Financial Economies with Restricted Participation

By

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To My Parents

Abstract

An abundant literature is concerned with the existence of equilibrium in incomplete markets where participation to financial markets is not restricted. To mention a few, Cass (1984), Werner (1985), Geanakoplos and Polemarchakis (1986), Duffie (1987), Duffie and Shafer (1985) and Magill and Quinzii (1996). Financial economies with incomplete markets assume (in general) a symmetric participation structure, i.e. each consumer is confronted with the same restrictions on her portfolio trades. This is a very limiting assumption: in reality, people get to know about different investment opportunities and not all investors are able to trade in the same markets. In this work we consider institutional restrictions on trading activity in the financial markets. Following Siconolfi (1989), Angeloni and Cornet (2006), and Hahn and Won (2007), the broadest formulation of such restricted participation is to assume that households face financial constraints modeled by closed convex subsets of the portfolio space.

Our contribution to the literature on general equilibrium of financial markets is threefold. In Chapter 2 we refine the definition of reduced financial structure to accommodate the case of financial structures with restricted participation. We then provide a characterization of reduced financial structures in terms of arbitrage-free prices and by the compactness of a set of "admissible" portfolio allocations. In Chapter 3 we introduce an equivalence relation on the set of financial structures and we show that, under mild assumptions, every financial structure is equivalent to a reduced financial structure, and that subsequently, all equilibria in a financial economy are in one-to-one correspondence with the equilibria of an economy where the financial structure is replaced by an equivalent reduced one. Finally, in Chapter 4 we prove a general existence result of equilibria for financial exchange economies with restricted participation in which agents may have nonordered preferences. Our result extends the results by Radner (1972), and Siconolfi (1989), and also extends to the restricted participation case the results by Cass (1984), Werner (1985), and Duffie (1987).

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Contents

Chapter 1

Introduction to financial markets

1.1 The financial exchange economy

1.1.1 The model of a stochastic economy

[1] We consider the basic model of a two time-date economy with nominal assets. It is also assumed that there are finite sets I, H, S, and J, respectively, of agents, divisible physical commodities, states of nature, and nominal assets.

In what follows, the first date will also be referred to as $t = 0$ and the second date, as $t = 1$. There is an a priori uncertainty at the first date ($t = 0$) about which of the states of nature $s \in S$ will prevail at the second date ($t = 1$). For the sake of unified notations of time and uncertainty, the non-random state at the first date is denoted by $s = 0$ and $\bar{S}$ stands for the set $\{0\} \cup S$.

[1] We shall use the following notation. If I and J are finite sets, the space $\mathbb{R}^I$ (identified to $\mathbb{R}^{\#I}$ whenever necessary) of functions $x : I \to \mathbb{R}$ (also denoted $x = (x(i))_{i \in I}$ or $x = (x_i)_i$) is endowed with the scalar product $x \cdot y := \sum_{i \in I} x(i)y(i)$, and we denote by $\|x\| := \sqrt{x \cdot x}$ the Euclidean norm. By $B(x, r)$ we denote the closed ball centered at $x \in \mathbb{R}^L$ of radius $r > 0$, namely $B(x, r) = \{y \in \mathbb{R}^L : \|y - x\| \leq r\}$. In $\mathbb{R}^I$, the notation $x \geq y$ (resp. $x > y$, $x \gg y$) means that, for every i, $x(i) \geq y(i)$ (resp. $x \geq y$ and $x \neq y$, $x(i) > y(i)$) and we let $\mathbb{R}^I_+ = \{x \in \mathbb{R}^I \mid x \geq 0\}$, $\mathbb{R}^I_{++} = \{x \in \mathbb{R}^I \mid x \gg 0\}$. An $I \times J$-matrix $A = (a^j_i)_{i \in I, j \in J}$ is an element of $\mathbb{R}^{I \times J}$ whose rows are denoted $A_i = (a^j_i)_{j \in J} \in \mathbb{R}^J$ ($i \in I$), and columns $A^j = (a^j_i)_{i \in I} \in \mathbb{R}^I$ ($j \in J$). To the matrix A, we associate the linear mapping, from $\mathbb{R}^J$ to $\mathbb{R}^I$, also denoted by A, defined by $Ax = (A_i \cdot x)_{i \in I}$. The span of the matrix A, also called the image of A, is the set $< A > := \{Ax \mid x \in \mathbb{R}^J\}$. The transpose matrix of A, denoted by A^T, is the $J \times I$-matrix whose rows are the columns of A, or equivalently, is the unique linear mapping $A^T : \mathbb{R}^I \to \mathbb{R}^J$, satisfying $Ax \cdot y = x \cdot A^T y$ for every $x \in \mathbb{R}^J$, $y \in \mathbb{R}^I$.

At each state of nature $s \in \bar{S}$, there is a spot market where the finite set H of physical commodities is available. We assume that each commodity does not last more than one period so that the commodity space is $\mathbb{R}^L$, with $L = H \times \bar{S}$ (in this model, a commodity is a couple $(h, s) \in H \times \bar{S}$ of a physical commodity, h, and a state of nature s, at which h will be available). An element $x \in \mathbb{R}^L$ is called a consumption (or a consumption plan), that is $x = (x(s))_{s \in \bar{S}} \in \mathbb{R}^L$, where $x(s) = (x(h, s))_{h \in H} \in \mathbb{R}^H$, for every $s \in \bar{S}$.

We denote by $p = (p(s))_{s \in \bar{S}} \in \mathbb{R}^L$ the vector of spot prices and $p(s) = (p(h, s))_{h \in H} \in \mathbb{R}^H$ is called the spot price at state s. The spot price $p(h, s)$ is the price paid, at date 0 if $s = 0$ and at date 1 if $s \in S$, for the delivery of one unit of commodity h at state s.

Each agent $i \in I$, also called a consumer, is endowed with a consumption set $X_i \subset \mathbb{R}^L$ which is the set of her possible consumptions. Typically we can take $X_i = \mathbb{R}^L_+$, but we allow for more general consumption sets. An allocation is an element $x = (x_i)_{i \in I} \in \Pi_i X_i$, and we denote by x_i the consumption of agent i.

The tastes of each consumer $i \in I$ are represented by a strict preference correspondence $P_i : \Pi_{k \in I} X_k \to X_i$, where $P_i(x)$ defines the set of consumptions that are strictly preferred by i to x_i, that is, given the consumptions x_k for other consumers $k \neq i$.

At each state of nature, $s \in \bar{S}$, every consumer $i \in I$ has a state-endowment of physical commodities, $e_i(s) \in \mathbb{R}^H$, contingent to the fact that s prevails and we denote by $e_i = (e_i(s))_{s \in \bar{S}} \in \mathbb{R}^L$ her endowment vector across the different states.

The consumption side of the economy, denoted $\mathcal{E}$, can be summarized by

$$\mathcal{E} = \Big(I, H, S, (X_i, P_i, e_i)_{i \in I}\Big).$$

Definition 1.1 *The economy $\mathcal{E}$ is said to be standard if it satisfies the following two standard assumptions* **C** *and* **LNS**.

Consumption Assumption C

- **(i)** *For every $i \in I$, X_i is a bounded below, closed, convex subset of $\mathbb{R}^{L(1+S)}$.*
- **(ii)** **Continuity of Preferences** *For every $i \in I$, the correspondence $P_i : \Pi_i X_i \to X_i$ is lower semicontinuous with convex open values in X_i for the relative topology of X_i.*
- **(iii)** **Irreflexive Preferences** *For every $i \in I$, for every $x = (x_i)_{i \in I} \in \Pi_i X_i$, $x_i \notin P_i(x)$.*

(iv) Strong Survival SS *For every* $i \in I$, $e_i \in \text{int} X_i$.

(v) Non-Satiation NS *For every* $i \in I$, *for every* $x \in \Pi_i X_i$, *for every* $s \in \bar{S}$, *there exists* $(x_i^n)_n \subset P_i(x)$ *such that* $x_i^n(s') = x_i(s')$ *for all* $s' \neq s$, *and* $x_i^n \underset{n\to\infty}{\longrightarrow} x_i$.

Agents may operate financial transfers across states in $\bar{S}$ (i.e. across the two dates and across the states of the second date) by exchanging a finite number of nominal assets $j \in J$, which define the financial structure of the model.[2] The nominal assets are traded at the first date ($t = 0$) and yield payoffs at the second date ($t = 1$), contingent on the realization of the state of nature $s \in S$. The payoff of the nominal asset $j \in J$, when state $s \in S$ is realized, is v_s^j, and we denote by V the $S \times J$-payoff matrix $V = (v_s^j)$, which does not depend upon the asset prices $q \in \mathbb{R}^J$ (and will not depend upon the commodity prices p in the associated equilibrium model). A portfolio $z = (z_j) \in \mathbb{R}^J$ specifies the quantities $|z_j|$ ($j \in J$) of each asset j (with the convention that the asset j is bought if $z_j > 0$ and sold if $z_j < 0$). Thus $Vz \in \mathbb{R}^S$ is the random financial payoff of portfolio z across states at time $t = 1$, and $V_s \cdot z$ is the payoff if state s prevails. Given an asset price vector $q \in \mathbb{R}^J$, we define the $(S+1) \times J$ matrix

$$W(q) = \begin{pmatrix} -q \\ V \end{pmatrix}$$

referred to as the full-payoff matrix.

We assume that each agent i is restricted to choose her portfolio within a portfolio set $Z_i \subset \mathbb{R}^J$, which represents the set of portfolios that are (institutionally) admissible for agent i. This general framework allows us to address, for example, the following important cases:

(i) $Z_i = \mathbb{R}^J$ (unconstrained portfolios).

(ii) $Z_i = \underline{z}_i + \mathbb{R}^J_+$, for some $\underline{z}_i \in -\mathbb{R}^J_+$ (exogenous bounds on short sales).

(iii) $Z_i = B_J(0, 1)$ (bounded portfolio sets).

(iv) Z_i is a vector space (linear equality constraints).

(v) Z_i is polyhedral and contains 0 (linear equality and inequality portfolio constraints).

[2]The case of no financial assets – i.e., J is empty – is called pure spot markets.

Note that the polyhedral case covers situations (i)-(iv) (with an appropriate choice of the norm in (iii)). In the sequel, we make the following assumption which covers all the above cases.

F1. For every $i \in I$, the set Z_i is closed, convex, and contains 0.

We summarize by $\mathcal{F} = \big(I, S, J, V, (Z_i)_{i\in I}\big)$ the financial characteristics, referred to as the financial structure. When there is no risk of confusion, the financial structure $\mathcal{F}$ will be denoted $\big(V, (Z_i)_i\big)$. When there are no constraints, $\mathcal{F}$ will be simply denoted by V.

The financial exchange economy $(\mathcal{E}, \mathcal{F})$ is a couple of an exchange economy $\mathcal{E}$ and a financial structure $\mathcal{F}$ as described above and it can be summarized by

$$(\mathcal{E}, \mathcal{F}) = \Big(I, H, S, (X_i, P_i, e_i)_{i\in I}, J, V, (Z_i)_{i\in I}\Big).$$

1.1.2 Financial equilibria and no-arbitrage

Consider a financial exchange economy $(\mathcal{E}, \mathcal{F})$. Given the spot price vector $p \in \mathbb{R}^L$ and the asset price vector $q \in \mathbb{R}^J$, the *budget set* of consumer $i \in I$ in this setting is defined as follows[3]

$$\begin{aligned} B_i(\mathcal{F}, p, q) &= \{(x_i, z_i) \in X_i \times Z_i : \ \forall s \in \bar{S}, \ \ p(s) \cdot [x_i(s) - e_i(s)] \leq [W(q)z_i](s)\} \\ &= \{(x_i, z_i) \in X_i \times Z_i : \ p \,\square\, (x_i - e_i) \leq W(q)z_i\}. \end{aligned}$$

An equilibrium in the financial exchange economy is then defined as a collection of commodity spot prices, consumptions (one for each agent), asset prices, and portfolios (one for each agent) such that each agent maximizes her preferences over her budget set, and all markets clear (commodity markets clear in all dates and states, and asset markets clear). Formally, we have

Definition 1.2 *An equilibrium in the financial exchange economy* $(\mathcal{E}, \mathcal{F})$ *is a list* $\big(\bar{p}, \bar{x}, \bar{q}, \bar{z}\big)$ $\in \mathbb{R}^L \backslash \{0\} \times (\mathbb{R}^L)^I \times \mathbb{R}^J \times (\mathbb{R}^J)^I$ *such that*

(a) *for every* $i \in I$, $(\bar{x}_i, \bar{z}_i)$ *maximizes the preferences* P_i *in the budget set* $B_i(\mathcal{F}, \bar{p}, \bar{q})$, *in the sense that*

$$(\bar{x}_i, \bar{z}_i) \in B_i(\mathcal{F}, \bar{p}, \bar{q}) \ \textit{and} \ [P_i(\bar{x}) \times Z_i] \cap B_i(\mathcal{F}, \bar{p}, \bar{q}) = \emptyset,$$

[3]For $x = (x(s))_{s\in\bar{S}}, p = (p(s))_{s\in\bar{S}}$ in $\mathbb{R}^L = \mathbb{R}^{H\bar{S}}$ (with $x(s), p(s)$ in $\mathbb{R}^H$ for each $s \in \bar{S}$) we let $p \,\square\, x = (p(s) \cdot x(s))_{s\in\bar{S}} \in \mathbb{R}^{\bar{S}}$.

where $\bar{x} = (\bar{x}_i)_{i \in I}$*, and*

(b) $\sum_{i \in I} \bar{x}_i = \sum_{i \in I} e_i$ *and* $\sum_{i \in I} \bar{z}_i = 0$.

A consumption equilibrium in the financial exchange economy $(\mathcal{E}, \mathcal{F})$ *is a list of commodity prices and consumptions* $(\bar{p}, \bar{x}) \in \mathbb{R}^L \backslash \{0\} \times (\mathbb{R}^L)^I$ *such that there exist asset prices and portfolios* $(\bar{q}, \bar{z}) \in \mathbb{R}^J \times (\mathbb{R}^J)^I$ *with* $\big(\bar{p}, \bar{x}, \bar{q}, \bar{z}\big)$ *is an equilibrium in* $(\mathcal{E}, \mathcal{F})$*.*

The following notion of no-arbitrage takes into account only arbitrage opportunities that might yield an infinite payoff.

Definition 1.3 *Given the financial structure* $\mathcal{F} = \big(V, (Z_i)_{i \in I}\big)$*, the portfolio* $v_i \in \mathbb{R}^J$ *is said to be an asymptotic arbitrage opportunity for agent* $i \in I$ *at the price* $\bar{q} \in \mathbb{R}^J$ *if* v_i *belongs to the asymptotic cone*[4] *of* Z_i*, denoted* $\boldsymbol{A}Z_i$*, and* $W(\bar{q})v_i > 0$*.*

Definition 1.4 *The asset price vector* $\bar{q}$ *is said to be asymptotic-arbitrage-free if for every agent* $i \in I$ *there is no asymptotic arbitrage opportunity at* $\bar{q}$*, that is, if*

$$W(\bar{q}) \ (\bigcup_{i \in I} \boldsymbol{A}Z_i) \cap \mathbb{R}_+^{S+1} = \{0\},$$

and we denote by Q *the set of asymptotic arbitrage-free prices.*

It is worth noticing that the set Q is a convex cone[5], a property which will be used throughout this thesis.

Proposition 1.1 *The set* Q *is a convex cone of* $\mathbb{R}^J$*.*

We recall that, under the non-satiation assumption **NS**, equilibrium asset price vectors are asymptotic-arbitrage-free. See [5] for a proof of Proposition 1.2.

Proposition 1.2 *Under* **NS***, if* $\big(\bar{p}, \bar{x}, \bar{q}, \bar{z}\big)$ *is an equilibrium of the economy* $(\mathcal{E}, \mathcal{F})$*, then* $\bar{q}$ *is asymptotic-arbitrage-free.*

[4]If C is a nonempty convex set in $\mathbb{R}^\alpha$, the asymptotic cone of C is $\boldsymbol{A}C = \{v \in \mathbb{R}^\alpha : v + \text{cl}C \subset \text{cl}C\}$. Note that the definition is given with "cl$C$" instead of C as in the cone $O^+(C)$ as defined by Rockafellar [35] so that $\boldsymbol{A}C = O^+(\text{cl}C)$.

[5]A set Q is a cone if for every $q \in Q$ and $\lambda > 0$, one has $\lambda q \in Q$.

1.1.3 Equivalent financial structures

We introduce an equivalence relation on the set of all financial structures. We will say that two financial structures are equivalent if they are indistinguishable in terms of consumption equilibria. The intuition behind this definition is the following. Financial structures allow agents to transfer wealth across states of nature and thereby give them the possibility to enlarge their budget set. Hence if, regardless of the standard exchange economy $\mathcal{E}$, equilibrium consumption allocations and equilibrium commodity price vectors are the same when agents carry out their financial activities through two different structures, then these two financial structures are said to be equivalent. Formally

Definition 1.5 *Consider two financial structures $\mathcal{F} = \big(V, (Z_i)_i\big)$ and $\mathcal{F}' = \big(V', (Z_i')_i\big)$. We say that $\mathcal{F} \sim \mathcal{F}'$ (read $\mathcal{F}$ is equivalent to $\mathcal{F}'$) if for every standard exchange economy $\mathcal{E}$, the financial exchange economies $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{E}, \mathcal{F}')$ have the same consumption equilibria.*

1.2 The unconstrained case

1.2.1 Existence of equilibria: the unconstrained case

In this section we consider an unconstrained financial structure $\mathcal{F} = \big(V, (\mathbb{R}^J)_i\big)$.

Unconstrained Portfolios For every $i \in I$, $Z_i = \mathbb{R}^J$.

The following theorem is the standard result of existence of equilibria in a financial exchange economy where agents portfolios are unrestricted (see for example Cass [10], Werner [37], Duffie and Schafer ([19], [20]), and Duffie [17]).

Theorem 1.1 *Let $(\mathcal{E}, \mathcal{F}) = \big((X_i, P_i, e_i)_{i\in I}, (V, (\mathbb{R}^J)_{i\in I})\big)$ be a financial exchange economy with unconstrained portfolios satisfying assumption* **C**, *then $(\mathcal{E}, \mathcal{F})$ admits an equilibrium.*

The standard proof of Theorem 1.1 is performed in three steps which are worth describing explicitly since we will follow the same scheme in the constrained case:

Step 1. Consider a reduced form (see Definition-Proposition 1.7). This is straightforward in the unconstrained case.

Step 2. Consider an equivalent reduced form $\mathcal{F}'$ of $\mathcal{F}$ (use Proposition 1.6 to eliminate redundant assets and get an equivalent reduced form).

Step 3. Show existence of equilibrium for the financial exchange economy $(\mathcal{E}, \mathcal{F}')$ using the compactness property of the set of "admissible" portfolio allocations (see Proposition 1.3). This would imply that $(\mathcal{E}, \mathcal{F})$ has an equilibrium since $\mathcal{F}$ is equivalent to $\mathcal{F}'$.

1.2.2 Reduced form: Eliminating redundant assets or useless portfolios?

In this section, we recall the notions of reduced financial structure and useless portfolio allocation when market participation is unrestricted. We will see later on that the definitions given in this section need to be refined to suit the case of restricted participation.

Definition 1.6 *We say that a financial structure $\mathcal{F} = \big(V, (Z_i)_i\big)$ is unconstrained if, for every $i \in I$, agent i's participation to the financial markets is unrestricted, that is $Z_i = \mathbb{R}^J$ where J is the number of assets available (J is the number of columns in the matrix V). An unconstrained financial structure $\mathcal{F} = (V, (\mathbb{R}^J)_i)$ will be simply denoted V.*

Proposition 1.3 *The unconstrained financial structure $\mathcal{F}$ is said to be reduced if it satisfies one of the following equivalent properties:*

(i) *The financial structure has no redundant[6] assets, that is, the return matrix V has full column rank (rank $V = J$).*

(ii) *The return matrix V is one-to-one ($\ker V = \{0\}$), hence there is no nonzero portfolio allocation $\zeta = (\zeta_1, \cdots, \zeta_I) \in (\ker V)^I$ (called useless hereafter).*

(iii) *The set Q of asymptotic arbitrage-free prices has full dimension ($\dim Q = J$) or equivalently, there is no nontrivial linear dependence between the asset prices, that is, there does not exist $\alpha \in \mathbb{R}^J, \alpha \neq 0$, such that $\sum_{j\in J} \alpha^j q^j = 0$ for every $q \in Q$.*

(iv) *For every $v = (v_i)_i \in (\mathbb{R}^S)^I$, the set $K_0(v)$ of "admissible" portfolio allocations is compact, where*

$$K_0(v) := \{(z_i)_i \in (\mathbb{R}^J)^I : \forall i \in I, \quad V z_i \geq v_i, \sum_{i\in I} z_i = 0\}.$$

[6] Recall that an asset is said to be redundant if its payoff is a linear combination of other assets payoffs.

Proof. The implication (i) $\Rightarrow$ (ii) is trivial. We show (ii) $\Rightarrow$ (iii). It is well known that in the unconstrained case, $Q = V^T(\mathbb{R}^S_{++})$. Since the linear map V is one-to-one, the linear map V^T is onto hence open. Therefore the set Q is open as the image of an open set by an open mapping. This shows that Q has full dimension. Now we show (iii) $\Rightarrow$ (iv). To this end, we show that $\boldsymbol{A}K_0(v) = \{0\}$. $K_0(v)$ is obviously closed, convex and

$$\boldsymbol{A}K_0(v) = \big\{(\zeta_i)_i \in (\mathbb{R}^J)^I, \forall i\ V\zeta_i \geq 0, \sum_{i\in I}\zeta_i = 0\big\}.$$

Let $(\zeta_1, \cdots, \zeta_I) \in \boldsymbol{A}K_0(v)$. From $V\zeta_i \geq 0$ for each i, and $\sum_{i\in I}\zeta_i = 0$, we get $V\zeta_i = 0$. Thus for every $i \in I$ and for every $q \in Q$, one has $q \cdot \zeta_i = 0$ (otherwise ζ_i or $-\zeta_i$ would be an asymptotic arbitrage opportunity at q which contradicts the fact that q is in Q). Hence for every $i \in I$, $\zeta_i \in Q^\perp = \{0\}$, that is $(\zeta_1, \cdots, \zeta_I) = (0, \cdots, 0)$. Finally we show that (iv) $\Rightarrow$ (i). If $z \in \ker V$, then $(z, -z, 0, \cdots, 0) \in \boldsymbol{A}K_0\big((0, \cdots, 0)\big) = \{0\}$. Hence $z = 0$. ∎

Definition 1.7 *The unconstrained financial structure $\mathcal{F}$ is said to be reduced if it satisfies one of the equivalent properties of Proposition 1.3.*

The concept of reduced financial structure is intimately related to the concept of useless portfolio allocation. We will say that a portfolio $z \in \mathbb{R}^J$ is useless if it has zero payoff, that is $Vz = 0$. Proposition 1.3 establishes that absence of nonzero useless portfolio allocations is equivalent to the financial structure being reduced when participation to financial markets is unrestricted. Therefore either concept can be taken as a primitive in the description of financial structures.

An important motivation to our interest in reducing financial structures is property (iv) in Proposition 1.3. We will call an opportunity of financial transfers to tomorrow ($t = 1$), any collection $(v_1, \cdots, v_I)$ of vectors in the space of returns $\mathbb{R}^S$. We will say that the opportunity of financial transfers to tomorrow $(v_1, \cdots, v_I)$ is achievable through or offered by (respectively, guaranteed by) the financial structure $\mathcal{F}$ if there exists a family of feasible and mutually compatible[7] portfolios $(z_1, \cdots, z_I)$ such that $Vz_i = v_i$ (respectively, $Vz_i \geq v_i$) for each $i \in I$. Proposition 1.3 states that a financial structure is reduced if and only if the set of mutually compatible portfolio allocations that guarantee a given level of returns is compact and, à fortiori, the set of mutually compatible portfolio allocations that achieve a given opportunity of financial transfers to tomorrow is compact.

[7] A portfolio allocation $(z_1, \cdots, z_I)$ is mutually compatible if it clears asset markets, that is, $\sum_{i\in I} z_i = 0$.

Consider an unconstrained financial structure $\mathcal{F} = \big(V, (\mathbb{R}^J)_i\big)$. Proposition 1.4 provides a first justification for the term "useless" used in Definition 1.7.

Proposition 1.4 (i) *For every $q \in Q$, for every $i \in I$, and for every useless portfolio ζ_i for agent i, that is, $\zeta_i \in \ker V$, one has $q \cdot \zeta_i = 0$.*

(ii) *Under the non-satiation assumption* **NS**, *if ζ_i is useless for agent i, then the following two assertions are equivalent.*

(a) *(x_i^*, z_i^*) maximizes the preferences of agent i in $B_i(\mathcal{F}, p^*, q^*)$.*

(b) *$(x_i^*, z_i^* + \zeta_i)$ maximizes the preferences of agent i in $B_i(\mathcal{F}, p^*, q^*)$.*

(iii) *Under the non-satiation assumption* **NS**, *for every mutually compatible useless portfolio allocation $\zeta = (\zeta_1, \cdots, \zeta_I)$, one has: $(\bar{p}, \bar{x}, \bar{q}, \bar{z})$ is an equilibrium in $(\mathcal{E}, \mathcal{F})$ if and only if $(\bar{p}, \bar{x}, \bar{q}, \bar{z} + \zeta)$ is an equilibrium in $(\mathcal{E}, \mathcal{F})$.*

Proof. Assertions (i) and (ii) are immediate. We show assertion (iii). Let $z = (z_1, \cdots, z_I) \in (\ker V)^I$ be such that $\sum_{i \in I} z_i = 0$. It suffices to show that, for every $i \in I$, $(\bar{x}_i, \bar{z}_i + z_i) \in B_i(\mathcal{F}, \bar{p}, \bar{q})$. This follows from

$$\begin{pmatrix} -\bar{q} \\ V \end{pmatrix} (\bar{z}_i + z_i) = \begin{pmatrix} -\bar{q} \\ V \end{pmatrix} \bar{z}_i$$

because **(1)** $z_i \in \ker V$ for each i, and **(2)** by Proposition 1.2, $\bar{q} \in Q$ (under **NS**), hence $-\bar{q} \cdot z_i = 0$ (if not, z_i would be an arbitrage opportunity at $\bar{q}$). ■

1.2.3 Equivalent financial structures

When portfolios are unconstrained, a sufficient condition for two financial structures to be equivalent, can be obtained in terms of income transfers to the second date.

Consider two unconstrained financial structures V and V'.

Proposition 1.5 *If $\mathrm{Im} V = \mathrm{Im} V'$ then the financial structures V and V' are equivalent.*

Proof. Let $\mathcal{E}$ be an exchange economy satisfying the non-satiation assumption **NS**, and let $(\bar{x}, \bar{p})$ be a consumption equilibrium in $(\mathcal{E}, \mathcal{F})$. Then there exist a portfolio allocation $\bar{z} = (\bar{z}_i)_i \in (\mathbb{R}^J)^I$ and an asset price vector $\bar{q} \in \mathbb{R}^J$ such that $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an equilibrium in $(\mathcal{E}, \mathcal{F})$. By **NS**, $\bar{q}$ is arbitrage free, hence there exists $\lambda \in \mathbb{R}^S_{++}$ such that $\bar{q} = V^T \lambda$. Define

$\bar{q}' = V'^T\lambda$. Since $\mathrm{Im}V = \mathrm{Im}V'$, one can find $\bar{z}'' = (\bar{z}_i'')_i \in (\mathbb{R}^{J'})^I$ such that $V\bar{z}_i = V'\bar{z}_i''$ for every $i \in I$. Note that since $\bar{z}$ is an equilibrium portfolio allocation, $\sum_{i\in I}\bar{z}_i = 0$ and therefore $\sum_{i\in I}\bar{z}_i'' \in \ker V'$. Let $\bar{z}_1' = \bar{z}_1'' - \sum_{i\in I}\bar{z}_i''$ and $\bar{z}_i' = \bar{z}_i''$ for $i \neq 1$. Then $V\bar{z}_i = V'\bar{z}_i'$ for every $i \in I$ (because $\sum_{i\in I}\bar{z}_i'' \in \ker V'$) and $\sum_{i\in I} z_i' = 0$. Now, it is easy to check that

$$\begin{pmatrix} -\bar{q} \\ V \end{pmatrix} \bar{z}_i = \begin{pmatrix} -\bar{q}' \\ V' \end{pmatrix} \bar{z}_i'.$$

We claim that $(\bar{x}, \bar{z}', \bar{p}, \bar{q}')$ is an equilibrium in $(\mathcal{E}, \mathcal{F}')$. Assume that for some agent i there exists $(x_i, z_i') \in B_i(\mathcal{F}', \bar{p}, \bar{q}')$ such that $x_i \in P_i(\bar{x})$. Let $z_i \in \mathbb{R}^J$ be such that $Vz_i = V'z_i'$ (this is possible since $\mathrm{Im}V = \mathrm{Im}V'$). Then

$$\begin{pmatrix} -\bar{q} \\ V \end{pmatrix} z_i = \begin{pmatrix} -\bar{q}' \\ V' \end{pmatrix} z_i'.$$

Hence $(x_i, z_i) \in B_i(\mathcal{F}, \bar{p}, \bar{q})$ which together with $x_i \in P_i(\bar{x})$ contradicts the fact that $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an equilibrium in $(\mathcal{E}, \mathcal{F})$. ∎

As the following proposition shows, when agents' portfolios in $\mathcal{F}$ are unrestricted, then removing redundant assets of $\mathcal{F}$ leads to a reduced financial structure $\mathcal{F}'$ which is equivalent to $\mathcal{F}$. This result is not robust to restrictions on agents participation to financial markets as we shall explain in the sequel. Note that properties (i) and (ii) in Definition 1.7 are independent of financial restrictions faced by agents. It is then this latter definition that we will need to change to accommodate the case of financial structures with restricted participation.

Proposition 1.6 (Equivalent reduced form) *Every unconstrained financial structure is equivalent to a reduced one (the latter can be chosen to be unconstrained).*

Proof. If $\mathcal{F} = \big(V, (\mathbb{R}^J)_i\big)$ is not already reduced then $\mathrm{rank}V < J$, and there exists a subset J' of J such that $\mathrm{rank}V' = J'$ where $V' = [V_j, j \in J']$. It is easy to check that $\mathcal{F}$ is equivalent to $\mathcal{F}' := \big(V', (\mathbb{R}^{J'})_i\big)$ (first notice that $\mathrm{Im}V = \mathrm{Im}V'$, and then use Proposition 1.5 to conclude). ∎

1.3 The constrained case

The main existence result in this thesis is a generalization of Theorem 1.1 to financial exchange economies with restricted participation. We will essentially follow the same scheme used to prove existence of equilibria in the unconstrained case.

We, therefore, need to modify the definition of a reduced financial structure to allow for financial constraints. This is done in Chapter 2.

We then show an analogous to Proposition 1.6, namely that every financial structure is equivalent to a reduced one. The proof of the latter result is constructive in the sense that we actually show how to transform a given financial structure (not necessarily unconstrained) into a reduced structure which is equivalent to the original one. This procedure is presented in Chapter 3.

Then, in Chapter 4, we state and prove the main existence result that we will present in the next section.

1.3.1 Main existence result

We make the same standard assumption **C** on the consumption side as in the unconstrained case. We make the following assumption **F** on the financial side. Given the financial structure $\mathcal{F} = \big(V, (Z_i)_{i\in I}\big)$, we denote $Z(\mathcal{F}) =< \sum_{i\in I} Z_i >$ the linear space spanned by $\sum_{i\in I} Z_i$, that is the space where financial activity takes place.

Assumption F

F1 For every $i \in I$, Z_i is closed, convex and $0 \in Z_i$.

F2 Closedness Assumption The following set $\mathcal{G}(\mathcal{F})$ is closed, where

$$\mathcal{G}(\mathcal{F}) := \{(Vz_1, \cdots, Vz_I, \sum_{i\in I} z_i) \in (\mathbb{R}^S)^I \times \mathbb{R}^J : \forall i \in I, z_i \in Z_i\}.$$

F3 FSSA For every $q \in \big(Q \cap Z(\mathcal{F})\big)\backslash\{0\}$, for every $i \in I$ there exists a portfolio $\zeta_i \in Z_i$ such that $q \cdot \zeta_i < 0$.

We can now state the main existence result of this thesis.

Theorem 1.2 *Let* $(\mathcal{E}, \mathcal{F}) = \big((X_i, P_i, e_i)_{i\in I}, (V, (Z_i)_{i\in I})\big)$ *be a financial exchange economy satisfying Assumptions* **C** *and* **F***, then it admits an equilibrium* $(\bar{p}, \bar{x}, \bar{q}, \bar{z})$ *such that* $\bar{p}(s) \neq 0$ *for every* $s \in \bar{S}$.

The proof of Theorem 1.2 will be performed in three steps corresponding to the three chapters of this thesis.

Step 1. Consider a reduced form. This step which was straightforward in the unconstrained case, will be treated in Chapter 2.

Step 2. Show the existence of a reduced financial structure $\mathcal{F}'$ which is equivalent to $\mathcal{F}$. This is the purpose of Chapter 3.

Step 3. Show the existence of equilibria for the financial exchange economy $(\mathcal{E}, \mathcal{F}')$ using the compactness property of the set of "admissible" portfolio allocations (see Chapter 4).

From Step 1-3, we then deduce that the financial exchange $(\mathcal{E}, \mathcal{F})$ has an equilibrium since $\mathcal{F}$ is equivalent to $\mathcal{F}'$ and $(\mathcal{E}, \mathcal{F}')$ has an equilibrium.

Remark 1.1 Under **NS**, $\bar{q} \in Q$ (by Proposition 1.2) and $\bar{p}(s) \neq 0$ for every $s \in \bar{S}$ (by assumption **NS**).

Remark 1.2 In Theorem 1.2, we can choose the equilibrium asset price $\bar{q}$ to be in $Q(\mathcal{F}) \cap Z(\mathcal{F})$. Indeed, if $q^* = \text{proj}_{Z(\mathcal{F})}\bar{q}$ then $(\bar{p}, q^*, \bar{x}, \bar{z})$ is also an equilibrium of $(\mathcal{E}, \mathcal{F})$ since for every $i \in I$, and for every $z_i \in Z_i$, one has $q^* \cdot z_i = \bar{q} \cdot z_i$.

Remark 1.3 Under the assumptions of Theorem 1.2, the equilibrium asset price vector may be zero, that is, we may have $\bar{q} = 0$ at equilibrium. A necessary and sufficient condition guaranteeing that $\bar{q} \neq 0$ is

$$\exists i \in I, \exists v_i \in \boldsymbol{A}Z_i, Vv_i > 0.$$

Indeed, under this assumption, $0 \notin Q$ and under the non-satiation assumption **NS**, $\bar{q} \in Q$, hence $\bar{q} \neq 0$.

1.3.2 Examples of restrictions satisfying assumption F2

As shown by the following Propositions 1.7 and 1.8, assumption **F2** holds true in many situations. Indeed, **F2** is fulfilled when the restrictions on portfolio choices are given by a finite number of linear inequalities, that is, when all portfolios sets are finite intersections of half spaces. In particular, **F2** is fulfilled when the portfolios sets are linear subspaces, when the portfolio sets are unconstrained, or when the portfolio sets are bounded from below. Furthermore, assumption **F2** holds true under the no mutually compatible potential arbitrage condition (Page [31]) that is when the family $\{\boldsymbol{A}Z_i \cap \ker V, i \in I\}$ is positively semi-independent (Siconolfi [36]), in particular **F2** holds true when the portfolio sets are bounded, or when there are no redundant assets i.e. rank$(V) = J$. The proofs of Proposition1.7 and Proposition 1.8 are given in [4].

Proposition 1.7 *Assumption* **F2** *holds true under anyone of the following conditions.*

(a) *For all $i \in I$, $Z_i = \mathbb{R}^J$ (unconstrained portfolios).*

(b) *For all $i \in I$, Z_i is a linear subspace (linear equality constraints).*

(c) *For all $i \in I$, $Z_i = \underline{z}_i + \mathbb{R}^J_+$, for some $\underline{z}_i \in -\mathbb{R}^J_+$ (exogenous bounds on short sales).*

(d) *For all $i \in I$, Z_i is polyhedral (linear equality and inequality constraints).*

(e) *For all $i \in I$, $Z_i = B_J(0,1)$ (bounded portfolio sets).*

(f) *For all $i \in I$, $Z_i = K_i + P_i$ where K_i is nonempty compact and convex, and P_i is polyhedral.*

Definition 1.8 *If C is a nonempty convex subset of $\mathbb{R}^J$, the lineality space of C is $\boldsymbol{L}(C) = \boldsymbol{A}C \cap -\boldsymbol{A}C$.*

Proposition 1.8 *Assumption* **F2** *holds true under anyone of the following conditions.*

(g) *There are no redundant assets i.e. rank $V = J$, or equivalently,* $\ker V = \{0\}$.

(h) $\forall i$, $\boldsymbol{A}Z_i \cap \ker V = \{0\}$.

(i1) $\boldsymbol{L}\Big(\sum_{i\in I}(Z_i \cap \{V \geq 0\})\Big) = \{0\}$.

(i2) $\boldsymbol{L}\Big(\sum_{i\in I}(Z_i \cap \ker V)\Big) = \{0\}$.

(i3) $\sum_{i\in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\}) \cap -\sum_{i\in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\}) = \{0\}$.

(i4) $\sum_{i\in I}(\boldsymbol{A}Z_i \cap \ker V) \cap -\sum_{i\in I}(\boldsymbol{A}Z_i \cap \ker V) = \{0\}$.

(j1) *The family $\{\boldsymbol{A}Z_i \cap \{V \geq 0\} : i \in I\}$ is positively semi-independent*[8].

(j2) *The family $\{\boldsymbol{A}Z_i \cap \ker V : i \in I\}$ is positively semi-independent.*

(k1) *The family $\{\boldsymbol{A}Z_i \cap \{V \geq 0\}, i \in I\}$ is weakly positively semi-independent*[9].

[8] A collection $\{C_i, i \in I\}$ of nonempty convex cones in $\mathbb{R}^\alpha$ is positively semi-independent if $c_i \in C_i$, for all $i \in I$ and $\sum_{i\in I} c_i = 0$, implies that for all $i \in I$, $c_i = 0$.

[9] A collection $\{C_i, i \in I\}$ of nonempty convex cones in $\mathbb{R}^\alpha$ is weakly positively semi independent if $v_i \in C_i$, for all $i \in I$ and $\sum_{i\in I} v_i = 0$, implies that for all $i \in I$, $v_i \in \boldsymbol{L}(C_i)$.

(k2) *The family* $\{\boldsymbol{A}Z_i \cap \ker V, i \in I\}$ *is weakly positively semi-independent.*

Proposition 1.9 (l1) *For every* $i \in I$, $V(Z_i)$ *is closed and there exists* i *such that* $Z_i = \mathbb{R}^J$.

(l2) *For every* $i \in I$, $V(Z_i)$ *is closed and* $\bigcup_i Z_i = \mathbb{R}^J$.

1.3.3 Some consequences of the existence result

Many results in the literature are now corollaries to Theorem 1.2.

Corollary 1.1 (Radner 1972 [34]) *The financial exchange economy* $(\mathcal{E}, \mathcal{F})$ *admits an equilibrium if it satisfies assumption* **C** *and*

F2' *For every* $i \in I$, Z_i *is the closed ball* $B(0, r_i)$, *for some* $r_i > 0$.

Corollary 1.2 (Radner 1972 [34]) *The financial exchange economy* $(\mathcal{E}, \mathcal{F})$ *admits an equilibrium if it satisfies assumption* **C** *and*

F2' *For every* $i \in I$, $Z_i = \{z \in \mathbb{R}^J, z \geq -\underline{z_i}\}$, *for some* $\underline{z_i} \gg 0$.

Corollary 1.3 *The financial exchange economy* $(\mathcal{E}, \mathcal{F})$ *admits an equilibrium if it satisfies assumptions* **C**, **F1**, **F3**, *and*

F2' $\ker V = \{0\}$.

Corollary 1.4 (Siconolfi 1987 [36]) *The financial exchange economy* $(\mathcal{E}, \mathcal{F})$ *admits an equilibrium if it satisfies assumptions* **C**, **F1**, **F3** *and*

F2' *For every* $i \in I$, $\boldsymbol{A}Z_i \cap \ker V = \{0\}$.

Corollary 1.5 *The financial exchange economy* $(\mathcal{E}, \mathcal{F})$ *admits an equilibrium if it satisfies assumptions* **C**, **F1**, **F3**, *and*

F2' *The family* $\{\boldsymbol{A}Z_i \cap \ker V, i \in I\}$ *is positively semi-independent.*

Corollary 1.6 (Aouani and Cornet 2007 [2]) *The financial exchange economy* $(\mathcal{E}, \mathcal{F})$ *admits an equilibrium if it satisfies assumptions* **C**, **F1**, **F3**, *and*

F2' *The family* $\{\boldsymbol{A}Z_i \cap \ker V, i \in I\}$ *is weakly positively semi-independent.*

Corollary 1.7 *The financial exchange economy* $(\mathcal{E}, \mathcal{F})$ *admits an equilibrium if it satisfies assumption* **C** *together with*

F2' *For every* i *in* I, Z_i *is a linear subspace of* $\mathbb{R}^J$ *and,*

F3' $-\text{cl}Q \cap \left(\cup_i Z_i^{\perp}\right) = \{0\}$.

Corollary 1.8 *The financial exchange economy* $(\mathcal{E}, \mathcal{F})$ *admits an equilibrium if it satisfies assumption* **C** *together with*

F2' *For every* $i \in I$, $Z_i = K_i + P_i$ *where* K_i *is nonempty compact and convex and* P_i *is polyhedral and,*

F3' $-\text{cl}Q \cap \left(\cup_i Z_i^o\right) \subset \{0\}$.

1.3.4 Reduced financial structures

Consider a financial structure $\mathcal{F} = \big(I, J, S, V, (Z_i)_{i\in I}\big)$ satisfying assumption **F1**. We will say that $\mathcal{F}$ is reduced if it satisfies one of the following equivalent conditions.

Theorem 1.3 *Under* **F1***, the following assertions are equivalent.*

(i) $\boldsymbol{A}(\sum_{i\in I} Z_i \cap \{V \geq 0\}) \cap -\boldsymbol{A}(\sum_{i\in I} Z_i \cap \{V \geq 0\}) = \{0\}$.

(ii) $(\sum_{i\in I} \boldsymbol{A}Z_i \cap \{V \geq 0\}) \cap -(\sum_{i\in I} \boldsymbol{A}Z_i \cap \{V \geq 0\}) = \{0\}$.

(iii) *The convex*[10] *set* Q *of asymptotic arbitrage-free prices has full dimension* ($\text{dim}Q = J$) *or equivalently, there is no nontrivial linear dependence between the asset prices, that is, there does not exist* $\alpha \in \mathbb{R}^J, \alpha \neq 0$, *such that* $\sum_{j\in J} \alpha^j q^j = 0$ *for every* $q \in Q$.

(iv) *For every* $v = (v_i)_i \in (\mathbb{R}^S)^I$, *the set* $K_1(v)$ *defined below is compact.*

$$K_1(v) := \{(z_i)_i \in \Pi_i Z_i : \forall i \; V z_i \geq v_i, -\sum_{i\in I} z_i \in \sum_{i\in I} (\boldsymbol{A}Z_i \cap \{V \geq 0\})\}.$$

Proof. See [3]. ∎

Definition 1.9 *The financial structure* $\mathcal{F}$ *is said to be reduced if one of the equivalent conditions of the above Proposition is satisfied.*

[10] The set Q is convex by Proposition 1.1.

We denote

$$\mathcal{L}(\mathcal{F}) := (\sum_{i\in I} \boldsymbol{A}Z_i \cap \{V \geq 0\}) \cap -(\sum_{i\in I} \boldsymbol{A}Z_i \cap \{V \geq 0\}),$$

and elements of $\mathcal{L}(\mathcal{F})$ will be called aggregate useless portfolios in $\mathcal{F}$ (as explained in [3]).

As shown by Theorem 1.3, the absence of aggregate useless portfolios in a financial structure is intimately related to the richness of the set of asymptotic-arbitrage-free prices. Indeed, a financial structure has no aggregate useless portfolios if and only if the associated set of asymptotic-arbitrage-free prices has full dimension.

Example 1.1 Assume there exists i such that $Z_i = \mathbb{R}^J$. Then $\mathcal{L}(\mathcal{F}) = \ker V$ and $\mathcal{F}$ is reduced if and only if $\ker V = \{0\}$, that is if and only if there are no redundant assets.

Proposition 1.10 *If assertion* (i) *of Theorem 1.3 holds then the set* $\sum_{i\in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\})$ *is closed*[11].

1.3.5 Equivalent reduced form of a financial structure

Under a suitable assumption, every financial structure, $\mathcal{F} = (V, (Z_i)_i)$, is equivalent to a reduced financial structure $\mathcal{F}'$.

Theorem 1.4 *Let* $\mathcal{F} = \big(V, (Z_i)_i\big)$ *be a financial structure satisfying assumptions* **F1** *and* **F2**. *Then there exists a financial structure* $\mathcal{F}'$ *satisfying* **F1**, *such that*

(i) $\mathcal{F}'$ *is reduced.*

(ii) *For every standard exchange economy* $\mathcal{E}$, *every consumption equilibrium of* $(\mathcal{E}, \mathcal{F}')$ *is a consumption equilibrium of* $(\mathcal{E}, \mathcal{F})$.

(iii) *The financial structures* $\mathcal{F}$ *and* $\mathcal{F}'$ *are equivalent if the financial structure* $\mathcal{F}$ *satisfies the following additional assumption: existence of a riskless asset.*

F0 *For every* $i \in I$, *there exists* $\zeta_i \in \boldsymbol{A}Z_i$ *such that* $V\zeta_i \gg 0$.

Moreover we can choose $\mathcal{F}'$ *so that the following property* **P** *is satisfied:*

P *For every* $(q, z) \in \Big(Q(\mathcal{F}') \cap Z(\mathcal{F}')\Big) \times \Pi_i Z_i$, *one has*

[11] Assertion (i) implies that the sets $\boldsymbol{A}Z_i \cap \{V \geq 0\}$ are positively semi-independent. Recall that a collection $\{C_i, i \in I\}$ of nonempty convex sets in $\mathbb{R}^\alpha$ is positively semi-independent if $c_i \in C_i$, for all $i \in I$ and $\sum_{i\in I} c_i = 0$, implies that $c_i = 0$ for all $i \in I$.

(i) $q \in Q(\mathcal{F}) \cap Z(\mathcal{F})$, *and*

(ii) *there exists* $z' = (z'_i)_i \in \Pi_i Z'_i$ *such that* $q \cdot z_i = q \cdot z'_i$ *for every* $i \in I$.

Theorem 1.4 is a consequence of the following Theorem 1.5. Before giving the statement of the theorem we need some definitions.

Definition 1.10 *Given the financial structure* $\mathcal{F} = \big(V, (Z_i)_{i\in I}\big)$, *we say that* $(q, z_i) \in \mathbb{R}^J \times Z_i$ *is arbitrage-free for agent* $i \in I$ *if there is no portfolio* $\bar{z}_i \in Z_i$ *such that* $W(q)\bar{z}_i > W(q)z_i$.

Let $z = (z_i)_{i\in I} \in \Pi_i Z_i$. *Then* (q, z) *is said to be arbitrage-free, if for every* $i \in I$, (q, z_i) *is arbitrage-free for agent* i.

The asset price vector $q \in \mathbb{R}^J$ *is said to be arbitrage-free if there exists* $z = (z_i)_{i\in I} \in \Pi_i Z_i$ *such that* (q, z) *is arbitrage-free.*

First, we introduce a preorder on the set of all financial structures. A financial opportunity is a collection $(w_1, \cdots, w_I)$ of vectors in the space $\mathbb{R}^{S+1}$. We will say that the financial opportunity $(w_1, \cdots, w_I)$ is achievable through (or offered by) the financial structure $\mathcal{F}$ if there exists an asset price vector $q \in \mathbb{R}^J$ and a family of mutually compatible portfolios $z = (z_1, \cdots, z_I) \in \Pi_i Z_i$ such that (q, z) is arbitrage-free in $\mathcal{F}$ and for every $i \in I$, $W(q)z_i = w_i$. Then the set, $W(\mathcal{F})$, of financial opportunities achievable through $\mathcal{F}$, is

$$W(\mathcal{F}) := \Big\{\big(W(q)z_1, \cdots, W(q)z_I\big) : (z_i)_i \in \Pi_i Z_i, \sum_{i\in I} z_i = 0, \text{ and } (q, z) \text{ is arbitrage-free}\Big\}.$$

Definition 1.11 *Consider two financial structures* $\mathcal{F} = \big(V, (Z_i)_i\big)$ *and* $\mathcal{F}' = \big(V', (Z'_i)_i\big)$. *We say that* $\mathcal{F}' \precsim \mathcal{F}$ *(read* $\mathcal{F}'$ *offers at most as many financial opportunities as those offered by* $\mathcal{F}$*) if*

$$W(\mathcal{F}') \subseteq W(\mathcal{F}).$$

When there is no risk of confusion, we simply denote the preorder defined in Definition 1.11 by $\precsim_W$ and, given the financial structure $\mathcal{F} = \big(V, (Z_i)_i\big)$, we denote

$$V(\mathcal{F}) := \Big\{\big(Vz_1, \cdots, Vz_I\big) : (z_i)_i \in \Pi_i Z_i, \sum_{i\in I} z_i = 0\Big\}.$$

Theorem 1.5 *Let* $\mathcal{F} = \big(V, (Z_i)_i\big)$ *be a financial structure satisfying assumptions* **F1** *and* **F2**, *and let* π *be a linear projection of* $\mathbb{R}^J$ *such that*

$$\ker \pi \subset \boldsymbol{L}(\mathcal{F}) := \boldsymbol{L}\Big(\sum_{i\in I}(Z_i \cap \{V \geq 0\})\Big).$$

Denote $\mathcal{F}_\pi := \big(V, (\mathrm{cl}\pi Z_i)_i\big)$. *We have*

(a) *The financial structure* $\mathcal{F}_\pi$ *satisfies* $V(\mathcal{F}) = V(\mathcal{F}_\pi)$.

(b) *If* $\ker \pi = \boldsymbol{L}(\mathcal{F})$, *then the financial structure* $\mathcal{F}_\pi$ *is reduced, that is* $\boldsymbol{L}(\mathcal{F}_\pi) = \{0\}$.

(c) *If* π *is orthogonal, then the financial structure* $\mathcal{F}_\pi$ *satisfies:*

for every standard exchange economy $\mathcal{E}$, *every consumption equilibrium of* $(\mathcal{E}, \mathcal{F}_\pi)$ *is a consumption equilibrium of* $(\mathcal{E}, \mathcal{F})$, *more precisely, if* $(\mathcal{E}, \mathcal{F}_\pi)$ *has an equilibrium* $(\bar{p}, \bar{q}, \bar{x}, \bar{y})$, *then there exists* $z^* \in \Pi_i Z_i$ *such that* $(\bar{p}, \pi\bar{q}, \bar{x}, z^*)$ *is an equilibrium of* $(\mathcal{E}, \mathcal{F})$.

(d) *If* π *is orthogonal and the financial structure* $\mathcal{F}$ *satisfies the following additional assumption* **F0**, *then the financial structures* $\mathcal{F}$ *and* $\mathcal{F}_\pi$ *are equivalent.*

F0 *For every* $i \in I$, *there exists* $\zeta_i \in \boldsymbol{A} Z_i$ *such that* $V\zeta_i \gg 0$.

(e) *If* π *is orthogonal, then the financial structure* $\mathcal{F}_\pi$ *satisfies the following property* **P**.

P *For every* $(q, z) \in \Big(Q(\mathcal{F}_\pi) \cap Z(\mathcal{F}_\pi)\Big) \times \Pi_i Z_i$, *one has*

(i) $q \in Q(\mathcal{F}) \cap Z(\mathcal{F})$, *and*

(ii) *there exists* $z' = (z'_i)_i \in \Pi_i \mathrm{cl}\pi Z_i$ *such that* $q \cdot z_i = q \cdot z'_i$ *for every* $i \in I$.

1.4 Sketch of the proof of the existence result

1.4.1 Existence result under additional assumptions

We make the following additional assumption.

SF2: The financial structure $\mathcal{F}$ is reduced, that is
$$\Big(\boldsymbol{A}\sum_{i\in I}(Z_i \cap \{V \geq 0\})\Big) \cap -\Big(\boldsymbol{A}\sum_{i\in I}(Z_i \cap \{V \geq 0\})\Big) = \{0\}.$$
Note that, by Proposition 1.8, assumption **SF2** implies **F2**.

Theorem 1.2 will be proved as a consequence of the following Theorem 1.6 in which the consumption structure $\mathcal{E}$ is standard and the financial structure $\mathcal{F}$ is reduced that is the financial exchange economy $(\mathcal{E}, \mathcal{F})$ satisfies the additional assumption **SF2**.

Theorem 1.6 *Let* $(\mathcal{E},\mathcal{F}) = \big((X_i, P_i, e_i)_{i\in I}, (V, (Z_i)_{i\in I})\big)$ *be a financial exchange economy satisfying Assumptions* **C***,* **F1***,* **SF2***,* **F3***, then it admits an equilibrium.*

The proof of Theorem 1.6 is in [5].

1.4.2 From Theorem 1.6 to Theorem 1.2

Now we show how to prove Theorem 1.2 as a consequence of Theorem 1.6. This is done in two steps. First, we transform the financial structure $\mathcal{F}$ to get **SF2**. Finally, we show how to get an equilibrium in $(\mathcal{E},\mathcal{F})$ once we have found an equilibrium in the transformed financial exchange economy.

