weighted average of the Engel curves of those R goods. The rank of an integrable demand system determines the number of price functions on which the indirect utility function and the cost or expenditure function depend. Lewbel calls the form in (52) Gorman

Engel curves.

We can see the implication that higher order terms in theoretically plausible polynomial demand systems must satisfy severe restrictions by considering the implications of the Slutsky symmetry conditions for third degree polynomial demand systems:

$$w_i = c_{i0}(\boldsymbol{p}) + c_{i1}(\boldsymbol{p}) \ln x + c_{i2}(\boldsymbol{p}) \ln^2 x + c_{i3}(\boldsymbol{p}) \ln^3 x.$$

It is straightforward to calculate the Slutsky terms corresponding to this demand system as

$$\frac{\partial w_i}{\partial \ln p_j} + w_j \frac{\partial w_i}{\partial \ln x},$$

in share term for $i, j = 1, ..., n$.[20] These Slutsky terms are polynomials of degree five in $\ln x$ and, because symmetry holds as an identity in the price-income space, symmetry must hold for the coefficients of like powers of $\ln x$. To illustrate our point, it suffices to calculate only that portion of the Slutsky term of degree four in $\ln x$. Note that the fifth polynomial term in $\ln x$ is given by $3c^{i3}c^{j3}$, so symmetry of this term imposes no restrictions on the coefficients of the demand system. The term of degree of four is given by

$$2c_{i2}c_{j3} + 3c_{i3}c_{j2},$$

or equivalently,

$$(2c_{i2}c_{j3} + 2c_{i3}c_{j2}) + c_{i3}c_{j2}.$$

[20] $\frac{\partial q_i}{\partial p_j} + q_j \frac{\partial q_i}{\partial x} = (x/p_i p_j)\left(\frac{\partial w_i}{\partial \ln p_j} + w_j \frac{\partial w_i}{\partial \ln x} + w_i w_j - \delta_{ij} w_i\right)$ where $\delta_{ij} = 1$ when $i = j$ and 0 otherwise.

Because the term in brackets is symmetric, symmetry of this equation implies

$$\frac{c_{i3}}{c_{i2}} = \frac{c_{j3}}{c_{j2}}.$$

Hence, the ratio of the coefficient of the cubic terms to the coefficients of the quadratic terms will be constant across equations since this condition is applied to any pair of equations. Hausman, Newly and Powell (1995) test whether or not a rank four specification gives any additional information by estimating,

$$w_i = \beta_0 + \beta_1 \ln x + \beta_2 \ln^2 x + \beta_3 \ln^3 x + \varepsilon_i.$$

If the demand system is rank three, the ratio β_2/β_3 (Gorman statistics) should be constant for any equations. They in fact find this ratio is almost perfectly constant despite considerable variation in the estimates of β_2's and the β_3's.

2.2.5.1 Demand Systems Proportional to Expenditure

Homothetic demand systems, with Engel curves being rays from the origin, have rank one. Rank one demand systems, such as the Cobb-Douglas, CES, and homothetic translog, exhibit expenditure proportionality (so that the budget share of every good is independent of total expenditure). This contradicts Engel's law, according to which the budget share of food is smaller for rich than for poor households.

Rank one demand systems can be written as

$$q_i(\boldsymbol{p}, x) = b_i(\boldsymbol{p})x$$

and are homothetic. For example, the demand system of the Cobb-Douglas utility function (3) is given by (6), that of the CES utility function (21) is

$$q_i(\boldsymbol{p}, x) = \frac{p_i^{1/(\gamma-1)}}{\sum_{k=1}^{n} p_k^{\gamma/(\gamma-1)}} x,$$

and that of the homothetic translog can be reduced from log TL in equation (50) after imposing the additional restriction $\sum_{k=1}^{n} \beta_{ik} = 0$ for all i.

Clearly, expenditure proportionality implies marginal budget shares that are constant and in fact equal to the average budget shares. Because of this, the assumption of expenditure proportionality has little relevance in empirical demand analysis.

2.2.5.2 Demand Systems Linear in Expenditure

A demand system that is linear in expenditure is of the form

$$q_i(\boldsymbol{p}, x) = c_i(\boldsymbol{p}) + b_i(\boldsymbol{p})x.$$

If $c_i(\boldsymbol{p}) = 0$ (for all i) then demands are homothetic. Gorman (1961) showed that any demand system is consistent with utility maximization and linear in expenditure must be of the form

$$\begin{aligned} q_i(\boldsymbol{p}, x) &= f_i(\boldsymbol{p}) - \frac{g_i(\boldsymbol{p})}{g(\boldsymbol{p})} f(\boldsymbol{p}) + \frac{g_i(\boldsymbol{p})}{g(\boldsymbol{p})} x \\ &= f_i(\boldsymbol{p}) + \frac{g_i(\boldsymbol{p})}{g(\boldsymbol{p})} \left[x - f(\boldsymbol{p}) \right], \end{aligned} \tag{53}$$

where $g(\boldsymbol{p})$ and $f(\boldsymbol{p})$ are functions homogeneous of degree one, and $g_i(\boldsymbol{p})$ and $f_i(\boldsymbol{p})$ denote the partial derivative of $g(\boldsymbol{p})$ and $f(\boldsymbol{p})$ with respect to the ith price. Such demand systems are generated by an indirect utility function of the "Gorman polar form,"

$$V(\boldsymbol{p}, x) = \frac{x - f(\boldsymbol{p})}{g(\boldsymbol{p})}. \tag{54}$$

To see this, apply Roy's identity (9) to (54) to get (53).

An example of a demand system linear in expenditure is the "linear expenditure system,"

$$q_i(\boldsymbol{p}, x) = b_i - \frac{a_i}{p_i}\sum_{k=1}^{n} p_k b_k + \frac{a_i}{p_i}x,$$

generated by the (Stone-Geary) utility function,

$$U(\boldsymbol{q}) = \sum_{k=1}^{n} a_k \ln(q_k - b_k), \qquad a_i > 0,\ q_i - b_i > 0,\ \sum_{k=1}^{n} a_k = 1$$

which is homothetic relative to the point $\boldsymbol{b} = [b_1, ..., b_n]$ as origin, or equivalently by an indirect utility function of the Gorman polar form, (54) with $f(\boldsymbol{p}) = \sum_{k=1}^{n} p_k b_k$ and $g(\boldsymbol{p}) = \prod_{k=1}^{n} p_k^{a_k}$, with $\sum_{k=1}^{n} a_k = 1$, so that $f_i(\boldsymbol{p}) = b_i$ and $g_i(\boldsymbol{p})/g(\boldsymbol{p}) = a_i/p_i$.

Demand systems linear in expenditure are rank two and have linear Engel curves, but not necessarily through the origin. Linearity in expenditure implies marginal budget shares that are independent of the level of expenditure, suggesting that poor and rich households spend the same fraction of an extra dollar on each good. This hypothesis, as well as the hypothesis of expenditure proportionality, are too restrictive for the analysis of household budget data.

2.2.5.3 Demand Systems Linear in the Logarithm of Expenditure

Muellbauer (1975) has studied "two-term" demand systems of the general form

$$q_i(\boldsymbol{p}, x) = c_i(\boldsymbol{p})x + b_i(\boldsymbol{p})f(x), \tag{55}$$

for any function $f(x)$. Homothetic demand is obtained, if $f(x) = 0$. He shows that if $f(x) \neq 0$, then $f(x)$ must be either equal to x^k with $k \neq 1$ (the price independent generalized linearity (PIGL) class) or equal to $x \ln x$ (the price independent generalized logarithmic (PIGLOG) class).

Hence, the PIGLOG class of demand systems is linear in the logarithm of total expenditure and has the form

$$q_i(\boldsymbol{p}, x) = c_i(\boldsymbol{p})x + b_i(\boldsymbol{p})x \ln x,$$

with expenditure entering linearly and as a logarithmic function of x. Muellbauer (1975b) has shown that theoretically plausible demand systems of the PIGLOG form must be written as

$$q_i(\boldsymbol{p}, x) = \frac{g_i(\boldsymbol{p})}{g(\boldsymbol{p})}x - \frac{G_i(\boldsymbol{p})}{G(\boldsymbol{p})}\left[\ln x - \ln g(\boldsymbol{p})\right]x, \tag{56}$$

where $G(\boldsymbol{p})$ is homogeneous of degree zero, $G(\boldsymbol{p}) = G(\lambda\boldsymbol{p})$, and $g(\lambda\boldsymbol{p})$ is homogeneous of degree one, $g(\lambda\boldsymbol{p}) = \lambda g(\boldsymbol{p})$. The indirect utility function associated with (56) is

$$V(\boldsymbol{p}, x) = G(\boldsymbol{p}) \left[\ln x - \ln g(\boldsymbol{p}) \right]. \tag{57}$$

To see this, apply Roy's identity (9) to (57) to get (56).

Examples of PIGLOG demand systems are the log-translog (log TL), a special case of the basic translog, and the AIDS model. It is to be noted, however, that PIGLOG specifications have a rank two, and thus have limited flexibility in modeling the curvature of Engel curves.

2.2.5.4 Demand Systems Quadratic in Expenditure

Lewbel (1987a) has studied "three-term" demand systems of the following form:

$$q_i(\boldsymbol{p}, x) = c_i(\boldsymbol{p}) + b_i(\boldsymbol{p})x + a_i(\boldsymbol{p})f(x). \tag{58}$$

Equation (58) is a special case of Gorman's (1981) equation (51), with r ranging from 1 to 3 and $\varphi_1(x) = 1$, $\varphi_2(x) = x$, and $\varphi_3(x) = f(x)$. Gorman's (1981) main result, that the matrix of Engel curve coefficients cannot have rank higher than three, is true in this case, since that matrix, $[c(\boldsymbol{p})\ b(\boldsymbol{p})\ a(\boldsymbol{p})]$, only has three columns. Lewbel (1987a) showed that in equation (58), $f(x)$ must be either 0, x^k, $x \ln x$, or $\ln x$, and that the only $f(x)$ that yields full rank-three demand systems is x^2. Hence, one way to relax the assumption that demand systems are linear in expenditure is to specify demand systems that are quadratic in expenditure, as follows:

$$q_i(\boldsymbol{p}, x) = c_i(\boldsymbol{p}) + b_i(\boldsymbol{p})x + a_i(\boldsymbol{p})x^2. \tag{59}$$

Ryan and Wales (1999), following Howe, Pollak and Wales (1979) and van Daal and Merkies (1989), argue that for a quadratic demand system to be theoretically plausible, the demand functions must be of the form

$$\begin{aligned} q_i(\boldsymbol{p}, x) \quad &= \quad \frac{1}{g(\boldsymbol{p})^2}\left(r_i(\boldsymbol{p}) - \frac{g_i(\boldsymbol{p})}{g(\boldsymbol{p})r(\boldsymbol{p})}\right)[x - f(\boldsymbol{p})]^2 \\ &+\frac{g_i(\boldsymbol{p})}{g(\boldsymbol{p})}[x - f(\boldsymbol{p})] + f_i(\boldsymbol{p}) \\ &+\chi\left(\frac{r(\boldsymbol{p})}{g(\boldsymbol{p})}\right)\left(r_i(\boldsymbol{p}) - \frac{g_i(\boldsymbol{p})}{g(\boldsymbol{p})}r(\boldsymbol{p})\right), \end{aligned} \tag{60}$$

where there are no restrictions on the function $\chi(\cdot)$, functions $f(\boldsymbol{p})$, $g(\boldsymbol{p})$, and $r(\boldsymbol{p})$ are restricted to be homogeneous of degree one in prices, and $f_i(\boldsymbol{p})$, $g_i(\boldsymbol{p})$, and $r_i(\boldsymbol{p})$ are the first partial derivatives of $f(\boldsymbol{p})$, $g(\boldsymbol{p})$, and $r(\boldsymbol{p})$ with respect to p_i. The demand function (60) can be simplified by assuming $\chi(\cdot) = 0$ and defining $r(\boldsymbol{p})$ to be the product of $g(\boldsymbol{p})$ and a function $h(\boldsymbol{p})$, that is homogeneous of degree zero in prices, so that the coefficient of the quadratic term in (60) becomes $h_i(\boldsymbol{p})/g(\boldsymbol{p})$. In that case (60) reduces to

$$q_i(\boldsymbol{p}, x) = \frac{h_i(\boldsymbol{p})}{g(\boldsymbol{p})}[x - f(\boldsymbol{p})]^2 + \frac{g_i(\boldsymbol{p})}{g(\boldsymbol{p})}[x - f(\boldsymbol{p})] + f_i(\boldsymbol{p}), \tag{61}$$

and its corresponding indirect utility function is

$$V(\boldsymbol{p}, x) = -\frac{g(\boldsymbol{p})}{x - f(\boldsymbol{p})} - h(\boldsymbol{p}). \tag{62}$$

To see this, apply Roy's identity (9) to (62) to get (61).

Equation (62) is the general form of the indirect utility function that can generate quadratic Engel curves (that is, rank-three demand systems).

The difference between the Gorman polar form indirect utility function (54) and the more general indirect utility function (62) is that the latter adds a term, $h(\boldsymbol{p})$, that is homogeneous of degree zero in prices, to the Gorman polar form indirect utility function (54).

The first functional form proposed along these lines is the quadratic AIDS of Bank, Blundell and Lewbel (1997) (known as QUAIDS), which we will use for the empirical study in chapter 4. The QUAIDS is an extension of simple AIDS, having expenditure shares linear in log income and in another smooth function of income. See Banks, Blundell and Lewbel (1997) for more details.

Following Banks, Blundell and Lewbel (1997), Ryan and Wales (1999) modified the translog, GL, and NQ demand systems and introduced three new demand systems called the translog-quadratic expenditure system, the GL-quadratic expenditure system, and the NQ-quadratic expenditure system.

To demonstrate, we consider the NQ expenditure function (NQEF), introduced by Diewert and Wales (1988b, 1993):

$$E(u, \boldsymbol{p}) = \boldsymbol{a}'\boldsymbol{p} + \left(\boldsymbol{b}'\boldsymbol{p} + \frac{1}{2}\frac{\boldsymbol{p}'\mathbf{B}\boldsymbol{p}}{\boldsymbol{\alpha}'\boldsymbol{p}} \right) u, \tag{63}$$

where the parameters of the model consist of $\boldsymbol{a}' = [a_1, ..., a_n]$, $\boldsymbol{b}' = [b_1, ..., b_n]$, and the elements of the $n \times n$ symmetric $\boldsymbol{B} = [B_{kl}]$ matrix. The nonnegative vector of predetermined parameters $\boldsymbol{\alpha}' = [\alpha_1, ..., \alpha_n]$ is assumed to satisfy

$$\boldsymbol{\alpha}'\boldsymbol{p}^* = 1, \qquad \alpha_i \geq 0 \quad \text{for} \quad i = 1, ..., n, \tag{64}$$

where p_i^* is the ith element of the reference vector. Moreover, the following restrictions are also imposed:

$$\sum_{k=1}^{n} a_k p_k^* = 0 \tag{65}$$

$$\sum_{k=1}^{n} B_{ik} p_k^* = 0, \qquad i = 1, ..., n. \tag{66}$$

Hence, there are $n(n+5)/2$ parameters in (63), but the imposition of the above restrictions reduces the number of parameters to $(n^2 + 3n - 2)/2$. The NQ expenditure function defined by (63)-(66) is a Gorman polar form, and the preferences that are dual to it are quasihomothetic. The quasi-homotheticity of the underlying preferences entails linear Engel curves which are not necessarily through the origin, but have restrictive implication in preferences. The income elasticity of goods with inelastic demand is forced to rise as the income level increases.

The indirect utility function corresponding to (63)-(65) is

$$V(\boldsymbol{p}, x) = \frac{x - \boldsymbol{a}'\boldsymbol{p}}{\boldsymbol{b}'\boldsymbol{p} + (1/2)\, \boldsymbol{p}'\mathbf{B}\boldsymbol{p}/\boldsymbol{a}'\boldsymbol{p}}, \tag{67}$$

and the consumer's system of Marshallian demand functions is obtained by applying the modified Roy's identity to (67):

$$\begin{aligned} w_i &= \frac{p_i a_i}{x} + \frac{1}{x} \times (x - \textstyle\sum_k a_k p_k) \\ &\quad \times \left(b_i p_i + \frac{p_i \sum_k B_{ik} p_k}{\sum_k \alpha_k p_k} - \frac{p_i \alpha_i}{2} \frac{\sum_k \sum_l B_{kl} p_k p_l}{(\sum_k \alpha_k p_k)^2} \right) \\ &\quad \div \left(\textstyle\sum_k b_k p_k + \frac{1}{2} \frac{\sum_k \sum_l B_{kl} p_k p_l}{\sum_k \alpha_k p_k} \right) \end{aligned} \tag{68}$$

for $i = 1, ..., n$.

Since the share equations in (68) are homogeneous of degree zero in the parameters, Diewert and Wales (1988b) impose the normalization $\sum_{k=1}^{n} b_k = 1$. Also, regarding the curvature properties of the NQ expenditure function, it is locally flexible in the class of expenditure function local money-metric scaling, and it retains this flexibility when concavity needs to be imposed. See Diewert and Wales (1988b, 1993) for more details.

On the other hand, following Diewert and Wales (1988b), the NQ reciprocal indirect utility function (NQRIUF) is defined as

$$V(\boldsymbol{\nu}) = b_0 + \boldsymbol{b}'\boldsymbol{\nu} + \frac{1}{2}\frac{\boldsymbol{\nu}'\mathbf{B}\boldsymbol{\nu}}{\boldsymbol{\alpha}'\boldsymbol{\nu}} + \boldsymbol{a}'\ln\boldsymbol{\nu}, \tag{69}$$

where b_0, $\boldsymbol{b} = [b_1, ..., b_n]'$, $\boldsymbol{a} = [a_1, ..., a_n]'$, and the elements of the $n \times n$ symmetric $\mathbf{B} = [B_{kl}]$ matrix are the unknown parameters to be estimated. It is important to note that the quadratic term in equation (69) is normalized by dividing through by a linear function, $\boldsymbol{\alpha}'\boldsymbol{\nu}$, and that the nonnegative vector of parameters $\boldsymbol{\alpha} = [\alpha_1, ..., \alpha_n]'$ is assumed to be predetermined and can be seen as weights on $\boldsymbol{\nu}$. Weighting can be based upon the index number theory. In principal, $\boldsymbol{\alpha}$ is estimable, but normally viewed as best selected in advance. Diewert and Wales (1988b) pick a reference (or base period) vector of expenditure normalized prices, $\boldsymbol{\nu}^* = \mathbf{1}$, and assume $\boldsymbol{\alpha}$ satisfies

$$\boldsymbol{\alpha}'\boldsymbol{\nu}^* = 1$$

where each of the elements of $\boldsymbol{\alpha}$ is nonnegative. Moreover, they assume that

$\mathbf{B}$ satisfies the following n restrictions:

$$\sum_{k=1}^{n} B_{ik}\nu_k^* = 0, \qquad i = 1, ..., n. \tag{70}$$

Using the modified version of Roy's identity in equation (11), the NQ demand system for $i = 1, ..., n$ is derived as

$$w_i = \frac{\nu_i b_i + (\boldsymbol{\alpha}'\boldsymbol{\nu})^{-1}\left(\sum_{k=1}^{n} B_{ik}\nu_k\right) - \frac{1}{2}\alpha_i(\boldsymbol{\alpha}'\boldsymbol{\nu})^{-2}\boldsymbol{\nu}'\mathbf{B}\boldsymbol{\nu} + a_i}{\boldsymbol{b}'\boldsymbol{\nu} + \frac{1}{2}(\boldsymbol{\alpha}'\boldsymbol{\nu})^{-1}\boldsymbol{\nu}'\mathbf{B}\boldsymbol{\nu} + \sum_{k=1}^{n} a_k}, \tag{71}$$

or

$$w_i = \frac{p_i b_i + p_i \sum_k B_{ik}p_k / \sum_k \alpha_k p_k - (p_i\alpha_i/2)\sum_k\sum_l B_{kl}p_k p_l / \left(\sum_k \alpha_k p_k\right)^2 + a_i x}{\sum_k b_k p_k + 1/2\sum_k\sum_l B_{kl}p_k p_l / \sum_k \alpha_k p_k + \sum_k a_k} \tag{72}$$

Finally, as the share equations are homogeneous of degree zero in the parameters, Diewert and Wales (1988b) suggest that we impose the normalization,

$$\textstyle\sum_{k=1}^{n} b_k = 1. \tag{73}$$

Hence, there are $n(n+5)/2$ parameters in equation (71) or (72), but the imposition of the $(n-1)$ restrictions in (70) and (73) reduces the number of parameters to be estimated to $(n^2 + 3n - 2)/2$.

In developing the NQ-QES, Ryan and Wales (1999) choose the $f(\boldsymbol{p})$,

$g(\boldsymbol{p})$, and $h(\boldsymbol{p})$ functions in (62) as follows:

$$\begin{aligned} f(\boldsymbol{p}) &= \boldsymbol{d}'\boldsymbol{p} && (74)\\ g(\boldsymbol{p}) &= \boldsymbol{b}'\boldsymbol{p} + \frac{1}{2}\left(\frac{\boldsymbol{p}'\mathbf{B}\boldsymbol{p}}{\boldsymbol{\alpha}'\boldsymbol{p}}\right) && (75)\\ h(\boldsymbol{p}) &= \boldsymbol{a}'\ln\boldsymbol{p}, \qquad \boldsymbol{\iota}'\boldsymbol{a} = 0. && (76) \end{aligned}$$

Substituting (74)-(76) in (62) and applying Roy's identity (9) to (62) yields the demand system for $i = 1, ..., n$:

$$\begin{aligned} q_i(\boldsymbol{p}, x) = {} & \frac{a_i}{p_i g(\boldsymbol{p})}\left(x - \textstyle\sum_{k=1}^n p_k d_k\right)^2 \\ & + \left(\frac{1}{g(\boldsymbol{p})}\right)\left(\frac{b_i + \sum_{k=1}^n B_{ik}p_k}{\sum_{k=1}^n \alpha_k p_k} - \frac{1}{2}\frac{\alpha_i \sum_{k=1}^n \sum_{l=1}^n B_{kl}p_k p_l}{\left(\sum_{l=1}^n \alpha_k p_k\right)^2}\right) \\ & \times \left(x - \textstyle\sum_{k=1}^n p_k d_k\right) + d_i, \end{aligned}$$

where a_k, b_k, d_k, and B_{kl} are unknown parameters, and the $\alpha_k > 0$ are predetermined parameters, $k, l = 1, ..., n$. The $\mathbf{B} = [B_{kl}]$ matrix also satisfies the following two restrictions, as in the common NQ model:

$$\begin{aligned} B_{kl} &= B_{lk} \quad \text{for all } k, l, \\ \mathbf{B}\boldsymbol{p}^* &= \mathbf{0} \quad \text{for some } \boldsymbol{p}^* > 0. \end{aligned}$$

The NQEF is nested in this model, and is obtained when all of a_i's are zero, which implies that $h(\boldsymbol{p}) = \boldsymbol{\alpha}'\ln\boldsymbol{p} = 0$ for all prices. This gives demands that are linear in total expenditure, and this proposition may be tested empirically in the usual procedure. A slightly restricted version of the NQRIUF is also nested in this model by noting that (71) and (72) can be rewritten using (74) to (76) as

$$q_i\left(\boldsymbol{p}, x\right) = \frac{h_i}{g} x^2 + \frac{g_i}{g} x.$$

Equation (61) becomes this form when $f(\boldsymbol{p}) = 0$.

The development of the GL-QES and TL-QES follows a similar pattern. See Ryan and Wales (1999) for more details.

The QUAIDS, translog-quadratic expenditure system, GL-quadratic expenditure system, and NQ-quadratic expenditure system are locally flexible in the Diewert sense and also are rank-three demand systems, thereby allowing more flexibility in modeling income distribution than the AIDS, translog, GL, and NQ models.

2.2.5.5 Demand Systems Quadratic in the Logarithm of Expenditure

The indirect utility function of the models that are quadratic in the logarithm of expenditure is

$$\ln V(\boldsymbol{p}, x) = \left[\left(\frac{\ln x - \ln a(\boldsymbol{p})}{b(\boldsymbol{p})} \right)^{-1} + c(\boldsymbol{p}) \right]^{-1} \tag{77}$$

where a, b are homogeneous of degree zero in $\boldsymbol{p}$, and c is homogeneous of degree one. This class of demand systems is predominantly known by the quadratic AIDS of Bank, Blundell and Lewbel (1997) with specifications of a, b, and c:

$$c\left(\boldsymbol{p}\right) = \sum_{k=1}^{n} c_k \ln p_k \qquad \text{where } \sum_{k=1}^{n} c_k = 0$$

$$\ln a(\boldsymbol{p}) = \alpha_0 + \sum_{k=1}^{n} \alpha_k \ln p_k + \frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} \gamma_{kl} \ln p_k \ln p_l$$

$$\ln b(\boldsymbol{p}) = \sum_{k=1}^{n} \beta_k \ln p_k.$$

By Roy's identity, the budget shares of the QUAIDS for $i = 1, ..., n$ are given by

$$w_i = \alpha_i + \sum_{k=1}^{n} \gamma_{ik} \ln p_k + \beta_i \ln [x/a(\boldsymbol{p})] + \frac{c_i}{b(\boldsymbol{p})} (\ln [x/a(\boldsymbol{p})])^2 . \tag{78}$$

The QUAIDS model has the income flexibility with rank three. The AIDS model is nested within it as a special case when $c(\boldsymbol{p}) = 0$.

2.2.5.6 Demand System Linear in Trigonometric Functions of Expenditure

Gorman (1981) first conjectured that utility-derived trigonometric demand systems would exist, without giving any examples of them. Lewbel (1988) proved the existence of the trigonometric class of demand systems, and characterized its general form alone with the underlying indirect utility function. This class of models has the maximum rank of an exactly aggregable demand system and offers the attractive features mentioned in the previous section. It is important to note that the Fourier flexible demand systems do not belong to the trigonometric class in terms of forms of Engel curves.

The trigonometric demand systems are derived from the indirect utility function of the form:

$$V(\boldsymbol{p},x)=b(\boldsymbol{p})+\frac{c(\boldsymbol{p})\cos\{\tau\log[a(\boldsymbol{p})/x]\}}{1+\sin\{\tau\log[a(\boldsymbol{p})/x]\}}\qquad \tau\neq 0,$$

where $a(\boldsymbol{p})$ is homogeneous of degree one, $b(\boldsymbol{p})$ and $c(\boldsymbol{p})$ are homogeneous of degree zero in $\boldsymbol{p}$, and τ is a non-zero parameter whose size should be reasonable enough to prevent many oscillations over the range of the data. For $i=1,...,n$ the straightforward application of Roy's identity yields

$$\begin{aligned} q_i &= \left(\frac{a_i}{a}-\frac{b_i}{\tau c}\right)x+\left[\frac{b_i}{\tau c}\cos(\tau\ln a)-\frac{c_i}{\tau c}\sin(\tau\ln a)\right]x\sin(\tau\ln x) \\ &\quad -\left[\frac{b_i}{\tau c}\sin(\tau\ln a)+\frac{c_i}{\tau c}\cos(\tau\ln a)\right]x\cos(\tau\ln x), \end{aligned}$$

and in share form,

$$w_i=\frac{p_i a_i}{a}-\frac{p_i}{\tau c}\left\{b_i+b_i\sin(\tau\ln[a/x])+c_i\cos(\tau\ln[a/x])\right\}. \tag{79}$$

To derive the estimable trigonometric demand functions, Matsuda (2006) provides the specification of these price functions such that:

$$a(\boldsymbol{p}) = \prod_{k=1}^{n} p_k^{\alpha_k}, \tag{80}$$

$$b(\boldsymbol{p}) = \sum_{k=1}^{n}\beta_k\ln p_k, \tag{81}$$

$$c(\boldsymbol{p}) = \sum_{k=1}^{n}\alpha_k+\sum_{k=1}^{n}\sum_{l=1}^{n}\gamma_{kl}\ln p_k\ln p_l, \tag{82}$$

with $\sum_{k=1}^{n}\alpha_k=1$, $\sum_{k=1}^{n}\beta_k=1$, $\sum_{k=1}^{n}\gamma_{kl}=0$ for $k=1,...,n$ to fulfil adding-up and homogeneity. Slutsky symmetry is guaranteed by $\gamma_{kl} = \gamma_{lk}$ for $k, l = 1, ..., n$.

