

Some Properties of Realcompact Subspaces and Coarser Normal Spaces

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To Nora, Albert, Arvin and Ali

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Abstract

In this work we obtain results in two areas of topology, normal condensations of products and size of realcompact subspaces of a space.

In 2000 Swardson proved that every uncountable compact space has a realcompact subspace of size ω_1 . In the Chapter 2 the work of Swardson in [Swa00] is continued to prove that realcompact spaces with pseudocharacter no greater than ω_1 have realcompact subspaces of size ω_1 . Under continuum hypothesis, a consequence is that every uncountable realcompact space has a realcompact subspace of size ω_1 . We also prove that every realcompact right-separated set of size larger than continuum has a realcompact subspace of size of any cardinal less or equal to continuum. A corollary is that every compact set of size bigger or equal to continuum has a realcompact set of size less or equal to continuum, answering a question by Professor William Fleissner.

In 1997 Buzjakova in [Buz] proved that for a pseudocompact space X , $X \times (\kappa + 1)$ condenses onto a normal space if and only if X condenses onto a compact space, where $\kappa = |\beta X|^+$. In the Chapter 3 we extend Buzjakovas's method to prove if $X \times (\kappa + 1)$ condenses onto a normal space, then X condenses onto a countably paracompact space, where $\kappa = 2^{2^{|X|^+}}$.

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This work is for my mother and for her love and dedication in making the roads smooth for my education and in memory of my dad who never stopped believing in me all his life.

Chapter 1

Introduction

In this dissertation, two distinct but connected areas of topology are studied. The first area is realcompact spaces. Realcompact spaces are those spaces that are homeomorphic to closed subsets of products of the real line (with the usual topology). Every Lindelöf (in particular, compact) space is realcompact. It follows that every subspace of the real line is realcompact. Realcompactness, like Lindelöfness and compactness, is closed hereditary but not hereditary. Swardson has proved that every uncountable compact space has a realcompact subspace of size ω_1 . Her result is a partial answer to one of the main problems of interest to topologists. This is the problem of determining those spaces with property P that have a subspace of a certain size and property Q. Since a countable space is Lindelöf, we seek realcompact subspaces of size ω_1 or larger.

It is known that the statement that an uncountable compact space has a compact subspace of size ω_1 is not true. In fact, in $\beta\omega$, the Stone-Čech compactification of positive integers, every uncountable closed set is isomorphic to $\beta\omega$ itself and $|\beta\omega| = 2^{(2^\omega)} > 2^\omega \geq \omega_1$. Therefore all the uncountable compact subspaces of $\beta\omega$ are of size larger than ω_1 and 2^ω .

It is unknown whether the statement that uncountable Lindelöf spaces have Lindelöf subspaces of size ω_1 is consistent with ZFC. In one direction, Tall and Koszmider have showed, with additional axioms, that there exists a Lindelöf space which does not contain any Lindelöf subspace of size ω_1 .

In the second chapter, we show that every uncountable realcompact space has a realcompact subspace of size ω_1 that has a coarser topology homeomorphic to a subspace of the real line, a discrete subspace of size ω_1 , or a zero-dimensional right-separated subspace of size ω_1 . In all three cases, the subspace of size ω_1 has a coarser topology that is Lindelöf. Therefore, an uncountable Lindelöf space has a subspace of size ω_1 with a coarser Lindelöf topology.

One of the problems that remains, is to determine the weakest property P for space X to possess to imply that X has a realcompact subspace of size ω_1 or 2^ω . In the second chapter, we discuss different possibilities for property P and also the possibility of independence from ZFC.

In the third chapter we explore a variation of a well-known theorem under a weaker hypothesis. The property of a space X that $X \times \mathbb{I}$ is normal (X is said to be *binormal*) had long been a hypothesis for certain homotopy extension theorems. The obvious question is whether X being binormal means more than X being normal. In 1951, Dowker [Dow] examined this problem and proved for a normal space X , $X \times \mathbb{I}$ is normal iff $X \times (\omega + 1)$ is normal iff X is also countably paracompact. (Definition of *paracompact* can be found in the third chapter.) The result by Dowker gave a useful property for normal spaces that are not binormal. Normal spaces that are not binormal are called *Dowker spaces*, and the question is do they exist. There have been a few Dowker spaces with extra set theoretic assumptions but the first real Dowker space was constructed by Mary Ellen Rudin [Rud] in 1971. In spite of the existence of Dowker spaces, Morita and Starbird showed, in 1974, that the “ $X \times \mathbb{I}$ is normal” assumption in

the old homotopy extension theorem was unnecessary. In other ways, Dowker spaces are part of broader question of normality of the products.

There is also a long history of topologists investigating the relationship between topological properties and continuous bijections, called condensations, i.e., the properties of coarser topologies. It is possible that the condensation topology is stronger. For example, Pytkeev proved that any σ -compact Borel space can be condensed onto a compact set. Buzjakova [Buz] presented some sufficient conditions for a space to condense onto a compact space; she provided a solution to a question of Arhangel'skii concerning coarser normal products. We extend her result by starting with spaces with a weaker hypothesis. There exists an example witnessing the fact that our weaker hypothesis can not imply condensation onto a compactum. Also, Pavlov [Pav99] proved that for any infinite compact set K , there exists a normal space X such that $X \times K$ can not be mapped one-to-one and onto a normal space; showing that the assumption that a Cartesian product can condense onto a normal space, the assumption which was used both by Buzjakova and the author, is not in vain.

Condensations can be useful in different areas of topology. Knowing condensation results are useful in understanding the properties of the original space.

Chapter 2

Realcompact Subspaces of Size Less than Continuum

2.1 Introduction

In 2000 Swardson proved that compact spaces of size $> \omega_1$ have a realcompact subspace of size ω_1 . In 2003 Professor Fleissner asked :

- Is it true that compact spaces of size $> \mathfrak{c}$ have realcompact subspaces of size \mathfrak{c} ?
- Is it true that all Lindelöf spaces of size $> \omega_1$ have realcompact subspace of size ω_1 ?
- Is it true that the property called CRV , which we define shortly, on X is equivalent to no realcompact subset of size ω_1 for X , in a model of set theory with no S -spaces?

We answer the first question in affirmative by proving that every compact space of size $> \kappa$ where $\kappa \leq \mathfrak{c}$ has a realcompact subspace of size κ .

For the second question, we obtain partial results by showing that if a realcompact space of size $> \omega_1$ has pseudocharacter $\leq \omega_1$ then there is a realcompact subspace of size ω_1 . As a corollary, the Continuum Hypothesis implies every uncountable realcompact space has a realcompact subspace of size ω_1 .

The third question was answered by Justin Moore. In 2005 he constructed an L-space with the CRV property. As we progress into this chapter, the properties of CRV and realcompact subspaces of size ω_1 become related. We show if a Tychonoff space X does not have the CRV property, then X has a realcompact subspace of size ω_1 . The L-space constructed by Justin Moore demonstrates that the converse is not true.

2.2 Preliminaries about Realcompact Spaces

Most of the results in this section are well-known. The proofs are included as they are a variation of the proofs given in [PW] and [GJ].

Lemma 1. *If S and T are Hausdorff extensions of X and $f : S \rightarrow T$ is a continuous extension of the identity function on X , then $f[S \setminus X] \subseteq T \setminus X$.*

Proof. By the way of contradiction, assume $x \in X$ and $y \in S \setminus X$ be such that $f(x) = f(y)$. Since S is Hausdorff, there are disjoint open sets U_x and U_y in S such that $x \in U_x$ and $y \in U_y$. Let W be an open set in T such that $W \cap f[X] = f[U_x \cap X]$. There is an open set R in S such that $y \in R \subseteq U_y$ and $f[R] \subseteq W$. It follows that $f[R \cap X] \subseteq f[U_x \cap X]$. As f is one-to-one on S , $\emptyset \neq R \cap X \subseteq U_x \cap X$, a contradiction as $R \cap U_x = \emptyset$. \square

Definition 2. A Tychonoff space X is called *realcompact* if it can be embedded as a closed subspace of a product of reals.

Definition 3. For a topological space X , $C(X)$ is defined to be the set of continuous functions $f : X \rightarrow \mathbb{R}$.

Fact 4. A Tychonoff space X is realcompact if and only if the embedding $e : X \rightarrow \prod_{C(X)} \mathbb{R}$ defined by $e(x)(f) = f(x)$ for $x \in X$ and $f \in C(X)$, is closed.

Proof. The sufficiency is obvious so we prove only the necessity. Let X be a Tychonoff space and $f : X \rightarrow \prod_{\mathcal{A}} \mathbb{R}$ an embedding so that $f[X]$ is closed in $\prod_{\mathcal{A}} \mathbb{R}$. For $a \in \mathcal{A}$, let $\pi_a : \prod_{\mathcal{A}} \mathbb{R} \rightarrow \mathbb{R}$ be the projection map onto the a^{th} coordinate and $g_a = \pi_a \circ f$. Now, $g_a : X \rightarrow \mathbb{R}$ is continuous. The function $h : \mathcal{A} \rightarrow C(X) : a \mapsto g_a$ has the property that $\{h^{-1}(g_a) : a \in \mathcal{A}\}$ is a partition of \mathcal{A} . Let $\mathcal{B} \subseteq \mathcal{A}$ such that $|h^{-1}(g_a) \cap \mathcal{B}| = 1$ for each $a \in \mathcal{A}$. So, $h|_{\mathcal{B}} : \mathcal{B} \rightarrow C(X)$ is an one-to-one function and for each $b \in \mathcal{B}$, $h(b) = g_b$. Define $i : \prod_{\mathcal{B}} \mathbb{R} \rightarrow \prod_{\mathcal{A}} \mathbb{R}$ by $i(x)(a) = x(b)$, where $b \in h^{-1}(g_a)$; i is a closed embedding. Let $\pi_{\mathcal{B}} : \prod_{C(X)} \mathbb{R} \rightarrow \prod_{\mathcal{B}} \mathbb{R}$ be the projection map. With this notation, $i \circ \pi_{\mathcal{B}} \circ e : X \rightarrow \prod_{\mathcal{A}} \mathbb{R}$ is a commutative diagram as shown below.

$$\begin{array}{ccc} X & \xrightarrow{i \circ \pi_{\mathcal{B}} \circ e} & \prod_{\mathcal{A}} \mathbb{R} \\ e \downarrow & & \uparrow i \\ \prod_{C(X)} \mathbb{R} & \xrightarrow{\pi_{\mathcal{B}}} & \prod_{\mathcal{B}} \mathbb{R} \end{array}$$

Next we show that $f = i \circ \pi_{\mathcal{B}} \circ e$. Let $x \in X$ and $a \in \mathcal{A}$. Now, $(i \circ \pi_{\mathcal{B}} \circ e)(x)(a) = e(x)(b)$ where $b \in h^{-1}(g_a)$ or $g_b = g_a$. So, $e(x)(b) = e(x)(g_b) = e(x)(g_a) = g_a(x) = (\pi_a \circ f)(x) = f(x)(a)$. This shows that $f = i \circ \pi_{\mathcal{B}} \circ e$, $f[X] \subseteq i[\prod_{\mathcal{B}} \mathbb{R}]$, and the map $i \circ \pi_{\mathcal{B}} : \prod_{C(X)} \mathbb{R} \rightarrow \prod_{\mathcal{A}} \mathbb{R}$ is a continuous extension of the homeomorphism $f \circ e^{-1} : e[X] \rightarrow f[X]$. By continuity,

$$i \circ \pi_{\mathcal{B}}[cl_{\prod_{C(X)} \mathbb{R}} e[X]] \subseteq cl_{\prod_{\mathcal{A}} \mathbb{R}}[i \circ \pi_{\mathcal{B}}[e[X]]] \subseteq cl_{\prod_{\mathcal{A}} \mathbb{R}}[f[X]] = f[X].$$

By Lemma 1, this shows that $cl_{\prod_{C(X)} \mathbb{R}} e[X] = e[X]$. □

Definition 5. Let X be Tychonoff space and $e : X \rightarrow Y := \prod_{C(X)} \mathbb{R}$ be the canonical embedding defined by $e(x)(f) = f(x)$ for all $f \in C(X)$. Then the *Hewitt realcompactification* of X is denoted by νX and is defined to be $cl_Y e[X]$.

Fact 6. Let X be a Tychonoff space and $f : X \rightarrow S$ be a continuous function into a realcompact space S . Then there exists a unique continuous extension of f , denote it by $\nu f : \nu X \rightarrow S$, such that $\nu f|_X = f$.

Proof. Let $e_X : X \rightarrow \prod_{C(X)} \mathbb{R}$ and $e_S : S \rightarrow \prod_{C(S)} \mathbb{R}$ be the canonical embeddings of X and S into a product of reals.

It is straightforward to show that the function $\pi : \prod_{C(X)} \mathbb{R} \rightarrow \prod_{C(S)} \mathbb{R}$ defined by $\pi(p)(g) = p(g \circ f)$ is continuous; note that for $x \in X$ and $g \in C(S)$, $\pi(e_X(x))(g) = e_X(x)(g \circ f) = g \circ f(x) = e_S(f(x))(g)$, i.e. $\pi \circ e_X = e_S \circ f$

Thus the composition function $\pi|_{e_X[X]} = e_S \circ f \circ e_X^{\leftarrow} : e_X[X] \rightarrow e_S[S]$ is noted in the following commutative diagram.

$$\begin{array}{ccc} e_X[X] & \xrightarrow{\pi|_{e_X[X]}} & e_S[X] \\ e^{\leftarrow} \downarrow & & \uparrow e_S \\ X & \xrightarrow{f} & S \end{array}$$

For $\nu f = \pi|_{cl_{\prod_{C(X)} \mathbb{R}} e_X[X]}$, $\nu f : cl_{\prod_{C(X)} \mathbb{R}} [e_X[X]] \rightarrow cl_{\prod_{C(S)} \mathbb{R}} e_S[S]$ is the continuous extension of f . As $e_S[S]$ is closed in $\prod_{C(S)} \mathbb{R}$ by Fact 4 and $\nu X = cl_{\prod_{C(X)} \mathbb{R}} [e_X[X]]$, we have that $\nu f : \nu X \rightarrow S$ is a continuous extension of f . Note that νf is unique because for any other continuous extension of f , denoted by $\hat{f} : \nu X \rightarrow S$, νf and \hat{f} agree on X , a dense subset of their domain, thus they are equal. \square

Definition 7. A closed subset $Z \subseteq X$ is called a *zero-set* or *z-set* of X if $Z = f^{\leftarrow}[\{0\}]$, for some continuous function $f : X \rightarrow \mathbb{R}$. The collection of all zero-sets of X is denoted by $\mathcal{Z}(X)$.

Definition 8. A *z-filter* \mathcal{F} on X is a subfamily of $\mathcal{Z}(X)$ with the following properties,

1. $\emptyset \notin \mathcal{F}$.
2. If Z_1 and $Z_2 \in \mathcal{F}$, then $Z_1 \cap Z_2 \in \mathcal{F}$.

3. If $Z \in \mathcal{F}$, $Z' \in \mathcal{L}(X)$, and $Z \subseteq Z'$, then $Z' \in \mathcal{F}$.

Definition 9. A z -ultrafilter \mathcal{F} on X is a z -filter with the additional property that if Z_1 and $Z_2 \in \mathcal{L}(X)$ where $Z_1 \cup Z_2 = X$, then $Z_1 \in \mathcal{F}$ or $Z_2 \in \mathcal{F}$.