Step 1: Transforming the financial structure to get SF2

Let π be the orthogonal projection of $\mathbb{R}^J$ such that

$$\ker\pi = \boldsymbol{A}\Big(\sum_{i\in I}(Z_i\cap\ker V)\Big)\cap -\boldsymbol{A}\Big(\sum_{i\in I}(Z_i\cap\ker V)\Big).$$

Let $\mathcal{F}_\pi = \Big(V, (\mathrm{cl}\pi Z_i)_i\Big)$. Obviously, the sets $\mathrm{cl}(\pi Z_i)$ are closed, convex and contain 0, that is $\mathcal{F}_\pi$ satisfies **F1** when $\mathcal{F}$ satisfies **F1**. We recall the definition of the sets $Q(\mathcal{F})$ (denoted Q previously) and $Q(\mathcal{F}_\pi)$ of arbitrage-free prices for $\mathcal{F}$ and $\mathcal{F}_\pi$, respectively.

$$\begin{aligned} Q(\mathcal{F}) &= \{q\in\mathbb{R}^J, W(q)(\cup_i \boldsymbol{A}Z_i)\cap\mathbb{R}^{\bar S}_+ = \{0\}\}, \\ Q(\mathcal{F}_\pi) &= \{q\in\mathbb{R}^J, W(q)(\cup_i \boldsymbol{A}\big(\mathrm{cl}\pi Z_i)\big)\cap\mathbb{R}^{\bar S}_+ = \{0\}\}. \end{aligned}$$

Proposition 1.11 *Let* $\mathcal{F}$ *satisfy Assumption* **F1***. Then*

(i) *If* $\mathcal{F}$ *satisfies Assumption* **F3***, then the financial structure* $\mathcal{F}_\pi$ *satisfies Assumption* **F3***.*

(ii) *If* $\mathcal{F}$ *satisfies Assumption* **F2***, then the financial structure* $\mathcal{F}_\pi$ *satisfies Assumption* **SF2***, that is*

$$\boldsymbol{A}\Big(\sum_{i\in I}(\mathrm{cl}\pi Z_i\cap\{V\geq 0\})\Big)\cap -\boldsymbol{A}\Big(\sum_{i\in I}(\mathrm{cl}\pi Z_i\cap\{V\geq 0\})\Big) = \{0\}.$$

(iii) *Under* **NS***,* **LNS***,* **F1***, and* **F2***, if* $(\mathcal{E},\mathcal{F}_\pi)$ *has an equilibrium* $(\bar p,\bar q,\bar x,\bar z)$*, then there exists* $z^*\in\Pi_i Z_i$ *such that* $(\bar p,\pi\bar q,\bar x,z^*)$ *is an equilibrium of* $(\mathcal{E},\mathcal{F})$*.*

Assertions (ii) and (iii) are consequences of Theorem 1.5. The proof of assertion (i) is in [5].

Step 3: Proof of Theorem 1.2

We start with a financial exchange economy $(\mathcal{E}, \mathcal{F})$ satisfying **C** and **F**. We perform the following steps.

1. We project the financial structure $\mathcal{F}$ as in 1.4.2 step 1, to obtain a financial structure $\mathcal{F}_\pi$ satisfying **F1**, **SF2**, and **F3**.

2. We apply Theorem 1.6 to find an equilibrium $(\bar{p}, \bar{q}, \bar{x}, \bar{z})$ of $(\mathcal{E}, \mathcal{F}_\pi)$.

3. We apply Proposition 1.11(iii) to find an equilibrium $(\bar{p}, \pi\bar{q}, \bar{x}, z^*)$ of $(\mathcal{E}, \mathcal{F})$. ■

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Chapter 2

Characterizing reduced financial structures

We refine the definition of reduced financial structure to accommodate the case of financial structures with restricted participation. We provide a characterization of reduced financial structures in terms of arbitrage-free prices and by the compactness of a set of "admissible" portfolio allocations.

2.1 Introduction

When participation to financial markets is unconstrained, it is customary to say that an asset is redundant if its payoff is a linear combination of other assets' payoffs, and that a portfolio is useless if it has zero payoff. An unconstrained financial structure is said to be reduced if it has no redundant assets. It is then obvious that an unconstrained financial structure is reduced if and only if it has no useless portfolios, which is also equivalent to the set of arbitrage-free prices having full dimension. We will call an opportunity of financial transfers to tomorrow ($t = 1$), any collection $(v_1, \cdots, v_I)$ of vectors in the space of returns $\mathbb{R}^S$. We will say that the opportunity of financial transfers to tomorrow $(v_1, \cdots, v_I)$ is achievable through or offered by (respectively, guaranteed by) the financial structure $\mathcal{F}$ if there exists a family of feasible and mutually compatible[1] portfolios $(z_1, \cdots, z_I)$ such that $Vz_i = v_i$ (respectively, $Vz_i \geq v_i$) for each $i \in I$. It is then easy to show that an unconstrained financial structure is reduced if and only if the set of mutually compatible

[1] A portfolio allocation $(z_1, \cdots, z_I)$ is mutually compatible if it clears asset markets, that is, $\sum_{i \in I} z_i = 0$.

portfolio allocations that guarantee a given level of returns is compact and, à fortiori, the set of mutually compatible portfolio allocations that achieve a given opportunity of financial transfers to tomorrow is compact.

In this chapter, we refine the definition of reduced financial structure to suit the case of restricted financial structures. By doing so, we identify the set of useless portfolios. It is worth noticing that, when there are financial restrictions, a redundant asset is not necessarily useless.

2.2 The model and the main result

2.2.1 The model of a stochastic economy

[2] We consider the basic model of a two time-date economy with nominal assets. It is also assumed that there are finite sets I, H, S, and J, respectively, of agents, divisible physical commodities, states of nature, and nominal assets.

In what follows, the first date will also be referred to as $t = 0$ and the second date, as $t = 1$. There is an a priori uncertainty at the first date ($t = 0$) about which of the states of nature $s \in S$ will prevail at the second date ($t = 1$). For the sake of unified notations of time and uncertainty, the non-random state at the first date is denoted by $s = 0$ ($S_0 = \{0\}$) and $\bar{S}$ stands for the set $\{0\} \cup S$.

At each state of nature $s \in \bar{S}$, there is a spot market where the finite set H of physical commodities is available. We assume that each commodity does not last more than one period so that the commodity space is $\mathbb{R}^L$, with $L = H \times \bar{S}$ (in this model, a commodity

[2]We shall use the following notation. If I and J are finite sets, the space $\mathbb{R}^I$ (identified to $\mathbb{R}^{\#I}$ whenever necessary) of functions $x : I \to \mathbb{R}$ (also denoted $x = (x(i))_{i \in I}$ or $x = (x_i)$) is endowed with the scalar product $x \cdot y := \sum_{i \in I} x(i)y(i)$, and we denote by $\|x\| := \sqrt{x \cdot x}$ the Euclidean norm. By $B(x, r)$ we denote the closed ball centered at $x \in \mathbb{R}^L$ of radius $r > 0$, namely $B(x, r) = \{y \in \mathbb{R}^L : \|y - x\| \leq r\}$. In $\mathbb{R}^I$, the notation $x \geq y$ (resp. $x > y$, $x \gg y$) means that, for every i, $x(i) \geq y(i)$ (resp. $x \geq y$ and $x \neq y$, $x(i) > y(i)$) and we let $\mathbb{R}^I_+ = \{x \in \mathbb{R}^I \mid x \geq 0\}$, $\mathbb{R}^I_{++} = \{x \in \mathbb{R}^I \mid x \gg 0\}$. An $I \times J$-matrix $A = (a_i^j)_{i \in I, j \in J}$ is an element of $\mathbb{R}^{I \times J}$ whose rows are denoted $A_i = (a_i^j)_{j \in J} \in \mathbb{R}^J$ ($i \in I$), and columns $A^j = (a_i^j)_{i \in I} \in \mathbb{R}^I$ ($j \in J$). To the matrix A, we associate the linear mapping, from $\mathbb{R}^J$ to $\mathbb{R}^I$, also denoted by A, defined by $Ax = (A_i \cdot x)_{i \in I}$. The span of the matrix A, also called the image of A, is the set $< A >:= \{Ax \mid x \in \mathbb{R}^J\}$. The transpose matrix of A, denoted by A^T, is the $J \times I$-matrix whose rows are the columns of A, or equivalently, is the unique linear mapping $A^T : \mathbb{R}^I \to \mathbb{R}^J$, satisfying $Ax \cdot y = x \cdot A^T y$ for every $x \in \mathbb{R}^J$, $y \in \mathbb{R}^I$.

is a couple $(h, s) \in H \times \bar{S}$ of a physical commodity, h, and a state of nature s, at which h will be available). An element $x \in \mathbb{R}^L$ is called a consumption (or a consumption plan), that is $x = (x(s))_{s \in \bar{S}} \in \mathbb{R}^L$, where $x(s) = (x(h, s))_{h \in H} \in \mathbb{R}^H$, for every $s \in \bar{S}$.

We denote by $p = (p(s))_{s \in \bar{S}} \in \mathbb{R}^L$ the vector of spot prices and $p(s) = (p(h, s))_{h \in H} \in \mathbb{R}^H$ is called the spot price at state s. The spot price $p(h, s)$ is the price paid, at date 0 if $s = 0$ and at date 1 if $s \in S$, for the delivery of one unit of commodity h at state s.

Each agent $i \in I$, also called a consumer, is endowed with a consumption set $X_i \subset \mathbb{R}^L$ which is the set of her possible consumptions. An allocation is an element $x \in \Pi_i X_i$, and we denote by x_i the consumption of agent i, that is the projection of x onto X_i.

The tastes of each consumer $i \in I$ are represented by a strict preference correspondence $P_i : \Pi_{k \in I} X_k \to X_i$, where $P_i(x)$ defines the set of consumptions that are strictly preferred by i to x_i, that is, given the consumptions x_k for other consumers $k \neq i$.

each state of nature, $s \in \bar{S}$, every consumer $i \in I$ has a state-endowment $e_i(s) \in \mathbb{R}^H$ contingent to the fact that s prevails and we denote by $e_i = (e_i(s))_{s \in \bar{S}} \in \mathbb{R}^L$ her endowment vector across the different states.

The consumption structure, denoted $\mathcal{E}$, can be summarized by

$$\mathcal{E} = \Big(I, H, S, (X_i, P_i, e_i)_{i \in I}\Big).$$

Agents may operate financial transfers across states in $\bar{S}$ (i.e. across the two dates and across the states of the second date) by exchanging a finite number of nominal assets $j \in J$[3], which define the financial structure of the model. The nominal assets are traded at the first date ($t = 0$) and yield payoffs at the second date ($t = 1$), contingent on the realization of the state of nature. The payoff of the nominal asset $j \in J$, when state $s \in S$ is realized, is V_s^j, and we denote by V the $S \times J$-return matrix $V = (V_s^j)$, which does not depend upon the asset prices $q \in \mathbb{R}^J$. A portfolio $z = (z_j) \in \mathbb{R}^J$ specifies the quantities $|z_j|$ ($j \in J$) of each asset j (with the convention that it is bought if $z_j > 0$ and sold if $z_j < 0$), thus Vz is its random financial return across states at time $t = 1$, and $< V[s], z >_J$ is its return if state s prevails.

We assume that each agent i is restricted to choose her portfolio within a portfolio set $Z_i \subset \mathbb{R}^J$, which represents the set of portfolios that are (institutionally) admissible for agent i. This general framework allows us to address, for example, the following important

[3]The case of no financial assets – i.e., J is empty – is called pure spot markets.

cases:

(i) For every i, $Z_i = \mathbb{R}^J$ (unconstrained portfolios).

(ii) For every i, $Z_i = \underline{z}_i + \mathbb{R}^J_+$, for some $\underline{z}_i \in -\mathbb{R}^J_+$ (exogenous bounds on short sales).

(iii) For every i, $Z_i = B_J(0,1)$ (bounded portfolio sets).

(iv) For every i, Z_i is a vector space.

(v) For every i, Z_i is polyhedral and contains 0 (linear equality and inequality portfolio constraints).

Note that the polyhedral case covers the cases (i)-(iv) (with an appropriate choice of the norm in (iii)). In the sequel, we make the following assumption which covers all the above cases:

F1. For every $i \in I$, the set Z_i is closed, convex, and contains 0.

We summarize by $\mathcal{F} = \Big(I, S, J, V, (Z_i)_{i\in I}\Big)$ the financial characteristics, referred to as the financial structure. When there is no risk of confusion, the financial structure $\mathcal{F}$ will be denoted $\big(V, (Z_i)_i\big)$.

The financial exchange economy is thus summarized by

$$(\mathcal{E}, \mathcal{F}) = \Big(I, H, S, (X_i, P_i, e_i)_{i\in I}, J, V, (Z_i)_{i\in I}\Big).$$

2.2.2 Financial equilibria and no-arbitrage

Consider a financial exchange economy $(\mathcal{E}, \mathcal{F})$, where $\mathcal{E}$ is an exchange economy and $\mathcal{F}$ a financial structure. Given the spot price vector $p \in \mathbb{R}^L$ and the asset price vector $q \in \mathbb{R}^J$, the *budget set* of consumer $i \in I$ in this setting is defined as follows[4]

$$\begin{aligned} B_i(\mathcal{F}, p, q) &= \{(x_i, z_i) \in X_i \times Z_i : \ \forall s \in \bar{S}, \ \ p(s) \cdot [x_i(s) - e_i(s)] \leq [W(q) z_i](s)\} \\ &= \{(x_i, z_i) \in X_i \times Z_i : \ p \,\square\, (x_i - e_i) \leq W(q) z_i\}. \end{aligned}$$

Where $W(q)$ is the $(S+1) \times J$ matrix $\begin{pmatrix} -q \\ V \end{pmatrix}$, referred to as the full-return matrix.

[4]For $x = (x(s))_{s\in\bar{S}}, p = (p(s))_{s\in\bar{S}}$ in $\mathbb{R}^L = \mathbb{R}^{H\bar{S}}$ (with $x(s), p(s)$ in $\mathbb{R}^H$ for each $s \in \bar{S}$) we let $p \,\square\, x = (p(s) \cdot x(s))_{s\in\bar{S}} \in \mathbb{R}^{\bar{S}}$.

An equilibrium in the financial exchange economy is then defined as a collection of commodity spot prices, consumption strategies (one for each agent), asset prices, and asset trade strategies (one for each agent) such that each agent maximizes her preferences over her budget set, and all markets clear (commodity markets clear in all dates and states, and asset markets clear).

Definition 2.1 *An equilibrium in the financial exchange economy* $(\mathcal{E}, \mathcal{F})$ *is a list* $\big(\bar{p}, \bar{x}, \bar{q}, \bar{z}\big) \in \mathbb{R}^L\backslash\{0\} \times (\mathbb{R}^L)^I \times \mathbb{R}^J \times (\mathbb{R}^J)^I$ *such that*

(a) *for every* $i \in I$, $(\bar{x}_i, \bar{z}_i)$ *maximizes the preferences* P_i, *in the sense that*

$$(\bar{x}_i, \bar{z}_i) \in B_i(\mathcal{F}, \bar{p}, \bar{q}) \text{ and } [P_i(\bar{x}) \times Z_i] \cap B_i(\mathcal{F}, \bar{p}, \bar{q}) = \emptyset$$

where $\bar{x} = (\bar{x}_i)_{i \in I}$, *and*

(b) $\sum_{i \in I} \bar{x}_i = \sum_{i \in I} e_i$ *and* $\sum_{i \in I} \bar{z}_i = 0.$

A consumption equilibrium in the financial exchange economy $(\mathcal{E}, \mathcal{F})$ *is a list of commodity prices and consumption strategies* $(\bar{p}, \bar{x}) \in \mathbb{R}^L\backslash\{0\} \times (\mathbb{R}^L)^I$ *such that there exist asset prices and trade strategies* $(\bar{q}, \bar{z}) \in \mathbb{R}^J \times (\mathbb{R}^J)^I$ *with* $\big(\bar{p}, \bar{x}, \bar{q}, \bar{z}\big)$ *is an equilibrium in* $(\mathcal{E}, \mathcal{F})$.

Our notion of no-arbitrage takes into account only arbitrage opportunities that might yield an infinite payoff (the intuition underlying this definition is that the market will be able to rule out any arbitrage opportunity with finite payoff).

Definition 2.2 *If* C *is a nonempty convex set in* $\mathbb{R}^J$, *we let*

$$\begin{aligned} \boldsymbol{A}C &:= \{\zeta \in \mathbb{R}^J : \zeta + \mathrm{cl}C \subset \mathrm{cl}C\} && \textit{be the asymptotic cone of } C \\ \boldsymbol{L}(C) &:= \boldsymbol{A}C \cap (-\boldsymbol{A}C) && \textit{be the lineality space of } C. \end{aligned}$$

Definition 2.3 *The set of arbitrage-free prices of* $\mathcal{F} = \big(V, (Z_i)_{i \in I}\big)$ *is*

$$Q = \big\{q \in \mathbb{R}^J : W(q)(\bigcup_i \boldsymbol{A}Z_i) \cap \mathbb{R}^{S+1}_+ = \{0\}\big\}.$$

where $\boldsymbol{A}Z_i$ *denotes the asymptotic cone of the set* Z_i.

Proposition 2.1 *The set* Q *is a convex cone with vertex 0.*

Proof. The set Q is obviously a cone. We show that Q is convex. Let $q_1, q_2 \in Q$ and $\alpha \in (0, 1)$. Assume $\alpha q_1 + (1 - \alpha) q_2 \notin Q$. Then there exists $i \in I$ and $v \in \boldsymbol{A}Z_i$ such that

$W(\alpha q_1 + (1-\alpha)q_2)v > 0$. Hence

$$\text{either} \begin{cases} -(\alpha q_1 + (1-\alpha)q_2) \cdot v > 0 \\ Vv \geq 0 \end{cases} \quad \text{or} \quad \begin{cases} -(\alpha q_1 + (1-\alpha)q_2) \cdot v \geq 0 \\ Vv > 0 \end{cases}$$

In the first case, we conclude that either $-q_1 \cdot v > 0$ or $-q_2 \cdot v > 0$ which, together with $Vv \geq 0$, implies that $W(q_i)v > 0$ for $i = 1$ or $i = 2$ contradicting the fact that q_1 and q_2 are both in Q. Similarly, in the second case, we conclude that either $-q_1 \cdot v \geq 0$ or $-q_2 \cdot v \geq 0$ which, together with $Vv > 0$, contradicts the fact that q_1 and q_2 are both in Q. ■

2.2.3 The main result

Let $\mathcal{F} = \big(V, (Z_i)_i\big)$ be a financial structure satisfying assumption **F1**. The aim of this paper is to prove the following result.

Theorem 2.1 *Under assumption* **F1***, the following assertions are equivalent.*

(i1) $\boldsymbol{A}\Big(\sum_{i\in I} Z_i \cap \{V \geq 0\}\Big) \cap -\boldsymbol{A}\Big(\sum_{i\in I} Z_i \cap \{V \geq 0\}\Big) = \{0\}$.

(i2) $(\sum_{i\in I} \boldsymbol{A}Z_i \cap \{V \geq 0\}) \cap -(\sum_{i\in I} \boldsymbol{A}Z_i \cap \{V \geq 0\}) = \{0\}$.

(ii) *The convex set Q of asymptotic arbitrage-free prices has full dimension (*$\dim Q = J$*) or equivalently, there is no nontrivial linear dependence between the asset prices, that is, there is no $\alpha = (\alpha^j)_j \in \mathbb{R}^J, \alpha \neq 0$, such that $\sum_{j\in J} \alpha^j q^j = 0$ for every $q \in Q$.*

(iii1) *For every $v = (v_i)_i \in (\mathbb{R}^S)^I$, the set $K_1(v)$ defined below is compact.*

$$K_1(v) := \{(z_i)_i \in \Pi_i Z_i : \forall i \ Vz_i \geq v_i, -\sum_{i\in I} z_i \in \sum_{i\in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\})\}.$$

Definition 2.4 *The financial structure $\mathcal{F}$ is said to be reduced if one of the equivalent conditions of the above Theorem is satisfied.*

Each of the three above equivalent conditions defining a reduced financial structure has an economic interpretation that will be developed in the following sections. We now consider the case of unconstrained portfolio sets for which we can deduce the following (known) result.

Corollary 2.1 (The unconstrained case) *If we assume that $Z_i = \mathbb{R}^J$ for some $i \in I$, then the following assertions are equivalent.*

(i0) *The financial structure has no redundant*[5] *assets, that is, there is no* $\alpha = (\alpha^j)_j \in \mathbb{R}^J, \alpha \neq 0$ *such that* $\sum_{j\in J}\alpha^j V^j = 0$.

(i') $\ker V = \{0\}$.

(ii) *The convex set* Q *of asymptotic arbitrage-free prices has full dimension (*$\dim Q = J$*) or equivalently, there is no nontrivial linear dependence between the asset prices, that is, there is no* $\alpha = (\alpha^j)_j \in \mathbb{R}^J, \alpha \neq 0$*, such that* $\sum_{j\in J}\alpha^j q^j = 0$ *for every* $q \in Q$.

(iii2) *For every* $v = (v_i)_i \in (\mathbb{R}^S)^I$*, the set* $K_1(v)$ *of "admissible" portfolio allocations is compact*[6].

If we further assume that $Z_i = \mathbb{R}^J$ *for every* $i \in I$*, then the above properties are also equivalent to the compactness of the standard set of "admissible" allocations.*

(iii0) *For every* $v = (v_i)_i \in (\mathbb{R}^S)^I$*, the following set* $K_0(v)$ *is compact, where*

$$K_0(v) := \{(z_i)_i \in \Pi_i Z_i : \forall i \; V z_i \geq v_i, \sum_{i\in I} z_i = 0\}.$$

Proof. Conditions (i'), (ii), and (iii2) are straightforwardly obtained from conditions (i1), (ii), and (iii1) of Theorem 2.1 by replacing one of the Z_i's, say Z_1, by $\mathbb{R}^J$. Condition (i0) is obviously equivalent to (i'). We prove that (iii2) and (iii0) are equivalent. Since for every $v = (v_i)_i \in (\mathbb{R}^S)^I$, the set $K_0(v)$ is a subset of $K_1(v)$, we have (iii2) $\Rightarrow$ (iii0). Finally, we show that (iii0) implies (iii2). First, notice that condition (iii") implies that $\ker V = \{0\}$. Indeed, if $z \in \ker V$, then

$$(z, -z, 0, \cdots, 0) \in \{(\zeta_i)_i \in (\mathbb{R}^J)^I, \forall i \; V\zeta_i \geq 0, \sum_{i\in I}\zeta_i = 0\} = \boldsymbol{A}K_0\big((0, \cdots, 0)\big).$$

But $\boldsymbol{A}K_0\big((0, \cdots, 0)\big) = \{0\}$ by compactness of $K_0\big((0, \cdots, 0)\big)$, hence $z = 0$. Therefore condition (iii0) implies (i') which is equivalent to (iii2) from above. ∎

[5] Recall that an asset is said to be redundant if its payoff is a linear combination of other assets payoffs.

[6] Note that in this case $K_1(v)$ reduces to

$$K_1(v) := \{(z_i)_i \in \Pi_i Z_i : \forall i \; V z_i \geq v_i, \sum_{i\in I} V z_i \leq 0\}.$$

2.2.4 Some remarks on Conditions (i)-(iii)

We let

$$\begin{aligned}\boldsymbol{L}(\mathcal{F}) &:= \boldsymbol{A}\Big(\sum_{i\in I} Z_i \cap \{V \geq 0\}\Big) \cap -\boldsymbol{A}\Big(\sum_{i\in I} Z_i \cap \{V \geq 0\}\Big),\\ \mathcal{L}(\mathcal{F}) &:= (\sum_{i\in I} \boldsymbol{A}Z_i \cap \{V \geq 0\}) \cap -(\sum_{i\in I} \boldsymbol{A}Z_i \cap \{V \geq 0\}).\end{aligned}$$

Conditions (i1) and (i2) can then be written as follows

$$\begin{aligned}\boldsymbol{L}(\mathcal{F}) &= \{0\},\\ \mathcal{L}(\mathcal{F}) &= \{0\}.\end{aligned}$$

The set $\boldsymbol{L}(\mathcal{F})$ plays a crucial role in showing the existence of a reduced financial structure that is equivalent to $\mathcal{F}$ (see [1]). The set $\mathcal{L}(\mathcal{F})$ will be interpreted in the following Section 2.3 in terms of aggregate useless portfolios.

In Theorem 2.1 and its corollary we have defined the following "admissible sets"

$$\begin{aligned}K_0(v) &= \{(z_i)_i \in \Pi_i Z_i : \forall i\ V z_i \geq v_i, \sum_{i\in I} z_i = 0\}\\ K_1(v) &= \{(z_i)_i \in \Pi_i Z_i : \forall i\ V z_i \geq v_i, -\sum_{i\in I} z_i \in \sum_{i\in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\})\}.\end{aligned}$$

Remark 2.1 It is worth noticing that the condition "$Vz_i \geq v_i$" for some v_i is mild since equilibrium portfolios' payoffs at $t = 1$ are bounded below and the lower bound is uniform i.e. it depends only on the characteristics of the economy.

Indeed, assuming that $X_i = \mathbb{R}^L_+$ (or more generally, X_i is bounded below), denote by $\mathcal{A}(\mathcal{E})$ the set of attainable allocations of the economy, that is

$$\mathcal{A}(\mathcal{E}) = \Big\{(x_i)_{i\in I} \in \prod_{i\in I} X_i \mid \sum_{i\in I} x_i = \sum_{i\in I} e_i\Big\},$$

and by $\widehat{X}_i$ the projection of $\mathcal{A}(\mathcal{E})$ on X_i. Note that for every $i \in I$, $e_i \in \widehat{X}_i$. Then the equilibrium portfolios satisfy

$$\text{for every } s \in S,\ v_i(s) = \inf\{p \cdot (x_i(s) - e_i(s)), p \in \mathbb{R}^L,\ ||p|| \leq 1, x_i \in \widehat{X}_i\} \leq (Vz_i)(s).$$

Remark 2.2 We always have the following inclusion $K_0(v) \subset K_1(v)$, so the following implication always holds:

$$\text{(iii1) } [K_1(v) \text{ is compact}] \ \Rightarrow\ \text{(iii0) } [K_0(v) \text{ is compact}].$$

We have seen that the converse is true when $Z_i = \mathbb{R}^J$, for every $i \in I$. The following remark gives a characterization of condition (iii0) and the next example 2.1 shows that the converse [(iii0) $\Rightarrow$ (iii1)] may not hold true if only one of the Z_i's is equal to $\mathbb{R}^J$.

Remark 2.3 Condition (iii0) is easily shown to be equivalent to the following condition

(i0) The sets $\boldsymbol{A}Z_i \cap \{V \geq 0\}$ are positively semi-independent **PSI**[7].

Indeed, it is easy to check that the sets $\boldsymbol{A}Z_i \cap \{V \geq 0\}$ are positively semi-independent if and only if $\boldsymbol{A}K_0(v) = \{0\}$ which in turn is equivalent to $K_0(v)$ being bounded (see [2]).

Example 2.1 Let $V = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $Z_1 = \mathbb{R}^2$, and $Z_2 = \{0\} \times \mathbb{R}$.

Then $\boldsymbol{A}Z_1 \cap \{V \geq 0\} = \mathbb{R} \times \mathbb{R}_+$, $\boldsymbol{A}Z_2 \cap \{V \geq 0\} = \{0\} \times \mathbb{R}_+$, and it is easy to check the latter two cones are positively semi-independent, hence $K_0(v)$ is compact for every $v = (v_i)_i \in (\mathbb{R}^S)^I$, that is condition (iii0) is fulfilled.

Furthermore, $\sum_{i \in I}(Z_i \cap \{V \geq 0\}) = \mathbb{R} \times \mathbb{R}_+$, hence $\boldsymbol{L}\big(\sum_{i \in I} Z_i \cap \{V \geq 0\}\big) = \mathbb{R} \times \{0\} \neq \{0\}$, that is condition (i1) is violated, therefore so is condition (iii1).

Remark 2.4 We always have

$$\alpha = (\alpha^j)_{j \in J} \in Q^\perp \Rightarrow \sum_{j \in J} \alpha^j V^j = 0.$$

The converse is true if there exists $i \in I$ such that $Z_i = \mathbb{R}^J$.

Definition 2.5 *The asset j_o is redundant if there exists $\alpha_{-j_o} = (\alpha_1, \cdots, \check{\alpha}_{j_o}, \cdots, \alpha_J) \in \mathbb{R}^{J-1}$ such that $q^{j_o} = \sum_{j \neq j_o} \alpha^j q^j$ for every $q = (q^j)_{j \in J} \in Q$.*

The above definition implies the classic definition of a redundant asset when participation to financial markets is not restricted, that is $V^{j_o} = \sum_{j \neq j_o} \alpha^j V^j$ (since $Q^\perp \subset \ker V$). The converse is true if there exists at least one agent whose participation to the financial markets is not restricted.

2.3 Eliminating useless portfolios

We start by transposing the notion of useless commodity bundles, introduced by Werner (1987)[3], to the setting of financial structures. Werner (1987)[3] distinguishes among all

[7] The family $(C_i)_{i \in I}$ of closed convex cones of $\mathbb{R}^J$ is said to be *positively semi-independent* **PSI** if $\forall i \in I,\ c_i \in C_i, \sum_{i \in I} c_i = 0 \Rightarrow c_i \in C_i \cap -C_i, \forall i \in I$.

bundles of commodities (including assets and securities) available in the economy, those which are useful or useless according to the taste of an individual. A commodity bundle $\hat{x}_i$ is useless for agent i when she starts at the consumption bundle x_i, if $\hat{x}_i$ is a direction in which agent i's utility function is constant. A direct application of Werner's definition to financial structures is not possible since it presumes the existence of a preference relation over agents portfolio choice sets. However, if an agent starts with a portfolio ζ_i, one can restrict attention to a smaller set of portfolios: those which are "naturally preferred" by agent i to the portfolio ζ_i. Subsection 2.3.1 is devoted the construction of useless portfolios à la Werner and their properties. A new definition of useless portfolio allocations in the case of financial structures with restrictions is then given in Subsection 2.3.2.

2.3.1 Useless portfolios à la Werner

Consider a financial structure $\mathcal{F} = \big(V, (Z_i)_{i\in I}\big)$. Given a family of portfolios $\zeta = (\zeta_i)_i \in \Pi_i Z_i$, a set of important interest for agent i is the set of all portfolios z_i's that generate a flow of return at $t = 1$ at least as much as the flow of return generated by ζ_i, that is the set

$$\{z_i \in Z_i : Vz_i \geq V\zeta_i\} = Z_i \cap \{V \geq V\zeta_i\}.$$

Definition 2.6 *If C is a nonempty convex set in $\mathbb{R}^J$, the lineality space of C is the set*

$$\boldsymbol{L}(C) := \boldsymbol{A}C \cap (-\boldsymbol{A}C).$$

Following Werner (1987)[3], we define useless portfolios as follows.

Definition 2.7 *A portfolio $\xi_i \in \mathbb{R}^J$ is said to be a potential asymptotic arbitrage opportunity for agent i at ζ_i, if for every portfolio $z_i \in Z_i \cap \{V \geq V\zeta_i\}$, we have $z_i + \xi_i \in Z_i \cap \{V \geq V\zeta_i\}$. Thus the set of potential asymptotic arbitrage opportunities for i at ζ_i is the asymptotic cone of $Z_i \cap \{V \geq V\zeta_i\}$, denoted $\boldsymbol{A}(Z_i \cap \{V \geq V\zeta_i\})$.*

The portfolio $\xi_i \in \mathbb{R}^J$ is said to be (individually) Werner *useless to agent i at ζ_i, if for every portfolio $z_i \in Z_i \cap \{V \geq V\zeta_i\}$, we have $z_i + \xi_i \in Z_i \cap \{V \geq V\zeta_i\}$, and $z_i - \xi_i \in Z_i \cap \{V \geq V\zeta_i\}$. The set of useless portfolios for i at ζ_i is then the lineality space of $Z_i \cap \{V \geq V\zeta_i\}$, denoted $\boldsymbol{L}(Z_i \cap \{V \geq V\zeta_i\})$.*

When the set Z_i is closed, convex and contains 0, the set of potential asymptotic arbitrage opportunities for i at ζ_i does not depend on ζ_i and therefore, neither does the set of useless portfolios, which turns out to be the largest linear subspace contained in the set of zero-return portfolios for agent i, that is the lineality space of the set $Z_i \cap \ker V$.

Proposition 2.2 *Under* **F1***, the set of potential asymptotic arbitrage opportunities for i at ζ_i and the set of useless portfolios for i at ζ_i are, respectively,*

$$\begin{aligned} \boldsymbol{A}(Z_i \cap \{V \geq V\zeta_i\}) &= \boldsymbol{A}Z_i \cap \{V \geq 0\}, \\ \boldsymbol{L}(Z_i \cap \{V \geq V\zeta_i\}) &= \boldsymbol{L}(Z_i) \cap \{V = 0\} = \boldsymbol{L}(Z_i \cap \ker V). \end{aligned}$$

Proof. Follows straightforwardly from the definition and Corollary 8.3.3 in [2]. ∎

Definition 2.8 *A portfolio allocation $(\xi_1, \cdots, \xi_I)$ is said to be* Werner *useless if for every $i \in I$, the portfolio ξ_i is* Werner *useless to agent i.*

The portfolio $\xi \in \mathbb{R}^J$ is said to be Werner *aggregate useless in $\mathcal{F}$ if $\xi = \sum_{i\in I}\xi_i$ with ξ_i is* Werner *useless to agent i, for every $i \in I$.*

The set of Werner *aggregate useless portfolios in $\mathcal{F}$ will be denoted*

$$\boldsymbol{L}_W(\mathcal{F}) = \sum_{i\in I}\big(\boldsymbol{L}(Z_i) \cap \ker V\big).$$

Proposition 2.3 *There are no nonzero* Werner *aggregate useless portfolios in $\mathcal{F}$ if and only if there are no nontrivial* Werner *useless portfolio allocations. That is the following two conditions are equivalent*

(i) $\sum_{i\in I}\big(\boldsymbol{L}(Z_i) \cap \ker V\big) = \{0\}$

(ii) $\boldsymbol{L}(Z_i) \cap \ker V = \{0\}, \forall i \in I$.

Proof. For each $i \in I$, $\boldsymbol{L}(Z_i) \cap \ker V \subset \sum_{i\in I}\big(\boldsymbol{L}(Z_i) \cap \ker V\big)$, hence if $\sum_{i\in I}\big(\boldsymbol{L}(Z_i) \cap \ker V\big) = \{0\}$, we must have $\boldsymbol{L}(Z_i) \cap \ker V = \{0\}, \forall i \in I$, that is (i) implies (ii). Conversely, if $\boldsymbol{L}(Z_i) \cap \ker V = \{0\}$, for each $i \in I$, then $\sum_{i\in I}\big(\boldsymbol{L}(Z_i) \cap \ker V\big) = \sum_{i\in I}\{0\} = \{0\}$. ∎

The first part of the following proposition states that all individually useless portfolios are free (their value is equal to zero) at asymptotic-arbitrage-free prices. The second part gives another justification for the term "individually useless". When an agent is maximizing her preferences using a given portfolio, then trading a useless portfolio does not yield any benefit. Finally, the last part shows that if the financial exchange economy is at equilibrium then it remains at equilibrium after trading in mutually compatible[8] potential asymptotic arbitrage opportunities.

Proposition 2.4 (i) *For every $q \in Q$, for every $i \in I$, and for every* Werner *useless portfolio ζ_i for agent i, that is, $\zeta_i \in \boldsymbol{L}(Z_i) \cap \ker V$, one has $q \cdot \zeta_i = 0$.*

(ii) *Under the non-satiation assumption* **NS**, *if ζ_i is* Werner *useless for agent i, then the following two assertions are equivalent.*

(a) *(x_i^*, z_i^*) solves agent i's problem in $B_i(\mathcal{F}, p^*, q^*)$.*

(b) *$(x_i^*, z_i^* + \zeta_i)$ solves agent i's problem in $B_i(\mathcal{F}, p^*, q^*)$.*

(iii) *Under the non-satiation assumption* **NS**, *for every mutually compatible* Werner *useless portfolio allocation $\zeta = (\zeta_1, \cdots, \zeta_I)$, one has: if (x^*, z^*, p^*, q^*) is an equilibrium in $(\mathcal{E}, \mathcal{F})$ then $(x^*, z^* + \zeta, p^*, q^*)$ is an equilibrium in $(\mathcal{E}, \mathcal{F})$.*

Proof. (i) Since $\zeta_i \in \boldsymbol{L}(Z_i) \subset \boldsymbol{A}Z_i$, $V\zeta_i = 0 \geq 0$, and $q \in Q$ we must have $-q \cdot \zeta_i \leq 0$. But $\zeta_i \in \boldsymbol{L}(Z_i)$ implies that $-\zeta_i \in \boldsymbol{L}(Z_i)$, hence we must also have $-q \cdot (-\zeta_i) \leq 0$. Therefore $q \cdot \zeta_i = 0$.

(ii) $z_i^* + \zeta_i \in Z_i$ since $\zeta_i \in \boldsymbol{L}(Z_i) \subset \boldsymbol{A}Z_i$. Moreover $V(z_i^* + \zeta_i) = V z_i$ because $\zeta_i \in \ker V$, and $-q \cdot (z_i + \zeta_i) = -q \cdot z_i^*$ because $q \cdot \zeta_i = 0$ from (i).

(iii) Follows easily from (ii) bearing in mind that $\sum_{i \in I} \zeta_i = 0$. ∎

2.3.2 Useless portfolios continued

Definition 2.9 *For every $i \in I$, we define the set L_i of useless portfolios for agent i by*

$$L_i := \Big(\boldsymbol{A}Z_i \cap \{V \geq 0\}\Big) \cap -\Big(\sum_{i \in I} \boldsymbol{A}Z_i \cap \{V \geq 0\}\Big).$$

A portfolio allocation $(\zeta_1, \cdots, \zeta_I)$ is said to be useless if for every $i \in I$, the portfolio ζ_i is useless to agent i.

The portfolio $\xi \in \mathbb{R}^J$ is said to be aggregate useless in $\mathcal{F}$ if $\xi = \sum_{i \in I} \xi_i$ with ξ_i is useless to agent i, for every $i \in I$.

[8] A family of portfolios $(z_1, \cdots, z_I) \in \Pi_i Z_i$ is mutually compatible if $\sum_{i \in I} z_i = 0$.

Denote by $\mathcal{L}(\mathcal{F})$ the set of aggregate useless portfolios in $\mathcal{F}$, that is

$$\mathcal{L}(\mathcal{F}) := \sum_{i \in I} L_i.$$

Proposition 2.5 (i) *The set L_i is a cone which is not necessarily a linear space.*

(ii) *The set L_i contains the set of* Werner *useless portfolios for agent i, that is $\boldsymbol{L}(Z_i) \cap \ker V \subset L_i$ for each $i \in I$.*

(iii) *We have*

$$\mathcal{L}(\mathcal{F}) = \sum_{i \in I} L_i = (\sum_{i \in I} \boldsymbol{A} Z_i \cap \{V \geq 0\}) \cap -(\sum_{i \in I} \boldsymbol{A} Z_i \cap \{V \geq 0\}) = \boldsymbol{L}\Big(\sum_{i \in I} \boldsymbol{A} Z_i \cap \{V \geq 0\}\Big).$$

Proof. Assertions (i) and (ii) are immediate. We show assertion (iii). Denote by $C_i = \boldsymbol{A} Z_i \cap \{V \geq 0\}$ and $C = \sum_{i \in I} C_i$. Then $L_i = C_i \cap -C$, hence $L_i \subset C \cap -C$ and $\sum_{i \in I} L_i \subset C \cap -C$. Conversely, let $z = \sum_{i \in I} c_i$ with $c_i \in C_i$ for each $i \in I$, and $z \in -C$. Then, for each $i \in I$,

$$c_i = z - \sum_{k \neq i} c_k \in -C - C \subset -C.$$

Therefore $c_i \in C_i \cap -C = L_i$ for each $i \in I$, and we conclude that $z = \sum_{i \in I} c_i \in \sum_{i \in I} L_i$. ∎

Proposition 2.6 *There are no nonzero useless portfolios if and only if there are no non-trivial useless portfolio allocations. That is, the following two conditions are equivalent.*

(i) $\boldsymbol{L}(\mathcal{F}) = \{0\}$.

(ii) $L_i = \{0\}, \forall i \in I$.

Proof. For each $i \in I$, $L_i \subset \sum_{i \in I} L_i = \mathcal{L}(\mathcal{F}) \subset \boldsymbol{L}(\mathcal{F})$, hence if $\boldsymbol{L}(\mathcal{F}) = \{0\}$, then we must have $L_i = \{0\}, \forall i \in I$, that is (i) implies (ii). Conversely, if $L_i = \{0\}$, for each $i \in I$, then since $\mathcal{L}(\mathcal{F}) = \sum_{i \in I} L_i$ (by Proposition 2.5), we have $\mathcal{L}(\mathcal{F}) = \sum_{i \in I} \{0\} = \{0\}$. Therefore $\boldsymbol{L}(\mathcal{F}) = \{0\}$ by Theorem 2.1. ∎

Proposition 2.7 (i) *For every asymptotic arbitrage-free price vector $q \in Q$, for every $i \in I$, and for every useless portfolio ζ_i for agent i, that is, $\zeta_i \in L_i$, one has $q \cdot \zeta_i = 0$. Hence For every asymptotic-arbitrage-free asset price vector $q \in Q$ and for every aggregate useless portfolio ζ in $\mathcal{F}$, one has $q \cdot \zeta = 0$.*

(ii) *Under the non-satiation assumption* **NS**, *if ζ_i is useless for agent i, then for every $q \in Q$, assertion* (a) *implies assertion* (b).

(a) $(x_i, z_i) \in B_i(\mathcal{F}, p, q)$.

(b) $(x_i, z_i + \zeta_i) \in B_i(\mathcal{F}, p, q)$.

(iii) *Under the non-satiation assumption* **NS**, *for every useless portfolio allocation* $\zeta = (\zeta_1, \cdots, \zeta_I)$, *one has: if* (x^*, z^*, p^*, q^*) *is an equilibrium in* $(\mathcal{E}, \mathcal{F})$ *then there exists a useless portfolio allocation* $\xi = (\xi_1, \cdots, \xi_I)$ *such that* $(x^*, z^* + \zeta + \xi, p^*, q^*)$ *is an equilibrium in* $(\mathcal{E}, \mathcal{F})$.

Proof. (i) Since $\zeta_i \in L_i \subset \boldsymbol{A}Z_i \cap \{V \geq 0\}$, and $q \in Q$ we must have $-q \cdot \zeta_i \leq 0$. But $\zeta_i \in \boldsymbol{L}(Z_i)$ implies that $-\zeta_i \in \boldsymbol{L}(Z_i)$, hence we must also have $-q \cdot (-\zeta_i) \leq 0$. Therefore $q \cdot \zeta_i = 0$.

(ii) $z_i + \zeta_i \in Z_i$ since $\zeta_i \in \boldsymbol{A}Z_i$. Moreover $V(z_i + \zeta_i) = Vz_i$ because $\zeta_i \in \ker V$ (since $V\zeta_i \geq 0$ and $V\zeta_i \leq 0$), and $-q \cdot (z_i + \zeta_i) = -q \cdot z_i$ because $q \cdot \zeta_i = 0$ since $q \in Q$ and $\zeta_i \in L_i \subset \boldsymbol{L}(\mathcal{F})$.

(iii) Follows from (ii). ∎

2.3.3 Equivalence between the two definitions of useless portfolios

Remark 2.5 The set of Werner useless portfolios can be strictly contained in the set of useless portfolios. Indeed, let $V = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $I = 2$, $Z_1 = \mathbb{R}^2_+$, and $Z_2 = \mathbb{R}_- \times \mathbb{R}_+$. Then $L_1 = \mathbb{R}_+ \times \{0\}$, $L_2 = \mathbb{R}_- \times \{0\}$, and $\boldsymbol{L}(\boldsymbol{A}Z_1 \cap \{V \geq 0\} + \boldsymbol{A}Z_2 \cap \{V \geq 0\}) = \mathbb{R} \times \{0\}$. Note that there is no nontrivial Werner useless portfolio since $\boldsymbol{L}(Z_1) \cap \ker V = \boldsymbol{L}(Z_2) \cap \ker V = \{0\}$.

The following proposition provides a necessary and sufficient condition for the definitions of useless portfolios to coincide. We recall the following definition.

Definition 2.10 *The family* $(C_i)_{i \in I}$ *of closed convex cones of* $\mathbb{R}^J$ *is said to be weakly positively semi-independent* **WPSI** *if*

$$\forall i \in I,\ c_i \in C_i, \sum_{i \in I} c_i = 0 \Rightarrow c_i \in C_i \cap -C_i, \forall i \in I.$$

Proposition 2.8 *The following two assertions are equivalent.*

(i) *For every* $i \in I$, $\boldsymbol{L}(Z_i) \cap \ker V = L_i$.

(ii) *The sets* $\boldsymbol{A}Z_i \cap \{V \geq 0\}$ *are weakly positively semi-independent.*

Proof. For $i \in I$, denote $C_i = \boldsymbol{A}Z_i \cap \{V \geq 0\}$. Hence $\boldsymbol{L}(Z_i) \cap \ker V = \boldsymbol{L}(C_i)$ and $L_i = C_i \cap -\sum_{i\in I} C_i = C_i \cap \boldsymbol{L}(\sum_{i\in I} C_i)$.

We show (i) $\Rightarrow$ (ii). Let $c_i \in C_i, \sum_{i\in I} c_i = 0$. Then $c_1 = -\sum_{i\neq 1} c_i \in \sum_{i\in I} C_i \cap -\sum_{i\in I} C_i$. Hence $c_1 \in C_1 \cap \boldsymbol{L}(\sum_{i\in I} C_i)$ and by (i) one has $c_1 \in \boldsymbol{L}(C_1)$.

We show (ii) $\Rightarrow$ (i). We claim that if the sets C_i are weakly positively semi-independent then

$$\sum_{i\in I}[C_i \cap -C_i] = \sum_{i\in I} C_i \cap -\sum_{i\in I} C_i.$$

Indeed, let $c \in \sum_{i\in I} C_i \cap -\sum_{i\in I} C_i$ then $c = \sum_{i\in I} c_i = -\sum_{i\in I} c'_i$ for some c_i, c'_i in C_i. Consequently, $\sum_{i\in I} c_i + c'_i = 0$, with $c_i + c'_i \in C_i$ (because C_i is convex). Since the C_i are weakly positively semi-independent, we deduce that for all i, $c_i + c'_i \in C_i \cap -C_i$, hence $c_i \in C_i \cap -C_i$. This shows that $c = \sum_{i\in I} c_i \in \sum_{i\in I}[C_i \cap -C_i]$ and ends the proof of the claim.

Let $v_1 \in C_1 \cap \boldsymbol{L}(\sum_{i\in I} C_i)$. Let $v_2 = \cdots = v_I = 0$. Then $\sum_{i\in I} v_i = v_1 \in \boldsymbol{L}(\sum_{i\in I} C_i)$ and $\forall i, v_i \in C_i$. Therefore, by the above claim, $\forall i, v_i \in \boldsymbol{L}(C_i)$ in particular $v_1 \in \boldsymbol{L}(C_1)$. ■

Remark 2.6 Condition (i) implies that

(i') $\sum_{i\in I} \boldsymbol{L}(Z_i) \cap \ker V = \sum_{i\in I} L_i$.

But the converse is not true as shown by the following example. That is having the same useless portfolio allocations is stronger than having the same aggregate useless portfolios.

Example 2.2 Consider $C_1 = \mathbb{R}_+ \times \mathbb{R}$ and $C_2 = \mathbb{R} \times \{0\}$. We have $L_i := C_i \cap -\sum_{i\in I} C_i = C_i$ and $\boldsymbol{L}(C_1) = \{0\} \times \mathbb{R}$, and $\boldsymbol{L}(C_2) = \mathbb{R} \times \{0\}$. Then $\boldsymbol{L}(C_1) + \boldsymbol{L}(C_2) = \mathbb{R}^2 = L_1 + L_2$, but $\boldsymbol{L}(C_1) \neq C_1$, that is, $\boldsymbol{L}(C_1) \neq L_1$.

2.4 Polar Characterization of Q and consequences

Theorem 2.2 *Assume that for every* i, Z_i *is closed convex and contains 0, then the following holds*

$$\begin{aligned}
\text{cl}Q &= -\Big(\sum_{i\in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\})\Big)^o \\
&= -\bigcap_i (\boldsymbol{A}Z_i \cap \{V \geq 0\})^o \\
&= -\bigcap_i \text{cl}\Big((\boldsymbol{A}Z_i)^o + V^T(-\mathbb{R}^S_+)\Big). \\
-Q^o &= \text{cl}\sum_{i\in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\}).
\end{aligned}$$

Before giving the proof of Theorem 2.2 we provide some consequences.

Corollary 2.2 *The following holds:*

$$\begin{aligned}
Q^\perp &= \text{cl}\sum_{i\in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\}) \cap -\text{cl}\sum_{i\in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\}) \\
&= \boldsymbol{L}\big(\sum_{i\in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\})\big), \\
\text{Aff}(Q) &= < Q >= \Big(\boldsymbol{L}\big(\sum_{i\in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\})\big)\Big)^\perp, \\
\text{dim}\mathbb{R}^J &= \text{dim}Q + \text{dim}\boldsymbol{L}\big(\sum_{i\in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\})\big).
\end{aligned}$$

Proof. Note that $Q^\perp = Q^o \cap -Q^o$. Hence from the last assertion of Theorem 2.2,

$$Q^\perp = \text{cl}\sum_{i\in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\}) \cap -\text{cl}\sum_{i\in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\}),$$

and from the definition of the lineality space, we get $Q^\perp \boldsymbol{L}\big(\sum_{i\in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\})\big)$. Now, using the bipolar theorem,

$$< Q >= Q^{\perp\perp} = \Big(\boldsymbol{L}\big(\sum_{i\in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\})\big)\Big)^\perp$$

and the equality $\text{dim}Q + \text{dim}\boldsymbol{L} = \text{dim}\mathbb{R}^J$ follows. ∎

Corollary 2.3 *The following two assertions are equivalent.*

(i) *The financial structure $\mathcal{F}$ is reduced, that is,*

$$\boldsymbol{L}\Big(\sum_{i\in I}\boldsymbol{A}Z_i \cap \{V \geq 0\}\Big) = \{0\}.$$

(ii) *The convex set Q of asymptotic arbitrage-free prices has full dimension ($\text{dim}Q = J$) or equivalently, there is no nontrivial linear dependence between the asset prices, that is, there does not exist $\alpha = (\alpha^j)_j \in \mathbb{R}^J, \alpha \neq 0$, such that $\sum_{j\in J}\alpha^j q^j = 0$ for every $q \in Q$.*

Proof. From Corollary 2.2, we have $\boldsymbol{L}\Big(\sum_{i\in I}\boldsymbol{A}Z_i\cap\{V\geq 0\}\Big)=\{0\}$ if and only if $\dim Q=J$. ∎

Corollary 2.4 *Assume one of the following assumptions holds.*

(i) **WPSI** *The cones $\boldsymbol{A}Z_i\cap\{V\geq 0\}$ satisfy weak positive semi-independence*[9].

(ii) **Unconstrained** *There exists i_o such that $Z_{i_o}=\mathbb{R}^J$.*

(iii) **Cass** *There exists i_o such that for every i, $Z_i\subset Z_{i_o}$.*

Then the set $\sum_{i\in I}(\boldsymbol{A}Z_i\cap\{V\geq 0\})$ is closed and one has

$$\begin{aligned}
-Q^o &= \sum_{i\in I}(\boldsymbol{A}Z_i\cap\{V\geq 0\})=\boldsymbol{A}(\sum_{i\in I}Z_i\cap\{V\geq 0\}), \qquad (2.4.1)\\
Q^\perp &= \boldsymbol{L}\Big(\sum_{i\in I}(\boldsymbol{A}Z_i\cap\{V\geq 0\})\Big)\\
&= \boldsymbol{L}\Big(\sum_{i\in I}(Z_i\cap\{V\geq 0\})\Big)\\
&= \sum_{i\in I}\boldsymbol{L}(Z_i\cap\{V\geq 0\}).
\end{aligned}$$

Proof. (i) If the cones $\boldsymbol{A}Z_i\cap\{V\geq 0\}$ satisfy **WPSI** then by Theorem 2.3 (in the appendix), $C=\sum_{i\in I}(\boldsymbol{A}Z_i\cap\{V\geq 0\})$ is closed, $\sum_{i\in I}(\boldsymbol{A}Z_i\cap\{V\geq 0\})=\boldsymbol{A}(\sum_{i\in I}Z_i\cap\{V\geq 0\})$ and $\boldsymbol{L}\Big(\sum_{i\in I}(Z_i\cap\{V\geq 0\})\Big)=\boldsymbol{L}\Big(\sum_{i\in I}(\boldsymbol{A}Z_i\cap\{V\geq 0\})\Big)=\sum_{i\in I}\boldsymbol{L}(Z_i\cap\{V\geq 0\})$. Hence the desired result follows from Theorem 2.2 and Corollary 2.2.

(ii) Notice that (ii) $\Rightarrow$ (iii), hence it suffices to show that the result of Corollary 2.4 holds true under assumption (iii).