The share equations in (79) is rewritten using (80) to (82) as[21]

$$w_i = \alpha_i - \frac{\beta_i + f_i(\boldsymbol{p}) \sin(\tau \log x) + g_i(\boldsymbol{p}) \cos(\tau \ln x)}{\tau c(\boldsymbol{p})}, \tag{83}$$

where $f_i(\boldsymbol{p})$ and $g_i(\boldsymbol{p})$ are functions specified as

$$\begin{aligned} f_i(\boldsymbol{p}) &= 2\left(\sum_{l=1}^{n} \gamma_{il} \ln p_k\right) \sin\left[\tau \ln a(\boldsymbol{p})\right] - \beta_i \cos\left[\tau \ln a(\boldsymbol{p})\right], \ \ i = 1, ...n, \\ g_i(\boldsymbol{p}) &= 2\left(\sum_{l=1}^{n} \gamma_{il} \ln p_k\right) \cos\left[\tau \ln a(\boldsymbol{p})\right] + \beta_i \sin\left[\tau \ln a(\boldsymbol{p})\right], \ \ i = 1, ...n. \end{aligned}$$

Matsuda (2006) called the new model represented by (83) the TDS, which stands for the trigonometric demand system. Linearity in trigonometric functions of share equations (83) allows for the aggregation of consumers without involving linear Engel curves. Rank of the TDS is three, whereas that of the AIDS is two. Although a flexible rank-three demand system, the TDS has $n-1$ fewer free parameters than known rank-three models such as the QUAIDS.

2.2.5.7 Fractional demand systems

Lewbel (1987b) has studied demand systems of the "fractional" form

[21] The following results from the angel difference fourmulas were used to derive the equation (83):

$$\sin\{\tau \log [a(\boldsymbol{p})/x]\} = \sin\{\tau \log [a(\boldsymbol{p})/x]\} \sin\{\tau \log [a(\boldsymbol{p})/x]\} - \sin\{\tau \log [a(\boldsymbol{p})/x]\} \sin\{\tau \log [a(\boldsymbol{p})/x]\},$$

$$\cos\{\tau \log [a(\boldsymbol{p})/x]\} = \cos\{\tau \log [a(\boldsymbol{p})/x]\} \cos\{\tau \log [a(\boldsymbol{p})/x]\} + \cos\{\tau \log [a(\boldsymbol{p})/x]\} \cos\{\tau \log [a(\boldsymbol{p})/x]\}.$$

$$q_i(\boldsymbol{p}, x) = \frac{c_i(\boldsymbol{p})f(x) + b_i(\boldsymbol{p})g(x)}{c(\boldsymbol{p})F(x) + b(\boldsymbol{p})G(x)}, \tag{84}$$

where $f(x)$, $g(x)$, $F(x)$, and $G(x)$ are differentiable functions of expenditure and $c_i(\boldsymbol{p})$, $b_i(\boldsymbol{p})$, $c(\boldsymbol{p})$, and $b(\boldsymbol{p})$ are differentiable functions of prices only. He shows that the budget shares of fractional demand systems can always be written as

$$w_i(\boldsymbol{p}, x) = \frac{c_i(\boldsymbol{p}) + b_i(\boldsymbol{p})f(x)}{1 + b(\boldsymbol{p})f(x)}, \tag{85}$$

where $f(x)$ must be either 0, $\ln x$, y^k, or $\tan(k \ln x)$ for $k \neq 0$. As can be seen, fractional demands are proportional to two-term demands. Moreover, if $f(x) = 0$ in equation (85), homothetic demands obtain and if $b(x) = 0$, Gorman polar form demands obtain, either PIGL demands or PIGLOG demands, corresponding to $f(x) = x^k$ or $f(x) = \ln x$, respectively. For $f(x) = x^2$, equation (85) reduces to what Lewbel (1987b) refers to as "EXP" demands; the minflex Laurent demand system is a member of the EXP class of demand systems.

As Lewbel (1987b) argues, fractional demand systems provide a parsimonious way of increasing the range of income response patterns. In fact, an advantage of fractional demands (84) over three-term demands (59) is that they require the estimation of only one more function of prices, $b(\boldsymbol{p})$, than two-term demands (55), whereas three-term demands require the estimation of one more function of income, $f(x)$, and $n - 1$ functions of prices, $a_i(\boldsymbol{p})$, than two-term demands.

For analysis involving substantial variation in income levels across individuals, increased flexibility in global Engel curve shapes is required and fractional demand systems in the form of equation (85) are likely to be superior to two-term demand systems (such as homothetic, PIGL, and PIGLOG systems) and three-term demand systems (such as the quadratic AIDS, GL-QES, TL-QES, and NQ-QES). Moreover, as already noted, fractional demand systems, like the minflex Laurent, have larger regular regions than two- and three-term demand systems.

2.2.5.8 Deflated Income Demand Systems

Let $a(\boldsymbol{p})$ be some homogeneous of degree one function of prices, and define $x/a(\boldsymbol{p})$ to be "deflated" income. A natural generalization of Gorman Engel curve systems in (52) is

$$w_i = \sum_{s \in S} c^{is}(\boldsymbol{p}) F^s \left[x/a(\boldsymbol{p}) \right], \qquad i = 1, ...n, \tag{86}$$

for some function $c^{is}(\boldsymbol{p})$ of prices. Equation (86) describes all demands that are linear in deflated, instead of nominal, income. Lewbel (1989, 1990) argues that the deflated income demand systems have two advantages over Gorman Engel curve systems in (52). First, they can achieve higher rank, and second, they permit at least one of the income functions to be completely unrestricted (except for smoothness).

Lewbel (1989) proposed this class of models to relax the constraints on Engel curves that are imposed by homogeneity in exactly aggregable models, and he showed that any Gorman Engel curve systems in (52) have representation as deflated income demand system. He gives a proof that the maximum possible rank of a rational deflated income demand system is not three, but four.

Lewbel (2003) considered a nearly log polynomial rational rank four demand system,

$$V(\boldsymbol{p}, x) = \left[\left(\frac{\ln [x - d(\boldsymbol{p})] - a(\boldsymbol{p})}{b(\boldsymbol{p})} \right)^{-1} + c(\boldsymbol{p}) \right]^{-1}, \tag{87}$$

where a, b, c and d are functions of prices. Homogeneity requires that c and d be homogenous of degree zero and that b and a be homogeneous of degree one in $\boldsymbol{p}$. Application of Roy's identity to this indirect utility function yields demands of the form

$$w = \frac{A_4(\boldsymbol{p})}{x} + \left(1 - \frac{d(\boldsymbol{p})}{x}\right) \sum_{r=1}^{3} A_r(\boldsymbol{p}) \left(\ln [x - d(\boldsymbol{p})]\right)^{r-1}, \tag{88}$$

where

$$\begin{aligned}
A_1(\boldsymbol{p}) &= \frac{\partial a(\boldsymbol{p})}{\partial \ln \boldsymbol{p}} - \frac{\partial b(\boldsymbol{p})}{\partial \ln \boldsymbol{p}} \frac{a(\boldsymbol{p})}{b(\boldsymbol{p})} + \frac{\partial c(\boldsymbol{p})}{\partial \ln \boldsymbol{p}} \frac{a(\boldsymbol{p})^2}{b(\boldsymbol{p})}, \\
A_2(\boldsymbol{p}) &= \frac{\partial b(\boldsymbol{p})}{\partial \ln \boldsymbol{p}} \frac{1}{b(\boldsymbol{p})} - 2 \frac{\partial c(\boldsymbol{p})}{\partial \ln \boldsymbol{p}} \frac{a(\boldsymbol{p})}{b(\boldsymbol{p})}, \\
A_3(\boldsymbol{p}) &= \frac{\partial c(\boldsymbol{p})}{\partial \ln \boldsymbol{p}} \frac{1}{b(\boldsymbol{p})}, \\
A_4(\boldsymbol{p}) &= \frac{\partial d(\boldsymbol{p})}{\partial \ln \boldsymbol{p}}.
\end{aligned}$$

This is a deflated income demand system and its rank is four, provided that no one of the functions a, b, c, or d can be written as a function of the other three.

If $d(\boldsymbol{p}) = 0$ for all $\boldsymbol{p}$, then $A_4(\boldsymbol{p})$ also equals zero, and equation (88) reduces to the quadratic logarithmic model, which is equation (78). Moreover, applying results in Banks, Blundell, and Lewbel (1997), all rank three quadratic logarithmic demands have utility functions given by equation (77) with $d(\boldsymbol{p}) = 0$, so (87) nests all possible rank three quadratic logarithmic demand systems.

To provide an explicit example of a utility derived rank four system, consider the following specification of a, b, c, and d:

$$\begin{aligned}
\ln d(\boldsymbol{p}) &= \boldsymbol{\delta}' \ln \boldsymbol{p} \\
a(\boldsymbol{p}) &= \alpha_0 + \boldsymbol{\alpha}' \ln \boldsymbol{p} + \ln \boldsymbol{p}' \boldsymbol{\Gamma} \ln \boldsymbol{p} \\
\ln b(\boldsymbol{p}) &= \boldsymbol{\beta}' \ln \boldsymbol{p} \\
c(\boldsymbol{p}) &= \boldsymbol{\lambda}' \ln \boldsymbol{p}
\end{aligned}$$

where $\boldsymbol{\alpha}'\mathbf{1} = 1$, $\boldsymbol{\beta}'\mathbf{1} = 0$, $\boldsymbol{\Gamma}\mathbf{1} = \mathbf{0}$, $\boldsymbol{\delta}'\mathbf{1} = 0$, and $\boldsymbol{\lambda}'\mathbf{1} = 0$. This model has budget shares given by equation (88) with

$$\begin{aligned}
A_1(\boldsymbol{p}) &= \boldsymbol{\alpha} + \boldsymbol{\Gamma} \ln \boldsymbol{p} - \frac{a(\boldsymbol{p})}{b(\boldsymbol{p})} \boldsymbol{\beta} + \frac{a(\boldsymbol{p})^2}{b(\boldsymbol{p})} \boldsymbol{\lambda}, \\
A_2(\boldsymbol{p}) &= \frac{1}{b(\boldsymbol{p})} \boldsymbol{\beta} - \frac{2a(\boldsymbol{p})}{b(\boldsymbol{p})} \boldsymbol{\lambda}, \\
A_3(\boldsymbol{p}) &= \frac{1}{b(\boldsymbol{p})} \boldsymbol{\lambda}, \\
A_4(\boldsymbol{p}) &= \boldsymbol{\delta},
\end{aligned}$$

or

$$\begin{aligned}
\boldsymbol{w} &= \frac{d(\boldsymbol{p})}{x}\boldsymbol{\delta} + \left(\frac{x - d(\boldsymbol{p})}{x}\right) \\
&\quad \times \left(\boldsymbol{\alpha} + \boldsymbol{\Gamma} \ln \boldsymbol{p} + \frac{\ln (x - d(\boldsymbol{p})) - a(\boldsymbol{p})}{b(\boldsymbol{p})}\boldsymbol{\beta} + \frac{(\ln (x - d(\boldsymbol{p})) - a(\boldsymbol{p}))^2}{b(\boldsymbol{p})}\boldsymbol{\lambda}\right).
\end{aligned}$$

In empirical applications with heterogeneous households, one might let the $\boldsymbol{\alpha}$ or $\boldsymbol{\delta}$ parameters vary by demographic characteristics of the household. The rank three QUAIDS model equals the special case of this model in which $a(\boldsymbol{p}) = 0$, and the rank two AIDS equals the special case of $a(\boldsymbol{p}) = 0$ and $d(\boldsymbol{p}) = 0$.

Chapter 3

The Theoretical Regularity Properties of the Normalized Quadratic Consumer Demand Model

3.1 Introduction

Uzawa (1962) proved that the constant elasticity of substitution (CES) model cannot attain arbitrary elasticities with more than two goods. As a result, the development of locally flexible functional forms evolved as a new approach to modeling specifications of tastes and technology. Flexible functional forms were defined by Diewert (1971) to be the class of functions that have enough free parameters to provide a local second-order approximation to any twice continuously differentiable function. If a flexible functional form has no more parametric freedom than needed to satisfy that definition, then the flexible functional form is called "parsimonious." Barnett (1983) proved that a functional form satisfies Diewert's definition if and only if it can attain any arbitrary elasticities at any one predetermined point in price-income space. Most of the available flexible functional forms are based on quadratic forms derived from second-order series expansions. The translog model of Christensen, Jorgenson and Lau (1971) and the AIDS model of Deaton and Muellbauer (1980) use Taylor series expansions in logarithms; the generalized Leontief model of Diewert (1971) uses a Taylor series expansion in square roots; and the Laurent models of Barnett (1983) use the Laurent series expansion.

As these flexible functional form models became available, applied researchers tended to overlook the maintained regularity conditions required by the theory. Regularity requires satisfaction of both curvature and monotonicity conditions. Simultaneous imposition of both of these conditions on a parsimonious flexible functional form destroys the model's local flexibility property. For instance, Lau (1978) showed that imposition of global regularity reduces the translog model to Cobb-Douglas, which is not a flexible functional form anymore and has no estimable elasticities. When regularity is not imposed, most of the estimated flexible functional forms in empirical applications exhibit frequent violations of regularity conditions at many data points.[22] Since that fact became evident, information about violations of regularity conditions in empirical applications have become hard to find.[23]

An exception to the common neglect of regularity conditions was Diewert and Wales' (1987, 1988b) work on the Normalized Quadratic model. That model permits imposition of curvature globally, while remaining flexible. Since violations of curvature have more often been reported than violations of monotonicity, the imposition of curvature alone seems to merit consideration. In subsequent papers of Diewert and Wales (1993, 1995) and others, imposition of curvature globally, without imposition of monotonicity, has become a common practice with the Normalized Quadratic functional form.[24]

But once curvature is imposed without the imposition of monotonicity, the earlier observation may no longer apply. When global curvature is im-

[22] See, e.g., Manser (1974) and Humphrey and Moroney (1975).

[23] A noteworthy exception is Moroney and Trapani (1981), who confirmed the earlier findings of frequent violations of maintained regularity conditions.

[24] Quah (2000) studies the monotonicity of individual and market demand with the aid of the indirect utility function and identifies sufficient conditions on an agent's indirect utility function to guarantee that the demand function is monotonic.

posed, the loss of model-fit may induce spurious improvements in fit through violations of monotonicity. This problem could be especially common with quadratic models, which can have bliss points. It is possible that violations of monotonicity could be induced by imposition of curvature.

With this model, it has become common not to check for monotonicity after imposing global curvature. Diewert and Wales (1995) and Ryan and Wales (1998) have expected that monotonicity will be satisfied as a result of the non-negativity of the dependent variables. But non-negativity of observed dependent variables does not assure non-negativity of fitted dependent variables. In Kohli (1993) and Diewert and Fox (1999), the curvature condition is treated as the sole regularity condition. But without satisfaction of both curvature and monotonicity, the second-order condition for optimizing behavior fails, duality theory fails, and inferences resulting from derived estimating equations become invalid.[25] Hence the common practice of equating regularity solely with curvature is not justified.

Barnett (2002) and Barnett and Pasupathy (2003) confirmed the potential problem and found further troublesome consequences when they checked regularity violations in their own previously published estimation of technology in the financial sector in Barnett, Kirova and Pasupathy (1995). Initially, they imposed curvature globally, but monotonicity only at a central data point with the Normalized Quadratic production model. In addition to violations of monotonicity, they encountered induced curvature reversals of composite functions, along with nonunique isoquants and complex valued

[25] The damage done to the inference has been pointed out by Basmann, Molina and Slottje (1983); Basmann, Diamond, Frentrup and White (1985); Basmann, Fawson and Shumway (1990); and Basmann, Hayes and Slottje (1994).

solutions. Even if curvature is imposed on both inner (category) production functions and weakly separable outer functions, the composite technology still can violate curvature if monotonicity is violated. The evidence suggested the need for a thorough investigation of the global regularity property of the Normalized Quadratic model. We undertake that task in this chapter.

A well-established approach to exploring regularity properties of a neoclassical function is to set the parameters of the model to produce various plausible elasticities and then plot the regular regions within which the model satisfies monotonicity and curvature. We do so by setting the parameters at various levels to produce elasticities that span the plausible range and then plot the regular region of the model when curvature is imposed while monotonicity is not imposed. The intent is to explore the common practice with the Normalized Quadratic model. Such experiments have been conducted with the translog and the generalized Leontief by Caves and Christensen (1980) and with newer models by Barnett, Lee and Wolfe (1985, 1987) and Barnett and Lee (1985).[26]

In our experiment, we obtain the parameter values of the Normalized Quadratic model by estimation of those parameters with data produced by another model at various settings of the elasticities. Jensen (1997) devised the experimental design, which closely follows that of Caves and Christensen, but Jensen applied the approach to estimate the coefficients of the Asymptotically Ideal Model (AIM) of Barnett and Jonas (1983).[27] We adopt a

[26] Other relevant papers include Wales (1977); Blackorby, Primont and Russell (1977); Guilkey and Lovell (1980); White (1980a); Guilkey, Lovell and Sickles (1983); Barnett and Choi (1989).

[27] The AIM is a seminonparametric model produced from a class of globally flexible series expansions. See also Barnett, Geweke and Wolfe (1991a, 1991b). Gallant's Fourier

similar experimental design in which (1) artificial data is generated, (2) the Normalized Quadratic model then is estimated with that data, and (3) its regular regions are displayed.

The flexible functional form that we investigate is the Normalized Quadratic reciprocal indirect utility function (Diewert and Wales, 1988b; Ryan and Wales, 1998). Globally correct curvature can be imposed on the model by imposing negative semi-definiteness on a particular coefficient matrix and non-negativity on a particular coefficient vector, but at the cost of losing flexibility. It has been argued that global curvature imposition forces the Slutsky matrix to be "too negative semi-definite." In that sense, the method imposes too much concavity, and thereby damages the flexibility. Since concavity is required by economic theory, the model's inability to impose full concavity without loss of flexibility is a serious defect of the model. If, instead of the indirect utility function, the Normalized Quadratic functional form is used to model the expenditure function, global concavity can be imposed without loss of flexibility. But the underlying preferences then are quasihomothetic and therefore produce linear Engel curves. Because of that serious restriction on tastes, we exclude that model from our experiment.

Ryan and Wales (1998) suggested a procedure for imposing negative semi-definiteness on the Slutsky matrix, as is necessary and sufficient for the curvature requirement of economic theory. But to avoid the loss of flexibility, Ryan and Wales apply the condition only at a single point of approximation.[28] With their data, they successfully found a data point

flexible functional form (1981) is also globally flexible.

[28] Moschini (1996) independently developed the identical procedure to impose local curvature on the semiflexible AIDS model. See Diewert and Wales (1988a) for the definition

such that imposition of curvature at that point results in full regularity (both curvature and monotonicity) at every sample point. They also applied the procedure to the linear translog and the AIDS demand systems.[29] By imposing correct curvature at a point, the intent with this procedure is to attain, without imposition, the curvature and monotonicity conditions at all data points. We explore the regular regions of the models with these two methods of curvature imposition. The objective is to determine the extent to which imposition of global, local, or no curvature results in regularity violations. Imposing curvature locally may induce violations of curvature at other points as well as violations of monotonicity at the point or other points.

We find monotonicity violations to be common. With these models, the violations exist widely within the region of the data, even when neither global curvature nor local curvature is imposed. We believe that this problem is common with many non-globally-regular flexible functional forms, and is not a problem specific to the Normalized Quadratic model. For example, one of the graphs for the AIM cost function in Jensen (1997), without regularity imposition, looks similar to the one of the graphs we produced.

Imposing curvature globally corrected the monotonicity violations globally in a case with one pair of complementary goods. But that imposition produced some overestimation of cross elasticities of substitutions in absolute values. A pair of complementary goods became more complementary

of the semiflexibility.

[29] Moreover, Ryan and Wales (2000) showed the effectiveness of the procedure when estimating the translog and the generalized Leontief cost functions with the data utilized by Berndt and Khaled (1979).

and a pair of substitute goods became stronger substitutes. Diewert and Wales (1993) similarly found that some method of imposing regularity can produce upper bounds on certain elasticities for the AIM and translog models.[30]

This chapter is organized as follows: Section 3.2 presents the models using the two methods of curvature imposition. Section 3.3 illustrates our experimental design, by which the artificial data is simulated, the model is estimated, and the regular region is displayed. Section 3.4 provides our results and discussion. Section 3.5 presents our conclusions.

3.2 The Model Description

Central to the imposition of curvature is a quadratic term of the form $\boldsymbol{\nu}'\mathbf{B}\boldsymbol{\nu}$, where $\boldsymbol{\nu}$ is a vector of the variables, and $\mathbf{B}$ is a symmetric matrix containing unknown parameters. With the Normalized Quadratic model, the quadratic term is normalized by a linear function of the form $\boldsymbol{\alpha}'\boldsymbol{\nu}$, where $\boldsymbol{\alpha}$ is a non-negative predetermined vector, and replaced with the normalized quadratic term $\boldsymbol{\nu}'\mathbf{B}\boldsymbol{\nu}/\boldsymbol{\alpha}'\boldsymbol{\nu}$. According to Diewert and Wales (1987), the normalized quadratic term is globally concave if the matrix $\mathbf{B}$ is negative semi-definite. In addition, the appropriate parameter restriction on $\boldsymbol{\alpha}'\boldsymbol{\nu}$ makes the term homogeneous of degree zero. Imposition of the constraint ensures concavity of the normalized quadratic term. As a result, imposition of global curvature starts with imposition of negative semi-definiteness on the matrix $\mathbf{B}$. We

[30] See also Terrell (1995). But it should be observed that more sophisticated methods of imposing regularity on AIM do not create that problem. In fact, it is provable that imposition of global regularity on seminonparametric models, such as AIM, cannot reduce the span, if imposition is by the most general methods.

reparameterize the matrix $\mathbf{B}$ by replacing it by the inverse of the product of a lower triangular matrix, $\mathbf{K}$, multiplied by its transpose, yielding $\mathbf{B} = -\mathbf{K}\mathbf{K}'$. Diewert and Wales (1987, 1988a, 1988b) have frequently used this technique developed by Wiley, Schmidt, and Bramble (1973) and generalized by Lau (1978) in producing models with correct curvature.

The Normalized Quadratic reciprocal indirect utility function of Diewert and Wales (1988b) and Ryan and Wales (1998) is defined as

$$V(\boldsymbol{\nu}) = \boldsymbol{b}'\boldsymbol{\nu} + \frac{1}{2}\left[\frac{\boldsymbol{\nu}'\mathbf{B}\boldsymbol{\nu}}{\boldsymbol{\alpha}'\boldsymbol{\nu}}\right] + \boldsymbol{a}'\ln(\boldsymbol{\nu}), \tag{89}$$

where $\boldsymbol{b}$ is a vector containing unknown parameters, and $\boldsymbol{\nu}$ is a vector of prices, $\boldsymbol{p}$, normalized by a scalar of total expenditure x, so that $\boldsymbol{\nu} \equiv \boldsymbol{p}/x$.[31] A fixed reference point $\boldsymbol{\nu}^*$ is chosen, such that the matrix $\mathbf{B}$ satisfies

$$\mathbf{B}\boldsymbol{\nu}^* = \mathbf{0}, \tag{90}$$

and the predetermined vector $\boldsymbol{\alpha}$ satisfies

$$\boldsymbol{\alpha}'\boldsymbol{\nu}^* = 1. \tag{91}$$

Using Diewert's (1974) modification of Roy's Identity, the system of share equations is derived as

$$\boldsymbol{w}(\boldsymbol{\nu}) = \frac{\widehat{\mathbf{V}}\boldsymbol{b} + \widehat{\mathbf{V}}\frac{\mathbf{B}\boldsymbol{\nu}}{\boldsymbol{\alpha}'\boldsymbol{\nu}} - \frac{1}{2}\widehat{\mathbf{V}}\left[\frac{\boldsymbol{\nu}'\mathbf{B}\boldsymbol{\nu}}{(\boldsymbol{\alpha}'\boldsymbol{v})^2}\right]\boldsymbol{\alpha} + \boldsymbol{a}}{\boldsymbol{\nu}'\boldsymbol{b} + \frac{1}{2}\left[\frac{\boldsymbol{\nu}'\mathbf{B}\boldsymbol{\nu}}{\boldsymbol{\alpha}'\boldsymbol{\nu}}\right] + \boldsymbol{\iota}'\boldsymbol{a}}, \tag{92}$$

[31] Diewert and Wales (1988b) include a level parameter b_0 additively in (89). But it is nonidentifiable and not estimable since it vanishes during the derivation of the estimating equations.

where $\boldsymbol{w} = \boldsymbol{w}(\boldsymbol{\nu})$ is a vector of budget shares, $\boldsymbol{\iota}$ is a unit vector with one in each entry, and $\widehat{\mathbf{V}}$ is a $n \times n$ diagonal matrix with normalized prices on the main diagonal and zeros on the off-diagonal. Homogeneity of degree zero in all parameters of the share equations (92) requires use of an identifying normalization. The normalization usually used is

$$\boldsymbol{b}'\boldsymbol{\nu}^* = 1. \tag{93}$$

The functional form (89) subject to restrictions (90), (91), and (93) will be globally concave over the positive orthant if the matrix $\mathbf{B}$ is negative semi-definite and all elements of the parameter vector $\boldsymbol{a}$ are non-negative. Global concavity can be imposed during the estimation by setting $\boldsymbol{B} = -\mathbf{K}\mathbf{K}'$ with $\mathbf{K}$ lower triangular, while setting $a_i = c_i^2$ for each i, where $\boldsymbol{c}$ is a vector of the same dimension as $\boldsymbol{a}$.[32] We then estimate the elements of $\mathbf{K}$ and $\boldsymbol{c}$ instead of those of $\mathbf{B}$ and $\boldsymbol{a}$. As mentioned above, this procedure for imposing global concavity damages flexibility.

Imposition of curvature locally is at the point of approximation. Without loss of generality, we choose $\boldsymbol{v}^* = \mathbf{1}$ to be that point. For ease of estimation we impose the following additional restriction:

$$\boldsymbol{a}'\boldsymbol{\nu}^* = 0. \tag{94}$$

With the additional restriction given in (94), the Slutsky matrix at the point

[32] The nonnegativity of a_i's can be also imposed by setting $a_i = \exp(c_i)$.

$\boldsymbol{\nu}^* = \mathbf{1}$ can be written as

$$\mathbf{S} = \mathbf{B} - \widehat{\mathbf{A}} + \boldsymbol{a}\boldsymbol{b}' + \boldsymbol{b}\boldsymbol{a}' + \mathbf{2}\boldsymbol{a}\boldsymbol{a}', \tag{95}$$

where $\widehat{\mathbf{A}}$ =diag($\boldsymbol{a}$), a diagonal matrix whose diagonal entries consist of $\boldsymbol{a}$. Imposing curvature locally is attained by setting $\mathbf{S} = -\mathbf{K}\mathbf{K}'$ with $\mathbf{K}$ lower triangular, and solving for $\mathbf{B}$ as

$$\mathbf{B} = -\left(\mathbf{K}\mathbf{K}'\right) + \widehat{\mathbf{A}} - \boldsymbol{a}\boldsymbol{b}' - \boldsymbol{b}\boldsymbol{a}' - \mathbf{2}\boldsymbol{a}\boldsymbol{a}'. \tag{96}$$

Ryan and Wales (1998) showed that the demand system described above is flexible. Also note that if Caves and Christensen's method is used, the regular regions of this model and the unconstrained model with equation (94) imposed will be exactly identical. Therefore, we cannot tell the effectiveness of this local curvature imposition if we use the Caves and Christensen method to investigate the regular region.