Definition 10. A z -ultrafilter has countable intersection property (*denoted by cip*) if every countable subset of it has non-empty intersection.

Fact 11. A z -ultrafilter \mathcal{F} with countable intersection property includes any countable intersection of its elements.

Proof. Let $\{Z_i : i \in \omega\} \subseteq \mathcal{F}$. Since the intersections of finitely many z -sets are z -sets and finite intersection of members of a z -filter belongs to that filter, we can assume there exists a descending chain $\{Z(f_i) : i \in \omega\} \subseteq \mathcal{F}$ where $Z(f_i)$ is the zero-set of f_i and $f_i : X \rightarrow [0, 1]$ is continuous ; and $\bigcap_{i \in \omega} Z_i = \bigcap_{i \in \omega} Z(f_i)$. Then $Z := \bigcap \{Z(f_i) : i \in \omega\}$ is the zero-set of the function $f(x) := \sum_{i \in \omega} f_i(x)/2^{i+1}$. For $Z' \in \mathcal{F}$, $Z' \cap Z(f_i) \neq \emptyset$ for all $i \in \omega$; and \mathcal{F} has cip. Thus, $Z' \cap Z \neq \emptyset$ for all $Z' \in \mathcal{F}$, and as \mathcal{F} is a z -ultrafilter therefore, $Z \in \mathcal{F}$. □

Note 12. For a Tychonoff space X , let βX denote the Stone-Ćech compactification of X . For each $p \in \beta X$, $\mathcal{F}_p = \{Z \in \mathcal{L}(X) : p \in cl_{\beta X} Z\}$ is the unique z -ultrafilter that converges to p (see [GJ], [PW]). For Z_1 and $Z_2 \in \mathcal{L}(X)$, $cl_{\beta X} Z_1 \cap cl_{\beta X} Z_2 = cl_{\beta X} (Z_1 \cap Z_2)$.

Lemma 13. A space X is realcompact if and only if every z -ultrafilter with countable intersection property on X is fixed.

Proof. Suppose $e : X \rightarrow \prod_{C(X)} \mathbb{R}$ is a closed embedding, $Y = \prod_{C(X)} \mathbb{R}$ and let $\alpha \mathbb{R}$ be the one-point compactification of \mathbb{R} . Then there exists a continuous extension $\beta e : \beta X \rightarrow \widehat{Y}$ where $\widehat{Y} = \prod_{\xi \in C(X)} \alpha \mathbb{R}_\xi$. Now Let $p \in \beta X \setminus X$ and \mathcal{F}_p be the free

z -ultrafilter on X that converges to p . Since $e[X]$ is closed in Y , $cl_{\widehat{Y}}e[X] \subseteq e[X] \cup \widehat{Y} \setminus Y$. By Lemma 1, $\beta e(p) \in \beta e[\beta X \setminus X] \subseteq cl_{\widehat{Y}}Y \setminus Y \subseteq \widehat{Y} \setminus Y$. There is $f \in C(X)$ such that $\pi_f(\beta e(p)) \in \alpha\mathbb{R} \setminus \mathbb{R}$.

For every $n \in \omega$, let $Z_n = (\alpha\mathbb{R} \setminus (-n, n)) \times \prod_{C(X) \setminus \{f\}} \alpha\mathbb{R}$ and $K_n = [-n, n] \times \prod_{C(X) \setminus \{f\}} \alpha\mathbb{R}$; both Z_n and K_n are zero-sets in \widehat{Y} with $\beta e(p) \in Z_n$. Also, $K_n \cup Z_n = \widehat{Y}$. Now, $e^{\leftarrow}[Z_n]$ and $e^{\leftarrow}[K_n]$ are zero-sets in X and $X = e^{\leftarrow}[Z_n] \cup e^{\leftarrow}[K_n]$. So, $p \in cl_{\beta X}e^{\leftarrow}[Z_n] \cup cl_{\beta X}e^{\leftarrow}[K_n]$. If $p \in cl_{\beta X}e^{\leftarrow}[K_n]$, then $\beta e(p) \in \beta e[cl_{\beta X}e^{\leftarrow}[K_n]] \subseteq cl_{\widehat{Y}}\beta e[e^{\leftarrow}[K_n]] \subseteq cl_{\widehat{Y}}K_n$, a contradiction. So, $p \in cl_{\beta X}e^{\leftarrow}[Z_n]$ and $e^{\leftarrow}[Z_n] \in \mathcal{F}_p$. Then $\bigcap_{n \in \omega} e^{\leftarrow}[Z_n] = \emptyset$ which means \mathcal{F}_p does not have countable intersection property.

Conversely suppose every z -ultrafilter on X with countable intersection property is fixed. Consider the embedding $e : X \rightarrow \prod_{C(X)} \mathbb{R}$ where $Y = \prod_{C(X)} \mathbb{R}$. It suffices to show that $e[X]$ is closed. Again there is a continuous extension of e , $\beta e : \beta X \rightarrow \widehat{Y}$ where $\widehat{Y} = \prod_{C(X)} \alpha\mathbb{R}$. Now $cl_{\widehat{Y}}e[X]$ is compact. It suffices to show that $cl_{\widehat{Y}}e[X] \cap Y = e[X]$. Let $p \in \beta X \setminus X$. So, \mathcal{F}_p is a free z -ultrafilter on X without cip. There exists a collection of z -sets $\{Z_n : n \in \omega\} \subseteq \mathcal{F}_p$ in X such that $\bigcap \{Z_n : n \in \omega\} = \emptyset$. Without loss of generality we assume that $Z_0 \supseteq Z_1 \supseteq Z_2 \supseteq \dots$. For each $n \in \omega$, let $f_n \in C(X)$ be such that $Z_n = Z(f_n)$ and $f_n[X] \subseteq [0, 1]$. We define a function $g : X \rightarrow [0, 1]$ by $g(x) = \sum_{i \in \omega} \frac{f_i(x)}{2^{i+1}}$. and note that g is a continuous function and positive on X . On the other hand for every $i \in \omega$, $Z_i \subseteq g^{\leftarrow}[0, 1/2^i]$. Thus $p \in cl_{\beta X}g^{\leftarrow}[0, 1/2^i]$ for every $i \in \omega$. Let $\beta g : \beta X \rightarrow [0, 1]$ be the continuous extension of g . Then $\beta g(p) \in [0, 1/2^i]$ for every $i \in \omega$. It follows that $\beta g(p) = 0$. Now the function $\frac{1}{g} : X \rightarrow [1, \infty)$ is also continuous and has a continuous extension to βX . But $\beta(1/g)(p) > n$ for all $n \in \omega$ and therefore $\beta(1/g)(p) \in \alpha\mathbb{R} \setminus \mathbb{R}$. This implies that $e(p) \notin Y$. □

Corollary 14. *Lindelöf spaces are realcompact.*

Proof. Every z -filter, with cip on a Lindelöf space has non-empty intersection. Therefore, every z -ultrafilter with countable intersection property has non-empty intersection. Thus by Lemma 13, every Lindelöf space is realcompact. □

Corollary 15. *Let T be a realcompact extension of a Tychonoff space X . Then every z -ultrafilter with countable intersection property on X has an adherent point in T .*

Proof. Let \mathcal{F} be a z -ultrafilter with countable intersection property on X . Then $\mathcal{G} = \{Z \in \mathcal{Z}(T) : Z \cap X \in \mathcal{F}\}$ is a z -ultrafilter with countable intersection property on T and therefore by Lemma 13 it is fixed. Denote the intersection by $\{p\} = \bigcap \mathcal{G}$. By the way of contradiction assume there exists $Z \in \mathcal{F}$ such that $p \notin \text{cl}_T Z$. Then there exists a continuous function $f : T \rightarrow [0, 1]$ such that $f(p) = 0$ and $f[\text{cl}_T Z] = 1$. Let $Z_1 = f^{-1}[[0, 1/2]]$ and $Z_2 = f^{-1}[[1/2, 1]]$. $Z_1 \notin \mathcal{G}$ therefore $Z_2 \in \mathcal{G}$, a contradiction to $p \in \bigcap \mathcal{G}$. □

Fact 16. A closed subspace of a realcompact space is realcompact.

Proof. Let A be a closed subspace of a realcompact space X and let $e : X \rightarrow \prod_{C(X)} \mathbb{R}$ be the canonical embedding. $e[A]$ is closed in $e[X]$ and $e[X]$ is closed in $\prod_{C(X)} \mathbb{R}$. Therefore $e|_A : A \rightarrow \prod_{C(X)} \mathbb{R}$ is an embedding of A onto a closed subspace of $\prod_{C(X)} \mathbb{R}$. □

Fact 17. Let $X \subseteq A \subseteq \upsilon X$. If A is realcompact then $A = \upsilon X$.

Proof. Let $i : X \rightarrow A$ and $j : A \rightarrow \upsilon X$ be the inclusion maps and let $\upsilon i : \upsilon X \rightarrow A$ be the continuous extension defined by Fact 6. Now let $h = j \circ \upsilon i : \upsilon X \rightarrow \upsilon X$, $h|_X : X \rightarrow \upsilon X$ is the identity map and by Fact 6 the extension has to be unique. Therefore h is the identity map and also υi has to be identity map. □

Fact 18. A product of a collection of realcompact spaces is realcompact.

Proof. Let $\mathcal{C} := \{X_\xi : \xi \in \mathcal{A}\}$ be a collection of realcompact spaces and let e_ξ denote the canonical embedding $e_\xi : X_\xi \rightarrow \prod_{C(X_\xi)} \mathbb{R}$. Then let $e : X := \prod_{\mathcal{A}} X_\xi \rightarrow \prod_{\xi \in \mathcal{A}} (\prod_{C(X_\xi)} \mathbb{R})$ defined by $e(\prod_{\xi \in \mathcal{A}} x_\xi) = \prod_{\xi \in \mathcal{A}} (\prod_{C(X_\xi)} e_\xi(x_\xi))$. Now $e[X]$ is closed in every coordinate of the product over \mathcal{A} , therefore it is closed in the product space. \square

Corollary 19. *An intersection of a family of realcompact subspaces is realcompact.*

Proof. Let $\{X_\xi : \xi \in \mathcal{A}\}$ be a family of realcompact subspaces of a space Y and let $X = \bigcap_{\xi \in \mathcal{A}} X_\xi$. The function $e : X \rightarrow \prod_{\xi \in \mathcal{A}} X_\xi$ defined by $e(x)(\xi) = x$ is a closed embedding into the product space and the product space is realcompact, therefore $e[X]$ is realcompact. \square

Fact 20. A Tychonoff space X is compact if and only if X is both countably compact and realcompact.

Proof. Let X be compact. Then every z -ultrafilter on X converges. Therefore any z -ultrafilter with cip converges, which, by Lemma 13, implies X is realcompact. On the other hand since X is compact, every countable cover on X has a finite subcover; thus X is countably compact.

Now, let X be realcompact and countably compact. Let \mathcal{F} be a z -ultrafilter on X . If we assume \mathcal{F} has cip, by X being realcompact, \mathcal{F} converges in X . Without loss of generality assume \mathcal{F} does not have cip, then there exists a countable family of z -sets $\mathcal{Z} := \{Z_i : i \in \omega\}$ with empty intersection. Now, $\{X \setminus Z_i : i \in \omega\}$ is a countable cover of X and since X is countably compact there exists a finite subcover, which implies there exists a finite subset of \mathcal{Z} with empty intersection, a contradiction. \square

Fact 21. A Tychonoff space that is the union of a realcompact subspace and a compact subspace is realcompact.

Proof. Let $Y = X \cup K$ be a Tychonoff space where X is realcompact and K is compact. Assume that Y is not realcompact. There is a z -ultrafilter \mathcal{F} on Y with cip such that $\bigcap \mathcal{F} = \emptyset$. There is $Z \in \mathcal{F}$ such that $Z \cap K = \emptyset$. Moreover, there is a zero-set Z' in Y such that $K \subseteq \text{int}_Y Z'$ and $Z' \cap Z = \emptyset$.

The goal is to show that $\mathcal{F}_X = \{T \cap X : T \in \mathcal{F}\}$ is a z -ultrafilter on X . Since $Z \in \mathcal{F}$, \mathcal{F}_X is a z -filter on X with cip. We need to show that \mathcal{F}_X is a z -ultrafilter on X . Let W be a zero-set in X such that W meets \mathcal{F}_X . Note that $W \cap Z$ is a zero-set in X . There is $f \in C(X)$ such that $W \cap Z = f^{-1}(0)$ and $f[X] \subseteq [0, 1]$. Also, $Z' \cap X$ is a zero-set in X . There is $g \in C(X)$ such that $Z' \cap X = g^{-1}(0)$ and $g[X] \subseteq [0, 1]$. Let $h = \frac{f}{f+g}$. Note that $Z(h) = Z(f) = W \cap Z$ and $h^{-1}(1) = Z(g) = Z' \cap X$. Define $H : Y \rightarrow \mathbb{R}$ by $H|_{X \setminus \text{int}_Y Z'} = h|_{X \setminus \text{int}_Y Z'}$ and $H|_{Z'} = 1$. By the Pasting Lemma, H is continuous. Now $H^{-1}(0) = Z(h) = W \cap Z$. Since $W \cap Z$ meets \mathcal{F}_X , $W \cap Z$ meets \mathcal{F} . As \mathcal{F} is a z -ultrafilter on Y , $W \cap Z \in \mathcal{F}$ and hence $W \cap Z \in \mathcal{F}_X$. This shows that \mathcal{F}_X is a z -ultrafilter on X . As \mathcal{F}_X has cip and $\bigcap \mathcal{F}_X = (\bigcap \mathcal{F}) \cap X = \emptyset$. This shows that X is not realcompact, a contradiction. Thus, every z -ultrafilter on Y with cip has non-empty intersection. It follows that Y is realcompact. \square

The next result is a well-known fact in basic topology.

Lemma 22. *Let $f : X \rightarrow Y$ be a continuous function and $\pi_X : X \times Y \rightarrow X$ the projection map. Then $\pi_X|_{gr(f)} : gr(f) \rightarrow X : (x, f(x)) \mapsto x$ is a homeomorphism.*

Fact 23. Let $f : X \rightarrow Y$ be a continuous function from a realcompact space X to a Tychonoff space Y . Then the preimage of a realcompact subspace of Y is realcompact.

Proof. Let Z be a realcompact subspace of Y . Then $X \times Z$ is also realcompact. Now, $gr(f)$ is a closed subspace of $X \times Y$; so, $gr(f) \cap (X \times Z)$ is realcompact as a closed subspace of a realcompact space. But $gr(f) \cap (X \times Z) = gr(f|_{f^{-1}[Z]})$. By previous lemma, $f^{-1}[Z]$ is homeomorphic to $gr(f|_{f^{-1}[Z]})$. Thus, $f^{-1}[Z]$ is realcompact. \square

Fact 24. Let X be a Tychonoff space and $f : X \rightarrow Y$ be a continuous, one-to-one function into a space Y that is hereditary realcompact. Then X is also hereditary realcompact.