If there exists i_o such that for every i, $Z_i\subset Z_{i_o}$, then for every i, $\boldsymbol{A}Z_i\cap\{V\geq 0\}\subset \boldsymbol{A}Z_{i_o}\cap\{V\geq 0\}$. But $\boldsymbol{A}Z_{i_o}\cap\{V\geq 0\}$ is a convex cone, hence $\sum_{i\in I}(\boldsymbol{A}Z_i\cap\{V\geq 0\})=\boldsymbol{A}Z_{i_o}\cap\{V\geq 0\}$. On the other hand, $\sum_{i\in I}(Z_i\cap\{V\geq 0\})\subset(\sharp I)(Z_{i_o}\cap\{V\geq 0\})$ hence $\boldsymbol{A}(\sum_{i\in I}Z_i\cap\{V\geq 0\})\subset\boldsymbol{A}(Z_{i_o}\cap\{V\geq 0\})$ and we have equality since the reverse inclusion is immediate. Therefore

$$\sum_{i\in I}(\boldsymbol{A}Z_i\cap\{V\geq 0\})=\boldsymbol{A}(\sum_{i\in I}Z_i\cap\{V\geq 0\}),$$

which shows that $\sum_{i\in I}(\boldsymbol{A}Z_i\cap\{V\geq 0\})$ is closed and we get (2.4.1) from Theorem 2.2.

[9] A collection $\{C_i, i\in I\}$ of nonempty cones in $\mathbb{R}^\alpha$ is weakly positively semi-independent (WPSI), if $v_i\in C_i$, for all $i\in I$ and $\sum_{i\in I}v_i=0$, implies that for all $i\in I$, $v_i\in\boldsymbol{L}(C_i)$.

Now we show the last part of the corollary. If there exists i_o such that for every i, $Z_i \subset Z_{i_o}$, then for every i, $\boldsymbol{L}(Z_i \cap \{V \geq 0\}) \subset \boldsymbol{L}(Z_{i_o} \cap \{V \geq 0\})$ hence $\sum_{i \in I} \boldsymbol{L}(Z_i \cap \{V \geq 0\}) = \boldsymbol{L}(Z_{i_o} \cap \{V \geq 0\})$. But, from above $\sum_{i \in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\}) = \boldsymbol{A}Z_{i_o} \cap \{V \geq 0\}$ hence $\boldsymbol{L}\Big(\sum_{i \in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\})\Big) = \boldsymbol{L}(Z_{i_o} \cap \{V \geq 0\}) = \sum_{i \in I} \boldsymbol{L}(Z_i \cap \{V \geq 0\})$. ∎

2.5 Proof of the main result

2.5.1 Proof of Theorem 2.1

1. The implication (i1) ⇒ (i2) is a consequence of the following inclusion

$$\sum_{i \in I} \boldsymbol{A}Z_i \cap \{V \geq 0\} \subset \boldsymbol{A}\Big(\sum_{i \in I} Z_i \cap \{V \geq 0\}\Big).$$

2. The equivalence (i2) ⇔ (ii) is a consequence of Corollary 2.3.

3. We show (i2) ⇒ (iii1). To show that the closed set $K_1(v)$ is compact, it suffices to show that $\boldsymbol{A}K_1(v) = \{0\}$ (see [2]). We have

$$\boldsymbol{A}K_1(v) = \Big\{(\zeta_1, \ldots, \zeta_I) \in \Pi_{i \in I} \boldsymbol{A}Z_i : V\zeta_i \geq 0, -\sum_{i \in I} \zeta_i \in \sum_{i \in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\})\Big\}.$$

Let $(\zeta_1, \ldots, \zeta_I) \in \boldsymbol{A}K_1(v)$. Then from $\zeta_i \in \boldsymbol{A}Z_i$, $V\zeta_i \geq 0$ for every i, and $-\sum_{i \in I} \zeta_i \in \sum_{i \in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\})$ we deduce that

$$\sum_{i \in I} \zeta_i \in \Big(\sum_{i \in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\})\Big) \cap -\Big(\sum_{i \in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\})\Big) = \{0\},$$

that is, $\sum_{i \in I} \zeta_i = 0$. Hence

$$\zeta_1 = -\sum_{i \neq 1} \zeta_i \in \sum_{i \in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\}) \cap -\sum_{i \in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\}) = \{0\}.$$

Similarly, $\zeta_i = 0$ for every i. Hence $(\zeta_1, \cdots, \zeta_I) = (0, \cdots, 0)$.

4. We show (iii1) ⇒ (i2). Since $K_1(v)$ is compact, we have (by [2]) $\boldsymbol{A}K_1(v) = \{0\}$. But

$$\begin{aligned} \boldsymbol{A}K_1(v) &= \Big\{(\zeta_1, \ldots, \zeta_I) \in \Pi_{i \in I} \boldsymbol{A}Z_i : V\zeta_i \geq 0, -\sum_{i \in I} \zeta_i \in \sum_{i \in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\})\Big\} \\ &= \Big\{\zeta \in \Pi_{i \in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\}) : -\sum_{i \in I} \zeta_i \in \sum_{i \in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\})\Big\}. \end{aligned}$$

Let $\xi \in \sum_{i \in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\}) \cap -\sum_{i \in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\})$. Then $\xi = \sum_{i \in I} \xi_i$ where $\xi_i \in \boldsymbol{A}Z_i \cap \{V \geq 0\}$ for each $i \in I$. Hence $(\xi_1, \cdots, \xi_I) \in \boldsymbol{A}K_1(v) = \{0\}$, that is, $\xi_i = 0$ for each $i \in I$. Consequently, $\xi = \sum_{i \in I} \xi_i = 0$.

5. Finally we show (i2) ⇒ (i1). Let $\xi \in \mathbb{R}^J$. We claim that the closed set κ_ξ defined below is compact.

$$\kappa_\xi := \{(z_i)_i \in \Pi_i Z_i : \forall i \; V z_i \geq 0, \sum_{i\in I} z_i = \xi\}.$$

Indeed, for every $\underline{v} = (\underline{v}_i)_i \in (\mathbb{R}^S)^I$, consider the following closed set

$$K_0(v) := \{(z_i)_i \in \Pi_i Z_i : \forall i \; V z_i \geq \underline{v}_i, \sum_{i\in I} z_i = 0\}.$$

Assertion (i2) implies assertion (iii1) which in turn implies that $K_0(v)$ is compact since $K_0(v) \subset K_1(v)$. Hence $\boldsymbol{A}K_0(v) = \{0\}$ (see [2]). But

$$\boldsymbol{A}K_0(v) = \left\{(\zeta_1, \ldots, \zeta_I) \in \Pi_{i\in I}\boldsymbol{A}Z_i : V\zeta_i \geq 0, \sum_{i\in I} \zeta_i = 0\right\} = \boldsymbol{A}\kappa_\xi.$$

Therefore $\boldsymbol{A}\kappa_\xi = \{0\}$ and since κ_ξ is closed, we conclude that κ_ξ is compact (see [2]). This ends the proof of the claim.

We come back to the proof of (iii1) $\Rightarrow$ (i1). Let $\xi \in \boldsymbol{A}(\sum_{i\in I} Z_i \cap \{V \geq 0\}) \cap -\boldsymbol{A}(\sum_{i\in I} Z_i \cap \{V \geq 0\})$, then for every integer n, there exists $z_i^n \in Z_i \cap \{V \geq 0\}$ such that $n\xi = \sum_{i\in I} z_i^n$ or equivalently $\xi = \sum_{i\in I} z_i^n/n$ and we notice that $z_i^n/n \in Z_i \cap \{V \geq 0\}$ (since Z_i is convex and contains 0). From the compactness of κ_ξ one deduces that, without any loss of generality each sequence (z_i^n/n) converges to some $\xi_i \in \boldsymbol{A}Z_i \cap \{V \geq 0\}$. Hence $\xi = \sum_{i\in I} \xi_i \in \sum_{i\in I} \boldsymbol{A}Z_i \cap \{V \geq 0\}$. Similarly we prove that $-\xi \in \sum_{i\in I} \boldsymbol{A}Z_i \cap \{V \geq 0\}$. Therefore $\xi \in \sum_{i\in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\}) \cap -\sum_{i\in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\}) = \{0\}$ (from (iv)). ■

2.5.2 Proof of Theorem 2.2

The proof of Theorem 2.2 is a consequence of Lemma 2.1, also of interest in itself, and Claims 2.5.1 and 2.5.2.

Lemma 2.1 *Let C be a nonempty convex cone in $\mathbb{R}^J$ and let $\boldsymbol{L} = \mathrm{cl}C \cap -\mathrm{cl}C$.*

(a) $\mathrm{Aff}(C^o) =< C^o >= \boldsymbol{L}^\perp$, *hence* $\dim C^o = \dim \boldsymbol{L}^\perp = J - \dim \boldsymbol{L}$.

(b) $\mathrm{ri}(C^o) = \{q \in \boldsymbol{L}^\perp \mid q \cdot c < 0, \forall c \in \mathrm{cl}C \backslash \boldsymbol{L}\}$. *(Therefore* $\mathrm{ri}(C^o) = \boldsymbol{L}^\perp$ *if* $\mathrm{cl}C = \boldsymbol{L}$*).*

(c) *Moreover if* $\mathrm{cl}C \neq \boldsymbol{L}$*, one also has* $\mathrm{ri}(C^o) = \{q \in \mathbb{R}^J \mid q \cdot c < 0, \forall c \in \mathrm{cl}C \backslash \boldsymbol{L}\}$.

The proof of Lemma 2.1 is given in the Subsection 2.5.3. In the remaining of this section, we let

$$C = \sum_{i\in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\}).$$

Claim 2.5.1 $\mathrm{cl}Q \subset -C^o$.

Proof. Let $q \in Q$, let $v_i \in \boldsymbol{A}Z_i \cap \{V \geq 0\}$ $(i \in I)$. Then $-q \cdot v_i \leq 0$ for every $i \in I$ (otherwise $-q \cdot v_i > 0$ which together with $Vv_i \geq 0$ contradicts $q \in Q$). Consequently $-q \cdot \sum_{i \in I} v_i \leq 0$, hence $-q \in C^o = (\sum_{i \in I} \boldsymbol{A}Z_i \cap \{V \geq 0\})^o$. ■

Claim 2.5.2 $-\text{ri}(C^o) \subset Q$.

Proof. Let $q \in -\text{ri}(C^o) = \{q \in \boldsymbol{L}^\perp \mid q \cdot v > 0, \forall v \in \text{cl}C \backslash \boldsymbol{L}\}$ (we have the equality from the Lemma above), and assume that $q \notin Q$. Then there exists $i \in I$ and there exists $v_i \in \boldsymbol{A}Z_i$ such that $\begin{pmatrix} -q \\ V \end{pmatrix} v_i > 0$. Thus $v_i \in \boldsymbol{A}Z_i \cap \{V \geq 0\} \subset C$.

We show that $v_i \in \boldsymbol{L}$. Indeed, otherwise $v_i \in C \backslash \boldsymbol{L} \subset \text{cl}C \backslash \boldsymbol{L}$ and we must have $q \cdot v_i > 0$ since $q \in -\text{ri}(C^o)$. A contradiction to $-q \cdot v_i \geq 0$.

From $v_i \in \boldsymbol{L}$, we deduce that $Vv_i = 0$ (since $\boldsymbol{L} \subset \ker V$), moreover from $q \in \boldsymbol{L}^\perp$ and $v_i \in \boldsymbol{L}$, we get $q \cdot v_i = 0$ therefore $\begin{pmatrix} -q \\ V \end{pmatrix} v_i = 0$ which obviously contradicts the inequalities $\begin{pmatrix} -q \\ V \end{pmatrix} v_i > 0$. ■

Proof of Theorem 2.2. (i) First we show $\text{cl}Q = -C^o$. We have

$$\begin{array}{rcll} -C^o & = & -\text{clri}(C^o) & \text{by Theorem 6.3 page 46 in [2]} \\ & \subset & \text{cl}Q & \text{by Claim 2.5.2} \\ & \subset & -C^o & \text{by Claim 2.5.1.} \end{array}$$

Hence $\text{cl}Q = -C^o$.

Now we show that $\text{cl}C = -Q^o$. From $\text{cl}Q = -C^o$ we get $-C^{oo} = (\text{cl}Q)^o = Q^o$. Furthermore $C^{oo} = \text{cl}C$ since $\text{cl}C$ is a closed convex cone from The Bipolar Theorem. Therefore $\text{cl}C = -Q^o$.

(ii) $Q^\perp = Q^o \cap -Q^o$. Hence $Q^\perp = \boldsymbol{L}\big(\sum_{i \in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\})\big)$.

(iii) $\text{ri}Q = -\text{ri}(C^o)$: From (i), we have $-\text{ri}(C^o) = \text{ricl}Q$ and since Q is convex (by Proposition 2.1), $\text{ricl}Q = \text{ri}Q$ (by Theorem 6.3 page 46 in [2]). Hence $-\text{ri}(C^o) = \text{ri}Q$. ■

2.5.3 Proof of Lemma 2.1

Claim 2.5.3 *Let C be a nonempty convex cone in $\mathbb{R}^J$ and let $\boldsymbol{L} = \mathrm{cl}C \cap -\mathrm{cl}C$. Then the linear space spanned by the polar cone C^o of C is exactly $\boldsymbol{L}^\perp$, that is*

$$< C^o >= \boldsymbol{L}^\perp.$$

Proof. Since $L \subset C$ we get $C^o \subset L^\perp$, hence $< C^o > \subset \boldsymbol{L}^\perp$. Assume that $C^o \subset < C^o > \subsetneq \boldsymbol{L}^\perp$. Then

$$\mathrm{cl}C = C^{oo} \supset < C^o >^\perp \supsetneq \boldsymbol{L}^{\perp\perp} = \boldsymbol{L} \qquad \text{by the Bipolar Theorem [2].}$$

That is $< C^o >^\perp$ is a linear subspace contained in $\mathrm{cl}C$ and strictly contains the lineality space of $\mathrm{cl}C$. This contradicts the definition of $\boldsymbol{L}$. ■

Claim 2.5.4 *Let C be a nonempty convex subset of $\mathbb{R}^J$, and let $\boldsymbol{L} \subset \boldsymbol{A}C \cap -\boldsymbol{A}C$. Then*

$$\mathrm{cl}C = \mathrm{cl}C \cap \boldsymbol{L}^\perp + \boldsymbol{L}.$$

Proof. Denote by π the orthogonal projection of $\mathbb{R}^J$ on $\boldsymbol{L}^\perp$. If $c \in \mathrm{cl}C$ then $c = \pi c + (c - \pi c)$. The second term $c - \pi c$ is in $\ker \pi = \boldsymbol{L}$ and the first term πc is in $\pi \mathrm{cl}C \subset \boldsymbol{L}^\perp$ and can be written $\pi c = c - (c - \pi c) \in \mathrm{cl}C + \boldsymbol{L} \subset \mathrm{cl}C$, hence $\pi c \in \mathrm{cl}C \cap \boldsymbol{L}^\perp$. For the converse, $\mathrm{cl}C \cap \boldsymbol{L}^\perp + \boldsymbol{L} \subset \mathrm{cl}C + \boldsymbol{L} \subset \mathrm{cl}C$ by definition of $\boldsymbol{L}$. This ends the proof of the claim. ■

Proof of Lemma 2.1. (a) From Claim 2.5.3 we have $< C^o >= \boldsymbol{L}^\perp$. Hence $\dim C^o = \dim \boldsymbol{L}^\perp = J - \dim \boldsymbol{L}$.

(b) Suppose first that $\mathrm{cl}C = \boldsymbol{L}$ then $C^o = \boldsymbol{L}^\perp$ and $\mathrm{ri}(C^o) = \boldsymbol{L}^\perp$.

Assume now that $\mathrm{cl}C \neq \boldsymbol{L}$. First we show $\mathrm{ri}(C^o) \subset \{q \in \boldsymbol{L}^\perp \mid q \cdot c < 0, \forall c \in \mathrm{cl}C \backslash \boldsymbol{L}\}$. Let $q \in \mathrm{ri}(C^o)$, then $q \in \boldsymbol{L}^\perp$ (from Claim 2.5.3) and there exists $\varepsilon > 0$ such that $B(q, 2\varepsilon) \cap \boldsymbol{L}^\perp \subset C^o$. Let $c \in \mathrm{cl}C \backslash \boldsymbol{L}$ with $||c|| \leq 1$. Using Claim 2.5.4, write $c = \check{c} + \ell$ with $\check{c} \neq 0, \check{c} \in \mathrm{cl}C \cap \boldsymbol{L}^\perp$ and $\ell \in L$. Then $||\check{c}|| \leq ||c|| \leq 1$ and $(q + \varepsilon\check{c}) \cdot c \leq 0$ (since $q + \varepsilon\check{c} \in B(q, 2\varepsilon) \cap \boldsymbol{L}^\perp \subset C^o$ and $c \in \mathrm{cl}C$). Therefore $q \cdot c \leq -\varepsilon\check{c} \cdot c = -\varepsilon||\check{c}||^2 < 0$ (since $\check{c} \neq 0$). Thus $q \in \{q \in \boldsymbol{L}^\perp \mid q \cdot c < 0, \forall c \in \mathrm{cl}C \backslash \boldsymbol{L}\}$.

Conversely, let $q \in \{q \in \boldsymbol{L}^\perp \mid q \cdot c < 0, \forall c \in \mathrm{cl}C \backslash \boldsymbol{L}\}$ and assume that $q \notin \mathrm{ri}(C^o)$. Recall that $C^o \subset \boldsymbol{L}^\perp$. Then by means of a separation theorem in $\boldsymbol{L}^\perp$, there exists $c^* \in \boldsymbol{L}^\perp \backslash \{0\}$

(hence $c^* \notin \boldsymbol{L}$) such that

$$\sup_{v \in C^o} c^* \cdot v = \sup_{v \in \mathrm{ri}(C^o)} c^* \cdot v \leq c^* \cdot q \quad \text{because } C^o = \mathrm{clri} C^o \quad \text{by [2].}$$

Since $0 \in C^o$, we get $c^* \cdot q \geq 0$. Moreover, since C^o is a cone and the linear form $v \mapsto c^* \cdot v$ is bounded above on C^o, we have $\sup_{v \in C^o} c^* \cdot v \leq 0$. Then $c^* \in C^{oo} = \mathrm{cl} C$ (from the Bipolar Theorem) and since $c^* \notin \boldsymbol{L}$ we must have $q \cdot c^* < 0$ (by definition of q). This contradicts the fact that $c^* \cdot q \geq 0$.

(c) Let $q \in \{q \in \mathbb{R}^J \mid q \cdot c < 0, \forall c \in \mathrm{cl} C \backslash \boldsymbol{L}\}$ and let $\ell \in \boldsymbol{L}$. We show that $q \cdot \ell = 0$. Let $\overline{v} \in \mathrm{cl} C \backslash \boldsymbol{L}$ (which is nonempty by assumption) and consider $v^n := \frac{1}{n}\overline{v} + (1 - \frac{1}{n})\ell$. Notice that $v^n \in \mathrm{cl} C \backslash \boldsymbol{L}$ (since $v^n \in \mathrm{cl} C$ because $\overline{v}, \ell \in \mathrm{cl} C$ and $\mathrm{cl} C$ is convex, and $v^n \notin \boldsymbol{L}$ because $\overline{v} \notin \boldsymbol{L}, \ell \in \boldsymbol{L}$ and $\boldsymbol{L}$ is a linear space). Consequently $q \cdot v^n < 0$, and by taking the limit when n goes to infinity, we get $q \cdot \ell \leq 0$. Since $\boldsymbol{L}$ is a linear space we also have $q \cdot)(-\ell) \leq 0$. Hence $q \cdot \ell = 0$. ■

Corollary 2.5 (a) *Let C be a nonempty convex cone in $\mathbb{R}^J$. Then the following assertions are equivalent.*

(i) $\mathrm{int}(C^o) \neq \emptyset$.

(ii) $\dim C^o = \dim \mathbb{R}^J$.

(iii) $\mathrm{cl} C \cap -\mathrm{cl} C = \{0\}$.

(b) *Let C be a nonempty convex cone in $\mathbb{R}^J$ such that* $\mathrm{cl} C \cap -\mathrm{cl} C = \{0\}$. *Then*

$$\mathrm{int}(C^o) = \{q \in \mathbb{R}^J \mid q \cdot c < 0, \forall c \in \mathrm{cl} C \backslash \{0\}\}.$$

Proof. (a) Assertions (i) and (ii) are obviously equivalent since a convex set has full dimension if and only if it has nonempty interior. The fact that assertions (ii) and (iii) are equivalent is an immediate consequence of Lemma 2.1.

(b) $\mathrm{cl} C \cap -\mathrm{cl} C = \{0\}$ implies that $\mathrm{int}(C^o) \neq \emptyset$, which in turn implies that $\mathrm{int}(C^o) = \mathrm{ri}(C^o)$. The desired result follows from Lemma 2.1. ■

2.6 Appendix

Let X_i ($i \in I$) be convex subsets of $\mathbb{R}^J$ containing 0 and denote $\boldsymbol{L}_i = \boldsymbol{L}(X_i) = \boldsymbol{L}(\mathrm{cl} X_i)$.

Theorem 2.3 *Let X_i ($i \in I$) be convex subsets of $\mathbb{R}^J$ containing* 0. *Then*

(a) *The following hold:*

(i) $\sum_{i\in I} \boldsymbol{A}X_i \subset \boldsymbol{A}(\sum_{i\in I} X_i)$,

(ii) $\sum_{i\in I} \boldsymbol{L}(X_i) \subset \boldsymbol{L}(\sum_{i\in I} \boldsymbol{A}X_i) \subset \boldsymbol{L}(\sum_{i\in I} X_i)$.

(b) *If we additionally assume that the sets $\boldsymbol{A}X_i$ are weakly positively semi-independent then the above inclusions are equalities, that is*

(i) $\sum_{i\in I} \boldsymbol{A}X_i = \boldsymbol{A}(\sum_{i\in I} X_i)$,

(ii) $\sum_{i\in I} \boldsymbol{L}(X_i) = \boldsymbol{L}(\sum_{i\in I} \boldsymbol{A}X_i) = \boldsymbol{L}(\sum_{i\in I} X_i)$.

For the proof of Theorem 2.3, we need a claim. Let B be a compact set of $\mathbb{R}^J$ and

$$\begin{aligned} K &:= \{(x_1, \cdots, x_I) \in \Pi_i \text{cl}X_i : \sum_{i\in I} x_i \in B\}, \\ K_w &:= \{(\text{proj}_{\boldsymbol{L}_1^\perp} x_1, \cdots, \text{proj}_{\boldsymbol{L}_I^\perp} x_I) \in \Pi_i(\text{cl}X_i \cap \boldsymbol{L}_i{}^\perp) : (x_1, \cdots, x_I) \in K\}. \end{aligned}$$

Note that $K_w = F(K)$ where $F : (\mathbb{R}^J)^I \to (\mathbb{R}^J)^I$ is defined by

$$F(x_1, \cdots, x_I) = (\text{proj}_{\boldsymbol{L}_1^\perp} x_1, \cdots, \text{proj}_{\boldsymbol{L}_I^\perp} x_I).$$

Claim 2.6.1 *The following assertions are equivalent.*

(i) *The sets $\boldsymbol{A}X_i$ are weakly positively semi-independent.*

(ii) *The set K_w is bounded.*

Moreover the set K_w is closed (without assuming (i)*).*

Proof. [*(i)* ⇒ *(ii)*] By contradiction, assume K_w is not bounded and let $((x_i^{\perp n})_i)_n$ be a sequence in K_w (each $x_i^{\perp n}$ is in $\text{cl}X_i \cap \boldsymbol{L}_i^\perp$) such that $\sum_{i\in I} ||x_i^{\perp n}|| \underset{n\to\infty}{\longrightarrow} \infty$. Let $\hat{x}_i^n \in \boldsymbol{L}_i$ be such that $(x_i^{\perp n} + \hat{x}_i^n)_i \in K$. Then, without loss of generality (taking subsequences if necessary), one can assume that for every i,

$$\frac{x_i^{\perp n}}{\sum_{i\in I} ||x_i^{\perp n}|| + \sum_{i\in I} ||\hat{x}_i^n||} \underset{n\to\infty}{\longrightarrow} x_i^\perp \in \boldsymbol{A}X_i \cap \boldsymbol{L}_i^\perp$$

and

$$\frac{\sum_{i\in I} \hat{x}_i^n}{\sum_{i\in I} ||x_i^{\perp n}|| + \sum_{i\in I} ||\hat{x}_i^n||} \underset{n\to\infty}{\longrightarrow} \alpha \in \boldsymbol{A}(\sum_{i\in I} X_i) \cap \sum_{i\in I} \boldsymbol{L}_i.$$

Write $\alpha = \sum_{i\in I}\alpha_i$ where for each i, $\alpha_i \in \boldsymbol{L}_i$. Then $\sum_{i\in I}(x_i^{\perp} + \alpha_i) = 0$ since $\sum_{i\in I}(x_i^{\perp n} + \hat{x}_i^n) \in B$ and $\sum_{i\in I}||x_i^{\perp n}|| \underset{n\to\infty}{\longrightarrow} \infty$. But $x_i^{\perp} \in \boldsymbol{A}X_i$ and $\alpha_i \in \boldsymbol{L}_i$ then $x_i^{\perp} + \alpha_i \in \boldsymbol{A}X_i$ hence, by WPSI, for every i, $x_i^{\perp} + \alpha_i \in \boldsymbol{L}_i$ that is $x_i^{\perp} = 0$. So, $\sum_{i\in I}\alpha_i = 0$. But for every n,

$$1 = \frac{\sum_{i\in I}||x_i^{\perp n}||}{\sum_{i\in I}||x_i^{\perp n}|| + \sum_{i\in I}||\hat{x}_i^n||} + \frac{||\sum_{i\in I}\hat{x}_i^n||}{\sum_{i\in I}||x_i^{\perp n}|| + \sum_{i\in I}||\hat{x}_i^n||}$$

implies $1 = ||\sum_i \alpha_i||$. A contradiction.

[$(ii) \Rightarrow (i)$] Conversely, if $v_i \in \boldsymbol{A}X_i$, and $\sum_{i\in I}v_i = 0$, then for each i, $v_i = v_i^{\perp} + \hat{v}_i$ with $v_i^{\perp} \in \boldsymbol{A}X_i \cap \boldsymbol{L}_i^{\perp}$ and $\hat{v}_i \in \boldsymbol{L}_i$. Let $(x_i)_i \in K$, then for every $t \geq 0$, $\sum_{i\in I}(x_i + tv_i) = \sum_{i\in I}x_i \in B$. Therefore $(\text{proj}_{\boldsymbol{L}_i^{\perp}} x_i^{\perp} + tv_i^{\perp})_i \in K_w$ for every $t \geq 0$. Since K_w is bounded we must have $v_i^{\perp} = 0$ for every i, that is $v_i \in \boldsymbol{L}_i$ for each i.

Now we show that K_w is closed. Let $((\text{proj}_{\boldsymbol{L}_i^{\perp}} x_i^n)_i)_n$ be a sequence in K_w (the sequence $((x_i^n)_i)_n$ is in K) such that $\text{proj}_{\boldsymbol{L}_i^{\perp}} x_i^n \underset{n\to\infty}{\longrightarrow} x_i^{\perp} \in \boldsymbol{L}_i^{\perp} \cap \text{cl}X_i$ for each i. For each n, let $(\hat{x}_i^n)_i \in \Pi_i \boldsymbol{L}_i$ be such that $(\text{proj}_{\boldsymbol{L}_i^{\perp}} x_i^n)_i + (\hat{x}_i^n)_i \in K$. That is

$$\sum_{i\in I}\text{proj}_{\boldsymbol{L}_i^{\perp}} x_i^n + \sum_{i\in I}\hat{x}_i^n \in B.$$

The first term, $\sum_{i\in I}\text{proj}_{\boldsymbol{L}_i^{\perp}} x_i^n$, converges to $\sum_{i\in I}x_i^{\perp}$, and since B is compact we can assume that the second term, $\sum_{i\in I}\hat{x}_i^n$, converges to some α. The limit α is in $\sum_{i\in I}\boldsymbol{L}_i$, hence $\alpha = \sum_{i\in I}\alpha_i$ where, for each i, $\alpha_i \in \boldsymbol{L}_i$. Since $x_i^{\perp} \in \boldsymbol{L}_i^{\perp} \cap \text{cl}X_i$ and $\alpha_i \in \boldsymbol{L}_i$, and $\sum_{i\in I}(x_i^{\perp} + \alpha_i) \in B$, we get $(x_i^{\perp} + \alpha_i)_i \in K$ hence $(x_i^{\perp})_i \in K_w$. ■

Proof of Theorem 2.3

(a) We first notice that, for all i, $\boldsymbol{L}(X_i) \subset \boldsymbol{A}X_i \subset X_i$. Hence

$$\sum_{i\in I}\boldsymbol{L}(X_i) \subset \sum_{i\in I}\boldsymbol{A}X_i \subset \sum_{i\in I}X_i.$$

Using the fact that $\boldsymbol{L}(A) \subset \boldsymbol{L}(B)$ if $A \subset B$ we get

$$\sum_{i\in I}\boldsymbol{L}(X_i) \subset \boldsymbol{L}(\sum_{i\in I}\boldsymbol{A}X_i) \subset \boldsymbol{L}(\sum_{i\in I}X_i).$$

(b) (i) Let $v \in \boldsymbol{A}(\sum_{i\in I}X_i)$. Write

$$v = \sum_{i\in I}\frac{1}{n}x_i^n = \sum_{i\in I}\frac{1}{n}x_i^{\perp n} + \sum_{i\in I}\frac{1}{n}\hat{x}_i^n$$

where, for each i, $x_i^n \in X_i$, $x_i^{\perp n} \in X_i \cap \boldsymbol{L}_i^{\perp} \subset \text{cl}X_i \cap \boldsymbol{L}_i^{\perp}$, and $\hat{x}_i^n \in \boldsymbol{L}_i$. Then (by Claim 2.6.1), for each i,

$$\frac{1}{n}x_i^{\perp n} \underset{n\to\infty}{\longrightarrow} x_i^{\perp} \in \boldsymbol{A}X_i \cap \boldsymbol{L}_i^{\perp}$$

and

$$\sum_{i\in I}\frac{1}{n}\hat{x}_i^n \underset{n\to\infty}{\longrightarrow} \beta \in \sum_{i\in I}\boldsymbol{L}_i.$$

Write $\beta = \sum_{i\in I}\beta_i$ with $\beta_i \in \boldsymbol{L}_i$ for each i. Then $v = \sum_{i\in I}(x_i^{\perp} + \beta_i) \in \sum_{i\in I}\boldsymbol{A}X_i$.

(b)(ii) From (i) above, we get $\boldsymbol{L}(\sum_{i\in I}X_i) \subset \boldsymbol{L}(\sum_{i\in I}\boldsymbol{A}X_i)$. We show that $\boldsymbol{L}(\sum_{i\in I}\boldsymbol{A}X_i) \subset \sum_{i\in I}\boldsymbol{L}(X_i)$. Let $\xi \in \boldsymbol{L}(\sum_{i\in I}\boldsymbol{A}X_i)$. Write

$$\xi = \sum_{i\in I}\xi_i = -\sum_{i\in I}\xi_i'$$

with ξ_i and ξ_i' in $\boldsymbol{A}X_i$. Then $0 = \sum_{i\in I}(\xi_i + \xi_i')$ and for each $i \in I$, $\xi_i + \xi_i' \in \boldsymbol{A}X_i$ which implies (by definition of **WPSI**) that for every $i \in I$, $\xi_i + \xi_i' \in \boldsymbol{L}(X_i)$. Hence

$$\xi_i = -\xi_i' + (\xi_i + \xi_i') \in -\boldsymbol{A}X_i + \boldsymbol{L}(X_i) \subset -\boldsymbol{A}X_i.$$

Therefore for every $i \in I$, $\xi_i \in \boldsymbol{L}(X_i)$ that is $\xi = \sum_{i\in I}\xi_i \in \sum_{i\in I}\boldsymbol{L}(X_i)$. ∎

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Chapter 3

Reduced equivalent form of a financial structure

We show that, under mild assumptions, every financial structure is equivalent to its reduced form, and that subsequently, all equilibria in a financial economy are in one-to-one correspondence with the equilibria of an economy where the financial structure is replaced by the reduced one.

3.1 Introduction

When market participation is unrestricted, there is no loss of generality in assuming that there are no redundant assets[1], otherwise, i.e. in the presence of redundant assets, agents would be indifferent between the financial possibilities offered by the set of all assets and those offered by a strictly smaller set of linearly independent assets. Furthermore, in the absence of arbitrage opportunities (an arbitrage opportunity is a feasible portfolio that generates non-negative return in all states and positive return in some state but has non-positive value), the pricing of redundant assets and more generally redundant portfolios is simple and done by arbitrage, that is the value of an asset is equal to the value of any replicating portfolio (a portfolio is said to replicate an asset if it yields the same returns as the asset).

[1]Some assets pay returns that are linearly dependent on those of other available assets. Such assets are redundant in a frictionless market in that they can be replicated by a portfolio containing other assets.

This is not the case, however, when market participation is restricted and if redundant assets are able to have a real effect on risk sharing opportunities, which is generally the case in the presence of market frictions. To illustrate this, we give a few examples where frictions force agents to make distinct use of redundant assets.

We work in a basic two time-date (today and tomorrow) economy with I agents and an a priori uncertainty about the future represented by S states of nature in the second date. Financial transfers across today and tomorrow and across the states of the world are allowed by means of a set J of financial assets that agents can trade in. The assets payoffs are given by the return matrix V. Restrictions on agents trade possibilities are represented by the sets $Z^i, i \in I$.

Example 3.1 Suppose $I = 3, S = 2, J = 3$, and $V = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. Let $Z^1 = \{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_1 + z_3 = 0, z_2 + z_3 = 0\}$, $Z^2 = \{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_3 = 0\}$, and $Z^3 = \{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_1 = z_2 = 0\}$. Then, removing the risk-free asset, i.e. the third asset which is redundant, would "kill" the market. No financial activity would take place.

Example 3.2 Suppose $I \neq \emptyset, S = 2, J = 3$, and $V = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{pmatrix}$. We do not give the constraints faced by agents explicitly, but we can think of the second asset as being an illiquid stock and the third asset is the combination of a long position in a put option on the stock with strike equal to 1 and a short position in a call option on the stock with the same strike. Assume an agent possesses 4 units of the stock (that she cannot get rid of immediately since the stock is supposed to be illiquid) and wants to guarantee a payoff of 6 tomorrow no matter what the state of the world is. In the presence of the third asset, the agent can achieve the payoff $(6, 6)$, by purchasing 2 units of the risk-free asset and 4 units of the third asset. Removing the third asset from the market would prevent the agent from efficiently hedging her position against the downward movement of the stock return.

An interesting example illustrating the fact that portfolio constraints generally generate mispricing between redundant assets and that some arbitrage portfolios might persist at equilibrium (with fixed prices) is given in [4].

As shown by the examples above, simply removing redundant assets in the presence of portfolio restrictions would considerably change the nature of the market by altering wealth transfer sets. In this paper we provide a different approach to the problem posed by dealing with redundant assets. Instead of removing all redundant assets, we show that actually, there is no harm in removing some of the redundant portfolios (the useless ones). More precisely, we show that every financial structure is equivalent (in terms of financial possibilities) to another structure in which there are no useless portfolios (its reduced form), and that subsequently, all equilibria in a financial economy are in one-to-one correspondence with the equilibria of an economy where the financial structure is replaced by the reduced one.

The paper is organized as follows. In the next section, we describe the financial structure, we define useless portfolios and the reduced form of a financial structure, and we state our main result. Section 3.3 is devoted to the proof of our main result, some examples are gathered in Section 3.4 and some proofs are deferred to the appendix.

3.2 The two-period model and the main result

3.2.1 The stochastic exchange economy

[2] We consider the basic model of a two time-date economy with nominal assets. It is also assumed that there are finite sets I, H, S, and J, respectively, of agents, divisible physical commodities, states of nature, and nominal assets.

In what follows, the first date will also be referred to as $t = 0$ and the second date, as $t = 1$. There is an a priori uncertainty at the first date ($t = 0$) about which of the states of nature $s \in S$ will prevail at the second date ($t = 1$). For the sake of unified notations of time and

[2]We shall use hereafter the following notations. If I and J are finite sets, the space $\mathbb{R}^I$ (identified to $\mathbb{R}^{\#I}$ whenever necessary) of functions $x : I \to \mathbb{R}$ (also denoted $x = (x(i))_{i\in I}$ or $x = (x_i)$) is endowed with the scalar product $x \cdot y := \sum_{i\in I} x(i)y(i)$, and we denote by $\|x\| := \sqrt{x \cdot x}$ the Euclidean norm. By $B(x, r)$ we denote the closed ball centered at $x \in \mathbb{R}^L$ of radius $r > 0$, namely $B(x, r) = \{y \in \mathbb{R}^L : \ \|y - x\| \leq r\}$. In $\mathbb{R}^I$, the notation $x \geq y$ (resp. $x > y$, $x \gg y$) means that, for every i, $x(i) \geq y(i)$ (resp. $x \geq y$ and $x \neq y$, $x(i) > y(i)$) and we let $\mathbb{R}^I_+ = \{x \in \mathbb{R}^I \mid x \geq 0\}$, $\mathbb{R}^I_{++} = \{x \in \mathbb{R}^I \mid x \gg 0\}$. An $I \times J$-matrix $A = (a^j_i)_{i\in I, j\in J}$ (identified with a classical $(\#I) \times (\#J)$-matrix if necessary) is an element of $\mathbb{R}^{I\times J}$ whose rows are denoted $A_i = (a^j_i)_{j\in J} \in \mathbb{R}^J$ ($i \in I$), and columns $A^j = (a^j_i)_{i\in I} \in \mathbb{R}^I$ ($j \in J$). To the matrix A, we associate the linear mapping, from $\mathbb{R}^J$ to $\mathbb{R}^I$, also denoted by A, defined by $Ax = (A_i \cdot x)_{i\in I}$. The span of the matrix A, also called the image of A, is the set $< A >:= \{Ax \mid x \in \mathbb{R}^J\}$. The transpose matrix of A, denoted by A^T, is the $J \times I$-matrix whose rows are the columns of A, or equivalently, is the unique linear mapping $A^T : \mathbb{R}^I \to \mathbb{R}^J$, satisfying $Ax \cdot y = x \cdot A^T y$ for every $x \in \mathbb{R}^J$, $y \in \mathbb{R}^I$.

uncertainty, the non-random state at the first date is denoted by $s = 0$ ($S_0 = \{0\}$) and $\bar{S}$ stands for the set $\{0\} \cup S$.

At each state of nature $s \in \bar{S}$, there is a spot market where the finite set H of physical commodities is available. We assume that each commodity does not last more than one period so that the commodity space is $\mathbb{R}^L$, with $L = H \times \bar{S}$ (in this model, a commodity is a couple $(h, s) \in H \times \bar{S}$ of a physical commodity, h, and a state of nature s, at which h will be available). An element $x \in \mathbb{R}^L$ is called a consumption (or a consumption plan), that is $x = (x(s))_{s \in \bar{S}} \in \mathbb{R}^L$, where $x(s) = (x(h, s))_{h \in H} \in \mathbb{R}^H$, for every $s \in \bar{S}$.

We denote by $p = (p(s))_{s \in \bar{S}} \in \mathbb{R}^L$ the vector of spot prices and $p(s) = (p(h, s))_{h \in H} \in \mathbb{R}^H$ is called the spot price at state s. The spot price $p(h, s)$ is the price paid, at date 0 if $s = 0$ and at date 1 if $s \in S$, for the delivery of one unit of commodity h at state s.

Each agent $i \in I$, also called a consumer, is endowed with a consumption set $X_i \subset \mathbb{R}^L$ which is the set of her possible consumptions. An allocation is an element $x \in \Pi_i X_i$, and we denote by x_i the consumption of agent i, that is the projection of x onto X_i.

The tastes of each consumer $i \in I$ are represented by a strict preference correspondence $P_i : \Pi_{k \in I} X_k \to X_i$, where $P_i(x)$ defines the set of consumptions that are strictly preferred by i to x_i, that is, given the consumptions x_k for other consumers $k \neq i$.

At each state of nature, $s \in \bar{S}$, every consumer $i \in I$ has a state-endowment $e_i(s) \in \mathbb{R}^H$ contingent to the fact that s prevails and we denote by $e_i = (e_i(s))_{s \in \bar{S}} \in \mathbb{R}^L$ her endowment vector across the different states.

The consumption structure, denoted $\mathcal{E}$, can be summarized by

$$\mathcal{E} = \Big(I, H, S, (X_i, P_i, e_i)_{i \in I}\Big).$$

Definition 3.1 *The consumption structure $\mathcal{E}$ is said to be standard if it satisfies the following two standard assumptions* **C** *and* **LNS**.

Consumption Assumption C

(i) *For every $i \in I$, X_i is a bounded below, closed, convex subset of $\mathbb{R}^{L(1+S)}$.*

(ii) Continuity of Preferences *For every $i \in I$, the correspondence $P_i : \Pi_i X_i \to X_i$ is lower semicontinuous with convex open values in X_i for the relative topology of X_i.*

(iii) Irreflexive Preferences *For every $i \in I$, for every $x = (x_i)_{i \in I} \in \Pi_i X_i$, $x_i \notin P_i(x)$.*

(iv) Strong Survival SS *For every* $i \in I$, $e_i \in \text{int}X_i$.

(v) Non-Satiation NS *For every* $i \in I$, *for every* $x \in \Pi_i X_i$, *for every* $s \in \bar{S}$, *there exists* $x'_i \in P_i(x)$ *such that* $x'_i(s') = x_i(s')$ *for all* $s' \neq s$.

Local Non-Satiation LNS *For every* $i \in I$, *for every* $\bar{x} \in \prod_{i \in I} X_i$, *and for every* $x_i \in P_i(\bar{x})$, $(\bar{x}_i, x_i] \subset P_i(\bar{x})$.

3.2.2 The Financial Structure

Agents may operate financial transfers across states in S (i.e. across the two periods and across the states of the second period) by exchanging a finite number of nominal assets $j \in J$, which define the financial structure of the model.[3] The nominal assets are traded at the first period ($t = 0$) and yield payoffs at the second period ($t = 1$), contingent on the realization of the state of nature $s \in S_1$. The payoff of the nominal asset $j \in J$, when state $s \in S$ is realized, is V_s^j, and we denote by V the $S \times J$-return matrix $V = (V_s^j)$, which does not depend upon the asset prices $q \in \mathbb{R}^J$ (and will not depend upon the commodity prices p in the associated equilibrium model). A portfolio $z = (z_j) \in \mathbb{R}^J$ specifies the quantities $|z_j|$ ($j \in J$) of each asset j (with the convention that the asset j is bought if $z_j > 0$ and sold if $z_j < 0$). Thus Vz is its random financial return across states at time $t = 1$, and $V_s \cdot z$ is its return if state s prevails.

We assume that each agent i is restricted to choose her portfolio within a portfolio set $Z_i \subset \mathbb{R}^J$, which represents the set of portfolios that are (institutionally) admissible for agent i. This general framework allows us to address, for example, the following important cases:

(i) $Z_i = \mathbb{R}^J$ (unconstrained portfolios),

(ii) $Z_i = \underline{z}_i + \mathbb{R}^J_+$, for some $\underline{z}_i \in -\mathbb{R}^J_+$ (exogenous bounds on short sales),

(iii) $Z_i = B_J(0, 1)$ (bounded portfolio sets).

(iv) Z_i is a vector space.

(v) Z_i is polyhedral and contains 0 (linear equality and inequality portfolio constraints).

[3]The case of no financial assets – i.e., J is empty – is called pure spot markets.

Note that the polyhedral case covers the cases (i)-(iv) (with an appropriate choice of the norm in (iii)). Throughout the paper we make the following assumption which covers all the above cases:

F1. For every $i \in I$, Z_i is closed, convex, and contains 0.

We summarize by $\mathcal{F} = \big(I, J, S, V, (Z_i)_{i\in I}\big)$ the financial characteristics, referred to as the financial structure. When there is no risk of confusion, the financial structure $\mathcal{F}$ will be denoted $\mathcal{F} := \big(V, (Z_i)_i\big)$. We use the following notation when there is no risk of confusion.

- If $Z_i = C$ for every $i \in I$, we denote $\mathcal{F} = (V, C)$. In particular when $C = \mathbb{R}^J$, we drop the dependence of $\mathcal{F}$ on C, that is we write $\mathcal{F} = V$.
- If $\mathcal{F} = \big(V, (Z_i)_i\big)$ and $\mathcal{F}' = \big(V', (Z'_i)_i\big)$, we denote[4] $\mathcal{F} \otimes \mathcal{F}' := \big([V, V'], (Z_i \times Z'_i)_i\big)$.

3.2.3 Financial equilibria and no-arbitrage

Consider a financial exchange economy $(\mathcal{E}, \mathcal{F})$, where $\mathcal{E}$ is an exchange economy and $\mathcal{F}$ a financial structure. Given the spot price vector $p \in \mathbb{R}^L$ and the asset price vector $q \in \mathbb{R}^J$, the *budget set* of consumer $i \in I$ in this setting is defined as follows[5]

$$\begin{aligned} B_i(\mathcal{F}, p, q) &= \{(x_i, z_i) \in X_i \times Z_i : \ \forall s \in \bar{S}, \ \ p(s) \cdot [x_i(s) - e_i(s)] \leq [W(q)z_i](s)\} \\ &= \{(x_i, z_i) \in X_i \times Z_i : \ p \,\Box\, (x_i - e_i) \leq W(q)z_i\}. \end{aligned}$$

Where $W(q)$ is the $(S+1) \times J$ matrix $\begin{pmatrix} -q \\ V \end{pmatrix}$, referred to as the full-return matrix.

An equilibrium in the financial exchange economy is then defined as a collection of strategies (a consumption and an asset trade strategy for each agent) and prices (commodity spot prices and asset prices) such that each agent maximizes her preferences over her budget set, and all markets clear (commodity markets clear in all dates and states, and asset markets clear).

Definition 3.2 *An equilibrium in the financial exchange economy $(\mathcal{E}, \mathcal{F})$ is a list of strategies and prices $\big(\bar{x}, \bar{z}, \bar{p}, \bar{q}\big) \in (\mathbb{R}^L)^I \times (\mathbb{R}^J)^I \times \mathbb{R}^L \backslash \{0\} \times \mathbb{R}^J$ such that*

[4]The matrix $[V, V']$ is the $(S \times (J + J'))$ matrix whose first J columns are those of V and the last J' columns are those of V'.

[5]For $x = (x(s))_{s\in\bar{S}}, p = (p(s))_{s\in\bar{S}}$ in $\mathbb{R}^L = \mathbb{R}^{H\bar{S}}$ (with $x(s), p(s)$ in $\mathbb{R}^H$ for each $s \in \bar{S}$) we let $p \,\Box\, x = (p(s) \cdot x(s))_{s\in\bar{S}} \in \mathbb{R}^{\bar{S}}$.

(a) *for every* $i \in I$, $(\bar{x}_i, \bar{z}_i)$ *maximizes the preferences* P_i, *in the sense that*

$$(\bar{x}_i, \bar{z}_i) \in B_i(\mathcal{F}, \bar{p}, \bar{q}) \text{ and } [P_i(\bar{x}) \times Z_i] \cap B_i(\mathcal{F}, \bar{p}, \bar{q}) = \emptyset$$

where $\bar{x} = (\bar{x}_i)_{i \in I}$, *and*

(b) $\sum_{i \in I} \bar{x}_i = \sum_{i \in I} e_i$ *and* $\sum_{i \in I} \bar{z}_i = 0$.

A consumption equilibrium in the financial exchange economy $(\mathcal{E}, \mathcal{F})$ *is a list of consumption strategies and commodity prices* $(\bar{x}, \bar{p}) \in (\mathbb{R}^L)^I \times \mathbb{R}^L \backslash \{0\}$ *such that there exist trade strategies and asset prices* $(\bar{z}, \bar{q}) \in (\mathbb{R}^J)^I \times \mathbb{R}^J$ *with* $\big(\bar{x}, \bar{z}, \bar{p}, \bar{q}\big)$ *is an equilibrium in* $(\mathcal{E}, \mathcal{F})$.

Our notion of no-arbitrage takes into account only arbitrage opportunities that might yield an infinite payoff (the intuition underlying this definition is that the market will be able to rule out any arbitrage opportunity with finite payoff).

Definition 3.3 *If* C *is a nonempty convex set in* $\mathbb{R}^J$, *we let*

$$\begin{aligned} \boldsymbol{A}C &:= \{\zeta \in \mathbb{R}^J : \zeta + \mathrm{cl}C \subset \mathrm{cl}C\} && \text{be the asymptotic cone of } C \\ \boldsymbol{L}(C) &:= \boldsymbol{A}C \cap (-\boldsymbol{A}C) && \text{be the lineality space of } C. \end{aligned}$$

Definition 3.4 *The set of arbitrage-free prices of* $\mathcal{F} = \big(V, (Z_i)_{i \in I}\big)$ *is*

$$Q(\mathcal{F}) = \big\{q \in \mathbb{R}^J : W(q)(\bigcup_i \boldsymbol{A}Z_i) \cap \mathbb{R}^{S+1}_+ = \{0\}\big\}.$$

where $\boldsymbol{A}Z_i$ *denotes the asymptotic cone of the set* Z_i.

3.2.4 Equivalent and reduced financial structures

We introduce an equivalence relation on the set of all financial structures. We will say that two financial structures are equivalent if they are indistinguishable in terms of consumption equilibria. The intuition behind this definition is the following. Financial structures allow agents to transfer wealth across states of nature and thereby give them the possibility to enlarge their budget set. Hence if, regardless of the standard exchange economy $\mathcal{E}$, equilibrium consumption allocations and equilibrium commodity price vectors are the same when agents carry out their financial activities through two different structures, then we say that these two financial structures are equivalent.

Definition 3.5 *Consider two financial structures* $\mathcal{F} = \big(V, (Z_i)_i\big)$ *and* $\mathcal{F}' = \big(V', (Z'_i)_i\big)$.

We say that $\mathcal{F} \sim \mathcal{F}'$ (read $\mathcal{F}$ is equivalent to $\mathcal{F}'$) if for every standard exchange economy $\mathcal{E}$, the financial exchange economies $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{E}, \mathcal{F}')$ have the same consumption equilibria.

Definition 3.6 *The financial structure $\mathcal{F}$ is said to be reduced if one of the following equivalent conditions is satisfied.*

(i1) $\boldsymbol{L}(\mathcal{F}) := \boldsymbol{A}\Big(\sum_{i\in I} Z_i \cap \{V \geq 0\}\Big) \cap -\boldsymbol{A}\Big(\sum_{i\in I} Z_i \cap \{V \geq 0\}\Big) = \{0\}$.

(i2) $\mathcal{L}(\mathcal{F}) := (\sum_{i\in I} \boldsymbol{A}Z_i \cap \{V \geq 0\}) \cap -(\sum_{i\in I} \boldsymbol{A}Z_i \cap \{V \geq 0\}) = \{0\}$.

(ii) *The convex set Q of asymptotic arbitrage-free prices has full dimension ($\dim Q = J$) or equivalently, there is no nontrivial linear dependence between the asset prices, that is, there is no $\alpha = (\alpha^j)_j \in \mathbb{R}^J, \alpha \neq 0$, such that $\sum_{j\in J} \alpha^j q^j = 0$ for every $q \in Q$.*

(iii1) *For every $v = (v_i)_i \in (\mathbb{R}^S)^I$, the set $K_1(v)$ defined below is compact.*

$$K_1(v) := \{(z_i)_i \in \Pi_i Z_i : \forall i \ V z_i \geq v_i, -\sum_{i\in I} z_i \in \sum_{i\in I} (\boldsymbol{A}Z_i \cap \{V \geq 0\})\}.$$

The equivalence between the above conditions is established in [1].

3.2.5 The main results

Before stating our first result we introduce an assumption that will be discussed in the next section.

F2 Closedness Assumption The following set $\mathcal{G}(\mathcal{F})$ is closed, where

$$\mathcal{G}(\mathcal{F}) := \{(V z_1, \cdots, V z_I, \sum_{i\in I} z_i) \in (\mathbb{R}^S)^I \times \mathbb{R}^J : \forall i \in I, z_i \in Z_i\}.$$

We can now state the first result of this paper.

Theorem 3.1 *Let $\mathcal{F} = \big(V, (Z_i)_i\big)$ be a financial structure satisfying assumptions* **F1** *and* **F2**. *Then there exists a financial structure $\mathcal{F}'$ satisfying* **F1**, *such that*

(i) *$\mathcal{F}'$ is reduced.*

(ii) *For every standard exchange economy $\mathcal{E}$, every consumption equilibrium of $(\mathcal{E}, \mathcal{F}')$ is a consumption equilibrium of $(\mathcal{E}, \mathcal{F})$.*

(iii) *If the financial structure $\mathcal{F}$ satisfies the following additional assumption* **F0**, *then the financial structures $\mathcal{F}$ and $\mathcal{F}_\pi$ are equivalent.*

F0 *For every $i \in I$, there exists $\zeta_i \in \boldsymbol{A}Z_i$ such that $V\zeta_i \gg 0$.*

Moreover we can choose $\mathcal{F}'$ so that the following property **P** *is satisfied:*

P *For every $(q, z) \in \Big(Q(\mathcal{F}') \cap Z(\mathcal{F}')\Big) \times \Pi_i Z_i$, one has*

(i) *$q \in Q(\mathcal{F}) \cap Z(\mathcal{F})$, and*

(ii) *there exists $z' = (z'_i)_i \in \Pi_i Z'_i$ such that $q \cdot z_i = q \cdot z'_i$ for every $i \in I$.*

The proof of Theorem 3.1 is postponed to Section 3.3

To state our second result, we need the following assumption. Given the financial structure $\mathcal{F} = \big(V, (Z_i)_{i\in I}\big)$, we denote $Z(\mathcal{F}) =< \sum_{i\in I} Z_i >$ the linear space spanned by $\sum_{i\in I} Z_i$, that is the space where financial activity takes place.