During estimation, the matrix $\mathbf{B}$ is replaced by the right-hand side of (96) to guarantee that the Slutsky matrix is negative semi-definite at the point of approximation. To see why imposing curvature globally damages the flexibility while imposing curvature locally does not, recall that the Slutsky matrix $\mathbf{S}$ is symmetric and satisfies $\mathbf{S}\boldsymbol{p} = \mathbf{0}$ or equivalently $\mathbf{S}\boldsymbol{\nu} = \mathbf{0}$. As a result, the rank of $\mathbf{S}$ is reduced by one, so that the number of the independent elements of $\mathbf{S}$ becomes equal to that of $\mathbf{B}$. Therefore $\mathbf{S}$ in equation (95) can be arbitrarily determined by $\mathbf{B}$, independently of $\boldsymbol{a}$ and $\boldsymbol{b}$. But the Hessian matrix of the indirect utility function is usually full rank unless linear homogeneity is imposed or attained empirically. In Diewert and Wales' (1988b) approach to prove the local flexibility, the second partial derivatives at the point of approximation depend on both $\mathbf{B}$ and $\boldsymbol{a}$. However, imposition of non-negativity on $\boldsymbol{a}$ to attain global curvature reduces the number of independent parameters and limits the span of $\mathbf{B}$ and $\boldsymbol{a}$. As a result, imposing curvature globally on this model damages flexibility. Regarding local curvature imposition, the condition that the Slutsky matrix be negative semi-definite is both necessary and sufficient for correct curvature at the point of approximation.

3.3 Experimental Design

Our Monte Carlo experiment is conducted with a model of three demand goods to permit different pairwise complementarities and substitutabilities. The design of the experiment is described below.

3.3.1 Data Generation

The data set employed in the actual estimation process includes data for normalized prices and budget shares, defined as $\boldsymbol{\nu} \equiv \boldsymbol{p}/x$ and $\boldsymbol{w} \equiv \widehat{\mathbf{V}}\boldsymbol{q}$ where $\boldsymbol{q}$ is a vector of demand quantities and $\widehat{\mathbf{V}}$ is as defined previously. The data for demand quantities are produced from the demand functions induced by two globally regular utility functions: the CES functional form and the linearly-homogeneous Constant-Differences of Elasticities-of-Substitution (CDE) functional form.[33]

[33] See Hanoch (1975), Jensen (1997), or section 2.2.1.1 for details of this model.

The CES indirect utility function with three goods is

$$V(\boldsymbol{p}, x) = x \left[\sum_{k=1}^{3} p_k^r \right]^{-1/r}, \tag{97}$$

where $r = \rho(\rho - 1)$. By applying Roy's identity to (97), Marshallian demand functions are derived as

$$q_i(\boldsymbol{p}, x) = x p_i^{r-1} / \textstyle\sum_{k=1}^{3} p_k^r, \tag{98}$$

for $i = 1$, 2, and 3. The CES utility function is globally regular if $\rho \leq 1$. The values of ρ are chosen so that the elasticity of substitution $\sigma = 1/(1-\rho)$ covers a sufficiently wide range. Fleissig, Kastens and Terrell (2000) also used this data generation model in comparing the performance of the Fourier flexible form, the AIM form, and a neural network in estimating technologies.

The CDE indirect reciprocal utility function $1/u = g(\boldsymbol{p}, x)$ is defined implicitly by an identity of the form:

$$G(\boldsymbol{p}, x, u) = \sum_{k=1}^{3} G^k (p_k, x, u) \equiv 1, \tag{99}$$

with $G^k = \phi_k u^{\theta_k} (p_k/x)^{\theta_k}$. Parametric restrictions required for the implicit utility function (99) to be globally regular are $\phi_k > 0$ and $\theta_k < 1$ for all k, and either $\theta_k \leq 1$ for all k or $0 < \theta_k < 1$ for all k. In all cases, the ϕ_k's equal the corresponding budget shares at $(\boldsymbol{p}^*, x^*) = (\mathbf{1}, 1)$. Applying the Roy's

Identity to (99), we derive the demand functions,

$$q_i(\boldsymbol{p}, x) = \frac{\theta_i \phi_i u^{\theta_i} \left(p_i/x\right)^{\theta_i - 1}}{\sum_{k=1}^{3} \theta_k \phi_k u^{\theta_k} \left(p_k/x\right)^{\theta_k}} \tag{100}$$

for i = 1,2, and 3. The utility level u is set to unity without loss of generality when artificial data is generated.

Our test bed consists of eight cases. Cases 1 to 4 use data simulated from demand functions of CES form in (98), and cases 5 to 8 use data from demand functions of CDE form in (100). Table 1 describes each case in terms of the elasticity of substitution and budget share settings at the reference point.[34] It is convenient to construct the data such that the mean of the normalized price of each good is one. We draw from a continuous uniform distribution over the interval $[0.5, 1.5]$ for price data and $[0.8, 1.2]$ for total expenditure data. The sample size is 100, as would be a typical sample size with annual data.[35]

The stochastic data, adding noise to the model's solved series, are constructed in the following manner: The noise vector $\boldsymbol{\varepsilon}$ is generated from a multivariate normal distribution of mean zero and covariance matrix $\mu\text{cov}(\boldsymbol{p})$, where μ is a constant and $\mu\text{cov}(\boldsymbol{p})$ is the covariance matrix of a generated price series given a draw of $\boldsymbol{p}$. We arbitrarily set $\mu \in [0.0, 1.0]$ to adjust the influence of noise on the estimation. We construct the price series in-

[34] The values of those elasticities are computed as Allen-Uzawa elasticities of substitution. The Allen-Uzawa elasticity of substitution is the commonly used elasticity of substitution measure. More complicated substitutability can be captured by the Morishima elasticity of substitution. See Blackorby and Russell (1989) or section 2.1.4.

[35] Unlike our experimental design, Jensen's (1997) design used price series of length 1000 with a factorial design at discrete points in the interval $[0.5, 2.0]$. Terrell (1995) used a grid of equally spaced data in evaluating the performance of the AIM production model.

corporating noise as $\widetilde{\boldsymbol{p}} = \boldsymbol{p} + \boldsymbol{\varepsilon}$ while making sure that the resulting prices are strictly positive with each setting of μ. We then use equations (98) and (100), together with the $\widetilde{\boldsymbol{p}}$ and a draw of total expenditure x, to generate the data for quantities demanded, $\widetilde{\boldsymbol{q}}(\widetilde{\boldsymbol{p}}, x)$. Using the noise-added data, we compute total expenditure $\widetilde{x} = \widetilde{\boldsymbol{q}}'\widetilde{\boldsymbol{p}}$, normalized prices $\widetilde{\boldsymbol{\nu}} = \widetilde{\boldsymbol{p}}/\widetilde{x}$, and budget shares $\widetilde{\boldsymbol{w}}$ =diag$(\widetilde{\boldsymbol{\nu}})\widetilde{\boldsymbol{q}}$. We then have the data for the dependent variable $\widetilde{\boldsymbol{w}}$ and the noise-free independent variable $\boldsymbol{\nu} = \boldsymbol{p}/x$. It is easier to ensure strictly positive noise-added data by this procedure, than by adding directly to the estimating budget share functions.[36]

3.3.2 Estimation

Using our simulated data, we estimate the system of budget share equations (92) with a vector of added disturbances $\boldsymbol{e}$. We assume that the $\boldsymbol{e}$'s are independently multivariate normally distributed with $E(\boldsymbol{e}) = \boldsymbol{0}$ and $E(\boldsymbol{e}\boldsymbol{e}') = \boldsymbol{\Omega}$, where $\boldsymbol{\Omega}$ is constant across observations. Since the budget constraint causes $\boldsymbol{\Omega}$ to be singular, we drop one equation, impose all restrictions by substitution, and compute the maximum likelihood estimates of the reparameterized model. Barten (1969) proved that consistent estimates can be obtained in this manner, with the estimates being invariant to the equation omitted. The unconstrained optimization is computed by MATLAB's Quasi-Newton algorithm. A complete set of parameters is recovered using the associated restrictions.

A priori, there is no known optimal method for choosing the vector

[36] Gallant and Golub (1984) used the same procedure for stochastic data generation with a production model.

	Budget shares			Elasticities of substitutions		
	w_1	w_2	w_3	σ_{12}	σ_{13}	σ_{23}
Case 1	0.333	0.333	0.333	0.200	0.200	0.200
Case 2	0.333	0.333	0.333	0.700	0.700	0.700
Case 3	0.333	0.333	0.333	2.000	2.000	2.000
Case 4	0.333	0.333	0.333	4.000	4.000	4.000
Case 5	0.300	0.300	0.400	1.500	1.500	1.500
Case 6	0.300	0.300	0.400	0.500	0.500	0.500
Case 7	0.395	0.395	0.211	0.061	0.561	0.561
Case 8	0.409	0.409	0.182	-0.010	0.591	0.591

Table 1: True underlying preferences at $(\boldsymbol{p}^*, x^*) = (\mathbf{1}, 1)$ in terms of budget shares and elasticities of substitution.

$\boldsymbol{\alpha}$, so we choose all elements of the vector to be equal. Some authors have experimented with alternative settings, such as setting $\boldsymbol{\alpha}$ as weights to form a Laspeyres-like price index, but with no clear gain over our choice. Hence, all elements of the vector $\boldsymbol{\alpha}$ are set at 1/3 as a result of equation (91) with $\boldsymbol{\nu}^* = \mathbf{1}$.[37] The number of Monte Carlo repetitions is 1000 for each case. We use boxplots to summarize the distribution of the estimated elasticities across the 1000 replicates. We follow the standard procedure for drawing boxplots. The box has lines at the lower quartile, median, and upper quartile values. The "whisker" is a line going through each end of the box, above and below the box. The length of the whisker above and below the box equals 1.5$\times$ (the upper quartile value $-$ the lower quartile value). Estimates above or below the whisker are considered outliers. Since we find these distributions of estimates to be asymmetric, standard errors alone cannot capture what is displayed in the boxplots. The average values of each parameter across replications are used to produce the regular regions of the models.

To begin our iterations, we start as follows: We compute the gradient vector and the Hessian matrix of the data-generating function at the point $(\boldsymbol{p}^*, x^*) = (\mathbf{1}, 1)$. We set that vector and that Hessian of the Normalized Quadratic model to be the same as those of the data generating model at that point. We then solve for corresponding parameter values of the Normalized Quadratic model and use the solution as the starting values for the optimization procedure. Those starting parameter values produce a local second order approximation of the Normalized Quadratic to the generating function. Our starting values facilitate convergence to the global maximum

[37] See Kohli (1993) and Diewert and Wales (1992).

of the likelihood function, since the global maximum is likely to be near the starting point.

3.3.3 Regular Region

Following Jensen (1997), we plot two-dimensional sections of the regular region in the Cartesian plane. The x-axis represents the natural logarithm of (p_2/p_1), and the y-axis the natural logarithm of (p_3/p_1). Each axis ranges between $\ln(0.2/5.0) \approx -3.2189$ and $\ln(5.0/0.2) \approx 3.2189$. The sample range is defined as the convex hull of possible prices within the above intervals of relative values and is displayed in our figures as a rectangle in the center of the graph. Accordingly, each of the four sides of the rectangle range from $\ln(0.5/1.5) \approx -1.0986$ to $\ln(1.5/0.5) \approx 1.0986$ since our data is generated from the interval of $[0.5, 1.5]$. The entire section is divided into 150×150 grid points at which the monotonicity and curvature conditions are evaluated. Each plot can be viewed as a display of a two-dimensional hyperplane through the three-dimensional space having dimensions $\ln(p_2/p_1)$, $\ln(p_3/p_1)$, and $\ln(x)$. We section the regular region perpendicular to the $\ln(x)$ axis at the reference point of $x = 1.0$. It is desirable to plot several hyperplanes at different settings of x, as done by Barnett, Lee and Wolfe (1985, 1987) to investigate the full three-dimensional properties of the model's regular region. But, since regularity is usually satisfied at the reference point and violations increase as data points move away from the reference point, the emergence of regularity violations on the single hyperplane with x fixed at the reference setting is sufficient to illustrate deficiencies of the Normalized Quadratic Model.

The monotonicity condition is evaluated using the gradient vector of the estimated equation in (89), $\nabla V(\boldsymbol{\nu})$. The model is required by theory to be strictly increasing in $\boldsymbol{\nu}$. For each grid point at which the gradient is evaluated, the monotonicity condition is satisfied, if $\nabla V > \mathbf{0}$.

Our approach to evaluation of the curvature condition differs from that used in most studies. In those other studies, the curvature condition is judged to be satisfied if the Allen elasticity of substitution matrix or the Slutsky substitution matrix is negative semi-definite.[38] The problem is that satisfaction of the monotonicity condition is required for those matrix conditions to be necessary and sufficient for satisfaction of the curvature condition. Hence, we do not use substitution matrices to evaluate the curvature condition. We directly evaluate the quasiconcavity of the equation (89) using the method proposed by Arrow and Enthoven (1961). Quasiconcavity is checked by confirming alternating signs of the principal minors of the bordered Hessian matrix, which contains the second partial derivatives bordered by first derivatives.[39] This approach is general, regardless of whether there are any monotonicity violations.

Each grid is filled with different gradations of black and white, designating the evaluation results for the regularity conditions. The completely black grid designates violations of both curvature and monotonicity; the very dark grey grid designates violation of only curvature; and the very light grey grid specifies violation of only monotonicity. The completely white regions are

[38] For example, Serletis and Shahmoradi (2005) computed the Cholesky values of the Slutsky substitution matrix to evaluate its negative semi-definiteness. A matrix is negative semi-definite, if its Cholesky factors are non-positive (Lau, 1978).

[39] The procedure is described in detail in section 2.1.5.

fully regular. There are 8 cases with 3 models each, resulting in 24 plots of regular regions.

3.4 Results and Discussion

We confirm convergence to a global maximum of the likelihood function by comparing the estimated elasticity values with the true ones. If the discrepancy is large, we discard that run and rerun the program. We do not seek to explain the cases of large discrepancies, other than to conclude that under such circumstances, an unresolved problem exists. Based on this criterion, we encounter substantial difficulty in the estimation of the model with local curvature imposed. When data are generated with elasticities of substitution greater than unity (cases 3, 4 and 5), the estimates converge to values far from the true ones. But when we try the somewhat lower elasticities of $\sigma 12 = \sigma 13 = \sigma 23 = 1.10$ or 1.20, convergence to reasonable estimates is more successful although some replications still often yield unreasonable estimates.

The boxplots of case 4 in figure 1 describe the distributions of 1000 estimates of the elasticities of substitution by the models with no curvature imposed (left), local curvature imposed (middle), and global curvature imposed (right).[40] The estimates of the elasticities from the local curvature imposed model not only have larger variations (described as longer whiskers) than those from the no curvature imposed and global curvature imposed models, but also include a number of severe outliers especially in the positive direc-

[40] All data used for boxplots are generated with $\mu = 0.20$ for the noise adjustment constant.

tion. Even when estimates of cross elasticities are reasonable, estimates of the own elasticities were often found to be far from the true values. In fact, the unconstrained and global-curvature-imposed models both require very high values of $\boldsymbol{a}$ (around $400,000$'s to $600,000$'s for all elements) to attain maximization of the likelihood function. We suspect that constraint (94) is too restrictive. For the local-curvature-imposed model to approximate a symmetric function, as in cases 3 and 4, the optimal values of $\boldsymbol{a}$ should all be zeros while satisfying the restriction (94). However, any statistical tests would reject the hypothesis that the restriction is valid. Moreover, as Diewert and Wales (1988b) observed, equation (94) often renders global concavity to be impossible. The only way to achieve globally correct curvature with (94) is to set $\boldsymbol{a} = \mathbf{0}$, since global concavity requires non-negativity of all elements of $\boldsymbol{a}$. In addition, since $\mathbf{B}$ is a function of $\boldsymbol{a}$ as well as of $\boldsymbol{b}$ and $\mathbf{K}$, any poor estimate of $\boldsymbol{a}$ will produce poor estimation of $\mathbf{B}$ which contains a set of parameters that explain the model's curvature property. With all such problems and nonlinearity embedded in the likelihood function, the optimization algorithm can search for wrong local maxima. We tried a few global optimization techniques, but without success.[41] When the point estimates of elasticities are themselves poor, we view regularity violations to be a "higher order problem" of lesser concern. For reference purposes, we plot in figure 2 the regular region of the local curvature imposed model (top) along with that of the global curvature imposed model (bottom). However,

[41] The genetic algorithm and the pattern search method were tried. But the lack of convergence of the former and the very slow convergence of the latter is a substantial computational burden for any study that requires a large number of repeated simulations. Dorsey and Mayer (1995) provide an empirical evaluation in econometric applications of the performance of genetic algorithms versus other global optimization techniques.

the validity of conclusions drawn from that figure is questionable.

A display of the regular region of the model with global curvature imposed (bottom) in figure 2, the regularity violations occupy a large part of the area. Within the very light grey areas of the plots, the regularity violations are attributed entirely to monotonicity violations. Severe regularity violations resulted from using data produced with high elasticities of substitution, and therefore, the most severe violations occurred in case 4. The plot of case 3 displays similar shape of regularity violation regions to case 4, but with a somewhat wider regular region. In case 4, we obtained an almost identical figure with the unconstrained model. Regularity violations indued by the monotonicity violations may indeed be common with many nonglobally regular flexible functional forms and should not be viewed as exclusive to the Normalized Quadratic model.

In case 4, the model substantially underestimates true elasticities. The boxplot (right) in figure 1 shows that the median of the 1000 sample estimates is near 3.0, while the true elasticity is 4.0. This finding is similar to those of Guilkey, Lovell and Sickles (1983) and Barnett and Lee (1985), which showed that the generalized Leontief model performs poorly when approximating the function with high elasticities of substitution. We find that the Normalized Quadratic model performs poorly both in estimating elasticities and in maintaining regularity conditions when the data used was produced with high elasticities of substitution.

For cases 1, 2, and 6, all models perform very well with no regularity violations. The Normalized Quadratic model performs well when the data is characterized by low elasticities of substitution (below unity) and pairwise elasticities are relatively close to each other. The plot with case 5 is omitted since the result is similar to case 4.[42]

With case 7, the plot on the top in figure 3 displays a very dark grey cloud on the top-right part of the plot, designating curvature violations for the unconstrained model. In this case, imposing curvature locally as well as globally eliminates all of the curvature violations within the region of the data, as shown by the entirely white region in the bottom plot. Figure 4 describes the distributions of estimates in case 7. For all three models, the median estimates are satisfactorily close to true elasticities. With global curvature imposed, elasticity estimates are severely downward-biased when the true elasticities are high, and upward-biased when the true elasticities are low as outliers. This suggests that all pairwise elasticities become close to each other. As should be expected by the outliers, the cause is not easy to determine, and we do not impute much importance to results with outliers. These problems do not arise when we impose curvature locally in case 7 in which the local curvature imposition succeeds in producing global regularity within the region of our data.

Figure 5 displays plots for case 8 of the regular regions for the unconstrained model (top) and the local-curvature constrained model (bottom). The top of the upper plot has a wide, thick cloud of curvature violations. That region intersects a small monotonicity-violation area on the right side.

[42] With case 5 there are regularity violations inside the sample range as in case 4, but to a milder extent, as in case 3, since both case 3 and 5 use data with lower elasticities of substitutions than case 4. The plot of case 5 is slightly shifted from the center as a result of the fact that the simulated budget shares at the center point are slightly asymmetric. The plot is very similar to the case I of Jensen (1997). Our case 5 and his case I use the same data-generating setting.

The resulting small intersection region designates the set within which both violations occur. Imposing local curvature does not shrink those regions, but rather expands them. On the bottom plot, the region of curvature violations now covers much of the two-dimensional section, with the exception of the white convex regular region and a wide thick pillar of monotonicity violations on the left side. In the intersection of the two non-regular regions, both violations occur. Notice that regularity is satisfied at the center point, at which correct curvature is imposed. This pattern of expansion and change of regularity violation regions is hard to explain.

Another disadvantage of the model is its failure to represent complementarity among goods. A middle boxplot in figure 6 shows that the lower whisker for σ_{12} is strictly above zero. A typical estimate of $\boldsymbol{a}$ for the unconstrained model was $\boldsymbol{a}' = (0.014, 0.014, 0.262)$.[43] Although all elements are strictly positive, they are not substantially different from $\boldsymbol{a} = \mathbf{0}$, in contrast to case 4. Hence, we do not believe that the inability to characterize complementarity was caused by the restrictiveness of equation (94). In case 8, a plot for the model with global curvature imposed is identical to the bottom plot in figure 3, achieving the global regularity within the range of plot. However, imposing global regularity may decrease the ability to produce the accurate approximation of the underlying preferences (Diewert and Wales 1987, Terrell 1996). Comparing the figure 6 boxplot of the unconstrained model's elasticity estimates (left) with those of the global-curvature imposed model (right), relative to the true elasticity values, we see that global cur-

[43] The estimate was obtained using noise-free data with a sample size of 500 instead of 100.

vature imposed model overestimates the elasticity of substitution, σ_{12}, for complementarity and the elasticities of substitution, σ_{13} and σ_{23} for substitutes. A possible cause is the imposition of negative semi-definiteness on the Hessian matrix. As in figure 4 the outlier estimates cause the pairwise elasticities of substitution estimates to become closer to each other.

3.5 Conclusion

We conducted a Monte Carlo study of the global regularity properties of the Normalized Quadratic model. We particularly investigated monotonicity violations as well as the performance of methods of locally and globally imposing curvature. We found that monotonicity violations are especially likely to occur when elasticities of substitution are greater than unity. We also found that imposing curvature locally produces difficulty in the estimation, smaller regular regions, and the poor elasticity estimates in many of the cases considered in this chapter.

When imposing curvature globally, our results were better. Although violations of monotonicity remain common in some of our cases, those violations do not appear to be induced solely by the global curvature imposition, but rather by the nature of the Normalized Quadratic model itself. However, imposition of global curvature does induce a problem with complementary goods by biasing the estimates towards over complementarity and substitute goods towards over substitutability.

With the Normalized Quadratic model, we find that both curvature and monotonicity must be checked with the estimated model, as has previously been shown to be the case with many other flexible functional forms. Imposition of curvature alone does not assure regularity, and imposing local curvature alone can have very adverse consequences.

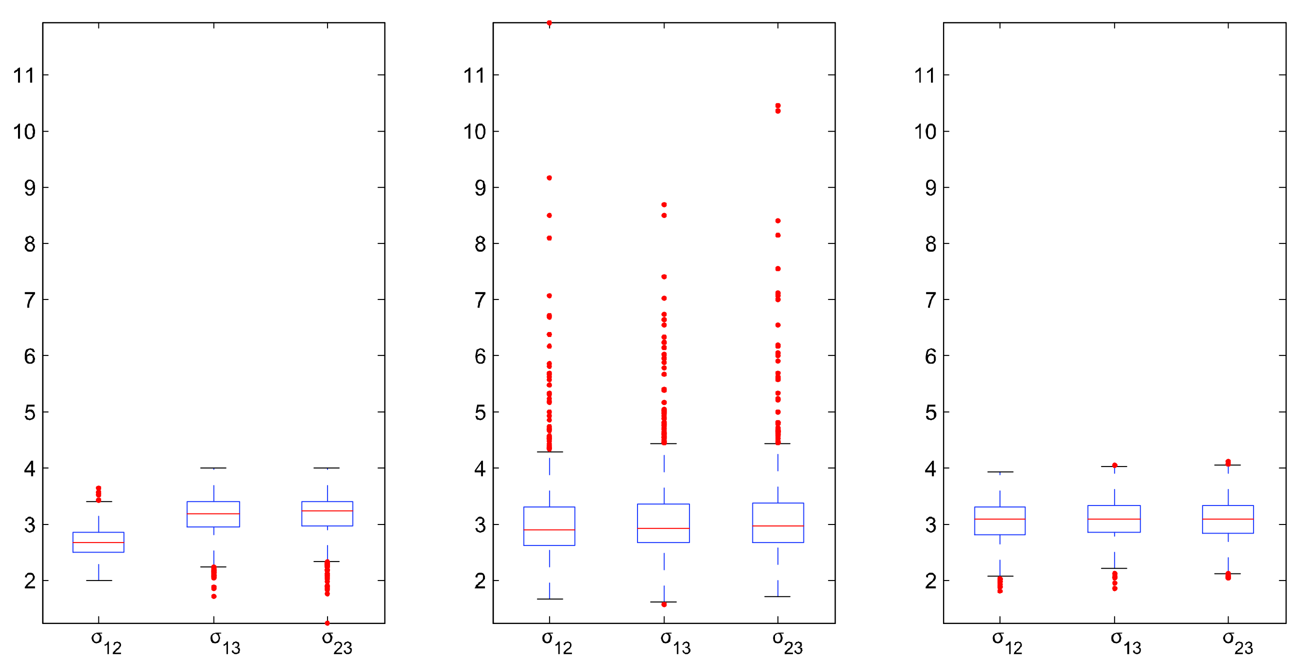

Figure 1: Boxplots of the distributions of estimates for case 4 with the model with no curvature imposed (left), with local curvature imposed (middle), and with global curvature imposed (right).

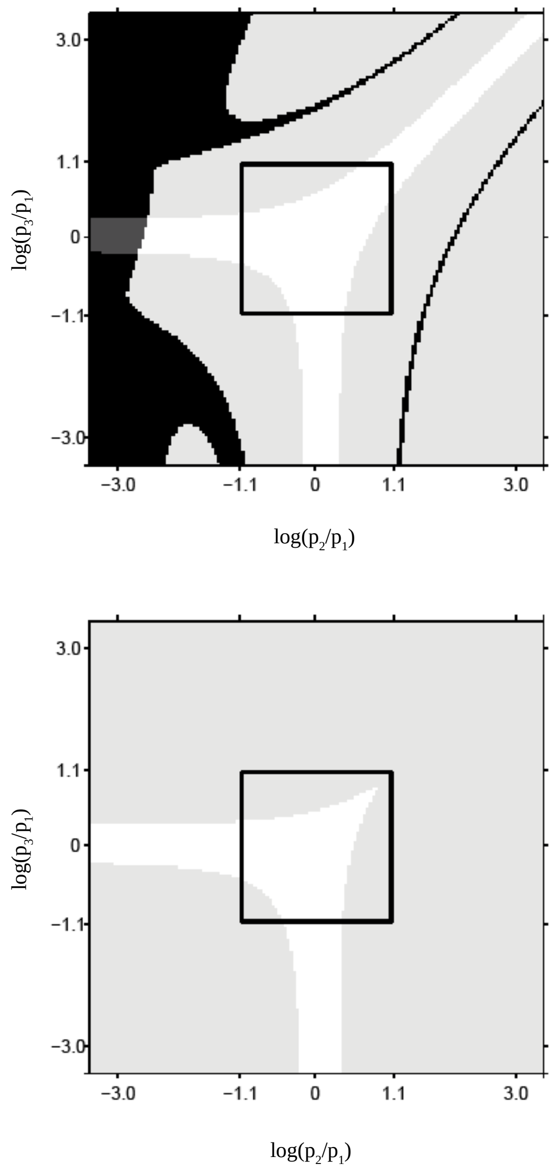

Figure 2: Section through regular regions of the model at $x = 1.0$ with local curvature imposed (top) and with global curvature imposed (bottom) for case 4.

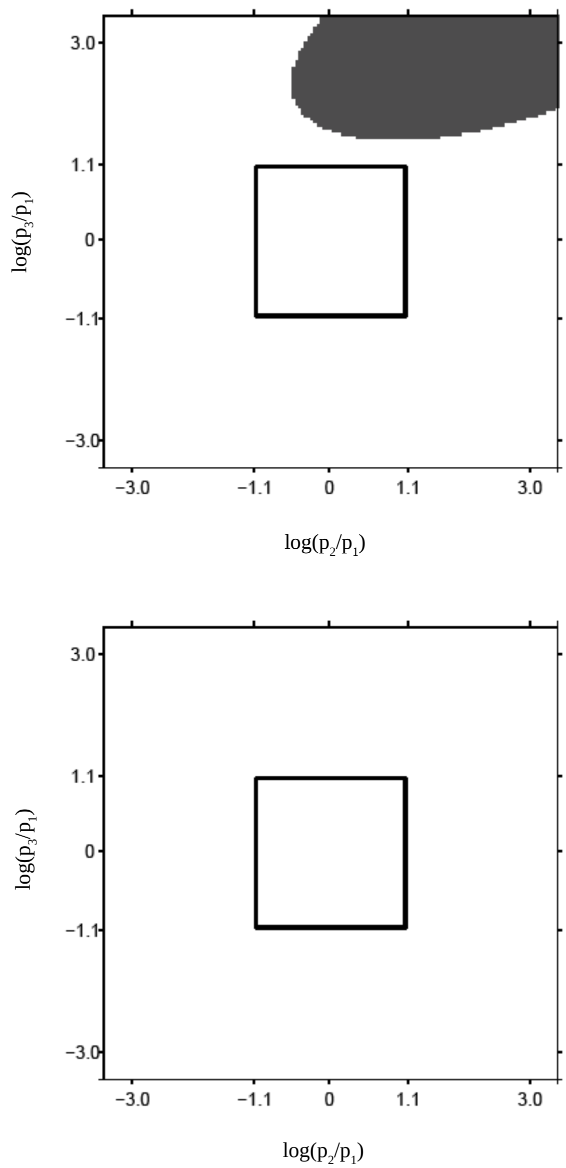

Figure 3: Section through regular regions of the model at $x = 1.0$ with no curvature imposed (top) and with local and global curvature imposed (bottom) for case 7.