Proof. There is a continuous extension $\nu f : \nu X \rightarrow Y$ that extends f by Fact 6. Let $p \in Y$. By Fact 23, $\nu f^{-1}[Y \setminus \{p\}] = \nu X \setminus \nu f^{-1}(p)$ is realcompact. So, by Fact 21, $\nu X \setminus \nu f^{-1}(p) \cup \{f^{-1}(p)\}$ is realcompact. As $X \subseteq (\nu X \setminus \nu f^{-1}(p)) \cup \{f^{-1}(p)\} \subseteq \nu X$, it follows that $\nu X \setminus \nu f^{-1}(p) \cup \{f^{-1}(p)\} = \nu X$. That is, for each $p \in Y$, $\nu f^{-1}(p) = f^{-1}(p)$. This shows that $\nu X = X$ and X is realcompact. By Fact 23, since f is one-to-one, X is hereditary realcompact. \square

2.3 Size of Realcompact Spaces

The following result is an extension of a theorem used by Swardson in [Swa00] from ω_1 to arbitrary infinite cardinal $\kappa \leq 2^\omega$.

Theorem 25. *If X is a space with a subspace $Y \subseteq X$ and $f \in C(Y)$ such that $|f[Y]| \geq \kappa$, then X has a realcompact subspace of size κ .*

Proof. Pick $B \subseteq f[Y]$ such that $|B| = \kappa$, then by Axiom of Choice pick $A \subseteq f^{-1}[B]$ such that $f|_A$ is one-to-one. By Fact 24, A is realcompact. \square

Definition 26. A space X is *right-separated* (resp. *left-separated*) if it can be enumerated as $\{x_\xi : \xi \in \eta\}$ where η an ordinal and the initial segment $\{x_\xi : \xi \in \alpha\}$ is open (resp. closed) for any $\alpha \in \eta$. We consider X ordered by this enumeration.

Definition 27. A cardinal κ is called *measurable* if a non-trivial $\{0, 1\}$ -valued countably additive measure can be defined on $\mathcal{P}(\kappa)$.

Lemma 28. [GJ] Every discrete space of size κ , where κ is a non-measurable cardinal, is realcompact.

Note that measurable cardinals are very large cardinals and ω_1 and \mathfrak{c} are non-measurable cardinals. We are going to use Lemma 28 in the next Lemma and in Lemma 49.

Lemma 29. If a Tychonoff space X is not hereditary separable then X contains a realcompact subspace of size ω_1 .

Proof. Let $A \subseteq X$ be a non-separable subspace. Then we can recursively form a left-separated subset of A by inductively picking a point $x_\alpha \in A \setminus cl_A\{x_\xi : \xi \in \alpha\}$ for each $\alpha \in \omega_1$. If $B := \{x_\xi : \xi \in \omega_1\}$ is Lindelöf then we are done. If B is not Lindelöf, we can build a right-separated subset as follows: Since B is not Lindelöf, it has a cover $\mathcal{U} := \{U_\xi : \xi \in \omega_1\}$ with no countable subcover.

At level α , let $U_{\delta_\alpha} \in \mathcal{U}$ be such that

$$V_\alpha := U_{\delta_\alpha} \setminus (\cup_{\xi \in \alpha} U_{\delta_\xi} \cup \{x_\zeta : \zeta \in (\sup\{\gamma_\xi : \xi \in \alpha\} + 1)\}) \neq \emptyset.$$

Pick $x_{\gamma_\alpha} \in V_\alpha$.

Now $\{x_{\gamma_\alpha} : \alpha \in \omega_1\}$ is both right-separated and left-separated. Thus $\{x_{\gamma_\alpha} : \alpha \in \omega_1\}$ is discrete and discrete spaces of size ω_1 are realcompact.

This proof can be simplified by using $\omega_1 \rightarrow (\omega_1, \omega)^2$, as follows:

Let B be a left-separated space as above. Denote the left-separated order by $<_L$. If B is not Lindelöf we build a right-separated subset $C \subseteq B$; denote the ordering by $<_R$. Now, by $\omega_1 \rightarrow (\omega_1, \omega)^2$, either there exists a subset $D \subseteq C$ of size ω_1 such that $<_L = <_R$ on D , or there exists a countable subset on which $<_L = >_R$ which is a contradiction to the fact that there exists no infinite descending chain in any well-ordered set. So there

exists a subset D of size ω_1 that is both left-separated and right-separated. Thus D is discrete and realcompact. \square

Definition 30. A space X is $< \kappa$ *real-valued* if for all $Y \subseteq X$ and for $f \in C(Y)$, $|f[X]| < \kappa$.

Definition 31. A Tychonoff space X has the *countable real valued property* or *CRV* if for all $Y \subseteq X$ and all $f \in C(Y)$, f has a countable image.

The following two results are well-known and proofs are included for completeness.

Fact 32. A Tychonoff space with countable real valued property is zero-dimensional.

Proof. Let $x \in U$ an open set in X then there exist a function $f : X \rightarrow [0, 1]$ such that $f(x) = 0$ and $f[X \setminus U] = \{1\}$. Since f has countable image let $r \in [0, 1] \setminus f[X]$ then $x \in f^{-1}[[0, r]] \subset U$ is a clopen set. So X has a clopen base. \square

Fact 33. A countably compact, Tychonoff space X with no isolated points has a closed subspace A and a continuous $f \in C(A)$ such that $|f[A]| = 2^\omega$.

Proof. We build a binary tree of size ω of regular-closed sets such that each branch has size ω and the collection of intersections of the branches map continuously onto a Cantor space $2^\omega \subseteq \mathbb{R}$: Let U_0 and U_1 be two open set with disjoint closure in X then let $U_{0,0}$ and $U_{0,1}$ be two open sets with disjoint closure in U_0 and $U_{1,0}$ and $U_{1,1}$ be two open sets with disjoint closure in U_1 .

At stage $n + 1$, $U_{h \cap 0}$ and $U_{h \cap 1}$ have disjoint closure inside U_h for all $h \in 2^n$. Let $A_g = \bigcap_{n \in \omega} clU_{g|_n}$ for $g \in 2^\omega$. $A_g \neq \emptyset$, for X is countably compact. Let $A = \bigcup_{g \in 2^\omega} A_g$. Define $f : \bigcup_{g \in 2^\omega} A_g \rightarrow 2^\omega$ by $f(x) = g$ if $x \in A_g$. f is well-defined and continuous:

$$f^{-1}[g|_n \times \prod_{m>n} 2] = clU_{g|_n} \cap A \subseteq (X \setminus \bigcup_{l \in 2^\omega, l \neq g} clU_{l|_n})$$

which means preimage of every basic open set is clopen.

A is closed because it is an intersection of closed sets $A = \bigcap_{n \in \omega} \bigcup_{h \in 2^n} clU_h$. \square

Definition 34. An uncountable regular Hausdorff space is sub-Ostaszewski if every open subset of it is either countable or co-countable.

Definition 35. A sub-Ostaszewski space is called Ostaszewski if it is countably compact and non-compact.

Definition 36. \clubsuit is the statement that there exists a collection $\{A_\delta : \delta \in \omega_1 \cap lim\}$ of countable subsets of ω_1 such that

1. $A_\delta \subseteq \delta$,
2. for any uncountable subset $A \subseteq \omega_1$, there exists a $\xi \in \omega_1$ so that $A_\xi \subset A$.

In 1975, assuming \clubsuit Ostaszewski [AO] constructed an Ostaszewski space of size ω_1 that is locally compact but is not realcompact.

Definition 37. A point p is a κ -accumulation point of a subset A of a space X , if $|A| \geq \kappa$ and for all open sets $U \ni p$, $|A \cap U| = \kappa$.

The following result is a basic property in topology.

Lemma 38. *An uncountable subset $A \subseteq \mathbb{R}$ has at least two ω_1 -accumulation points.*

Proof. Define $A_i := A \cap (i, i+1)$ for $i \in \mathbb{Z}$. A is uncountable and $A = \bigcup_{i \in \mathbb{Z}} A_i \cup (A \cap \mathbb{Z})$.

Therefore, there exists $i \in \mathbb{Z}$ such that A_i is uncountable.

Let

$$r_1 := \sup(\{q \in \mathbb{Q} : |A_i \cap (i, q)| < \omega\} \cup \{i\})$$

and let

$$r_2 := \inf(\{q \in \mathbb{Q} : |A_i \cap (q, i+1)| < \omega\} \cup \{i+1\}).$$

$r_1 < r_2$: By the way of contradiction, assume $r_1 > r_2$. It follows from the definition of r_1 and r_2 that for every $q \in \mathbb{Q} \cap (r_2, r_1)$, $|A_i \cap (i, q]| \leq \omega$ and $|A_i \cap [q, i + 1)| \leq \omega$ which implies $A_i = (A_i \cap (i, q]) \cup (A_i \cap [q, i + 1))$ is countable, a contradiction. Thus $r_1 < r_2$.

Now pick $p_1, p_2 \in (r_1, r_2)$ such that $p_1 < p_2$. Then both sets $[i, p_1]$ and $[p_2, i + 1]$ are compact and their intersection with A_i is uncountable. Every uncountable subspace of a compact space has at least one ω_1 -accumulation point and therefore $A_i \cap [i, p_1]$ and $A_i \cap [p_2, i + 1]$ each have at least one ω_1 -accumulation point. Therefore A_i has at least two ω_1 -accumulation points. \square

Fact 39. A sub-Ostaszewski space X has the countable real valued property.

Proof. By the way of contradiction, assume $Y \subseteq X$, and $f : Y \rightarrow \mathbb{R}$ is a continuous function that has uncountable image. By the Lemma 38, $f[Y]$ has two ω_1 -accumulation points, denote them by p_1 and p_2 where $p_1 < p_2$. Let $r_1, r_2 \in (p_1, p_2)$ such that $r_1 < r_2$.

Then $f^{-1}[(p_1, r_1)]$ is an open set in Y , and there exists an open set in X , U , such that $U \cap Y = f^{-1}[(p_1, r_1)]$. Therefore U is uncountable and since $f^{-1}[(r_2, p_2)] \subseteq X \setminus U$, U is also not co-countable, a contradiction. Thus f has countable image. \square

In light of Theorem 25, the question is whether "a realcompact space X has a realcompact subspace of size ω_1 " is equivalent to "X does not have CRV property". The one-point compactification of locally compact Ostaszewski space is a realcompact space of size ω_1 with CRV property which contradicts that fact with assuming some extra axioms. Justin Moore constructed a hereditary Lindelöf, non-separable space that has the CRV property. This shows that having a realcompact subspace of size ω_1 is not equivalent to not having the CRV property in ZFC. So, the converse of Theorem 25 is not true.

2.4 Realcompact Subspaces of Size ω_1

Fact 40. Let X be an uncountable space with countable real-value property and $A \subseteq X$. If \mathcal{F}_A is a z -ultrafilter on A with the countable intersection property (denoted as *cip*), then $\mathcal{F}_X = \{Z \in \mathcal{L}(X) : Z \cap A \in \mathcal{F}_A\}$ is a z -ultrafilter on X with *cip*.

Proof. Let V be a clopen set in X that meets \mathcal{F}_X , we show that $V \in \mathcal{F}_X$:

If $V \cap A = \emptyset$, then $X \setminus V \supseteq A$. Then $(X \setminus V) \cap A = A \in \mathcal{F}_A$ and hence $X \setminus V \in \mathcal{F}_X$, a contradiction to V meeting \mathcal{F}_X . This implies that $V \cap A \neq \emptyset$.

Now if $V \cap A \neq \emptyset$ and $(X \setminus V) \cap A \in \mathcal{F}_A$, then $X \setminus V \in \mathcal{F}_X$, a contradiction to V meeting \mathcal{F}_X . So $V \cap A \in \mathcal{F}_A$ which implies $V \in \mathcal{F}_X$.

Now let $Z \in \mathcal{L}(X)$. Then $Z = \bigcap_{i \in \omega} V_i$ where V_i is clopen in X (follows from the CRV property). If $Z \in \mathcal{L}(X)$ meets \mathcal{F}_X , then each V_i meets \mathcal{F}_X and therefore $V_i \in \mathcal{F}_X$ and $V_i \cap A \in \mathcal{F}_A$. By *cip* for \mathcal{F}_A , $\bigcap_{i \in \omega} V_i \cap A \in \mathcal{F}_A$ which means $Z \in \mathcal{F}_X$. \square

For the rest of this section for $A \subseteq X$ and \mathcal{F}_A a z -ultrafilter with *cip* on A we use \mathcal{F}_X as defined above.

Definition 41. The *pseudocharacter* of a point x in a T_1 space X is the cardinal $\inf\{|\mathcal{U}| : \mathcal{U} \subseteq \tau(X), \bigcap \mathcal{U} = \{x\}\}$ and is denoted by $\psi(x, X)$. The pseudocharacter of a space X is the supremum of all infinite cardinals $\psi(x, X)$ for $x \in X$ and is denoted by $\psi(X)$.

Lemma 42. Let X be an uncountable realcompact space with CRV and let $x \in X$ be such that $\psi(x, X) = \omega_1$. Then X contains a realcompact subspace of size ω_1 .

Proof. Let $\{U_\xi : \xi \in \omega_1\}$ be a collection of clopen neighborhoods of x such that $\bigcap_{\xi \in \omega_1} U_\xi = \{x\}$ and for all $\alpha \in \omega_1$,

$$\bigcap_{\xi < \alpha} U_\xi \setminus U_\alpha = \bigcap_{\xi < \alpha} U_\xi \setminus \left(\bigcap_{\xi \leq \alpha} U_\xi \right) \neq \emptyset.$$

Now pick $x_\xi \in \bigcap_{\xi < \alpha} U_\xi \setminus U_\alpha$. Note that $x_\alpha = x_\beta$ if and only if $\alpha = \beta$.

We claim that $A = \{x_\xi : \xi \in \omega_1\} \cup \{x\}$ is realcompact.

Let \mathcal{F}_A be a z -ultrafilter on A with cip. There are two cases:

Case 1) There exists $\alpha \in \omega_1$ such that $U_\alpha \cap A \notin \mathcal{F}_A$. Thus, $(X \setminus U_\alpha) \cap A \in \mathcal{F}_A$. Let α be least such ordinal. Then, $U_\xi \cap A \in \mathcal{F}_A$ for all $\xi < \alpha$. It follows that $\bigcap_{\xi < \alpha} U_\xi \cap (X \setminus U_\alpha) \cap A = \{x_\alpha\}$ and \mathcal{F}_A has cip. So $\{x_\alpha\} \in \mathcal{F}_A$ and $\bigcap \mathcal{F}_A = \{x_\alpha\}$.

Case 2) For all $\xi \in \omega_1$, $U_\xi \cap A \in \mathcal{F}_A$. Then, for all $\xi \in \omega_1$, $U_\xi \in \mathcal{F}_X$. Since \mathcal{F}_X is a z -ultrafilter with cip on X , $\bigcap \mathcal{F}_X \neq \emptyset$. But $\bigcap \mathcal{F}_X \subseteq \bigcap_{\xi \in \omega_1} U_\xi$ therefore $\bigcap \mathcal{F}_X = \{x\}$ which implies $\bigcap \mathcal{F}_A \subseteq \{x\}$.

But if $Z \in \mathcal{L}(A)$ and $x \notin Z$, then since $x \in A$ and Z is closed in A , $x \notin cl_X Z$. So, there is a clopen set U in X such that $x \in U$ and $U \cap cl_X Z = \emptyset$. As $\bigcap \mathcal{F}_X = \{x\}$ and \mathcal{F}_X is a z -ultrafilter, $U \in \mathcal{F}_X$ which implies $U \cap A \in \mathcal{F}_A$ and $Z \notin \mathcal{F}_A$. \square

In [GJ], Lindelöf spaces whose points are G_δ sets are shown to be hereditary realcompact. Here is an improvement of their result.