F3 FSSA For every $q \in \big(Q(\mathcal{F}) \cap Z(\mathcal{F})\big)\backslash\{0\}$, for every $i \in I$ there exists a portfolio $\zeta_i \in Z_i$ such that $q \cdot \zeta_i < 0$.

Theorem 3.2 *Let $(\mathcal{E}, \mathcal{F}) = \big((X_i, P_i, e_i)_{i\in I}, (V, (Z_i)_{i\in I})\big)$ be a financial exchange economy such that $\mathcal{E}$ is standard and $\mathcal{F}$ satisfies* **F1**, **F2**, *and* **F3**, *then it admits an equilibrium $(\bar{p}, \bar{x}, \bar{q}, \bar{z})$ such that $\bar{p}(s) \neq 0$ for every $s \in \bar{S}$.*

3.2.6 Examples of restrictions satisfying assumption F2

As shown by the following Propositions 3.1 and 3.2, assumption **F2** holds true in many situations. Indeed, **F2** is fulfilled when the restrictions on portfolio choices are given by a finite number of linear inequalities, that is, when all portfolios sets are finite intersections of half spaces. In particular, **F2** is fulfilled when the portfolios sets are linear subspaces, when the portfolio sets are unconstrained, or when there is an exogenous bound on portfolio short sales. Furthermore, assumption **F2** holds true under the no mutually compatible potential arbitrage condition (Page [5]) that is when the family $\{\boldsymbol{A}Z_i \cap \ker V, i \in I\}$ is positively semi-independent[6] (Siconolfi [7]), in particular **F2** holds true when the portfolio sets are bounded, or when there are no redundant assets i.e. $Rank(V) = J$.

Proposition 3.1 *Assumption* **F2** *holds true under anyone of the following conditions.*

(a) *For all $i \in I$, $Z_i = \mathbb{R}^J$ (unconstrained portfolios).*

[6] A collection $\{C_i, i \in I\}$ of nonempty convex sets in $\mathbb{R}^\alpha$ is positively semi-independent if $v_i \in C_i$, for all $i \in I$ and $\sum_{i\in I} v_i = 0$, implies that $v_i = 0$ for all $i \in I$.

(b) *For all* $i \in I$, Z_i *is a linear subspace.*

(c) *For all* $i \in I$, $Z_i = \underline{z}_i + \mathbb{R}^J_+$, *for some* $\underline{z}_i \in -\mathbb{R}^J_+$ *(exogenous bounds on short sales).*

(d) *For all* $i \in I$, Z_i *is polyhedral.*

(e) *For all* $i \in I$, $Z_i = B_J(0,1)$ *(bounded portfolio sets).*

(f) *For all* $i \in I$, $Z_i = K_i + P_i$ *where* K_i *is nonempty compact and convex, and* P_i *is polyhedral.*

The proof of Proposition 3.1 is given in the Appendix.

Proposition 3.2 *Assumption* **F2** *holds true under each of the following conditions.*

(g) *There are no redundant assets i.e. rank*$(V) = J$, *or equivalently,* $\ker V = \{0\}$.

(h) $\forall i$, $\boldsymbol{A}Z_i \cap \ker V = \{0\}$.

(i1) $\boldsymbol{L}\Big(\sum_{i\in I}(Z_i \cap \{V \geq 0\})\Big) = \{0\}$.

(i2) $\boldsymbol{L}\Big(\sum_{i\in I}(Z_i \cap \ker V)\Big) = \{0\}$.

(i3) $\sum_{i\in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\}) \cap -\sum_{i\in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\}) = \{0\}$.

(i4) $\sum_{i\in I}(\boldsymbol{A}Z_i \cap \ker V) \cap -\sum_{i\in I}(\boldsymbol{A}Z_i \cap \ker V) = \{0\}$.

(j1) *The family* $\{\boldsymbol{A}Z_i \cap \{V \geq 0\} : i \in I\}$ *is positively semi-independent.*

(j2) *The family* $\{\boldsymbol{A}Z_i \cap \ker V : i \in I\}$ *is positively semi-independent.*

(k1) *The family* $\{\boldsymbol{A}Z_i \cap \{V \geq 0\}, i \in I\}$ *is weakly positively semi-independent*[7].

(k2) *The family* $\{\boldsymbol{A}Z_i \cap \ker V, i \in I\}$ *is weakly positively semi-independent.*

The proof of Proposition 3.2 is given in the Appendix.

[7] A collection $\{C_i, i \in I\}$ of nonempty convex cones in $\mathbb{R}^\alpha$ is weakly positively semi independent if $v_i \in C_i$, for all $i \in I$ and $\sum_{i\in I} v_i = 0$, implies that for all $i \in I$, $v_i \in \boldsymbol{L}(C_i)$.

3.2.7 Proof of the equilibrium existence result

Let $(\mathcal{E},\mathcal{F}) = \big((X_i, P_i, e_i)_{i\in I}, (V, (Z_i)_{i\in I})\big)$ be a financial exchange economy such that $\mathcal{E}$ is standard and $\mathcal{F}$ satisfies **F1**, **F2**, and **F3**. By Theorem 3.1, the financial structure $\mathcal{F}$ is equivalent to a reduced financial structure $\mathcal{F}'$ satisfying **F1**, and **P**. Claim 3.2.1 below, shows that since $\mathcal{F}$ satisfies **F3**, $\mathcal{F}'$ also satisfies **F3**. This allows us to apply the existence result in [2] to the financial exchange economy $(\mathcal{E},\mathcal{F}')$ in which $\mathcal{E}$ is standard and $\mathcal{F}'$ is reduced and satisfies **F1** and **F3**, to conclude to the existence of an equilibrium in $(\mathcal{E},\mathcal{F}')$. Then $(\mathcal{E},\mathcal{F})$ has an equilibrium since $\mathcal{F}$ and $\mathcal{F}'$ are equivalent.

Claim 3.2.1 *If $\mathcal{F}$ satisfies assumption* **F3**, *then the financial structure $\mathcal{F}'$ provided by Theorem 3.1, satisfies assumption* **F3**.

Proof. Assume $\mathcal{F}$ satisfies **F3** and let $q \in Q(\mathcal{F}') \cap Z(\mathcal{F}')\backslash\{0\}$. Then, by Theorem 3.1 (more precisely, by property **P**(i)), $q \in Q(\mathcal{F}) \cap Z(\mathcal{F})\backslash\{0\}$ and by **F3** in $\mathcal{F}$, for every $i \in I$, there exists $z_i \in Z_i$ such that $q \cdot z_i < 0$. Hence, again by Theorem 3.1 (more precisely, by property **P**(ii)), for each $i \in I$, there exists $z_i' \in Z_i'$ such that $q \cdot z_i' = q \cdot z_i < 0$. ■

3.3 Proof of Theorem 3.1

3.3.1 A sharper result

Definition 3.7 *Given the financial structure $\mathcal{F} = \big(J, V, (Z_i)_{i\in I}\big)$, we say that $(q, z_i) \in \mathbb{R}^J \times Z_i$ is arbitrage-free for agent $i \in I$ if there is no portfolio $\bar{z}_i \in Z_i$ such that $W(q)\bar{z}_i > W(q)z_i$. A list of portfolios $z = (z_i)_{i\in I} \in \Pi_i Z_i$ is said to be arbitrage-free at q, or (q, z) is said to be arbitrage-free, if for every $i \in I$, (q, z_i) is arbitrage-free for agent i. The asset price vector $q \in \mathbb{R}^J$ is said to be arbitrage-free if there exists $z = (z_i)_{i\in I} \in \Pi_i Z_i$ such that (q, z) is arbitrage-free.*

First, we introduce a preorder on the set of all financial structures. We will call a financial opportunity any collection $(w_1, \cdots, w_I)$ of vectors in the space R^{S+1}. We will say that the financial opportunity $(w_1, \cdots, w_I)$ is achievable through (or offered by) the financial structure $\mathcal{F}$ if there exists an asset price vector $q \in \mathbb{R}^J$ and a family of feasible and mutually compatible[8] portfolios $z = (z_1, \cdots, z_I)$ such that (q, z) is arbitrage-free in $\mathcal{F}$ and for every

[8]The portfolio allocation $z = (z_1, \cdots, z_I)$ is said to be mutually compatible if $\sum_{i\in I} z_i = 0$.

$i \in I$, $W(q)z_i = w_i$. Let us denote $W(\mathcal{F})$ the set of financial opportunities achievable through $\mathcal{F}$. Then

$$W(\mathcal{F}) := \Big\{ \big(W(q)z_1, \cdots, W(q)z_I\big) : (z_i)_i \in \Pi_i Z_i, \sum_{i \in I} z_i = 0, \text{ and } (q, z) \text{ is arbitrage-free} \Big\}.$$

Definition 3.8 *Consider two financial structures* $\mathcal{F} = \big(V, (Z_i)_i\big)$ *and* $\mathcal{F}' = \big(V', (Z_i')_i\big)$. *We say that* $\mathcal{F}' \precsim \mathcal{F}$ *(read* $\mathcal{F}'$ *offers at most as many financial opportunities as those offered by* $\mathcal{F}$*) if*

$$W(\mathcal{F}') \subseteq W(\mathcal{F}).$$

For the sake of clarity and to avoid lengthy sentences we denote the preorder defined in Definition 3.8 by $\precsim_W$ and, given the financial structure $\mathcal{F} = \big(V, (Z_i)_i\big)$, we denote

$$V(\mathcal{F}) := \Big\{ \big(Vz_1, \cdots, Vz_I\big) : (z_i)_i \in \Pi_i Z_i, \sum_{i \in I} z_i = 0 \Big\}.$$

Theorem 3.1 is a consequence of the following theorem.

Theorem 3.3 *Let* $\mathcal{F} = \big(V, (Z_i)_i\big)$ *be a financial structure satisfying assumptions* **F1** *and* **F2**, *and let* π *be a linear projection of* $\mathbb{R}^J$ *such that*

$$\ker \pi \subset \boldsymbol{L}(\mathcal{F}) := \boldsymbol{L}\Big(\sum_{i \in I}(Z_i \cap \{V \geq 0\})\Big).$$

Denote $\mathcal{F}_\pi := \big(V, (\mathrm{cl}\pi Z_i)_i\big)$. *We have*

(a) *The financial structure* $\mathcal{F}_\pi$ *satisfies* $V(\mathcal{F}) = V(\mathcal{F}_\pi)$.

(b) *If* $\ker \pi = \boldsymbol{L}(\mathcal{F})$, *then the financial structure* $\mathcal{F}_\pi$ *is reduced, that is* $\boldsymbol{L}(\mathcal{F}_\pi) = \{0\}$.

(c) *If* π *is orthogonal, then the financial structure* $\mathcal{F}_\pi$ *satisfies:*

  (i) *if* (q, y) *is arbitrage-free in* $\mathcal{F}_\pi$ *and* $\sum_{i \in I} y_i = 0$, *then there exists a mutually compatible portfolio allocation* $z^* \in \Pi_i Z_i$ *such that* $(\pi q, z^*)$ *is arbitrage-free, and* $W(q)y_i = W(\pi q)z_i^*$ *for every* $i \in I$. *That is* $\mathcal{F}_\pi \precsim_W \mathcal{F}$.

  (ii) *for every standard exchange economy* $\mathcal{E}$, *if* $(\mathcal{E}, \mathcal{F}_\pi)$ *has an equilibrium* $(\bar{p}, \bar{q}, \bar{x}, \bar{y})$, *then there exists* $z^* \in \Pi_i Z_i$ *such that* $(\bar{p}, \pi\bar{q}, \bar{x}, z^*)$ *is an equilibrium of* $(\mathcal{E}, \mathcal{F})$.

(d) *If* π *is orthogonal and the financial structure* $\mathcal{F}$ *satisfies the following additional assumption* **F0**, *then the financial structures* $\mathcal{F}$ *and* $\mathcal{F}_\pi$ *are equivalent.*

  F0 *For every* $i \in I$, *there exists* $\zeta_i \in \boldsymbol{A}Z_i$ *such that* $V\zeta_i \gg 0$.

(e) *If π is orthogonal, then the financial structure $\mathcal{F}_\pi$ satisfies the following property* **P**.

P *For every* $(q, z) \in \Big(Q(\mathcal{F}_\pi) \cap Z(\mathcal{F}_\pi)\Big) \times \Pi_i Z_i$*, one has*

(i) $q \in Q(\mathcal{F}) \cap Z(\mathcal{F})$*, and*

(ii) *there exists* $z' = (z'_i)_i \in \Pi_i Z'_i$ *such that* $q \cdot z_i = q \cdot z'_i$ *for every* $i \in I$.

Notation Let $\mathcal{F} = \big(V, (Z_i)_i\big)$ be a financial structure and denote

- $\boldsymbol{L}(\mathcal{F}) \;:=\; \boldsymbol{L}\Big(\sum_{i\in I}(Z_i \cap \{V \geq 0\})\Big)$,
- $\mathcal{G}(\mathcal{F}) \;:=\; \{(Vz_1, \cdots, Vz_I, \sum_{i\in I} z_i) \in (\mathbb{R}^S)^I \times \mathbb{R}^J : \forall i \in I, z_i \in Z_i\}$,
- $\mathcal{G}'(\mathcal{F}) \;:=\; \{(v, \sum_{i\in I} z_i) \in (\mathbb{R}^S)^I \times \mathbb{R}^J : \forall i \in I, z_i \in Z_i, Vz_i \geq v_i\}$,
- $\pi \;:=\;$ a linear projection of $\mathbb{R}^J$ such that $\ker\pi \subset \boldsymbol{L}(\mathcal{F})$.

3.3.2 Preliminary results

Lemma 3.1 *The set $\mathcal{G}(\mathcal{F})$ is closed if and only if the set $\mathcal{G}'(\mathcal{F})$ is closed.*

Proof. Assume $\mathcal{G}'(\mathcal{F})$ is closed and let $(w^n)_n$ be a sequence in $\mathcal{G}(\mathcal{F})$ which converges to some $w \in (\mathbb{R}^S)^I \times \mathbb{R}^J$ i.e. $w^n = (Vz_1^n, \cdots, Vz_I^n, \sum_{i\in I} z_i^n) \underset{n\to\infty}{\longrightarrow} w = (v_1, \cdots, v_I, z)$, with $z_i^n \in Z_i$ for each $i \in I$ and for every $n \in \mathbb{N}$. Then $w^n \in \mathcal{G}'(\mathcal{F})$ for every n, and since $\mathcal{G}'(\mathcal{F})$ is closed, we have $w \in \mathcal{G}'(\mathcal{F})$. That is $z = \sum_{i\in I} z_i$ with $z_i \in Z_i$ and $Vz_i \geq v_i$ for every $i \in I$. But $\sum_{i\in I} v_i = \sum_{i\in I} \lim_n Vz_i^n = \lim_n V(\sum_{i\in I} z_i^n) = Vz = V(\sum_{i\in I} z_i) = \sum_{i\in I} Vz_i$, hence $v_i = Vz_i$ for each $i \in I$, and consequently, $w = (Vz_1, \cdots, Vz_i, \sum_{i\in I} z_i) \in \mathcal{G}(\mathcal{F})$.

Conversely, assume $\mathcal{G}(\mathcal{F})$ closed and let $(w'^n)_n$ be a sequence in $\mathcal{G}'(\mathcal{F})$ which converges to some $w' \in (\mathbb{R}^S)^I \times \mathbb{R}^J$ i.e. $w'^n = (v_1'^n, \cdots, v_I'^n, \sum_{i\in I} z_i^n) \underset{n\to\infty}{\longrightarrow} w' = (v'_1, \cdots, v'_I, z)$, with $z_i^n \in Z_i$ and $Vz_i^n \geq v_i'^n$ for each $i \in I$ and for every $n \in \mathbb{N}$. For each $i \in I$, the sequence $(v_i'^n)_n$ converges hence is bounded, therefore the sequence $(Vz_i^n)_n$ is bounded below (since $Vz_i^n \geq v_i'^n$ for every n). Moreover the sequence $(\sum_{i\in I} Vz_i^n)_n$ converges (towards Vz), hence for each $i \in I$, the sequence $(Vz_i^n)_n$ is bounded. We can therefore assume that for each $i \in I$, the sequence $(Vz_i^n)_n$ converges (use subsequences if necessary) to $v_i \in \mathbb{R}^S$ satisfying $v_i \geq v'_i$. Now we consider the sequence $(w^n)_n \subset \mathcal{G}(\mathcal{F})$ where $w^n = (Vz_1^n, \cdots, Vz_I^n, \sum_{i\in I} z_i^n)$. Then from above, $w^n \underset{n\to\infty}{\longrightarrow} w = (v_1, \cdots, v_I, z) \in \mathcal{G}(\mathcal{F})$ (since $\mathcal{G}(\mathcal{F})$ is assumed to be closed). Hence z can be written as $z = \sum_{i\in I} z_i$ with $z_i \in Z_i$ and $Vz_i = v_i$ for each $i \in I$. Recall that $Vz_i = v_i \geq v'_i$ for each $i \in I$ and that $w' = w' = (v'_1, \cdots, v'_I, z) = (v'_1, \cdots, v'_I, \sum_{i \in I} z_i)$, hence $w' \in \mathcal{G}'(\mathcal{F})$. ∎

Lemma 3.2 *Under* **NS** *and* **LNS**, *if $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an equilibrium of the financial exchange economy $(\mathcal{E}, \mathcal{F})$, then for every $i \in I$, there is no $z_i \in Z_i$ such that $W(\bar{q})z_i > W(\bar{q})\bar{z}_i$. That is $(\bar{q}, \bar{z})$ is arbitrage-free in $\mathcal{F}$.*

Proof. By contradiction. Assume that for some $i \in I$, there exists $z_i \in Z_i$ such that $W(\bar{q})z_i > W(\bar{q})\bar{z}_i$, namely $[W(\bar{q})z_i](s) \geq [W(\bar{q})\bar{z}_i](s)$, for every $s \in \bar{S}$, with at least one strict inequality, say for $\bar{s} \in \bar{S}$. Then, since $\sum_{i \in I}(\bar{x}_i - e_i) = 0$, from Assumption **NS**, there exists $x \in \prod_{i \in I} X_i$ such that, for each $s \neq \bar{s}$, $x_i(s) = \bar{x}_i(s)$ and $x_i \in P_i(\bar{x})$. Consider $\lambda \in (0, 1)$ and define $x_i^\lambda := \lambda x_i + (1 - \lambda)\bar{x}_i$. Then, by Assumption **LNS**, $x_i^\lambda \in (x_i, \bar{x}_i) \subset P_i(\bar{x})$. Now, we claim that for $\lambda > 0$ small enough, $(x_i^\lambda, z_i) \in B_i(\mathcal{F}, \bar{p}, \bar{q})$, which contradicts the fact that $[P_i(\bar{x}) \times Z_i] \cap B_i(\mathcal{F}, \bar{p}, \bar{q}) = \emptyset$ (since $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an equilibrium). Indeed, since $(\bar{x}_i, \bar{z}_i) \in B_i(\mathcal{F}, \bar{p}, \bar{q})$, and for every $s \neq \bar{s}$, $x_i^\lambda(s) = \bar{x}_i(s)$ we have

$$\bar{p}(s) \cdot [x_i^\lambda(s) - e_i(s)] = \bar{p}(s) \cdot [\bar{x}_i(s) - e_i(s)] \leq [W(\bar{q})\bar{z}_i](s) \leq [W(\bar{q})z_i](s).$$

Now, for $s = \bar{s}$, we have

$$\bar{p}(\bar{s}) \cdot [\bar{x}_i(\bar{s}) - e_i(\bar{s})] \leq [W(\bar{q})\bar{z}_i](\bar{s}) < [W(\bar{q})z_i](\bar{s}).$$

But, when $\lambda \to 0$, $x_i^\lambda \to \bar{x}_i$, hence for $\lambda > 0$ small enough we have

$$\bar{p}(\bar{s}) \cdot [x_i^\lambda(\bar{s}) - e_i(\bar{s})] < [W(\bar{q})z_i](\bar{s}).$$

Consequently, $(x_i^\lambda, z_i) \in B_i(\mathcal{F}, \bar{p}, \bar{q})$. ∎

3.3.3 Proof of Part (a) of Theorem 3.3

We prepare the proof by some claims.

Claim 3.3.1 *Under* **F1** *and* **F2** *we have, for all $v = (v_i)_{i \in I} \in (\mathbb{R}^S)^I$,*

(a) $\boldsymbol{A}\Big(\sum_{i \in I}(Z_i \cap \{V \geq 0\})\Big) \subset \boldsymbol{A}\Big(\sum_{i \in I}(Z_i \cap \{V \geq v_i\})\Big)$.

(b) $\boldsymbol{L}(\mathcal{F}) := \boldsymbol{L}\Big(\sum_{i \in I}(Z_i \cap \{V \geq 0\})\Big) \subset \boldsymbol{L}\Big(\sum_{i \in I}(Z_i \cap \{V \geq v_i\})\Big)$.

Proof. (a) Let $\zeta \in \boldsymbol{A}\Big(\sum_{i \in I}(Z_i \cap \{V \geq 0\})\Big)$, then for every $n \in \mathbb{N}$, $n\zeta = \sum_{i \in I} z_i^n$ for some $z_i^n \in Z_i \cap \{V \geq 0\}$. We need to show that $\zeta \in \boldsymbol{A}\Big(\sum_{i \in I}(Z_i \cap \{V \geq v_i\})\Big)$, that is, for

$z_i \in Z_i$ $(i \in I)$ such that $Vz_i \geq v_i$, we have

$$\zeta + \sum_{i \in I} z_i \in \sum_{i \in I} (Z_i \cap \{V \geq v_i\}).$$

From above,

$$\zeta + \sum_{i \in I} z_i = \lim_{n \to \infty} \sum_{i \in I} \Big(\frac{1}{n} z_i^n + (1 - \frac{1}{n}) z_i \Big).$$

Notice that, for $n \geq 1$, $\frac{1}{n} \in [0,1]$ hence $y_i^n := \frac{1}{n} z_i^n + (1 - \frac{1}{n}) z_i$ belongs to Z_i (because z_i^n and z_i belong to Z_i, and Z_i is convex). Furthermore $Vy_i^n \geq (1 - \frac{1}{n}) Vz_i \geq (1 - \frac{1}{n}) v_i$. Consequently, $y_i^n \in Z_i \cap \{V \geq (1 - \frac{1}{n}) v_i\}$. Therefore

$$\Big((1 - \frac{1}{n}) v_1, \cdots, (1 - \frac{1}{n}) v_I, \sum_{i \in I} y_i^n \Big) \in \mathcal{G}(\mathcal{F}).$$

Now, since the set $\mathcal{G}(\mathcal{F})$ is closed (by assumption **F2**) and $\sum_{i \in I} y_i^n \underset{n \to \infty}{\longrightarrow} \zeta + \sum_{i \in I} z_i$, and for each $i \in I$, $(1 - \frac{1}{n}) v_i \underset{n \to \infty}{\longrightarrow} v_i$, we conclude that $(v_1, \cdots, v_I, \zeta + \sum_{i \in I} z_i) \in \mathcal{G}(\mathcal{F})$, that is $\zeta + \sum_{i \in I} z_i \in \sum_{i \in I} Z_i \cap \{V \geq v_i\}$.

(b) This is a direct consequence of Part (a) of this claim, and the definition of the lineality space. ∎

Claim 3.3.2 $\boldsymbol{L}\Big(\sum_{i \in I} (Z_i \cap \ker V)\Big) = \boldsymbol{L}\Big(\sum_{i \in I} (Z_i \cap \{V \geq 0\})\Big)$. *In particular* $\boldsymbol{L}(\mathcal{F}) \subset \ker V$.

Proof. The first inclusion is immediate. We show

$$\boldsymbol{L}\Big(\sum_{i \in I} (Z_i \cap \{V \geq 0\})\Big) \subset \boldsymbol{L}\Big(\sum_{i \in I} (Z_i \cap \ker V)\Big).$$

Let $\zeta \in \boldsymbol{L}\Big(\sum_{i \in I} (Z_i \cap \{V \geq 0\})\Big)$, then for every $\lambda \in \mathbb{R}_+$, both vectors $\lambda \zeta$ and $-\lambda \zeta$ belong to $\sum_{i \in I} (Z_i \cap \{V \geq 0\})$, that is, there exist vectors $z_1, \cdots, z_I$, $z'_1, \cdots, z'_I$ such that z_i and z'_i are both in $Z_i \cap \{V \geq 0\}$ for every $i \in I$, and

$$\lambda \zeta = \sum_{i \in I} z_i = -\sum_{i \in I} z'_i.$$

Hence $\sum_{i \in I} (z_i + z'_i) = 0$ which together with the inequalities $Vz_i \geq 0$, $Vz'_i \geq 0$ for every $i \in I$ implies $Vz_i = Vz'_i = 0$, for every $i \in I$. Therefore, for every $\lambda \in \mathbb{R}_+$, both vectors $\lambda \zeta$ and $-\lambda \zeta$ are in $\sum_{i \in I} (Z_i \cap \ker V)$, that is, $\zeta \in \boldsymbol{L}\Big(\sum_{i \in I} (Z_i \cap \ker V)\Big)$. ∎

Claim 3.3.3 *Under* **F1** *and* **F2**, *we have, for all* $v = (v_i)_{i \in I} \in (\mathbb{R}^S)^I$,

$$\sum_{i \in I} \mathrm{cl}\Big(\pi Z_i \cap \{V \geq v_i\} \Big) \subset \mathrm{cl} \sum_{i \in I} \big(Z_i \cap \{V \geq v_i\} \big).$$

Proof. (a) First, we show that

$$\begin{aligned}\sum_{i\in I}\big(\pi Z_i\cap\{V\geq v_i\}\big) &= \sum_{i\in I}\pi\big(Z_i\cap\{V\geq v_i\}\big) && (3.3.1)\\ &\subset \ker\pi+\sum_{i\in I}\big(Z_i\cap\{V\geq v_i\}\big) && (3.3.2)\\ &\subset \mathrm{cl}\Big(\sum_{i\in I}\big(Z_i\cap\{V\geq v_i\}\big)\Big). && (3.3.3)\end{aligned}$$

To prove the equality (3.3.1), it suffices to notice that for every $i\in I$, $\pi Z_i\cap\{V\geq v_i\}=\pi(Z_i\cap\{V\geq v_i\})$. Indeed, let $y_i\in\pi Z_i\cap\{V\geq v_i\}$, then there exists $z_i\in Z_i$ such that $y_i=\pi z_i$, and $Vy_i\geq v_i$. But $Vz_i=Vy_i+V(z_i-\pi z_i)=Vy_i$ (since $z_i-\pi z_i\in\ker\pi\subset\boldsymbol{L}(\mathcal{F})$ and $\boldsymbol{L}(\mathcal{F})\subset\ker V$ from Claim 3.3.2). Then $z_i\in Z_i\cap\{V\geq v_i\}$ and Consequently $y_i\in\pi(Z_i\cap\{V\geq v_i\})$. The proof of the converse inclusion is similar.

To prove the inclusion (3.3.2), let $y=\sum_{i\in I}\pi z_i$ with $z_i\in Z_i\cap\{V\geq v_i\}$. Then $y=\pi z=(\pi z-z)+z$ with $\pi z-z\in\ker\pi$ and $z=\sum_{i\in I}z_i\in\sum_{i\in I}(Z_i\cap\{V\geq v_i\})$. This ends the proof of the inclusion of (3.3.2).

The second inclusion (3.3.3) comes from the fact that

$$\begin{aligned}\ker\pi &\subset \boldsymbol{L}(\mathcal{F}) && \text{by assumption}\\ \boldsymbol{L}(\mathcal{F}) &\subset \boldsymbol{L}\Big(\sum_{i\in I}(Z_i\cap\{V\geq v_i\})\Big) && \text{by Claim 3.3.1}\\ \boldsymbol{L}\Big(\sum_{i\in I}(Z_i\cap\{V\geq v_i\})\Big) &\subset \boldsymbol{A}\Big(\sum_{i\in I}(Z_i\cap\{V\geq v_i\})\Big) && \text{by definition of the lineality space,}\end{aligned}$$

consequently,

$$\begin{aligned}\ker\pi+\sum_{i\in I}(Z_i\cap\{V\geq v_i\}) &\subset \boldsymbol{A}\Big(\sum_{i\in I}(Z_i\cap\{V\geq v_i\})\Big)+\sum_{i\in I}(Z_i\cap\{V\geq v_i\})\\ &\subset \mathrm{cl}\sum_{i\in I}(Z_i\cap\{V\geq v_i\}).\end{aligned}$$

Using the above result (3.3.3) and recalling that for a finite family of sets $A_i\subset\mathbb{R}^k,(i\in I)$, one always has $\sum_{i\in I}\mathrm{cl}A_i\subset\mathrm{cl}(\sum_{i\in I}A_i)$, we get

$$\sum_{i\in I}\mathrm{cl}\Big(\pi Z_i\cap\{V\geq v_i\}\Big)\subset\mathrm{cl}\sum_{i\in I}\Big(\pi Z_i\cap\{V\geq v_i\}\Big)\subset\mathrm{cl}\sum_{i\in I}\Big(Z_i\cap\{V\geq v_i\}\Big).$$

This ends the proof of the claim. ∎

Claim 3.3.4 *Under* **F1** *and* **F2***, we have, for all* $v=(v_i)_{i\in I}\in(\mathbb{R}^S)^I$,

$$\sum_{i\in I}\Big((\mathrm{cl}\pi Z_i)\cap\{V\geq v_i\}\Big)\subset\sum_{i\in I}\big(Z_i\cap\{V\geq v_i\}\big).$$

Proof. Let $y_i \in (\mathrm{cl}\pi Z_i) \cap \{V \geq v_i\}$. Take $v_i^n \uparrow v_i$ such that $v_i \gg v_i^n$ for every n. Pick $\bar{y}_i \in \mathrm{ri}\pi Z_i$ and consider $y_i^n = (1-\lambda^n)y_i + \lambda^n \bar{y}_i$ with $0 < \lambda^n < \frac{1}{n}$ small enough so that $Vy_i^n \gg v_i^n$. Then $y_i^n \in [\bar{y}_i, y_i) \subset \mathrm{ri}\pi Z_i$ since $y_i \in \mathrm{cl}\pi Z_i$ and $\bar{y}_i \in \mathrm{ri}\pi Z_i$ (Theorem 6.1 page 45 in [6]). Thus $y_i^n \in \pi Z_i \cap \{V \geq v_i^n\}$.

Therefore, by Claim 3.3.3,

$$\sum_{i\in I} y_i^n \in \mathrm{cl}\sum_{i\in I}\Big(Z_i \cap \{V \geq v_i^n\}\Big).$$

Now, since the set $\mathcal{G}(\mathcal{F})$ is closed (by assumption **F2**) and $v_i^n \underset{n\to\infty}{\longrightarrow} v_i$, $\sum_{i\in I} y_i^n \underset{n\to\infty}{\longrightarrow} \sum_{i\in I} y_i$, we get $\sum_{i\in I} y_i \in \sum_{i\in I}\big(Z_i \cap \{V \geq v_i\}\big)$. ■

Proof of Part (a) of Theorem 3.3

Step 1. $V(\mathcal{F}) \subset V(\mathcal{F}_\pi)$: Since $\ker \pi \subset \boldsymbol{L}(\mathcal{F}) \subset \ker V$ (by Claim 3.3.2), we always have $\mathcal{F} \preceq \mathcal{F}_\pi$. Indeed, if $z_i \in Z_i$ ($i \in I$) are such that $\sum_{i\in I} z_i = 0$, then $y_i = \pi z_i \in \mathrm{cl}\pi Z_i$ ($i \in I$) satisfy $\sum_{i\in I} y_i = \sum_{i\in I} \pi z_i = \pi(\sum_{i\in I} z_i) = 0$, and for each $i \in I$, $Vy_i = V(\pi z_i - z_i) + Vz_i = Vz_i$ since $\ker \pi \subset \ker V$. ■

Step 2. $V(\mathcal{F}_\pi) \subset V(\mathcal{F})$: Let $y := (y_i)_i \in \Pi_i \mathrm{cl}\pi Z_i$ be such that $\sum_{i\in I} y_i = 0$. Then, by Claim 3.3.4,

$$0 = \sum_{i\in I} y_i \in \sum_{i\in I}\Big((\mathrm{cl}\pi Z_i) \cap \{V \geq Vy_i\}\Big) \subset \sum_{i\in I}\big(Z_i \cap \{V \geq Vy_i\}\big).$$

Hence $0 = \sum_{i\in I} z_i$ for some $z_i \in Z_i \cap \{V \geq Vy_i\}$, that is, $z_i \in Z_i$ and $V(z_i - y_i) \geq 0$ for every i. Noticing that $\sum_{i\in I}(z_i - y_i) = 0$, we get $\sum_{i\in I} V(z_i = y_i) = 0$ and we conclude that $Vz_i - Vy_i = 0$ for every i. ■

3.3.4 Proof of Part (b) of Theorem 3.3

First, we need a claim.

Claim 3.3.5 $\boldsymbol{L}(\mathcal{F}_\pi) \subset \boldsymbol{L}(\mathcal{F}) \cap \mathrm{Im}\pi$.

Proof. We clearly have $\boldsymbol{L}(\mathcal{F}_\pi) \subset \mathrm{Im}\pi$ since $\sum_{i\in I}\big(\mathrm{cl}\pi Z_i \cap \{V \geq 0\}\big) \subset \mathrm{Im}\pi$. It remains to show that $\boldsymbol{L}(\mathcal{F}_\pi) \subset \boldsymbol{L}(\mathcal{F})$. By Claim 3.3.4, taking $v_i = 0$ for each $i \in I$, we have

$$\sum_{i\in I}\big((\mathrm{cl}\pi Z_i) \cap \{V \geq 0\}\big) \subset \sum_{i\in I}(Z_i \cap \{V \geq 0\}).$$

Thus $\boldsymbol{L}\Big(\sum_{i\in I}\big((\mathrm{cl}\pi Z_i) \cap \{V \geq 0\}\big)\Big) \subset \boldsymbol{L}\Big(\sum_{i\in I}(Z_i \cap \{V \geq 0\})\Big)$, that is, $\boldsymbol{L}(\mathcal{F}_\pi) \subset \boldsymbol{L}(\mathcal{F})$. ■

Proof of Part (b) of Theorem 3.3

If $\boldsymbol{L}(\mathcal{F}) = \ker \pi$, then from Claim 3.3.5, we get

$$\boldsymbol{L}(\mathcal{F}_\pi) \subset \boldsymbol{L}(\mathcal{F}) \cap \mathrm{Im}\pi = \ker \pi \cap \mathrm{Im}\pi = \{0\}.$$

This ends the proof of Part (b) of Theorem 3.3. ∎

3.3.5 Proof of Part (c) of Theorem 3.3

First, we need a claim.

Claim 3.3.6 *If the projection π is orthogonal and $V(\mathcal{F}_\pi) \subset V(\mathcal{F})$, then for (q, y) arbitrage-free in $\mathcal{F}_\pi$ such that $\sum_{i \in I} y_i = 0$, there exists $z^* \in \Pi_i Z_i$ such that*

(a) *For every i, $\begin{pmatrix} -\pi q \\ V \end{pmatrix} z_i^* = \begin{pmatrix} -q \\ V \end{pmatrix} y_i$, and $\sum_{i \in I} z_i^* = 0$,*

(b) *$(\pi q, z^*)$ is arbitrage-free in $\mathcal{F}$.*

Proof. (a) Let (q, y) be arbitrage-free in $\mathcal{F}_\pi$ and such that $\sum_{i \in I} y_i = 0$. Since $V(\mathcal{F}_\pi) \subset V(\mathcal{F})$, there exists $z^* \in \Pi_i Z_i$ such that $\sum_{i \in I} z_i^* = 0$ and $V y_i = V z_i^*$, for every $i \in I$. We show that, for every i, $\pi q \cdot z_i^* = \pi q \cdot y_i$ ($= q \cdot \pi y_i$ since the projection π is orthogonal).

Let us first note that it suffices to show that for every $i \in I$, $-\pi q \cdot z_i^* \leq -\pi q \cdot y_i$. In this case $-\pi q \cdot (\sum_{i \in I} z_i^* - \sum_{i \in I} y_i) = 0$ implies for every i, $\pi q \cdot z_i^* = \pi q \cdot y_i$.

By contraposition suppose that for some i, $-\pi q \cdot z_i^* > -\pi q \cdot y_i$. We have $-\pi q \cdot z_i^* = -q \cdot \pi z_i^*$ (since the projection π is orthogonal), $V z_i^* = V y_i$ and $V y_i = V \pi y_i$ (since $y_i - \pi y_i \in \ker \pi \subset \ker V$). Hence $\pi z_i^* \in \pi Z_i \subset \mathrm{cl}\pi Z_i$ and $\begin{pmatrix} -q \\ V \end{pmatrix} \pi z_i^* > \begin{pmatrix} -q \\ V \end{pmatrix} \pi y_i$. It thus suffices to show that $\pi y_i = y_i$ and we will contradict the assumption that (q, y_i) is arbitrage-free in $\mathcal{F}_\pi$ for agent i.

Since $y_i \in \mathrm{cl}\pi Z_i$, $y_i = \lim_n \pi y_i^n$ with $y_i^n \in Z_i$. Then $\pi y_i = \pi \lim_n \pi y_i^n = \lim_n \pi(\pi y_i^n) = \lim_n \pi y_i^n = y_i$.

(b) If $(\pi q, z_i^*)$ is not arbitrage-free, then there exists $\bar{z}_i \in Z_i$ such that $\begin{pmatrix} -\pi q \\ V \end{pmatrix} \bar{z}_i > \begin{pmatrix} -\pi q \\ V \end{pmatrix} z_i^*$. But $\begin{pmatrix} -\pi q \\ V \end{pmatrix} \bar{z}_i = \begin{pmatrix} -q \\ V \end{pmatrix} \pi \bar{z}_i$ since π is an orthogonal projection and $\ker \pi \subset$

$\ker V$. Then $\begin{pmatrix} -q \\ V \end{pmatrix} \pi \bar{z}_i > \begin{pmatrix} -q \\ V \end{pmatrix} y_i$. Contradiction to the fact that (q, y) is arbitrage-free in $\mathcal{F}_\pi$. ∎

Proof of Part (c) of Theorem 3.3

(i) Assume the projection π is orthogonal and let (q, y) be arbitrage-free in $\mathcal{F}_\pi$ such that $\sum_{i \in I} y_i = 0$. From Part (a) we know that $V(\mathcal{F}_\pi) \subset V(\mathcal{F})$, hence by Claim 3.3.6, there exists $z^* \in \Pi_i Z_i$ such that $\sum_{i \in I} z_i^* = 0$, $(\pi q, z^*)$ is arbitrage-free in $\mathcal{F}$, and for every $i \in I$, $W(\pi q) z_i^* = W(q) y_i$. Therefore $\mathcal{F}_\pi \precsim_W \mathcal{F}$. ∎

(ii) Assume $\mathcal{E}$ satisfies **NS** and **LNS**, and $\mathcal{F}$ satisfies **F1** and **F2**. We show that if $(\mathcal{E}, \mathcal{F}_\pi)$ has an equilibrium $(\bar{p}, \bar{q}, \bar{x}, \bar{y})$, then there exists $z^* \in \Pi_i Z_i$ such that $(\bar{p}, \pi\bar{q}, \bar{x}, z^*)$ is an equilibrium of $(\mathcal{E}, \mathcal{F})$.

Let $(\bar{p}, \bar{q}, \bar{x}, \bar{y})$ be an equilibrium in $(\mathcal{E}, \mathcal{F}_\pi)$. By Lemma 3.2, $(\bar{q}, \bar{y})$ is arbitrage-free in $\mathcal{F}_\pi$, and by Part (c)(i) of Theorem 3.3, for each i, there exists $z_i^* \in Z^i$ such that $W(\pi\bar{q}) z_i^* = W(\bar{q}) \bar{y}_i$, $\sum_{i \in I} z_i^* = 0$, and $(\pi\bar{q}, z^*)$ is arbitrage-free. We show that $(\bar{p}, \pi\bar{q}, \bar{x}, z^*)$ is an equilibrium of $(\mathcal{E}, \mathcal{F})$.

First, from $W(\pi\bar{q}) z_i^* = W(\bar{q}) \bar{y}_i$ for each $i \in I$, we conclude that $(\bar{x}_i, z_i^*) \in \mathcal{B}_i(\mathcal{F}, \bar{p}, \pi\bar{q})$ since $(\bar{x}_i, \bar{y}_i) \in \mathcal{B}_i(\mathcal{F}_\pi, \bar{p}, \bar{q})$).

To complete the proof, we need only show that for each i,

$$\mathcal{B}_i(\mathcal{F}, \bar{p}, \pi\bar{q}) \cap [P_i(\bar{x}) \times Z_i] = \emptyset.$$

Since $(\bar{p}, \bar{q}, \bar{x}, \bar{z})$ is an equilibrium of $(\mathcal{E}, \mathcal{F}_\pi)$, we have

$$\mathcal{B}_i(\mathcal{F}_\pi, \bar{p}, \bar{q}) \cap [P_i(\bar{x}) \times \mathrm{cl}\pi Z_i] = \emptyset.$$

In view of the above, the proof will be completed if we show that if $(x_i, z_i) \in \mathcal{B}_i(\mathcal{F}, \bar{p}, \pi\bar{q})$, then $(x_i, \pi z_i) \in \mathcal{B}_i(\mathcal{F}_\pi, \bar{p}, \bar{q})$, which is true if $W(\pi\bar{q}) z_i \leq W(\bar{q}) \pi z_i$. Recalling that for every i, $V z_i = V \pi z_i$ (since $z_i - \pi z_i \in \ker \pi \subset \boldsymbol{L}(\mathcal{F}) \subset \ker V$), we only need to show that $\pi\bar{q} \cdot z_i = \bar{q} \cdot \pi z_i$. But $\pi\bar{q} \in \mathrm{Im}\pi = (\ker \pi)^\perp$ (since the projection π is orthogonal) implies, $\pi\bar{q} \cdot z_i = \pi\bar{q} \cdot \pi z_i$, and again since $\bar{q} - \pi\bar{q} \in (\mathrm{Im}\pi)^\perp$, we have $\pi\bar{q} \cdot \pi z_i = \bar{q} \cdot \pi z_i$. Hence $\pi\bar{q} \cdot z_i = \bar{q} \cdot \pi z_i$. ∎

3.3.6 Proof of Part (d) of Theorem 3.3

First, we need some claims.

Claim 3.3.7 (a) *Under assumption* **F0***, if q is an arbitrage-free asset price then*

$$q \in -\big(\boldsymbol{A}\sum_{i\in I}(Z_i \cap [V \geq 0])\big)^o \subset \big(\boldsymbol{L}(\mathcal{F})\big)^\perp.$$

(b) *Under assumptions* **F0** *and* **NS***, and LNS, if $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an equilibrium of the economy $(\mathcal{E}, \mathcal{F})$, then $\bar{q}$ is arbitrage-free hence*

$$\bar{q} \in -\big(\boldsymbol{A}\sum_{i\in I}(Z_i \cap [V \geq 0])\big)^o \subset \big(\boldsymbol{L}(\mathcal{F})\big)^\perp.$$

Proof. (a) By contraposition. Let q be an arbitrage-free asset price vector and suppose that $q \notin -\big(\boldsymbol{A}\sum_{i\in I}(Z_i \cap \{V \geq 0\})\big)^o$. Then there is $\zeta \in \boldsymbol{A}\sum_{i\in I}(Z_i \cap \{V \geq 0\})$ such that $-q \cdot \zeta > 0$. Then for all $n \in \mathbb{N}$, $n^2\zeta = \sum_{i\in I} z_i^n$ with $z_i^n \in (Z_i \cap \{V \geq 0\})$, for every $i \in I$. Therefore $-q \cdot \sum_{i\in I} \frac{z_i^n}{n} = -nq \cdot \zeta \to +\infty$ when $n \to \infty$. Hence, without any loss of generality, we can assume that for some agent, say $i = 1$, $-q \cdot \frac{z_1^n}{n} \to +\infty$ when $n \to \infty$.

By **F0**, there exists $\xi_1 \in \boldsymbol{A}Z_1$ such that $V\xi_1 \gg 0$. Consider $\bar{z} = (\bar{z}_i)_i \in \Pi_i Z_i$ such that q is arbitrage-free at $\bar{z}$, and define

$$\zeta_1^n := \frac{1}{n} z_1^n + (1 - \frac{1}{n})(\bar{z}_1 + \xi_1).$$

We end the proof by showing that, for n large enough, ζ_1^n is an arbitrage opportunity for agent 1 at $\bar{z}_1$, that is **(i)** $\zeta_1^n \in Z_1$, and **(ii)** $W(q)\zeta_1^n > W(q)\bar{z}_1$ (which contradicts that q is arbitrage-free at $\bar{z}$). Indeed, first $\bar{z}_1 + \xi_1 \in Z_1$ since $\xi_1 \in \boldsymbol{A}Z_1$, hence ζ_1^n belongs to Z_1 since it is a convex combination of $z_1^n \in Z_1$, $\bar{z}_1 + \xi_1 \in Z_1$ and Z_1 is convex.

Second, one has $-q \cdot \zeta_1^n = -q \cdot \frac{1}{n} z_1^n + -q \cdot (1 - \frac{1}{n})(\bar{z}_1 + \xi_1) > -q \cdot \bar{z}_1$ for n large enough (since $-q \cdot \frac{z_1^n}{n} \to +\infty$).

Finally, since $z_1^n \in \{V \geq 0\}$ and $V\xi_1 \gg 0$ one has, for n large enough,

$$V\zeta_1^n = V[\frac{1}{n} z_1^n + (1 - \frac{1}{n})(\bar{z}_1 + \xi_1)] \geq (1 - \frac{1}{n})V(\bar{z}_1 + \xi_1) \gg V\bar{z}_1.$$

(b) If $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an equilibrium of the economy $(\mathcal{E}, \mathcal{F})$ then, under **NS**, and **LNS**, $\bar{q}$ is arbitrage-free (from Claim 3.2), and under **F0** we have the result from Part (a). ■

Claim 3.3.8 *Assume that for all $s \in \bar{S}$, $p(s) \neq 0$ and for all $i \in I$, $e_i \in \text{int}X_i$, then*

$$\mathcal{B}_i(\mathcal{F}_\pi, p, q) = \text{cl}\Big\{(x, v) \in X_i \times \pi Z_i : p \,\square\, (x - e_i) \ll W(q)v\Big\}.$$

Proof. We first claim that there exists $\delta = (\delta(s))_{s\in\bar{S}} \in \mathbb{R}^L$ such that **(i)** $e_i - \delta \in X_i$ and **(ii)** $p(s) \cdot \delta(s) > 0$ for every $s \in \bar{S}$. Indeed, take $\delta = \lambda p$ for $\lambda > 0$ small enough so that $e_i + \delta \in X_i$, using the fact that $e_i \in \mathrm{int} X_i$, and for all $s \in \bar{S}$, $p(s) \cdot p(s) > 0$, since $p(s) \neq 0$.

Let $(x_i, v_i) \in \mathcal{B}_i(\mathcal{F}_\pi, p, q)$. For all $\alpha \in (0,1)$, we claim that
$p \,\Box\, (\alpha x_i + (1-\alpha)(e_i - \delta) - e_i) - W(q)(\alpha v_i) = \alpha\Big(p \,\Box\, (x_i - e_i) - W(q)v_i\Big) - (1-\alpha)p \,\Box\, \delta \ll 0.$
Indeed, we first have $p \,\Box\, (x_i - e_i) - W(q)v_i \leq 0$ because $(x_i, v_i) \in \mathcal{B}_i(\mathcal{F}_\pi, p, q)$. Second, $-(1-\alpha)p \,\Box\, \delta \ll 0$ from above. Furthermore,

$x_i^\alpha := \alpha x_i + (1-\alpha)(e_i - \delta) \in X_i$ since $x_i \in X_i$ and $e_i - \delta \in X_i$, and

$\alpha v_i \in \mathrm{cl}\pi Z_i$ since $\mathrm{cl}\pi Z_i$ is convex and $0 \in \mathrm{cl}\pi Z_i$.

Consequently, there exists $v_i^\alpha \in \pi Z_i$ such that $||v_i^\alpha - v_i|| \leq (1-\alpha)||v_i||$ and

$$p \,\Box\, (x_i^\alpha - e_i) - W(q)v_i^\alpha \ll 0.$$

Noticing that, when $\alpha \to 1$, $(x_i^\alpha, v_i^\alpha) \to (x_i, v_i)$, we thus get the result. ∎

Proof of Part (d) of Theorem 3.3

Assume $\mathcal{E}$ satisfies **NS** and **LNS**, and $\mathcal{F}$ satisfies **F0**, **F1** and **F2**. Assume that for all $i \in I$, $e_i \in \mathrm{int} X_i$. We show that if $(\mathcal{E}, \mathcal{F})$ has an equilibrium (p^*, q^*, x^*, z^*) such that $P_i(x^*)$ is open for every i, then $(p^*, \pi q^*, x^*, \pi z^*)$ is an equilibrium of $(\mathcal{E}, \mathcal{F}_\pi)$.

Let (p^*, q^*, x^*, z^*) be an equilibrium in $(\mathcal{E}, \mathcal{F})$. The asset market clearing condition in $(\mathcal{E}, \mathcal{F}_\pi)$: $\sum_{i\in I} \pi z_i^* = 0$ is a direct consequence of $\sum_{i\in I} z_i^* = 0$. First, we show that for each $i \in I$, $(x_i^*, \pi z_i^*) \in B_i(\mathcal{F}_\pi, p^*, \pi q^*)$. We have $W(\pi q^*)\pi z_i^* = W(q) z_i^*$ since $\pi q^* \cdot \pi z_i^* = q^* \cdot \pi z_i^* = q^* \cdot z_i^*$ (the first equality because π is orthogonal and the second equality because, under assumption **F0**, $q^* \in \big(\boldsymbol{L}(\mathcal{F})\big)^\perp$ by Claim 3.3.7 and therefore $q^* \in (\ker \pi)^\perp$ since $\ker \pi \subset \boldsymbol{L}(\mathcal{F})$), and $V\pi z_i^* = V z_i^*$ because $\ker \pi \subset \ker V$.

We now show that for each $i \in I$, $(x_i^*, \pi z_i^*)$ solves agent i's problem in $(\mathcal{E}, \mathcal{F}_\pi)$. Suppose on the contrary that for some agent, say $i = 1$, there exists $(x_1, z_1) \in \mathcal{B}_1(\mathcal{F}_\pi, p^*, \pi q^*)$ such that $x_1 \in P_1(x^*)$. From the above Claim 3.3.8, $(x_1, z_1) = \lim_n (x_1^n, \pi z_1^n)$ for some sequences $(x_1^n)_n \subset X_1$ and $(z_1^n)_n \subset Z_1$ such that

$$p^* \,\Box\, (x_1^n - e_1) - W(\pi q^*)(\pi z_1^n) \leq 0.$$

We notice that (for the first equality we use the fact that, under **F0**, $q^* \in \big(\boldsymbol{L}(\mathcal{F})\big)^\perp$ hence

$q^* \in (\ker \pi)^\perp$)

$$W(q^*)z_1^n = W(q^*)\pi z_1^n = W(\pi q^*)\pi z_1^n$$

since $z_1^n - \pi z_1^n \in \ker \pi$, $\pi q^* \in \mathrm{Im}\pi = (\ker \pi)^\perp$ and $\ker \pi \subset \big(\boldsymbol{L}(\mathcal{F})\big) \subset \ker V$. Consequently, from above

$$p^* \square (x_1^n - e_1) - W(q^*)z_1^n = p^* \square (x_1^n - e_1) - W(\pi q^*)(\pi z_1^n) \leq 0.$$

Hence $(x_1^n, z_1^n) \in \mathcal{B}_1(\mathcal{F}, p^*, q^*)$. Recalling that $x_1 \in P_1(x^*)$, $x_1 = \lim_n x_1^n$ and using the fact that $P_1(x^*)$ is open, we deduce that for n large enough $x_1^n \in P_1(x^*)$. Then the two assertions $(x_1^n, z_1^n) \in \mathcal{B}_1(\mathcal{F}, p^*, q^*)$ and $x_1 \in P_1(x^*)$ contradict the optimality of (x_1^*, z_1^*) in $(\mathcal{E}, \mathcal{F})$. ■

3.3.7 Proof of Part (e) of Theorem 3.3

We prepare the proof by some claims.

Claim 3.3.9 *$\pi Q(\mathcal{F}_\pi) \subset Q(\mathcal{F})$.*

Proof. Let $q \in Q(\mathcal{F}_\pi)$ and assume that $\pi q \notin Q(\mathcal{F})$. Then there exists $i \in I$ and $v_i \in \boldsymbol{A}Z_i$ such that $W(\pi q)v_i > 0$. The vector $\pi v_i \in \pi(\boldsymbol{A}Z_i) \subset \boldsymbol{A}(\pi Z_i)$ and, since π is an orthogonal projection, $q \cdot \pi v_i = \pi q \cdot \pi v_i = \pi q \cdot v_i$ (because both $q - \pi q$ and $v_i - \pi v_i$ belong to $\ker \pi = (\mathrm{Im}\pi)^\perp$). Furthermore $V\pi v_i = V v_i$ since $v_i - \pi v_i \in \ker \pi \subset \ker V$. Hence $W(q)(\pi v_i) = W(\pi q)v_i > 0$ which contradicts the fact that $q \in Q(\mathcal{F}_\pi)$. This ends the proof of the claim. ■

Claim 3.3.10 *$Q(\mathcal{F}_\pi) \cap \mathrm{Im}\pi = \pi Q(\mathcal{F}_\pi)$.*

Proof. Let $q \in Q(\mathcal{F}_\pi) \cap \mathrm{Im}\pi$. Then $q = \pi q$ since $q \in \mathrm{Im}\pi$, hence $q \in \pi Q(\mathcal{F}_\pi)$. This shows that $Q(\mathcal{F}_\pi) \cap \mathrm{Im}\pi \subset \pi Q(\mathcal{F}_\pi)$.