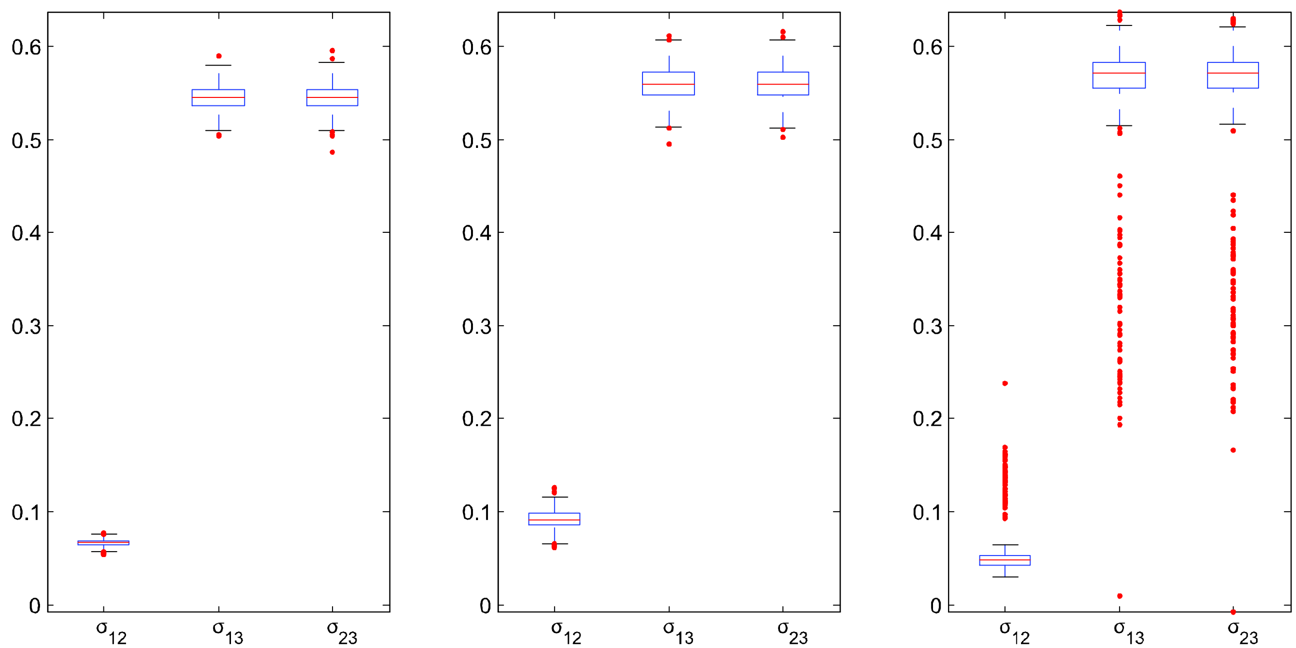

Figure 4: Boxplots of the distributions of estimates for case 7 with the unconstrained model (left), local-curvature-imposed model (middle) , and global-curvature imposed model (right).

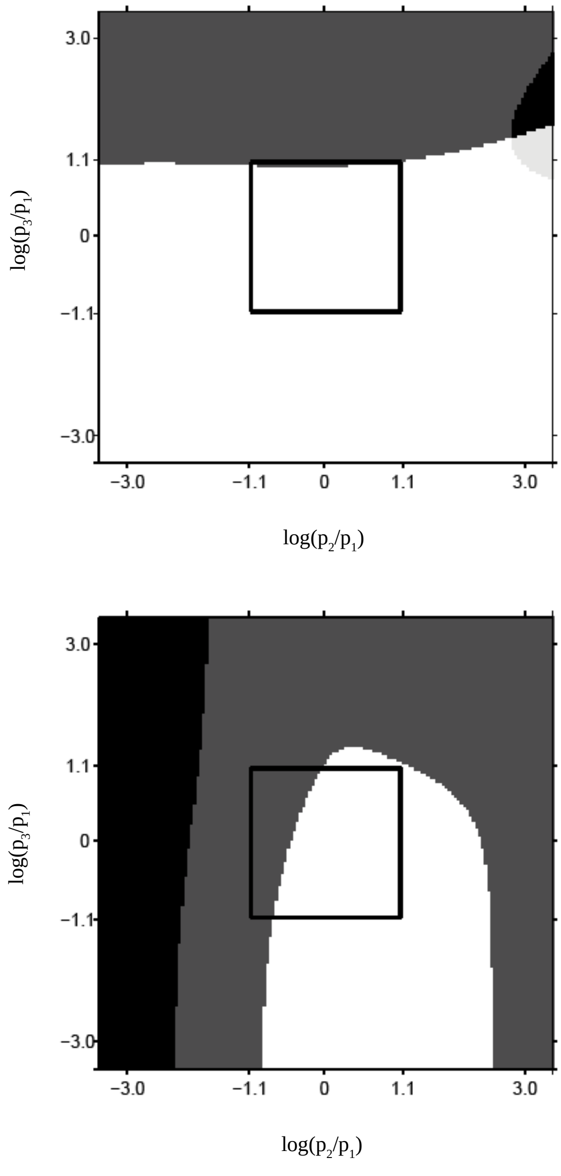

Figure 5: Section through regular regions of the model at $x = 1.0$ with no curvature imposed (top) and with local curvature imposed (bottom) for case 8.

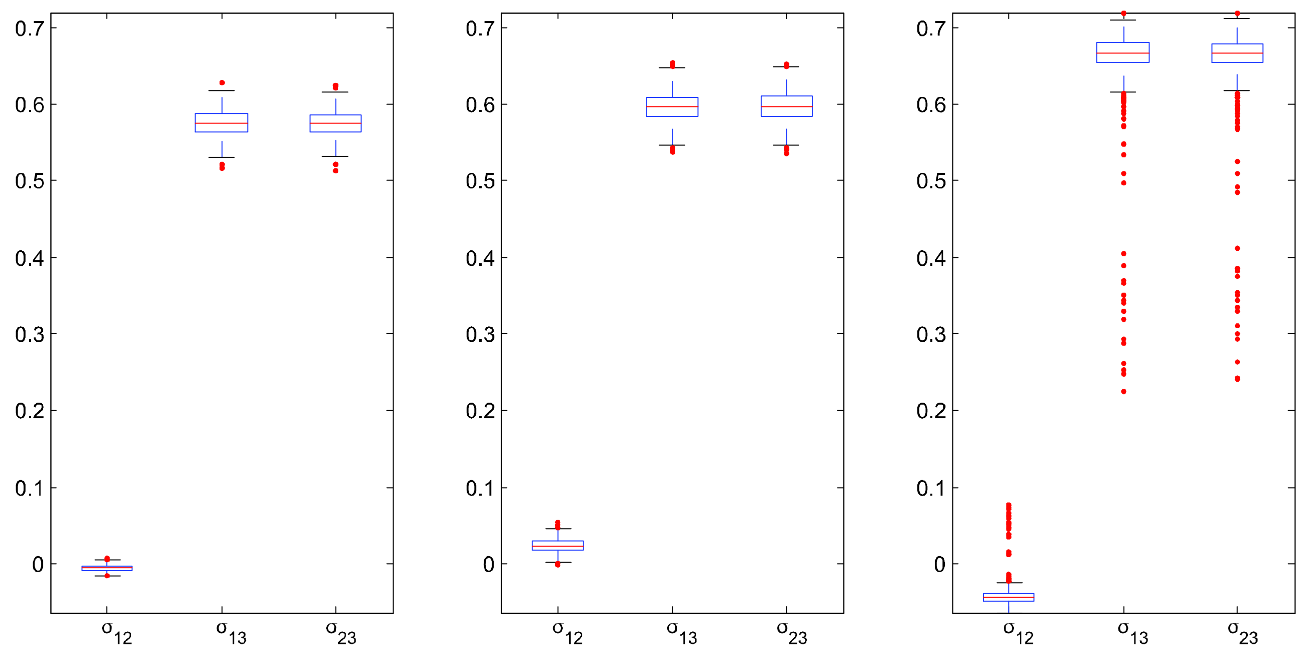

Figure 6: Boxplots of the distributions of estimates for case 8 with the unconstrained model (left), local-curvature-imposed model (middle), and global-curvature-imposed model (right).

Chapter 4

Estimation of Expenditure-Dependent Equivalence Scales and the Cost of a Child Using Japanese Household Expenditure Data

4.1 Introduction

This empirical section demonstrates the use of rank three and rank four consumer demand models which can characterize more complicated Engel curve shapes, and it investigates household consumption behaviors using household expenditure survey data, especially for attempts to estimate the cost of a child using Japanese household expenditure data. An Engel curve is the function describing how consumer's expenditures on some goods or services relate to her total expenditure while keeping prices fixed. The rank of a demand system can be defined as the dimension of the space spanned by its Engel curves. A famous result by Gorman (1981) tells that exactly aggregable demand systems have at most rank three or less. Lewbel (1991) extended Gorman's result and defines the rank M of any demand system to be the maximum dimension of the function space spanned by the Engel curves of the demand system.

We employ a rank three demand system of QUAIDS (Bank, Blundell and Lewbel, 1997) and a rank four demand system of translated QUAIDS (Lewbel 2003; Donaldson and Pendakur 2006) in this chapter for empirical parametric specifications of demand models. Given the large set of household expenditure data within which observations of total expenditure vary substantially, higher rank demand systems are necessary tools to account for the observed expenditure patterns at different income levels than the lower rank, say, rank two demand systems such as models of the PIGLOG type which are linear in logarithm of total expenditure.[44] For analyses involving only a modest range of income levels, such richness in global Engel curve shapes may be unnecessary, just as local flexibility (in Diewert's sense) may adequately capture a full range of price effects when prices do not change much over a sample. However, income levels usually vary widely across households, and analysis of welfare or cost-of-living involves comparisons of demands between these income levels. So, in studies involving cross-sectional data, cost-of-living analyses, or welfare comparisons, richness in Engel curve shapes across a wide range of income level is required. The results of nonparametric kernel regressions of Engel curves for UK households (Bank, Blundell and Lewbel, 1997) showed that the linear relationship is not sufficient to characterize the shape of the Engel curves on some goods which exhibit nonlinear shapes of Engel curves. The nonparametric rank test (Gill and Lewbel 1992; Gragg and Donald, 1997) also suggested that a rank three relationship is required, as would be the case in the second-order polynomial. Blundell, Chen and Kristensen (2007) found that some goods have Engel curves that are close to linear or quadratic, while others are more S-shaped.

There are a number of studies which concern the rank of demand systems, and a number of tests of rank have been conducted either parametrically, semiparametrically or nonparametrically (Grodal and Hildenbrand

[44] The AIDS model, which has rank two, performed poorly in the Monte Carlo study of Barnett and Seck (2007) using several linear approximation methods.

1992; Hildenbrand 1994; Kneip 1994; Lewbel 1991; Donald 1997; Hausman, Newey and Powell 1995; and Lyssiotou, Pashardes and Stengos 1999). Nicol (2001) finds that conditional demands for some demographic groups are rank three, which implies that unconditional demands for those groups are likely to be rank four. Lyssiotou, Pashardes and Stengos (2008) suggest the possibility of higher rank than three.[45]

Therefore, contemporary studies on household demand behaviors are expected to adopt at least rank three demand systems if no (nonparametric) rank test is carried out or no knowledge about the shape of its Engel curves is available a priori. The rank three parametric demand system used almost exclusively is the quadratic AIDS model. It nests a classic AIDS model, and hence, provides a simple way to test against the specification of the additional quadratic term. The linearization method for QUAIDS, as for AIDS, is also made available by Matsuda (2006) to help ease the complicated nonlinear estimation.

Lewbel (1991) showed that a class of deflated income demand systems which has Engel functions as being a function of "deflated" income which is the total expenditure divided by some homogeneous of degree one function of prices, can have rank up to four. In fact, the classic AIDS and QUAIDS are members of this class, as can be seen by their form.[46] Lewbel (2003) and Donaldson and Pendakur (2006) specified the four price functions of a rank four deflated income demand system and estimated the household demands using the UK Family expenditure survey and the Canadian Family

[45] See Lewbel (2002).
[46] See section 2.2.5.8.

Expenditure Surveys, respectively. The model replaces x in QUAIDS with $x - d(\boldsymbol{p})$, where $d(\boldsymbol{p})$ is a "translation" function and is homogeneous of degree one in prices to maintain the homogeneity of the original QUAIDS. This method of introducing translation or overhead term into demand systems is well known (see Samuelson, 1948; and Gorman, 1976), and typically has the effect of increasing the rank of the demand system. The system is called "translated" QUAIDS by Donaldson and Pendakur (2006). The rank four model allows one to test for the specification of rank three.

Having specified the model of demand functions, one needs to consider how to incorporate the demographic characteristics into the individual consumer demand model to represent different types of households. Lewbel (1985) provided a general approach to do this by modifying the expenditure function through two modifying functions. This method leads to the representation that the welfare level (or utility) of the households with some characteristics is equal to the utility level of a reference household with income and prices adjusted through these modifying functions. We show that the particular specifications of the demographic modification give rise to the equivalent-expenditure functions which are used to conduct the welfare analysis. The equivalent-expenditure function depends on prices and household characteristics, and is that expenditure which, if enjoyed by a reference household facing the same prices, would result in a utility level equal to that of each household member. The equivalence scales are the ratio of expenditure to equivalent expenditure (relative) or the difference between expenditure and equivalent expenditure (absolute).

We limit our attention to the identifiable equivalent-expenditure functions (and equivalence scales) by demand data alone. Equivalent-expenditure functions and equivalence scales are identifiable by demand data if there is a one-to-one correspondence between the preference ordering and the observed demands which it generates.[47] This identification requires either strong, untestable assumptions regarding preferences or unusual types of data. Our identification focuses on imposing as little restriction as possible on the household preference structure.

Lewbel (1989) and Blackorby and Donaldson (1993) consider the case where the equivalence scale function is independent of utility, which they call "independence of base" (IB) and "equivalence-scale exactness (ESE), respectively.[48] The ESE property assumes that household expenditure functions across families with different demographic compositions are proportional with respect to reference expenditure, hence equivalence scales are, a priori, independent of reference income. Blackorby and Donaldson (1993) showed that the exact equivalence scales are identified by observed demand data if the preference of the reference household is not PIGLOG (not log-linear in utility). This is an unfortunate result because PIGLOG specification includes the popular AIDS model of Deaton and Muellbauer and exactly aggregable translog of Jorgenson, Lau and Stoker (1982). Despite the identifiability and convenience of econometric approaches, exact equivalence scales may not hold in practice, due to the restrictive form of household preferences that they impose, as can be found in the later section. Tests of the

[47] For a detailed explanation of the identifiablility of equivalence scales, see Pollak and Wales (1979), Blundell and Lewbel (1991), Lewbel and Pendakur (2006). A more technical presentation is given by Blackorby and Donaldson (1991, 1993), Lewbel (1989d).

[48] We use the term of Blackorby and Donaldson in this dissertation.

exactness of equivalence scales have produced mixed results.[49]

Exact equivalence scales have a critical disadvantage for policy purposes as well as for economic interpretation. Conniffe (1992) argues that equivalence scales are not constant as regards income and that the current practice of assuming the exactness of equivalence scales should be reconsidered. Koulovatianos, Schröder and Schmidt (2005) finds strong evidence from research surveys conducted in Germany and France that equivalence scales are significantly decreasing in reference income, and strongly encourage the use of parametric or semi-parametric demand systems that can produce equivalence scales which are decreasing in reference income. Yet scales used in comparative and policy-related welfare studies have almost always used the exactness property of equivalence scales. The exact equivalence scales suggest that a rich household with a child has higher equivalent expenditures than a poor household with a child since exact equivalent expenditures are proportional to the reference income of that particular household without a child. No researcher or policy-maker seriously suggests that within a particular society higher income groups should be deliberately granted greater monetary compensation for having children (through child allowances, or whatever) than lower income groups. In the formulation of policy in the areas of welfare benefits and tax allowances, the assumption of exact equivalence scales could have serious and unacceptable implications.

[49] See Blundell and Lewbel, 1991, Dickens, Fry and Pashardes 1993, Pashardes 1995, Blundell, Duncan and Pendakur 1998, Gozalo 1997, Pendakur 1999, Koulovatianos, Schroder and Schmidt 2005. Semi- or non-parametric specification of demand systems tends to satisfy the equivalence-scale exactness, suggesting that the validity of the ESE condition may be attributed to the parametric specifications of the demand system. Also note that it is possible to statistically reject ESE, but if the the demand restrictions implied by ESE are not rejected, it remains impossible to infer that ESE actually holds.

An approach to relax the exact equivalence scales is suggested by Donaldson and Pendakur (2004). In particular, they introduce a property named "Generalized Relative Equivalence Scale Exactness "(GRESE) which implies a linear relationship between the log of equivalence scales and the log of reference incomes. Therefore, if the slope of this linear relationship is negative with a positive intercept term, then the equivalence scales are consistent with the recommendation of Koulovatianos, Schröder and Schmidt (2005). They show that if GRESE is a maintained hypothesis and the reference expenditure function is not PIGLOG, the equivalent-expenditure function can be identified from demand behavior. Since GRESE nests ESE, the estimation of demand systems allows for an easy specification test for a more appropriate equivalence scale structure.

Donaldson and Pendakur (2006) also relax the exact "absolute" scales and name the property "Generalized Absolute Equivalence Scale Exactness (GAESE) which implies a linear relationship between the equivalence scales and the reference incomes. GAESE allows for formulation of the equivalent-expenditure functions with a fixed component and a variable component that is proportional to the reference income.

We use the Japanese Panel Survey on Consumers (JPSC) conducted and administered by the Institute for the Research on Household Economics for household expenditures and characteristics data. Estimation of child cost in Japan itself is an interesting subject in the context of the super-low fertility rate of Japan. In addition, discussion of a child subsidy program became one of the biggest political hooking points during the recent general elections, which took place in the summer of 2009. Despite the evidence documented by Koulovatianos, Schröder and Schmidt (2005), it may not be evident that the equivalence scales are decreasing in the income given seemingly generous existing child-suppot welfare programs because the universally available medical insurance in Japan may lesson the out-of-pocket cost of additional children compared to the U.S., where a large number of households do not have any medical insurance (Phipps and Garner, 1994). The discussion on the low fertility trend in Japan and the ongoing discussion of child-support programs is placed in the next section.

We have encountered the existence of a mass of zero-expenditure observations for some of the goods in our data set. With the presence of a large pile of budge share observations at zero, the usual estimation procedure of appending a normal error vector to the demand system to solve the nonlinear simultaneous system of equations becomes difficult to apply. It makes the distribution of error terms hardly look normal, and the resulting parameter estimates become biased. There are several methods to deal with the zero expenditure issue, depending on what kind of underlying causes are expected to be generating zero expenditures. We assume that the underlying cause is the infrequent purchase of goods (IFP). [50] Although the commodities are actually consumed, purchases may not be recorded because the purchase interval is longer than the survey period. We believe that this is an appropriate assumption in wealthy countries like Japan, as opposed to developing countries where many households do not purchase certain goods because they are too expensive. We adopted the Amemiya-Tobit approach, specifically, a variant of Heckman's (1979) sample selection model in which

[50] See Deaton and Irish (1982).

the consumer decides to purchase a good based on some exogenous variables other than prices and income, and then allocates her budget to decide how much to purchase. We estimated the model using a two-step procedure proposed by Shonkwiler and Yen (1999) that consists of the first step of the Probit estimations of each equation, which determines the purchase decision, and the second step of estimating the augmented demand system, based on the parameter estimates obtained from the first step.

We find that the GRESE specification with the QUAIDS preferences of the households is statistically preferred to any of the more restricted specifications and that the equivalence scales are negatively correlated with the total expenditure levels. The results are appealing since the equivalence scales are identified by rejecting AIDS specification which is PIGLOG, and hence, there is no ambiguity with the result in that sense. Moreover, the implication is that the equivalence scales increase (decrease) as the total expenditures decrease (increase), which is consistent with the findings of Koulovatianos, Schröder and Schmidt (2005) and Conniffe (1992). It gives the straightforward policy design implication that poor households should be more compensated than rich households when they have additional children if a child-support policy is pursued by the Japanese government at all.

This chapter is organized as follows: Section 2 presents the current discussion on the low birthrate trend in Japan and the public policies that are designed to tackle this issue. Section 3 shows how to construct the demographically modified expenditure function using Lewbel's (1985) procedure. Section 4 shows the definition of equivalence scales and ESE property, and it presents two types of characterizations of Donaldson and Pendakur's equivalent-expenditure function. Section 5 describes our empirical parametric model specification including the construction of our datasets. Section 6 describes the estimation procedure with emphasis on how to deal with a piled-up of expenditure observations at zero in the data. Section 7 presents the results, and the conclusion follows.

4.2 Discussion on Low Fertility Rate, Relevant Public Policies, and Empirical Equivalent Scales in Japan

Japan has been recording very low fertility rates over the years, 1.27 as of 2009.[51] Although South Korea now has the lowest fertility rate, 1.22, the low fertility rate is more problematic for Japan in view of the fact that Japan is the "oldest"country meaning that the median age of the whole population is the highest among all countries (44.4 years old). The dependency ratio, an age-population ratio of those typically not in the labor force (the dependent part) and those typically in the labor force (the productive part) has increased to 22.7 in 2009 whereas it was 4.9 in 1950. Therefore, this prolonged low birthrate level creates an unbalanced demographic composition between productive and dependent populations. This trend may result in a greater burden per person regarding social security and have a negative effect on Japan's long-term economic performance.

There are several government programs that have existed and have been expanded in response to the declining trend of birthrate. The child subsidy program (Jidouteate) is definitely one of the most important of the policies designed to help families with child-rearing expense. Japanese households

[51] See Date and Shimizutani (2007) for the up-to-date survey paper on the cause of the declining fertility rate of Japan written in English.

currently (as of 2009) receive 5,000 or 10,000 yen per child with the cutoff amount dependent upon household income level and the age of the children. Beginning in 1993, the amount of allowance which applied to children under three years of age was 5,000 yen each for a first and second child, and 10,000 yen for each additional child. In 2000, the age cutoff increased to 6 years of age (or before attending elementary school age), and the amount for all children under 3 years of age increased to 10,000 yen in 2007, even for a first or second child. After April 2006, the cutoff income level for national health insurance holders, for instance, starts at 4,600,000 yen and increases by 380,000 yen each as the number of dependent household members increases. In October 2006, to offset medical expenses incurred from childbirth, the one-time government allowance paid to families with newborn children was increased from 300,000 yen to 350,000 yen, and to 420,000 yen in October 2009. Tanaka and Kouno (2009) found that an increase of 100,000 yen in the one-time government childbirth allowance raises the fertility rate by 0.017. A tax credit for dependent family members also helps decrease the cost of raising children. An income tax credit of 380,000 and a residential tax credit of 330,000 yen per dependant are given as of 2009. The dramatic increase in the amount of child support allowance proposed as a new child subsidy program by DPJ (Minshuto) was one of the biggest issues debated during the general election campaign in Japan prior to the the election held on August 30, 2009. The party currently holding the cabinet office is advocating the allowance amount of 26,000 yen per child younger than 15 year of age (or junior highschoolers) without an income threshold. The issue of no income threshold has been controversial in the current policy debate.

Despite the seemingly generous monetary compensation offered by the government welfare program to compensate for child-birth and child-rearing expenses, when couples are asked the reason why their planned number of children is lower than their preferred number of children, the majority reports that cost is the major issue. According to the latest Thirteenth Basic Survey on Birth Trend, 65.9% of couples report that "child-rearing and education costs are too high."[52] Therefore, it is of particular interest to see how the obtained equivalence scales are compared to scales found in other studies using different data sets and to see if the result of the Basic Survey on Birth Trend is consistent with the scales obtained from the estimation results. The papers that actually calculated the equivalence scales include Nagase (2001), Suruga (1993, 1995), and Suruga and Nishimoto (2001). They all used Engel or Rothbarth methods which impose very restricted assumptions on preferences and equivalence scales.[53] Oyama (2006) used psychometric data in the form of an Income Evaluation Question to estimate the equivalence scales.[54] She found that the cost of a child aged 0-18 years is between 1.386 and 1.475; between 1.280 and 1.454 for a child aged 0-6; between 1.277 to 1.407 for the age range 7-13 years; and between 2.090 to 4.329 for the age range 14-18. Recently, Hasegawa, Ueda and Mori (2008) undertook estimation of equivalence scales with the system of Working-Lesser equations

[52] The source for this information is the National Institute of Population and Social Security Research, Dai 12 kai shussei doukou kihon chousa [Thirteen Basic Survey on Birth Trends] (www.ipss.go.jp).

[53] The Rothbarth method using the consumption pattern of adult goods requires separability assumption between the parents' and the children's consumption. See Gronau (1988) and Nelson (1992).

[54] See Van Praag and Warnaar (2003) Van Praag and Kapteyn (1973), and Van Praag and Van der Sar, (1989).

using the same data source as ours. They produced an equivalence scale of 1.3695.

Other studies related to this topic include Matsuura (2007), who finds that the marginal utility of having children is higher for males in the context of the hypothesis that education cost is a major cause of reluctance to have more children. Aoki and Konishi (2009a, 2009b) show that improved quality of consumption has a negative impact on the birthrate in relation to female labor supply. Abe (2005) studied the effects of child-related benefits and found that these benefits contribute little to the reduction of poverty in households with children. Yamaguchi (2005a, 2005b, 2006) provides a sociological view of how the society overall can help raise the fertility rate.

4.3 Demographically Modified Demand System

Lewbel (1985) proposed a unified approach to incorporate demographic effects into any demand systems. His method generalizes the existing approaches of demographic translating, demographic scaling (Barten scale), and the modified Prais-Houthakker procedure (Pollak and Wales 1981, 1992). It offers a large class of possible modifications that have both the universal applicability of demographic scaling and translating, and the flexibility to allow demographic variables to interact with price and expenditure terms in the demand system in an almost unlimited variety of ways.

Lewbel's procedure modifies the expenditure function by first replacing each price by a function that depends on all prices and demographic variables and then subjecting the resulting expenditure function to a further transformation that depends on all price and demographic variables. He gives theorems to guarantee the integrability of the modified system, regardless of to what initial system these two transformations are applied. Using his theorems, we can derive characteristics of the GRESE equivalent-expenditure function identical to the ones derived by construction in Donaldson and Pendakur (2004).

We show how the modified demand system may be written directly as a function of the original system, so the effect of modifying functions on the demand equations may be directly assessed, without consideration of the cost or utility functions involved. Donaldson and Pendakur (2004) show how the demand function for the child good is restricted under ESE property by writing the modified demand function as a function of the reference household demand function.

Let E^* be a legitimate expenditure function, $\boldsymbol{p}$ be an n-vector of prices, and $\boldsymbol{z}$ be any vector of demographic variables. Lewbel (1985) considers a new expenditure function by the general transformation given by

$$E(u, \boldsymbol{p}, \boldsymbol{z}) = f\left[E^*\left(u, \boldsymbol{h}\left[\boldsymbol{p}, \boldsymbol{z}\right]\right), \boldsymbol{p}, \boldsymbol{z}\right]. \tag{101}$$

The properties of two modifying functions f, and $\boldsymbol{h}$ determine the types of transformation.