Lemma 43. *Let X be a realcompact space such that $\psi(X) = \omega$. Then X is hereditary realcompact.*

Proof. Let A be a subspace of X and \mathcal{F} be a z -ultrafilter with cip on A . Since X is realcompact, $cl_X A$ is a realcompactification of A . Therefore, there exists a continuous onto function $\beta i_d : \nu A \rightarrow cl_X A$ such that $\beta i_d|_A = i_d$. $\beta i_d(\mathcal{F}) = x$ for some $x \in cl_X A$. Now since $\psi(x, X) = \omega$, $\{x\} = \bigcap_{i \in \omega} U_i$ where U_i 's are open. Let $V_n = \bigcap_{i \in n} U_i$ and $f_i : X \rightarrow [0, 1/2^{i+1}]$ be a continuous function such that $f_i(x) = 0$ and $f_i[X \setminus V_i] = \{1/2^{i+1}\}$, then $f : X \rightarrow [0, 1]$ defined by $f(y) = \sum_{i \in \omega} f_i(y)$ is a continuous function and $Z(f) = \{x\}$. Let $Z_i := f^{-1}[[0, 1/i]]$ for $i \in \omega$, Z_i s are zero-sets and $\{x\} = \bigcap_{i \in \omega} Z_i$. Let

$Z'_i := f^{-1}[[1/i + 1, 1]]$ for all $i \in \omega$. The Z'_i 's are also zero-sets and $Z_i \cup Z'_i = X$ so for all $i \in \omega$ either $Z_i \cap A \in \mathcal{F}$ or $Z'_i \cap A \in \mathcal{F}$.

Let us assume there exists an $n \in \omega$ such that $Z_n \cap A \notin \mathcal{F}$. Then $\beta_{i_d}(\mathcal{F}) \neq x \notin Z'_n \cap A$, a contradiction.

So for all $i \in \omega$ $Z_i \cap A \in \mathcal{F}$ and by countable intersection property of \mathcal{F} , $\{x\} \supseteq \bigcap_{i \in \omega} (Z_i \cap A) \neq \emptyset$. Therefore $x \in A$. Which means \mathcal{F} is fixed in A . \square

Now by the Lemma 42 and Lemma 43 and the fact that any space without CRV property has a realcompact subspace of size ω_1 (Theorem 25) we have the following:

Corollary 44. *If X is an uncountable realcompact space and $\psi(X) \leq \omega_1$, then X has a realcompact subspace of size ω_1 .*

Now by the above corollary we show the following:

Corollary 45. *[CH] Every uncountable realcompact space has a realcompact subspace of size ω_1 .*

Proof. By Lemma 29 we can assume that the uncountable realcompact space X is separable. By the cardinality formula $\chi(X) \leq 2^{d(X)}$, the fact that $\psi(X) \leq \chi(X)$, and CH, we have that $\psi(X) \leq \omega_1$. By above corollary, we are done. \square

2.5 Realcompact Subspaces of Size less or equal to Continuum

Definition 46. A Hausdorff space is called *scattered* if every subspace of X has isolated points.

Fact 47. A space X is scattered if and only if X is right-separated.

Proof. Let X be scattered and let x_0 be an isolated point of X . At stage α if $X \setminus \{x_\xi : \xi < \alpha\} \neq \emptyset$, let x_α be an isolated point of $X \setminus \{x_\xi : \xi < \alpha\} \neq \emptyset$. For some η , $X \setminus \{x_\xi : \xi \in \eta\} = \emptyset$. Then $\{x_\xi : \xi \in \eta\}$ witnesses that X is right-separated.

Conversely suppose $X = \{x_\xi : \xi < \alpha\}$ is right-separation of X and $A \subseteq X$ is a non-empty subset. Now $A = \{x_\xi : \xi \in \Omega\}$ where $\Omega \subseteq \alpha$. Let $\eta = \min \Omega$, x_η is an isolated point of A .

□

Fact 48. Let X be right-separated of order type κ^+ . If X has no dense subspace of size $< \kappa$, then X has a discrete set of size κ .

Proof. Let $X = \{x_\xi : \xi < \kappa^+\}$ be a right-separation of order type κ^+ . Define $\phi : \kappa \rightarrow \kappa^+$ as follows: $\phi(0) = 0$. At stage $\alpha < \kappa$, pick $x_\zeta \in X \setminus cl\{x_{\phi(\xi)} : \xi < \alpha\}$ such that $\zeta > \sup\{\phi(\xi) : \xi < \alpha\}$. Let $\phi(\alpha) = \zeta$. Now $\{x_{\phi(\xi)} : \xi < \kappa\}$ is the desired set.

□

Lemma 49. Let κ be a cardinal and X be a right-separated, realcompact Hausdorff space such that $|X| \geq \kappa$. Then X has a realcompact subspace of size κ .

Proof. If $|X| = \kappa$, then we are done so let's assume $|X| \geq \kappa^+$. Enumerate $X := \{x_\xi : \xi \leq \eta\}$, for some ordinal η ($\eta \geq \kappa^+$), where the indices correspond to the right-separation of X . Let $Y := \{x_\xi : \xi < \kappa^+\} \subset X$. Regarding Y we have three cases:

Case 1: There exists $\xi < \kappa^+$ and an open set U such that $x_\xi \in U \subseteq clU \subseteq [x_0, x_{\xi+1})$ and $|clU| \geq \kappa$. In this case we are done as clU is realcompact and has size κ .

Case 2: There is a subspace $S \subseteq Y$ such that $|S| > \kappa$ and S has no dense subset of size $< \kappa$. By Fact 48, S has a discrete subset of size κ .

Case 3: Neither Case 1 nor Case 2. Let's build B_ξ by the way of induction. B_1 is a set of size $< \kappa$ that is dense in Y . B_δ is a set of size $< \kappa$ that is dense in $Y \setminus \bigcup\{B_\xi : \xi < \delta\}$. Then $B := \bigcup\{B_\xi : \xi < \kappa\}$ is of size κ . Let $\alpha = \sup\{\xi : x_\xi \in B\}$, $\alpha < \kappa^+$. Then there is an open set U such that $x_\alpha \in U \subseteq clU \subseteq [x_0, x_\alpha]$. However, as U has to intersect all B_ξ 's, $|U| \geq \kappa$. A contradiction to not Case 1. \square

Theorem 50. *Let $\omega \leq \kappa \leq 2^\omega$. If X is a compact space of size $\geq \kappa$, then X has a realcompact subspace of size κ .*

Proof. If there exists $A \subseteq X$ and $f \in C(A)$ such that $|f[A]| \geq \kappa$ then we are done by Theorem 25. Suppose for all $A \subseteq X$ and for all $f \in C(A)$, $|f[A]| < \kappa$ (X has $< \kappa$ real-valued property). By the way of contradiction let $B \subseteq X$ such that B has no isolated point. Then $cl_X B$ is countably compact and since B has no isolated point, $cl_X B$ has no isolated point. By Fact 33, there exists $f \in C(cl_X B)$ such that $|f[cl_X B]| = 2^\omega$, a contradiction. Now, scattered implies right-separated, and X is compact, Hausdorff, right-separated, and $|X| \geq \kappa$. By Lemma 49, X has to have a realcompact subspace of size κ . \square

Chapter 3

Normal images of the product and countably paracompact condensation

3.1 Introduction and Preliminaries.

3.1.1 Background and Introduction

There is a long history of examining when the product of two normal spaces is also normal. In homotopy theory, it is useful to know when $X \times \mathbb{I}$ is normal when X is normal and \mathbb{I} is the unit interval with the usual topology. In 1951, Dowker [Dow] examined this problem and proved for a normal space X , $X \times \mathbb{I}$ is normal iff $X \times (\omega + 1)$ is normal iff X is also countably paracompact.

The question of whether every normal space is countably paracompact was solved in 1971 by Mary Ellen Rudin [Rud] who constructed a normal space that is not countably paracompact.

In 1960, Tamano [Tam] proved this amazing result: For a normal space X , $X \times \beta X$ is normal if and only if X is paracompact.

We list a few results that are variations of Tamano's theorem. One is the result by Kunen discussed in 3.1.3. In 1984, Kunen [Kun] proved for a normal space X , $X \times (|X| + 1)$ is normal iff X is paracompact.

In 1997, Buzjakova [Buz] proved another variation of Tamano's theorem. She proved:

Let X be a pseudocompact Tychonoff space and $\kappa = |\beta X|^+$. Then X condenses onto a compact space if and only if $X \times (\kappa + 1)$ condenses onto a normal space.

Comparing the Buzjakova's result to Kunen's result the natural question is whether it is possible to prove a condensation variation of Kunen's result without the extra assumption that X is pseudocompact.

The final background result is due to Pavlov [Pav99]. In 1999, he proved that for every compact space K , there exists a space X such that $X \times K$ can not be condensed onto a normal space. This shows that not all spaces have coarser normal products and assuming that a space X has this property is not vacuous.

In this chapter we show for a Tychonoff space X , if $X \times (\kappa + 1)$ condenses onto a normal space, then X condenses onto a countably paracompact space, where $\kappa = 2^{2^{|X|^+}}$. Let all spaces be Tychonoff and for chosen κ , let $Y := X \times (\kappa + 1)$.

3.1.2 Preliminaries

Definition 51. Let \mathcal{U} be an open cover of space X . A cover \mathcal{C} of X is a locally finite refinement of \mathcal{U} if for every $V \in \mathcal{C}$ there exists a $U \in \mathcal{U}$ such that $V \subseteq U$ and each $x \in X$ has a neighborhood that intersects only finitely many members of \mathcal{C} .

Definition 52. A space X is called *paracompact* if every open cover has a locally finite open refinement.

Definition 53. A space X is called *countably paracompact* if every countable open cover has a locally finite open refinement.

Definition 54. A *Dowker space* is a normal space that is not countably paracompact.

Definition 55. A continuous function $f : X \rightarrow Y$ is a *condensation* if f is one-to-one and onto Y . If f is the identity, we can say that Y has a coarser topology than X .

3.1.3 Normal Product and Paracompactness

Theorem 56. [Kun] Let X be a Tychonoff space and let $|X| + 1$ have the order topology. If $X \times (|X| + 1)$ is normal, then X is paracompact.

Proof. Let $\{U_\alpha\}_{\alpha \in \lambda}$ be an open cover of X and $\{V_\alpha\}_{\alpha \in \lambda}$ be a collection of open sets in βX such that $V_\alpha \cap X = U_\alpha$. Note: without loss of generality We may assume that the open covers are indexed by a cardinal $\lambda \leq |X|$. If $\bigcap_{\alpha \in \lambda} (\beta X \setminus V_\alpha) = \emptyset$, then since βX is compact, we are done. So let us assume

$$\emptyset \neq A = \bigcap_{\alpha \in \lambda} (\beta X \setminus V_\alpha) \subseteq \beta X \setminus X$$

.

Now let $A_\alpha = X \setminus (\bigcup_{\beta < \alpha} V_\beta)$. $\bigcup_{\alpha \in \lambda} (A_\alpha \times \{\alpha\})$ is a closed set in $X \times (|X| + 1)$ because $\bigcap_{\alpha \in \lambda} A_\alpha = \emptyset$ and A_α is decreasing. Also $\bigcup_{\alpha \in \lambda} (A_\alpha \times \{\alpha\})$ is disjoint from the closed set $X \times \{\lambda\}$

So there exists a continuous function $f : X \times (|X| + 1) \rightarrow [0, 1]$ such that

- $f[X \times \{\lambda\}] = 0$, and
- $f[\bigcup_{\alpha \in \lambda} (A_\alpha \times \{\alpha\})] = 1$.

Use f to define a pseudometric d on X by:

$$d(x, y) = \sup_{\alpha \in \lambda + 1} |f(x, \alpha) - f(y, \alpha)|.$$

We verify the pseudometric properties.

For $x, y, z \in X$:

1. $d(x, x) = 0$,
2. $d(x, y) = d(y, x)$,
3. $d(x, y) = \sup_{\alpha \in \lambda + 1} |f(x, \alpha) - f(y, \alpha)| \leq \sup_{\alpha \in \lambda + 1} (|f(x, \alpha) - f(z, \alpha)| + |f(z, \alpha) - f(y, \alpha)|) \leq \sup_{\alpha \in \lambda + 1} |f(x, \alpha) - f(z, \alpha)| + \sup_{\alpha \in \lambda + 1} |f(z, \alpha) - f(y, \alpha)| = d(x, z) + d(z, y)$.

Claim: $\tau(d) \subseteq \tau(X)$.

Let $x \in X$ and $\varepsilon > 0$. First we show there is a $T \in \tau(X)$ such that $x \in T \subseteq B_d(x, \varepsilon)$.

Let $\alpha \in \lambda + 1$ and let $B_\alpha = W_\alpha \times O_\alpha$ (where O_α is open in $\alpha + 1$ containing α and $x \in W_\alpha$ is open in X) be a basic open set containing (x, α) such that for all $(y, \beta) \in B_\alpha$ ($\beta \in O_\alpha$), $|f(y, \beta) - f(x, \alpha)| < \varepsilon/2$.

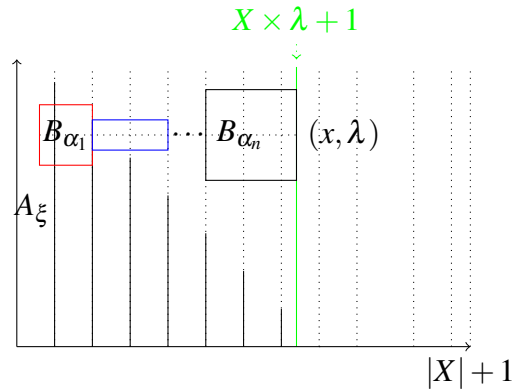


Figure 3.1: An illustration of a finite open cover of $\lambda + 1$.

Since $|f(y, \beta) - f(x, \beta)| < |f(y, \beta) - f(x, \alpha)| + |f(x, \alpha) - f(x, \beta)|$,

$$\sup_{\beta \in A_\alpha} |f(y, \beta) - f(x, \beta)| < \varepsilon.$$

Now $\lambda + 1$ is compact; therefore, it can be covered by finitely many $\{O_{\alpha_i}\}_{i < k}$. Then $(\cap_{i < k} W_{\alpha_i}) \times (|\lambda| + 1)$ is open in $X \times (|X| + 1)$. For all $y \in \cap_{i < k} W_{\alpha_i}$ and for $\alpha \in \lambda + 1$, $\sup_{\alpha \in |\lambda| + 1} |f(y, \alpha) - f(x, \alpha)| \leq \varepsilon$ implies $d(y, x) < \varepsilon$. $T := \cap_{i < k} W_{\alpha_i}$ is open in X and $x \in \cap_{i < k} W_{\alpha_i} \subseteq B_d(x, \varepsilon)$. Now to show $B_d(x, \varepsilon)$ is open in X , let $x' \in B_d(x, \varepsilon)$. There is $T' \in \tau(X)$ such that $x' \in T' \subseteq B_d(x', \varepsilon - d(x, x')) \subseteq B_d(x, \varepsilon)$. This shows that $B_d(x, \varepsilon) \in \tau(X)$.

By continuity of f at (x, λ) , there is $\alpha < \lambda$ such that $|f(x, \beta) - f(x, \lambda)| < 1/4$ for all $\beta > \alpha$.