Let $q \in Q(\mathcal{F}_\pi)$, and write $\pi q = q - (q - \pi q) \in Q(\mathcal{F}_\pi) + \ker \pi$. And we claim that $Q(\mathcal{F}_\pi) + \ker \pi \subset Q(\mathcal{F}_\pi)$ (which will end the proof of Claim 3.3.10).

Indeed, let $\alpha \in \ker \pi, q \in Q(\mathcal{F}_\pi)$ and suppose $q + \alpha \notin Q(\mathcal{F}_\pi)$. Then there exists $i \in I$ and there exists $v_i \in \boldsymbol{A}(\pi Z_i)$ such that $\begin{pmatrix} -q - \alpha \\ V \end{pmatrix} v_i > 0$. But $v_i \in \mathrm{Im}\pi$ and $\alpha \in \ker \pi$, hence $\alpha \cdot v_i = 0$. Therefore $\begin{pmatrix} -q \\ V \end{pmatrix} v_i > 0$ which contradicts the fact that $q \in Q(\mathcal{F}_\pi)$. ■

Claim 3.3.11 $Z(\mathcal{F}_\pi) \subset Z(\mathcal{F})$.

Proof. First we show that $Z(\mathcal{F}_\pi) \subset \pi Z(\mathcal{F})$. For each $i \in I$, $Z_i \subset Z(\mathcal{F})$. Then for each i, $\pi Z_i \subset \pi Z(\mathcal{F})$, which implies $\text{cl}\pi Z_i \subset \pi Z(\mathcal{F})$, therefore $Z(\mathcal{F}_\pi) \subset \pi Z(\mathcal{F})$ (since $Z(\mathcal{F}_\pi) =< \cup_i \text{cl}\pi Z_i >$).

Second we show that $\pi Z(\mathcal{F}) \subset Z(\mathcal{F})$. Let $z \in Z(\mathcal{F})$ and write $\pi z = (\pi z - z) + z \in \ker\pi + Z(\mathcal{F}) \subset Z(\mathcal{F})$ since $\ker\pi = \boldsymbol{L}\Big(\sum_{i\in I} Z_i \cap \{V \geq 0\}\Big) \subset Z(\mathcal{F})$. ∎

Claim 3.3.12 $Q(\mathcal{F}_\pi) \cap Z(\mathcal{F}_\pi) \subset Q(\mathcal{F}) \cap Z(\mathcal{F})$.

Proof. First, we show $Q(\mathcal{F}_\pi) \cap Z(\mathcal{F}_\pi) \subset Q(\mathcal{F})$.

$$\begin{aligned} Q(\mathcal{F}_\pi) \cap Z(\mathcal{F}_\pi) &\subset Q(\mathcal{F}_\pi) \cap \text{Im}\pi && \text{because } Z(\mathcal{F}_\pi) \subset \text{Im}\pi \\ Q(\mathcal{F}_\pi) \cap \text{Im}\pi &= \pi Q(\mathcal{F}_\pi) && \text{by Claim 3.3.10} \\ \pi Q(\mathcal{F}_\pi) &\subset Q(\mathcal{F}) && \text{by Claim 3.3.9.} \end{aligned}$$

Second, we show $Q(\mathcal{F}_\pi) \cap Z(\mathcal{F}_\pi) \subset Z(\mathcal{F})$.

$$\begin{aligned} Q(\mathcal{F}_\pi) \cap Z(\mathcal{F}_\pi) &\subset Z(\mathcal{F}_\pi) \\ Z(\mathcal{F}_\pi) &\subset Z(\mathcal{F}) && \text{by Claim 3.3.11.} \end{aligned}$$

Hence $Q(\mathcal{F}_\pi) \cap Z(\mathcal{F}_\pi) \subset Q(\mathcal{F}) \cap Z(\mathcal{F})$. ∎

Proof of Part (e) of Theorem 3.3

Claim 3.3.12 implies that $\mathcal{F}_\pi$ satisfies property $\mathbf{P}$(i). We need only show that $\mathcal{F}_\pi$ satisfies property $\mathbf{P}$(ii). Let $q \in Q(\mathcal{F}_\pi) \cap Z(\mathcal{F}_\pi)$, $i \in I$, and $z_i \in Z_i$. Then $\pi z_i \in \text{cl}\pi Z_i$ and $q \cdot \pi z_i = q \cdot z_i$ since the projection π is orthogonal, $q \in Q(\mathcal{F}_\pi) \cap Z(\mathcal{F}_\pi) \subset \text{Im}\pi$, and $z_i - \pi z_i \in \ker\pi$. This ends the proof of Theorem 3.3. ∎

3.3.8 Final Remark

Proposition 3.3 *Let π be a linear projection of $\mathbb{R}^J$ such that $\ker\pi \subset \boldsymbol{L}\Big(\sum_{i\in I}(Z_i \cap \ker V)\Big)$, and consider the following assertions.*

(i) $\ker\pi = \boldsymbol{L}\Big(\sum_{i\in I}(Z_i \cap \ker V)\Big)$.

(i') $\ker\pi = \boldsymbol{L}\Big(\sum_{i\in I}(\boldsymbol{A}Z_i \cap \ker V)\Big)$.

(ii) $\boldsymbol{L}\Big(\sum_{i\in I}\big(\mathrm{cl}\pi Z_i \cap \ker V\big)\Big) = \{0\}$.

(ii') $\boldsymbol{L}\Big(\sum_{i\in I}\big(\boldsymbol{A}\mathrm{cl}\pi Z_i \cap \ker V\big)\Big) = \{0\}$.

(iii) $\forall i \in I, \boldsymbol{L}\big(\mathrm{cl}\pi Z_i \cap \ker V\big) = \{0\}$.

Then the following hold[9]

(a) (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (ii') $\Rightarrow$ (iii).

(b) (i') $\not\Rightarrow$ (ii).

(c) *If the cones* $\boldsymbol{A}Z_i \cap \ker V$ *satisfy* **WPSI** *then* (i) $\Leftrightarrow$ (i') $\Leftrightarrow$ (ii) $\Leftrightarrow$ (ii') $\Leftrightarrow$ (iii).

Proof. *(a)* [(i) $\Rightarrow$ (ii)]. This is Part (d) of Theorem 3.3.

[(ii) $\Leftrightarrow$ (ii')]. The implication (ii) $\Rightarrow$ (ii') is immediate. We show (ii') $\Rightarrow$ (ii). We will show the following more general result

$$\boldsymbol{L}\Big(\sum_{i\in I}\big(\boldsymbol{A}Z_i \cap \ker V\big)\Big) = \{0\} \Rightarrow \boldsymbol{L}\Big(\sum_{i\in I}\big(Z_i \cap \ker V\big)\Big) = \{0\}.$$

Let $\zeta \in \mathbf{A}(\sum_{i\in I} Z_i \cap \ker V) \cap -\mathbf{A}(\sum_{i\in I} Z_i \cap \ker V)$, then for every integer n, there exists $z_i^n \in Z_i \cap \ker V$ such that $n\zeta = \sum_{i\in I} z_i^n$ or equivalently $\zeta = \sum_{i\in I} z_i^n/n$ and we notice that $z_i^n/n \in Z_i \cap \ker V$ (since Z_i is convex and contains 0). Consider now the set

$$K := \{(z_1, \ldots, z_I) \in \Pi_{i\in I} Z_i : \sum_{i\in I} z_i = \zeta, V z_i = 0\}.$$

We claim that the set K is compact. Indeed, K is obviously closed and we only need to show that it is bounded. To this end, we show that the asymptotic cone $\boldsymbol{A}K$ of K is equal to $\{0\}$ (see [6]). We have

$$\mathbf{A}K := \{(\xi_1, \ldots, \xi_I) \in \Pi_{i\in I}\mathbf{A}Z_i : \sum_{i\in I} \xi_i = 0, V\xi_i = 0\}.$$

Hence, if $(\xi_1, \ldots, \xi_I) \in \boldsymbol{A}K$, from $V\xi_i = 0$ for every $i \in I$ and $\sum_{i\in I}\xi_i = 0$ we deduce that $\xi_1 = -\sum_{i\neq 1} \xi_i \in \sum_{i\in I}(\boldsymbol{A}Z_i \cap \ker V) \cap -\sum_{i\in I}(\boldsymbol{A}Z_i \cap \ker V) = \{0\}$. Therefore $\xi_1 = 0$ and similarly, $\xi_i = 0$ for every $i \in I$. That is $\boldsymbol{A}K = \{0\}$. This ends the proof of the claim.

From the compactness of K one deduces that, without any loss of generality each sequence (z_i^n/n) converges to some $\zeta_i \in \mathbf{A}Z_i \cap \ker V$. Hence $\zeta = \sum_{i\in I} \zeta_i \in \sum_{i\in I} \mathbf{A}Z_i \cap \ker V$. Similarly we prove that $-\zeta \in \sum_{i\in I} \mathbf{A}Z_i \cap \ker V$. Therefore $\zeta = 0$.

[9] The implication (3) $\Rightarrow$ (1) holds true under WPSI.

[(ii') $\Rightarrow$ (iii)]. This is obvious since, for each $i \in I$, we have

$$\boldsymbol{L}\big(\mathrm{cl}\pi Z_i \cap \ker V\big) = \boldsymbol{L}\big(\boldsymbol{A}\mathrm{cl}\pi Z_i \cap \ker V\big) \subset \boldsymbol{L}\Big(\sum_{i\in I}\big(\boldsymbol{A}\mathrm{cl}\pi Z_i \cap \ker V\big)\Big) = \{0\}.$$

(b) [(i') $\not\Rightarrow$ (ii')]. This is an example of a financial structure where

$$\ker \pi = \boldsymbol{L}(\sum_{i\in I}(\boldsymbol{A}Z_i \cap \ker V)) \not\Rightarrow \boldsymbol{L}(\sum_{i\in I}(\boldsymbol{A}\pi Z_i \cap \ker V)) = \{0\}.$$

Let $V = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$, $I = 2$, and

$$\begin{aligned} Z_1 &= \{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_1 \geq 0, z_2 \geq 0, z_3 \in \mathbb{R} \text{ or } z_1 \leq 0, z_2 \geq z_1^2, z_3 \in \mathbb{R}\}, \\ Z_2 &= \{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_1 \geq 0, z_2 \leq 0, z_3 \in \mathbb{R} \text{ or } z_1 \leq 0, z_2 \leq -z_1^2, z_3 \in \mathbb{R}\}. \end{aligned}$$

Then $\ker V = \mathbb{R} \times \mathbb{R} \times \{0\}$, $\boldsymbol{A}Z_1 = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$, $\boldsymbol{A}Z_2 = \mathbb{R}_+ \times \mathbb{R}_- \times \mathbb{R}$, and

$$\begin{aligned} \boldsymbol{A}Z_1 \cap \ker V &= \mathbb{R}_+ \times \mathbb{R}_+ \times \{0\}, \\ \boldsymbol{A}Z_2 \cap \ker V &= \mathbb{R}_+ \times \mathbb{R}_- \times \{0\}, \\ \sum_{i\in I}(\boldsymbol{A}Z_i \cap \ker V) &= \mathbb{R}_+ \times \mathbb{R} \times \{0\}, \\ \ker \pi = \boldsymbol{L}(\sum_{i\in I}(\boldsymbol{A}Z_i \cap \ker V)) &= \{0\} \times \mathbb{R} \times \{0\}, \\ \mathrm{Im}\pi = (\ker \pi)^\perp &= \mathbb{R} \times \{0\} \times \mathbb{R}, \\ \boldsymbol{A}\pi Z_1 = \pi Z_1 &= \mathbb{R} \times \{0\} \times \mathbb{R}, \\ \boldsymbol{A}\pi Z_2 = \pi Z_2 &= \mathbb{R} \times \{0\} \times \mathbb{R}, \\ \boldsymbol{A}\pi Z_1 \cap \ker V = \boldsymbol{A}\pi Z_2 \cap \ker V &= \mathbb{R} \times \{0\} \times \{0\}. \end{aligned}$$

Hence

$$\boldsymbol{L}\Big(\sum_{i\in I}(\boldsymbol{A}\pi Z_i \cap \ker V)\Big) = \mathbb{R} \times \{0\} \times \{0\} \neq \{0\}.$$

(c) We need only show that, under **WPSI**, (i) is equivalent to (i') and (iii) implies (i).

[(i) $\Leftrightarrow$ (i')]. Follows from Lemma 3.4.

[(iii) $\Rightarrow$ (i) When the cones $\boldsymbol{A}Z_i \cap \ker V$ satisfy **WPSI**]. Let $\zeta \in \boldsymbol{L}\Big(\sum_{i\in I}(Z_i \cap \ker V)\Big)$, then by **WPSI** (see Theorem 3.4(b)(ii) in the appendix), $\zeta \in \sum_{i\in I}\big(\boldsymbol{L}(Z_i) \cap \ker V\big)$, that is $\zeta = \sum_{i\in I}\zeta_i$ with $\zeta_i \in \boldsymbol{L}(Z_i) \cap \ker V$ for each $i \in I$. Thus $\pi\zeta = \sum_{i\in I}\pi\zeta_i$ and for each $i \in I$,

$$\pi\zeta_i \in \pi\Big(\boldsymbol{L}(Z_i) \cap \ker V\Big) \subset \pi\big(\boldsymbol{L}(Z_i)\big) \cap \ker V \subset \boldsymbol{L}(\pi Z_i) \cap \ker V$$

(notice that for the first inclusion we used the following fact: $\pi(\ker V) \subset \ker V$). Recall that, by assumption (iii), $\boldsymbol{L}(\pi Z_i) \cap \ker V = \{0\}$ for every i. Hence $\pi\zeta_i = 0$ for each i and consequently $\pi\zeta = 0$, that is $\zeta \in \ker \pi$. ■

3.4 Examples

Example 3.3 The financial structures $\mathcal{F} = (V, (Z_i)_i)$ and $\mathcal{F}' = (V', (Z_i')_i)$, where[10] $V' = [V, \mathbf{1}]$ and for $i \in I$, $Z_i' = Z_i \times \mathbb{R}_+$, are equivalent. That is $\mathcal{F} \sim \mathcal{F} \otimes (\mathbf{1}, \mathbb{R}_+)$.

Proof. (a) $\mathcal{F} \preceq \mathcal{F}'$. Indeed if $z_i \in Z_i, \forall i$ and $\sum_{i \in I} z_i = 0$ then $z_i' := (z_i, 0)$ are in Z_i' and satisfy $\sum_{i \in I} z_i' = 0$ and $V' z_i' = V z_i$.

(b) $\mathcal{F}' \preceq \mathcal{F}$. Indeed if $z_i' = (z_i, \alpha_i) \in Z_i', \forall i$ and $\sum_{i \in I} z_i' = 0$ then necessarily $\alpha_i = 0, \forall i$ and $\sum_{i \in I} z_i = 0$, hence $V' z_i' = V z_i$. ■

Example 3.4 The financial structures $\mathcal{F} = V$ and $\mathcal{F}' = (V, (\ker V)^{\perp})$ are equivalent.

In the following we will construct equivalent financial structures using the same scheme, which relies on linear projection as defined hereafter.

Definition 3.9 *Let* Π_V *be the set of all linear projections* $\pi : \mathbb{R}^J \to \mathbb{R}^J$ *such that* $\ker \pi \subset \ker V$. *Let* $\mathcal{F} = \big(V, (Z_i)_i\big)$, *and let* $\pi \in \Pi_V$, *we denote* $\mathcal{F}_\pi := \Big(V, \big(\mathrm{cl}(\pi Z_i)\big)_i\Big)$.

The following example gives simple cases under which $\mathcal{F}$ and $\mathcal{F}_\pi$ are equivalent.

Example 3.5 For all $\pi \in \Pi_V$, $\mathcal{F} = \big(V, (Z_i)_i\big) \preceq \big(V, (\pi Z_i)_i\big) \preceq \mathcal{F}_\pi := \Big(V, \big(\mathrm{cl}(\pi Z_i)\big)_i\Big)$. Moreover, for $\pi \in \Pi_V$, $\mathcal{F} \sim \mathcal{F}_\pi$ in each of the following cases.

(1) For all $i, Z_i = Z$ is a linear subspace.

(2) For all i, Z_i is a linear subspace and $\ker \pi \subset \sum_{i \in I} (Z_i \cap \ker V)$.

(3) For all $i, Z_i = Z$ is closed, convex, and contains 0 and $\ker \pi \subset \boldsymbol{L}(Z \cap \ker V)$.[11]

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[10]The symbol $\mathbf{1}$ represents the vector of $\mathbb{R}^{S'}$ whose components are all equal to one. The matrix $[V, \mathbf{1}]$ is the $(S' \times (J+1))$ matrix whose first J columns are those of V and the last column is the vector $\mathbf{1}$.

[11]For the sake of completeness, the proof of these assertions goes as follows. Let $(z_i)_i \subset Z^i$ be such that $\sum_{i \in I} z_i = 0$. Then $V z_i = V \pi z_i + V(z_i - \pi z_i) = V \pi z_i$ with $\sum_{i \in I} \pi z_i = \pi(\sum_{i \in I} z_i) = 0$. Hence $\mathcal{F} \preceq \mathcal{F}_\pi$.

(1) We need only show that $\mathcal{F}_\pi \preceq \mathcal{F}$. Let $y_i = \pi z_i, z_i \in Z$ be such that $0 = \sum_{i \in I} y_i = \sum_{i \in I} \pi z_i$. Then $\sum_{i \in I} z_i \in \ker \pi$ and $\forall i, V \pi z_i = V z_i + V(\pi z_i - z_i) = V z_i$. Let $\hat{z}_i = z_i - \frac{1}{I} \sum_{i \in I} z_i$, then $\hat{z}_i \in Z$ (since Z is a linear subspace), $\sum_{i \in I} \hat{z}_i = 0$ and $V \hat{z}_i = V z_i = V y_i$.

Remark 3.1 *If* $\ker \pi \subset \boldsymbol{L}(Z \cap \ker V)$ *then* $\boldsymbol{A}Z \cap \ker \pi \subset \boldsymbol{L}(Z)$ *and* πZ *is closed when* Z *is closed by Theorem 9.1 page 73 in Rockafellar [6] which is (by the way) valid in our setting.*

3.4.1 Symmetric linear portfolio sets: for all i, $Z_i = Z$ is a linear subspace

Example 3.6 (a) If Z' is a linear subspace, then the following assertions are equivalent.

(1) $(V, (Z')_i) \sim (V, (\mathbb{R}^J)_i)$, (2) $VZ' = V\mathbb{R}^J$, (3) $Z' + \ker V = \mathbb{R}^J$.

(4) $Z' = \pi(\mathbb{R}^J)$ for some $\pi \in \Pi_V$.

(b) Moreover, if (1) is satisfied, the following assertions (5)-(8) are equivalent.

(5) $Z' \cap \ker V = \{0\}$, (6) $\dim\ Z' = \text{rank}\ V$, (7) $Z' \bigoplus \ker V = \mathbb{R}^J$.

(8) $Z' = \pi(\mathbb{R}^J)$ for some $\pi \in \Pi_V$ such that $\ker \pi = \ker V$.[12]

Example 3.7 (Eliminating redundant assets) Consider $V = [V^1, V^2, \cdots, V^J]$ and let $\tilde{V} = [V^1, \cdots, V^r], (r \leq J)$ and assume that rank$V$ = rank$\tilde{V}$. Let $Z = \mathbb{R}^r \times \{0\}^{J-r}$.

---

(2) We need only show that $\mathcal{F}_\pi \preceq \mathcal{F}$. Let $y_i = \pi z_i, z_i \in Z^i$ be such that $0 = \sum_{i\in I} y_i = \sum_{i\in I} \pi z_i$. Then $\sum_{i\in I} z_i \in \ker \pi$ and $\forall i, V\pi z_i = Vz_i + V(\pi z_i - z_i) = Vz_i$. Write $\sum_{i\in I} z_i = \sum_{i\in I} v_i$ where $\forall i, v_i \in Z^i \cap \ker V$, and define $\hat{z}_i = z_i - v_i$. Then $\forall i, \hat{z}_i \in Z^i, \sum_{i\in I} \hat{z}_i = 0$ and $V\hat{z}_i = Vz_i = Vy_i$.

(3) We need only show that $\mathcal{F}_\pi \preceq \mathcal{F}$. Let $y_i = \pi z_i, z_i \in Z$ be such that $0 = \sum_{i\in I} y_i = \sum_{i\in I} \pi z_i$. Then $\sum_{i\in I} z_i \in \ker \pi$ and $\forall i, V\pi z_i = Vz_i + V(\pi z_i - z_i) = Vz_i$. Let $v \in \boldsymbol{L}(Z \cap \ker V)$ be such that $\sum_{i\in I} z_i = v$ and define $\hat{z}_i = z_i - \frac{v}{I}$. Then $\forall i, \hat{z}_i \in Z, \sum_{i\in I} \hat{z}_i = 0$ and $V\hat{z}_i = V(z_i - \frac{v}{I}) = Vz_i = Vy_i$.

[12] The proof goes as follows. *Part (a).* (1)$\Rightarrow$ (2). Obvious.

(2)$\Rightarrow$ (3). Let $y \in \mathbb{R}^J$, then $\exists z \in Z'$ s.t. $Vy = Vz$. Hence $y - z \in \ker V$, that is $y \in Z' + \ker V$.

(3)$\Rightarrow$ (4). To be written.

(4)$\Rightarrow$ (1). Apply the result of Example 3.5.

Part (b). (5) $\Rightarrow$ (6). We have

$$\begin{aligned} \dim\ Z' &= \dim\ VZ' \text{ by (5)} \\ &= \dim\ V\mathbb{R}^J \text{ by (2)} \\ &= \text{rank}\ V \text{ by definition of the rank.} \end{aligned}$$

(6) $\Rightarrow$ (7). Then $\dim\ Z' + \dim\ \ker V = \text{rank}\ V + \dim\ \ker V = \dim\ \mathbb{R}^J$. Combining this result about the dimensions with (3), we get $Z' \bigoplus \ker V = \mathbb{R}^J$.

(7) $\Rightarrow$ (8). This a consequence of the next Example.

(8) $\Rightarrow$ (5). $Z' \cap \ker V = p(\mathbb{R}^J) \cap \ker V = \{0\}$.

(a) Then $(V, (\mathbb{R}^J)_i) \sim (V, (Z)_i) \sim (\tilde{V}, \mathbb{R}^r)$.

(b) Moreover, the following conditions are equivalent.

(1) $\mathbb{R}^r \times \{0\}^{J-r} \cap \ker V = \{0\}$, (2) $\mathbb{R}^r \times \{0\}^{J-r} \bigoplus \ker V = \mathbb{R}^J$,

(3) $r = \operatorname{rank} V$, (4) The vectors $(V^1, \cdots, V^r)$ are linearly independent.

Proof of Part (a). Indeed,

$$V(\mathbb{R}^J) = \{\sum_{j\in J} z^j V^j : z^j \in \mathbb{R}, j \in J\} = \{\sum_{j=1}^{r} z^j V^j : z^j \in \mathbb{R}, j \in [|1, r|]\} = V(Z) = \tilde{V}(\mathbb{R}^r).$$

Proof of Part (b). To be written.[13]

3.4.2 Symmetric nonlinear portfolio sets: for all i, $Z_i = Z$ is closed convex

Example 3.8 Let Z be a closed, convex subset of $\mathbb{R}^J$ containing 0. If $\ker \pi \subset \boldsymbol{L}(Z \cap \ker V)$ then $\mathcal{F} = \big(V, (Z)_i\big) \sim \mathcal{F}_\pi := \Big(V, \big(\operatorname{cl}(\pi Z_i)\big)_i\Big)$. (Notice that $\boldsymbol{L}(Z \cap \ker V) = \boldsymbol{L}(Z) \cap \ker V$, and that $\boldsymbol{A}Z \cap \ker \pi \subset \ker \pi \subset \boldsymbol{L}(Z)$. Therefore, by Theorem 9.1 page 73 in Rockafellar [6], πZ is closed.)

Furthermore, under the assumption $\ker \pi \subset \boldsymbol{L}(Z \cap \ker V)$, we have $\boldsymbol{L}\big(\pi Z \cap \ker V\big) = \{0\}$, that is $\boldsymbol{L}(\mathcal{F}_\pi) = \{0\}$, if and only if $\ker \pi = \boldsymbol{L}(Z \cap \ker V)$.

Proof. We need only show that $\mathcal{F}_\pi \preceq \mathcal{F}$. Let $y_i = \pi z_i, z_i \in Z$ be such that $0 = \sum_{i\in I} y_i = \sum_{i\in I} \pi z_i$. Then $\sum_{i\in I} z_i \in \ker \pi$ and $\forall i, V\pi z_i = V z_i + V(\pi z_i - z_i) = V z_i$. Let $v \in \boldsymbol{L}(Z \cap \ker V)$ be such that $\sum_{i\in I} z_i = v$ and define $\hat{z}_i = z_i - \frac{v}{I}$. Then $\forall i, \hat{z}_i \in Z, \sum_{i\in I} \hat{z}_i = 0$ and $V\hat{z}_i = V(z_i - \frac{v}{I}) = V z_i = V y_i$. ∎

3.4.3 Linear portfolio sets: for all i, Z_i is a linear subspace

Example 3.9 Let Z_i be a linear subspace for every i.

(a) If $\ker \pi \subset \sum_{i\in I}(Z_i \cap \ker V)$, then $\mathcal{F} = \big(V, (Z_i)_i\big) \sim \mathcal{F}_\pi := \Big(V, \big(\operatorname{cl}(\pi Z_i)\big)_i\Big)$ and

$$\pi\big(\sum_{i\in I}(Z_i \cap \ker V)\big) = \sum_{i\in I}\big(\pi(Z_i) \cap \ker V\big).$$

Notice that since the Z_i's are linear subspaces, for each i, one has πZ_i is closed.

[13] We show (4)⇒ (1). Let $z \in \mathbb{R}^r \times \{0\}^{J-r} \cap \ker V$. Then $0 = Vz = \sum_{j\in J} z^j V^j$ and $z^j = 0$ for $j > r$. Hence $\sum_{j=1}^{r} z^j V^j = 0$, and since the vectors $V^1, \cdots, V^r$ are independent, we conclude that $z = 0$.

(b) Under the above condition, the following assertions are equivalent.

(1) $\sum_{i\in I}\big(\pi(Z_i)\cap\ker V\big)=\{0\}$,

(2) $\ker\pi=\sum_{i\in I}(Z_i\cap\ker V)$.

Proof. (a) To show the equivalence of $\mathcal{F}$ and $\mathcal{F}_\pi$, we need only show that $\mathcal{F}_\pi\preceq\mathcal{F}$. Let $y_i=\pi z_i, z_i\in Z_i$ be such that $0=\sum_{i\in I}y_i=\sum_{i\in I}\pi z_i$. Then $\sum_{i\in I}z_i\in\ker\pi$ and $\forall i, V\pi z_i=Vz_i+V(\pi z_i-z_i)=Vz_i$. Write $\sum_{i\in I}z_i=\sum_{i\in I}v_i$ where $\forall i, v_i\in Z_i\cap\ker V$, and define $\hat{z}_i=z_i-v_i$. Then $\forall i, \hat{z}_i\in Z_i, \sum_{i\in I}\hat{z}_i=0$ and $V\hat{z}_i=Vz_i=Vy_i$.

Let $z_i\in Z_i\cap\ker V$, then $\pi(\sum_{i\in I}z_i)=\sum_{i\in I}\pi z_i$ with $V\pi z_i=Vz_i=0$ hence $\pi z_i\in\pi Z_i\cap\ker V$ therefore $\pi\big(\sum_{i\in I}(Z_i\cap\ker V)\big)\subset\sum_{i\in I}\big(\pi Z_i\cap\ker V\big)$. Conversely, let $z_i\in Z_i$ be such that $V\pi z_i=0$. Then $Vz_i=V\pi z_i=0$ i.e. $z_i\in Z_i\cap\ker V$ and $\sum_{i\in I}\pi z_i=\pi(\sum_{i\in I}z_i)$, which shows that $\sum_{i\in I}\big(\pi Z_i\cap\ker V\big)\subset\pi\big(\sum_{i\in I}(Z_i\cap\ker V)\big)$.

(b) Obvious (by the second part of (a)). ∎

Example 3.10 *The projection used in Balasko, Cass, and Siconolfi [3].*

Let $N=\sum_{i\in I}(Z_i\cap\ker V)$. Then N has a supplementary space of the form $\mathbb{R}^{J\setminus A}\times\{0\}^A$ with $A\subset J$ and $|A|=\dim N$. Using this result we can get the existence of a linear projector $\pi\in\Pi_V$ such that $\ker\pi=N$ and $\pi(\mathbb{R}^J)=\mathbb{R}^{J\setminus A}\times\{0\}^A$.[14]

Let $\mathcal{F}=(V,(Z_i)_i)$, let $N=\sum_{i\in I}(Z_i\cap\ker V)$, then there exist linear subspaces $(Z_i')_i$ of $\mathbb{R}^J$ such that

(1) $\mathcal{F}=(V,(Z_i)_i)\sim\mathcal{F}'=(V,(Z_i')_i)$,

(2) $\dim V(Z_i')=\dim Z_i'$,

(3) $\sum_{i\in I}(Z_i'\cap\ker V)=\{0\}$.

[14] Indeed, let M be the matrix of coordinates of a basis $\{n_1,\cdots,n_k\}$ of N. Clearly $\text{rank}M=k=\dim N$, and there exists a subset A of J such that $|A|=\dim N$ and the family $(M_i)_{i\in A}$ of rows of M is linearly independent. We first claim that $N\cap(\mathbb{R}^{J\setminus A}\times\{0\}^A)=\{0\}$. Indeed, without any loss of generality, we can assume that M can be written $M=\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ where α is a $(J-k)\times k$ matrix, β is a $k\times k$ matrix, and β is invertible. Let $x\in N\cap\mathbb{R}^{J\setminus A}\times\{0\}^A$ then there exists $\lambda\in\mathbb{R}^A$, $x=M\lambda$ (because $x\in N$) hence $0=x_A=M_A\lambda$ which implies that $\lambda=0$ since M_A is invertible, therefore $x=0$. Consequently, since the sum of dimensions of the two spaces, $N$and $\mathbb{R}^{J\setminus A}\times\{0\}^A$, is equal to J, they are supplementary spaces.

Choosing π as the linear projection such that $\ker \pi = N$ and $\pi(\mathbb{R}^J) = \mathbb{R}^{J \setminus A} \times \{0\}^A$ allows us to define $\hat{V} = [V^j \mid j \in J \setminus A]$, $\hat{Z}_i$ by $Z'_i = \pi(Z_i) = \hat{Z}_i \times \{0\}^A$, and we have

$$\mathcal{F} = (V, (Z_i)_i) \sim \mathcal{F}' = (V, (Z'_i)_i) \sim (\hat{V}, (\hat{Z}_i)_i).$$

Example 3.11 If $\forall i, Z_i$ is a linear subspace, and $\cup_i Z_i$ is a linear subspace, or equivalently there exists $i_0 \in I$ such that $\forall i, Z_i \subset Z_{i_0}$, then the following assertions are equivalent.

(1) $(V, (Z_i)_i) \sim (V, (Z'_i)_i)$,

(2) $\forall i, V Z_i = V Z'_i$,

(3) $\forall i, Z^i + \ker V = Z'_i + \ker V$.

Proof. (1)$\Rightarrow$ (2). Obvious.

(2)$\Rightarrow$ (3). Let $y \in Z'_i + \ker V$, then $Vy \in VZ'_i$ hence there exists $z \in Z^i$ such that $Vy = Vz$. Therefore $y - z \in \ker V$, that is $y \in Z^i + \ker V$.

(3)$\Rightarrow$ (1). Let $(\zeta_i)_i \in \Pi_i Z'_i$ be such that $\sum_{i\in I} \zeta_i = 0$. For each i, write $\zeta_i = z_i + n_i$ with $z_i \in Z^i$ and $n_i \in \ker V$. Then $\forall i, V\zeta_i = Vz_i$. Let $\bar{z}_{i_0} = z_{i_0} - \sum_{i\in I} z_i$ and $\bar{z}_i = z_i$ for $i \neq i_0$, then $\bar{z}_i \in Z^i$, $V\bar{z}_i = Vz_i$ for each i, and $\sum_{i\in I} \bar{z}_i = 0$. Therefore, $(V, (Z'_i)_i) \preceq (V, (Z_i)_i)$. ■

3.5 Appendix

3.5.1 Counter-example

Hereafter we give an explicit example of a correspondence Φ satisfying

- $\Phi(\zeta)$ is closed for every $\zeta \in (\mathbb{R}^J)^I$
- The inclusion $\Phi(\zeta) \subset \boldsymbol{L}(\mathcal{F}) + \mathrm{proj}_{\boldsymbol{L}(\mathcal{F})^\perp} \Phi(\zeta)$ holds at every ζ and is not an equality at some ζ.
- The correspondence Φ from $(\mathbb{R}^J)^I$ to $\mathbb{R}^J$ does not have a closed graph.

Note that the third property is a consequence of the second one.

We let $I = S = 2$, $J = 3$, $V = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, and

$$
\begin{aligned}
Z_1 &= \{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_1 \geq -1, z_2 \geq 0, z_3^2 \leq (z_1+1)z_2\},\\
Z_2 &= \{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_1 \geq 0, z_2 \leq 0, z_3 = 0\}.
\end{aligned}
$$

$\bar{\zeta} = (-1, 0, 0)$ and $\bar{\xi} = (0, 0, 0)$.

Then we easily see

$$
\begin{aligned}
\ker V &= \{0\} \times \mathbb{R} \times \mathbb{R},\\
Z_1 \cap \ker V &= \{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_1 = 0, z_2 \geq 0, z_3^2 \leq z_2\},\\
Z_2 \cap \ker V &= \{0\} \times \mathbb{R}_- \times \{0\},\\
\boldsymbol{L}(\mathcal{F}) := \boldsymbol{L}(\sum_{i\in I} Z_i \cap \ker V) &= \{0\} \times \mathbb{R} \times \mathbb{R} = \ker V,\\
\big(\boldsymbol{L}(\mathcal{F})\big)^{\perp} &= \mathbb{R} \times \{0\} \times \{0\},\\
\{z \in Z_1 : Vz \geq V\bar{\zeta}\} &= \{-1\} \times \mathbb{R}_+ \times \{0\},\\
\{z \in Z^i : Vz \geq V\bar{\xi}\} &= \{0\} \times \mathbb{R}_- \times \{0\},\\
\Phi(\bar{\zeta}, \bar{\xi}) &= \{-1\} \times \mathbb{R} \times \{0\},\\
\mathrm{proj}_{\left(\boldsymbol{L}(\mathcal{F})\right)^{\perp}} \Phi(\bar{\zeta}, \bar{\xi}) &= \{(-1, 0, 0)\},\\
\boldsymbol{L}(\mathcal{F}) + \mathrm{proj}_{\left(\boldsymbol{L}(\mathcal{F})\right)^{\perp}} \Phi(\bar{\zeta}, \bar{\xi}) &= \{-1\} \times \mathbb{R} \times \mathbb{R}.
\end{aligned}
$$

Furthermore, for every $(\zeta, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3$,

$$
\Phi(\zeta, \xi) = \begin{cases} (\zeta_1 + \xi_1) \times \mathbb{R} \times \mathbb{R} & \text{if } \zeta_1 \geq -1, \xi_1 \geq 0, \zeta_1 + \xi_1 > -1,\\ \{-1\} \times \mathbb{R} \times \{0\} & \text{if } \zeta_1 \geq -1, \xi_1 \geq 0, \zeta_1 + \xi_1 = -1,\\ \emptyset & \text{if } \zeta_1 < -1 \text{ or } \xi_1 < 0. \end{cases}
$$

Define $\widehat{\Phi}(\zeta) := \{y \in \mathbb{R}^S : \exists (y^n)_n \subset \mathbb{R}^S, \exists (\zeta^n)_n \subset \mathbb{R}^J, y^n \underset{n\to\infty}{\longrightarrow} y, \zeta^n \underset{n\to\infty}{\longrightarrow} \zeta, y^n \in \Phi(\zeta^n)\}$. Then, in the example above, $\widehat{\Phi}(\bar{\zeta}) = \{-1\} \times \mathbb{R} \times \mathbb{R}$. Thus, from above

- $\Phi(\zeta, \xi)$ is closed for every $(\zeta, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3$.
- $\boldsymbol{L}(\mathcal{F}) + \mathrm{proj}_{\left(\boldsymbol{L}(\mathcal{F})\right)^{\perp}} \Phi(\bar{\zeta}, \bar{\xi}) \supset \Phi(\bar{\zeta}, \bar{\xi})$ and the inclusion is strict.
- $\Phi(\bar{\zeta}, \bar{\xi}) \subset \widehat{\Phi}(\bar{\zeta}, \bar{\xi})$ and the inclusion is strict. Hence Φ does not have a closed graph.

3.5.2 Proof of Proposition 3.1

Notice that assertions (a)-(e) are special cases of (f). Hence, we will prove only (f).

(f) First, we prove the result when for every $i \in I$, $K_i = \{0\}$, i.e. when Z_i is polyhedral for every i. Let

$$f : \mathbb{R}^{SI} \times \mathbb{R}^{JI} \to \mathbb{R}^{SI}, (v_1, \cdots, v_I, z_1, \cdots, z_I) \mapsto (Vz_1 - v_1, \cdots, Vz_I - v_I),$$

and

$$g : \mathbb{R}^{SI} \times \mathbb{R}^{JI} \to \mathbb{R}^{SI} \times \mathbb{R}^{J}, (v_1, \cdots, v_I, z_1, \cdots, z_I) \mapsto (v_1, \cdots, v_I, \sum_{i \in I} z_i).$$

Then f and g are linear and the set $\mathcal{G}'(\mathcal{F})$ is

$$\mathcal{G}'(\mathcal{F}) = g\Big((\mathbb{R}^S \times \cdots \times \mathbb{R}^S \times Z_1 \times \cdots \times Z_I) \cap f^{-1}(\mathbb{R}^S_+ \times \cdots \times \mathbb{R}^S_+)\Big).$$

Since $\mathbb{R}^S_+ \times \cdots \times \mathbb{R}^S_+$ and $\mathbb{R}^S \times \cdots \times \mathbb{R}^S \times Z_1 \times \cdots \times Z_I$ are polyhedral, Corollary 19.2.2 and Theorem 19.3 page 174 in [6] allow to conclude.

Now, we show the result in the general case. Let (v^n, y^n) be a sequence in the set $\mathcal{G}'(\mathcal{F})$ such that $(v^n, y^n) \underset{n \to \infty}{\longrightarrow} (v, y)$. Write $y^n = \sum_{i \in I} y_i^n$ where $\forall i, \forall n, y_i^n \in Z_i$ and $V y_i^n \geq v_i^n$. By assumption, $\forall i, \forall n$, $y_i^n = k_i^n + p_i^n$ where $k_i^n \in K_i$ and $p_i^n \in P_i$. Since the K_i's are compact, we can assume $k_i^n \underset{n \to \infty}{\longrightarrow} k_i$ for every $i \in I$. Denote $k_n = \sum_{i \in I} k_i^n$. Then, the sequence $\big((v_i^n - V k_i^n)_i, y^n - k^n\big)_n$ is in the set $\mathcal{H}$, where $\mathcal{H}$ is the set defined in the same manner as $\mathcal{G}'(\mathcal{F})$ with the Z_i's replaced by the P_i's. Hence, by the first part of the proof, $\big((v_i - V k_i)_i, y - \sum_{i \in I} k_i\big)$ is in $\mathcal{H}$, that is, for all i there exists $p_i \in P_i$ such that $V p_i \geq v_i - V k_i$ and $y - \sum_{i \in I} k_i = \sum_{i \in I} p_i$. Therefore $y = \sum_{i \in I}(k_i + p_i)$ and $(v, y) = \big((v_i)_i, \sum_{i \in I}(k_i + p_i)\big)$ with $k_i + p_i \in Z_i$ for each i and $V(k_i + p_i) \geq v_i$. ■

3.5.3 Proof of Proposition 3.2

Notice that assertions (g) to (k1) are special cases of (k2). Hence, we will prove only (k2). We show that if the sets $\boldsymbol{A} Z_i \cap \ker V$ are **WPSI** then the set $\mathcal{G}'(\mathcal{F})$ is closed.

We have

$$\mathcal{G}'(\mathcal{F}) = \{(v_1, \cdots, v_I, \sum_{i \in I} z_i) : \forall i \; v_i \in \mathbb{R}^S, z_i \in Z_i, V z_i \geq v_i\} = \sum_{i \in I} X_i$$

with

$$X_i = \{(0, \cdots, 0, \cdots v_i, 0, \cdots, 0, z_i) : v_i \in \mathbb{R}^S, z_i \in Z_i, V z_i \geq v_i\}.$$

Then

$$\boldsymbol{A} X_i = \{(0, \cdots, 0, \cdots t_i, 0, \cdots, 0, \zeta_i) : t_i \in \mathbb{R}^S, \zeta_i \in \boldsymbol{A} Z_i, V \zeta_i \geq t_i\}.$$

Now we show that WPSI of the $\boldsymbol{A} X_i$ is a consequence of WPSI of the sets $\boldsymbol{A} Z_i \cap \ker V$ (this would end the proof by Claim 3.5.2).

If $\sum_{i\in I} w_i = \sum_{i\in I}(0,\cdots,0,\cdots t_i,0,\cdots,0,\zeta_i) = 0$ with $t_i \in \mathbb{R}^S, \zeta_i \in \boldsymbol{A}Z_i, V\zeta_i \geq t_i$, then for every i, $t_i = 0, \zeta_i \in \boldsymbol{A}Z_i, V\zeta_i \geq 0$, and $\sum_{i\in I}\zeta_i = 0$. Hence for each i, $\zeta_i \in \boldsymbol{A}Z_i \cap \ker V$ and $\sum_{i\in I}\zeta_i = 0$. By WPSI of the sets $\boldsymbol{A}Z_i \cap \ker V$, we get $\zeta_i \in \boldsymbol{L}(Z_i)$ for each i. Hence $w_i \in \boldsymbol{L}(X_i)$ for each $i \in I$. ■

3.5.4 Statement and proof of Theorem 3.4

Let X_i $(i \in I)$ be convex subsets of $\mathbb{R}^J$ containing 0 and denote $\boldsymbol{L}_i = \boldsymbol{L}(X_i) = \boldsymbol{L}(\mathrm{cl}X_i)$.

Theorem 3.4 *Let X_i $(i \in I)$ be convex subsets of $\mathbb{R}^J$ containing 0. Then*

(a) *The following hold:*

(i) $\sum_{i\in I} \boldsymbol{A}X_i \subset \boldsymbol{A}(\sum_{i\in I} X_i)$,

(ii) $\sum_{i\in I} \boldsymbol{L}(X_i) \subset \boldsymbol{L}(\sum_{i\in I} \boldsymbol{A}X_i) \subset \boldsymbol{L}(\sum_{i\in I} X_i)$.

(b) *If we additionally assume that the sets $\boldsymbol{A}X_i$ are weakly positively semi-independent then the above inclusions are equalities, that is*

(i) $\sum_{i\in I} \boldsymbol{A}X_i = \boldsymbol{A}(\sum_{i\in I} X_i)$,

(ii) $\sum_{i\in I} \boldsymbol{L}(X_i) = \boldsymbol{L}(\sum_{i\in I} \boldsymbol{A}X_i) = \boldsymbol{L}(\sum_{i\in I} X_i)$,

and the set $\sum_{i\in I} X_i$ is closed.

For the proof of Theorem 3.4, we need a claim. Let B be a compact set of $\mathbb{R}^J$ and

$$\begin{aligned} K &:= \{(x_1,\cdots,x_I) \in \Pi_i \mathrm{cl}X_i : \sum_{i\in I} x_i \in B\}, \\ K_w &:= \{(\mathrm{proj}_{\boldsymbol{L}_1^\perp} x_1,\cdots,\mathrm{proj}_{\boldsymbol{L}_I^\perp} x_I) \in \Pi_i(\mathrm{cl}X_i \cap {\boldsymbol{L}_i}^\perp) : (x_1,\cdots,x_I) \in K\}. \end{aligned}$$

Note that $K_w = F(K)$ where $F : (\mathbb{R}^J)^I \to (\mathbb{R}^J)^I$ is defined by

$$F(x_1,\cdots,x_I) = (\mathrm{proj}_{\boldsymbol{L}_1^\perp} x_1,\cdots,\mathrm{proj}_{\boldsymbol{L}_I^\perp} x_I).$$

Claim 3.5.1 *The following assertions are equivalent.*

(i) *The sets $\boldsymbol{A}X_i$ are weakly positively semi-independent.*

(ii) *The set K_w is bounded.*

Moreover the set K_w is closed (without assuming (i)*).*

Proof. $[(i) \Rightarrow (ii)]$ By contradiction, assume K_w is not bounded and let $((x_i^{\perp n})_i)_n$ be a sequence in K_w (each $x_i^{\perp n}$ is in $\text{cl}X_i \cap \boldsymbol{L}_i^\perp$) such that $\sum_{i\in I}||x_i^{\perp n}|| \underset{n\to\infty}{\longrightarrow} \infty$. Let $\hat{x}_i^n \in \boldsymbol{L}_i$ be such that $(x_i^{\perp n} + \hat{x}_i^n)_i \in K$. Then, without loss of generality (taking subsequences if necessary), one can assume that for every i,

$$\frac{x_i^{\perp n}}{\sum_{i\in I}||x_i^{\perp n}|| + \sum_{i\in I}||\hat{x}_i^n||} \underset{n\to\infty}{\longrightarrow} x_i^\perp \in \boldsymbol{A}X_i \cap \boldsymbol{L}_i^\perp$$

and

$$\frac{\sum_{i\in I}\hat{x}_i^n}{\sum_{i\in I}||x_i^{\perp n}|| + \sum_{i\in I}||\hat{x}_i^n||} \underset{n\to\infty}{\longrightarrow} \alpha \in \boldsymbol{A}(\sum_{i\in I}X_i) \cap \sum_{i\in I}\boldsymbol{L}_i.$$

Write $\alpha = \sum_{i\in I}\alpha_i$ where for each i, $\alpha_i \in \boldsymbol{L}_i$. Then $\sum_{i\in I}(x_i^\perp + \alpha_i) = 0$ since $\sum_{i\in I}(x_i^{\perp n} + \hat{x}_i^n) \in B$ and $\sum_{i\in I}||x_i^{\perp n}|| \underset{n\to\infty}{\longrightarrow} \infty$. But $x_i^\perp \in \boldsymbol{A}X_i$ and $\alpha_i \in \boldsymbol{L}_i$ then $x_i^\perp + \alpha_i \in \boldsymbol{A}X_i$ hence, by WPSI, for every i, $x_i^\perp + \alpha_i \in \boldsymbol{L}_i$ that is $x_i^\perp = 0$. So, $\sum_{i\in I}\alpha_i = 0$. But for every n,

$$1 = \frac{\sum_{i\in I}||x_i^{\perp n}||}{\sum_{i\in I}||x_i^{\perp n}|| + \sum_{i\in I}||\hat{x}_i^n||} + \frac{||\sum_{i\in I}\hat{x}_i^n||}{\sum_{i\in I}||x_i^{\perp n}|| + \sum_{i\in I}||\hat{x}_i^n||}$$

implies $1 = ||\sum_i \alpha_i||$. A contradiction.

$[(ii) \Rightarrow (i)]$ Conversely, if $v_i \in \boldsymbol{A}X_i$, and $\sum_{i\in I}v_i = 0$, then for each i, $v_i = v_i^\perp + \hat{v}_i$ with $v_i^\perp \in \boldsymbol{A}X_i \cap \boldsymbol{L}_i^\perp$ and $\hat{v}_i \in \boldsymbol{L}_i$. Let $(x_i)_i \in K$, then for every $t \geq 0$, $\sum_{i\in I}(x_i + tv_i) = \sum_{i\in I}x_i \in B$. Therefore $(\text{proj}_{\boldsymbol{L}_i^\perp}x_i^\perp + tv_i^\perp)_i \in K_w$ for every $t \geq 0$. Since K_w is bounded we must have $v_i^\perp = 0$ for every i, that is $v_i \in \boldsymbol{L}_i$ for each i.

Now we show that K_w is closed. Let $((\text{proj}_{\boldsymbol{L}_i^\perp}x_i^n)_i)_n$ be a sequence in K_w (the sequence $((x_i^n)_i)_n$ is in K) such that $\text{proj}_{\boldsymbol{L}_i^\perp}x_i^n \underset{n\to\infty}{\longrightarrow} x_i^\perp \in \boldsymbol{L}_i^\perp \cap \text{cl}X_i$ for each i. For each n, let $(\hat{x}_i^n)_i \in \Pi_i\boldsymbol{L}_i$ be such that $(\text{proj}_{\boldsymbol{L}_i^\perp}x_i^n)_i + (\hat{x}_i^n)_i \in K$. That is

$$\sum_{i\in I}\text{proj}_{\boldsymbol{L}_i^\perp}x_i^n + \sum_{i\in I}\hat{x}_i^n \in B.$$

The first term, $\sum_{i\in I}\text{proj}_{\boldsymbol{L}_i^\perp}x_i^n$, converges to $\sum_{i\in I}x_i^\perp$, and since B is compact we can assume that the second term, $\sum_{i\in I}\hat{x}_i^n$, converges to some α. The limit α is in $\sum_{i\in I}\boldsymbol{L}_i$, hence $\alpha = \sum_{i\in I}\alpha_i$ where, for each i, $\alpha_i \in \boldsymbol{L}_i$. Since $x_i^\perp \in \boldsymbol{L}_i^\perp \cap \text{cl}X_i$ and $\alpha_i \in \boldsymbol{L}_i$, and $\sum_{i\in I}(x_i^\perp + \alpha_i) \in B$, we get $(x_i^\perp + \alpha_i)_i \in K$ hence $(x_i^\perp)_i \in K_w$. ∎

Claim 3.5.2 *Let X_i $(i \in I)$ be closed convex subsets of $\mathbb{R}^J$ containing 0. If the cones $\mathbf{A}X_i$ are weakly positively semi-independent then the set $\sum_{i\in I}X_i$ is closed.*

Proof. Let $\sum_{i\in I}x_i^n \underset{n\to\infty}{\longrightarrow} \alpha$ where $x_i^n \in X_i$. Then

$$\sum_{i\in I}x_i^{\perp n} + \sum_{i\in I}\hat{x}_i^n \underset{n\to\infty}{\longrightarrow} \alpha.$$

Notice that (by the previous Claim 3.5.1) for each i, $x_i^{\perp n} \underset{n\to\infty}{\longrightarrow} x_i^{\perp} \in X_i \cap \boldsymbol{L}_i^{\perp}$ and we can assume that $\sum_{i\in I}\hat{x}_i^n \underset{n\to\infty}{\longrightarrow} \beta = \sum_{i\in I}\beta_i$ where $\beta_i \in \boldsymbol{L}_i$ for each i. Then $\alpha = \sum_{i\in I}(x_i^{\perp}+\beta_i) \in \sum_{i\in I}X_i$. ∎

Proof of Theorem 3.4

(a) We first notice that, for all i, $\boldsymbol{L}(X_i) \subset \boldsymbol{A}X_i \subset X_i$. Hence

$$\sum_{i\in I}\boldsymbol{L}(X_i) \subset \sum_{i\in I}\boldsymbol{A}X_i \subset \sum_{i\in I}X_i.$$

Using the fact that $\boldsymbol{L}(A) \subset \boldsymbol{L}(B)$ if $A \subset B$ we get

$$\sum_{i\in I}\boldsymbol{L}(X_i) \subset \boldsymbol{L}(\sum_{i\in I}\boldsymbol{A}X_i) \subset \boldsymbol{L}(\sum_{i\in I}X_i).$$

(b) (i) Let $v \in \boldsymbol{A}(\sum_{i\in I}X_i)$. Write

$$v = \sum_{i\in I}\frac{1}{n}x_i^n = \sum_{i\in I}\frac{1}{n}x_i^{\perp n} + \sum_{i\in I}\frac{1}{n}\hat{x}_i^n$$

where, for each i, $x_i^n \in X_i$, $x_i^{\perp n} \in X_i \cap \boldsymbol{L}_i^{\perp} \subset \mathrm{cl}X_i \cap \boldsymbol{L}_i^{\perp}$, and $\hat{x}_i^n \in \boldsymbol{L}_i$. Then (by Claim 2.6.1), for each i,

$$\frac{1}{n}x_i^{\perp n} \underset{n\to\infty}{\longrightarrow} x_i^{\perp} \in \boldsymbol{A}X_i \cap \boldsymbol{L}_i^{\perp}$$

and

$$\sum_{i\in I}\frac{1}{n}\hat{x}_i^n \underset{n\to\infty}{\longrightarrow} \beta \in \sum_{i\in I}\boldsymbol{L}_i.$$

Write $\beta = \sum_{i\in I}\beta_i$ with $\beta_i \in \boldsymbol{L}_i$ for each i. Then $v = \sum_{i\in I}(x_i^{\perp} + \beta_i) \in \sum_{i\in I}\boldsymbol{A}X_i$.