Considering the case where $\boldsymbol{p} = \boldsymbol{h}$, and f is a positive log-affine transformation, the new expenditure function (101) is written as

$$\ln E(u, \boldsymbol{p}, \boldsymbol{z}) = K(\boldsymbol{p}, \boldsymbol{z}) \ln E^*(u, \boldsymbol{p}) + \ln G(\boldsymbol{p}, \boldsymbol{z}), \tag{102}$$

where $K(\boldsymbol{p},\boldsymbol{z}) > 0$ and $G(\boldsymbol{p},\boldsymbol{z}) > 0$.

The integrability of the new expenditure function in (102) requires E to be homogeneous of degree one and concave. Symmetry and monotonicity are already satisfied by construction.

Assuming that E is differentiable in $\boldsymbol{p}$, we get by differentiation of (102) with respect to p_j,

$$\begin{aligned} \frac{\partial \ln E(u,\boldsymbol{p},\boldsymbol{z})}{\partial p_j} &= \frac{\partial K(\boldsymbol{p},\boldsymbol{z})}{\partial p_j} \ln E^*(u,\boldsymbol{p}) \\ &\quad + K(\boldsymbol{p},\boldsymbol{z}) \frac{\partial \ln E^*(u,\boldsymbol{p})}{\partial p_j} + \frac{\partial \ln G(\boldsymbol{p},\boldsymbol{z})}{\partial p_j}. \end{aligned} \tag{103}$$

Multiplying both sides of (103) by p_j and summing over j to n gives

$$\sum_{j=1}^{n} \frac{\partial \ln E}{\partial p_j} p_j = \ln E^* \sum_{j=1}^{n} \frac{\partial K(\boldsymbol{p},\boldsymbol{z})}{\partial p_j} p_j + K(\boldsymbol{p},\boldsymbol{z}) \sum_{j=1}^{n} \frac{\partial \ln E^*}{\partial p_j} p_j + \sum_{j=1}^{n} \frac{\partial \ln G(\boldsymbol{p},\boldsymbol{z})}{\partial p_j} p_j,$$

and then

$$1 = \ln E^* \sum_{j=1}^{n} \frac{\partial K(\boldsymbol{p},\boldsymbol{z})}{\partial p_j} p_j + K(\boldsymbol{p},\boldsymbol{z}) + \sum_{j=1}^{n} \frac{\partial \ln G(\boldsymbol{p},\boldsymbol{z})}{\partial p_j} p_j, \tag{104}$$

after applying Euler's theorem to the linear homogeneous expenditure function E^*.[55]

[55] If a function g is homogeneous of degree ρ,

$$\sum_j \frac{\partial g(\boldsymbol{\omega})}{\partial \omega_j} \omega_j = \rho g(\boldsymbol{\omega}),$$

which implies

$$\sum_j \frac{\partial g(\boldsymbol{\omega})}{\partial \omega_j} \frac{\omega_j}{g(\boldsymbol{\omega})} = \sum_j \frac{\partial \ln g(\boldsymbol{\omega})}{\partial \ln \omega_j} \omega_j = \rho.$$

The left-hand side of equation (104) is constant while the right-hand side is unbounded above in u through $E^*(u, \boldsymbol{p})$ which is assumed to be an increasing function of u. It implies that $\sum_j p_j \partial K(\boldsymbol{p}, \boldsymbol{z})/\partial p_j = 0$, and therefore K is homogeneous of degree zero, and

$$K(\boldsymbol{p}, \boldsymbol{z}) = 1 - \sum_{j=1}^{n} \frac{\partial \ln G(\boldsymbol{p}, \boldsymbol{z})}{\partial p_j} p_j = 1 - \sum_{j=1}^{n} \frac{\partial \ln G(\boldsymbol{p}, \boldsymbol{z})}{\partial \ln p_j}. \tag{105}$$

This means that K is functionally dependent on G: there is unique K for every G. The functional relationship between K and G in equation (105) is also derived in Donaldson and Pendakur (2004), and one can prove that this condition is sufficient for E to be homogeneous of degree one in $\boldsymbol{p}$.

Let $x^e = E^*(u, \boldsymbol{p})$, $x = E(u, \boldsymbol{p}, \boldsymbol{z})$, and $V^*(\boldsymbol{p}, x^e)$ be the corresponding indirect utility function of total expenditure x^e and prices $\boldsymbol{p}$, and let $V(\boldsymbol{p}, x, \boldsymbol{z})$ be the corresponding indirect utility function of total expenditure x and prices $\boldsymbol{p}$. Lewbel's (1985) theorem 4 says

$$V(\boldsymbol{p}, x, \boldsymbol{z}) = V^*(\boldsymbol{p}, x^e). \tag{106}$$

This representation implies that the utility level of a household with characteristics $\boldsymbol{z}$ is equal to that of the reference household with adjusted total expenditure x^e, and it gives a basis for the construction of the estimable demand functions by specifying the demand functions of the type of reference households. The representation in (106) can be further written as

$$\begin{aligned} V(\boldsymbol{p}, \ln x, \boldsymbol{z}) &= V^*\left(\boldsymbol{p}, \frac{\ln x - \ln G(\boldsymbol{p}, \boldsymbol{z})}{K(\boldsymbol{p}, \boldsymbol{z})}\right) \\ &= V^*\left(\boldsymbol{p}, \ln x - \frac{(K-1)\ln x - \ln G}{K}\right). \end{aligned}$$

Therefore, $\exp\{[(K-1)\ln x - \ln G]/K\}$ can be viewed as scaling down the expenditure of the original indirect utility function to preserve the equality of utility V and V^*, and in fact, $\exp\{[(K-1)\ln x - \ln G]/K\}$ is taken as an expenditure-dependent equivalence scale in a later section. It increases (decreases) in $\ln x$ if $K > 1$ $(K < 1)$.

We give a sketch of proof to derive the representation (106) . The indirect utility function V is the inverse of E, so $x = E[V(\boldsymbol{p}, x, \boldsymbol{z}), \boldsymbol{p}, \boldsymbol{z}]$. Using equation (101) as the definition of E, after specifying $\boldsymbol{h} = \boldsymbol{p}$, the equation for x becomes

$$x = f\left[E^*(V[\boldsymbol{p}, x, \boldsymbol{z}], \boldsymbol{p}), \boldsymbol{p}, \boldsymbol{z}\right]. \tag{107}$$

Defining the function X as the inverse of f on its first argument such that:

$$x^e = X(x, \boldsymbol{p}, \boldsymbol{z}),$$

which is well-defined since f is monotonic in its first argument as being a positive log-affine transformation, then

$$V^*(\boldsymbol{p}, x^e) = V^*\left[\boldsymbol{p}, X(x, \boldsymbol{p}, \boldsymbol{z})\right].$$

Substituting equation (107) into this equation yields

$$\begin{aligned} V^*(\boldsymbol{p}, x^e) &= V^*\left[\boldsymbol{p}, X(x, \boldsymbol{p}, \boldsymbol{z})\right] \\ &= V^*\left[\boldsymbol{p}, X(f\left[E^*(V[\boldsymbol{p}, x, \boldsymbol{z}], \boldsymbol{p}), \boldsymbol{p}, \boldsymbol{z}\right], \boldsymbol{p}, \boldsymbol{z})\right] \\ &= V^*\left[\boldsymbol{p}, E^*(V[\boldsymbol{p}, x, \boldsymbol{z}], \boldsymbol{p}), \boldsymbol{z})\right] \qquad \text{after cancelling out } f \text{ of } X \\ &= V[\boldsymbol{p}, x, \boldsymbol{z}]. \qquad \text{after canceling out } E^* \text{ of } V^* \end{aligned}$$

Lewbel used the theorem described in section 2.1.5 to derive the conditions under which the new expenditure function is concave in $\boldsymbol{p}$ given a legitimate expenditure function E^*. According to theorem 3 of Lewbel (1985), the new expenditure function in (102) is concave in $\boldsymbol{p}$ if the following conditions are satisfied:

$$K(K-1)GE^{*(K-2)}\xi_j^2 \leq 0, \tag{108}$$

for $j = 1, ..., n$, and

$$\frac{1}{n}\left[\frac{\partial f(E^*, \boldsymbol{p}, \boldsymbol{z})}{\partial p_i \partial p_j}\xi_i\xi_j\right] \leq 0, \tag{109}$$

for arbitrary n-vector $\boldsymbol{\xi}$.[56] While the condition in (109) turns out to be untractable after calculating the second derivative, the condition in (108) implies $K \leq 1$. Therefore, global concavity of the modified expenditure function implicitly assumes that the corresponding equivalence scales are decreasing functions of total expenditure. Although the household expenditure function does not technically have to be concave, the characteristics resulting from this particular transformation (which is actually GRESE) implies the still restrictive nature of GRESE imposed on the household preferences.

[56] See the complete theorem in Lewbel (1985) for the details.

A similar exercise can be done with the transformation associated with GAESE with:

$$E(u, \boldsymbol{p}, \boldsymbol{z}) = R(\boldsymbol{p}, \boldsymbol{z})E^*(u, \boldsymbol{p}) + A(\boldsymbol{p}, \boldsymbol{z}).$$

The next section presents the technical definition of equivalence scales and shows that GRESE and GAESE are generalizations of ESE from the viewpoint that they are equivalence scales.

4.4 Equivalence scales

To define equivalent-expenditure functions and equivalence scales, a reference household type is needed and, although other choices are possible, we use a childless married couple as the reference and denote its characteristics as $\boldsymbol{z}^r$. An relative equivalence scale S_R is defined by

$$v = V(\boldsymbol{p}, x, \boldsymbol{z}) = V(\boldsymbol{p}, x/s_R, \boldsymbol{z}^r) = V^r(\boldsymbol{p}, x/s_R), \tag{110}$$

where $V^r = V(\cdot, \cdot, \boldsymbol{z}^r)$ and s_R is a value of the scale. Remembering the representation in (106), equivalence-expenditure x^e can be written as the ratio of household expenditure to the scale, x/s_R. Let $x^e = X(\boldsymbol{p}, x, \boldsymbol{z})$ $(= E^r\,(V^r(\boldsymbol{p}, x^e), \boldsymbol{z}^r))$, where $E^r(u, \boldsymbol{p}) = E(u, \boldsymbol{p}, \boldsymbol{z}^r)$ and $V^r(\boldsymbol{p}, x) = V(\boldsymbol{p}, x, \boldsymbol{z}^r)$. Consequently, an equivalence scale is the ratio of household expenditure to equivalence expenditure and we can write

$$s_R = S_R(\boldsymbol{p}, x, \boldsymbol{z}) = \frac{x}{X(\boldsymbol{p}, x, \boldsymbol{z})}.$$

Because X is homogeneous of degree one in $(\boldsymbol{p}, x)$, S_R is homogeneous of degree of zero in $(\boldsymbol{p}, x)$. In addition, $S_R(\boldsymbol{p}, x, \boldsymbol{z}^r) = 1$ for all $(\boldsymbol{p}, x)$, which is intuitively consistent. The equation (110) can be solved for s_R using the expenditure function to obtain

$$s_R = \frac{E(u, \boldsymbol{p}, \boldsymbol{z})}{E^r(u, \boldsymbol{p})}.$$

In general, equivalence scales depend on expenditure, but those that do not depend on expenditure are called exact. A scale is exact such that:

$$S_R(\boldsymbol{p}, x, \boldsymbol{z}) = \overline{S}_R(\boldsymbol{p}, \boldsymbol{z}),$$

if and only if the expenditure function is multiplicatively decomposable (Blackorby and Donaldson, 1991, 1993; Lewbel, 1991), with the condition

$$E(u, \boldsymbol{p}, \boldsymbol{z}) = \overline{S}_R(\boldsymbol{p}, \boldsymbol{z}) E^r(u, \boldsymbol{p}), \tag{111}$$

which we call equivalence-scale exactness (ESE).

Absolute equivalence scales measure the amount of income over reference-household income needed by a particular type of household to preserve equal utility. It is implicitly defined by

$$v = V(\boldsymbol{p}, x, \boldsymbol{z}) = V^r(\boldsymbol{p}, x - s_A), \tag{112}$$

and is written

$$s_A = S_A(\boldsymbol{p}, x, \boldsymbol{z}) = x - X(\boldsymbol{p}, x, \boldsymbol{z}),$$

where s_A is a value of the scale, and $S_A(\boldsymbol{p}, x, \boldsymbol{z}^r) = 0$ for all $(\boldsymbol{p}, x)$. Because X is homogeneous of degree one in $(\boldsymbol{p}, x)$, S_A is homogeneous of degree one as well. Equation (112) implies that

$$s_A = E(u, \boldsymbol{p}, \boldsymbol{z}) - E^r(u, \boldsymbol{p}). \tag{113}$$

The absolute equivalence scale S_A is income dependent or, in the formulation of (113), utility dependent. The scale is independent of these variables such that:

$$S_A(\boldsymbol{p}, x, \boldsymbol{z}) = \overline{S}_A(\boldsymbol{p}, \boldsymbol{z}),$$

and, therefore, exact if and only if the expenditure function is additively decomposable (Blackorby and Donaldson, 1994) such that:

$$E(u, \boldsymbol{p}, \boldsymbol{z}) = E^r(u, \boldsymbol{p}) + \overline{S}_A(\boldsymbol{p}, \boldsymbol{z}). \tag{114}$$

We call this condition on E, Absolute Equivalence-Scale Exactness (AESE). The absolute scale $\overline{S}_A(\boldsymbol{p}, \boldsymbol{z})$ can be interpreted as the fixed costs of characteristics faced by different types of households.

Without additional assumptions, neither equivalent-expenditure functions nor equivalence scales can be identified by household demand behavior alone. Blackorby and Donaldson (1993, 1994) investigated theoretical identification when ESE or AESE is accepted as a maintained hypothesis. They found that, given ESE and AESE with a technical condition, estimation from demand behavior is possible if and only if the reference expenditure function is not log-affine for ESE and affine for AESE, that is, if it does not satisfy

$$\ln E^r(u, \boldsymbol{p}) = C(\boldsymbol{p})g(u) + \ln D(\boldsymbol{p}) \tag{115}$$

and

$$E^r(u, \boldsymbol{p}) = \widetilde{C}(\boldsymbol{p})\widetilde{g}(u) + \widetilde{D}(\boldsymbol{p}) \tag{116}$$

where C, g, D, $\widetilde{C}$, $\widetilde{g}$, and $\widetilde{D}$ are some functions with appropriate properties to make equations (115) and (116) legitimate expenditure functions. The equation in (115) takes a form of PIGLOG given in (48) after some algebraic arrangement, and the equation in (115) takes a Gorman Polar Form in (58). In any case, demand systems with lower rank should not be applied when using those equivalence scale exactness properties.

4.4.1 Generalized Relative Equivalence-Scale Exactness

If equivalence-scale exactness (ESE) is satisfied, given equation (111), one can write

$$\frac{dx^e}{x^e} = \frac{dx}{x},$$

which implies that an equal percentage increase in income preserves utility equality across household types. ESE is equivalent to a condition on the way that interpersonal comparisons are related to expenditures called income-ratio comparability (IRC), (Blackorby and Donaldson, 1991, 1993). If a household with arbitrary characteristics and a reference household facing the same prices have expenditures such that their utilities are equal, common scaling of their expenditures (which leaves the expenditure ratio unchanged) preserves utility equality. Thus, an increase in a household's expenditure of one per cent matched by an increase in the reference household's expenditure of one per cent preserves equality of well-being.

Suppose that, instead of requiring an equal percentage increase in the reference household's income to preserve utility equality, we require the matching percentage increase to be independent of income only. This requires the existence of a function such that:

$$\frac{dx^e}{x^e} = \kappa(\boldsymbol{p}, \boldsymbol{z})\frac{dx}{x}, \tag{117}$$

where $\kappa > 0$ for all $(\boldsymbol{p}, x, \boldsymbol{z})$. A one percent increase in household income requires an increase of κ percent to preserve equality of well-being. Integrating both sides of (117) gives

$$\ln x^e = \ln E^r(\boldsymbol{p}, x) = \kappa(\boldsymbol{p}, \boldsymbol{z}) \ln x + \ln \gamma(\boldsymbol{p}, \boldsymbol{z}), \tag{118}$$

where γ is some function independent of x. Consequently,

$$x^e = X(\boldsymbol{p}, x, \boldsymbol{z}) = \gamma(\boldsymbol{p}, \boldsymbol{z})x^{\kappa(\boldsymbol{p}, \boldsymbol{z})}.$$

Because $\kappa > 0$ and X is increasing in x, $\gamma > 0$ for all $(\boldsymbol{p}, x)$. In addition, because $X(\boldsymbol{p}, x, \boldsymbol{z}^r) = x$ for all $(\boldsymbol{p}, x)$, $\gamma(\boldsymbol{p}, \boldsymbol{z}^r) = \kappa(\boldsymbol{p}, \boldsymbol{z}^r) = 1$ for all $\boldsymbol{p}$.

Defining $K(\boldsymbol{p}, \boldsymbol{z}) = 1/\kappa(\boldsymbol{p}, \boldsymbol{z})$ and $G(\boldsymbol{p}, \boldsymbol{z}) = 1/\gamma(\boldsymbol{p}, \boldsymbol{z})^{1/\kappa(\boldsymbol{p}, \boldsymbol{z})}$, the equation (118) can be written as

$$\ln x^e = \ln X(\boldsymbol{p}, x, \boldsymbol{z}) = \frac{\ln x - \ln G(\boldsymbol{p}, \boldsymbol{z})}{K(\boldsymbol{p}, \boldsymbol{z})}.$$

Because $V(\boldsymbol{p}, x, \boldsymbol{z}) = V^r(\boldsymbol{p}, x^e)$, it follows that

$$V(\boldsymbol{p}, x, \boldsymbol{z}) = V^r\left(\boldsymbol{p}, \exp\left\{\frac{\ln x - \ln G(\boldsymbol{p}, \boldsymbol{z})}{K(\boldsymbol{p}, \boldsymbol{z})}\right\}\right),$$

and the expenditure function E is given by

$$\ln E(u, \boldsymbol{p}, \boldsymbol{z}) = K(\boldsymbol{p}, \boldsymbol{z}) \ln E^r(u, \boldsymbol{p}) + \ln G(\boldsymbol{p}, \boldsymbol{z}). \tag{119}$$

Given GRESE, the equivalence scale S_R satisfies

$$\ln S_R(\boldsymbol{p}, x, \boldsymbol{z}) = \frac{(K(\boldsymbol{p}, \boldsymbol{z}) - 1) \ln x - \ln G(\boldsymbol{p}, \boldsymbol{z})}{K(\boldsymbol{p}, \boldsymbol{z})}. \tag{120}$$

S_R is increasing (decreasing) in x if $K > 1$ $(K < 1)$.

If $V(\cdot, \cdot, \boldsymbol{z})$ is differentiable for all $\boldsymbol{z}$ with $\partial V(\boldsymbol{p}, x, \boldsymbol{z})/\partial x > 0$ for all $(\boldsymbol{p}, x, \boldsymbol{z})$, by Roy's identity, given GRESE, the share equations w_i for $i = 1, ..., n$, satisfy

$$\begin{aligned} w_i(\boldsymbol{p}, \ln x, \boldsymbol{z}) \quad = \quad & K(\boldsymbol{p}, \boldsymbol{z}) w_i^r\left(\boldsymbol{p}, \frac{\ln x - \ln G(\boldsymbol{p}, \boldsymbol{z})}{K(\boldsymbol{p}, \boldsymbol{z})}\right) \\ & + \frac{\partial K(\boldsymbol{p}, \boldsymbol{z})}{\partial \ln p_i}\left(\frac{\ln x - \ln G(\boldsymbol{p}, \boldsymbol{z})}{K(\boldsymbol{p}, \boldsymbol{z})}\right) && (121) \\ & + \frac{\partial \ln G(\boldsymbol{p}, \boldsymbol{z})}{\partial \ln p_i}, && (122) \end{aligned}$$

where $w_i^r = w_i(\cdot, \cdot, \boldsymbol{z}^r)$ is the reference household's share equations. Given ESE, $K(\boldsymbol{p}, \boldsymbol{z}) = 1$ (because $\kappa = 1$) and the second term of (121) vanishes. Thus the restrictions on budget share equations implied by GRESE are a generalization of those implied by ESE.

Donaldson and Pendakur (2004) investigate the relationship of behavior and equivalent-expenditure functions when GRESE is satisfied and show that estimation of the scale from demand behavior is possible if and only if the reference expenditure function is not PIGLOG. Equivalence scales depend on utility, which cannot be directly observed, and therefore, they must be inferred from consumer demand data, that is, from the quantities that consumers buy in varying price regimes and at various income levels. Here below, we show why the identification of equivalence scales requires strong assumptions regarding preferences.

Two indirect utility functions, $\widehat{V}$ and $\widetilde{V}$, represent the same preferences for each household type if and only if there exists a function ψ, increasing in its first argument, such that

$$\widehat{V}(\boldsymbol{p}, x, \boldsymbol{z}) = \psi(\widetilde{V}(\boldsymbol{p}, x, \boldsymbol{z}), \boldsymbol{z}) \tag{123}$$

for all $(\boldsymbol{p}, x, \boldsymbol{z})$. One can check that Roy's identity gives the same demand functions from $\widehat{V}$ and $\widetilde{V}$. Define φ to be the inverse of ψ in its first argument. Let $\widehat{E}$ and $\widetilde{E}$ be the expenditure function associated with $\widehat{V}$and $\widetilde{V}$ respectively. Applying φ to both sides of (123) gives

$$\varphi\left(\widehat{V}, \boldsymbol{z}\right) = \widetilde{V}(\boldsymbol{p}, x, \boldsymbol{z}). \tag{124}$$

Then, inverting (124) by $\widetilde{E}$ gives

$$\widetilde{E}\left(\widetilde{V}, \boldsymbol{p}, \boldsymbol{z}\right) = \widetilde{E}\left(\varphi\left(\widehat{V}, \boldsymbol{z}\right), \boldsymbol{p}, \boldsymbol{z}\right) = x. \tag{125}$$

On the other hand, inverting (123) by $\widehat{E}$, we get

$$\widehat{E}\left(\widehat{V},\boldsymbol{p},\boldsymbol{z}\right)=x. \tag{126}$$

Combining (125) and (126) yields

$$\widehat{E}\left(\widehat{V},\boldsymbol{p},\boldsymbol{z}\right)=\widetilde{E}\left(\varphi\left(\widehat{V},\boldsymbol{z}\right),\boldsymbol{p},\boldsymbol{z}\right). \tag{127}$$

The representation (127) implies generally that $\widehat{E}\left(v,\boldsymbol{p},\boldsymbol{z}\right)\neq\widetilde{E}(v,\boldsymbol{p},\boldsymbol{z})$ while attaining a utility level $v=\widehat{V}$. By revealed preference theory, demand data identifies the shape and ranking of a consumer's indifference curves over bundles of goods, but not the actual utility level associated with each indifference curve. Changing $\varphi\left(v,\boldsymbol{z}\right)$ just changes the utility level associated with each indifference curve. Therefore, one could get two different equivalence scales through different expenditure functions $\widehat{E}$ and $\widetilde{E}$ from the same demand data.

If GRESE is satisfied, Theorems 2 and 3 of Donaldson and Pendakur (2004) together imply that: (1) if the reference expenditure function is PIGLOG, there are infinitely many log-affine equivalent-expenditure functions that are consistent with behavior; and (2) if the reference expenditure function is not PIGLOG, there are infinitely many equivalent-expenditure functions that are consistent with behavior but only one of them is log-affine. It follows that, in order to identify equivalent-expenditure functions and equivalence scales from behavior alone, the reference expenditure function must not be PIGLOG. If this condition is met, the functions K and G

are unique and can be estimated from behavior.

4.4.2 Generalized Absolute Equivalence-Scale Exactness

If Absolute Equivalence-Scale Exactness (AESE) is satisfied, equal absolute increases of income preserve utility equality across household types. Given (114), one can write

$$dx^e = dx,$$

which implies an increase in a household's income of one dollar matched by an increase of one dollar in the reference household's income preserves equality of well-being. AESE is equivalent to a condition on the way that interpersonal comparisons are related to incomes, called Income-Difference Comparability (Blackorby and Donaldson, 1994). If a household with arbitrary characteristics and a reference household facing the same prices have incomes such that their utilities are equal, common absolute increases in their incomes (which leave the income difference unchanged) preserve utility equality.

Instead, suppose that we require the weaker condition that the change in the reference household's income that preserves utility equality is independent of income. This requires the existence of a function ρ such that

$$dx^e = \rho(\boldsymbol{p}, \boldsymbol{z})dx. \tag{128}$$

A one dollar increase in household income requires an increase of $\rho(\boldsymbol{p}, \boldsymbol{z})$ dollars to preserve equality of well-being. Integrating both sides of (128) gives

$$x^e = X(\boldsymbol{p}, x, \boldsymbol{z}) = \rho(\boldsymbol{p}, \boldsymbol{z})x + \alpha(\boldsymbol{p}, \boldsymbol{z}), \tag{129}$$

for some function α that is independent of total expenditure. Because X is increasing in x and homogeneous of degree one in $(\boldsymbol{p}, x)$, $\rho(\boldsymbol{p}, \boldsymbol{z}) > 0$ for all $(\boldsymbol{p}, z)$, ρ is homogeneous of degree zero, and α is homogeneous of degree one in $\boldsymbol{p}$. In addition, because $X(\boldsymbol{p}, x, \boldsymbol{z}^r) = x$ for all $(\boldsymbol{p}, x)$, $\rho(\boldsymbol{p}, \boldsymbol{z}^r) = 1$ and $\alpha(\boldsymbol{p}, \boldsymbol{z}^r) = 0$ for all $\boldsymbol{p}$.

Defining $R(\boldsymbol{p}, \boldsymbol{z}) = 1/\rho(\boldsymbol{p}, \boldsymbol{z})$ and $A(\boldsymbol{p}, \boldsymbol{z}) = -\alpha(\boldsymbol{p}, \boldsymbol{z})/\rho(\boldsymbol{p}, \boldsymbol{z})$, the equation in (129) can be rewritten as

$$x^e = X(\boldsymbol{p}, x, \boldsymbol{z}) = \frac{x - A(\boldsymbol{p}, \boldsymbol{z})}{R(\boldsymbol{p}, \boldsymbol{z})}. \tag{130}$$

Because $V(\boldsymbol{p}, x, \boldsymbol{z}) = V^r(\boldsymbol{p}, x^e)$, the indirect utility function can be written as

$$V(\boldsymbol{p}, x, \boldsymbol{z}) = V^r\left(p, \frac{x - A(\boldsymbol{p}, \boldsymbol{z})}{R(\boldsymbol{p}, \boldsymbol{z})}\right), \tag{131}$$

and the expenditure function E is given by

$$E(u, \boldsymbol{p}, \boldsymbol{z}) = R(\boldsymbol{p}, \boldsymbol{z})E^r(u, \boldsymbol{p}) + A(\boldsymbol{p}, \boldsymbol{z}). \tag{132}$$

Because $\rho(\boldsymbol{p}, \boldsymbol{z}^r) = 1$ and $\alpha(\boldsymbol{p}, \boldsymbol{z}^r) = 0$, $R(\boldsymbol{p}, \boldsymbol{z}^r) = 1$ and $A(\boldsymbol{p}, \boldsymbol{z}^r) = 0$. We call the condition expressed in (129) to (132), Generalized Absolute Equivalence-Scale Exactness (GAESE). A and R are the absolute and relative components for the equivalent-expenditure function.

For GAESE (and AESE), the cost of characteristics can be considered as the cost of maintaining a household with characteristics $\boldsymbol{z}$ at a particular utility level less the cost of maintaining a reference household at the same utility level, given by $E(u,\boldsymbol{p},\boldsymbol{z})-E^r(u,\boldsymbol{p})$. Given GAESE, this becomes

$$E(u,\boldsymbol{p},\boldsymbol{z})-E^r(u,\boldsymbol{p})=\left(R(\boldsymbol{p},\boldsymbol{z})-1\right)E^r(u,\boldsymbol{p})+A(\boldsymbol{p},\boldsymbol{z})\,. \tag{133}$$

The term $A(\boldsymbol{p},\boldsymbol{z})$ is a fixed cost that is the same at all utility levels. Therefore, the equation in (133) has the convenient interpretation that the expenditure function consists of a fixed component that is independent of household expenditure and a component that is proportional to household expenditure.