For $\xi > \alpha$, define a continuous function $g_\xi : X \rightarrow [0, 1]$ by $g_\xi(x) = f(x, \xi)$. Let $\overline{g_\xi}$ be the extension of g_ξ to βX . For $x' \in B_d(x, 1/4)$,

$$g_\xi(x') \leq g_\xi(x) + |g_\xi(x') - g_\xi(x)| \leq g_\xi(x) + 1/4 = f(x, \xi) + 1/4 < 1/2.$$

Therefore,

$$\overline{g_\xi}[cl_{\beta X} B_d(x, 1/4)] \subseteq cl_{\mathbb{R}} g_\xi[B_d(x, 1/4)] \subseteq cl_{\mathbb{R}} [0, 1/2] = [0, 1/2].$$

For $y \in cl_{\beta X} A_\xi$, $\overline{g_\xi}(y) = 1$. Hence, $cl_{\beta X} B_d(x, 1/4) \cap A_\xi = \emptyset$, for all $x \in X$. As $A \subseteq A_\xi$, $cl_{\beta X} (B_d(x, 1/4) \cap A) = \emptyset$.

Now $\{B_d(x, 1/4) : x \in X\}$ has a locally finite open refinement in $\tau(d)$ call it $\{W_\gamma : \gamma \in \delta\}$ and $cl_{\beta X} W_\gamma \cap A = \emptyset$ and $cl_{\beta X} W_\gamma$ is compact. Therefore, there exists a finite open cover, $\{V_\alpha\}_{\alpha \in F_\gamma}$, of $cl_{\beta X} W_\gamma$ and then $\{W_\gamma \cap V_\alpha : \gamma \in \delta, \alpha \in F_\gamma\}$ is the locally finite refinement of $\{U_\alpha\}_{\alpha \in \lambda}$. \square

3.2 Condensation of the Product onto a Normal Space and Countable Paracompactness

3.2.1 Structure of the Product

We will start with a few facts about Stone-Ćech compactification of product space $X \times (\kappa + 1)$:

Fact 57. Let X be a Tychonoff space, κ a cardinal, and C a closed subset of $\kappa + 1$. Then $X \times C$ is C^* -embedded in $X \times (\kappa + 1)$

Proof. Enumerate C , $C = \{\alpha_\xi : \xi \in \eta + 1\}$ where $\eta \leq \kappa$ is an ordinal and $\xi < \zeta$ implies $\alpha_\xi < \alpha_\zeta$ in $\kappa + 1$. Let $g : X \times C \rightarrow [0, 1]$ be a continuous function. Define the extension $\bar{g} : X \times (\kappa + 1) \rightarrow [0, 1]$ in the following manner:

$$\bar{g}(x, \beta) = \begin{cases} g(x, \alpha_0), & \text{if } \beta \leq \alpha_0; \\ g(x, \alpha_{\xi+1}), & \text{if } \beta \in (\alpha_\xi, \alpha_{\xi+1}]; \\ g(x, \alpha_\xi), & \text{if } \beta = \alpha_\xi, \text{ where } \xi \text{ is a limit ordinal}; \\ g(x, \alpha_\eta), & \text{if } \beta > \alpha_\eta. \end{cases}$$

\bar{g} is identical for each of these blocks of lines.

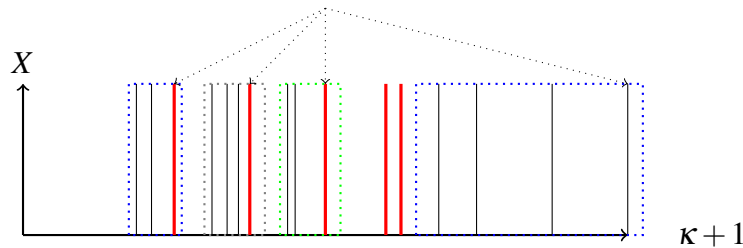


Figure 3.2: Extending g from $X \times C$ (the red set) to the whole space.

\bar{g} is continuous at (x, β) , where $\beta \in (\alpha_\xi, \alpha_{\xi+1}]$, because :

For basic open set $(r_1, r_2) \cap [0, 1] \ni \bar{g}(x, \beta)$ in $[0, 1]$,

$$\begin{aligned} & \pi_X[\bar{g}^{\leftarrow}[(r_1, r_2)] \cap (X \times \{\beta\})] \times (\alpha_\xi, \alpha_{\xi+1}] \\ &= \pi_X[g^{\leftarrow}[(r_1, r_2)] \cap (X \times \{\alpha_{\xi+1}\})] \times (\alpha_\xi, \alpha_{\xi+1}] \subseteq \bar{g}^{\leftarrow}[(r_1, r_2)] \end{aligned}$$

is an open set containing (x, β) . (It is a product $\pi_X[g^{\leftarrow}[(r_1, r_2)] \cap (X \times \{\alpha_{\xi+1}\})]$, an open set in X , and $(\alpha_\xi, \alpha_{\xi+1}]$, an open set in $\kappa + 1$.)

\bar{g} is continuous at (x, β) for $\beta \leq \alpha_0$:

Again for basic open set $(r_1, r_2) \cap [0, 1] \ni \bar{g}(x, \beta)$ in $[0, 1]$,

$$\begin{aligned} & \pi_X[\bar{g}^{\leftarrow}[(r_1, r_2)] \cap (X \times \{\beta\})] \times [0, \alpha_0] \\ &= \pi_X[g^{\leftarrow}[(r_1, r_2)] \cap (X \times \{\alpha_0\})] \times [0, \alpha_0] \subseteq \bar{g}^{\leftarrow}[(r_1, r_2)] \end{aligned}$$

is an open set. (It is a product $\pi_X[g^{\leftarrow}[(r_1, r_2)] \cap (X \times \{\alpha_0\})]$, an open set in X , and $[0, \alpha_0]$, an open set in $\kappa + 1$.)

\bar{g} is continuous at (x, β) for $\beta > \alpha_\eta$:

For basic open set $(r_1, r_2) \cap [0, 1] \ni \bar{g}(x, \beta)$ in $[0, 1]$,

$$\begin{aligned} & \pi_X[\bar{g}^{\leftarrow}[(r_1, r_2)] \cap (X \times \{\beta\})] \times (\alpha_\eta, \kappa] \\ &= \pi_X[g^{\leftarrow}[(r_1, r_2)] \cap (X \times \{\alpha_\eta\})] \times (\alpha_\eta, \kappa] \subseteq \bar{g}^{\leftarrow}[(r_1, r_2)] \end{aligned}$$

is an open set. (It is a product $\pi_X[g^{\leftarrow}[(r_1, r_2)] \cap (X \times \{\alpha_\eta\})]$, an open set in X , and $(\alpha_\eta, \kappa]$, an open set in $\kappa + 1$.)

\bar{g} is continuous at (x, β) , where $\beta = \alpha_\xi$ for ξ a limit ordinal:

For basic open set $(r_1, r_2) \cap [0, 1]$, let U open in X and $\zeta \leq \xi$ be such that

$$(U \times (\alpha_\zeta, \alpha_\xi]) \cap (X \times C) \subseteq g^{-1}[(r_1, r_2)],$$

then

$$(x, \beta) \in (U \times (\alpha_\zeta, \alpha_\xi]) \subseteq \bar{g}^{-1}[(r_1, r_2)].$$

□

Notation.

For a Tychonoff space X , a cardinal κ , and $\alpha \in \kappa + 1$, by Fact 57, $X \times \{\alpha\}$ is C^* -embedded in $X \times (\kappa + 1) =: Y$. As Y is C^* -embedded in βY , it follows that $X \times \{\alpha\}$ is C^* -embedded in βY . So $h_\alpha : \beta X \times \{\alpha\} \approx cl_{\beta Y}(X \times \{\alpha\})$. For $y \in \beta X$, $h_\alpha(y, \alpha) \in \beta Y$. To avoid confusion, we denote $h_\alpha(y, \alpha)$ by $e(y, \alpha)$ and

$$e[\beta X \times (\kappa + 1)] = \{e(y, \alpha) : y \in \beta X, \alpha \in (\kappa + 1)\}.$$

Using this notation, the Stone-Ćech compactification of $Y := X \times (\kappa + 1)$ is shown in Figure 3.3.

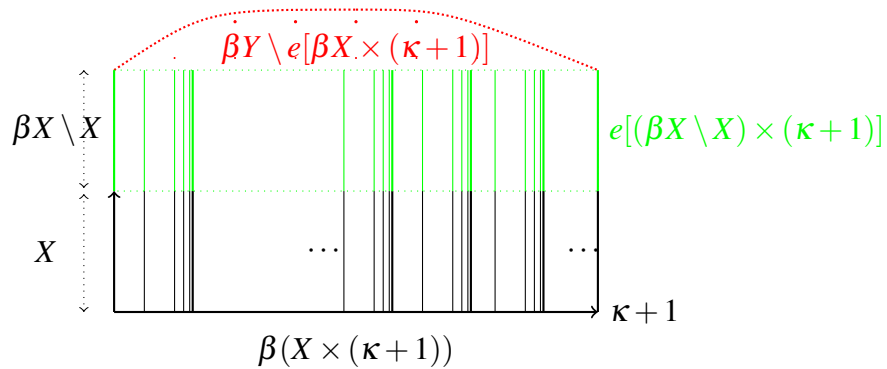


Figure 3.3: Demonstrating βY as a set.

Corollary 58. *Let X be a Tychonoff space and κ be a cardinal and $Y := X \times (\kappa + 1)$, then*

$$\beta(Y) \supseteq \bigcup_{\alpha \in \kappa + 1} e[\beta X \times \{\alpha\}]$$

Proof. By Fact 57, $X \times \{\alpha\}$ is C^* -embedded in Y . As Y is C^* -embedded in $\beta(Y)$, it follows that $X \times \{\alpha\}$ is C^* -embedded in $\beta(Y)$. So, $\beta X \times \{\alpha\} \simeq cl_{\beta(Y)}(X \times \{\alpha\})$. \square

Warning: With this notation, we have $\beta X \times (\kappa + 1) \hookrightarrow \beta Y$; however, $\beta X \times (\kappa + 1)$ does not have the product topology as a subspace of βY .

The above corollary enables us to think of $\beta(X \times (\kappa + 1))$ as a space containing a copy of $e[\beta X \times (\kappa + 1)]$, as a subset not as a subspace, and later on serves as a tool to work with the $\beta(X \times H_\alpha)$ where the H_α 's are isomorphic closed subsets of κ .

3.2.2 Convergence Tools

The final step of the proof needs convergence properties of sequences of the form $\{e(y, \xi) : \xi \in \alpha\}$ in βY for some ordinal $\alpha \in \kappa$ and $y \in \beta X \setminus X$, and the next fact helps us achieve convergence for some specific α 's.

Fact 59. Let $Y = X \times (\kappa + 1)$ and $\alpha \in \kappa + 1$ be such that $cof(\alpha) > |X|$. Then for $y \in \beta X \setminus X$ and $e(y, \alpha) \in U \in \tau(\beta Y)$, there exists $\beta < \alpha$ and $V \in \tau(\beta X)$ such that $e(y, \alpha) \in e[V \times (\beta, \alpha)] \subseteq U$.

Proof. There exists $W \in \tau(\beta Y)$ such that $e(y, \alpha) \in W \subseteq cl_{\beta Y} W \subseteq U$, and there exists $V \in \tau(\beta X)$ such that $e[V \times \{\alpha\}] \subseteq W$ where $y \in V$.

For each $x \in V \cap X$, as $(x, \alpha) \in W \cap (X \times \{\alpha\})$, there exists $\beta_x < \alpha$ such that $\{x\} \times (\beta_x, \alpha] \subseteq W$, let $\beta = \sup\{\beta_x : x \in V \cap X\}$. Then $(V \cap X) \times (\beta, \alpha] \subseteq W$.

For each $\gamma \in (\beta, \alpha]$, $(V \cap X) \times \{\gamma\} \subseteq W$; therefore,

$$cl_{\beta Y}((V \cap X) \times \{\gamma\}) \subseteq cl_{\beta Y} W \subseteq U.$$

But $cl_{\beta Y}((V \cap X) \times \{\gamma\}) = e[cl_{\beta X}(V \cap X) \times \{\gamma\}] \supseteq e[V \times \{\gamma\}]$.

Thus $e[V \times (\beta, \alpha]] \subseteq U$. □

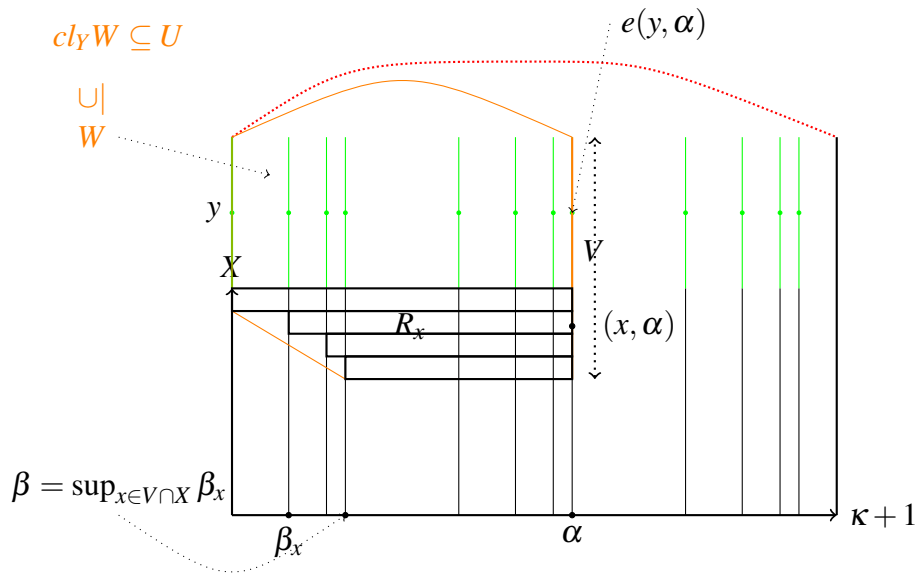


Figure 3.4: A neighbourhood of $e(y, \alpha)$ contains a product of two open sets.

Definition 60. A point p is a *complete accumulation point* of a subset A of a space X , if for all open sets $U \ni p$, $|A| = |U \cap A|$.

Definition 61. A point $p \in X$ has *large neighbourhoods* in $A \subseteq X$ if for any open set $U \ni p$, $|A \setminus U| < |A|$.

Note 62. For a Hausdorff space X , if $p \in X$ has large neighbourhoods in $A \subseteq X$, then p is the unique accumulation point of A .

Corollary 63. *Let $Y = X \times (\kappa + 1)$, let $y \in \beta X \setminus X$ and let $\alpha \in \kappa + 1$ be such that $\text{cof}(\alpha) > |X|$. If $\{\beta_\gamma : \gamma < \text{cof}(\alpha)\}$ is a cofinal sequence in α , then $\{e(y, \beta_\gamma) : \gamma < \text{cof}(\alpha)\} \rightarrow e(y, \alpha)$. In other words, for open set $U \ni e(y, \alpha)$,*

$$|\{e(y, \beta_\gamma) : \gamma < \text{cof}(\alpha)\} \setminus U| < |\{e(y, \beta_\gamma) : \gamma < \text{cof}(\alpha)\}|.$$

That is, $e(y, \alpha)$ is the unique complete accumulation point of the set $\{e(y, \beta_\gamma) : \gamma < \text{cof}(\alpha)\}$.