(b)(ii) From (i) above, we get $\boldsymbol{L}(\sum_{i\in I}X_i) \subset \boldsymbol{L}(\sum_{i\in I}\boldsymbol{A}X_i)$. We show that $\boldsymbol{L}(\sum_{i\in I}\boldsymbol{A}X_i) \subset \sum_{i\in I}\boldsymbol{L}(X_i)$. Let $\xi \in \boldsymbol{L}(\sum_{i\in I}\boldsymbol{A}X_i)$. Write

$$\xi = \sum_{i\in I}\xi_i = -\sum_{i\in I}\xi_i'$$

with ξ_i and ξ_i' in $\boldsymbol{A}X_i$. Then $0 = \sum_{i\in I}(\xi_i + \xi_i')$ and for each $i \in I$, $\xi_i + \xi_i' \in \boldsymbol{A}X_i$ which implies (by definition of **WPSI**) that for every $i \in I$, $\xi_i + \xi_i' \in \boldsymbol{L}(X_i)$. Hence

$$\xi_i = -\xi_i' + (\xi_i + \xi_i') \in -\boldsymbol{A}X_i + \boldsymbol{L}(X_i) \subset -\boldsymbol{A}X_i.$$

Therefore for every $i \in I$, $\xi_i \in \boldsymbol{L}(X_i)$ that is $\xi = \sum_{i\in I}\xi_i \in \sum_{i\in I}\boldsymbol{L}(X_i)$.

The last assertion of Theorem 3.4 is the result of Claim 3.5.2. ∎

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Chapter 4

Existence of financial equilibria with restricted participation

In this chapter we prove the existence of a financial equilibrium for an economy with restricted participation in the financial markets, and without monotonic or ordered preferences.

4.1 Introduction

There is a large body of literature on existence and optimality results for exchange economies with incomplete financial markets, see for example Cass [6], Werner [17], and Duffie [7] when assets are nominal and Geanakoplos and Polemarchakis [10] for the case of numéraire assets. A natural cause of market incompleteness is the so called notion of restricted participation to financial markets, where agents face asymmetric restrictions on their portfolio trades. In order to capture a wide range of imperfections in the financial markets (such as short selling constraints, collateral requirements, and more generally institutional constraints), restrictions are modeled by subsets of the space of financial assets. Cass [6] states that

> "A very significant analysis from an interpretive viewpoint ... is the imposition of institutional restrictions on trading activity in the bond (financial) markets. The broadest formulation of such restricted participation is to assume that in addition to the bugdet constraints, households face the financial con-

straints $z_i \in Z_i \subset \mathbb{R}^J$ for $i \in I$. The implications within this particular model of a financial equilibrium seems to me a problem well worth deeper analysis in its own right."

There is a growing body of literature on this subject, see for instance the seminal papers of Balasko, Cass and Siconolfi [4] for linear restrictions with nominal assets, and Polemarchakis and Siconolfi [13] for the case of linear restrictions with real assets. But apart from Siconolfi [16], very little has been said when restrictions are not necessarily linear even when assets are nominal, see e.g. Angeloni and Cornet [1], and Hahn and Won [11]. The goal of existence is out of reach in the general case of real assets and we will focus on nominal and numéraire assets.

The purpose of this chapter is to provide a "general" existence result of equilibria in a financial exchange economy with restricted participation in the financial markets. We work in a basic two time-date (today and tomorrow) financial exchange economy with a finite set of agents and an a priori uncertainty about the future represented by a finite set of states of nature tomorrow. Today and at each state of nature tomorrow there is a market for physical commodities. Financial transfers across today and tomorrow and across the states of the world are allowed by means of a finite set J of financial assets that the agents can trade in today and whose returns are continuous functions of commodities prices. Agents face asymmetric "institutional" constraints on their portfolio trades.

The remainder of the chapter is organized as follows. In Section 4.2, we describe the financial exchange economy, state our results, and discuss their assumptions. Section 4.3 is devoted to the proof of our main existence theorem. Some proofs are gathered in the appendix.

4.2 The model and the main result

4.2.1 The model of a financial exchange economy

[1]Let us consider two time periods $t = 0$ and $t = 1$. In the second period, there is a nonempty finite set S of states of the nature. In period 0 and in each state of nature of

[1]We shall use hereafter the following notations. If I and J are finite sets, the space $\mathbb{R}^I$ (identified to $\mathbb{R}^{\#I}$ whenever necessary) of functions $x : I \to \mathbb{R}$ (also denoted $x = (x(i))_{i \in I}$ or $x = (x_i)$) is endowed with the scalar product $x \cdot y := \sum_{i \in I} x(i)y(i)$, and we denote by $\|x\| := \sqrt{x \cdot x}$ the Euclidean norm. By $B_L(x, r)$

the second period, there is a nonempty finite set L of divisible goods. We assume that the commodities are perishable which means that no storage is possible. For convenience, $s = 0$ denotes the state of the world (known with certainty) at period 0 and $\bar{S} = \{0\} \cup S$. The commodity space of the model is then $(\mathbb{R}^L)^{\bar{S}}$.

On such a stochastic structure, we consider a pure exchange economy with a nonempty finite set I of consumers. Each consumer is characterized by a consumption set $X_i \subset (\mathbb{R}^L)^{\bar{S}}$, a preference correspondence $P_i : \prod_{i \in I} X_i \to X_i$ and an endowment vector $e_i \in (\mathbb{R}^L)^{\bar{S}}$. For $x \in X$, $P_i(x)$ is interpreted as the set of consumption plans in X_i which are strictly preferred to x_i by consumer i, given the consumption plans $(x_{i'})_{i' \neq i}$ of the other agents.

We denote by $\mathcal{A}(\mathcal{E})$ the set of attainable allocations of the economy, that is

$$\mathcal{A}(\mathcal{E}) = \{(x_i)_{i \in I} \in \prod_{i \in I} X_i \mid \sum_{i \in I} x_i = \sum_{i \in I} e_i\},$$

and by $\widehat{X}_i$ the projection of $\mathcal{A}(\mathcal{E})$ on X_i. Note that for every $i \in I$, $e_i \in \widehat{X}_i$.

There is a finite set J of nominal assets. An *asset* j is a contract which promises to deliver in each state s of period $t = 1$ the payoff V_s^j, so that asset j is described by the vector $(V_s^j)_{s \in S}$. The matrix $V = \left(V_s^j\right)_{\substack{s \in S \\ j \in J}}$, which gives the financial returns, summarizes the financial asset structure.

Let us call *portfolio* an asset bundle $z \in \mathbb{R}^J$ with the convention :

if $z_j > 0$, z_j represents a quantity of asset j bought at period 0,

if $z_j < 0$, $|z_j|$ represents a quantity of asset j sold at period 0.

We assume that portfolios may be constrained, that is, each agent i has a portfolio set $Z_i \subset \mathbb{R}^J$ which describes the portfolios available for her. Then the definition of a financial exchange economy is the following.

Definition 4.1 *A financial exchange economy* $(\mathcal{E}, \mathcal{F})$ *is a collection*

$$\left((X_i, P_i, e_i)_{i \in I}, (V, (Z_i)_{i \in I})\right),$$

---

we denote the closed ball centered at $x \in \mathbb{R}^L$ of radius $r > 0$, namely $B(x, r) = \{y \in \mathbb{R}^L : \|y - x\| \leq r\}$. In $\mathbb{R}^I$, the notation $x \geq y$ (resp. $x > y$, $x \gg y$) means that, for every i, $x(i) \geq y(i)$ (resp. $x \geq y$ and $x \neq y$, $x(i) > y(i)$) and we let $\mathbb{R}^I_+ = \{x \in \mathbb{R}^I \mid x \geq 0\}$, $\mathbb{R}^I_{++} = \{x \in \mathbb{R}^I \mid x \gg 0\}$. An $I \times J$-matrix $A = (a_i^j)_{i \in I, j \in J}$ (identified with a classical $(\#I) \times (\#J)$-matrix if necessary) is an element of $\mathbb{R}^{I \times J}$ whose rows are denoted $A_i = (a_i^j)_{j \in J} \in \mathbb{R}^J$ $(i \in I)$, and columns $A^j = (a_i^j)_{i \in I} \in \mathbb{R}^I$ (for $j \in J$). The span of a family of vectors $F \subset \mathbb{R}^J$ in $\mathbb{R}^J$ is the linear subspace of $\mathbb{R}^J$, $< F >:= \{\sum_k \alpha_k x_k$, the sum is finite and for all $k, \alpha_k \in \mathbb{R}, x_k \in \mathbb{R}^J\}$.

where $\mathcal{E} = (X_i, P_i, e_i)_{i\in I}$ *and* $\mathcal{F} = (V, (Z_i)_{i\in I})$.

4.2.2 Financial equilibria

Given commodity and asset prices $(p, q) \in (\mathbb{R}^L)^{\bar{S}} \times \mathbb{R}^J$, the budget set of consumer i is

$$B_i(\mathcal{F}, p, q) = \left\{ (x_i, z_i) \in X_i \times Z_i \,\middle|\, \begin{array}{l} p(0)\cdot x_i(0) + q\cdot z_i \leq p(0)\cdot e_i(0) \\ p(s)\cdot x_i(s) \leq p(s)\cdot e_i(s) + V_s\cdot z_i, \;\; \forall s\in S \end{array} \right\}$$

where V_s denotes the row s of the matrix V. If we adopt the compact notations

- $p \,\square\, x_i$ denotes the vector $\big(p(s)\cdot x_i(s)\big)_{s\in\bar{S}}$ and
- $W(q)$ denotes the $\bar{S}\times J$ matrix $\begin{pmatrix} -q \\ V \end{pmatrix}$,

the budget set can be equivalently written as:

$$B_i(\mathcal{F}, p, q) = \{(x_i, z_i) \in X_i \times Z_i \mid p \,\square\, (x_i - e_i) \leq W(q) z_i\}.$$

Definition 4.2 *An equilibrium of the financial exchange economy* $(\mathcal{E}, \mathcal{F})$ *is a list* $\big(\bar{p}, \bar{q}, \bar{x}, \bar{z}\big) \in \mathbb{R}^{L(1+S)} \times \mathbb{R}^J \times (\mathbb{R}^{L(1+S)})^I \times (\mathbb{R}^J)^I$ *such that*

(i) *for each* i, $(\bar{x}_i, \bar{z}_i)$ *maximizes the preference* P_i *under the budget constraint, that is*

$$(\bar{x}_i, \bar{z}_i) \in B_i(\mathcal{F}, \bar{p}, \bar{q}) \text{ and } (P_i(\bar{x}) \times Z_i) \bigcap B_i(\mathcal{F}, \bar{p}, \bar{q}) = \emptyset.$$

(ii) $\sum_{i\in I} \bar{x}_i = \sum_{i\in I} e_i$ *and* $\sum_{i\in I} \bar{z}_i = 0$.

4.2.3 The main existence result

We make the following standard assumption on the consumption side.

Consumption Assumption C

(i) For every $i \in I$, X_i is a bounded below, closed, convex subset of $\mathbb{R}^{L(1+S)}$.

(ii) Continuity of Preferences For every $i \in I$, the correspondence $P_i : \Pi_i X_i \to X_i$ is lower semicontinuous with convex open values in X_i for the relative topology of X_i.

(iii) Irreflexive Preferences For every $i \in I$, for every $x = (x_i)_{i\in I} \in \Pi_i X_i$, $x_i \notin P_i(x)$.

(iv) Strong Survival SS For every $i \in I$, $e_i \in \text{int} X_i$.

(v) Non-Satiation NS For every $i \in I$, for every $x \in \Pi_i X_i$, for every $s \in \bar{S}$, there exists $x'_i \in P_i(x)$ such that $x'_i(s') = x_i(s')$ for all $s' \neq s$.

Definition 4.3 *The set of arbitrage-free prices of* $\mathcal{F} = \big(V, (Z_i)_{i\in I}\big)$ *is*

$$Q = \big\{q \in \mathbb{R}^J : W(q)(\bigcup_i \boldsymbol{A}Z_i) \cap \mathbb{R}^{\bar{S}}_+ = \{0\}\big\}.$$

where $\boldsymbol{A}Z_i$ *denotes the asymptotic cone of the set* Z_i*.*

Proposition 4.1 *Under* **NS***, if* $(\bar{p}, \bar{q}, \bar{x}, \bar{z})$ *is an equilibrium of the economy* $(\mathcal{E}, \mathcal{F})$*, then* $\bar{q}$ *is a no-arbitrage price*[2]*.*

We will make some of the following assumptions on the financial side. Given the financial structure $\mathcal{F} = \big(V, (Z_i)_{i\in I}\big)$, denote $Z(\mathcal{F}) =< \sum_{i\in I} Z_i >$ the linear space where financial activity takes place.

Assumption F

F1 For every $i \in I$, Z_i is closed, convex and $0 \in Z_i$.

F2 Closedness Assumption The following set $\mathcal{G}(\mathcal{F})$ is closed, where

$$\mathcal{G}(\mathcal{F}) := \{(Vz_1, \cdots, Vz_I, \sum_{i\in I} z_i) \in (\mathbb{R}^S)^I \times \mathbb{R}^J : \forall i \in I, z_i \in Z_i\}.$$

F2' Weak Positive Semi-Independence[3] **WPSI:** The sets $\boldsymbol{A}Z_i \cap \{V \geq 0\}$ are weakly positively semi-independent.

F3 FSSA For every $q \in \big(Q \cap Z(\mathcal{F})\big) \setminus \{0\}$, for every $i \in I$ there exists a portfolio $\zeta_i \in Z_i$ such that $q \cdot \zeta_i < 0$.

Assumption **F1** is straightforward and both Assumptions **F2** and **F3** are discussed in the next section.

We can now state the main result of this paper.

[2]**Proof.** Assume that, for some $i \in I$, there exists a portfolio $v_i \in \boldsymbol{A}Z_i$ such that $W(\bar{q})v_i > 0$, namely $[W(\bar{q})v_i](s) \geq 0$, for every $s \in \bar{S}$, with at least one strict inequality, say for $\bar{s} \in \bar{S}$.
Since $\sum_{i\in I}(\bar{x}_i - e_i) = 0$, from Assumption **(NS)**, there exists $x \in \prod_{i\in I} X_i$ such that, for each $s \neq \bar{s}$, $x_i(s) = \bar{x}_i(s)$ and $x_i \in P_i(\bar{x})$.
For $t > 0$ large enough, $\bar{p} \,\square\, (x_i - e_i) \leq W(\bar{q})(\bar{z}_i + t\, v_i)$. Since $\bar{z}_i + t\, v_i \in Z_i$, we get $(x_i, \bar{z}_i + t\, v_i) \in B_i(\bar{p}, \bar{q})$ but since $x_i \in P_i(\bar{x})$, this contradicts the optimality of $(\bar{x}_i, \bar{z}_i)$ in $B_i(\bar{p}, \bar{q})$. ∎

[3]A collection $\{C_i, i \in I\}$ of nonempty convex cones in $\mathbb{R}^\alpha$ is weakly positively semi independent if $c_i \in C_i$, for all $i \in I$ and $\sum_{i\in I} c_i = 0$, implies that for all $i \in I$, $c_i \in \boldsymbol{L}(C_i)$.

Theorem 4.1 *Let* $(\mathcal{E},\mathcal{F}) = \big((X_i, P_i, e_i)_{i\in I}, (V, (Z_i)_{i\in I})\big)$ *be a financial exchange economy satisfying assumptions* **C**, **F1**, **F2'** *and* **F3**, *then it admits an equilibrium* $(\bar{p}, \bar{q}, \bar{x}, \bar{z})$.

Theorem 4.2 *Let* $(\mathcal{E},\mathcal{F}) = \big((X_i, P_i, e_i)_{i\in I}, (V, (Z_i)_{i\in I})\big)$ *be a financial exchange economy satisfying assumptions* **C**, **F1**, **F2** *and* **F3**, *then it admits an equilibrium* $(\bar{p}, \bar{q}, \bar{x}, \bar{z})$.

Remark 4.1 Under **NS** $\bar{q} \in Q$ (by Proposition 4.1) and $\bar{p}(s) \neq 0$ for every $s \in \bar{S}$.

Remark 4.2 We can choose the equilibrium asset price $\bar{q}$ to be in $Q(\mathcal{F}) \cap Z(\mathcal{F})$. Indeed, if $q^* = \text{proj}_{Z(\mathcal{F})}\bar{q}$ then $(\bar{p}, q^*, \bar{x}, \bar{z})$ is an equilibrium of $(\mathcal{E},\mathcal{F})$ since for every $i \in I$, and for every $z_i \in Z_i$, one has $q^* \cdot z_i = \bar{q} \cdot z_i$. Moreover, $q^* \in Q(\mathcal{F})$ under **NS** by Proposition 4.1.

Remark 4.3 Under the assumptions of Theorem 4.2, the equilibrium asset price vector may be zero, that is, we may have $\bar{q} = 0$ at equilibrium. A necessary and sufficient condition guaranteeing that $\bar{q} \neq 0$ is

$$\exists i \in I, \exists v_i \in \boldsymbol{A}Z_i, Vv_i > 0.$$

Indeed, under this assumption, $0 \notin Q$ and under the non-satiation assumption **NS**, $\bar{q} \in Q$, hence $\bar{q} \neq 0$.

4.2.4 Discussion of the Assumptions of Theorem 4.2

Discussion of Assumption F3

Remark 4.4 Condition **F2** can be equivalently written

$$-\text{cl}Q \cap \Big(\cup_i N_{Z_i}(0)\Big) \subset \{0\},$$

where $N_Z(0)$ is the normal cone to the convex Z at 0, that is

$$N_Z(0) := \{\alpha \in \mathbb{R}^J : \alpha \cdot z \leq 0, \forall z \in Z\}.$$

Remark 4.5 If for every $i \in I$, $0 \in \text{int}Z_i$, then **F3** is fulfilled.

Sufficient conditions for the closedness Assumption F2

As shown by the following Propositions 4.2 and 4.3, assumption **F2** holds true in many situations. Indeed, **F2** is fulfilled when the restrictions on portfolio choices are given by a finite number of linear inequalities, that is, when all portfolios sets are finite intersections of half spaces. In particular, **F2** is fulfilled when the portfolios sets are linear subspaces, when the portfolio sets are unconstrained, or when the portfolio sets are bounded from below. Furthermore, assumption **F2** holds true under the no mutually compatible potential arbitrage condition (Page [12]) that is when the family $\{\boldsymbol{A}Z_i \cap \ker V, i \in I\}$ is positively semi-independent[4] (Siconolfi [16]), in particular **F2** holds true when the portfolio sets are bounded, or when there are no redundant assets i.e. $\text{rank}(V) = J$. The proofs of Proposition 4.2 and Proposition 4.3 are in [3].

Proposition 4.2 *Assumption* **F2** *holds true under anyone of the following conditions.*

(a) *For all $i \in I$, $Z_i = \mathbb{R}^J$ (unconstrained portfolios).*

(b) *For all $i \in I$, Z_i is a linear subspace.*

(c) *For all $i \in I$, $Z_i = \underline{z}_i + \mathbb{R}^J_+$, for some $\underline{z}_i \in -\mathbb{R}^J_+$ (exogenous bounds on short sales).*

(d) *For all $i \in I$, Z_i is polyhedral.*

(e) *For all $i \in I$, $Z_i = B_J(0,1)$ (bounded portfolio sets).*

(f) *For all $i \in I$, $Z_i = K_i + P_i$ where K_i is nonempty compact and convex, and P_i is polyhedral.*

Proposition 4.3 *Assumption* **F2** *holds true under anyone of the following conditions.*

(g) *There are no redundant assets i.e. $Rank(V) = J$, or equivalently, $\ker V = \{0\}$.*

(h) $\forall i$, $\boldsymbol{A}Z_i \cap \ker V = \{0\}$.

(i1) $\boldsymbol{L}\Big(\sum_{i\in I}(Z_i \cap \{V \geq 0\})\Big) = \{0\}$.

(i2) $\boldsymbol{L}\Big(\sum_{i\in I}(Z_i \cap \ker V)\Big) = \{0\}$.

(i3) $\sum_{i\in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\}) \cap -\sum_{i\in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\}) = \{0\}$.

(i4) $\sum_{i\in I}(\boldsymbol{A}Z_i \cap \ker V) \cap -\sum_{i\in I}(\boldsymbol{A}Z_i \cap \ker V) = \{0\}$.

(j1) *The family $\{\boldsymbol{A}Z_i \cap \{V \geq 0\} : i \in I\}$ is positively semi-independent*[5].

[4] A collection $\{C_i, i \in I\}$ of nonempty convex sets in $\mathbb{R}^\alpha$ is positively semi-independent if $c_i \in C_i$, for all $i \in I$ and $\sum_{i\in I} c_i = 0$, implies that $c_i = 0$ for all $i \in I$.

[5] A collection $\{C_i, i \in I\}$ of nonempty convex cones in $\mathbb{R}^\alpha$ is positively semi-independent, (respectively weakly positively semi independent) if $v_i \in C_i$, for all $i \in I$ and $\sum_{i\in I} v_i = 0$, implies that for all $i \in I$ $v_i = 0$ (resp. $v_i \in \boldsymbol{L}(C_i)$).

(j2) *The family* $\{\boldsymbol{A}Z_i \cap \ker V : i \in I\}$ *is positively semi-independent.*

(k1) *The family* $\{\boldsymbol{A}Z_i \cap \{V \geq 0\}, i \in I\}$ *is weakly positively semi-independent.*

(k2) *The family* $\{\boldsymbol{A}Z_i \cap \ker V, i \in I\}$ *is weakly positively semi-independent.*

4.2.5 Some consequences of the existence result

Many results in the literature are now corollaries to Theorem 4.2.

Corollary 4.1 (Radner 1972 [14]) *The financial exchange economy* $(\mathcal{E}, \mathcal{F})$ *admits an equilibrium if it satisfies assumption* **C** *and*

F2' *For every* $i \in I$*,* Z_i *is the closed ball* $\text{cl}B(0, r_i)$*, for some* $r_i > 0$*.*

Corollary 4.2 (Radner (bis) 1972 [14]) *The financial exchange economy* $(\mathcal{E}, \mathcal{F})$ *admits an equilibrium if it satisfies assumption* **C** *and*

F2' *For every* $i \in I$*,* $Z_i = \{z \in \mathbb{R}^J, z \geq -\underline{z_i}\}$*, for some* $\underline{z_i} \gg 0$*.*

Corollary 4.3 *The financial exchange economy* $(\mathcal{E}, \mathcal{F})$ *admits an equilibrium if it satisfies assumptions* **C***,* **F1***,* **F3***, and*

F2' $\ker V = \{0\}$.

Corollary 4.4 (Siconolfi 1987 [16]) *The financial exchange economy* $(\mathcal{E}, \mathcal{F})$ *admits an equilibrium if it satisfies assumptions* **C***,* **F1***,* **F3** *and*

F2' *For every* $i \in I$*,* $\boldsymbol{A}Z_i \cap \ker V = \{0\}$*.*

Corollary 4.5 *The financial exchange economy* $(\mathcal{E}, \mathcal{F})$ *admits an equilibrium if it satisfies assumptions* **C***,* **F1***,* **F3***, and*

F2' *The family* $\{\boldsymbol{A}Z_i \cap \ker V, i \in I\}$ *is positively semi independent.*

Corollary 4.6 *The financial exchange economy* $(\mathcal{E}, \mathcal{F})$ *admits an equilibrium if it satisfies assumption* **C** *together with*

F2' *For every* i *in* I*,* Z_i *is a linear subspace of* $\mathbb{R}^J$ *and,*

F3' $-\mathrm{cl}Q \cap \left(\cup_i Z_i^{\perp}\right) = \{0\}$.

Corollary 4.7 *The financial exchange economy $(\mathcal{E}, \mathcal{F})$ admits an equilibrium if it satisfies assumption* **C** *together with*

F2' *For every $i \in I$, $Z_i = K_i + P_i$ where K_i is nonempty compact and convex and P_i is polyhedral and,*

F3' $-\mathrm{cl}Q \cap \left(\cup_i Z_i^{o}\right) \subset \{0\}$.

4.3 Proof of the Theorem 4.1

The proof will consist in two major steps. First, we prove the existence of a financial equilibrium when the economy $(\mathcal{E}, \mathcal{F})$ satisfies some additional assumptions. We shall use the Fixed-Point Theorem of Gale and Mas-Colell [8].

Second, we show how to transform the initial financial economy into an economy satisfying the additional assumptions and that from every financial equilibrium of the transformed financial economy one can "construct" a financial equilibrium of the original financial economy $(\mathcal{E}, \mathcal{F})$.

We make the following assumption.

Local Non Satiation LNS: For every $\bar{x} \in \prod_{i\in I} X_i$, for every $x_i \in P_i(\bar{x})$, $(\bar{x}_i, x_i] \subset P_i(\bar{x})$.

Theorem 4.1 will be proved as a consequence of the following Theorem 4.3 in which the financial economy $(\mathcal{E}, \mathcal{F})$ satisfies the additional assumption **LNS**.

Theorem 4.3 *Let $(\mathcal{E}, \mathcal{F}) = \big((X_i, P_i, e_i)_{i\in I}, (V, (Z_i)_{i\in I})\big)$ be a financial exchange economy satisfying Assumptions* **C**, **LNS**, **F1**, **F2'**, *and* **F3**, *then it admits an equilibrium.*

4.3.1 Preliminary Results

Lemma 4.1 *The set Q is a convex cone with vertex 0.*

Proof. The set Q is obviously a cone, and we now show that Q is convex. Indeed, let $q_1, q_2 \in Q$ and let $\alpha \in (0,1)$. Assume $\alpha q_1 + (1-\alpha) q_2 \notin Q$. Then there exists $v \in C$ such that $W(\alpha q_1 + (1-\alpha)q_2)v > 0$. Hence

$$\text{either} \begin{cases} -(\alpha q_1 + (1-\alpha)q_2) \cdot v > 0 \\ Vv \geq 0 \end{cases} \quad \text{or} \quad \begin{cases} -(\alpha q_1 + (1-\alpha)q_2) \cdot v \geq 0 \\ Vv > 0 \end{cases}$$

In the first case, we conclude that either $-q_1 \cdot v > 0$ or $-q_2 \cdot v > 0$ which, together with $Vv \geq 0$, implies that $W(q_i)v > 0$ for $i = 1$ or $i = 2$ contradicting the fact that q_1 and q_2 are both in Q. Similarly, in the second case, we conclude that either $-q_1 \cdot v \geq 0$ or $-q_2 \cdot v \geq 0$ which, together with $Vv > 0$, contradicts the fact that q_1 and q_2 are both in Q. ∎

Lemma 4.2 *Under Assumption* **WPSI**, $-Q^o = \sum_{i \in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\})$.

Proof. See [2]. ∎

4.3.2 Proof of Theorem 4.3

Transforming the economy

In the following we let

$$\begin{aligned} \boldsymbol{L} &= \sum_{i \in I}\Big(\boldsymbol{L}(Z_i) \cap \ker V\Big), \\ \pi &= \text{proj}_{\boldsymbol{L}^\perp}, \\ \mathcal{F}_\pi &= \Big(V, (\text{cl}\pi Z_i)_i\Big). \end{aligned}$$

Remark 4.6 Notice that $\text{cl}Q \subset \boldsymbol{L}^\perp$. Indeed, from Lemma 4.2 and the Bipolar Theorem,

$$-\text{cl}Q = \Big(\sum_{i \in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\})\Big)^o \subset \Big(\sum_{i \in I}\boldsymbol{L}(Z_i) \cap \ker V\Big)^o = \Big(\sum_{i \in I}\boldsymbol{L}(Z_i) \cap \ker V\Big)^\perp = \boldsymbol{L}^\perp.$$

Note that this implies $\text{cl}Q \subset (\boldsymbol{L}(Z_i) \cap \ker V)^\perp$ for every i since $\boldsymbol{L}^\perp = \bigcap_i (\boldsymbol{L}(Z_i) \cap \ker V)^\perp$.

In view of the following Theorem 4.4, Theorem 4.3 will be proven if we show that the economy $(\mathcal{E}, \mathcal{F}_\pi)$ has an equilibrium.

Theorem 4.4 *Assume* **NS**, **LNS**, *and* **F1**. *Under* **WPSI**, *if* $(\mathcal{E}, \mathcal{F}_\pi)$ *has an equilibrium* $(\bar{p}, \bar{q}, \bar{x}, \bar{z})$, *then there exists* $z^* \in \Pi_i Z_i$ *such that* $(\bar{p}, \pi\bar{q}, \bar{x}, z^*)$ *is an equilibrium of* $(\mathcal{E}, \mathcal{F})$.

We prepare the proof of Theorem 4.4 by some claims.

Claim 4.3.1 *If the sets* $\boldsymbol{A}Z_i \cap \ker V$ *satisfy* **WPSI**, *then for every* $i \in I$,

(i) $\boldsymbol{A}Z_i \cap \ker \pi \subset \boldsymbol{L}(Z_i)$,

(ii) $\mathrm{cl}\pi Z_i = \pi Z_i$, *and*

(iii) $\boldsymbol{A}\pi Z_i = \pi \boldsymbol{A} Z_i$.

Proof. (i) Let $\zeta_1 \in \boldsymbol{A}Z_1 \cap \ker \pi = \boldsymbol{A}Z_1 \cap \sum_{i \in I} \big(\boldsymbol{L}(Z_i) \cap \ker V\big)$. Then the vector ζ_1 belongs to $\ker V$ and can be written $\zeta_1 = \sum_{i \in I} \xi_i$, where $\xi_i \in \boldsymbol{L}(Z_i) \cap \ker V$ for each $i \in s$. Thus $0 = (\zeta_1 - \xi_1) + (-\xi_2) + \cdots + (-\xi_I)$ with $\zeta_1 - \xi_1 \in \boldsymbol{A}Z_1 \cap \ker V$ and $-\xi_i \in \boldsymbol{A}Z_i \cap \ker V$ for $i \geq 2$, therefore by weak positive semi-independence $\zeta_1 - \xi_1 \in \boldsymbol{L}(Z_1)$ and the fact that ξ_1 is already in $\boldsymbol{L}(Z_1)$ implies that ζ_1 belongs to $\boldsymbol{L}(Z_1)$.

(ii) and (iii): These two properties are immediate consequences of Theorem 4.5. ■

Claim 4.3.2 *For every $(y_i)_i \in \Pi_{i \in I} \mathrm{cl}\pi Z_i$ such that $\sum_{i \in I} y_i = 0$ there exists $(z_i^*)_i \in \Pi_{i \in I} Z_i$ such that*

(a) $V z_i^* = V y_i$ *for every $i \in I$, and*

(b) $\sum_{i \in I} z_i^* = 0$.

Proof. Let $y := (y_i)_i \in \Pi_i \mathrm{cl}\pi Z_i = \Pi_i \pi Z_i$ (we have the equality from Claim 4.3.1) be such that $\sum_{i \in I} y_i = 0$. For each i, let $z_i \in Z_i$ be such that $y_i = \pi z_i$. Then $\sum z_i \in \ker \pi = \sum_{i \in I} \Big(\boldsymbol{L}(Z_i) \cap \ker V\Big)$. Then $\sum_{i \in I} z_i = \sum_{i \in I} \ell_i$ with $\ell_i \in \boldsymbol{L}(Z_i) \cap \ker V$ for each i. Denote $z_i^* = z_i - \ell_i$. Hence $z_i^* \in Z_i$ for each $i \in I$, $\sum_{i \in I} z_i^* = 0$, and $V z_i^* = V z_i = V \pi z_i$ (since $\boldsymbol{L}(Z_i) \subset \ker \pi \subset \ker V$), therefore $V z_i^* = V y_i$. ■

Claim 4.3.3 *Under* **NS** *and* **LNS**, *if $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an equilibrium of the financial exchange economy $(\mathcal{E}, \mathcal{F})$, then for every $i \in I$, there is no $z_i \in Z_i$ such that $W(\bar{q}) z_i > W(\bar{q}) \bar{z}_i$.*

Proof. By contradiction. Assume that for some $i \in I$, there exists $z_i \in Z_i$ such that $W(\bar{q}) z_i > W(\bar{q}) \bar{z}_i$, namely $[W(\bar{q}) z_i](s) \geq [W(\bar{q}) \bar{z}_i](s)$, for every $s \in \bar{S}$, with at least one strict inequality, say for $\bar{s} \in \bar{S}$. Then, since $\sum_{i \in I} (\bar{x}_i - e_i) = 0$, from Assumption **NS**, there exists $x \in \prod_{i \in I} X_i$ such that, for each $s \neq \bar{s}$, $x_i(s) = \bar{x}_i(s)$ and $x_i \in P_i(\bar{x})$. Consider $\lambda \in (0, 1)$ and define $x_i^\lambda := \lambda x_i + (1 - \lambda) \bar{x}_i$. Then, by Assumption **LNS**, $x_i^\lambda \in (x_i, \bar{x}_i) \subset P_i(\bar{x})$. Now, we claim that for $\lambda > 0$ small enough, $(x_i^\lambda, z_i) \in B_i(\mathcal{F}, \bar{p}, \bar{q})$, which contradicts the fact that $[P_i(\bar{x}) \times Z_i] \cap B_i(\mathcal{F}, \bar{p}, \bar{q}) = \emptyset$ (since $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an equilibrium). Indeed, since $(\bar{x}_i, \bar{z}_i) \in B_i(\mathcal{F}, \bar{p}, \bar{q})$, and for every $s \neq \bar{s}$, $x_i^\lambda(s) = \bar{x}_i(s)$ we have

$$\bar{p}(s) \cdot [x_i^\lambda(s) - e_i(s)] = \bar{p}(s) \cdot [\bar{x}_i(s) - e_i(s)] \leq [W(\bar{q}) \bar{z}_i](s) \leq [W(\bar{q}) z_i](s).$$

Now, for $s = \bar{s}$, we have

$$\bar{p}(\bar{s}) \cdot [\bar{x}_i(\bar{s}) - e_i(\bar{s})] \le [W(\bar{q})\bar{z}_i](\bar{s}) < [W(\bar{q})z_i](\bar{s}).$$

But, when $\lambda \to 0$, $x_i^\lambda \to \bar{x}_i$, hence for $\lambda > 0$ small enough we have

$$\bar{p}(\bar{s}) \cdot [x_i^\lambda(\bar{s}) - e_i(\bar{s})] < [W(\bar{q})z_i](\bar{s}).$$

Consequently, $(x_i^\lambda, z_i) \in B_i(\mathcal{F}, \bar{p}, \bar{q})$. ■

Proof of Theorem 4.4

By Claim 4.3.2, for every i, there exists $z_i^* \in Z_i$ such that $Vz_i^* = V\bar{z}_i$ and $\sum z_i^* = 0$. We show that $(\bar{p}, \pi\bar{q}, \bar{x}, z^*)$ is an equilibrium of $(\mathcal{E}, \mathcal{F})$.

First, we show that for all i, $\pi\bar{q} \cdot z_i^* = \bar{q} \cdot \bar{z}_i$ (which, together with $Vz_i^* = V\bar{z}_i$, implies that $(\bar{x}_i, z_i^*) \in \mathcal{B}_i(\mathcal{F}, \bar{p}, \pi\bar{q})$ since $(\bar{x}_i, \bar{z}_i) \in \mathcal{B}_i(\mathcal{F}_\pi, \bar{p}, \bar{q})$). By Claim 4.3.3, from **NS**, **LNS**, and the fact that $(\bar{p}, \bar{q}, \bar{x}, \bar{z})$ is an equilibrium of $(\mathcal{E}, \mathcal{F}_\pi)$, we deduce that for each $i \in I$, there does not exist $\hat{z}_i \in \mathrm{cl}(\pi Z_i)$ such that $W(\bar{q})\hat{z}_i > W(\bar{q})\bar{z}_i$. This implies that there does not exist $\hat{z}_i \in \mathrm{cl}(\pi Z_i)$ such that $W(\pi\bar{q})\hat{z}_i > W(\pi\bar{q})\bar{z}_i$ (since $\pi\bar{q} \cdot z = \bar{q} \cdot z$ for every $z \in \mathrm{Im}\pi$). Hence for all i, there does not exist $z_i \in Z_i$ such that $W(\pi\bar{q})z_i > W(\pi\bar{q})\bar{z}_i$, otherwise $W(\pi\bar{q})z_i = W(\pi\bar{q})\pi z_i > W(\pi\bar{q})\bar{z}_i$ (since $z_i - \pi z_i \in \ker\pi$, $\pi\bar{q} \in \mathrm{Im}\pi = (\ker\pi)^\perp$ and $\ker\pi = \sum_{i\in I}\Big(\boldsymbol{L}(Z_i) \cap \ker V\Big) \subset \ker V$). In particular for all i, $W(\pi\bar{q})z_i^* \not> W(\pi\bar{q})\bar{z}_i$. Taking into account the fact that for all i, $Vz_i^* = V\bar{z}_i$, we deduce that for all i, $-\pi\bar{q} \cdot z_i^* \not> -\pi\bar{q} \cdot \bar{z}_i$, that is $\pi\bar{q} \cdot (z_i^* - \bar{z}_i) \ge 0$. Recalling that $\sum_{i\in I}(z_i^* - \bar{z}_i) = 0$, we deduce that for all i, $\pi\bar{q} \cdot z_i^* = \pi\bar{q} \cdot \bar{z}_i$.

To complete the proof, we need only show that for each $i \in I$,

$$\mathcal{B}_i(\mathcal{F}, \bar{p}, \pi\bar{q}) \cap [P_i(\bar{x}) \times Z_i] = \emptyset.$$

But $(\bar{p}, \bar{q}, \bar{x}, \bar{z})$ is an equilibrium of $(\mathcal{E}, \mathcal{F}_\pi)$, hence

$$\mathcal{B}_i(\mathcal{F}_\pi, \bar{p}, \bar{q}) \cap [P_i(\bar{x}) \times \mathrm{cl}\pi Z_i] = \emptyset.$$

In view of the above, the proof will be completed if we show that if $(x_i, z_i) \in \mathcal{B}_i(\mathcal{F}, \bar{p}, \pi\bar{q})$, then $(x_i, \pi z_i) \in \mathcal{B}_i(\mathcal{F}_\pi, \bar{p}, \bar{q})$, which is true if $W(\pi\bar{q})z_i \le W(\bar{q})\pi z_i$. Recalling that for every i, $Vz_i = V\pi z_i$ (since $z_i - \pi z_i \in \ker\pi = \sum_{i\in I}\Big(\boldsymbol{L}(Z_i) \cap \ker V\Big) \subset \ker V$), we only need to show that $\pi\bar{q} \cdot z_i = \bar{q} \cdot \pi z_i$. Since $\pi\bar{q} \in \mathrm{Im}\pi = (\ker\pi)^\perp$, we have $\pi\bar{q} \cdot z_i = \pi\bar{q} \cdot \pi z_i$. Since $\bar{q} - \pi\bar{q} \in (\mathrm{Im}\pi)^\perp$, we have $\pi\bar{q} \cdot \pi z_i = \bar{q} \cdot \pi z_i$. Hence $\pi\bar{q} \cdot z_i = \bar{q} \cdot \pi z_i$. ■

To end the proof of Theorem 4.3, we need to show that the financial exchange economy

$(\mathcal{E}, \mathcal{F}_\pi)$ admits an equilibrium.

Truncating the economy

Let $\Pi = \{(p,q) \in (\mathbb{R}^L)^{\bar{S}} \times \mathbb{R}^J \mid \forall s \in \bar{S},\ \|p(s)\| \leq 1, q \in \text{cl}Q \cap Z(\mathcal{F}) \text{ and } \|q\| \leq 1\}$ be the set of admissible prices for commodities and assets.

Lemma 4.3 *For all $v = (v_i)_i \in (\mathbb{R}^S)^I$, if the sets $\boldsymbol{A}Z_i \cap \{V \geq 0\})$ satisfy* **WPSI***, then the set K_v defined by*

$$K_v := \{(\pi z_1, \cdots, \pi z_I) \in \Pi_i \pi Z_i : \forall i, z_i \in Z_i, V z_i \geq v_i, \sum_{i \in I} z_i = 0\}$$

is bounded.

Proof. If $(\pi\zeta_1, \cdots, \pi\zeta_I) \in \boldsymbol{A}K_v$ with $\zeta_i \in \boldsymbol{A}Z_i \cap \{V \geq 0\}$, and $\sum_{i \in I} \zeta_i = 0$, then from **WPSI** we get $\zeta_i \in \boldsymbol{L}(Z_i) \cap \ker V \subset \boldsymbol{L}$ for every $i \in I$. That is $\pi\zeta_i = 0, \forall i$. Hence $\boldsymbol{A}K_v = \{0\}$ and K_v is bounded (see [15]). ■

For $i \in I$, let $\underline{v}_i \in \mathbb{R}^S$ be defined by

$$\text{for every } s \in S,\ \underline{v}_i(s) = \inf\{p \cdot (x_i(s) - e_i(s)) - 1, p \in B_L(0,1), x_i \in \widehat{X}_i\}. \qquad (4.3.1)$$

The existence of $\underline{v}_i$ follows from Assumption **C(i)** and from the compactness of $B_L(0,1)$. We denote by $\widehat{Z}_i$ the projection of $K_{\underline{v}}$ on the i-th component.

It follows from Assumption **C(i)** and from Lemma 4.3 that the sets $\mathcal{A}(\mathcal{E})$ and $K_{\underline{v}}$ are compact. Hence the sets $\widehat{X}_i$ and $\widehat{Z}_i$ are bounded, for every $i \in I$. Consequently, one can choose $r > 0$ large enough such that

$$\widehat{X}_i \subset \text{int} B_{L\bar{S}}(0,r) \text{ and } \widehat{Z}_i \subset \text{int} B_J(0,r) \text{ for every } i \in I.$$

We let for every $i \in I$,

$$\begin{aligned} X_i^r &= X_i \cap B_{L\bar{S}}(0,r), \\ P_i^r(x) &= P_i(x) \cap \text{int} B_{L\bar{S}}(0,r), \text{ and} \\ Z_i^r &= \text{cl}\pi Z_i \cap B_J(0,r), \end{aligned}$$

and we define a new financial economy $(\mathcal{E}^r, \mathcal{F}_\pi^r)$ where the consumption sets are X_i^r, the preference correspondences are P_i^r, and the portfolio sets are Z_i^r. To summarize, we let

$$(\mathcal{E}^r, \mathcal{F}_\pi^r) := \Big(\big(X_i^r, P_i^r, e_i\big)_{i \in I}, \big(V, (Z_i^r)_{i \in I}\big) \Big).$$

Note that, for every $i \in I$, $e_i \in \widehat{X}_i$ hence from Assumption **C(iv)**, $e_i \in \text{int} X_i^r$.

Definition of correspondences and the fixed-point argument

Given $(p,q) \in \Pi$, following ideas originating from Bergstrom ([5]), we define the "modified" budget sets of consumer i as follows:

$$\begin{aligned} B_i^{r\varepsilon}(p,q) &= \{(x_i,z_i) \in X_i^r \times Z_i^r, p \,\square\, (x_i - e_i) \leq W(p,q)z_i + \varepsilon(p,q)\}, \\ \breve{B}_i^{r\varepsilon}(p,q) &= \{(x_i,z_i) \in X_i^r \times Z_i^r, p \,\square\, (x_i - e_i) \ll W(p,q)z_i + \varepsilon(p,q)\}. \end{aligned}$$

where $\varepsilon(p,q) \in \mathbb{R}^{\bar{S}}$ is defined by

$$\begin{aligned} \varepsilon_0(p,q) &= 1 - \min\{1, \|p(0)\| + \|q\|\} \\ \varepsilon_s(p,q) &= 1 - \|p(s)\|, \; s \in S. \end{aligned}$$

Denote $\Pi' = \Big\{(p,q) \in \Pi : q \in (Q \cap Z(\mathcal{F})) \cup \{0\}\Big\}$.

Claim 4.3.4 *For all $(p,q) \in \Pi'$, $\breve{B}_i^{r\varepsilon}(p,q) \neq \emptyset$ and $B_i^{r\varepsilon} = \mathrm{cl}\breve{B}_i^{r\varepsilon}$. Moreover, for all $(p,q) \in \Pi$, $B_i^{r\varepsilon}(p,q) \neq \emptyset$.*

Proof. Let $(p,q) \in \Pi'$. Since $e_i \in \mathrm{int}X_i$, there exists $x_i \in X_i^r$ such that $p \,\square\, (x_i - e_i) \leq 0$ with a strict inequality at each state $s \in \bar{S}$ such that $p(s) \neq 0$. Now, if $p(0) \neq 0$ or $q = 0$, $(x_i, 0) \in \breve{B}_i^{r\varepsilon}(p,q)$. If $p(0) = 0$ and $q \neq 0$, we claim that there exists $y_i \in \pi Z_i$ such that $q \cdot y_i < 0$. Indeed, by Assumption **F2**, for every $i \in I$ there exists $z_i \in Z_i$ such that $q \cdot z_i < 0$, hence the vector $y_i = \pi z_i \in \pi Z_i$ and satisfies $q \cdot y_i = q \cdot z_i < 0$ because $q \in \boldsymbol{L}^{\perp} = \mathrm{Im}\pi = (\ker \pi)^{\perp}$ by Remark 4.6. Now, recalling that $p(s) \cdot \big(x_i(s) - e_i(s)\big) - \varepsilon_s(p,q) < 0$ for all $s \in S$, we can choose $z \in Z_i^r$ such that $q \cdot z < 0$ and $V_s \cdot z > p(s) \cdot \big(x_i(s) - e_i(s)\big) - \varepsilon_s(p,q)$ for all $s \in S$ (take $z = ty_i$ for $t > 0$ small enough). Then, $(x_i, z) \in \breve{B}_i^{r\varepsilon}(p,q)$.

The last assertion of the claim follows from the fact that $(e_i, 0) \in B_i^{r\varepsilon}(p,q)$ for every $(p,q) \in \Pi$. ■

Claim 4.3.5 *For all $i \in I$, $B_i^{r\varepsilon}$ is lower semicontinuous on Π' and upper semicontinuous on Π with closed convex values.*

Proof. From Claim 4.3.4, $B_i^{r\varepsilon}$ is the closure of $\breve{B}_i^{r\varepsilon}$ on Π'. We then notice that $\breve{B}_i^{r\varepsilon}$ has an open graph hence is lower semicontinuous. Consequently $B_i^{r\varepsilon}$ which is the closure of a lower semicontinuous correspondence is also lower semicontinuous. Furthermore, $B_i^{r\varepsilon}$ has a closed graph with convex values in the compact convex set $X_i^r \times Z_i^r$. ■

We now introduce an additional agent and, as in Gale and Mas-Colell ([8], [9]), we set the following reaction correspondences defined on $\Pi \times \prod_{i\in I} X_i^r \times Z_i^r$.

$$\psi_i(p,q,x,z) = \begin{cases} B_i^{r\varepsilon}(p,q) \text{ if } (x_i,z_i) \notin B_i^{r\varepsilon}(p,q) \\ \breve{B}_i^{r\varepsilon}(p,q) \cap (P_i^r(x) \times Z_i^r) \text{ if } (x_i,z_i) \in B_i^{r\varepsilon}(p,q) \end{cases}$$

$$\psi_0(p,q,x,z) = \{(p',q') \in \Pi \mid (p'-p)\cdot \sum_{i\in I}(x_i - e_i) + (q'-q)\cdot \sum_{i\in I} z_i > 0\}.$$

Claim 4.3.6 *The correspondence ψ_0 is lower semicontinuous with convex values on $\Pi \times \prod_{i\in I} X_i^r \times Z_i^r$ and for all $i \in I$, ψ_i is lower semicontinuous with convex values on $\Pi' \times \prod_{i\in I} X_i^r \times Z_i^r$.*

Proof. The correspondence ψ_0 has an open graph thus it is lower semicontinuous and one easily checks that it has convex values. If $i \neq 0$, it follows from the lower semicontinuity and the upper semicontinuity of $B_i^{r\varepsilon}$ that ψ_i is lower semicontinuous at (p,q,x,z) if $(x_i,z_i) \notin B_i^{r\varepsilon}(p,q)$ since $\psi_i = B_i^{r\varepsilon}$ on a neighborhood of (x_i,z_i) which does not intersect the graph of $B_i^{r\varepsilon}$. If $(x_i,z_i) \in B_i^{r\varepsilon}(p,q)$, note that $\breve{B}_i^{r\varepsilon} \cap (P_i^r \times Z_i^r)$ is lower semicontinuous since $\breve{B}_i^{r\varepsilon}$ has an open graph and $P_i^r \times Z_i^r$ is lower semicontinuous. Thus, ψ_i is lower semicontinuous at (p,q,x,z) since $\breve{B}_i^{r\varepsilon}(p,q) \subset B_i^{r\varepsilon}(p,q)$ which clearly implies that $\psi_i(p,q,x,z) \subset B_i^{r\varepsilon}(p,q)$. The convexity of the values of ψ_i is a consequence of the convexity of $\breve{B}_i^{r\varepsilon}(p,q)$, $B_i^{r\varepsilon}(p,q)$, Z_i^r and $P_i^r(x)$. ∎

Now, fix $q_o \in \mathrm{ri}Q \cap Z(\mathcal{F}) \cap B_J(0,1)$, and for $i \in I$ and $n > 0$, define the correspondences $B_i^{r\varepsilon n}$, and $\breve{B}_i^{r\varepsilon n}$ on Π by

$$B_i^{r\varepsilon n}(p,q) = B_i^{r\varepsilon}\big((p,(1-\frac{1}{n})q + \frac{1}{n}q_o\big),$$
$$\breve{B}_i^{r\varepsilon n}(p,q) = \breve{B}_i^{r\varepsilon}\big(p,(1-\frac{1}{n})q + \frac{1}{n}q_o\big),$$

and for $i \in I \cup \{0\}$, define the correspondences ψ_i^n on $\Pi \times \prod_{i\in I} X_i^r \times Z_i^r$ by

$$\psi_i^n(p,q,x,z) = \psi_i\big(p,(1-\frac{1}{n})q + \frac{1}{n}q_o, x, z\big),$$
$$\psi_0^n = \psi_0.$$

Note that for every $q \in \mathrm{cl}Q \cap Z(\mathcal{F}) \cap B_J(0,1)$ and for every $n > 0$, one has $(1-\frac{1}{n})q + \frac{1}{n}q_o \in \mathrm{ri}Q \cap Z(\mathcal{F}) \subset \big(Q \cap Z(\mathcal{F})\big) \cup \{0\}$. Then, by Claim 4.3.6, ψ_i^n is lower semicontinuous with convex values on $\Pi \times \prod_{i\in I} X_i^r \times Z_i^r$ for each $i \in I \cup \{0\}$.

Remark that, by construction, $(p,q) \notin \psi_0(p,q,x,z)$, and that for every $i \in I$, whenever $(x_i,z_i) \notin B_i^{r\varepsilon n}(p,q) = B_i^{r\varepsilon}\big(p,(1-\frac{1}{n})q + \frac{1}{n}q_o\big)$ then one has $\psi_i^n(p,q,x,z) \neq \emptyset$ and $(x_i,z_i) \notin \psi_i^n(p,q,x,z)$.

It follows from the Fixed-Point Theorem of Gale and Mas-Colell [8] that there exists $(\bar p^n, \bar q^n, \bar x^n, \bar z^n) \in \Pi \times \prod_{i\in I}(X_i^r \times Z_i^r)$ such that for all $i \in I$, either $(\bar x_i^n, \bar z_i^n) \in \psi_i^n(\bar p^n, \bar q^n, \bar x^n, \bar z^n)$ or $\psi_i^n(\bar p^n, \bar q^n, \bar x^n, \bar z^n) = \emptyset$, and for $i = 0$, either $(\bar p^n, \bar q^n) \in \psi_0(\bar p^n, \bar q^n, \bar x^n, \bar z^n)$ or $\psi_0(\bar p^n, \bar q^n, \bar x^n, \bar z^n) = \emptyset$.

From the above remark, one deduces that for all $i \in I$, $(\bar x_i^n, \bar z_i^n) \in B_i^{r\varepsilon n}(\bar p^n, \bar q^n)$ and

$$\text{either } \breve B_i^{r\varepsilon n}(\bar p^n, \bar q^n) \cap \left(P_i^r(\bar x^n) \times Z_i^r\right) = \emptyset \quad \text{or} \quad (\bar x_i^n, \bar z_i^n) \in \breve B_i^{r\varepsilon n}(\bar p^n, \bar q^n) \cap (P_i^r(\bar x^n) \times Z_i^r)$$

and

$$p \cdot \sum_{i\in I}(\bar x_i^n - e_i) + q \cdot \sum_{i\in I} \bar z_i^n \le \bar p^n \cdot \sum_{i\in I}(\bar x_i^n - e_i) + \bar q^n \cdot \sum_{i\in I} \bar z_i^n, \ \forall (p,q) \in \Pi. \tag{4.3.2}$$

From the irreflexivity of P_i^r for each i, we conclude that for every i and for every n, $(\bar x_i^n, \bar z_i^n) \notin \breve B_i^{r\varepsilon n}(\bar p^n, \bar q^n) \cap (P_i^r(\bar x^n) \times Z_i^r)$, hence for every $i \in I$,

$$\breve B_i^{r\varepsilon n}(\bar p^n, \bar q^n) \cap \left(P_i^r(\bar x^n) \times Z_i^r\right) = \emptyset. \tag{4.3.3}$$

Claim 4.3.7 *The sequence* $\left((\bar p^n, \bar q^n, \bar x^n, \bar z^n)\right)_n$ *has a subsequence which converges to a point* $(\bar p, \bar q, \bar x, \bar z) \in \Pi \times \prod_{i\in I}(X_i^r \times Z_i^r)$ *satisfying:*

$$\text{for all } i \in I, \ (\bar x_i, \bar z_i) \in B_i^{r\varepsilon}(\bar p, \bar q), \quad \breve B_i^{r\varepsilon}(\bar p, \bar q) \cap \left(P_i^r(\bar x) \times Z_i^r\right) = \emptyset, \tag{4.3.4}$$

and

$$p \cdot \sum_{i\in I}(\bar x_i - e_i) + q \cdot \sum_{i\in I} \bar z_i \le \bar p \cdot \sum_{i\in I}(\bar x_i - e_i) + \bar q \cdot \sum_{i\in I} \bar z_i, \ \forall (p,q) \in \Pi. \tag{4.3.5}$$

Proof. The fact that the sequence $\left((\bar p^n, \bar q^n, \bar x^n, \bar z^n)\right)_n$ is bounded implies that it has a subsequence which converges to a point $(\bar p, \bar q, \bar x, \bar z) \in \Pi \times \prod_{i\in I}(X_i^r \times Z_i^r)$.