Given GAESE, the absolute equivalence scale S_A is affine in income and is given by

$$S_A=\frac{(R(\boldsymbol{p},\boldsymbol{z})-1)x+A(\boldsymbol{p},\boldsymbol{z})}{R(\boldsymbol{p},\boldsymbol{z})}. \tag{134}$$

It is increasing (decreasing) in x if $R>1$ ($R<1$). Because $\lim_{x\to 0}S_A(\boldsymbol{p},x,\boldsymbol{z})=A(\boldsymbol{p},\boldsymbol{z})/R(\boldsymbol{p},\boldsymbol{z})$, AESE is approximately satisfied for small x.

If $V(\cdot,\cdot,\boldsymbol{z})$ is differentiable for all $\boldsymbol{z}$ with $\partial V(\boldsymbol{p},x,\boldsymbol{z})/\partial x>0$ for all $(\boldsymbol{p},x,\boldsymbol{z})$, by Roy's identity, given GAESE, the demand equations q_i for $i=1,...,n$, satisfy

$$\begin{aligned}q_i(\boldsymbol{p},x,\boldsymbol{z}) \quad = \quad & R(\boldsymbol{p},\boldsymbol{z})q_i^r\left(\boldsymbol{p},\frac{x-A(\boldsymbol{p},\boldsymbol{z})}{R(\boldsymbol{p},\boldsymbol{z})}\right) \\ & +\frac{\partial R(\boldsymbol{p},\boldsymbol{z})}{\partial p_i}\left(\frac{x-A(\boldsymbol{p},\boldsymbol{z})}{R(\boldsymbol{p},\boldsymbol{z})}\right)+\frac{\partial A(\boldsymbol{p},\boldsymbol{z})}{\partial p_i},\end{aligned} \tag{135}$$

where $q_i^r=q_i(\cdot,\cdot,\boldsymbol{z}^r)$ is the reference household's budget share equations.

4.5 Empirical Procedure

Parametric estimation of equivalent-expenditure functions requires specification of the demand system and parametric expressions of the restrictions embodying GRESE/GAESE. The econometric strategy we employ exploits the following convenient characteristic of GRESE/GAESE: given GRESE/GAESE, if any household type has demand functions that are quadratic in the natural logarithm of income, then all household types have commodity demands that are quadratic in the natural logarithm of income and if any household type has demand functions that are of the form of the translated QUAIDS, then all household types have commodity demands that are of the form of the translated QUAIDS. Therefore, the reference expenditure function can be maintained to be not PIGLOG or affine. The QUAIDS and the translated QUAIDS satisfy this requirement. The specific parameterization of each price function and household characteristic variables are illustrated in later subsections.

In order to obtain consistent parameter estimates, we use an Amemiya-Tobit type estimation method to take care of a large portion of zero expenditure observations on some goods. Among several estimation methods available, we employ the two-step estimation procedure of Shonkwiler and Yen (1999). The next subsection describes the construction of our data from three data sources.

4.5.1 Data

Our data sets are constructed from three sources: the Japanese Panel Survey on Consumers (JPSC) conducted and administered by the Institute for the Research on Household Economics, and the Consumer Price Index Survey and Family Expenditure Survey Index administered by the Statistics Bureau and the Director-General for Policy Planning (Statistical Standards) of Japan.

JPSC is panel data that tracks the personal lives of young women in Japan over a number of years. The survey consists of a number of questionnaires, topics ranged from family members and social activities to opinions on current issues and their personalities. The first survey was conducted in 1993 when a group of 1,500 women between the ages of 24 and 34 years was surveyed. On top of that group of women, labeled as cohort A, another 500 women between the ages of 24 and 27 were added in 1997 (cohort B). Another 836 women with ages between 24 and 29 were added in 2003 (cohort C). As of 2010, the survey is still continuing. It takes place in October of each year, and most of the information about household expenditures is collected for expenditures made in September. It was not until 1998 that the survey began collecting information on expenditures of subaggregated categories of goods. The following six categories are selected in the analysis: Food; Fuel, light and water charges; Furniture and household utensils; Clothes and footwear; Transportation; Communication.[57]

[57] All expenditure categories are Food; Housing; Fuel, light and water charges; Furniture and household utensils; Clothes and footwear; Medical care; Transportation; Communication; Education; Reading and recreation; Social activity; Allowance for family members; and Miscellaneous.

Housing expenditure may be an important expenditure category, but we decided not to include it in our analysis. Because Japanese housing accommodations usually are characterized as far less spacious and far more expensive than their U.S. counterparts, a shift of expenditures toward housing resulting from the need of growing children for more space, may have a big impact on the household's budgeting decisions. For renters, expenditure on housing is simply defined as expenditure on rent. For homeowners, the rent equivalent value of the flow of the service from an owner-occupied dwelling must be calculated. We could impute the rental-equivalent of housing for homeowners by following the procedure of Phipps (1998) because JPSC contains enough details about housing characteristics for homeowners. But, JPSC contains no information on the characteristics of rental housing other than building type, floor size, and number of rooms. Phipps obtained rental-equivalent housing expenditures by regressing the rental charge on several characteristics of the accommodation: age of the building, type of house (single, semidetached, row house or duplex), number of rooms and bathrooms, regional location, time period). They then produced the rental-equivalent expenditures as predicted values from the estimated model using variables of house of homeowners. The limited amount of information on characteristics of rental housing units in JPSC makes it difficult to trust the reliability of estimates obtained by this method. Another method of imputing rent for owner-occupied dwellings includes user cost approach. The user cost approach may be the ideal way to impute rental-equivalent cost, but it requires more details on family financial matters such as the costs of property taxes on house and land, and the costs of the monthly mortgage.

We tried to make use of a variety of information on housing and real estate for the years 1998 and 2000 complied in the Housing and Land Survey. But, we could not extract any useful information from this source. Despite our efforts to extract useful housing expenditure data for each household, the lack of information on the characteristics of rented dwellings in JPSC hinders the use of available methods to produce satisfactorily accurate values for imputed rental-equivalence.

We define each good as follows: Food includes expenditures on eating-out and children's school lunches as well as food in general. Utility includes expenditures for electricity, gas, water, and sewer charges. Furniture includes expenditures on durable goods such as electronic appliances and household utensils as well as bedding. Clothing includes expenditures on clothing and footwear. Transportation includes vehicle purchase expense, gasoline expense, and public transportation fees. Communication consists of postage fees, and charges for telephone and Internet usage. Survey respondents are made aware of these definitions.

We created variables for the number of children in the household classified by their age group: Youji represents children under mandatory school-age, usually less than 7 years old. Children who attend kindergarten are included in this variable. Shou represents children who attend elementary school, usually those between 7 and 12 years old. Chu represents children who attend junior high school, usually those between 13 and 15 years old. Kou represents children who attend high school, usually those between 16 and 18 years old. Since there are not enough observations of households with high schoolers, we combine Chu and Kou to create a variable Shonen for the analysis.

We estimate the child cost for typical married couples who are financially independent of their parents, supportng both themselves and their children. Based on this criteria, single women were eliminated from the sample because single-parent families headed by unmarried mothers are uncommon in Japan. We further eliminated samples with children older than high school age. We also eliminated those households in which all members are not cohabitating, or with other adult members (such as grandparents).

Households that consist of several generations of family members are able to have family members other than parents to take care of the children. It is considered as a traditional fom of family composition. Unfortunately, the sample selection criteria prevents us from estimating the effect of having other co-dwelling adult family members in the household. Having other members of the family live in the household, for example, those who are retired from market production is helpful to cover the opportunity cost associated with child-rearing. This is especially true in the case of infants because they require constant care from parents who would be likely to be more productive in the labor market than engaging in household production. It may be the case that households which include the children's grandparents tend to have more children on that basis. This discussion is undoubtedly related to the topic of female labor supply and fertility, another important field in economics by itself. The arguments, however, regarding child cost as one of the major reasons for the low birth-rate trend are usually centered on the aforementioned nuclear family households which, in 2005, comprised

57.9 percent of the total number of Japanese household units.[58] A public policy to encourage married couples to live with their parents does not seem appropriate as a political stance because of the extent to which it would involve the government in private family matters. Yamaguchi (2006) recommends making child-care institutions more readily available in the workplace itself to offset the opportunity cost associated with child rearing.

We have prepared two sets of price data. One is the yearly average price index, stratified by the level of household income for the given year, obtained from the Consumer Price Index. The income level is classified into five quintiles. Price data for income class is presented in Table 2 along with the value of the threshold of the quintiles. We had to construct price data for Transportation since the original data gives public transportation and private transportation price data separately. We aggregated the two series as the weighted average by the expenditure share of the average Japanese household. The data on expenditures made by average Japanese households on a variety of goods are available in the Family Expenditure Survey. Looking at Table 2, the overall deflationary trend in prices can be seen during the data period. It is especially noticeable in Clothing and Communication. It probably reflects the recent major global shift of textile manufacturing to China for cheaper production costs and the on-going rapid development in the Internet and associated information technology. The other dataset is the price index for the month of September from the same source. Since more price variation is preferred, we used the price data mentioned earlier.

We selected a number of demographic variables which is used in the first

[58] One-person households made up a 29.5 percent of the population.

stage estimation of the Probit model. After considering the appropriateness of exogenous variables and the parsimoniousness of the model specifications to explain the effect on the occurrence of zero expenditures, we chose the following variables: age of the husband, the number of computer and vehicles held in the household, total income, and dummy variables indicating urbanization of residence (13 big cities), credit card usage (occasional use), survey years, wife's employment status (employed), homeownership (public rental housing), and education achievement levels of wife and husband. Other than the variables selected, the household characteristics we have considered include: price index for private residential rent; price index for public residential rent; price index for cost of repair and maintenance (from the Consumer Price Index); typical daily time allocation of wife and husband such as how many hours are spent on commuting etc.; units of dishwasher; units of clothes dryers; units of air conditioners; units of TV sets; and units of cell phones; household family financial management style, which has 18 different patterns.

It required some thought to construct the homeowership dummy. It is not an unusual practice in the Japanese real estate business for a house building itself to be owned by the resident while the land beneath it may be rented or shared with neighbors. We have tried cases where homeownership means that the building and land are both owned and cases where only the building is owned. After checking the statistical significance of the variables, we found that a dummy indicating the public rental housing works better than a dummy indicating homeownership. This indicator may be important since living in public funded housing in Japan is a popular housing choice

	Food	Clothing	Transportation	Utility	Furniture	Communication
1998						
<405	104.2	107.2	96.3	99.8	121.4	120.4
405-568	104.1	107.8	96.5	99.7	120.8	120.8
568-746	104.0	107.5	96.6	99.9	124.2	121.0
746-1004	104.0	107.4	96.5	99.9	123.5	120.7
>1004	104.1	107.5	97.1	100.3	122.9	121.1
1999						
<391	103.7	107.2	96.2	98.4	120.0	120.0
391-552	103.6	107.7	96.3	98.3	119.3	120.5
552-732	103.6	107.3	96.6	98.4	122.7	120.6
732-993	103.6	107.3	96.4	98.3	122.2	120.3
>993	103.6	107.2	96.8	98.8	121.4	120.8
2000						
<382	101.7	106.2	97.3	99.9	116.4	116.4
382-533	101.7	106.5	97.3	99.8	116.1	117.0
533-711	101.6	106.2	97.3	100.0	119.0	117.1
711-964	101.6	106.0	97.4	99.9	118.1	116.8
>964	101.6	105.9	97.9	100.4	117.6	117.4
2001						
<375	101.0	103.8	98.1	100.7	112.5	109.5
375-524	101.0	104.2	97.9	100.6	112.3	109.9
524-692*	100.9	103.8	98.1	100.7	114.4	110.0
692-936	100.9	103.7	98.1	100.7	113.7	109.9
>936	100.9	103.6	98.5	101.0	113.3	110.3
2002						
<366	100.3	101.6	97.6	99.5	108.5	108.0
366-507	100.3	101.7	97.5	99.4	108.5	108.3
507-675	100.2	101.5	97.7	99.5	109.9	108.2
675-920	100.2	101.4	97.8	99.4	109.3	108.2
>920	100.2	101.3	98.2	99.8	109.2	108.6
2003						
<364	100.1	99.8	97.6	99.1	105.7	107.9
364-500	100.1	99.8	97.5	99.0	105.6	108.1
500-657	100.0	99.7	97.7	99.1	106.4	108.0
657-890	100.0	99.6	97.8	99.0	106.0	108.1
>890	100.0	99.5	98.2	99.3	105.9	108.5

Table 2: Price data.

for many young couples in suburban areas. The idea of public housing is viewed more positively in Japanese society than in the U.S.

Yearly dummy variables do not appear to be good predictors of zero expenditure occurrence. Nevertheless, a dummy for year 2001 was found to be statistically significant for some goods. We may interpret this as "911 terrorist attack" effect. If the year 2002 dummy turned out to be significant, then we may interpreted it as the "World Cup" effect since it was co-hosted by Japan and Korea in July of 2002. The husband income and wife income data on JPSC are marked by a number of missing records and this incident is often encountered in microdata. We have avoided having to discard some of the observations by assigning an income of zero for wives who appear to be unemployed, housewives or students. The total income is the sum of incomes from both husband and wife.

We have observed that there exit a few observations with unit budget share. Of course, this can occur in only one good since, in that case, budget shares of all other goods will be zero. This is perfectly legitimate since a household can spend on goods outside of the six categories of goods which we analyze. A household can also have zero expenditure on certain goods even while consuming them: ones does not go naked during a time when no clothing purchases have been made. Nevertheless, these outliers are not favored by statistical analysis and are likely to contaminate the precision of the parameters we would like to estimate. We found one sample of unit value of budget share in Food, two samples in Utility, none in Furniture, Clothing and Transportation, and one in communication. After inspecting the histograms of the budget share data shown in Figure 7, we concluded

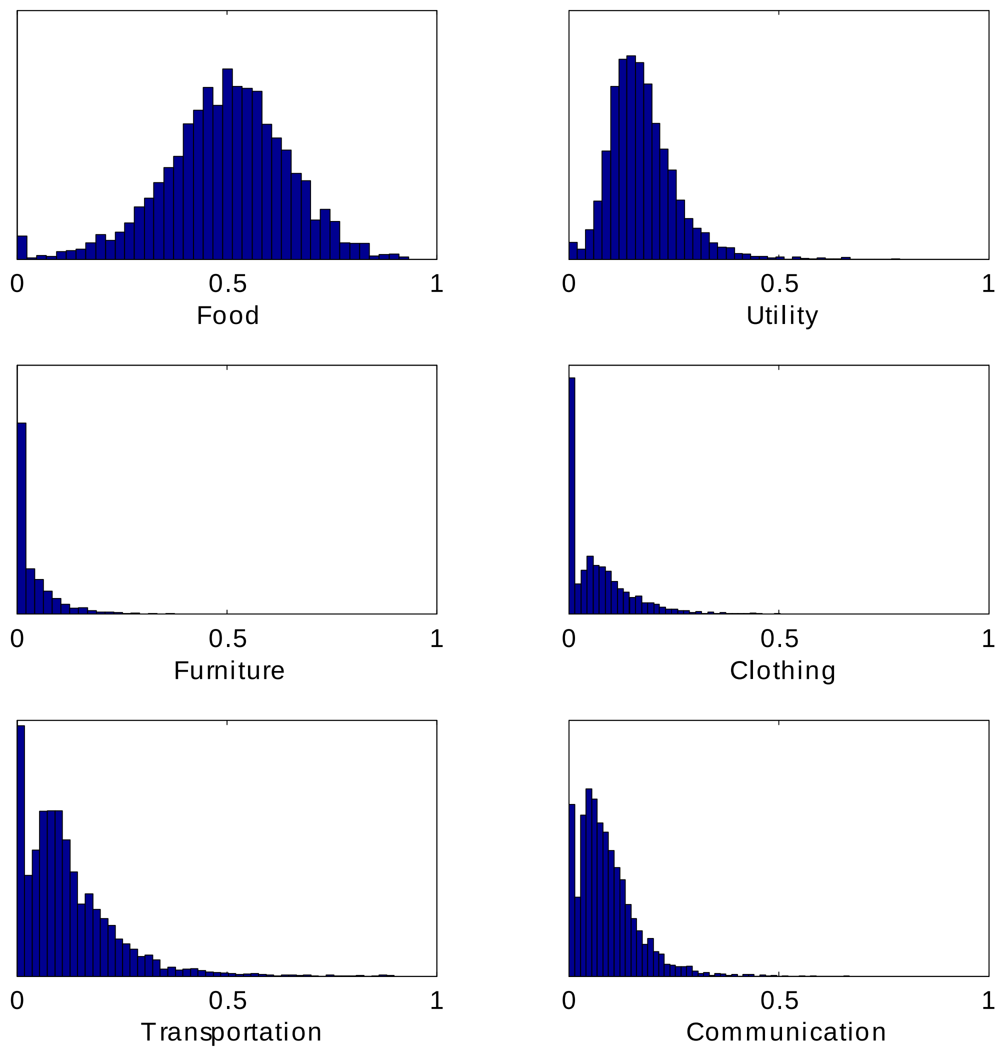

Figure 7: Histograms of the budget share for six consumption items.

that the observation with unit budget share in Communication is an outlier. We eliminated that observation for the reason that the second highest value is only 0.67. For Food and Utility, seveal observations can be seen lying at the right tail of the histograms including those at one. This implies that there is a negligible, but positive probability that the budget shares of Food and Utility take very high values. So, we decided to keep the observations. Decisions on keeping or eliminating these observations with unit budget share may affect the estimates of parameters and its precision greatly since discarding these observations also eliminates the occurrence of zero expenditures on all other goods. The effect is unclear and we will not consider this aspect further.

Accounting for all these sample selection procedures and missing records, we have a total of 3,914 observations used in our main econometric analysis.

4.5.2 Parametric Demand System Specifications

The demographically modified indirect utility function of QUAIDS of Bank, Blundell and Lewbel (1997) is written as

$$V(\boldsymbol{p}, x, \boldsymbol{z}) = \left[\left(\frac{\ln x - \ln a(\boldsymbol{p}, \boldsymbol{z})}{b(\boldsymbol{p}, \boldsymbol{z})} \right)^{-1} - c\left(\boldsymbol{p}, \boldsymbol{z}\right) \right]^{-1}, \tag{136}$$

where a is homogeneous of degree one in $\boldsymbol{p}$ and b and c are homogeneous of degree zero in $\boldsymbol{p}$, and $a(\boldsymbol{p}, \boldsymbol{z}) > 0$ is required to maintain that V is increasing in x for all $(\boldsymbol{p}, x)$. Denoting $a^r(\boldsymbol{p}) = a(\boldsymbol{p}, \boldsymbol{z}^r)$, $b^r(\boldsymbol{p}) = b(\boldsymbol{p}, \boldsymbol{z}^r)$ and $c^r(\boldsymbol{p}) = c(\boldsymbol{p}, \boldsymbol{z}^r)$, and assuming that the reference indirect utility function is

QUAIDS, GRESE implies that

$$\begin{aligned}
V(\boldsymbol{p}, x, \boldsymbol{z}) &= V^r(\boldsymbol{p}, x^e) = V^r\left(\boldsymbol{p}, \exp\left\{\frac{\ln x - \ln G(\boldsymbol{p}, \boldsymbol{z})}{K(\boldsymbol{p}, \boldsymbol{z})}\right\}\right) \\
&= \left[\left(\frac{\frac{\ln x - \ln G(\boldsymbol{p}, \boldsymbol{z})}{K(\boldsymbol{p}, \boldsymbol{z})} - \ln a^r(\boldsymbol{p})}{b^r(\boldsymbol{p})}\right)^{-1} - c^r(\boldsymbol{p})\right]^{-1} \\
&= \left[\begin{array}{c} \left(\frac{\ln x - \ln G(\boldsymbol{p}, \boldsymbol{z}) - K(\boldsymbol{p}, \boldsymbol{z}) \ln a^r(\boldsymbol{p})}{K(\boldsymbol{p}, \boldsymbol{z}) b^r(\boldsymbol{p})}\right)^{-1} \\ -c^r(\boldsymbol{p}) \end{array}\right]^{-1}.
\end{aligned}$$

Thus, if reference preferences satisfy QUAIDS then, under GRESE, all households have QUAIDS preferences. In this case, GRESE implies that

$$\ln a(\boldsymbol{p}, \boldsymbol{z}) = \ln G(\boldsymbol{p}, \boldsymbol{z}) + K(\boldsymbol{p}, \boldsymbol{z}) \ln a^r(\boldsymbol{p}),$$

$$b(\boldsymbol{p}, \boldsymbol{z}) = K(\boldsymbol{p}, \boldsymbol{z}) b^r(\boldsymbol{p}),$$

and

$$c(\boldsymbol{p}, \boldsymbol{z}) = c^r(\boldsymbol{p}).$$

Thus, we can estimate equivalent-expenditure functions given GRESE by requiring $c(\boldsymbol{p}, \boldsymbol{z}) = c^r(\boldsymbol{p})$ and calculating

$$K(\boldsymbol{p}, \boldsymbol{z}) = \frac{b(\boldsymbol{p}, \boldsymbol{z})}{b^r(\boldsymbol{p})},$$

and

$$\ln G(\boldsymbol{p}, \boldsymbol{z}) = \ln a(\boldsymbol{p}, \boldsymbol{z}) - K(\boldsymbol{p}, \boldsymbol{z}) \ln a^r(\boldsymbol{p}).$$

In addition to estimating equivalent-expenditure functions, use of the QUAIDS demand system allows a simple parametric test of the behavioral restrictions of GRESE against an unrestricted QUAIDS alternative. In particular, if preferences do not satisfy $c(\boldsymbol{p},\boldsymbol{z}) = c^r(\boldsymbol{p})$, then GRESE cannot hold. It is also possible to test down from GRESE in this framework. We can test ESE against GRESE by asking whether $K(\boldsymbol{p},\boldsymbol{z}) = 1$ or, equivalently, $b(\boldsymbol{p},\boldsymbol{z}) = b^r(\boldsymbol{p})$.

The demographically modified indirect utility function of translated QUAIDS of Lewbel (2003) is written as

$$V(\boldsymbol{p},x,\boldsymbol{z}) = \left[\left(\frac{\ln\left[x - d(\boldsymbol{p},\boldsymbol{z})\right] - \ln a(\boldsymbol{p},\boldsymbol{z})}{b(\boldsymbol{p},\boldsymbol{z})}\right)^{-1} - c\left(\boldsymbol{p},\boldsymbol{z}\right)\right]^{-1}, \qquad (137)$$

where d and a are homogeneous of degree one in $\boldsymbol{p}$, b and c are homogeneous of degree zero in $\boldsymbol{p}$, and $a(\boldsymbol{p},\boldsymbol{z}) > 0$ is required to maintain that V is increasing in x for all $(\boldsymbol{p},x)$.

The translated QUAIDS is a rank four demand system that is almost polynomial in log-expenditure. As expenditure grows large relative to d, the translated QUAIDS has expenditure share equations that approach a function that is quadratic in the natural logarithm of expenditure. However, as expenditure becomes small relative to d, expenditure share equations asymptote to plus or minus infinity, and the equations are undefined for expenditures less than d.

Denoting $a^r(\boldsymbol{p}) = a(\boldsymbol{p},\boldsymbol{z}^r)$, $b^r(\boldsymbol{p}) = b(\boldsymbol{p},\boldsymbol{z}^r)$, $c^r(\boldsymbol{p}) = c(\boldsymbol{p},\boldsymbol{z}^r)$, as before, and $d^r(\boldsymbol{p}) = d\left(\boldsymbol{p},\boldsymbol{z}^r\right)$, and assuming that the reference indirect utility function is translated QUAIDS, GAESE implies that

$$
\begin{aligned}
V(\boldsymbol{p}, x, \boldsymbol{z}) &= V^r(\boldsymbol{p}, x^e) = V^r\left(\boldsymbol{p}, \frac{x - A(\boldsymbol{p}, \boldsymbol{z})}{R(\boldsymbol{p}, \boldsymbol{z})}\right) \\
&= \left[\left(\frac{\ln\left[\frac{x - A(\boldsymbol{p}, \boldsymbol{z})}{R(\boldsymbol{p}, \boldsymbol{z})} - d^r(\boldsymbol{p})\right] - \ln a^r(\boldsymbol{p})}{b^r(\boldsymbol{p})}\right)^{-1} - c^r(\boldsymbol{p})\right]^{-1} \\
&= \left[\begin{array}{c}\left(\frac{\ln[x - A(\boldsymbol{p}, \boldsymbol{z}) - d^r(\boldsymbol{p}) R(\boldsymbol{p}, \boldsymbol{z})] - \ln R(\boldsymbol{p}, \boldsymbol{z}) - \ln a^r(\boldsymbol{p})}{b^r(\boldsymbol{p})}\right)^{-1} \\ -c^r(\boldsymbol{p})\end{array}\right]^{-1} \qquad (138)
\end{aligned}
$$

Assuming that the reference indirect utility function is translated QUAIDS, GAESE implies that all households have translated QUAIDS preferences, and

$$
\begin{aligned}
d(\boldsymbol{p}, \boldsymbol{z}) &= R(\boldsymbol{p}, \boldsymbol{z}) d^r(\boldsymbol{p}) + A(\boldsymbol{p}, \boldsymbol{z}), \\
\ln a(\boldsymbol{p}, \boldsymbol{z}) &= \ln R(\boldsymbol{p}, \boldsymbol{z}) + \ln a^r(\boldsymbol{p}), \\
b(\boldsymbol{p}, \boldsymbol{z}) &= b^r(\boldsymbol{p}),
\end{aligned}
$$

and

$$c(\boldsymbol{p}, \boldsymbol{z}) = c^r(\boldsymbol{p}).$$

Thus, we can estimate equivalent-expenditure functions given GAESE by restricting $b(\boldsymbol{p}, \boldsymbol{z}) = b^r(\boldsymbol{p})$ and $c(\boldsymbol{p}, \boldsymbol{z}) = c^r(\boldsymbol{p})$ and calculating

$$R(\boldsymbol{p}, \boldsymbol{z}) = \frac{a(\boldsymbol{p}, \boldsymbol{z})}{a^r(\boldsymbol{p})},$$

and

$$A(\boldsymbol{p}, \boldsymbol{z}) = d(\boldsymbol{p}, \boldsymbol{z}) - R(\boldsymbol{p}, \boldsymbol{z}) d^r(\boldsymbol{p}). \tag{139}$$

With these specifications for d, a, b, and c, A can be either positive or negative, but R is positive.

In addition to estimating equivalent-expenditure functions under GAESE, using the translated QUAIDS allows a simple parametric test of GAESE against an unrestricted translated QUAIDS alternative. In particular, if preferences do not satisfy $c(\boldsymbol{p}, \boldsymbol{z}) = c^r(\boldsymbol{p})$ and $b(\boldsymbol{p}, \boldsymbol{z}) = b^r(\boldsymbol{p})$, then GAESE cannot hold. It is also possible to test down from GAESE in this framework. We can test AESE against GAESE by asking whether $R(\boldsymbol{p}, \boldsymbol{z}) = 1$ or, equivalently, $a(\boldsymbol{p}, \boldsymbol{z}) = a^r(\boldsymbol{p})$. Furthermore, we can test ESE against GAESE by asking whether $A(\boldsymbol{p}, \boldsymbol{z}) = 0$, which is true only if $d(\boldsymbol{p}, \boldsymbol{z}) = 0$.

To estimate QUAIDS demand system, we specify the functions, a, b, and c as

$$\ln a(\boldsymbol{p}, \boldsymbol{z}) = a_0(\boldsymbol{z}) + \sum_{k=1}^{n} a_k(\boldsymbol{z}) \ln p_k + \frac{1}{2} \sum_{k=1}^{n} \sum_{k=1}^{n} a_{kl} \ln p_k \ln p_l, \tag{140}$$

where $\sum_{k=1}^{n} a_k(\boldsymbol{z}) = 1$, $\sum_{l=1}^{n} a_{kl}(\boldsymbol{z}) = 0$ for all k, and $a_{kl} = a_{lk}$ for all k and l,

$$b(\boldsymbol{p}, \boldsymbol{z}) = \frac{1}{1 - b_0(\boldsymbol{z})} \exp\left\{ \sum_{k=1}^{n} b_k(\boldsymbol{z}) \ln p_k \right\}, \tag{141}$$

where $\sum_{k=1}^{n} b_k(\boldsymbol{z}) = 0$, and

$$c(\boldsymbol{p}, \boldsymbol{z}) = \sum_{k=1}^{n} c_k(\boldsymbol{z}) \ln p_k, \tag{142}$$

where $\sum_{k=1}^{n} c_k(\boldsymbol{z}) = 0$.