Proof. Obviously, by Fact 59, $e(y, \alpha)$ is a complete accumulation point of the set $\{e(y, \beta_\gamma) : \gamma < \text{cof}(\alpha)\}$. If $z \in \text{cl}_{\beta Y} \{e(y, \beta_\gamma) : \gamma < \text{cof}(\alpha)\}$ and $z \neq e(y, \alpha)$, then there is an open set W in βY such that $z \in W$ and $e(y, \alpha) \notin \text{cl}_{\beta Y} W$. By Fact 59, there is $\beta < \alpha$ and $V \in \tau(\beta X)$ such that $y \in V$ and $e[V \times (\beta, \alpha)] \subseteq \beta Y \setminus \text{cl}_{\beta Y} W$. There is $\delta < \text{cof}(\alpha)$ such that $W \cap \{e(y, \beta_\gamma) : \gamma < \text{cof}(\alpha)\} \subseteq \{e(y, \beta_\gamma) : \gamma < \delta\}$ and $|\{e(y, \beta_\gamma) : \gamma < \delta\}| < \text{cof}(\alpha)$. So, z is not a complete accumulation point of $\{e(y, \beta_\gamma) : \gamma < \text{cof}(\alpha)\}$. \square

Note 64. Let $A \subseteq T$ be a C^* -embedded subset. Then $\beta A = \text{cl}_{\beta T} A \subseteq \beta T$. Therefore, every free z -ultrafilter on A converges to a point in $\beta A \setminus A = (\text{cl}_{\beta T} A \setminus A) \subseteq (\beta T \setminus A)$. When we talk about a free z -ultrafilter on A as a point we are talking about the adherence point of that filter in βT .

We still need more convergences statements about points other than the points of $e[\beta X \times \kappa]$; these other points appear in the Stone-Ćech compactification of infinitely many $X \times \{\xi\}$'s which is a requirement in our proof. To understand the repetitive behavior of such points, we need to use isomorphic subspaces of Y of the form $X \times C_\alpha$, so we can have a large number of isomorphic free z -ultrafilter defined on them. Then

we need to make convergence statement about those isomorphic z -ultrafilters and the Fact 66. is a crucial tool to accomplish that.

Before Stating the Fact 66, we prove the following well-known Lemma:

Lemma 65. *The projection map $\pi_X : X \times C \rightarrow X$ is closed, where C is a compact set.*

Proof. Let $F \subseteq X \times C$ be a closed subset and $y \notin \pi_X[F]$. Then $\{y\} \times C \subseteq (X \times C) \setminus F$. Since C is compact, $\{y\} \times C$ can be covered by finitely many basic open sets in the product. Therefore, there exists an open set $U \in \tau(X)$ such that $\{y\} \times C \subseteq U \times C$ and $(U \times C) \cap F = \emptyset$. Then $y \in \pi_X[U \times C] = U$ and $U \cap \pi_X[F] = \emptyset$. Thus $y \notin cl_X \pi_X[F]$. \square

Note that for z -ultrafilter \mathcal{F} on $X \times C$, if $y \in \bigcap_{F \in \mathcal{F}} cl_{\beta X} \pi_X[F] \cap X$, then there exists $\xi \in C$ such that $(y, \xi) \in F$ for all $F \in \mathcal{F}$. By the way of contradiction suppose for all $\xi \in C$ there exists $F_\xi \in \mathcal{F}$ such that $(y, \xi) \notin F_\xi$. Then since F_ξ is closed, we can find an open set containing (y, ξ) missing F_ξ . On the other hand, $\{y\} \times C$ is compact, so it can be covered by finitely many of these open sets. Hence, there exists an $F \in \mathcal{F}$ such that $(\{y\} \times C) \cap F = \emptyset$ and by Lemma 65 $y \notin cl_X \pi_X[F]$. A contradiction. We conclude that for free z -ultrafilter \mathcal{F} on $X \times C$,

$$\bigcap_{F \in \mathcal{F}} cl_{\beta X} \pi_X[F] \cap X = \emptyset$$

Fact 66. For every free z -ultrafilter \mathcal{F} on $X \times C$, where C is closed in $\kappa + 1$, there exists a unique $y_{\mathcal{F}} \in \beta X \setminus X$ such that $\bigcap_{F \in \mathcal{F}} cl_{\beta X} \pi_X[F] = \{y_{\mathcal{F}}\}$. Denote $y_{\mathcal{F}}$ as the corresponding βX - element of \mathcal{F} .

Proof. $\bigcap_{F \in \mathcal{F}} cl_{\beta X} \pi_X[F]$ is nonempty because it's an intersection of compact sets with nonempty finite intersection. The intersection can not have more than one point since

\mathcal{F} is a z -ultrafilter. If there are two points y_1 and y_2 in $\bigcap_{F \in \mathcal{F}} cl_{\beta X} \pi_X[F]$, there are two disjoint z -set neighborhood, Z_1 and Z_2 , in βX such that $y_1 \in Z_1$ and $y_2 \in Z_2$. Now for $F \in \mathcal{F}$, $F \cap e[Z_1 \times C]$ and $F \cap e[Z_2 \times C]$ are two disjoint z -sets and they both intersect \mathcal{F} , a contradiction to the maximality of the z -ultrafilter. So, $\bigcap_{F \in \mathcal{F}} cl_{\beta X} \pi_X[F] = \{y_{\mathcal{F}}\}$.

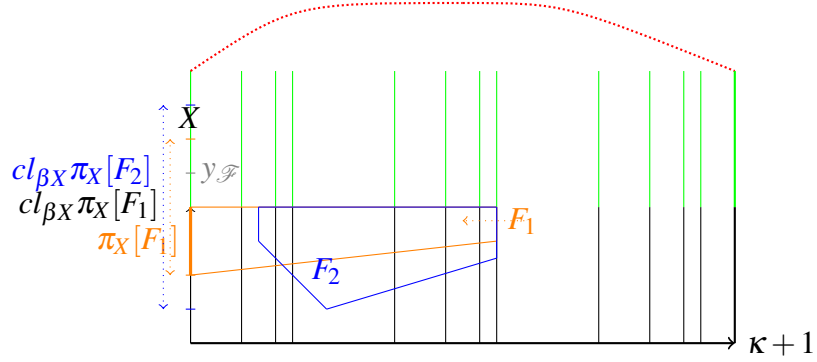


Figure 3.5: The projection of the filter \mathcal{F} .

□

We introduce the last two convergence tools of this section:

Fact 67. Let C be a closed set in κ such that $C \simeq \eta + 1$ for some $\eta \in \kappa$. Let $g : \eta + 1 \rightarrow C$ be the homeomorphism and let \mathcal{F} be a free z -ultrafilter on $X \times (\eta + 1)$. Then $\mathcal{F}' := \{\{(x, g(\xi)) : (x, \xi) \in F\} : F \in \mathcal{F}\}$ is a z -ultrafilter on $X \times C$ and has the same corresponding βX -element as \mathcal{F} . In other words, $y_{\mathcal{F}'} = y_{\mathcal{F}}$.

Proof. It is easy to see that \mathcal{F}' is a z -ultrafilter. It suffices to show $y_{\mathcal{F}'} = y_{\mathcal{F}}$:

$$\begin{aligned} \{y_{\mathcal{F}'}\} &:= \bigcap_{F \in \mathcal{F}'} cl_{\beta X} \pi_X[F] = \\ &\bigcap_{F \in \mathcal{F}'} cl_{\beta X} \{x : (x, g(\xi)) \in F \text{ for some } \xi \in C\} = \end{aligned}$$

$$\bigcap_{F \in \mathcal{F}} cl_{\beta X} \{x : (x, \xi) \in F \text{ for some } \xi \in C\} = \{y_{\mathcal{F}}\}.$$

□

Fact 68. Let $\{C_{\alpha}^{\eta} : \alpha \in \kappa\}$ be a collection of closed sets in $\kappa + 1$, where $\eta < \kappa$, with these properties:

1. Each C_{α}^{η} is closed and bounded in $\kappa + 1$.
2. $\min C_{\alpha}^{\eta} > \sup C_{\beta}^{\eta}$ if $\alpha > \beta$.
3. $C_{\alpha}^{\eta} \cong \eta + 1$ and $g_{\alpha} : \eta + 1 \longrightarrow C_{\alpha}^{\eta}$ is the isomorphism.

Let \mathcal{F} be a free z -ultrafilter on $X \times (\eta + 1)$, then $\mathcal{F}_{(\alpha)} := \{\{(x, g_{\alpha}(\xi)) : (x, \xi) \in F\} : F \in \mathcal{F}\}$ is a free z -ultrafilter on $X \times C_{\alpha}$, and $y_{\mathcal{F}_{(\alpha)}} = y_{\mathcal{F}}$ for $\alpha \in \kappa$.

Furthermore, for ζ , a regular cardinal, such that $|X| < \zeta \leq \kappa$, $e(y_{\mathcal{F}}, \lambda)$ has large neighbourhoods in the set $\{ad_{\beta Y} \mathcal{F}_{(\alpha_v)} : v \in \zeta\}$, where $\lambda := \sup_{v \in \zeta} C_{\alpha_v}^{\eta}$ and $y_{\mathcal{F}}$ is the corresponding βX -element of $\mathcal{F}_{(\alpha_v)}$.

Proof. Let $e(y_{\mathcal{F}}, \lambda) \in V$, where V is open in βY . By the Tychonoff property of βY , there is an open set U in βY such that $e(y_{\mathcal{F}}, \lambda) \in U \subseteq cl_{\beta(X \times (\kappa+1))} U \subseteq V$, where $cl_{\beta Y} U$ is a z -set in βY . By Fact 59 there exists $\mu \in \lambda$ and $y_{\mathcal{F}} \in W \in \tau(\beta X)$ such that $e[W \times (\mu, \lambda)] \subseteq U$. Now, for all $v \in \zeta$ such that $\min C_{\alpha_v}^{\eta} > \mu$, $[U \cap (X \times C_{\alpha_v}^{\eta})] \cap F \supseteq [e[W \times (\mu, \lambda)] \cap (X \times C_{\alpha_v}^{\eta}) \cap F] \neq \emptyset$ for all $F \in \mathcal{F}_{(\alpha_v)}$. Thus $[cl_{\beta Y} U \cap (X \times C_{\alpha_v}^{\eta})] \in \mathcal{F}_{(\alpha_v)}$. Now, $\mathcal{F}_{(\alpha_v)} = \bigcap_{F \in \mathcal{F}_{(\alpha_v)}} cl_{\beta Y} F \in cl_{\beta Y} U$ (which means $ad_{\beta Y} \mathcal{F}_{(\alpha)}$ is in $cl_{\beta Y} U \subseteq V$), Therefore, $(y_{\mathcal{F}}, \lambda)$ is a complete accumulation point of the set $\{\mathcal{F}_{(\alpha_v)} : v \in \zeta\} := \{ad_{\beta Y} \mathcal{F}_{(\alpha_v)} : v \in \zeta\}$. On the other hand, since βY is Hausdorff and $\beta Y \setminus V$ contains at most $< \zeta$ many elements of the set $\{\mathcal{F}_{(\alpha_v)} : v \in \zeta\}$, $e(y_{\mathcal{F}}, \lambda)$ has large neighbourhoods in that set.

□

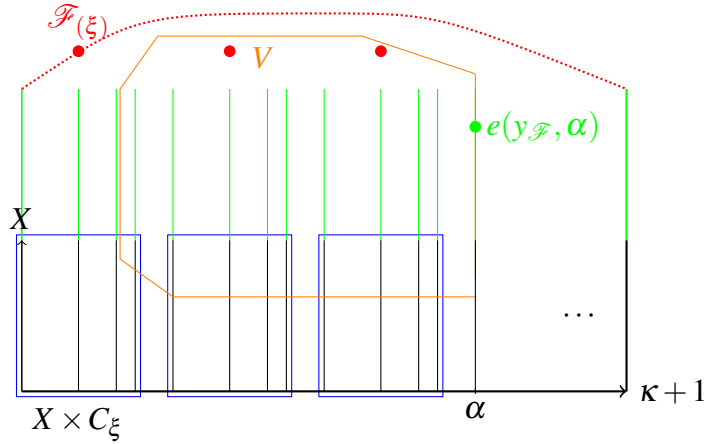


Figure 3.6: A neighbourhood of $e(y_{\mathcal{F}}, \alpha)$.

3.2.3 Skirting around Pseudocompactness

In this section we are going to expand various notions in Buzjakova's proof. Here is a Lemma to clarify the method used in finding closures in the rest of this chapter.

Lemma 69. *Let T be a Tychonoff space and $f : T \rightarrow S$ be a one-to-one, continuous function, then for closed set $A \subseteq T$*

1) *If $z \in cl_S f[A] \setminus f[A]$ then $z = \bar{f}(y)$ for some $y \in cl_{\beta T} A \setminus T$ where $\bar{f} : \beta T \rightarrow \beta S$ is the extension of f .*

2) *For $y \in cl_{\beta T} A \setminus A$ if $f(y) \in S \setminus f[A]$ then $f(y) \in cl_S f[A]$.*

Henceforth, let f denote a condensation from $X \times (\kappa + 1) \rightarrow Z$, as shown in the figure below, where Z is normal. Let $\bar{f} : \beta Y \rightarrow \beta Z$ be the continuous extension of $f : Y \rightarrow Z$ to Stone-Čech compactification of Y .

In the Definition that follows, we present a variation of Buzjakova's definition, which suits our case, of the sets C_1 and C_2 . Different properties of these sets help us find the ordinals ξ 's for which closures of $f[X \times \{\xi\}]$'s in βZ are isomorphic to each other and have minimal points, compared to images of other such lines, in some section of Z with large ordinals.

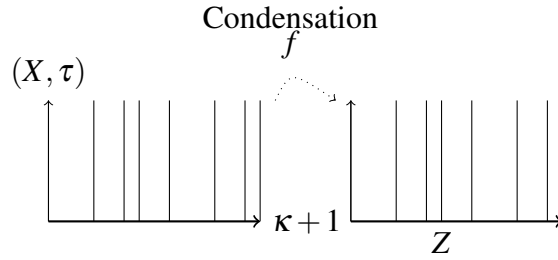


Figure 3.7: The condensation $f : X \times \kappa \rightarrow Z$.

Definition 70. Let X be a Tychonoff space and κ be a cardinal such that $\text{cof}(\kappa) > |\beta X|$ and let $f : X \times (\kappa + 1) \rightarrow Z$ be a condensation onto a normal space Z , then let

$$C_1 := \{y \in \beta X \setminus X : |\{\alpha \in \kappa : \bar{f}(e(y, \alpha)) \in Z\}| = \kappa \text{ and } |\bar{f}[e[\{y\} \times \kappa]] \cap Z| = \kappa\}$$

and

$$C_2 := \{y \in \beta X \setminus X : |\{\alpha \in \kappa : \bar{f}(e(y, \alpha)) \in Z\}| = \kappa \text{ and } |\bar{f}[e[\{y\} \times \kappa]] \cap Z| < \kappa\}.$$

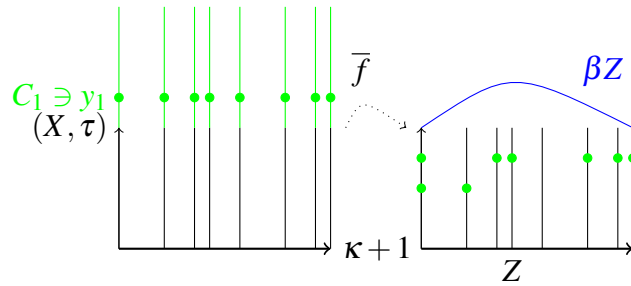


Image of a spine on an element of C_1 under \bar{f} .