Passing to the limit in (4.3.2) we get (4.3.5). The fact that for each i, $(\bar x_i, \bar z_i) \in B_i^{r\varepsilon}(\bar p, \bar q)$ follows from the upper semicontinuity of $B_i^{r\varepsilon}$ on Π.

Now we show that $\breve B_i^{r\varepsilon}(\bar p, \bar q) \cap \left(P_i^r(\bar x) \times Z_i^r\right) = \emptyset$. By contradiction, assume that there exists $(x_i, z_i) \in \breve B_i^{r\varepsilon}(\bar p, \bar q) \cap \left(P_i^r(\bar x) \times Z_i^r\right) \neq \emptyset$. Then for n large enough, $(x_i, z_i) \in \breve B_i^{r\varepsilon n}(\bar p^n, \bar q^n)$, and from the lower semicontinuity of P_i^r and the fact that P_i^r has open and convex values, we deduce that the sets $(P_i^r)^{-1}(x_i) := \{x \mid x_i \in P_i^r(x)\}$ are open, therefore for n large enough, $x_i \in P_i^r(\bar x^n)$. Hence, for n large enough, $(x_i, z_i) \in \breve B_i^{r\varepsilon n}(\bar p^n, \bar q^n) \cap \left(P_i^r(\bar x^n) \times Z_i^r\right)$, a contradiction to (4.3.3). ∎

Checking the market clearing conditions

Claim 4.3.8 *For every i $V\bar{z}_i \geq \underline{v}_i$, $\sum_{i\in I}\bar{z}_i \in -\sum_{i\in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\})$, and $\bar{q}\cdot\sum_{i\in I}\bar{z}_i = 0$.*

Proof. The fact that $V\bar{z}_i \geq \underline{v}_i$ follows straightforwardly from $(\bar{x}_i, \bar{z}_i) \in B_i^{r\varepsilon}(\bar{p}, \bar{q})$ (which is a consequence of (4.3.4)) and the definition of $\underline{v}_i$.

To show that $\sum_{i\in I}\bar{z}_i \in -\sum_{i\in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\})$, we show that $\sum_{i\in I}\bar{z}_i \in Q^o$ and use Lemma 4.2 to conclude. Assume there exists $q \in Q \cap B_J(0,1)$ such that $q\cdot(\sum_{i\in I}\bar{z}_i) > 0$. Then, from (4.3.5), $\bar{q}\cdot(\sum_{i\in I}\bar{z}_i) \geq q\cdot(\sum_{i\in I}\bar{z}_i) > 0$. We claim that $||\bar{q}|| = 1$. Indeed, we have $0 < \frac{\bar{q}}{||\bar{q}||}\cdot\sum_{i\in I}\bar{z}_i \leq \bar{q}\cdot\sum_{i\in I}\bar{z}_i$, which implies $||\bar{q}|| \geq 1$, and since $\bar{q} \in \mathrm{cl}B(0,1)$, we get $||\bar{q}|| = 1$. Hence $\varepsilon_0(\bar{p},\bar{q}) = 0$ and since $(\bar{x}_i, \bar{z}_i) \in B_i^{r\varepsilon}(\bar{p}, \bar{q})$, we have for every $i \in I$, $\bar{p}(0)\cdot(\bar{x}_i(0) - e_i(0)) + \bar{q}\cdot\bar{z}_i \leq 0$. Summing up over i we get $\bar{p}(0)\cdot\sum_{i\in I}(\bar{x}_i(0) - e_i(0)) + \bar{q}\cdot\sum_{i\in I}\bar{z}_i \leq 0$. On the other hand, we have, from (4.3.5), $\bar{p}(0)\cdot\sum_{i\in I}(\bar{x}_i(0)-e_i(0))+\bar{q}\cdot\sum_{i\in I}\bar{z}_i \geq \bar{q}\cdot\sum_{i\in I}\bar{z}_i > 0$. Contradiction. Therefore $\sum_{i\in I}\bar{z}_i \in Q^o = -\sum_{i\in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\})$, by Lemma 4.2.

Finally, we show that $\bar{q}\cdot\sum_{i\in I}\bar{z}_i = 0$. Since $\bar{q} \in \mathrm{cl}Q$ we conclude that $\bar{q}\cdot\sum_{i\in I}\bar{z}_i \leq 0$ and since $0 \in \mathrm{cl}Q$ we have $0 = 0\cdot\sum_{i\in I}\bar{z}_i \leq \bar{q}\cdot\sum_{i\in I}\bar{z}_i$ (from (4.3.5)). Hence $\bar{q}\cdot\sum_{i\in I}\bar{z}_i = 0$. ■

Claim 4.3.9 $\sum_{i\in I}\bar{x}_i = \sum_{i\in I}e_i$.

Proof. If $\sum_{i\in I}\bar{x}_i \neq \sum_{i\in I}e_i$, we deduce from (4.3.5) that for some $s \in \bar{S}$: $\|\bar{p}(s)\| = 1$, $\varepsilon_s(\bar{p},\bar{q}) = 0$, $\bar{p}(s)\cdot\sum_{i\in I}\big(\bar{x}_i(s) - e_i(s)\big) > 0$. Since $(\bar{x}_i, \bar{z}_i) \in B_i^{r\varepsilon}(\bar{p},\bar{q})$, we have $\bar{p}(s)\cdot\big(\bar{x}_i(s) - e_i(s)\big) \leq w(\bar{q}, s)\cdot\bar{z}_i$, $i \in I$, where $w(\bar{q}, s)$ denotes the row s of the matrix $W(\bar{q})$. Summing up over i we get; if $s = 0$, $\bar{p}(0)\cdot\sum_{i\in I}\big(\bar{x}_i(0) - e_i(0)\big) \leq -\bar{q}\cdot\sum_{i\in I}\bar{z}_i = 0$, a contradiction, and if $s \neq 0$, $\bar{p}(s)\cdot\sum_{i\in I}\big(\bar{x}_i(s) - e_i(s)\big) \leq \sum_{i\in I}V[s]\cdot\bar{z}_i \leq 0$ (since, by Claim 4.3.8, $-\sum_{i\in I}\bar{z}_i \in \{V \geq 0\}$), a contradiction. ■

From Claim 4.3.8, $\sum_{i\in I}\bar{z}_i \in -\sum_{i\in I}(\boldsymbol{A}Z_i \cap \{V \geq 0\})$. Then there exist $(\zeta_1, \cdots, \zeta_I) \in \Pi_i(\boldsymbol{A}Z_i \cap \{V \geq 0\})$ such that $\sum_{i\in I}\bar{z}_i = -\sum_{i\in I}\zeta_i$. We let $\bar{\bar{z}}_i = \pi(\bar{z}_i + \zeta_i) = \bar{z}_i + \pi\zeta_i$ (because $\pi\bar{z}_i = \bar{z}_i$ since $\bar{z}_i \in \pi Z_i^r \subset \mathrm{cl}\pi Z_i = \pi Z_i$).

Recall the definition of the set

$$K_{\underline{v}} := \{(\pi z_i)_i \in \Pi_i(\pi Z_i \cap \{V \geq \underline{v}_i\}) : \sum_{i\in I} z_i = 0\}.$$

Claim 4.3.10 *$\bar{\bar{z}} \in K_{\underline{v}}$ hence for every i, $\bar{\bar{z}}_i \in \widehat{Z}_i \subset Z_i^r \subset \mathrm{int}B_J(0,r)$, and $\sum_{i\in I}\bar{\bar{z}}_i = 0$.*

Proof. Clearly $\sum_{i\in I}(\bar{z}_i+\zeta_i)=0$ and for each i, $\pi\bar{z}_i+\pi\zeta_i=\bar{z}_i+\pi\zeta_i\in\pi Z_i$ since $\bar{z}_i\in\pi Z_i$ and $\pi\zeta_i\in\pi\boldsymbol{A}Z_i=\boldsymbol{A}\pi Z_i$ (from Claim 4.3.1), and we need only show that $V(\bar{z}_i+\zeta_i)\geq\alpha_i$. Indeed, $\zeta_i\in\boldsymbol{A}Z_i\cap\{V\geq 0\}$ then $V\zeta_i\geq 0$. Hence $V(\bar{z}_i+\zeta_i)\geq V\bar{z}_i\geq\underline{v}_i$ from Claim 4.3.8. The equality $\sum_{i\in I}\bar{\bar{z}}_i=0$ is straightforward. ■

The list $(\bar{x},\bar{\bar{z}},\bar{p},\bar{q})$ is an equilibrium of $(\mathcal{E}^r,\mathcal{F}^r_\pi)$

Claim 4.3.11 *For each $i\in I$, $(\bar{x}_i,\bar{z}_i)$ and $(\bar{x}_i,\bar{\bar{z}}_i)$ belong to $B_i^{r\varepsilon}(\bar{p},\bar{q})$, and $B_i^{r\varepsilon}(\bar{p},\bar{q})\cap\big(P_i^r(\bar{x})\times Z_i^r\big)=\emptyset$.*

Proof. For each $i\in I$, $(\bar{x}_i,\bar{z}_i)$ belongs to $B_i^{r\varepsilon}(\bar{p},\bar{q})$. This is a consequence of (4.3.4).

We now prove that $B_i^{r\varepsilon}(\bar{p},\bar{q})\cap\big(P_i^r(\bar{x})\times Z_i^r\big)=\emptyset$. From the irreflexivity assumption, $\bar{x}_i\notin P_i^r(\bar{x})$ for each i. Therefore, from (4.3.4), one deduces that $\breve{B}_i^{r\varepsilon}(\bar{p},\bar{q})\cap\big(P_i^r(\bar{x})\times Z_i^r\big)=\emptyset$. Since P_i^r has open values and since $B_i^{r\varepsilon}(\bar{p},\bar{q})=\mathrm{cl}\breve{B}_i^{r\varepsilon}(\bar{p},\bar{q})$ (by Claim 4.3.4), this implies $B_i^{r\varepsilon}(\bar{p},\bar{q})\cap\big(P_i^r(\bar{x})\times Z_i^r\big)=\emptyset$.

Finally, we prove that for each $i\in I$, $(\bar{x}_i,\bar{\bar{z}}_i)$ belongs to $B_i^{r\varepsilon}(\bar{p},\bar{q})$. From Claim 4.3.10, we know that $\bar{\bar{z}}_i\in Z_i^r$ and since $(\bar{x}_i,\bar{z}_i)\in B_i^{r\varepsilon}(\bar{p},\bar{q})$ it suffices to show $W(\bar{q})\bar{\bar{z}}_i\geq W(\bar{q})\bar{z}_i$ for every i, or equivalently $W(\bar{q})\zeta_i\geq 0$, for every $i\in I$. Note that because $\bar{q}\in\boldsymbol{L}^\perp=(\ker\pi)^\perp$ and $\ker\pi=\boldsymbol{L}\subset\ker V$, one has $W(\bar{q})\bar{\bar{z}}_i=W(\bar{q})(\bar{z}_i+\zeta_i)$.

Recall that for each i, $\zeta_i\in\boldsymbol{A}Z_i\cap\{V\geq 0\}$, hence $V\zeta_i\geq 0$ for every i. It remains to shaw that $-\bar{q}\cdot\zeta_i\geq 0$ for every $i\in I$. We claim that $-\bar{q}\cdot\zeta_i=0$ for every i. Indeed, $\bar{q}\in\mathrm{cl}Q=-\Big(\sum_{i\in I}(\boldsymbol{A}Z_i\cap\{V\geq 0\})\Big)^o$ (by Lemma 4.2) and for each $i\in I$, $\zeta_i\in\boldsymbol{A}Z_i\cap\{V\geq 0\}\subset\sum_{k\in I}(\boldsymbol{A}Z_k\cap\{V\geq 0\})$, then $-\bar{q}\cdot\zeta_i\leq 0$ for every i. Now recalling that $-\bar{q}\cdot\sum_{i\in I}\zeta_i=\bar{q}\cdot\sum_{i\in I}\bar{z}_i=0$ from Claim 4.3.8, we deduce that $\bar{q}\cdot\zeta_i=0$ for every $i\in I$. Therefore $W(\bar{q})\zeta_i\geq 0$ for every $i\in I$. ■

Claim 4.3.12 $\varepsilon(\bar{p},\bar{q})=0$. *Hence $B_i^{r\varepsilon}(\bar{p},\bar{q})=B_i^r(\bar{p},\bar{q})$.*

Proof. From Claim 4.3.11 $(\bar{x}_i,\bar{\bar{z}}_i)\in B_i^{r\varepsilon}(\bar{p},\bar{q})$ for each $i\in I$. Hence

$$\bar{p}\,\Box\,(\bar{x}_i-e_i)\leq W(\bar{p},\bar{q})\bar{\bar{z}}_i+\varepsilon(\bar{p},\bar{q}).$$

Moreover, Assumption **NS** together with **LNS** implies that,

$$\bar{p}\,\Box\,(\bar{x}_i-e_i)=W(\bar{p},\bar{q})\bar{\bar{z}}_i+\varepsilon(\bar{p},\bar{q}),\text{ for all } i\in I. \tag{4.3.6}$$

Indeed, if it is not true then there exists $s\in\bar{S}$ such that $\bar{p}(s)\cdot(\bar{x}_i(s)-e_i(s))<[W(\bar{q})\bar{\bar{z}}_i](s)+\varepsilon_s(\bar{p},\bar{q})$. From **NS**, there exists $x_i\in P_i^r(\bar{x})$ such that $x_i(s')=\bar{x}_i(s')$ for every $s\neq s'$, and from **LNS**, for every $t \in (0,1]$, $tx_i + (1-t)\bar{x}_i \in]\bar{x}_i, x_i] \subset P_i^r(\bar{x})$. Since $tx_i + (1-t)\bar{x}_i \underset{t\to 0}{\longrightarrow} \bar{x}_i$, it is possible to choose t small enough so that $(tx_i + (1-t)\bar{x}_i, \bar{\bar{z}}_i) \in B_i^{r\varepsilon}(\bar{p}, \bar{q})$. Hence $(tx_i + (1-t)\bar{x}_i, \bar{\bar{z}}_i)$ belongs to $B_i^{r\varepsilon}(\bar{p}, \bar{q}) \cap \left(P_i^r(\bar{x}) \times Z_i^r\right)$ which is empty by Claim 4.3.11, a contradiction.

Summing up over i the equalities (4.3.6) and using the two facts (i) $\sum_{i\in I}\bar{x}_i = \sum_{i\in I} e_i$ from Claim 4.3.9 and (ii) $\sum_{i\in I}\bar{\bar{z}}_i = 0$ from Claim 4.3.10, we get $(\sharp I)\varepsilon(\bar{p}, \bar{q}) = 0$, where $\sharp I$ is the cardinality of I, hence $\varepsilon(\bar{p}, \bar{q}) = 0$. ∎

Claims 4.3.9-4.3.12 show that $(\bar{x}, \bar{\bar{z}}, \bar{p}, \bar{q})$ is an equilibrium of $(\mathcal{E}^r, \mathcal{F}_\pi^r)$ and we will now prove that it is an equilibrium of $(\mathcal{E}, \mathcal{F}_\pi)$.

The list $(\bar{x}, \bar{\bar{z}}, \bar{p}, \bar{q})$ is an equilibrium of $(\mathcal{E}, \mathcal{F}_\pi)$

Proof. Market clearing conditions clearly hold from Claims 4.3.9 and 4.3.10, and we only have to prove that

$$(P_i(\bar{x}) \times \pi Z_i) \cap B_i(\mathcal{F}_\pi, \bar{p}, \bar{q}) = \emptyset \text{ for every } i \in I.$$

Assume that it is not true then for some i, there exists $(x_i, \pi z_i) \in B_i(\mathcal{F}_\pi, \bar{p}, \bar{q}) \cap (P_i(\bar{x}) \times \pi Z_i)$. Consequently, $\bar{p} \,\square\, (x_i - e_i) \leq W(\bar{q})\pi z_i$. Since $\bar{x}$ is an attainable allocation and $\bar{\bar{z}} \in K_\alpha$, the definition of r implies that $\bar{x}_i \in \widehat{X}_i \subset \text{int}B_{L\bar{S}}(0,r)$ and $\bar{\bar{z}}_i \in \widehat{Z}_i \subset \text{int}B_J(0,r)$. Since $(\bar{x}, \bar{\bar{z}}, \bar{p}, \bar{q})$ is an equilibrium of $(\mathcal{E}^r, \mathcal{F}_\pi^r)$, $(\bar{x}_i, \bar{\bar{z}}_i) \in B_i^r(\bar{p}, \bar{q}) = \{(x_i, z_i) \in X_i^r \times Z_i^r, \bar{p} \,\square\, (x_i - e_i) \leq W(\bar{q}) z_i\}$. Thus for α small enough, $(\bar{x}_i + \alpha(x_i - \bar{x}_i), \bar{\bar{z}}_i + \alpha(\pi z_i - \bar{\bar{z}}_i)) \in B_i^r(\bar{p}, \bar{q})$. On the other hand, by Assumption **LNS**, for every $\alpha \in (0,1]$, $\bar{x}_i + \alpha(x_i - \bar{x}_i) \in (\bar{x}_i, x_i] \subset P_i(\bar{x})$. Therefore for $\alpha > 0$ small enough, $(\bar{x}_i + \alpha(x_i - \bar{x}_i), \bar{\bar{z}}_i + \alpha(\pi z_i - \bar{\bar{z}}_i)) \in B_i^r(\bar{p}, \bar{q}) \cap (P_i^r(\bar{x}) \times Z_i^r)$, a contradiction to the fact that agent i maximizes her preferences in her budget set in $(\mathcal{E}^r, \mathcal{F}_\pi^r)$. ∎

Remark 4.7 At this stage, it is important to emphasize that the equilibrium asset price vector $\bar{q}$ can be equal to 0.

4.3.3 From Theorem 4.3 to Theorem 4.1

Now we show how to prove Theorem 4.1 as a consequence of Theorem 4.3. This is done by transforming the consumption economy $\mathcal{E}$ to get local non-satiation **LNS**.

Following Gale and Mas-Colell ([8], [9]), for $x \in \prod_{i\in I} X_i$, we define the "augmented preferences" $\widetilde{P}_i$ by

$$\widetilde{P}_i(x) = \cup_{x'_i \in P_i(x)}(x_i, x'_i] = \{x_i + \alpha(x'_i - x_i) \mid 0 < \alpha \leq 1,\ x'_i \in P_i(x)\}$$

and we notice that we have $P_i(x) \subset \widetilde{P}_i(x) \subset X_i$.

We now define a new economy $\widetilde{\mathcal{E}}$ which only differs from the original one $\mathcal{E}$ by the fact that the original preferred sets $P_i(x)$ are replaced by the larger ones $\widetilde{P}_i(x)$ defined above. To summarize, we let

$$\widetilde{\mathcal{E}} := \left(X_i, \widetilde{P}_i, e_i\right)_{i \in I}.$$

The interest of the economy $\widetilde{\mathcal{E}}$, instead of $\mathcal{E}$, is twofold. First, $\widetilde{\mathcal{E}}$ satisfies more properties than $\mathcal{E}$, as shown in the following Proposition. Second, every equilibrium of $(\widetilde{\mathcal{E}}, \mathcal{F})$ is a financial equilibrium of $(\mathcal{E}, \mathcal{F})$. The proof of Proposition 4.4 and Proposition 4.5 are routine and therefore are omitted.

Proposition 4.4 *Let* $x \in \prod_{i \in I} X_i$.

(a) *If* P_i *is lower semicontinuous, then* $\widetilde{P}_i$ *is lower semicontinuous.*

(b) *If* $P_i(x)$ *is convex, then* $\widetilde{P}_i(x)$ *is convex.*

(c) *If* $P_i(x)$ *is open in* X_i *then* $\widetilde{P}_i(x)$ *is open in* X_i.

(d) *If* $x_i \in \widetilde{P}_i(\bar{x})$ *then* $(\bar{x}_i, x_i] \subset \widetilde{P}_i(\bar{x})$.

(e) *If* $x_i \notin P_i(x)$, *then* $x_i \notin \widetilde{P}_i(x)$.

Proposition 4.5 *Every equilibrium of* $(\widetilde{\mathcal{E}}, \mathcal{F})$ *is an equilibrium of* $(\mathcal{E}, \mathcal{F})$.

4.4 Appendix

Theorem 4.5 *Let* C *be a nonempty convex set in* $\mathbb{R}^n$ *and let* $f : \mathbb{R}^n \to \mathbb{R}^m$ *be linear. If* $\boldsymbol{A}C \cap \ker f \subset \boldsymbol{L}(C)$, *then*

(a) $\mathrm{cl} f(C) = f(\mathrm{cl} C)$.

(b) $\boldsymbol{A}(f(\mathrm{cl}C)) = f(\boldsymbol{A}(\mathrm{cl}C))$.

Claim 4.4.1 *Let* C *be a nonempty convex set in* $\mathbb{R}^n$ *and let* L *be a linear subspace contained in the lineality space of* C *i.e.* $L \subset L(C)$. *Then* $\mathrm{cl}C = L + (L^{\perp} \cap \mathrm{cl}C)$.

Proof. Let $\ell \in L$ and $x \in L^{\perp} \cap \mathrm{cl}C$, then $\ell \in \boldsymbol{L}(C) = \boldsymbol{L}(\mathrm{cl}C)$ and $x \in \mathrm{cl}C$ which implies $\ell + x \in \mathrm{cl}C$, hence $L + (L^{\perp} \cap \mathrm{cl}C) \subset \mathrm{cl}C$. Conversely, let $x \in \mathrm{cl}C$, then $x = \mathrm{proj}_L x + \mathrm{proj}_{L^{\perp}} x$, where $\mathrm{proj}_L x$ (respectively, $\mathrm{proj}_{L^{\perp}} x$) is the orthogonal projection of x on L (respectively, $L^{\perp}$). Hence $\mathrm{proj}_{L^{\perp}} x = x + (-\mathrm{proj}_L x) \in \mathrm{cl}C + L \subset \mathrm{cl}C + \boldsymbol{L}(\mathrm{cl}C) \subset \mathrm{cl}C$. This ends the proof of the claim. ∎

Proof of Theorem 4.5

(a) We always have $f(\mathrm{cl}C) \subset \mathrm{cl}f(C)$. Let $y \in \mathrm{cl}f(C)$, we will show that $y = f(x)$ for some $x \in \mathrm{cl}C$.

Let $L := \boldsymbol{L}(\mathrm{cl}C) \cap \ker f = -\boldsymbol{A}C \cap \boldsymbol{A}C \cap \ker f$. Notice that $L \subset \boldsymbol{A}C \cap \ker f$ and by assumption $\boldsymbol{A}C \cap \ker f \subset \boldsymbol{L}(C) = \boldsymbol{L}(\mathrm{cl}C)$ hence $\boldsymbol{A}C \cap \ker f \subset \boldsymbol{L}(\mathrm{cl}C) \cap \ker f = L$. Therefore $L = \boldsymbol{A}C \cap \ker f$.

First, we claim that $f(L^{\perp} \cap \mathrm{cl}C) = f(\mathrm{cl}C)$. Indeed, $L^{\perp} \cap \mathrm{cl}C \subset \mathrm{cl}C$ hence $f(L^{\perp} \cap \mathrm{cl}C) \subset f(\mathrm{cl}C)$. Conversely, let $x \in \mathrm{cl}C$. From the previous claim, x can be written $x = \ell + z$ with $\ell \in L$ and $z \in L^{\perp} \cap \mathrm{cl}C$. Then $f(x) = f(\ell) + f(z) = f(z) \in f(L^{\perp} \cap \mathrm{cl}C)$ since $\ell \in L \subset \ker f$. This ends the proof of the claim.

Now, since $y \in \mathrm{cl}f(C) \subset \mathrm{cl}f(\mathrm{cl}C) = \mathrm{cl}f(L^{\perp} \cap \mathrm{cl}C)$, for every $k \in \mathbb{N}^*$ the following set, C_k, is not empty

$$C_k = L^{\perp} \cap \mathrm{cl}C \cap \{x : ||y - f(x)|| \leq \frac{1}{k}\}.$$

Notice that $\boldsymbol{A}(C_k) = L^{\perp} \cap \boldsymbol{A}(\mathrm{cl}C) \cap \ker f = L^{\perp} \cap L = \{0\}$. Hence $(C_k)_k$ is a family of bounded closed (hence compact) nonempty sets in $\mathbb{R}^n$, satisfying the finite intersection property (actually the sequence $(C_k)_k$ is non-increasing). Therefore $\cap_k C_k \neq \emptyset$, and every $x \in \cap_k C_k$ satisfies $x \in \mathrm{cl}C$ and $y = f(x)$.

(b) Since $\boldsymbol{A}(f(\mathrm{cl}C)) = \boldsymbol{A}(\mathrm{cl}f(C)) = \boldsymbol{A}(f(C))$ and $f(\boldsymbol{A}(\mathrm{cl}C)) = f(\boldsymbol{A}C)$, we need to show $\boldsymbol{A}(f(C)) = f(\boldsymbol{A}C)$. Define the convex cone

$$K = \{(\lambda, x) : \lambda > 0, x \in \lambda C\} \subset \mathbb{R}^{n+1},$$

and the linear mapping

$$g : \mathbb{R}^{n+1} \to \mathbb{R}^{m+1}, (\lambda, x) \mapsto (\lambda, f(x)).$$

Then $\boldsymbol{A}K = \boldsymbol{A}(\mathrm{cl}K) = \mathrm{cl}K = K \cup \{(0, z) : z \in \boldsymbol{A}C\}$, and $\ker g = \{0\} \times \ker f$. Hence

$$\boldsymbol{A}K \cap \ker g = \{(0, z) : z \in \boldsymbol{A}C \cap \ker f\} \subset \{ (0, z) : z \in \boldsymbol{L}(C)\} = \boldsymbol{L}(K).$$

Therefore, by Part (a), $g(\mathrm{cl}K) = \mathrm{cl}g(K)$. We have

$$\begin{aligned} g(\mathrm{cl}K) &= \{(\lambda, f(x)) : \lambda > 0, x \in \lambda C\} \cup \{ (0, f(z)) : z \in \boldsymbol{A}C\}, \\ \mathrm{cl}g(K) &= \mathrm{cl}\{(\lambda, y) : \lambda > 0, y \in \lambda f(C)\}, \\ &= \{(\lambda, y) : \lambda > 0, y \in \lambda f(C)\} \cup \{ (0, y) : y \in \boldsymbol{A}(f(C))\}. \end{aligned}$$

Therefore $\boldsymbol{A}f(C) = f(\boldsymbol{A}C)$. ∎

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Chapter 5

Appendix

5.1 Asymptotic Cones

5.1.1 Definition and first properties in the non convex case

We need to use the notion of asymptotic cone for subsets of $\mathbb{R}^m$, which may not be closed (and in most cases will be convex). Our definition will follow Debreu [1] and departs from Rockafellar[1] [2] when the set will be convex but not closed. It coincices with the standard definition when the set is convex AND closed.

Definition 5.1 *(Debreu [1]) Let C be a nonempty subset of $\mathbb{R}^m$. The asymptotic cone of C, denoted by $\boldsymbol{A}C$, is the set of vectors $v \in \mathbb{R}^m$ satisfying one of the two equivalent conditions:*

- $\exists \lambda_n \downarrow 0, \exists (x_n)_n \subset C,\ v = \lim_n \lambda_n x_n$.
- $v \in \cap_{k \geq 0} \mathrm{cl}(\cup_{\lambda \geq 0} \lambda C^k)$ *where* $C^k := \{x \in C : ||x|| \geq k\}$.

Proof. Let $v \in \{v \in \mathbb{R}^m : \exists \lambda_n \downarrow 0, \exists (x_n)_n \subset C, v = \lim_n \lambda_n x_n\}$. Assume $v \neq 0$ and fix $k \geq 0$. We need to show that $v \in \mathrm{cl}(\cup_{\lambda \geq 0} \lambda C^k)$.

Write $v = \lim_n \lambda_n x_n$ with $\lambda_n \downarrow 0$ and $(x_n)_n \subset C$. Then $(x_n)_n$ is not bounded and we can assume $||x_n|| \underset{n \to \infty}{\longrightarrow} +\infty$ so that $\exists n_0 \in \mathbb{N}$ such that $(x_n)_{n \geq n_0} \subset C^k$. Then $(\lambda_n x_n)_{n \geq n_0} \subset \cup_{\lambda \geq 0} \lambda C^k$ and consequently $v = \lim_n \lambda_n x_n \in \mathrm{cl}(\cup_{\lambda \geq 0} \lambda C^k)$.

[1] Take $C = \mathbb{R}^2_{++} \cup \{0\}$, then the asymptotic cone called recession cone is C (see page 60-63 in [2]) whereas for us it is the closure of C (see below).

Conversely, let $v \in \cap_{k\geq 0}\mathrm{cl}(\cup_{\lambda\geq 0}\lambda C^k)$ and assume that $v \neq 0$. Then $\forall k \geq 0$, $v \in \mathrm{cl}(\cup_{\lambda\geq 0}\lambda C^k)$, thus $\forall k \geq 0$, $\mathrm{int}B(v, \frac{1}{k}) \cap (\cup_{\lambda\geq 0}\lambda C^k) \neq \emptyset$. Hence $\forall k \geq 0$, $\exists \lambda_k > 0, \exists x_k \in C^k$ (therefore $||x_k|| \geq k$) such that $||v - \lambda_k x_k|| < \frac{1}{k}$. Therefore $v = \lim_k \lambda_k x_k$ with $x_k \in C$ and $\lambda_k \underset{k\to\infty}{\longrightarrow} 0$ (since $||x_k|| \underset{k\to\infty}{\longrightarrow} +\infty$). ■

Definition 5.2 *The lineality space of a nonempty subset C of $\mathbb{R}^m$ is the set $\boldsymbol{L}(C) = \boldsymbol{A}C \cap [-\boldsymbol{A}C]$.*

Proposition 5.1 *Let C be a non-empty subset of $\mathbb{R}^k$. Then*

(1) $\boldsymbol{A}(C) = \boldsymbol{A}(\mathrm{cl}C)$, and $\boldsymbol{L}(C) = \boldsymbol{L}(\mathrm{cl}C)$,

(2) $\boldsymbol{A}(C)$ is a closed cone, and $\boldsymbol{L}(C)$ is a (closed vector subspace),

(3) If $C \subset D$, then $\boldsymbol{A}C \subset \boldsymbol{A}D$,

(4) Let $(C_i)_{i\in I}$ be a family of nonempty subsets of $\mathbb{R}^m$ whose intersection is nonempty. Then $\boldsymbol{A}(\cap_i C_i) \subset \cap_i \boldsymbol{A}C_i$ and the equality does not hold in general.

Proposition 5.2 *If $(C_i)_{i\in I}$ is a family of subsets of $\mathbb{R}^m$ such that $\cap_i \boldsymbol{A}C_i = \{0\}$, then $\cap_i C_i$ is bounded.*

5.1.2 Definition and first properties in the convex case

When C is additionally assumed to be convex, we have more characterizations.

Definition 5.3 *Let C be a non-empty convex subset of $\mathbb{R}^m$. Let $v \in \mathbb{R}^m$. The recession cone of C is*

$$O^+C = \{v \in \mathbb{R}^m : v + C \subset C\}.$$

The set $O(C)$ is then defined as $O(C) = O^+C \cap -O^+C$.

Note that O^+C is a convex cone containing the origin but may not be closed, and asymptotic cone as defined here satisfies $\boldsymbol{A}C = O^+(\mathrm{cl}C)$. Thus the two notions coincide when C is closed but they may differ otherwise as shown by the following example.

Example 5.1 *Let $C = \mathbb{R}^2_{++} \cup \{(0,0)\}$. Then $O^+C = C$ and $\boldsymbol{A}C = \mathbb{R}^2_+$.*

Remark 5.1 *$v \in O^+(C)$ if and only if*

$$\forall x \in C, \forall \lambda \geq 0, x + \lambda v \in C.$$

Remark 5.2 *Note that $O^+(C) \subset \boldsymbol{A}C$ and in general $O^+(C) \subsetneq \boldsymbol{A}C$.*

The following proposition will show that

$$\boldsymbol{A}C = O^+(\mathrm{cl}C).$$

Proposition 5.3 *Let C be a non-empty convex subset of $\mathbb{R}^m$. Let $v \in \mathbb{R}^m$. Then the following assertions are equivalent.*

(1) $\forall x \in C,\ x + v \in \mathrm{cl}C$.

(2) $\forall x \in \mathrm{cl}C, x + v \in \mathrm{cl}C$.

(3) $\forall x \in C, \forall \lambda \geq 0,\ x + \lambda v \in \mathrm{cl}C$.

(4) $\exists x \in C, \forall \lambda \geq 0,\ x + \lambda v \in \mathrm{cl}C$.

(5) $\forall x \in \mathrm{ri}C, x + v \in \mathrm{ri}C$.

(6) $\exists \lambda_n \downarrow 0, \exists (x_n)_n \subset C,\ v = \lim_n \lambda_n x_n$.

(7) (Debreu) $v \in \cap_{k \geq 0} \mathrm{cl}(\cup_{\lambda \geq 0} \lambda C^k)$ where $C^k := \{x \in C : ||x|| \geq k\}$.

Proof. (1) $\Rightarrow$ (2): Let $x \in \mathrm{cl}C$, then $x = \lim_n x_n$ for some $(x_n)_n \subset C$, thus $x_n + v \in \mathrm{cl}C$ for every n. Hence $x + v = \lim_n (x_n + v) \in \mathrm{cl}C$.

(2) $\Rightarrow$ (3): We first notice $\forall x \in C, x + nv \in \mathrm{cl}C$ for every $n \in \mathbb{N}$. Hence for every $t \in [0, n]$, we have

$x + t\ v = (1 - \frac{t}{n})x + \frac{t}{n}(x + n\ v) \in \mathrm{cl}C$ (since $\mathrm{cl}C$ is convex).

(3) $\Rightarrow$ (4): Obvious.

(4) $\Rightarrow$ (6): For every $n \in \mathbb{N}, x + n\ v \in \mathrm{cl}C$. Then $\forall n, \exists x_n \in C$ s.t. $||x + n\ v - x_n|| < \frac{1}{n}$ and we deduce that $v = \lim_n \frac{x_n}{n}$ since

$$||v - \frac{x_n}{n}|| \leq ||v - \frac{x_n}{n} + \frac{x}{n}|| + ||\frac{x}{n}|| \leq \frac{1}{n^2} + \frac{||x||}{n} \underset{n \to \infty}{\longrightarrow} 0.$$

(6) $\Rightarrow$ (1): For $x \in C$, $x + v = \lim_n \Big((1 - \lambda_n)x + \lambda_n x_n\Big) \in \mathrm{cl}C$.

The equivalence between (6) and (7) has already been shown at the beginning of this chapter. ∎

Proposition 5.4 *Let C be a non-empty convex subset of $\mathbb{R}^m$. Then*

(1) $\boldsymbol{A}C$ is a closed convex cone (with vertex 0) of $\mathbb{R}^m$ and $\boldsymbol{L}(C)$ is a linear subspace of $\mathbb{R}^m$.

(2) $\boldsymbol{A}(C) = \boldsymbol{A}(\mathrm{cl}C)$, and $\boldsymbol{L}(C) = \boldsymbol{L}(\mathrm{cl}C)$,

(3) If $f : \mathbb{R}^k \to \mathbb{R}^p$ is linear, then $f(\boldsymbol{A}C) \subset \boldsymbol{A}(f(C))$.

Proof. (1) Let $v \in \boldsymbol{A}(C)$. From Proposition 5.3 (3), there exists $x \in C \subset D$ such that $\forall \lambda \geq 0, x + \lambda\, v \in \mathrm{cl}C \subset \mathrm{cl}D$. Hence from Proposition 5.3 (3), $v \in \boldsymbol{A}(D)$.

(2) Since $C \subset \mathrm{cl}C$, from (1) we deduce that $\boldsymbol{A}(C) \subset \boldsymbol{A}(\mathrm{cl}C)$. Conversely, let $v \in \boldsymbol{A}(\mathrm{cl}C)$. Then, from Proposition 5.3 (1), $x + v \in \mathrm{cl}C$ for every $x \in \mathrm{cl}C$, hence in particular for every $x \in C$. This shows that $v \in \boldsymbol{A}(C)$.

(3) This is a consequence of (2) $\boldsymbol{A}(C) = \boldsymbol{A}(\mathrm{cl}C)$ and [2]. ∎

Proposition 5.5 *Let $(C_i)_{i \in I}$ be a family of nonempty convex subsets of $\mathbb{R}^k$ whose intersection is nonempty. Then $\boldsymbol{A}(\cap_i C_i) = \cap_i \boldsymbol{A}C_i$, and $\boldsymbol{L}(\cap_i C_i) = \cap_i \boldsymbol{L}C_i$*

Proof. For every $i \in I$, $\cap_i C_i \subset C_i$ then $\boldsymbol{A}(\cap_i C_i) \subset \boldsymbol{A}C_i$ for each i. Hence $\boldsymbol{A}(\cap_i C_i) \subset \cap_i \boldsymbol{A}C_i$. Conversely, let $v \in \cap_i \boldsymbol{A}C_i$, and let $x \in \mathrm{cl} \cap_i C_i$ then for every i, $x + v \in \mathrm{cl}C_i$ (since $v \in \boldsymbol{A}C_i$ and $x \in \mathrm{cl}C_i$ because $x \in \mathrm{cl} \cap_i C_i \subset \cap_i \mathrm{cl}C_i$). Hence $x + v \in \cap_i \mathrm{cl}C_i = \mathrm{cl}(\cap_i C_i)$ since the C_i's are convex with nonempty intersection (see e.g. [2]). Therefore $v \in \boldsymbol{A}(\cap_i C_i)$. ∎

Proposition 5.6 *Let C and D be nonempty convex sets such that $C \cap V^{-1}(D) = \{x \in C : Vx \in D\} \neq \emptyset$. Then*

$$\boldsymbol{A}(C \cap V^{-1}(D)) = \boldsymbol{A}C \cap [Vx \in \boldsymbol{A}D] = \boldsymbol{A}C \cap V^{-1}(\boldsymbol{A}D).$$

5.1.3 A closedness property

Theorem 5.1 *Let C be a nonempty convex set in $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}^m$ be linear. If $\boldsymbol{A}C \cap \ker f \subset \boldsymbol{L}(C)$, then*

(a) $\mathrm{cl}f(C) = f(\mathrm{cl}C)$.

(b) $\boldsymbol{A}(f(\mathrm{cl}C)) = f(\boldsymbol{A}(\mathrm{cl}C))$.

Claim 5.1.1 *Let C be a nonempty convex set in $\mathbb{R}^n$ and let L be a linear subspace contained in the lineality space of C i.e. $L \subset L(C)$. Then $\mathrm{cl}C = L + (L^{\perp} \cap \mathrm{cl}C)$.*

Proof. Let $\ell \in L$ and $x \in L^{\perp} \cap \mathrm{cl}C$, then $\ell \in \boldsymbol{L}(C) = \boldsymbol{L}(\mathrm{cl}C)$ and $x \in \mathrm{cl}C$ which implies $\ell + x \in \mathrm{cl}C$, hence $L + (L^{\perp} \cap \mathrm{cl}C) \subset \mathrm{cl}C$. Conversely, let $x \in \mathrm{cl}C$, then $x = \mathrm{proj}_L x + \mathrm{proj}_{L^{\perp}} x$, where $\mathrm{proj}_L x$ (respectively, $\mathrm{proj}_{L^{\perp}} x$) is the orthogonal projection of x on L (respectively, $L^{\perp}$). Hence $\mathrm{proj}_{L^{\perp}} x = x + (-\mathrm{proj}_L x) \in \mathrm{cl}C + L \subset \mathrm{cl}C + \boldsymbol{L}(\mathrm{cl}C) \subset \mathrm{cl}C$. This ends the proof of the claim. ∎

Proof of Theorem 5.1 (a) We always have $f(\mathrm{cl}C) \subset \mathrm{cl}f(C)$. Let $y \in \mathrm{cl}f(C)$, we will show that $y = f(x)$ for some $x \in \mathrm{cl}C$.

Let $L := \boldsymbol{L}(\mathrm{cl}C) \cap \ker f = -\boldsymbol{A}C \cap \boldsymbol{A}C \cap \ker f$. Notice that $L \subset \boldsymbol{A}C \cap \ker f$ and by assumption $\boldsymbol{A}C \cap \ker f \subset \boldsymbol{L}(C) = \boldsymbol{L}(\mathrm{cl}C)$ hence $\boldsymbol{A}C \cap \ker f \subset \boldsymbol{L}(\mathrm{cl}C) \cap \ker f = L$. Therefore $L = \boldsymbol{A}C \cap \ker f$.

First, we claim that $f(L^{\perp} \cap \mathrm{cl}C) = f(\mathrm{cl}C)$. Indeed, $L^{\perp} \cap \mathrm{cl}C \subset \mathrm{cl}C$ hence $f(L^{\perp} \cap \mathrm{cl}C) \subset f(\mathrm{cl}C)$. Conversely, let $x \in \mathrm{cl}C$. From the previous claim, x can be written $x = \ell + z$ with $\ell \in L$ and $z \in L^{\perp} \cap \mathrm{cl}C$. Then $f(x) = f(\ell) + f(z) = f(z) \in f(L^{\perp} \cap \mathrm{cl}C)$ since $\ell \in L \subset \ker f$. This ends the proof of the claim.

Now, since $y \in \mathrm{cl}f(C) \subset \mathrm{cl}f(\mathrm{cl}C) = \mathrm{cl}f(L^{\perp} \cap \mathrm{cl}C)$, for every $k \in \mathbb{N}^*$ the following set, C_k, is not empty

$$C_k = L^{\perp} \cap \mathrm{cl}C \cap \{x : ||y - f(x)|| \leq \frac{1}{k}\}.$$

Notice that $\boldsymbol{A}(C_k) = L^{\perp} \cap \boldsymbol{A}(\mathrm{cl}C) \cap \ker f = L^{\perp} \cap L = \{0\}$. Hence $(C_k)_k$ is a family of bounded closed (hence compact) nonempty sets in $\mathbb{R}^n$, satisfying the finite intersection property (actually the sequence $(C_k)_k$ is non-increasing). Therefore $\cap_k C_k \neq \emptyset$, and every $x \in \cap_k C_k$ satisfies $x \in \mathrm{cl}C$ and $y = f(x)$.

(b) Since $\boldsymbol{A}(f(\mathrm{cl}C)) = \boldsymbol{A}(\mathrm{cl}f(C)) = \boldsymbol{A}(f(C))$ and $f(\boldsymbol{A}(\mathrm{cl}C)) = f(\boldsymbol{A}C)$, we need to show $\boldsymbol{A}(f(C)) = f(\boldsymbol{A}C)$.

Define the convex cone

$$K = \{(\lambda, x) : \lambda > 0, x \in \lambda C\} \subset \mathbb{R}^{n+1},$$

and the linear mapping

$$g : \mathbb{R}^{n+1} \to \mathbb{R}^{m+1}, (\lambda, x) \mapsto (\lambda, f(x)).$$

Then $\boldsymbol{A}K = \boldsymbol{A}(\mathrm{cl}K) = \mathrm{cl}K = K \cup \{(0, z) : z \in \boldsymbol{A}C\}$, and $\ker g = \{0\} \times \ker f$. Hence

$$\boldsymbol{A}K \cap \ker g = \{(0, z) : z \in \boldsymbol{A}C \cap \ker f\} \subset \{(0, z) : z \in \boldsymbol{L}(C)\} = \boldsymbol{L}(K).$$

Therefore, by Part (a), $g(\mathrm{cl}K) = \mathrm{cl}g(K)$. We have

$$g(\mathrm{cl}K) = \{(\lambda, f(x)) : \lambda > 0, x \in \lambda C\} \cup \{\, (0, f(z)) : z \in \boldsymbol{A}C\}$$

$$\begin{aligned}\mathrm{cl}g(K) &= \mathrm{cl}\{(\lambda, y) : \lambda > 0, y \in \lambda f(C)\} \\ &= \{(\lambda, y) : \lambda > 0, y \in \lambda f(C)\} \cup \{\, (0, y) : y \in \boldsymbol{A}(f(C))\}.\end{aligned}$$

Therefore $\boldsymbol{A}f(C) = f(\boldsymbol{A}C)$. ■

Another proof for part (b) of the previous Theorem if C contains 0

We always have $f(\boldsymbol{A}C) \subset \boldsymbol{A}(f(C))$. Indeed, let $v \in \boldsymbol{A}C$. For every $c \in C$, $f(c)+f(v) = f(c+v) \in f(C)$ since $c+v \in C$. Hence $f(v) \in \boldsymbol{A}(f(C))$.

We show that if $\boldsymbol{A}C \cap \ker f \subset \boldsymbol{L}(C)$ then $\boldsymbol{A}(f(C)) \subset f(\boldsymbol{A}C)$.

Let $L = \boldsymbol{L}(C) \cap \ker V$ (from above $L = \boldsymbol{A}C \cap \ker V$).

Let $K := \{x \in \mathrm{cl}C : f(x) \in \mathrm{cl}B(v, 1)\}$ and $K_w := \{\mathrm{proj}_{L^\perp} x \in L^\perp : x \in K\}$. Then $\boldsymbol{A}K = \boldsymbol{A}C \cap \ker f$ and $\boldsymbol{A}K_w = L^\perp \cap \boldsymbol{A}C \cap \ker f = \{0\}$ hence K_w is bounded.

Let $v \in \boldsymbol{A}(f(C))$. Then $v = \lim_n \lambda^n f(c^n) = \lim_n f(\lambda^n c^n)$ where $\lambda^n \downarrow 0$ and $c^n \in C$ for every n. Then for n large enough, $\lambda^n c^n \in K$ (since C contains 0 and convex, $\lambda^n c^n \in C$ for n large enough) and

$$\lambda^n c^n = \lambda^n c^{\perp n} + \lambda^n \hat{c}^n$$

with $c^{\perp n} \in L^\perp \cap C$, $\hat{c}^n \in L \subset \ker f$, and $\lambda^n c^{\perp n} \in K_w$ therefore the sequence $(\lambda^n c^{\perp n})_n$ is bounded and we can assume that it converges to a vector $t \in L^\perp \cap \boldsymbol{A}C$. Hence $v = f(t) \in f(L^\perp \cap \boldsymbol{A}C) = f(\boldsymbol{A}C)$ (because $\boldsymbol{A}C = L + (L^\perp \cap \boldsymbol{A}C)$ and $L \subset \ker f$). ■

5.1.4 Polyhedral convex sets

Theorem 5.2 *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be linear.*

If C is a polyhedral convex set in $\mathbb{R}^n$ then $f(\boldsymbol{A}C) = \boldsymbol{A}(f(C))$.

Proof. (a) Let C be a polyhedral convex set in $\mathbb{R}^n$. From Theorem 19.1 page 171 in Rockafellar [2], C is finitely generated, that is there exist vectors $a_1, \cdots, a_k, a_{k+1}, \cdots, a_r$ such that

$$C = \{\sum_{i=1}^{r} \lambda_i a_i \mid \lambda_1 + \cdots + \lambda_k = 1, \lambda_i \geq 0 \text{ for } i = 1, \cdots, r\}.$$

Therefore $C = K + P$ where K is polyhedral and compact and P is a finitely generated convex cone. Indeed

$$\begin{aligned} K &= \{\sum_{i=1}^{k} \lambda_i a_i \mid \lambda_1 + \cdots + \lambda_k = 1, \lambda_i \geq 0 \text{ for } i = 1, \cdots, k\}, \\ P &= \{\sum_{i=k+1}^{r} \lambda_i a_i \mid \lambda_i \geq 0 \text{ for } i = k+1, \cdots, r\}. \end{aligned}$$

We claim that $\boldsymbol{A}C = P$. Indeed, on the one hand $C + P \subset K + P + P \subset K + P = C$, hence $P \subset \boldsymbol{A}C$. On the other hand, if $v \in \boldsymbol{A}C$ then $v = \lim_\ell \alpha^\ell c^\ell$ where $c^\ell \in C$ and $\alpha^\ell \downarrow 0$. Since $C = K + P$, we have $c^\ell = k^\ell + p^\ell$ with $k^\ell \in K$ and $p^\ell \in P$. From the boundedness of K we get $v = \lim_\ell \alpha^\ell p^\ell$, and since $\alpha^\ell \downarrow 0$ and $p^\ell \in P$ we conclude that $v \in \boldsymbol{A}P$. Recall that P is a convex cone, hence $\boldsymbol{A}P = P$. Therefore $v \in P$ and $\boldsymbol{A}C \subset P$. This ends the proof of the claim.

Let $b_i = f(a_i)$. Then

$$f(C) = \{\sum_{i=1}^{r} \lambda_i b_i \mid \lambda_1 + \cdots + \lambda_k = 1, \lambda_i \geq 0 \text{ for } i = 1, \cdots, r\}.$$

Thus $f(C)$ is finitely generated hence polyhedral and $f(C) = \widetilde{K} + \widetilde{P}$ where $\widetilde{K}$ is the compact $f(K)$ (this set is compact as the image of a compact set by a continuous function) and $\widetilde{P}$ is the finitely generated convex cone $f(P)$. From the claim above we get

$$\boldsymbol{A}f(C) = \widetilde{P} = f(P) = f(\boldsymbol{A}C),$$

which is the desired result. ∎

Corollary 5.1 *If the sets C_i $(i \in I)$ are polyhedral convex sets in $\mathbb{R}^n$ then $\boldsymbol{A}(\sum_{i \in I} C_i) = \sum_{i \in I} \boldsymbol{A}C_i$.*

Proof. Consider the map $f : (\mathbb{R}^n)^I \to \mathbb{R}^n, (x_1, \cdots, x_I) \mapsto \sum_{i \in I} x_i$, and let $C = \times_i C_i$. Then it is easy to check that f is linear, C is polyhedral, and $\boldsymbol{A}C = \Pi_i \boldsymbol{A}C_i$. Applying the result of Theorem 5.2 to f and C yields

$$\boldsymbol{A}f(\Pi_i C_i) = \boldsymbol{A}f(C) = f(\boldsymbol{A}C) = f(\Pi_i \boldsymbol{A}C_i).$$

Since $f(\Pi_i C_i) = \sum_{i \in I} C_i$ and $f(\Pi_i \boldsymbol{A}C_i) = \sum_{i \in I} \boldsymbol{A}C_i$, we get $\boldsymbol{A}(\sum_{i \in I} C_i) = \sum_{i \in I} \boldsymbol{A}C_i$. ∎

5.1.5 The cylindric decomposition of a convex set

Proposition 5.7 *Let X be a nonempty convex subset of $\mathbb{R}^J$, L a subspace of $\mathbb{R}^J$. The following two assertions are equivalent.*

(a) $L + X \subset X$, *that is* $L \subset O^+(X) \cap -O^+(X)$.

(b) $X = L + (X \cap L^\perp)$.

Claim 5.1.2 *Let X be a nonempty convex subset of $\mathbb{R}^J$, L a subspace of $\mathbb{R}^J$, and denote $\pi = \mathrm{proj}_{L^\perp}$. If $L + X \subset X$ then $\pi X = X \cap L^\perp$.*

Proof. Let $x \in X$, then $\pi x = (\pi x - x) + x \in \ker \pi + X = L + X \subset X$, and $\pi x \in \mathrm{Im}\pi = L^\perp$. Hence $\pi x \in X \cap L^\perp$. Conversely, if $x \in X \cap L^\perp$ then $x = \pi x$ (since $x \in L^\perp = \mathrm{Im}\pi$), hence $x \in \pi X$. ■

Proof of Proposition 5.7 Denote $\pi = \mathrm{proj}_{L^\perp}$.

(a) $\Rightarrow$ (b): First we show $X \subset L + (X \cap L^\perp)$. Let $x \in X$, write $x = (x - \pi x) + \pi x \in L + \pi X = L + (X \cap L^\perp)$ (by Claim 5.1.2).

Second we show $L + (X \cap L^\perp) \subset X$. Let $\ell \in L$ and $x \in X \cap L^\perp$. Then $\ell + x \in L + X \subset X$ (by (a)).

(b) $\Rightarrow$ (a): Let $\ell \in L$ and $x \in X$. Use the inclusion $X \subset L + (X \cap L^\perp)$ in assertion (b) to decompose x as $x = x_1 + x_2$ with $x_1 \in L$ and $x_2 \in X \cap L^\perp$. Then $\ell + x = (\ell + x_1) + x_2 \in L + L + (X \cap L^\perp) \subset L + (X \cap L^\perp) \subset X$ by assertion (b). ■

Corollary 5.2 *Let X be a nonempty convex subset of $\mathbb{R}^J$, L a subspace of $\mathbb{R}^J$, and denote $\pi = \mathrm{proj}_{L^\perp}$.* (a') *is equivalent to* (b') *and they both imlpy* (c').

(a') $L + \mathrm{cl}X \subset \mathrm{cl}X$, *that is* $L \subset O^+(\mathrm{cl}X) \cap -O^+(\mathrm{cl}X) = \boldsymbol{L}(X)$.

(b') $\mathrm{cl}X = L + \mathrm{cl}X \cap L^\perp$.

(c') $\pi\mathrm{cl}X = \mathrm{cl}X \cap L^\perp$.