The functions a_k, b_k, and c_k depend on z, and we assume the linear specification such that:

$$\begin{aligned} a_k(\boldsymbol{z}) &= a_k^r + a_k^{\text{Youji}}\text{Youji} + a_k^{\text{Shou}}\text{Shou} + a_k^{\text{Shonen}}\text{Shonen}, & (143) \\ b_k(\boldsymbol{z}) &= b_k^r + b_k^{\text{Youji}}\text{Youji} + b_k^{\text{Shou}}\text{Shou} + b_k^{\text{Shonen}}\text{Shonen}, & (144) \end{aligned}$$

for $k = 0, ..., n$, and

$$c_k(\boldsymbol{z}) = c_k^r + c_k^{\text{Youji}}\text{Youji} + c_k^{\text{Shou}}\text{Shou} + c_k^{\text{Shonen}}\text{Shonen}, \qquad (145)$$

for $k = 1, ..., n$. Youji is the natural logarithm of the number of children younger than 7 plus one, Shou indicates the natural logarithm of the number of children between the ages of 7 and 12 plus one, and Shoen is the natural logarithm of the number of children between the ages of 13 and 18 plus one.[59] Other household characteristics can be incorporated into the equations (143), (144) , and (145) such as geographical location, housing tenure, and education level of husband and wife, etc. Since those demographic characteristics are used for the Probit estimation of sample selection equations, we do not include them in the demand system.

Two parameters are set rather than estimated. We set a_0^r so that $\ln a(\boldsymbol{p}, \boldsymbol{z}^r)$ is equal to the average expenditure of the reference household (a household without any children) in 2001 (the middle year of data).[60] We set $b_0^r = 0$

[59] Including the number of children in different age bands may induces discontinuities that are certainly spurious. Browning (1992) suggests using as a variable the average age of the children in the household.
[60] The average total expenditure for the 2001 panel is 1.303 (in unit of 100,000 yen).

for identification purpose.

The restrictions in (140) to (142) are carried over to the restrictions imposed on (143) to (145) such that:

$$\sum_{k=1}^{n} a_k^r = 1, \tag{146}$$

$$\sum_{k=1}^{n} a_k^{\text{Youji}} = \sum_{k=1}^{n} a_k^{\text{Shou}} = \sum_{k=1}^{n} a_k^{\text{Shonen}} = 0, \tag{147}$$

$$\sum_{k=1}^{n} b_k^r = \sum_{k=1}^{n} b_k^{\text{Youji}} = \sum_{k=1}^{n} b_k^{\text{Shou}} = \sum_{k=1}^{n} b_k^{\text{Shonen}} = 0, \tag{148}$$

$$\sum_{k=1}^{n} c_k^r = \sum_{k=1}^{n} c_k^{\text{Youji}} = \sum_{k=1}^{n} c_k^{\text{Shou}} = \sum_{k=1}^{n} c_k^{\text{Shonen}} = 0. \tag{149}$$

Applying the logarithm form of Roy's identity produces the log-quadratic expenditure budget share equations

$$\begin{aligned} w_i(\boldsymbol{p}, x, \boldsymbol{z}) &= \frac{\partial \ln a(\boldsymbol{p}, \boldsymbol{z})}{\partial \ln p_i} + \frac{\partial \ln b(\boldsymbol{p}, \boldsymbol{z})}{\partial \ln p_i} (\ln x - \ln a(\boldsymbol{p}, \boldsymbol{z})) \\ &\quad + \frac{\partial q(\boldsymbol{p}, \boldsymbol{z})}{\partial \ln p_i} \frac{(\ln x - \ln a(\boldsymbol{p}, \boldsymbol{z}))^2}{b(\boldsymbol{p}, \boldsymbol{z})}, \end{aligned} \tag{150}$$

for $i = 1, ..., n$. Substituting equations (140) - (142) into (150), we get

$$\begin{aligned} w_i(\boldsymbol{p}, x, \boldsymbol{z}) &= \left(a_i(\boldsymbol{z}) + \sum_{k=1}^{n} a_{ik}^r \ln p_k \right) + b_i(\boldsymbol{z})(\ln x - \ln a(\boldsymbol{p}, \boldsymbol{z})) \\ &\quad + c_i(\boldsymbol{z})(1 - b_0(\boldsymbol{z})) \frac{(\ln x - \ln a(\boldsymbol{p}, \boldsymbol{z}))^2}{\exp \left\{ \sum_{k=1}^{n} b_k(\boldsymbol{z}) \ln p_k \right\}}, \end{aligned} \tag{151}$$

for $i = 1, ..., n$.

GRESE requires $c_k(\boldsymbol{z}) = c_k^r$ for all $k = 1, ..., n$. Thus, given QUAIDS demands, GRESE restricts preferences in such a way that the coefficients on $(\ln x)^2$ are proportional across household types. To test up from GRESE, we estimate a model in which GRESE is not maintained so that these proportionality restrictions are relaxed. In that case, the share equations are homogeneous in $(b_0(\boldsymbol{z}), c_1(\boldsymbol{z}), ..., c_n(\boldsymbol{z}))$, so that these functions are not separately identifiable. Doubling $b_0(\boldsymbol{z})$ halves all $c_k(\boldsymbol{z})$'s. To identify $c_k(\boldsymbol{z})$ in the unrestricted QUAIDS, we impose the restriction $b_0(\boldsymbol{z}) = 0$. However, under GRESE, this restriction is not necessary because $c_k(\boldsymbol{z}) = c_k^r$, and we use the weaker restriction that $b_0(\boldsymbol{z}^r) = b_0^r = 0$.

GRESE can be also tested down. One can test whether or not $K(\boldsymbol{p}, \boldsymbol{z}) = 1$ which gives ESE by testing the additional restriction that $b_0(\boldsymbol{z}) = b_0^r = 0$ in a GRESE-restricted model.

Given GRESE, equivalent-expenditure functions are uniquely identifiable if and only if GRESE is maintained a priori and the reference expenditure function is not PIGLOG. The QUAIDS model is PIGLOG if and only if $c(\boldsymbol{p}, \boldsymbol{z}) = 0$. Thus, we may test identification restrictions of GRESE of testing $c_k^r = 0$ for all k, in a GRESE-restricted model.

With the GRESE-restricted QUAIDS model, the GRESE functions K and G take on relatively simple forms in terms of the parameters if evaluated at an n-vector of equal prices $\boldsymbol{p}^* = \mathbf{1}$, with

$$K(\boldsymbol{p}^*, \boldsymbol{z}) = \frac{1}{1 - b_0(\boldsymbol{z})}$$

and

$$\ln G\left(\boldsymbol{p}^{*}, \boldsymbol{z}\right) = a_0(\boldsymbol{z}) - \frac{a_0^r}{1 - b_0(\boldsymbol{z})}.$$

Substituting these expressions into equation (120) and manipulating, we get a simple expression for the log-equivalence scale evaluated at a vector of unit prices:

$$\ln S_R(\boldsymbol{p}^{*}, x, \boldsymbol{z}) = b_0(\boldsymbol{z})\left(\ln x - a_0(\boldsymbol{z})\right) + \left(a_0(\boldsymbol{z}) - a_0^r\right). \tag{152}$$

The first term is zero if $\ln x = a_0(\boldsymbol{z})$, which by construction is true for households whose equivalent expenditure is equal to the average expenditure of the reference households in the base year 2001. Thus, $\ln S\left(\boldsymbol{p}^{*}, a(\boldsymbol{p}^{*}, \boldsymbol{z}), \boldsymbol{z}\right) = a_0(\boldsymbol{z}) - a_0^r$. If ESE is true, then $b_0(\boldsymbol{z}) = 0$, so that the equivalence scale takes this form at all levels of expenditure.

To estimate the translated QUAIDS demand systems, we specify the functions d, a, b, and c as

$$d(\boldsymbol{p}, \boldsymbol{z}) = \sum_{k=1}^{n} d_k(\boldsymbol{z}) p_k,$$

$$a(\boldsymbol{p}, \boldsymbol{z}) = a_0(\boldsymbol{z}) + \sum_{k=1}^{n} a_k(\boldsymbol{z}) \ln p_k + \frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} a_{kl} \ln p_k \ln p_l, \tag{153}$$

where $\sum_{k=1}^{n} a_k(\boldsymbol{z}) = 1$, $\sum_{l=1}^{n} a_{kl}(\boldsymbol{z}) = 0$ for all k and $a_{kl} = a_{lk}$ for all k and l,

$$\ln b(\boldsymbol{p}, \boldsymbol{z}) = \sum_{k=1}^{n} b_k(\boldsymbol{z}) \ln p_k, \tag{154}$$

where $\sum_{k=1}^{n} b_k(\boldsymbol{z}) = 0$, and

$$c(\boldsymbol{p}, \boldsymbol{z}) = \sum_{k=1}^{n} c_k(\boldsymbol{z}) \ln p_k, \tag{155}$$

where $\sum_{k=1}^{n} c_k(\boldsymbol{z}) = 0$.

It is convenient to denote

$$d_0(\boldsymbol{z}) = \sum_{k=1}^{n} d_k(\boldsymbol{z}). \tag{156}$$

The functions d_k, a_k, b_k, and c_k depend on $\boldsymbol{z}$, and we assume that

$$\begin{aligned} d_k(\boldsymbol{z}) &= d_k^r + a_k^{\text{Youji}}\text{Youji} + a_k^{\text{Shou}}\text{Shou} + a_k^{\text{Shonen}}\text{Shonen}, & (157) \\ a_k(\boldsymbol{z}) &= a_k^r + a_k^{\text{Youji}}\text{Youji} + a_k^{\text{Shou}}\text{Shou} + a_k^{\text{Shonen}}\text{Shonen}, & (158) \\ b_k(\boldsymbol{z}) &= b_k^r + b_k^{\text{Youji}}\text{Youji} + b_k^{\text{Shou}}\text{Shou} + b_k^{\text{Shonen}}\text{Shonen}, & (159) \\ c_k(\boldsymbol{z}) &= c_k^r + c_k^{\text{Youji}}\text{Youji} + c_k^{\text{Shou}}\text{Shou} + c_k^{\text{Shonen}}\text{Shonen}, & (160) \end{aligned}$$

for $k = 1, ..., n$. Youji is the natural logarithm of the number of children younger than 6 plus one, Shou indicates the natural logarithm of the number of children between the ages of 7 and 12 plus one, and Shonen is the natural logarithm of the number of children between the ages of 13 and 18 plus one. Other household characteristics can be incorporated into the equations in (157), (158), (159), and (160) such as geographical location, housing tenure, and education level of husband and wife, etc. Since those demographic characteristics are used first in the Probit estimation for the two-step estimation procedure, we do not include them in the demand system.

The restrictions in (153) to (155) are carried over to the restrictions imposed on (157) to (160) such that:

$$\sum_{k=1}^{n} a_k^r = 1, \tag{161}$$

$$\sum_{k=1}^{n} a_k^{\text{Youji}} = \sum_{k=1}^{n} a_k^{\text{Shou}} = \sum_{k=1}^{n} a_k^{\text{Shonen}} = 0, \tag{162}$$

$$\sum_{k=1}^{n} b_k^r = \sum_{k=1}^{n} b_k^{\text{Youji}} = \sum_{k=1}^{n} b_k^{\text{Shou}} = \sum_{k=1}^{n} b_k^{\text{Shonen}} = 0, \tag{163}$$

$$\sum_{k=1}^{n} c_k^r = \sum_{k=1}^{n} c_k^{\text{Youji}} = \sum_{k=1}^{n} c_k^{\text{Shou}} = \sum_{k=1}^{n} c_k^{\text{Shonen}} = 0. \tag{164}$$

There are no restrictions on d_k's.

We note that, with the translated QUAIDS specification, the GAESE functions A and R take on relatively simple forms in terms of the parameters if evaluated at an n-vector of unit prices $\boldsymbol{p}^* = \mathbf{1}$ with

$$R(\boldsymbol{p}^*, \boldsymbol{z}) = \exp(a_0(\boldsymbol{z}) - a_0(\boldsymbol{z}^r))$$

and

$$A(\boldsymbol{p}^*, \boldsymbol{z}) = (d_0(\boldsymbol{z}) - \exp(a_0(\boldsymbol{z}) - a_0(\boldsymbol{z}^r))d_0(\boldsymbol{z}^r)).$$

The GAESE in (134) becomes

$$S_A = \frac{(\exp(a_0(\boldsymbol{z}) - a_0^r) - 1)x + (d_0(\boldsymbol{z}) - \exp(a_0(\boldsymbol{z}) - a_0^r)d_0^r)}{\exp(a_0(\boldsymbol{z}) - a_0^r)}. \tag{165}$$

Applying Roy's theorem in its logarithmic form to (138) generates the expenditure share equations

$$w_i = \frac{p_i d_i(\boldsymbol{z})}{x} + \left(1 - \frac{d(\boldsymbol{p},\boldsymbol{z})}{x}\right) \times \left(\begin{array}{c} a_i(\boldsymbol{z}) + \sum_{k=1}^{n} a_{ik} \ln p_k + b_i(\boldsymbol{z}) \left(\ln \frac{x-d(\boldsymbol{p},\boldsymbol{z})}{a(\boldsymbol{p},\boldsymbol{z})}\right) \\ + \frac{c_i(\boldsymbol{z})}{b(\boldsymbol{p},\boldsymbol{z})} \left(\ln \frac{x-d(\boldsymbol{p},\boldsymbol{z})}{a(\boldsymbol{p},\boldsymbol{z})}\right)^2 \end{array} \right), \tag{166}$$

for $i = 1, .., n$. Equation (166) does not necessarily satisfy GAESE, which requires that $b_k(\boldsymbol{z}) = b_k^r$ and $c_k(\boldsymbol{z}) = c_k^r$. The translated QUAIDS demand system corresponds to an affine expenditure function if and only if $b(\boldsymbol{p}, \boldsymbol{z})$ and $c(\boldsymbol{p}, \boldsymbol{z})$ are both 0, and corresponds to a log-affine expenditure function if and only if $d(\boldsymbol{p}, \boldsymbol{z})$ and $c(\boldsymbol{p}, \boldsymbol{z})$ are both 0. Because equivalent-expenditure functions given GAESE are uniquely identifiable if the expenditure function is neither affine nor log-affine, identification is possible given GAESE if c is nonzero or if both d and b are nonzero.

4.5.3 Estimation

The usual procedure to estimate the demand systems is to append an error term $\boldsymbol{\varepsilon}$ to the right-hand side of the share equations and then estimate the model by ML or iterative SUR, assuming the joint normal distribution of the errors term. As mentioned, this usual estimation procedure is not applicable for our data due to the presence of a large pile of zero expenditures in some of the goods.

In microeconomics data on consumer expenditure, it is frequently the case that some units do not purchase some of the commodities, alcohol and tobacco being the standard examples.[61] Microeconomics theory predicts it as the case of a corner solution of the optimization problem as opposed to the interior solution. When the relative price of goods is too high, the consumer may choose not to consumer the good at all as the optimal consumption allocation. Although the zero expenditure on some goods is entirely consistent with the theory of consumer behavior, the estimation faces a serious problem in dealing with it. In the presence of a mass of zero expenditure observations, a joint normal distribution of error structure does not allow for a significant proportion of realization at zero expenditure.[62] Standard estimation methods for this model such as SUR or maximum likelihood estimator do not take special account of zero expenditures, and consequently yield inconsistent estimates of the parameters. Even if observations containing zero expenditures on one or more goods were eliminated for purposes of estimation, these standard estimators would be biased and inconsistent. Moreover, excluding these observations might significantly reduce the sample size. Regardless of whether or not the complete sample is used, the bias and inconsistency occur because the random disturbances have expectations which are not zero and which depend upon the exogenous variables. The easiest way to deal with zero expenditure is to form a broader aggregate commodity category. Hoderlein (2008) grouped his housing goods category

[61] Fry and Pashardes (1994) studied the tobacco expenditures, Su and Yen (2000) studied the demands for alcohol and tobacco products with a number of zero-expenditure samples in their data, and Unayama (2006) examined the rank of demand function for alcohol.

[62] Woodland (1979) proposed that the error terms be modelled as Dirichlet or log-normal distribution for the purpose of restricting the deterministic part and disturbance part of budget share to the closed unit interval $[0, 1]$. His method does not solve zero expenditure issue since Dirichlet (and log-normal) distribution cannot model a sharp increase of the probability of expenditure at zero.

as a combination of rent or mortgage payment, furniture, and household goods and services which reduces the occurrence of zero expenditures on each of these subcategories.

Assuming that the zero expenditure occurs as a result of the corner solution, the econometric methods based on the Kuhn-Tucker conditions (Wales and Woodland, 1983) and on the virtual prices, which is dual to the Kuhn-Tucker conditions (Lee and Pitt, 1986, 1987), have been developed.[63] However, because any of the methods usually requires the evaluation of multiple integration to calculate the value of likelihood function in each iteration, the application is limited to very restrictive forms of the demand model. The simulation and Bayesian methods have been proposed to circumvent the curse of dimensionality (Kao, Lee and Pitt, 2001, Millimet and Tchernis, 2008, Pitt and Millimet, 1999).[64]

Households may appear to have bought none of some particular goods not because they are too expensive, but because the enumeriation period was too short. Storable or durable goods such as canned foods and clothing are frequently not purchased during the survey period even if they are used during that time. Zero purchases of clothing during the particular weeks or month of a survey period does not mean that people go naked during that period.[65] This problem had been noticed for a while, and Deaton and Irish (1982) finally attempted to estimate a econometric model of the household

[63] An earlier attempt to take a possible corner solution into account in demand analysis includes Barnett (1979).

[64] The estimation of demand systems by Bayesian methods are studied by Barnett, Geweke and Wolfe (1991a, 1991b), Terrell (1996).

[65] JPSC collects expenditures made during the entire month of September. We do not know whether a one-month enumeration period is long enough or is too short.

demand behavior with zero expenditure samples which are assumed to be generated by the infrequent purchase. Deaton and Irish (1984) present a p-tobit model that extends the tobit specification to model zero expenditures. Recorded data for expenditure on commodities are $1/p$ times consumption during the survey period, where p denotes the ratio of the survey period to the purchase period. This is applicable when goods are consumed during the survey period; however, expenditures are only observed with a probability p because of infrequent purchasing (Deaton and Irish, 1984). Kay, Keen and Morris (1984) extend the model proposed by Deaton and Irish (1984) by providing a stochastic relationship between expenditure and consumption in a sophisticated manner. Keen (1986) estimates a system of linear Engel functions that satisfy the adding-up condition and derives a consistent estimator based on the instrumental variables method. While the purchasing probabilities are constant parameters in Deaton and Irish (1984), Kay, Keen and Morris (1984) and Keen (1986), Blundell and Meghir (1987) propose a model with probit-type purchasing probabilities. Griffiths and Valenzuela (1998), Hasegawa, Ueda and Mori (2008) estimate a system of linear Engel functions and equivalence scales using a Bayesian method, and Pudney (1989, 1990) reviews several theoretical aspects associated with zero expenditures.

Another alternative approache is the Amemiya-Tobin approach which explicitly takes into account the censored nature of the negative quantities demanded. It has been applied where estimation is via full information maximum likelihood (Wales and Woodland, 1983; Yen and Lin, 2004), quasi-maximum likelihood (Yen, Lin and Smallwood, 2003), or various two-step estimators (Heien and Wessell, 1993; Perali and Chavas, 1998; Shonkwiler and Yen, 1999; Meyerhoefer, Ranney and Sahn, 2003).

In this empirical section, we employ a variant of the bivariate sample selection model of Heckman (1974). The model can be viewed as a sample selection generalization of Amemiya's Tobit system. The estimation is done in two steps instead of doing a full maximum likelihood estimation at all once.[66] The two-step procedure guarantees the consistency of the estimates as long as parameters of each step are consistent, but it results in the loss of efficient estimation. The next section illustrates the two-step estimation procedure in Shonkwiler and Yen (1999) which has corrected the inconsistency issue found in Heien and Wessell (1993).

The usual way to deal with the endogeniety likely to exist in demand systems is to use income or some functions of income as instruments and estimate the system by GMM. As long as we use this two-step estimation procedure, we cannot use GMM in the second step. We may literally replace the total expenditure with the projection of the total expenditure, but the theoretical validity of doing this is questionable.

4.5.4 The Two-Step Estimation Procedure

The idea of this procedure is that one equation describes the household's decision to participate in the market in the similar way to the original framework for female labor supply study of Heckman (1974), and a set of equations describes the consumption behavior provided that a decision has been made to participate in the market. A formal structure is considered in which

[66] The full ML estimation procedure is provided in Yen and Lin (2004)

censoring of each commodity i is governed by a separate stochastic process $\boldsymbol{y}_{it}'\boldsymbol{\tau}_i + \varsigma_{it}$ such that:

$$\begin{aligned} w_{it}^* &= w_{it}(\boldsymbol{p}, x, \boldsymbol{z}; \boldsymbol{\theta}) + \varepsilon_{it} && \text{if } \boldsymbol{y}_{it}'\boldsymbol{\tau}_i + \varsigma_{it} > 0 \\ &= 0 && \text{if } \boldsymbol{y}_{it}'\boldsymbol{\tau}_i + \varsigma_{it} \leq 0 \end{aligned}, \tag{167}$$

where w_{it}^* is the observed expenditure share, $\boldsymbol{y}_{it}$ is a vector of exogenous variables, $\boldsymbol{\tau}_i$ is a conformable parameter vector, and $\boldsymbol{\varepsilon}_{it}$ and ς_{it} are random errors. Assuming the $2n \times 1$ vector of disturbances $\boldsymbol{u}_t = [\boldsymbol{\varepsilon}_t', \boldsymbol{\varsigma}_t'] = [\varepsilon_{1t}', ..., \varepsilon_{nt}', \varsigma_{1t}', ..., \varsigma_{nt}']'$ is multivariate normal with

$$\begin{aligned} E(\boldsymbol{u}_t, \boldsymbol{u}_s) &= \begin{bmatrix} \boldsymbol{\Sigma}_{\varepsilon\varepsilon} & \boldsymbol{\Sigma}_{\varepsilon\varsigma} \\ \boldsymbol{\Sigma}_{\varepsilon\varsigma} & \mathbf{I}_n \end{bmatrix} && \text{if } t = s \\ &= 0, && \text{otherwise,} \end{aligned} \tag{168}$$

where

$$\boldsymbol{\Sigma}_{\varepsilon\varepsilon} = E\left[\varepsilon_t \varepsilon_t'\right] = [\sigma_{ij}],$$

and

$$\boldsymbol{\Sigma}_{\varepsilon\varsigma} = E\left[\varepsilon_t \varsigma_t'\right] = \text{diag}\left[\delta_1, \delta_2, ..., \delta_n\right].$$

Note that although the binary censoring mechanism $\boldsymbol{y}_{it}'\boldsymbol{\tau}_i + \varsigma_{it}$ is independent of $\boldsymbol{y}_{jt}'\boldsymbol{\tau}_j + \varsigma_{it}$ and the level equation $w_{it}(\boldsymbol{p}, x, \boldsymbol{z}; \boldsymbol{\theta}) + \varepsilon_{it}$ for $i \neq j$, correlations are allowed between $\boldsymbol{y}_{it}'\boldsymbol{\tau}_i + \varsigma_{it}$ and $w_{it}(\boldsymbol{p}, x, \boldsymbol{z}; \boldsymbol{\theta}) + \varepsilon_{it}$ for each commodity and, more importantly, among $w_{it}(\boldsymbol{p}, x, \boldsymbol{z}; \boldsymbol{\theta}) + \varepsilon_{it}$ and $w_{jt}(\boldsymbol{p}, x, \boldsymbol{z}; \boldsymbol{\theta}) + \varepsilon_{jt}$ through $\sigma_{ij} \neq 0$ for all i, j. Cross-equation restrictions and correlations among demand equations are the focus of existing censored system estimators. Using equation (167) and the bivariate normality of $[\varepsilon_i, \varsigma_i]'$, the mean of w_{it}^* conditional on a positive observation is

$$E\left[w_{it}^* | \varsigma_{it} > -\boldsymbol{y}_{it}'\boldsymbol{\tau}_i\right] = w_{it}\left(\boldsymbol{p}_t, x_t; \boldsymbol{\theta}\right) + \delta_i \phi\left(\boldsymbol{y}_{it}'\boldsymbol{\tau}_i\right) / \Phi\left(\boldsymbol{y}_{it}'\boldsymbol{\tau}_i\right), \qquad (169)$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the standard normal probability density and distribution functions, respectively. Since $\Pr[\varsigma_{it} > -\boldsymbol{y}_{it}'\boldsymbol{\tau}_i] = \Phi(\boldsymbol{y}_{it}'\boldsymbol{\tau}_i)$ and $E[w_{it}^* | \varsigma_{it} < -\boldsymbol{y}_{it}'\boldsymbol{\tau}_i]$, the unconditional mean of w_{it}^* is

$$E\left[w_{it}^*\right] = \Phi\left(\boldsymbol{y}_{it}'\boldsymbol{\tau}_i\right) w_{it}\left(p_t, x_t; \boldsymbol{\theta}\right) + \delta_i \phi\left(\boldsymbol{y}_{it}'\boldsymbol{\tau}_i\right). \qquad (170)$$

Based on this equation, the system of share equations can be written as

$$w_{it} = \Phi\left(\boldsymbol{y}_{it}'\boldsymbol{\tau}_i\right) w_{it}\left(\boldsymbol{p}_t, x_t; \boldsymbol{\theta}\right) + \delta_i \phi\left(\boldsymbol{y}_{it}'\boldsymbol{\tau}_i\right) + \xi_{it}, \qquad i = 1, ..., n \qquad (171)$$

where $\xi_{it} = w_{it} - E[w_{it}^*]$. It is easily seen that $E[\xi_{it}] = 0$ and that ξ_{it} is heteroscedastic. Drawing on Shonkwiler and Yen (1999), the system in (171) can be estimated with a two-step procedure: (1) obtain maximum-likelihood (ML) probit estimates $\widehat{\tau}_i$ of τ_i using the binary outcomes $w_{it}^* = 0$ and $w_{it}^* > 0$; (2) calculate $\Phi(\boldsymbol{y}_{it}'\boldsymbol{\tau}_i)$ and $\phi(\boldsymbol{y}_{it}'\boldsymbol{\tau}_i)$ for all i and estimate θ, $\delta_1, .., \delta_n$ in the augmented system

$$w_{it} = \Phi\left(\boldsymbol{y}_{it}'\widehat{\boldsymbol{\tau}}_i\right) w_{it}\left(\boldsymbol{p}_t, x_t; \boldsymbol{\theta}\right) + \delta_i \phi\left(\boldsymbol{y}_{it}'\widehat{\boldsymbol{\tau}}_i\right) + \xi_{it} \qquad (172)$$

by ML or SUR procedure.

Because the ML probit estimators $\widehat{\boldsymbol{\tau}}_i$ are consistent, applying ML or SUR estimation to equation (172) produces consistent estimates in the second step.[67] However, because error terms ξ_{it} are heteroskedastic, the second-step ML or SUR estimator obtained by the usual procedure is inefficient. Efficiency could be achieved by using a weighted system estimator to account for the specific type of heteroskedasticity. We used the heteroscedasticity-consistent covariance matrix estimator proposed by White (1980b). Another problem caused by the use of the estimated $\widehat{\boldsymbol{\tau}}_i$ in equation (172) is that the covariance matrix of the second-step estimator is incorrect. This covariance matrix can be adjusted by the procedure of Murphy and Topel (1985).