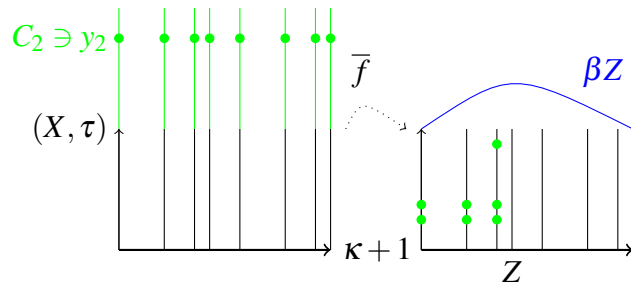


Figure 3.8: Image of a spine on an element of C_2 under \bar{f} .

In Fact 73, we are going to establish some properties of C_1 and C_2 by using convergence facts from previous section and the following lemmata.

Lemma 71. *Let R and T be Hausdorff spaces, $A \subseteq R$, and $f : R \rightarrow T$ be a continuous and onto function. If $p \in A$ has large neighbourhoods in A and $|A| = |f[A]|$, then $f(p)$ is the unique complete accumulation point of $f[A]$.*

Proof. Let $f(p) \in V \in \tau(T)$. It suffices to show that $|f[A] \setminus V| < |f[A]|$. Note that $A \cap f^{-1}[T \setminus V] = A \cap (f^{-1}[T] \setminus f^{-1}[V]) = A \cap (R \setminus f^{-1}[V]) = A \setminus f^{-1}[V]$. Now $p \in f^{-1}[V]$. Thus, by hypothesis, $|A \setminus f^{-1}[V]| < |A|$.

Therefore, $|f[A \setminus f^{-1}[V]]| \leq |A \setminus f^{-1}[V]| < |A| = |f[A]|$. On the other hand,

$$f[A \setminus f^{-1}[V]] = f[A \cap f^{-1}[T \setminus V]] = f[A] \cap (T \setminus V) = f[A] \setminus V.$$

Thus $|f[A] \setminus V| < |f[A]|$. □

Lemma 72. *Let R and T be Hausdorff spaces, $A \subseteq R$, let $f : R \rightarrow T$ be a continuous and onto function and let p be such that whenever $p \in U \in \tau(R)$, $|A \setminus U| < |A|$. If $|A| > |f[A]|$ and $|A|$ is a regular cardinal, then $f(p)$ is the unique point q in T such that $|f^{-1}[\{q\}] \cap A| = |A|$.*

Proof. Let $q \neq f(p)$. There exists open sets U and V in T such that $q \in U$ and $f(p) \in V$ and $U \cap V = \emptyset$. Then $f^{-1}[\{q\}] \subseteq f^{-1}[U]$ and $f^{-1}[\{f(p)\}] \subseteq f^{-1}[V]$ and $f^{-1}[U] \cap f^{-1}[V] = \emptyset$. Now, by hypothesis, $|A \setminus f^{-1}[V]| < |A|$.

Therefore, $|f^{-1}[U] \cap A| < |A|$. That is, $|f^{-1}[\{q\}] \cap A| < |A|$. On the other hand,

$$\left| \bigcup_{q \in f[A] \setminus \{f(p)\}} (f^{-1}[\{q\}] \cap A) \right| < \sum_{|f[A]|} |f^{-1}[\{q\}] \cap A| < |f[A]|.$$

Therefore, $|f^{-1}[\{f(p)\}] \cap A| = |A|$.

□

Fact 73. Let X be Tychonoff, κ a regular ordinal such that $\text{cof}(\kappa) > |X|$, then

(a) for $y \in C_1$,

(i) there exists $x_y \in X$ such that $\bar{f}(e(y, \kappa)) = f(x_y, \kappa)$.

(ii) for $x \in X \setminus \{x_y\}$, $|\{\bar{f}(e(y, \alpha)) : \alpha \in \kappa\} \cap \{f(x, \alpha) : \alpha \in \kappa\}| < \kappa$.

(iii) there exists $\beta'_y < \kappa$ such that if $\alpha > \beta'_y$ and $\bar{f}(e(y, \alpha)) \in Z$ there exists $\gamma_\alpha \leq \kappa$ such that $\bar{f}(e(y, \alpha)) = f(x_y, \gamma_\alpha)$.

(iv) furthermore if $\text{cof}(\kappa) > |X|^+$, $|\{\alpha \in \kappa : \bar{f}(e(y, \alpha)) = f(x_y, \gamma)\}| < \kappa$, for $\gamma \in \kappa + 1$.

Therefore, there exists $\beta_y < \kappa$ such that if $\alpha > \beta_y$ and $\bar{f}(e(y, \alpha)) \in Z$, then there exists $\gamma_\alpha < \kappa$ such that

$$\bar{f}(e(y, \alpha)) = f(x_y, \gamma_\alpha).$$

(b) For $y \in C_2$,

let $I_y := \bar{f}[e[\{y\} \times \kappa]] \cap Z$, note that $|I_y| < \kappa$, then

(i) there exists a unique $x_y \in X$ and $\alpha_y \in \kappa + 1$ such that $\bar{f}(e(y, \kappa)) = f(x_y, \alpha_y)$.

(ii) for $(x, \alpha_x) \in I_y \setminus \{(x_y, \alpha_y)\}$, $|\{\xi : \bar{f}(e(y, \xi)) = f(x, \alpha_x)\}| < \kappa$

(iii) there exists β_y such that for $\xi > \beta_y$ if $\bar{f}(e(y, \xi)) \in Z$ then $\bar{f}(e(y, \xi)) = f(x_y, \alpha_y)$.

(c) For $y \in \beta X \setminus C_1 \cup C_2$, there exists β_y such that for $\xi > \beta_y$, $\bar{f}(e(y, \xi)) \notin Z$.

Proof. (a) (i) As $\bar{f}[e[\{y\} \times \kappa]] \cap Z = \bigcup_{x \in X} \{\bar{f}[e[\{y\} \times \kappa]] \cap f[\{x\} \times \kappa]\}$, there exists $x_y \in X$ such that $|\bar{f}[e[\{y\} \times \kappa]] \cap f[\{x_y\} \times \kappa]| = \kappa$.

Note: for $\alpha < \kappa$, $|\bar{f}[e[\{y\} \times (\alpha, \kappa)]] \cap f[\{x_y\} \times (\alpha, \kappa)]| = \kappa$.

Let $B_\alpha = \{\beta \in \kappa : \alpha < \beta \text{ and } \bar{f}(e(y, \beta)) \in f[\{x_y\} \times (\alpha, \kappa)]\}$. Let C_α be defined by $\bar{f}[e[\{y\} \times B_\alpha]] = f[\{x_y\} \times C_\alpha]$.

$|B_\alpha| = \kappa$; therefore, by the one-to-one property of f , $|C_\alpha| = \kappa$.

By Fact 59 $e(y, \kappa) \in cl_{\beta_X} e[\{y\} \times B_\alpha]$. So,

$$\begin{aligned} \bar{f}(e(y, \kappa)) &\in \bar{f}[cl_{\beta_Y} e[\{y\} \times B_\alpha]] \subseteq cl_{\beta_Z} \bar{f}[e[\{y\} \times B_\alpha]] = cl_{\beta_Z} f[e[\{x_y\} \times C_\alpha]] \\ &\subseteq f[\{x_y\} \times (\alpha, \kappa)]. \end{aligned}$$

On the other hand, $f[\{x_y\} \times (\alpha, \kappa)]$ is the continuous image of a compact set. Therefore, it is closed. So, $\bar{f}(e(y, \kappa)) \in \bigcap_{\alpha < \kappa} f[\{x_y\} \times (\alpha, \kappa)] = \{f(x_y, \kappa)\}$. Thus, $\bar{f}(e(y, \kappa)) = f(x_y, \kappa)$.

The uniqueness follows from one-to-one property of f . If $\bar{f}(e(y, \kappa)) = f(x_y, \kappa)$ and $\bar{f}(e(y, \kappa)) = f(x', \kappa)$, then $f(x_y, \kappa) = f(x', \kappa)$. So, $x' = x_y$ as f is one-to-one.

(a) (ii) If for some $x \in X \setminus \{x_y\}$, $|\bar{f}[e[\{y\} \times \kappa]] \cap f[\{x\} \times \kappa]| = \kappa$, then by the proof of (i), $\bar{f}(e(y, \kappa)) = f(x, \kappa)$. But $f(x, \kappa) \neq f(x_y, \kappa)$ as f is one-to-one, a contradiction. In fact, $|\bar{f}[e[\{y\} \times \kappa]] \cap f[\{x\} \times \kappa]| < \kappa$.

(a)(iii) For $x \in X \setminus \{x_y\}$, there exists $\beta_{y,x} < \kappa$ such that $\bar{f}[e[\{y\} \times \kappa]] \cap f[\{x\} \times \kappa] \subseteq f[\{x\} \times [0, \beta_x]]$.

Let $\beta'_y = \sup_{x \in X \setminus \{x_y\}} \beta_{y,x}$. For $\alpha > \beta'_y$ and $\bar{f}(e(y, \alpha)) \in Z$, then $\bar{f}(e(y, \alpha)) \in f[\{x_y\} \times (\kappa + 1)]$, that is, there exists $\gamma_\alpha \leq \kappa$ such that $\bar{f}(e(y, \alpha)) = f(x_y, \gamma_\alpha)$.

(a) (iv) By the way of contradiction, assume $|\{\alpha : \bar{f}(e(y, \alpha)) = f(x, \gamma)\}| = \kappa$, for $\gamma \in \kappa + 1$. (Note that $|A_y(x, \gamma)| = \kappa$.)

Define $A_y(x, \gamma) := \{\alpha \in \kappa : \bar{f}(e(y, \alpha)) = f(x, \gamma)\}$,

and $B_y := \{\alpha \in \kappa : \bar{f}(e(y, \alpha)) = f(x_y, \nu) \text{ for some } \nu \in \kappa\}$,

and C_y be such that $\bar{f}[e[\{y\} \times B_y]] = f[\{x_y\} \times C_y]$.

It is easy to reach a contradiction from $(x, \gamma) \neq (x_y, \kappa)$, in fact $e(y, \kappa)$ has large neighbourhoods in both sets $e[\{y\} \times B_y]$ and $e[\{y\} \times A_y(x, \gamma)]$ which, by Lemma 71, means $f(x_y, \kappa)$ is the unique complete accumulation point of $\bar{f}[e[\{y\} \times B_y]] = f[\{x_y\} \times C_y]$

and $f(x, \gamma)$ is a complete accumulation point of $\bar{f}[e[\{y\} \times A_y(x, \gamma)]]$. Therefore, both $f(x_y, \kappa)$ and $f(x, \gamma)$ are complete accumulation points of the set $\bar{f}[e[\{y\} \times \kappa]]$ which is a contradiction to uniqueness of complete accumulation point of that set.

To reach a contradiction from the case of $(x, \gamma) = (x_y, \kappa)$, we inductively define α_ξ , β_ξ , δ_ξ for $\xi \in |X|$.

Let $\alpha_0 \in A_y(x_y, \kappa)$ and $\beta_0 \in B_y \cap (\alpha_0, \kappa)$ be such that $\delta_0 \in (\alpha_0, \kappa)$, where δ_0 is defined by $\bar{f}(y, \beta_0) = f(x_y, \delta_0)$.

At stage $\xi \in |X|^+$, where ξ is a limit ordinal, let $\alpha_\xi \in (\sup\{\alpha_\zeta : \zeta \in \xi\}, \kappa)$ and $\beta_\xi \in B \cap (\alpha_\xi, \kappa)$ such that $\delta_\xi \in (\alpha_\xi, \kappa)$, where δ_ξ is defined by $\bar{f}(e(y, \beta_\xi)) = f(x_y, \delta_\xi)$.

At stage $\xi = \zeta + 1$, let $\alpha_\xi \in A_y(x_y, \kappa) \cap (\max\{\beta_\zeta, \delta_\zeta\}, \kappa)$. Define β_ξ and δ_ξ as in the previous case.

Let $\lambda := \sup\{\alpha_\xi : \xi \in |X|^+\} = \sup\{\delta_\xi : \xi \in |X|^+\} < \kappa$.

1) By Corollary 63, $e(y, \lambda)$ has large neighbourhoods in $\{e(y, \beta_\xi) : \xi \in |X|^+, \text{ where } \bar{f}(e(y, \alpha_\xi)) = f(x_y, \delta_\xi), \text{ for } \delta_\xi \neq \kappa\}$.

So by Lemma 71 $\bar{f}(e(y, \lambda))$ is the complete accumulation point of $\{\bar{f}(e(y, \beta_\xi)) = f(x_y, \delta_\xi) : \xi \in |X|^+\} = \{f(x_y, \delta_\xi) : \xi \in |X|^+\} \subseteq \{f(x_y, \delta) : \delta \in (0, \lambda]\}$ in βZ . That is $\bar{f}(e(y, \lambda)) = f(x_y, \lambda)$.

2) On the other hand, by Lemma 72, $\bar{f}(e(y, \lambda))$ is the unique point in $\{\bar{f}(e(y, \alpha_\xi)) = f(x_y, \gamma) : \xi \in |X|^+\} = \{f(x_y, \kappa)\}$ such that $|\bar{f}^+[\{\bar{f}(e(y, \lambda))\}] \cap \{e(y, \beta_\xi) : \xi \in |X|^+, \text{ where } \bar{f}(e(y, \alpha_\xi)) = f(x_y, \delta_\xi)\}| = |X|^+$.

So $\bar{f}(e(y, \lambda)) = f(x_y, \kappa)$, a contradiction to the first case.

Let $\beta_y = \sup(\{\gamma : \bar{f}(e(y, \gamma)) = f(x_y, \kappa)\} \cup \{\beta'_y\})$.

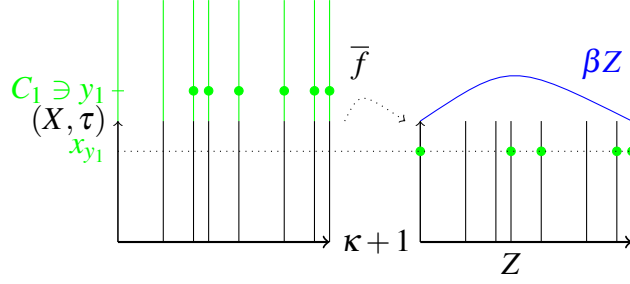


Figure 3.9: Image of a final segment of a spine on an element of C_1 under \bar{f} .

b) For $y \in C_2$:

(b)(i) Since $|\{\xi : \bar{f}(e(y, \xi)) \in Z\}| = \kappa$ and $\{\xi : \bar{f}(e(y, \xi)) \in Z\} = \bigcup_{f(x, \alpha) \in I_y} \{\xi : \bar{f}(e(y, \xi)) = f(x, \alpha)\}$, there exists $f(x_y, \alpha_y) \in I_y$ such that $|\{\xi : \bar{f}(e(y, \xi)) = f(x_y, \alpha_y)\}| = \kappa$.