Moreover, if one of the above holds, then

$$X \subset L + \pi X \subset L + \pi\mathrm{cl}X \subset L + \mathrm{cl}\pi X \subset \mathrm{cl}X.$$

Proposition 5.8 *Let X be convex (not assumed to be closed) and L a vector space such that $L \subset \boldsymbol{L}(X)$. Let π be a linear projection such that* $\ker \pi = L$. *Then*

$$X \subset L + \pi X \subset L + \pi\mathrm{cl}X \subset L + \mathrm{cl}\pi X \subset \mathrm{cl}(L + \pi X) \subset \mathrm{cl}X.$$

Moreover $\mathrm{cl}\pi X = (\mathrm{cl}X) \cap \mathrm{Im}\pi = \pi\mathrm{cl}X$, *hence* $\pi\mathrm{cl}X$ *is closed.*

Proof. • First inclusion: Let $x \in X$, write $x = \pi x + (x - \pi x)$ with $x - \pi x \in \ker \pi = L$ hence $x \in L + \pi X$.

• Second inclusion: Obvious since $X \subset \mathrm{cl}X$.

• Third inclusion: π is continuous hence $\pi \mathrm{cl}X \subset \mathrm{cl}\pi X$.

• Fourth inclusion: Comes from the fact that $\sum_{i\in I} \mathrm{cl}A_i \subset \mathrm{cl}(\sum_{i\in I} A_i)$.

• Fifth inclusion: We show $L + \pi X \subset \mathrm{cl}X$. Let $x \in X, \ell \in L$. Then $\ell + \pi x = (\ell + \pi x - x) + x \in L + X \subset \mathrm{cl}X$ since $L \subset \boldsymbol{L}(X)$.

To show $\mathrm{cl}\pi X \subset (\mathrm{cl}X) \cap \mathrm{Im}\pi$, it suffices to show that $\pi X \subset (\mathrm{cl}X) \cap \mathrm{Im}\pi$ (which is closed). Let $x \in X$, $\pi x = x - (x - \pi x) \in X - \ker \pi \subset X - \boldsymbol{L}(X) \subset \mathrm{cl}X$, hence $\pi x \in (\mathrm{cl}X) \cap \mathrm{Im}\pi$.

For the reverse inclusion, that is $(\mathrm{cl}X) \cap \mathrm{Im}\pi \subset \mathrm{cl}\pi X$, it suffices to show $X \cap \mathrm{Im}\pi \subset \pi X$. Let $x \in X \cap \mathrm{Im}\pi$, then $x - \pi x \in \ker \pi$ and $x - \pi x \in \mathrm{Im}\pi$ (since $x \in \mathrm{Im}\pi$), hence $x - \pi x \in \ker \pi \cap \mathrm{Im}\pi = \{0\}$, therefore $x = \pi x$ and $x \in \pi X$.

In fact we have shown that

$$X \cap \mathrm{Im}\pi \subset \pi X \subset \mathrm{cl}X \cap \mathrm{Im}\pi.$$

Applying this result to $\mathrm{cl}X$ we get

$$\mathrm{cl}X \cap \mathrm{Im}\pi \subset \pi\mathrm{cl}X \subset \mathrm{cl}X \cap \mathrm{Im}\pi,$$

thus $\pi\mathrm{cl}X = (\mathrm{cl}X) \cap \mathrm{Im}\pi$ ($= \mathrm{cl}\pi X$ from above). ■

Corollary 5.3 *Let X_i be convex $i \in I$ (finite), and let π be a linear projection such that $\ker \pi = L \subset \boldsymbol{L}(\sum_{i\in I} X_i)$. Then*

$$\sum_{i\in I} X_i \subset L + \pi(\sum_{i\in I} X_i) = L + \sum_{i\in I} \pi X_i \subset L + \sum_{i\in I} \mathrm{cl}\pi X_i \subset \mathrm{cl}(L + \sum_{i\in I} \pi X_i) \subset \mathrm{cl}(\sum_{i\in I} X_i).$$

Proof. Take $X = \sum_{i\in I} X_i$ and apply the above proposition to get

$$\sum_{i\in I} X_i \subset L + \pi \sum_{i\in I} X_i \subset L + \sum_{i\in I} \pi\mathrm{cl}X_i \subset L + \sum_{i\in I} \mathrm{cl}\pi X_i,$$

and notice that $\sum_{i\in I} \mathrm{cl}A_i \subset \mathrm{cl}(\sum_{i\in I} A_i)$, hence $L + \sum_{i\in I} \mathrm{cl}\pi X_i \subset \mathrm{cl}(L + \pi \sum_{i\in I} X_i) \subset \mathrm{cl}\sum_{i\in I} X_i$. ■

Example 5.2 *In $\mathbb{R}^2$, let $X_1 = \mathbb{R}^2_+$ and $X_2 = \mathbb{R}_- \times \mathbb{R}_+$ Then $X_1 + X_2 = \mathbb{R} \times \mathbb{R}_+$, $\boldsymbol{L} = \boldsymbol{L}(X_1 + X_2) = \mathbb{R} \times \{0\}$, and if $\pi = \mathrm{proj}_{\boldsymbol{L}^\perp}$ then $\pi X_1 = \pi X_2 = \{0\} \times \mathbb{R}_+$.*

5.2 Positive semi-independence and consequences

5.2.1 Different definitions of positive semi-independence

Definition 5.4 *The family $(C_i)_{i\in I}$ of closed convex cones of $\mathbb{R}^m$ is said to be*

(i) weakly positively semi-independent (WPSI) if

$$\forall i, v_i \in C_i, \sum_{i\in I} v_i = 0 \Rightarrow v_i \in C_i \cap -C_i, \forall i.$$

(ii) positively semi-independent (PSI) if

$$\forall i, v_i \in C_i, \sum_{i\in I} v_i = 0 \Rightarrow v_i = 0, \ \forall i.$$

(iii) strongly positively semi-independent (SPSI) if

$$\sum_{i\in I} C_i \cap - \sum_{i\in I} C_i = \{0\}.$$

Remark 5.3 If the C_i's are vector subspaces of $\mathbb{R}^m$ then

- Condition (i) is always satisfied.
- Condition (ii) means that the C_i's are linearly independent vector subspaces.
- Condition (iii) holds if and only if, for every i, $C_i = \{0\}$.
- If, for every i, $C_i \cap -C_i = \{0\}$ then the three notions of positive semi-independence coincide, that is, the C_i's are SPSI if and only if they are WPSI.

Proposition 5.9 *Let C_i $(i \in I)$ be finitely many closed convex cones of $\mathbb{R}^J$.*

(a) Then

$$\sum_{i\in I} [C_i \cap -C_i] \subset \sum_{i\in I} C_i \cap - \sum_{i\in I} C_i.$$

(b) If we additionally assume that the C_i are weakly positively semi-independent then

$$\sum_{i\in I} [C_i \cap -C_i] = \sum_{i\in I} C_i \cap - \sum_{i\in I} C_i.$$

Proof. *Part (a)* Straightforward.

Part (b) Let $c \in \sum_{i\in I} C_i \cap - \sum_{i\in I} C_i$ then $c = \sum_{i\in I} c_i = - \sum_{i\in I} c_i'$ for some c_i, c_i' in C_i. Consequently, $\sum_{i\in I} c_i + c_i' = 0$, with $c_i + c_i' \in C_i$ (because C_i is convex). Since the C_i are weakly positively semi-independent, we deduce that for all i, $c_i + c_i' \in C_i \cap -C_i$, hence $c_i \in C_i \cap -C_i$. This shows that $c = \sum_{i\in I} c_i \in \sum_{i\in I} [C_i \cap -C_i]$. ■

The following propositions will show the relationship between these three definitions. In particular it shows that a SPSI family is PSI, and that a PSI family is WPSI.

Proposition 5.10 *Let $(C_i)_{i\in I}$ be a family of closed convex cones of $\mathbb{R}^m$. Then the following assertions are equivalent.*

(i) The sets C_i are PSI.

(ii) The sets C_i are WPSI and the vector spaces $C_i \cap -C_i$ are linearly independent.

Proof. $[(i) \Rightarrow (ii)]$. Assume C_i are PSI. Let $v_i \in C_i$ be such that $\sum_{i\in I} v_i = 0$. Then, by PSI, for each i, $v_i = 0 \in C_i \cap -C_i$. Hence WPSI. Let $v_i \in C_i \cap -C_i$ be such that $\sum_{i\in I} v_i = 0$. Then for every i, $v_i \in C_i$ and hence $v_i = 0$ (by PSI), which shows that the sets $C_i \cap -C_i$ are independent.

$[(ii) \Rightarrow (i)]$. Conversely, if $0 = \sum_{i\in I} v_i$ where each $v_i \in C_i$. Then, from WPSI, $v_i \in C_i \cap -C_i$, but by assumption the vector spaces $C_i \cap -C_i$ are linearly independent, hence $v_i = 0$ for each i, that is the C_i's are PSI. ∎

Proposition 5.11 *Let C_i $(i \in I)$ be finitely many closed convex cones of $\mathbb{R}^J$. The following three properties are equivalent:*

(i) $\sum_{i\in I} C_i \cap -\sum_{i\in I} C_i = \{0\}$, (that is $\boldsymbol{L}[\sum_{i\in I} C_i] = \{0\}$),

(ii) for all i $C_i \cap -C_i = \{0\}$ (that is $\boldsymbol{L}C_i = \{0\}$) and the C_i are positively semi-independent,

(iii) for all i $C_i \cap -C_i = \{0\}$ (that is $\boldsymbol{L}C_i = \{0\}$) and the C_i are weakly positively semi-independent.

Proof. $(ii) \Leftrightarrow (iii)$ Straightforward.

$(i) \Rightarrow (ii)$. From Proposition 5.18 and from (i), we have

$$\sum_{i\in I}[C_i \cap -C_i] \subset \sum_{i\in I} C_i \cap -\sum_{i\in I} C_i = \{0\}.$$

We now prove that the C_i are positively semi-independent. Indeed, let $c_i \in C_i$ such that $\sum_i c_i = 0$, then

$$c_1 = -\sum_{i\neq 1} c_i \in C_1 \cap -\sum_{i\neq 1} C_i \subset \sum_{i\in I} C_i \cap -\sum_{i\in I} C_i = \{0\}.$$

Consequently, $c_1 = 0$ and similarly $c_i = 0$ for all i.

$(ii) \Rightarrow (i)$. This is a consequence of Part (b) of Proposition 5.18. A direct proof goes as follows. Let $c \in \sum_{i\in I} C_i \cap -\sum_{i\in I} C_i$ then $c = \sum_{i\in I} c_i = -\sum_{i\in I} c'_i$ for some c_i, c'_i in C_i. Consequently, $\sum_{i\in I} c_i + c'_i = 0$, with $c_i + c'_i \in C_i$. Since the C_i are positively semi-independent, we deduce that for all i, $c_i + c'_i = 0$, hence $c_i \in C_i \cap -C_i = \{0\}$. This shows that $c = \sum_{i\in I} c_i = 0$. ∎

Remark 5.4 If we remove in (ii) and (iii) the condition that the C_i are positively semi-independent the results may not hold, that is, the equality in (i) may not hold.

Consider $C_1 = -\mathbb{R}_+$ and $C_2 = \mathbb{R}_+$. Then the cones C_1 and C_2 are pointed but not weakly positively semi-independent and

$$\{0\} = (C_1 \cap -C_1) + (C_2 \cap -C_2) \subset (C_1 + C_2) \cap -(C_1 + C_2) = \mathbb{R} \neq \{0\}.$$

Remark 5.5 The above condition (i) is strictly stronger that the fact that the C_i are positively semi-independent. In other words the equivalence between (i) and (ii) does not hold in general if we remove in (ii) the condition that for all i, $C_i \cap -C_i = \{0\}$.

Consider $C_1 = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$ and $C_2 = \{(x, y) \in \mathbb{R}^2 : x = 0 \text{ and } y \geq 0\}$. Then the cones C_1 and C_2 are positively semi-independent but

$$(C_1 + C_2) \cap -(C_1 + C_2) = \{(x, y) \in \mathbb{R}^2 : y = 0\} \neq \{0\}.$$

5.2.2 Further properties of positive semi-independence

Proposition 5.12 *Let C_i $(i \in I)$ be finitely many closed convex cones of $\mathbb{R}^J$. The following assertions are equivalent.*

(i) The sets C_i are WPSI that is $\forall v_i \in C_i, \sum_{i\in I} v_i = 0 \Rightarrow \forall i, v_i \in \boldsymbol{L}(C_i)$.

(ii) For all $v_i \in C_i$, $\sum_{i\in I} v_i \in \sum_{i\in I} \boldsymbol{L}(C_i) \Rightarrow \forall i, v_i \in \boldsymbol{L}(C_i)$.

(ii') For all $v_i \in C_i \cap \boldsymbol{L}(C_i)^{\perp}$, $\sum_{i\in I} v_i \in \sum_{i\in I} \boldsymbol{L}(C_i) \Rightarrow \forall i, v_i = 0$.

(iii) For all $v_i \in C_i$, $\sum_{i\in I} v_i \in \boldsymbol{L}(\sum_{i\in I} C_i) \Rightarrow \forall i, v_i \in \boldsymbol{L}(C_i)$.

(iv) For all $i \in I$, $C_i \cap \boldsymbol{L}(\sum_{j\in I} C_j) \subset \boldsymbol{L}(C_i)$.

Proof. The implications $(iii) \Rightarrow (ii) \Rightarrow (i)$ are obvious. We show $(i) \Rightarrow (iii)$. Under WPSI, by Proposition 5.18, we have $\boldsymbol{L}(\sum_{i\in I} C_i) = \sum_{i\in I} \boldsymbol{L}(C_i)$. Let $v_i \in C_i$ with $\sum_{i\in I} v_i \in \boldsymbol{L}(\sum_{i\in I} C_i)$. Then $\sum_{i\in I} v_i \in \sum_{i\in I} \boldsymbol{L}(C_i)$, hence $\sum_{i\in I} v_i = \sum_{i\in I} \ell_i$ with $\ell_i \in \boldsymbol{L}(C_i) =$

$C_i \cap -C_i$. Thus $\sum_{i\in I}(v_i - \ell_i) = 0$ and $v_i - \ell_i \in C_i$ for each i. Therefore, by WPSI, $v_i - \ell_i \in \boldsymbol{L}(C_i)$ for each i. Consequently, for every i, $v_i \in \ell_i + \boldsymbol{L}(C_i) \subset \boldsymbol{L}(C_i)$.

(iii) $\Rightarrow$ *(iv)*: Let $v_1 \in C_1 \cap \boldsymbol{L}(\sum_{i\in I} C_i)$. Let $v_2 = \cdots = v_I = 0$. Then $\sum_{i\in I} v_i = v_1 \in \boldsymbol{L}(\sum_{i\in I} C_i)$ and $\forall i, v_i \in C_i$. Therefore, by *(iii)*, $\forall i, v_i \in \boldsymbol{L}(C_i)$ in particular $v_1 \in \boldsymbol{L}(C_1)$.

(iv) $\Rightarrow$ *(i)*: Let $v_i \in C_i, \sum_{i\in I} v_i = 0$. Then $v_1 = -\sum_{i\neq 1} v_i \in \sum_{i\in I} C_i \cap -\sum_{i\in I} C_i$. Hence $v_1 \in \boldsymbol{L}(\sum_{i\in I} C_i)$ and by *(iv)* one has $v_1 \in \boldsymbol{L}(C_1)$.

(ii) $\Rightarrow$ *(ii′)*: Let $v_i \in C_i \cap \boldsymbol{L}(C_i)^\perp$, $\sum_{i\in I} v_i \in \sum_{i\in I} \boldsymbol{L}(C_i)$. Then form *(ii)*, for every i, $v_i \in \boldsymbol{L}(C_i)$. But $v_i \in \boldsymbol{L}(C_i)^\perp$. Therefore $v_i = 0$ for every i.

(ii′) $\Rightarrow$ *(ii)*: Let $v_i \in C_i$, $\sum_{i\in I} v_i \in \sum_{i\in I} \boldsymbol{L}(C_i)$. Write $v_i = \ell_i + p_i$, with $\ell_i \in \boldsymbol{L}(C_i), p_i \in C_i \cap \boldsymbol{L}(C_i)^\perp$. Then from $\sum_{i\in I} v_i \in \sum_{i\in I} \boldsymbol{L}(C_i)$ we get $\sum_{i\in I} p_i \in \sum_{i\in I} \boldsymbol{L}(C_i)$ and form *(ii′)* we obtain $p_i = 0$ for every i. hence $v_i = \ell_i \in \boldsymbol{L}(C_i)$ for every i. ∎

Remark 5.6 Condition *(ii′)* in Proposition 5.12 says that the sets $C_i \cap \boldsymbol{L}(C_i)^\perp$ $(i \in I)$ and $\sum_{i\in I} \boldsymbol{L}(C_i)$ are positively semi-independent (PSI). In particular the sets $C_i \cap \boldsymbol{L}(C_i)^\perp$ $(i \in I)$ are PSI and the sets $C_i \cap \boldsymbol{L}(\sum_{i\in I} C_i)^\perp$ $(i \in I)$ are PSI.

Proposition 5.13 *If the family $\{C_i : i \in I\}$ is WPSI then*

(i) The family $\{C_i \cap \boldsymbol{L}_i^\perp : i \in I\}$ is PSI, where $\boldsymbol{L}_i = \boldsymbol{L}(C_i)$.

(ii) The family $\{\mathrm{proj}_{\boldsymbol{L}^\perp} C_i : i \in I\}$ is PSI, where $\boldsymbol{L} = \boldsymbol{L}(\sum_{i\in I} C_i)$.

(iii) The family $\{C_i \cap \boldsymbol{L}^\perp : i \in I\}$ is PSI, where $\boldsymbol{L} = \boldsymbol{L}(\sum_{i\in I} C_i)$.

Proof. Notice that *(i)* implies *(iii)* and *(ii)* implies *(iii)*. We show that if the C_i's are WPSI then *(i)* and *(ii)* hold.

For *(i)*, let $v_i \in C_i \cap \boldsymbol{L}_i^\perp$ such that $\sum_{i\in I} v_i = 0$. Then for each i, $v_i \in C_i$ and $\sum_{i\in I} v_i = 0$ thus by weak positive semi-independence, $v_i \in \boldsymbol{L}_i$, for every i. Therefore $v_i \in \boldsymbol{L}_i \cap \boldsymbol{L}_i^\perp = \{0\}$ i.e. $v_i = 0$ for each i.

To show *(ii)*, let $u_i \in \mathrm{proj}_{\boldsymbol{L}^\perp} C_i$ be such that $\sum_{i\in I} u_i = 0$. Write $u_i = \mathrm{proj}_{\boldsymbol{L}^\perp} v_i$ for some $v_i \in C_i$. Then each v_i can be written $v_i = u_i + \ell_i$ with $\ell_i \in \boldsymbol{L}$. Hence $v_i \in C_i$ for each i and $\sum_{i\in I} v_i = \sum_{i\in I} \ell_i \in \boldsymbol{L}$, therefore by weak positive semi-independence, $v_i \in \boldsymbol{L}_i \subset \boldsymbol{L}$ for each i, thus $u_i = v_i - \ell_i \in \boldsymbol{L} \cap \boldsymbol{L}^\perp = \{0\}$. ∎

Remark 5.7 None of the properties (i), (ii), and (iii) is sufficient for WPSI to hold.

In $\mathbb{R}^2$, let $C_1 = \mathbb{R}(1,0)$, $C_2 = \mathbb{R}(0,1)$, and $C_3 = \mathbb{R}_+(-1,-1)$. Then (i) is satisfied but the C_i's are not WPSI.

The above counter-example also shows that $\boldsymbol{L}(\sum_{i\in I} C_i) = \sum_{i\in I}\boldsymbol{L}(C_i)$ does not imply that the C_i's are WPSI.

In $\mathbb{R}^2$, let $C_1 = \mathbb{R}\times\mathbb{R}_+$, and $C_2 = \mathbb{R}\times\mathbb{R}_-$. Then (ii) and (iii) are satisfied but the C_i's are not WPSI.

Proposition 5.14 *Let C_i $(i \in I)$ be finitely many closed convex cones of $\mathbb{R}^J$. The following assertions are equivalent.*

(i) The sets C_i are SPSI that is $\sum_{i\in I} C_i \cap -\sum_{i\in I} C_i = \{0\}$.

(ii) $\sum_{i\in I} C_i \cap - \cup_i C_i = \{0\}$.

(iii) $\forall i, v_i \in C_i, \sum_{i\in I} v_i \in - \cup_i C_i \Rightarrow \forall i, v_i = 0$.

Proof. [$(i) \Rightarrow (ii)$.] This implication is obvious since $\cup_i C_i \subset \sum_{i\in I} C_i$.

[$(ii) \Rightarrow (iii)$.] Let $v_i \in C_i$ be such that $\sum_{i\in I} v_i \in - \cup_i C_i$. From (ii) we get $\sum_{i\in I} v_i = 0$. Hence $-v_1 = \sum_{i\neq 1} v_i \in -C_1 \cap \sum_{i\in I} C_i \subset - \cup_i C_i \cap \sum_{i\in I} C_i = \{0\}$. Therefore $v_1 = 0$, and similarly $v_i = 0$ for each i.

[$(iii) \Rightarrow (i)$.] Let $v_i \in C_i$ be such that $\sum_{i\in I} v_i \in -\sum_{i\in I} C_i$. Then $\sum_{i\in I} v_i = -\sum_{i\in I} c_i$ with $c_i \in C_i$ for each i. Hence $\sum_{i\in I}(v_i + c_i) = 0$ and from (iii) we get for each i, $v_i + c_i = 0$ (since $v_i + c_i \in C_i$ by convexity of C_i). Thus $v_i \in C_i \subset \sum_{i\in I} C_i$ and $v_i = -c_i \in -C_i \subset - \cup_i C_i$. Hence, by (iii), $v_i = 0$. ∎

5.2.3 Characterizing semi-independence by compactness

Let X_i $(i \in I)$ be convex subsets of $\mathbb{R}^J$ containing 0 and denote $\boldsymbol{L}_i = \boldsymbol{L}(X_i) = \boldsymbol{L}(\mathrm{cl} X_i)$. Let B be a compact set of $\mathbb{R}^J$ and

$$
\begin{aligned}
K &:= \{(x_1, \cdots, x_I) \in \Pi_i \mathrm{cl} X_i : \sum_{i\in I} x_i \in B\}, \\
K_w &:= \{(\mathrm{proj}_{\boldsymbol{L}_1^\perp} x_1, \cdots, \mathrm{proj}_{\boldsymbol{L}_I^\perp} x_I) \in \Pi_i {\boldsymbol{L}_i}^\perp : (x_1, \cdots, x_I) \in K\}, \\
K'_w &:= \{(y_1, \cdots, y_I) \in \Pi_i (X_i \cap \boldsymbol{L}_i^\perp) : \sum_{i\in I} y_i \in B + \sum_{i\in I} \boldsymbol{L}_i\}.
\end{aligned}
$$

Note that $K_w = F(K)$ where $F : (\mathbb{R}^J)^I \to (\mathbb{R}^J)^I$ is defined by

$$F(x_1, \cdots, x_I) = (\mathrm{proj}_{\boldsymbol{L}_1^\perp} x_1, \cdots, \mathrm{proj}_{\boldsymbol{L}_I^\perp} x_I).$$

Proposition 5.15 *The following assertions are equivalent.*

(i) The sets $\boldsymbol{A}X_i$ are weakly positively semi-independent.

(ii) The set K_w is bounded.

(iii) The set K'_w is bounded (in fact $K'_w = K_w$).

Moreover the set K_w is closed (without assuming (i)).

Here are two proofs. The first one uses Theorem 9.1 in [2]. The second one is a direct proof.

First Proof

[$(i) \Rightarrow (ii)$] If $(\mathrm{proj}_{\boldsymbol{L}_1^\perp} v_1, \cdots, \mathrm{proj}_{\boldsymbol{L}_I^\perp} v_I) \in \boldsymbol{A}K_w$ with $v_i \in \boldsymbol{A}X_i$, and $\sum_{i \in I} v_i = 0$, then from **WPSI** we get $v_i \in \boldsymbol{L}_i$ for every i. That is $\mathrm{proj}_{\boldsymbol{L}_i^\perp} v_i = 0, \forall i$. Hence $\boldsymbol{A}K_w = \{0\}$ and K_w is bounded.

[$(ii) \Rightarrow (i)$] Conversely, if K_w is compact then $\boldsymbol{A}K_w = \{0\}$. Thus if $v_i \in \boldsymbol{A}X_i$ and $\sum_{i \in I} v_i = 0$, we must have $\mathrm{proj}_{\boldsymbol{L}_i^\perp} v_i = 0$ for every i. That is $v_i \in \boldsymbol{L}_i, \forall i$. Hence WPSI.

Proof that the set K_w is closed. We have $\ker F = \Pi_i \boldsymbol{L}_i$ and

$$\begin{aligned} \boldsymbol{A}K &= \{(v_1, \cdots, v_I) \in \Pi_i \boldsymbol{A}X_i : \sum_{i \in I} v_i = 0\}, \\ \boldsymbol{L}(K) &= \{(v_1, \cdots, v_I) \in \Pi_i \boldsymbol{L}_i : \sum_{i \in I} v_i = 0\}. \end{aligned}$$

Then $\ker F \cap \boldsymbol{A}K \subset \boldsymbol{L}(K)$. Hence, by Theorem 9.1 page 73 in [2], $\boldsymbol{A}(F(K)) = F(\boldsymbol{A}K)$ and $F(K)$ is closed (since K is obviously closed). That is,

$$\boldsymbol{A}K_w = \{(\mathrm{proj}_{\boldsymbol{L}_1^\perp} v_1, \cdots, \mathrm{proj}_{\boldsymbol{L}_I^\perp} v_I) \in \Pi_i {\boldsymbol{L}_i}^\perp : v_i \in \boldsymbol{A}X_i, \sum_{i \in I} v_i = 0\},$$

and K_w is closed. ■

Second Proof

[$(i) \Rightarrow (ii)$] By contradiction, assume K_w is not bounded and let $((x_i^{\perp n})_i)_n$ be a sequence in K_w (each $x_i^{\perp n}$ is in $\mathrm{cl} X_i \cap \boldsymbol{L}_i^\perp$) such that $\sum_{i \in I} ||x_i^{\perp n}|| \underset{n \to \infty}{\longrightarrow} \infty$. Let $\hat{x}_i^n \in \boldsymbol{L}_i$ be such that

$(x_i^{\perp n} + \hat{x}_i^n)_i \in K$. Then, without loss of generality (taking subsequences if necessary), one can assume that for every i,

$$\frac{x_i^{\perp n}}{\sum_{i\in I}||x_i^{\perp n}|| + \sum_{i\in I}||\hat{x}_i^n||} \underset{n\to\infty}{\longrightarrow} x_i^{\perp} \in \boldsymbol{A}X_i \cap \boldsymbol{L}_i^{\perp}$$

and

$$\frac{\sum_{i\in I}\hat{x}_i^n}{\sum_{i\in I}||x_i^{\perp n}|| + \sum_{i\in I}||\hat{x}_i^n||} \underset{n\to\infty}{\longrightarrow} \alpha \in \boldsymbol{A}(\sum_{i\in I}X_i) \cap \sum_{i\in I}\boldsymbol{L}_i.$$

Write $\alpha = \sum_{i\in I}\alpha_i$ where for each i, $\alpha_i \in \boldsymbol{L}_i$. Then $\sum_{i\in I}(x_i^{\perp} + \alpha_i) = 0$ since $\sum_{i\in I}(x_i^{\perp n} + \hat{x}_i^n) \in B$ and $\sum_{i\in I}||x_i^{\perp n}|| \underset{n\to\infty}{\longrightarrow} \infty$. But $x_i^{\perp} \in \boldsymbol{A}X_i$ and $\alpha_i \in \boldsymbol{L}_i$ then $x_i^{\perp} + \alpha_i \in \boldsymbol{A}X_i$ hence, by WPSI, for every i, $x_i^{\perp} + \alpha_i \in \boldsymbol{L}_i$ that is $x_i^{\perp} = 0$. So, $\sum_{i\in I}\alpha_i = 0$. But for every n,

$$1 = \frac{\sum_{i\in I}||x_i^{\perp n}||}{\sum_{i\in I}||x_i^{\perp n}|| + \sum_{i\in I}||\hat{x}_i^n||} + \frac{||\sum_{i\in I}\hat{x}_i^n||}{\sum_{i\in I}||x_i^{\perp n}|| + \sum_{i\in I}||\hat{x}_i^n||}$$

implies $1 = ||\sum_i \alpha_i||$. A contradiction.

[$(ii) \Rightarrow (i)$] Conversely, if $v_i \in \boldsymbol{A}X_i$, and $\sum_{i\in I}v_i = 0$, then for each i, $v_i = v_i^{\perp} + \hat{v}_i$ with $v_i^{\perp} \in \boldsymbol{A}X_i \cap \boldsymbol{L}_i^{\perp}$ and $\hat{v}_i \in \boldsymbol{L}_i$. Let $(x_i)_i \in K$, then for every $t \geq 0$, $\sum_{i\in I}(x_i + tv_i) = \sum_{i\in I}x_i \in B$. Therefore $(\mathrm{proj}_{\boldsymbol{L}_i^{\perp}}x_i^{\perp} + tv_i^{\perp})_i \in K_w$ for every $t \geq 0$. Since K_w is bounded we must have $v_i^{\perp} = 0$ for every i, that is $v_i \in \boldsymbol{L}_i$ for each i.

Now we show that K_w is closed. Let $((\mathrm{proj}_{\boldsymbol{L}_i^{\perp}}x_i^n)_i)_n$ be a sequence in K_w (the sequence $((x_i^n)_i)_n$ is in K) such that $\mathrm{proj}_{\boldsymbol{L}_i^{\perp}}x_i^n \underset{n\to\infty}{\longrightarrow} x_i^{\perp} \in \boldsymbol{L}_i^{\perp} \cap \mathrm{cl}X_i$ for each i. For each n, let $(\hat{x}_i^n)_i \in \Pi_i\boldsymbol{L}_i$ be such that $(\mathrm{proj}_{\boldsymbol{L}_i^{\perp}}x_i^n)_i + (\hat{x}_i^n)_i \in K$. That is

$$\sum_{i\in I}\mathrm{proj}_{\boldsymbol{L}_i^{\perp}}x_i^n + \sum_{i\in I}\hat{x}_i^n \in B.$$

The first term, $\sum_{i\in I}\mathrm{proj}_{\boldsymbol{L}_i^{\perp}}x_i^n$, converges to $\sum_{i\in I}x_i^{\perp}$, and since B is compact we can assume that the second term, $\sum_{i\in I}\hat{x}_i^n$, converges to some α. The limit α is in $\sum_{i\in I}\boldsymbol{L}_i$, hence $\alpha = \sum_{i\in I}\alpha_i$ where, for each i, $\alpha_i \in \boldsymbol{L}_i$. Since $x_i^{\perp} \in \boldsymbol{L}_i^{\perp} \cap \mathrm{cl}X_i$ and $\alpha_i \in \boldsymbol{L}_i$, and $\sum_{i\in I}(x_i^{\perp} + \alpha_i) \in B$, we get $(x_i^{\perp} + \alpha_i)_i \in K$ hence $(x_i^{\perp})_i \in K_w$. ■

Proposition 5.16 *The following assertions are equivalent.*

(i) The sets $\boldsymbol{A}X_i$ are positively semi-independent.

(ii) The set $K := \{(x_1, \cdots, x_I) \in \Pi_i\mathrm{cl}X_i : \sum_{i\in I}x_i \in B\}$ is bounded.

Moreover the set K is closed (without assuming (i)).

Proof. The set K is obviously closed, convex and its asymptotic cone is

$$\boldsymbol{A}K = \{(v_1, \cdots, v_I) \in \Pi_i\boldsymbol{A}X_i : \sum_{i\in I}v_i = 0\}.$$

It is easy to check that the sets $\boldsymbol{A}X_i$ are positively semi-independent if and only if $\boldsymbol{A}K = \{0\}$ which in turn is equivalent to K being bounded. ■

Proposition 5.17 *Let B be a convex compact. The following assertions are equivalent.*

(i) The sets $\boldsymbol{A}X_i$ are strongly positively semi-independent.

(ii) The set $K_s = \{(x_i)_i \in \Pi_i \mathrm{cl} X_i : -\sum_{i\in I} x_i \in \sum_{i\in I} \mathrm{cl} X_i + B\}$ is bounded.

Moreover the set K_s is closed (without assuming (i)).

Proof. The set K_s is obviously closed (by Proposition 5.18) and convex, and its asymptotic cone is (note that $\boldsymbol{A}(\sum_{i\in I} X_i) \subset \boldsymbol{A}(\sum_{i\in I} \mathrm{cl} X_i) \subset \boldsymbol{A}(\mathrm{cl} \sum_{i\in I} X_i) = \boldsymbol{A}(\sum_{i\in I} X_i))$

$$\boldsymbol{A}K_s = \{(v_i)_i \in \Pi_i \boldsymbol{A}X_i, -\sum_{i\in I} v_i \in \boldsymbol{A}(\sum_{i\in I} X_i)\}.$$

We show that the sets $\boldsymbol{A}X_i$ are strongly positively semi-independent if and only if $\boldsymbol{A}K_s = \{0\}$. Assume SPSI of the sets $\boldsymbol{A}X_i$ and let $(v_1, \cdots, v_I) \in \boldsymbol{A}K_s$. From $v_i \in \boldsymbol{A}X_i$ for each i, and $-\sum_{i\in I} v_i \in \boldsymbol{A}(\sum_{i\in I} X_i)$, we get $\sum_{i\in I} v_i \in \sum_{i\in I} \boldsymbol{A}X_i \cap -\boldsymbol{A}(\sum_{i\in I} X_i)$. Hence, by strong semi-independence, $\sum_{i\in I} v_i = 0$. Recalling that for every i, $v_i \in \boldsymbol{A}X_i$ and that, by Proposition 5.11, the sets $\boldsymbol{A}X_i$ are positively-semi-independent, we conclude $v_i = 0$ for every i. The converse is immediate. ■

5.2.4 Closedness properties

Proposition 5.18 *Let X_i $(i \in I)$ be convex subsets of $\mathbb{R}^J$ containing 0.*

(a) Then

$\sum_{i\in I} \boldsymbol{A}X_i \subset \boldsymbol{A}(\sum_{i\in I} X_i)$.

$\sum_{i\in I} \boldsymbol{L}(X_i) \subset \boldsymbol{L}(\sum_{i\in I} \boldsymbol{A}X_i) \subset \boldsymbol{L}(\sum_{i\in I} X_i)$.

(b) If we additionally assume that the sets $\boldsymbol{A}X_i$ are weakly positively semi-independent then

(i) the set $\sum_{i\in I} \mathrm{cl} X_i$ is closed,

and the above inclusions are equalities, that is

(ii) $\sum_{i\in I} \boldsymbol{A}X_i = \boldsymbol{A}(\sum_{i\in I} X_i)$,

(iii) $\sum_{i\in I} \boldsymbol{L}(X_i) = \boldsymbol{L}(\sum_{i\in I} \boldsymbol{A}X_i) = \boldsymbol{L}(\sum_{i\in I} X_i)$.

Proof. (a) We first notice that, for all i, $\boldsymbol{L}(X_i) \subset \boldsymbol{A}X_i \subset X_i$. Hence

$$\sum_{i\in I} \boldsymbol{L}(X_i) \subset \sum_{i\in I} \boldsymbol{A}X_i \subset \sum_{i\in I} X_i.$$

Using the fact that $\boldsymbol{L}(A) \subset \boldsymbol{L}(B)$ if $A \subset B$ we get

$$\sum_{i\in I} \boldsymbol{L}(X_i) \subset \boldsymbol{L}(\sum_{i\in I} \boldsymbol{A}X_i) \subset \boldsymbol{L}(\sum_{i\in I} X_i).$$

Part (b) (i) Let $\sum_{i\in I} x_i^n \underset{n\to\infty}{\longrightarrow} \alpha$ where $x_i^n \in \mathrm{cl}X_i$. Then

$$\sum_{i\in I} x_i^{\perp n} + \sum_{i\in I} \hat{x}_i^n \underset{n\to\infty}{\longrightarrow} \alpha.$$

Notice that (by Proposition 5.15) for each i, $x_i^{\perp n} \underset{n\to\infty}{\longrightarrow} x_i^{\perp} \in \mathrm{cl}X_i \cap \boldsymbol{L}_i^{\perp}$ and we can assume that $\sum_{i\in I} \hat{x}_i^n \underset{n\to\infty}{\longrightarrow} \beta = \sum_{i\in I} \beta_i$ where $\beta_i \in \boldsymbol{L}_i$ for each i. Then $\alpha = \sum_{i\in I}(x_i^{\perp} + \beta_i) \in \sum_{i\in I} \mathrm{cl}X_i$.

Part (b) (ii) Let $v \in \boldsymbol{A}(\sum_{i\in I} X_i)$. Write

$$v = \sum_{i\in I} \frac{1}{n} x_i^n = \sum_{i\in I} \frac{1}{n} x_i^{\perp n} + \sum_{i\in I} \frac{1}{n} \hat{x}_i^n$$

where, for each i, $x_i^n \in X_i$, $x_i^{\perp n} \in X_i \cap \boldsymbol{L}_i^{\perp} \subset \mathrm{cl}X_i \cap \boldsymbol{L}_i^{\perp}$, and $\hat{x}_i^n \in \boldsymbol{L}_i$. Then (by Proposition 5.15), for each i,

$$\frac{1}{n} x_i^{\perp n} \underset{n\to\infty}{\longrightarrow} x_i^{\perp} \in \boldsymbol{A}X_i \cap \boldsymbol{L}_i^{\perp}$$

and

$$\sum_{i\in I} \frac{1}{n} \hat{x}_i^n \underset{n\to\infty}{\longrightarrow} \beta \in \sum_{i\in I} \boldsymbol{L}_i.$$

Write $\beta = \sum_{i\in I} \beta_i$ with $\beta_i \in \boldsymbol{L}_i$ for each i. Then $v = \sum_{i\in I}(x_i^{\perp} + \beta_i) \in \sum_{i\in I} \boldsymbol{A}X_i$.

Part (b) (iii) From (ii) above, we get $\boldsymbol{L}(\sum_{i\in I} X_i) \subset \boldsymbol{L}(\sum_{i\in I} \boldsymbol{A}X_i)$. We show that $\boldsymbol{L}(\sum_{i\in I} \boldsymbol{A}X_i) \subset \sum_{i\in I} \boldsymbol{L}(X_i)$. Let $v \in \boldsymbol{L}(\sum_{i\in I} \boldsymbol{A}X_i)$. Write

$$v = \sum_{i\in I} v_i = -\sum_{i\in I} v_i'$$

with $v_i, v_i' \in \boldsymbol{A}X_i$. Then $0 = \sum_{i\in I}(v_i + v_i')$ and for each i, $v_i + v_i' \in \boldsymbol{A}X_i$ which implies (under WPSI) that for every i, $v_i + v_i' \in \boldsymbol{L}(X_i)$. Hence

$$v_i = -v_i' + (v_i + v_i') \in -\boldsymbol{A}X_i + \boldsymbol{L}(X_i) \subset -\boldsymbol{A}X_i.$$

Therefore for every i, $v_i \in \boldsymbol{L}(X_i)$ that is $v = \sum_{i\in I} v_i \in \sum_{i\in I} \boldsymbol{L}(X_i)$. ■

Remark 5.8 $\boldsymbol{L}(\sum_{i\in I} X_i) \subset \boldsymbol{L}(\sum_{i\in I} \boldsymbol{A}X_i)$. Let $v \in \boldsymbol{L}(\sum_{i\in I} X_i) \subset \boldsymbol{A}(\sum_{i\in I} X_i)$, then for every integer n, there exists $x_i^n \in X_i$ such that $nv = \sum_{i\in I} x_i^n$ or equivalently $v = \sum_{i\in I} x_i^n/n$ and we notice that $x_i^n/n \in X_i$ (since X_i is convex and contains 0). Consider now the set

$$K := \{(x_1, \ldots, x_I) \in \Pi_{i\in I}\mathrm{cl}X_i : \sum_{i\in I} x_i = v\}.$$

Then K is compact since the fact that the sets $\boldsymbol{A}X_i$'s are positively semi-independent implies that

$$\boldsymbol{A}K := \{(v_1, \ldots, v_I) \in \Pi_{i\in I}\boldsymbol{A}X_i : \sum_{i\in I} v_i = 0\} = \{0\}.$$

From the compactness of K one deduces that, without any loss of generality each sequence (x_i^n/n) converges to some $v_i \in \boldsymbol{A}X_i$. Hence $v = \sum_{i\in I} v_i \in \sum_{i\in I} \boldsymbol{A}X_i$. Similarly we prove that $-v \in \sum_{i\in I} \boldsymbol{A}X_i$.

Proof of $\boldsymbol{L}(\sum_{i\in I} \boldsymbol{A}X_i) \subset \sum_{i\in I} \boldsymbol{L}(X_i)$. This is a consequence of Proposition 5.18 taking $C_i := \boldsymbol{A}X_i$.

Proposition 5.19 *Let X_i $(i \in I)$ be convex subsets of $\mathbb{R}^J$ containing 0. The following four assertions are equivalent*

(i) $\boldsymbol{L}(\sum_{i\in I} X_i) = \{0\}$,

(ii) $\boldsymbol{L}(\sum_{i\in I} \boldsymbol{A}X_i) = \{0\}$,

(iii) $\boldsymbol{L}(X_i) = \{0\}$ *for all i, and the sets $\boldsymbol{A}X_i$ are positively semi-independent.*

(iv) $\boldsymbol{L}(X_i) = \{0\}$ *for all i, and the sets $\boldsymbol{A}X_i$ are weakly positively semi-independent.*

Proof. From the above inclusions in (a) of Proposition 5.18, it is clear that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$. We now show that $(iv) \Rightarrow (i)$. Indeed, from Part (b) of Proposition 5.18, one has

$$\sum_{i\in I} \boldsymbol{L}(X_i) = \boldsymbol{L}(\sum_{i\in I} X_i).$$

Using the fact that $\boldsymbol{L}(X_i) = \{0\}$, by (iv) we deduce that $\boldsymbol{L}(\sum_{i\in I} X_i) = \{0\}$. ■

5.2.5 SPSI when the sets are not cones: once more

We need to reformulate the previous result to treat the case where $X_i := Z_i \cap \ker V$, when Z_i $(i \in I)$ are closed convex subsets of $\mathbb{R}^J$ containing 0. The following result is a consequence of the previous proposition noticing that for all i

$\boldsymbol{A}(Z_i \cap \ker V) = \boldsymbol{A}Z_i \cap \ker V$, and $\boldsymbol{L}(Z_i \cap \ker V) = \boldsymbol{L}(Z_i) \cap \ker V$.

Proposition 5.20 *Let Z_i $(i \in I)$ be closed convex subsets of $\mathbb{R}^J$ containing 0.*

(a) Then for all i $\boldsymbol{A}(Z_i \cap \ker V) = \boldsymbol{A}Z_i \cap \ker V$, $\boldsymbol{L}(Z_i \cap \ker V) = \boldsymbol{L}(Z_i) \cap \ker V$ and

$$\begin{aligned} \sum_{i\in I} \boldsymbol{L}(Z_i) \cap \ker V &\subset \boldsymbol{L}(\sum_{i\in I} \boldsymbol{A}Z_i \cap \ker V) \subset \boldsymbol{L}(\sum_{i\in I} Z_i \cap \ker V), \\ \sum_{i\in I} \boldsymbol{L}(Z_i) \cap \{V \geq 0\} &\subset \boldsymbol{L}(\sum_{i\in I} \boldsymbol{A}Z_i \cap \{V \geq 0\}) \subset \boldsymbol{L}(\sum_{i\in I} Z_i \cap \{V \geq 0\}), \\ \boldsymbol{L}(\sum_{i\in I} \boldsymbol{A}Z_i \cap \ker V) &= \boldsymbol{L}(\sum_{i\in I} \boldsymbol{A}Z_i \cap \{V \geq 0\}). \end{aligned}$$

The sets $\boldsymbol{A}Z_i \cap \ker V$ are positively semi-independent if and only if $\boldsymbol{A}Z_i \cap \{V \geq 0\}$ are positively semi-independent.

(b) If we additionally assume that the sets $\boldsymbol{A}Z_i \cap \ker V$ are positively semi-independent then the above inclusions are equalities, that is

$$\begin{aligned} \sum_{i\in I} \boldsymbol{L}(Z_i) \cap \ker V &= \boldsymbol{L}(\sum_{i\in I} \boldsymbol{A}Z_i \cap \ker V) = \boldsymbol{L}(\sum_{i\in I} Z_i \cap \ker V) \\ \sum_{i\in I} \boldsymbol{L}(Z_i) \cap \{V \geq 0\} &= \boldsymbol{L}(\sum_{i\in I} \boldsymbol{A}Z_i \cap \{V \geq 0\}) = \boldsymbol{L}(\sum_{i\in I} Z_i \cap \{V \geq 0\}), \\ \sum_{i\in I} \boldsymbol{L}(Z_i) \cap \ker V &= \sum_{i\in I} \boldsymbol{L}(Z_i) \cap \{V \geq 0\}. \end{aligned}$$

(c) The following assertions are equivalent

(i) $\boldsymbol{L}(\sum_{i\in I} Z_i \cap \ker V) = \{0\}$, *that is* $\boldsymbol{A}(\sum_{i\in I} Z_i \cap \ker V) \cap -\boldsymbol{A}(\sum_{i\in I} Z_i \cap \ker V) = \{0\}$,

(ii) $\boldsymbol{L}(\sum_{i\in I} \boldsymbol{A}Z_i \cap \ker V) = \{0\}$, *that is* $(\sum_{i\in I} \boldsymbol{A}Z_i \cap \ker V) \cap -(\sum_{i\in I} \boldsymbol{A}Z_i \cap \ker V) = \{0\}$,

(iii) $\sum_{i\in I} \boldsymbol{L}(Z_i) \cap \ker V = \{0\}$, *and the sets $\boldsymbol{A}Z_i \cap \ker V$ are positively semi-independent,*

(iv) $\boldsymbol{L}(Z_i) \cap \ker V = \{0\}$ *for all i, and the sets $\boldsymbol{A}Z_i \cap \ker V$ are positively semi-independent.*

(v) $\boldsymbol{L}(\sum_{i\in I} Z_i \cap \{V \geq 0\}) = \{0\}$, *that is* $\boldsymbol{A}(\sum_{i\in I} Z_i \cap [V \geq 0]) \cap -\boldsymbol{A}(\sum_{i\in I} Z_i \cap \{V \geq 0\}) = \{0\}$,

(vi) $\boldsymbol{L}(\sum_{i\in I} \boldsymbol{A}Z_i \cap \{V \geq 0\}) = \{0\}$, *that is* $(\sum_{i\in I} \boldsymbol{A}Z_i \cap \{V \geq 0\}) \cap -(\sum_{i\in I} \boldsymbol{A}Z_i \cap \{V \geq 0\}) = \{0\}$,

(vii) $\sum_{i\in I}(\boldsymbol{L}(Z_i) \cap \{V \geq 0\}) = \{0\}$, *and the sets $\boldsymbol{A}Z_i \cap \ker V$ are positively semi-independent,*

(viii) $\boldsymbol{L}(Z_i) \cap \{V \geq 0\} = \{0\}$ *for all* i, *and the sets* $\boldsymbol{A}Z_i \cap \ker V$ *are positively semi-independent.*

Proof. This is a consequence of the previous proposition, noticing that

$\boldsymbol{A}(Z_i \cap \ker V) = \boldsymbol{A}Z_i \cap \ker V$, $\boldsymbol{L}(Z_i \cap \ker V) = \boldsymbol{L}(Z_i) \cap \ker V$.

A direct proof is also given hereafter.

(a) We first notice that, for all i, $\boldsymbol{L}(Z_i) \subset \boldsymbol{A}Z_i \subset Z_i$. Hence

$$\sum_{i \in I} \boldsymbol{L}(Z_i) \cap \ker V \subset \sum_{i \in I} \boldsymbol{A}Z_i \cap \ker V \subset \sum_{i \in I} Z_i \cap \ker V.$$

Using the fact that $\boldsymbol{L}(A) \subset \boldsymbol{L}(B)$ if $A \subset B$ we get

$$\sum_{i \in I} \boldsymbol{L}(Z_i) \cap \ker V \subset \boldsymbol{L}(\sum_{i \in I} \boldsymbol{A}Z_i \cap \ker V) \subset \boldsymbol{L}(\sum_{i \in I} Z_i \cap \ker V).$$

Similarly the analogous inclusions hold when we replace $\ker V$ by $\{V \geq 0\}$.

To show the last equality we first notice that $\boldsymbol{L}(\sum_{i \in I} \boldsymbol{A}Z_i \cap \ker V) \subset \boldsymbol{L}(\sum_{i \in I} \boldsymbol{A}Z_i \cap [V \geq 0])$. Conversely let $v \in \boldsymbol{L}(\sum_{i \in I} \boldsymbol{A}Z_i \cap \{V \geq 0\})$. Then $v = \sum_{i \in I} v_i = -\sum_{i \in I} w_i$ with v_i, w_i in $\boldsymbol{A}Z_i \cap \{V \geq 0\}$. One easily checks that $Vv_i = Vw_i = 0$ and therefore $v \in \boldsymbol{L}(\sum_{i \in I} \boldsymbol{A}Z_i \cap \ker V)$.

Part (b) $\boldsymbol{L}(\sum_{i \in I} Z_i \cap \ker V) \subset \boldsymbol{L}(\sum_{i \in I} \boldsymbol{A}Z_i \cap \ker V)$. Let $v \in \boldsymbol{L}(\sum_{i \in I} Z_i \cap \ker V)$, then for every integer n, there exists $z_i^n \in Z_i \cap \ker V$ such that $nv = \sum_{i \in I} z_i^n$ or equivalently $v = \sum_{i \in I} z_i^n/n$ and we notice that $z_i^n/n \in Z_i \cap \ker V$ (since Z_i is convex and contains 0). Consider now the set

$$K := \{(z_1, \ldots, z_I \in \Pi_{i \in I} Z_i : \sum_{i \in I} z_i = v, Vz_i = 0\}.$$

Then K is compact since the fact that the sets $\boldsymbol{A}Z_i \cap \ker V$ are positively semi-independent implies that

$$\boldsymbol{A}K := \{(v_1, \ldots, v_I \in \Pi_{i \in I} \boldsymbol{A}Z_i : \sum_{i \in I} v_i = 0, Vv_i = 0\} = \{0\}.$$

From the compactness of K one deduces that, without any loss of generality each sequence (z_i^n/n) converges to some $v_i \in \boldsymbol{A}Z_i \cap \ker V$. Hence $v = \sum_{i \in I} v_i \in \sum_{i \in I} \boldsymbol{A}Z_i \cap \ker V$. Similarly we prove that $-v \in \sum_{i \in I} \boldsymbol{A}Z_i \cap \ker V$.

Proof of $\boldsymbol{L}(\sum_{i \in I} \boldsymbol{A}Z_i \cap \ker V) \subset \sum_{i \in I} \boldsymbol{L}(Z_i) \cap \ker V$. This is a consequence of the above proposition taking $C_i := \boldsymbol{A}Z_i \cap \ker V$.

Part (c) From the above inclusions in (a), it is clear that $(i) \Rightarrow (ii) \Rightarrow (iii)$ and clearly $(iii) \Leftrightarrow (iv)$. The implication $(iii) \Rightarrow (ii)$ is a consequence of Part (b). We now prove that $(ii) \Rightarrow (i)$. Indeed, if $\boldsymbol{L}(\sum_{i\in I} \boldsymbol{A}Z_i \cap \ker V) = \{0\}$, from the above proposition taking $C_i := \boldsymbol{A}Z_i \cap \ker V$ we deduce that the family $C_i := \boldsymbol{A}Z_i \cap \ker V$ is positively semi-independent. Consequently from Part (b)

$$\boldsymbol{L}(\sum_{i\in I} \boldsymbol{A}Z_i \cap \ker V) = \boldsymbol{L}(\sum_{i\in I} Z_i \cap \ker V) = \{0\}.$$

The implication $(v) \Rightarrow (ii)$ is immediate. For $(ii) \Rightarrow (v)$, let $v \in \boldsymbol{L}(\sum_{i\in I} \boldsymbol{A}Z_i \cap \{V \geq 0\})$. Then $v = \sum_{i\in I} v_i = -\sum_{i\in I} w_i$ where, for every i, $v_i, w_i \in \boldsymbol{A}Z_i \cap \{V \geq 0\}$. Hence $0 = \sum_{i\in I}(Vv_i + Vw_i)$ which implies $Vv_i = Vw_i = 0$ for every i. Therefore $v \in \boldsymbol{L}(\sum_{i\in I} \boldsymbol{A}Z_i \cap \ker V) = \{0\}$, that is $v = 0$. ■

Remark The implication $(iii) \Rightarrow (ii)$ may not be true if the sets $\boldsymbol{A}Z_i \cap \ker V$ are not assumed to be positively semi-independent. Consider in $\mathbb{R}^2$ a null matrix V (so that $\ker V = \{0\}$ and $Z_1 := \mathbb{R}^2_+$ and $Z_2 := \{(x, y) : y \geq x^2\}$. Then $\boldsymbol{L}Z_1 = \boldsymbol{L}Z_2 = \boldsymbol{L}Z_1 + \boldsymbol{L}Z_2) = \{0\}$ and $\boldsymbol{A}Z_1 = \mathbb{R}^2_+$, $\boldsymbol{A}Z_2 = \{(x, y) : x = 0, y \leq 0\}$, $\boldsymbol{A}Z_1 + \boldsymbol{A}Z_2 = \{\ (x, y) : x \geq 0\}$, and $\boldsymbol{L}(\boldsymbol{A}Z_1 + \boldsymbol{A}Z_2) = \{\ (x, y) : x = 0\}$.

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