Denote the log-likelihoods of the first-step probit models as $L_{11}(\boldsymbol{\tau}_1), ...,$ $L_{1n}(\boldsymbol{\tau}_n)$ and the log-likelihood of the second-step system as $L_2(\widehat{\boldsymbol{\tau}}_1, .., \widehat{\boldsymbol{\tau}}_n, \boldsymbol{\theta})$. Then, the covariance matrix of $\widehat{\boldsymbol{\theta}}$ is

$$\Sigma = R_2^{-1} + R_2^{-1}\left[R_3' R_1^{-1} R_3 - R_4' R_1^{-1} R_3 - R_3' R_1^{-1} R_4\right] R_2^{-1}$$

where by extending the results of Murphy and Topel (1985),

$$\begin{aligned} R_1 &= \operatorname{diag}\left[R_{11}(\boldsymbol{\tau}_1), ..., R_{1n}(\boldsymbol{\tau}_n)\right] \\ R_3 &= \left[R_{31}'(\boldsymbol{\tau}_1, \boldsymbol{\theta}) | \cdots | R_{3n}'(\boldsymbol{\tau}_n, \boldsymbol{\theta})\right]' \\ R_4 &= \left[R_{41}'(\boldsymbol{\tau}_1, \boldsymbol{\theta}) | \cdots | R_{4n}'(\boldsymbol{\tau}_n, \boldsymbol{\theta})\right]' \end{aligned}$$

[67] Estimation of the separate probit models implies that the restriction $E(\zeta_{it}\zeta_{jt}) = 0$ for $i \neq j$ in covariance matrix of (168), without which the multivariate probit model would have to be estimated. With some loss in efficiency (relative to multivariate probit) these separate probit estimates are nevertheless consistent.

and for $i = 1, ..., n$,

$$\begin{aligned} R_{1i}(\boldsymbol{\tau}_i) &= E\frac{\partial L_{1i}}{\partial \boldsymbol{\tau}_i}\left(\frac{\partial L_{1i}}{\partial \boldsymbol{\tau}_i}\right)' = -E\frac{\partial^2 L_{1i}}{\partial \boldsymbol{\tau}_i \partial \boldsymbol{\tau}_i'} \\ R_{3i}(\boldsymbol{\tau}_i, \boldsymbol{\theta}) &= E\frac{\partial L_2}{\partial \boldsymbol{\tau}_i}\left(\frac{\partial L_2}{\partial \boldsymbol{\theta}}\right)' = -E\frac{\partial^2 L_{1i}}{\partial \boldsymbol{\tau}_i \partial \boldsymbol{\theta}'} \\ R_{4i}(\boldsymbol{\tau}_i, \boldsymbol{\theta}) &= E\frac{\partial L_{1i}}{\partial \boldsymbol{\tau}_i}\left(\frac{\partial L_2}{\partial \boldsymbol{\theta}}\right)', \end{aligned}$$

and R_2 is the information matrix of the second-step estimation:

$$R_2 = E\frac{\partial L_2}{\partial \boldsymbol{\theta}}\left(\frac{\partial L_2}{\partial \boldsymbol{\theta}}\right)' = -E\frac{\partial^2 L_2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'}.$$

4.6 Results

For the first-step Probit estimation, because of only a small number of observations of zero expenditure on Food and Utility, we could not obtain precise parameter estimates for these items. Therefore, in the second-step estimation, we did not use share equations of the form (172) for Food and Utility. We failed to estimate the translated QUAIDS, possibly because of the term $x - d(\boldsymbol{p}, \boldsymbol{z})$ in (166), which is undefined if $d(\boldsymbol{p}, \boldsymbol{z})$ exceeds the minimum value of x. It is difficult to prevent this from happening "during" the estimation. Since d is a function of $\boldsymbol{p}$ and $\boldsymbol{z}$ with associated parameters, the function evaluation during the optimization routine may produce $x - d(\boldsymbol{p}, \boldsymbol{z}) < 0$. It may be easier to obtain succesful estimation with a smaller system, but the system of smaller than four goods ruins the advantage of using the rank "four" demand system.[68] Therefore, we could not evaluate the household

[68] See section 2.2.5 for the disscusion on the rank of demand system again.

preferences for GAESE and the possibility that there exists a fixed cost associated only with different number of children regardless the level of utility or income.[69]

We also found the singularity of covariance matrix when the full set of n equations is estimated in GRESE case. The two-step procedure allows for the estimation of the full system unlike usual demand system estimation procedure. Since the augmented system does not require the adding-up to unity, the covariance matrix is not singular by construction. We follow the plausible and simple approach, suggested by Pudney (1989), of treating the nth good as a residual category with no specific demand of its own and estimated the rest of $n-1$ equations; our nth item is transportation. Table 3 presents some parameter estimates associated with the calculation of equivalence scale, and the log-likelihood values with their number of parameters to calculate likelihood ratio test statistics. All specification tests are based on the likelihood ratio test.[70]

We look at the model statistics for four models: (1) unrestricted QUAIDS; (2) GRESE-restricted QUAIDS; (3) ESE-restricted QUAIDS; and (4) unrestricted AIDS as shown. For the case (4), the equivalence scale is not identified, but we can test it against the alternative that the model contains the quadratic term as QUAIDS. If AIDS model is accepted, then the equivalence scale obtained is not unique either under GRESE or ESE, and it is difficult to draw any conclusion from the resulting estimates of equivalence scales. Given the maintained assumption of GRESE restricted QUAIDS,

[69] The GAESE specification of household preferences can be implemented with QES. We will discuss this in the conclusion section of this dissertation.

[70] The likelihood test is known to have a tendency to over-reject the hypothesis.

Model	AIDS		QUAIDS		QUAIDS		QUAIDS	
Restriction	Unrestricted		ESE		GRESE		Unrestricted	
Param.	Est.	Std.error	Est.	Std.error	Est.	Std.error	Est.	Std.error
a_0^r	1.303	–	1.303	–	1.303	–	1.303	–
a_0^{youji}	0.472*	0.180	0.126*	0.037	0.174*	0.078	0.177*	0.064
a_0^{shou}	0.944*	0.202	0.420*	0.033	0.482*	0.154	0.401*	0.070
a_0^{shonen}	0.230*	0.228	0.469*	0.048	0.359*	0.106	0.347*	0.081
b_0^r	–	–	–	–	–	–	–	–
b_0^{youji}	–	–	–	–	−0.012	0.132	–	–
b_0^{shou}	–	–	–	–	−0.451*	0.141	–	–
b_0^{shonen}	–	–	–	–	−0.048	0.145	–	–
c_1^r	–	–	−0.122	0.006	−0.099*	0.011	−0.085*	0.016
c_2^r	–	–	0.033*	0.003	0.029*	0.004	0.030*	0.008
c_3^r	–	–	−0.004	0.004	−0.000	0.003	0.085*	0.010
c_4^r	–	–	0.042*	0.005	−0.019*	0.006	−0.015	0.015
c_5^r	–	–	−0.013*	0.006	0.038*	0.005	−0.020	0.020
Log-Likelihood	22780.84		23065.36		23089.47		23127.86	
# of params.	41		36		52		64	

Table 3: Model statistics and the selected parameter estimates with * if significant at 5% or better.

ESE instead requires that $b(\boldsymbol{p}, \boldsymbol{z}) = b(\boldsymbol{p})$. The likelihood ratio test statistics for this hypothesis is 48.22 and is distributed as a χ^2_{16} which has a one-sided one percent critical value of 32. Therefore, the test prefers GRESE specification of QUAIDS model. Given an unrestricted QUAIDS model, GRESE requires that $c(\boldsymbol{p}, \boldsymbol{z}) = c\,(\boldsymbol{p})$and that $b(\boldsymbol{p}, \boldsymbol{z}) = b(\boldsymbol{p})$. The likelihood ratio statistics is 76.78 and is distributed as a χ^2_{12} which has a one-sided one percent critical value of 26. Thus, the observable restrictions imposed by GRESE on the QUAIDS model are rejected. Demographic effects may affect the household demand behaviors in a more complicated way than those permitted by GRESE. The likelihood ratio statistics for the hypothesis that GRESE is true and the reference expenditure function is PIGLOG (AIDS in this case) against a GRESE-restricted QUAIDS alternative is important to reject. Without the maintained assumption that GRESE is true and that the reference expenditure function is not PIGLOG, the GRESE equivalent-expenditure functions are not identified. Our result shows that the unrestricted AIDS is strongly rejected; the log-likelihood value is smaller than the ESE-QUAIDS model with fewer parameters. It implies that the quadratic term is important to describe the Japanese households' consumption behaviors and suggests that the importance of using the rank three demand system to account for the more flexible Engel curves (Pashardes, 1995).

The positive parameter values of a_0's indicate that more children generate larger equivalence scales, and our estimates of a_0's are all positive and mostly significant in all models, which help validate the model specification. As can be seen in equation (152), the negative value of $b_0(\boldsymbol{z})$ is necessary to produce the expenditure-dependent equivalence scales decreasing in the total expenditure. The important parameters for this are b_0^r's in the column of GRESE restricted QUAIDS in Table 3. We managed to obtain negative coefficients for all of b_0^r's, but only one of them is significant.

Based on this result of the GRESE restricted QUAIDS and on that of the ESE restricted QUAIDS, figure 8 shows the expenditure-dependent equivalence scales with the price level normalized to unit prices in the base year. We consider six household composition scenarios; (i) 1 Youji (ii), 1 Youji and 1 Shou, (iii) 1 Shou and 1 Shonen, (iv) 2 Shou's, (v) 1 Shonen, (vi) 1 shou, and (vii) 1 shou based on the estimates of ESE restricted QUAIDS. We focus on (i), (v), (vi) and (vii) for one child cases. The equivalence scales for households with one child in any of the age groups take the moderate size in the mean expenditure level at 1.303 depicted by the dotted vertical line with equivalence scales of between 1.2 and 1.6, which is a reasonable range, and all of the equivalence scales decline as total expenditure rises, as indicated by the GRESE parameter estimates. Next, we focus on (vi) and (vii). The lower constant equivalence scale (1.338) based on the estimation of the ESE restricted QUAIDS than the expenditure-dependent counterpart based on the estimation of the GRESE restricted QUAIDS on the dotted vertical line implies that the demand model may underestimate the equivalence scales if it is restricted by the ESE specification.[71] The underestimated equivalence scales may undermine the perceived cost related to childbirth and child-rearing and may distort any policy decisions.

[71] Remember that a study by Hasegawa, Ueda and Mori (2008) obtained the scale of 1.3695 which is very close to our number (1.338) using the same data source. This indicates that our estimation result is as robust as their study.

However, the degree of decline of the scale for (vi) is still not enough to be free from the criticism cast upon the constant equivalence scales for the purpose of policy implementation. Looking at the scale for (vi), the values of the scale at total expenditure 1.0 is 1.75 and at 2.0 is 1.40, approximately. The corresponding equivalent expenditures are 0.75 and 0.80, and therefore, the higher income households are going to need to receive a bigger compensation to maintain the same utility level as before having a child. It is hardly justifiable in any society to implement a policy in which the higher income classes receive bigger welfare benefits.

4.7 Conclusion

This chapter demonstrated the use of an application of the flexible functional form with higher rank to study the Japanese household demand behaviors for the purpose of estimating the cost of a child. In doing so, we have used conditions that can unambiguously identify the equivalence scales which can vary in the total expenditure level. We also overcome the issue of the presence of a mass of zero-expenditure observations to correct the bias in the parameter estimates.

Our results show that the specification of demand systems with expenditure-dependent equivalence scales is statistically preferred to any of the more restricted models, and that the equivalence scales decrease in total expenditure. Moreover, the result implies that the implementation of the constant equivalence scale may underestimate it, which may result in undermining the cost associated with child-birth and child-rearing.

Based on our results, any public policies that benefit families with chil-

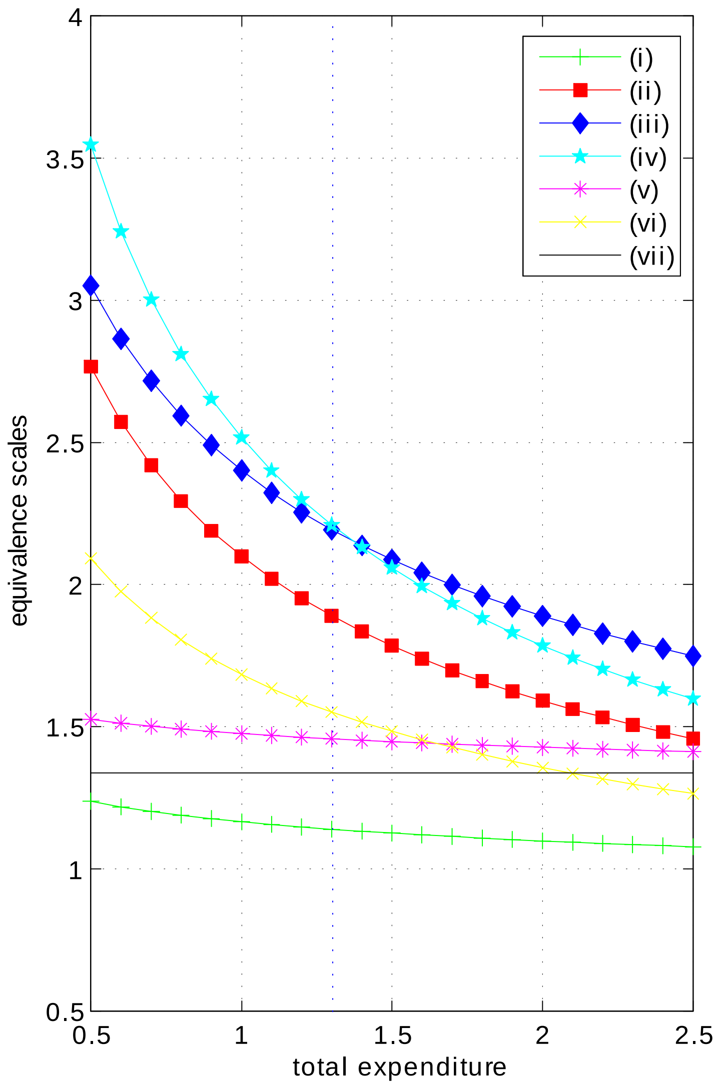

Figure 8: Plots of expenditure-dependent equivalence scales for households with 7 different characteristics: (i) 1 youji (ii) 1 youji and 1 shou (iii) 1 shou and 1 shonen (iv) 2 shou (v) 1 shonen (vi) 1 shou (vii) 1 shou. The dotted vertical line at 1.303 shows the level of total expenditure at the base point. Unit is 100,000 yen.

dren should consider the progressivity in the compensation. Poor households with children should be given bigger compensations than rich households with children to maintain the welfare equality. However, the lack of stronger degrees of decline in the equivalence scales as income rises and their statistical significance weaken the above assertion since our estimated equivalence scales still produce higher equivalent expenditures for higher-income class households.

5. Conclusion

In this dissertation, after reviewing relevant theoretical results on the consumer demand theory in chapter 2, chapter 3 investigates the regularity property of the normalized quadratic demand system of Diewert and Wales (1988b). The regularity conditions of monotonicity and concavity are two of the axioms which guarantee the existence of the utility maximizing consumers, and the fulfillment of all axioms makes direct utility, indirect utility, and expenditure functions equivalent representations of the underlying preferences. Without satisfaction of both curvature and monotonicity, the second-order condition for optimizating behavior fails, duality theory fails, and inferences resulting from derived estimating equations become invalid. Barnett (2002) commented that imposition of the monotonicity condition is especially overlooked when global curvature is imposed on the NQ model. To futher examine his observation, we displayed regular regions, differentiating regular and non-regualr regions by the types of violations to fully reveal the regularity property of the NQ model. We tried three methods of imposing curvature on the NQ model: global, local, and no curvature imposition, to see how the different methods affect the regular regions of the model. We found that monotonicity violations are especially likely to occur when elasticities of substitution are greater than one. We also found that imposing curvature locally produces difficulties in estimation, smaller regular regions, and poor elasticity estimates in many cases considered. When imposing curvature globally, our results were better. Although violations of monotonicity remained common in some of our cases, those violations do not appear to have been induced solely by the curvature imposition, but rather by the nature of the Normalized Quadratic model itself. However, imposition of global curvature makes complement goods more complement and substitute goods more substitute. With the Normalized Quadratic model, we find that both curvature and monotonicity must be checked with the estimated model, as has previously been shown to be the case with many other flexible functional forms. Imposition of curvature alone does not assure regularity, and imposing local curvature alone can have very adverse consequences.

One straightforward future research direction is to expand the analysis to other, even newer demand systems such as QUAIDS, which has been increasingly used as a better alternative to the classic AIDS model, and such as the translated QUAIDS, which has the unusually high rank of four. Lewbel (1988) suggested that a rank-three trigonometric demand system is able to produce an unusually large regular region, but its implementation is only found in Matsuda (2006) as far as we know.

Another direction is to try different methods of imposing correct curvature on demand functions. Diewert and Lawrence (2002) proposed imposing curvature at *two* points to ensure globally correct curvature without sacrificing the flexibility property of the normalized quadratic profit function. Diewert and Wales (1988a) proposed the concept of a semiflexible functional form in which the rank of the Slutsky matrix is reduced, but in a manner that does not restrict the price derivatives in any a priori undesirable way. This allows the researcher to systematically limit the parameter space while maintaining full income flexibility and partial price flexibility, an objective that may prove useful in large demand systems. The model is designed to estimate large demand systems by sacrificing the flexibility to attain globally correct curvature (Ryan and Wales, 1998). Moschini (1998) studied the semiflexible AIDS on which only the local curvature is imposed. Thus, it may be interesting to see whether this additional restrictiveness induces further monotonicity violations as the rank of the Slutsky matrix is reduced.

In chapter 4, we studied the expenditure-dependent equivalence scales to estimate the cost of a child in Japan. We found that the Japanese household equivalence scales are decreasing in the total expenditure as well as increasing in the number of children. We introduced the property called "Generalized Relative/Absolute Expenditure-Scale Exactness" (GRESE/GAESE) which can identify the estimated equivalence scales by observed demand data alone if the specification of the reference household expenditure function is neither affine nor log-affine (PIGLOG). Hence, we employ the functional forms of the demand model that are neither affine nor log-affine and that have a higher rank which can explain the more complicated shapes of the Engel curves. We paid special attention to correcting a possible bias in the parameter estimation induced by a pile of observations of zero expenditure on some goods. Our results suggest an intuitively straightfoward policy design for the child-support welfare program: the compensation should depend on their income level as well as on the number of children. Depending on what type of health insurance a household holds, Japan's current child-subsidy program sets income threshold levels for receiving child-support in the form of monetary compensation. But the new recently proposed program does not set an income level cutoff. Our results suggest that in terms of the welfare equality there is a need for a mechanism to decrease benefits as household income goes up and for a reasonable limit to be set on the income level for households to receive this entitlement.

Future research should focus on obtaining more robust estimation results in order to draw persuasive conclusions for the purpose of devising policy. Taking the case (iv) for the equivalence scale of a household with one child between the ages of 7 and 12 years old, we contrived to obtain one that declines relatively sharply in the total expenditure. However, this equivalence scale still produces higher equivalent expenditures for households with higher income. Hence, the criticism for the constant-equivalence scales still remains. In order to draw a strong conclusion, we need to have larger negative values of statistically significant parameter estimates for b_0^r's.

One direction to follow to modify our method is to try out other functional forms of demand systems that are neither log-affine nor affine and that possess ranks higher than two while we continue to search for a way to get the translated QUAIDS to work.

Donaldson and Pendakur (1999) used the Quadratic Expenditure System (QES) proposed by Howe, Pollak and Wales (1979).[72] The demographically modified indirect utility function of QES is written as

$$V(\boldsymbol{p}, x, \boldsymbol{z}) = \left[\left(\frac{x - a(\boldsymbol{p}, \boldsymbol{z})}{b(\boldsymbol{p}, \boldsymbol{z})} \right) - c\left(\boldsymbol{p}, \boldsymbol{z}\right) \right]^{-1}, \tag{173}$$

where a and b are homogeneous of degree one in $\boldsymbol{p}$ and c is homogeneous of

[72] See section 2.2.5.4.

degree zero in $\boldsymbol{p}$. Denoting $a^r(\boldsymbol{p}) = a(\boldsymbol{p}, \boldsymbol{z})$, $b^r(\boldsymbol{p}) = b(\boldsymbol{p}, \boldsymbol{z})$, and $c^r(\boldsymbol{p}) = c(\boldsymbol{p}, \boldsymbol{z})$ and assuming that the reference indirect utility function is QES, GAESE of (130) implies that

$$
\begin{aligned}
V(\boldsymbol{p}, x, \boldsymbol{z}) &= V^r(\boldsymbol{p}, x^e) = V^r\left(\boldsymbol{p}, \frac{x - A(\boldsymbol{p}, \boldsymbol{z})}{R(\boldsymbol{p}, \boldsymbol{z})}\right) \\
&= \left[\left(\frac{\frac{x-A(\boldsymbol{p},\boldsymbol{z})}{R(\boldsymbol{p},\boldsymbol{z})} - a^r(\boldsymbol{p})}{b^r(\boldsymbol{p})}\right) - c^r(\boldsymbol{p})\right]^{-1} \\
&= \left[\left(\frac{x - A(\boldsymbol{p}, \boldsymbol{z}) - R(\boldsymbol{p}, \boldsymbol{z}) a^r(\boldsymbol{p})}{R(\boldsymbol{p}, \boldsymbol{z}) b^r(\boldsymbol{p})}\right) - c^r(\boldsymbol{p})\right]^{-1}. \qquad (174)
\end{aligned}
$$

If reference preferences satisfy QES, then, given GAESE, all households have QES preferences. In this case, GAESE implies that

$$a(\boldsymbol{p}, \boldsymbol{z}) = R(\boldsymbol{p}, \boldsymbol{z}) a^r(\boldsymbol{p}) + A(\boldsymbol{p}, \boldsymbol{z}),$$

$$b(\boldsymbol{p}, \boldsymbol{z}) = R(\boldsymbol{p}, \boldsymbol{z}) b^r(\boldsymbol{p}),$$

and

$$q(\boldsymbol{p}, \boldsymbol{z}) = q^r(\boldsymbol{p}).$$

Thus, given GAESE, we can estimate equivalent-expenditures by requiring $c(\boldsymbol{p}, \boldsymbol{z}) = c^r(\boldsymbol{p})$ and calculating

$$R(\boldsymbol{p}, \boldsymbol{z}) = \frac{b(\boldsymbol{p}, \boldsymbol{z})}{b^r(\boldsymbol{p})}$$

and

$$A(\boldsymbol{p}, \boldsymbol{z}) = a(\boldsymbol{p}, \boldsymbol{z}) - R(\boldsymbol{p}, \boldsymbol{z}) a^r(\boldsymbol{p}).$$

One can check that GRESE cannot apply to the translated QUAIDS and QES since it is not possible to obtain a representation such as (174) corresponding to (173) or a representation such as (137) corresponding to (138). Similarly, GAESE cannot apply to the QUAIDS specification of demand system. We do not know of any other demand systems that allow for the implementation of GRESE.

For example, the exactly aggregable trigonometric Engel curve demand system (Lewbel, 1988) is rank three, but imposition of GRESE or GAESE is not feasible although the imposition of ESE is possible. The demographically modified indirect utility function of the trigonometric demand system is written as[73]

$$V(\boldsymbol{p}, x, \boldsymbol{z}) = b(\boldsymbol{p}, \boldsymbol{z}) + \frac{c(\boldsymbol{p}, \boldsymbol{z}) \cos\{\tau \ln a(\boldsymbol{p}, \boldsymbol{z}) - \tau \ln x\}}{1 + \sin\{\tau \ln a(\boldsymbol{p}, \boldsymbol{z}) - \tau \ln x\}}, \qquad \tau \neq 0, \quad (175)$$

where $a(\boldsymbol{p}, \boldsymbol{z})$ is homogeneous of degree one, and $b(\boldsymbol{p}, \boldsymbol{z})$ and $c(\boldsymbol{p}, \boldsymbol{z})$ are homogeneous of degree zero in $\boldsymbol{p}$. Denoting $a^r(\boldsymbol{p}) = a(\boldsymbol{p}, \boldsymbol{z}^r)$, $b^r(\boldsymbol{p}) = b(\boldsymbol{p}, \boldsymbol{z}^r)$, and $c^r(\boldsymbol{p}) = c(\boldsymbol{p}, \boldsymbol{z}^r)$ and assuming that the reference indirect utility function is the trigonometric demand system, ESE on (175) implies that

$$\begin{aligned} V(\boldsymbol{p}, x, z) &= V^r(\boldsymbol{p}, x^e) = V^r\left(\boldsymbol{p}, \frac{x}{G(\boldsymbol{p}, x)}\right) \\ &= b^r(\boldsymbol{p}) \\ &\quad + \frac{c^r(\boldsymbol{p}) \cos\{\tau(\ln a^r(\boldsymbol{p}) + \ln G(\boldsymbol{p}, \boldsymbol{z})) - \tau \ln x\}}{1 + \sin\{\tau(\ln a^r(\boldsymbol{p}) + \ln G(\boldsymbol{p}, \boldsymbol{z})) - \tau \ln x\}}. \end{aligned}$$

[73] See section 2.2.5.6.

If reference preferences satisfy TDS, then, given ESE, all households have TDS preferences. In this case, ESE implies that

$$b(\boldsymbol{p}, \boldsymbol{z}) = b^r(\boldsymbol{p}),$$

$$c(\boldsymbol{p}, \boldsymbol{z}) = c^r(\boldsymbol{p}),$$

and

$$\ln a(\boldsymbol{p}, \boldsymbol{z}) = \ln a^r(\boldsymbol{p}) + \ln G(\boldsymbol{p}, \boldsymbol{z}).$$

Thus, given ESE we can estimate equivalent-expenditures by requiring $b(\boldsymbol{p}, \boldsymbol{z}) = b^r(\boldsymbol{p})$ and $c(\boldsymbol{p}, \boldsymbol{z}) = c^r(\boldsymbol{p})$ and calculating

$$G(\boldsymbol{p}, \boldsymbol{z}) = \frac{a(\boldsymbol{p}, \boldsymbol{z})}{a^r(\boldsymbol{p})}.$$

ESE can be tested with an unconstrained form and a constrained form of TDS using the Likelihood test.

We also may be able to use a very new demand system, the Exact Affine Stone Index (EASI) implicit Marshallian demand system (Lewbel and Pendakur, 2009). EASI demand systems allow for the incorporation of both unobserved preference heterogeneity and complex Engel curves into demand functions. With the STATA computer code, Pendakur (2009) gives a less technical introduction to implicit Marshallian demands and to the EASI demand system in particular. Lewbel and Pendakur (2009) explains how to impose ESE on EASI demand systems, but we do not know how GRESE and GAESE are applicable to EASI demand systems.

Another direction to improve our procedure is to try out different methods to address the presence of a pile of zero-expenditure observations on some goods. Hasegawa, Ueda and Mori (2008) explicitly assume the mechanism of a number of the zero expenditure occurrence as infrequency of purchase (IFP), and they estimate a system of Engel functions using Markov chain Monte Carlo method to obtain the estimates of unobservable parameters. Golan, Perloff and Shen (2001) demonstrate a new approach to efficiently estimate a system of AIDS demand functions for Mexican meat demand with binding nonnegativity constraints. This approach, called generalized maximum entropy (GME) building on the entropy-information measure of Shannon (1948), the classical maximum entropy (ME) principle of Jaynes (1957a, 1957b) which was developed to recover information from underdetermined systems, and generalized maximum entropy theory of Golan, Judge and Miller (1996), is more practical and efficient than traditional maximum likelihood methods. The methods have been applied in Golan, Judge and Miller (2003); Golan, Judge and Karp (1996); and Arndt (1999). The implementation of these innovative procedures is technically difficult and computationally intensive. In any case, the ideal method will be the one that can utilize the household characteristic effects most effectively. The procedure in chapter 4 conveniently assumes that the explanatory variables of sample selection equations are any possible demographic characteristics variables other than prices, total expenditures, and the number of age types of the children. It is possible that some of the demographic characteristics variables may better explain the household behavior if included in the demand system. Another improvement will be to use the multivariate Probit model instead of the Probit estimation of each share equation to increase the efficiency of the first-step estimation procedure.

We may even be able to exploit the nature of our data set as panel data to avoid using any of the restrictive GRESE or GAESE to construct the expenditure-dependent equivalence scales. It is possible to incorporate unobservable individual heterogeneity into demand systems as random effects (Pollak and Wales, 1992, ch.5) or to use the EASI demand system which already has embeded the feature.[74] But we are not aware of any studies which successfully take advantage of panel data as "additional data" to identify the expenditure-dependent equivalence scales and which address the zero expenditure issue at the same time.

[74] Biørn and Jansen (1983) estimated a system of demand functions with individual effects using incomplete cross-section/time-series data.

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