Now for $\alpha < \kappa$, let $B_\alpha := \{\xi > \alpha : \bar{f}(e(y, \xi)) = f(x_y, \alpha_y)\}$.

By Fact 59, $e(y, \kappa) \in cl_{\beta_Y} e[\{y\} \times B_\alpha]$. So

$$\bar{f}(e(y, \kappa)) \in \bar{f}[cl_{\beta_Y} e[\{y\} \times B_\alpha]] \subseteq cl_{\beta_Z} \bar{f}[e[\{y\} \times B_\alpha]] = cl_{\beta_Z} \{f(x_y, \alpha_y)\} = \{f(x_y, \alpha_y)\}.$$

Therefore, $\bar{f}(e(y, \kappa)) = f(x_y, \alpha_y)$.

Uniqueness again follows from the one-to-one property of f .

(b)(ii) If there exists $(x', \alpha_{x'})$ such that $|\{\xi : \bar{f}(e(y, \xi)) = f(x', \alpha_{x'})\}| = \kappa$, then, by proof of (b)(i), $\bar{f}(e(y, \kappa)) = f(x', \alpha_{x'})$, which implies $(x', \alpha_{x'}) = (x_y, \alpha_y)$.

(b)(iii) For $(x_\xi, \alpha_\xi) \in I_y \setminus \{(x_y, \alpha_y)\}$, there exists $\beta_{(x_\xi, \alpha_\xi)} < \kappa$ such that for $\alpha > \beta_{(x_\xi, \alpha_\xi)}$, if $\bar{f}(e(y, \alpha)) \in Z$, then $\bar{f}(e(y, \alpha)) \neq f(x_\xi, \alpha_\xi)$. Let

$$\beta_y = \sup\{\beta_{(x_\xi, \alpha_\xi)} : (x_\xi, \alpha_\xi) \in I_y \setminus (x_y, \alpha_y)\}$$

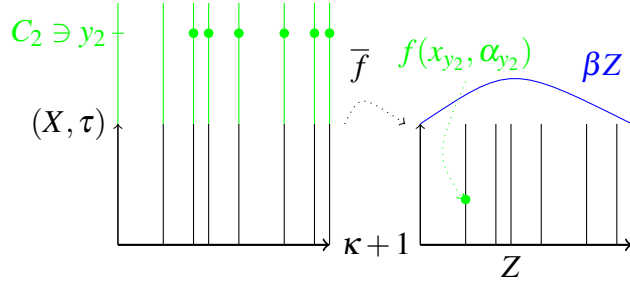


Figure 3.10: Image of a final segment of a spine on an element of C_2 under \bar{f}

c) This follows directly from the definition of C_1 and C_2 . \square

Note 74. Let κ be regular cardinal such that $\text{cof}(\kappa) > |\beta X|$ and let $\beta^* := \max\{\sup\{\beta_y : \text{for } y \in \beta X \setminus X\}, \sup\{\alpha_y : y \in C_2, \alpha_y \neq \kappa \text{ and } \bar{f}(e(y, \kappa)) = f(x_y, \alpha_y)\}\}$.

For all $\alpha > \beta^*$ and $y \in \beta X \setminus X$,

if $y \in C_1$ and $\bar{f}(e(y, \alpha)) \in Z$, then $\bar{f}(e(y, \alpha)) = f(x_y, \gamma_\alpha)$,

if $y \in C_2$ and $\bar{f}(e(y, \alpha)) \in Z$, then $\bar{f}(e(y, \alpha)) = f(x_y, \gamma_y)$, and

if $y \notin C_1 \cup C_2$, then $\bar{f}(e(y, \alpha)) \notin Z$.

Note 75. For a regular ordinal κ such that $\text{cof}(\kappa) > |X|^+$, for $\alpha \in \kappa$, and every $y \in C_1$ there exists $\beta_{(y, \alpha)}$ such that for $\xi > \beta_{(y, \alpha)}$, if $\bar{f}(e(y, \xi)) = f(x_y, \gamma)$, then $\gamma > \alpha$.

Proof. For $y \in C_1$ and $\delta \leq \alpha$, there exists $\gamma_{(y, \delta)} > \beta_y$ such that for $\xi > \gamma_{(y, \delta)}$, $\bar{f}(e(y, \xi)) \neq f(x_y, \delta)$. Now let $\beta_{(y, \alpha)} := \sup\{\gamma_{(y, \delta)} : \delta \in \alpha + 1\}$. \square

Let $\beta_{(\alpha)} := \sup\{\beta_{(y, \alpha)} : y \in C_1\}$. Let $\xi > \beta_{(\alpha)}$, for all $y \in C_1$, if $\bar{f}(e(y, \xi)) = f(x_y, \gamma)$, then $\gamma > \alpha$.

Using the concepts of β^* and $\beta_{(y)}$, we will define an unbounded subset, B , of κ , such that $f[X \times \{\xi\}]$ has the *optimal* closure in Z for large ξ 's, i.e., $f[X \times \{\xi\}]$ has the coarsest topology among large ξ 's, up to an isomorphism, and

$clzf[X \times \{\xi\}] \setminus f[X \times \{\xi\}]$ are the same, for all $\xi \in B$. This will produce enough vertical lines with the same topology but also enables us to predict the closure of particular triangle shape subset of a bunch of those lines.

Claim 76. There exists an unbounded subset B of κ such that

$$\text{for } \alpha \in B \text{ and } y \in C_1, \bar{f}(e(y, \alpha)) = f(x_y, \alpha)$$

$$\text{and for } \alpha \in B \text{ and } y \in C_2, \bar{f}(e(y, \alpha)) = f(x_y, \gamma_y)$$

and $\gamma_y \notin B$.

Proof. For the first element of B , we choose any ordinal above β^* .

Assuming that we have found the first $\gamma < \kappa$ elements of B , we start with an ordinal above all those elements of B and we follow the process below to find the next element of B .

Enumerate $C_1 := \{y_\alpha^1 : \alpha \in \eta_1\}$ and $C_2 := \{y_\alpha^2 : \alpha \in \eta_2\}$.

(C_1 and C_2 are defined in Definition 70.)

At every stage ξ pick $\gamma_{\xi, \alpha}^i$ in this manner:

For $\alpha = 0$, require $\gamma_{\xi, 0}^1 > \sup\{\gamma_{\delta, \alpha}^i : \delta < \xi \text{ and } \alpha \in \eta_i, i = 1, 2\}$.

For $y_\alpha^1 \in C_1$, $\bar{f}(e(y_\alpha^1, \gamma_{\xi, \alpha}^1)) = f(x_{y_\alpha^1}, \gamma_{\xi, \alpha}^1)$,

where $\gamma_{\xi, \alpha}^1 > \sup\{\beta_{(y_\alpha^1, \max\{\gamma_{\xi, \delta}^1, \gamma_{\xi, \delta}^2\})} : \delta < \alpha\}$.

So that

$$\gamma_{\xi, \alpha}^1 > \gamma_{\xi, \delta}^1 \text{ for all } \delta < \alpha,$$

$$\gamma_{\xi, \alpha}^1 > \gamma_{\xi, \delta}^1 \text{ for all } \delta < \alpha,$$

$$\gamma_{\xi, \alpha}^1 > \gamma_{\xi, \delta}^1 \text{ for all } \delta < \alpha,$$

$$\gamma_{\xi, \alpha}^1 > \gamma_{\xi, \delta}^1 \text{ for all } \delta < \alpha.$$

Let $\gamma_{\xi, 0}^2 > \max\{\sup\{\gamma_{\xi, \alpha}^1 : \alpha \in \eta_1\}, \sup\{\gamma_{\xi, \delta}^2 : \delta < \alpha\}\}$.

let $\gamma_{\xi, \alpha}^2$ be such that

$$\bar{f}(e(y_\alpha^2, \gamma_{\xi, \alpha}^2)) = f(x_{y_\alpha^2}, \gamma_{y_\alpha^2}).$$

Now let $\delta := \sup\{\gamma_{\xi,\alpha}^i : \alpha \in \eta_i \text{ for } i = 1, 2 \text{ and } \xi \in |X|^+\}$. We need to show that $\delta \in B$:

Let $y_\alpha \in C_1$, by Corollary 63, $e(y_\alpha, \delta)$ has large neighbourhoods in $\{e(y_\alpha, \gamma_{\xi,\alpha}^2) : \xi \in |X|^+\}$. Thus by Lemma 71, $\bar{f}(e(y_\alpha, \delta))$ is the complete accumulation point of

$$\{\bar{f}(e(y_\alpha, \gamma_{\xi,\alpha}^1)) : \xi \in |X|^+\} = \{f(x_{y_\alpha}, \gamma_{\xi,\alpha}^1) : \xi \in |X|^+\} \subseteq \{f(x_{y_\alpha}, \xi) : \xi \in \delta + 1\},$$

a compact set, and that means $\bar{f}(e(y_\alpha, \delta)) = f(x_{y_\alpha}, \delta)$.

For $y_\alpha \in C_2$, again by Corollary 63, $e(y_\alpha, \delta)$ has large neighbourhoods in $\{e(y_\alpha, \gamma_{\xi,\alpha}^2) : \xi \in |X|^+\}$. On the other hand,

$$\{\bar{f}(e(y_\alpha, \gamma_{\xi,\alpha}^2)) : \xi \in |X|^+\} = \{\bar{f}(e(y_\alpha, \gamma_{\xi,\alpha}^2)) : \xi \in |X|^+ + 1\} = \{f(x_{y_\alpha}, \alpha_{y_\alpha})\}.$$

Therefore, by Lemma 72, $\bar{f}(e(y_\alpha, \delta)) = f(x_{y_\alpha}, \alpha_{y_\alpha})$. □

Fact 77. For $\delta, \delta' \in B$, $f[X \times \{\delta\}] \cong f[X \times \{\delta'\}]$.

Proof. Let U be open in $f[X \times \{\delta\}]$, and let V be an open set in βX such that

$$e[V \times \{\delta\}] \cap (\bar{f}|_{e[(X \cup C_1) \times \{\delta\}]})^{\leftarrow} [f[X \times \{\delta\}]] = \bar{f}|_{e[(X \cup C_1) \times \{\delta\}]}^{\leftarrow} [U].$$

Then $V' := f[X \times \{\delta'\}] \setminus \bar{f}[e[(\beta X \setminus V) \times \{\delta'\}]]$ is open in $f[X \times \{\delta'\}]$.

We need to show that $f[e[(V \cap X) \times \{\delta'\}]] = V'$:

Obviously $V' \subseteq \bar{f}[e[V \times \{\delta'\}]]$ because $f(x, \delta') \in V'$, we have $f(x, \delta') \notin \bar{f}[e[(\beta X \setminus V) \times \{\delta'\}]]$. On the other hand, $f(x, \delta') \in \bar{f}[e[\beta X \times \{\delta'\}]]$, so $f(x, \delta') \in \bar{f}[e[V \times \{\delta'\}]]$ and we are done.

Next we have to prove the reverse inclusion:

Let $f(x, \delta') \in f[(V \cap X) \times \{\delta'\}] \setminus V'$. $f(x, \delta') \notin V'$ implies $f(x, \delta') \in \bar{f}[e[(\beta X \setminus V) \times \{\delta'\}]]$.

For $y \in \beta X \setminus V$ such that $\bar{f}(e(y, \delta')) \in \bar{f}[e[(\beta X \setminus V) \times \{\delta'\}]] \cap f[X \times \{\delta'\}]$ and $\bar{f}(e(y, \delta')) = f(x, \delta')$, $y \in C_1$ because $\bar{f}(e(y, \delta')) \in Z$ and $\bar{f}(e(y, \delta')) = f(x, \delta')$. $y \in C_1$ implies $\bar{f}(e(y, \delta)) = f(x, \delta)$. Thus $x \notin V$, a contradiction.

Therefore, $g: f[X \times \{\delta\}] \rightarrow f[X \times \{\delta'\}]$ defined by $g(f(x, \delta)) = f(\pi_X[f^{\leftarrow}(f(x, \delta))], \delta')$ is a homeomorphism. \square

Note 78. $cl_{\beta Z}\{\bar{f}(e(y, \delta)) : y \in C_2 \text{ and } \delta \in B\} \cap f[X \times \{\gamma\}] = \emptyset$ for all $\gamma \in B$.

Proof. $\{f(x_y, \alpha_y) : y \in C_2\} = \{\bar{f}(e(y, \delta)) : y \in C_2, \delta \in B\} \subseteq \bigcap_{\gamma \in B} \bar{f}[e[\beta X \times \{\gamma\}]]$ which is a compact set and does not include any of $f[X \times \{\gamma\}]$ for $\gamma \in B$. \square

Note 79. For $\delta \in B$, $\bar{f}[e[C_2 \times \{\delta\}]] \subseteq f[X \times \{\delta\}]$ and $cl_{\beta Z}\bar{f}[e[C_2 \times B]] \cap f[X \times B] = \emptyset$.

Note 80. $\bar{f}[e[(X \cup C_2) \times \{\delta\}]] = f[X \times \{\delta\}] \cup (cl_{\beta Z}\bar{f}[e[C_2 \times \{\delta\}]] \cap Z)$ is closed in Z for $\delta \in B$.

3.2.4 Using the Structural Facts about Stone-Ćech Compactification to Find a Good Subset of B

In this section, we prove that $f[X \times \{\delta\}] \cup (cl_{\beta Z}\bar{f}[e[C_2 \times \{\delta\}]] \cap Z)$ is countably paracompact but since it also is a good candidate for being paracompact, we keep the process so that it works with any cover of any size until very end. Then, since the same proof does not go through for covers of uncountable size, we prove the countable paracompact property. We begin with finding collections of subsets of κ , of order type η for every $\eta \leq |X|$, whose closure of image of their product with X under f becomes predictable for larger ordinals.

Fact 81. Let $\kappa > 2^{2^{|X|}}$ For every $\eta \leq |X|$ we can find a collection of sets $\{H_\alpha^\eta : \alpha \in \kappa\}$ where

1. $H_\alpha^\eta \subseteq B$.
2. $|H_\alpha^\eta| = \eta$.
3. $\sup H_\alpha^\eta < \inf H_\beta^\eta$ if and only if $\alpha < \beta$.
4. If $A \subset H_\alpha^\eta$ such that $|A| < \eta$ then $cl_{zf}[X \times A] \cap f[X \times (H_\alpha^\eta \setminus (\sup A + 1))] = \emptyset$.
5. For every cofinal subset $A \subset H_\alpha^\eta$, and for $y \in C_1$ $\bar{f}(y, \sup A) = f(x_y, \gamma)$, where $\gamma \geq \sup A$.

Proof. For $\alpha \in \kappa$ to define H_α^η , we start with $B \setminus (\sup(\cup_{\xi \in \alpha} H_\xi^\eta) + 1)$ and we pick the first element of $B \setminus (\sup(\cup_{\xi \in \alpha} H_\xi^\eta) + 1)$ to be the first element of H_α^η .

At every successor stage $\gamma + 1 \in \eta$, if γ is a successor ordinal let

$$h_{\gamma+1}^{\alpha, \eta} = \min(B \setminus (\max\{h_\gamma^{\alpha, \eta}, \beta_{(\gamma)}\} + 1)),$$

where $\beta_{(\gamma)}$ is the function described in Note 75.

At every successor stage $\gamma + 1 \in \eta$, if γ is a limit ordinal then let

$$h_{\gamma+1}^{\alpha, \eta} = \min(B \setminus (\sup L + 1))$$

where $L := \pi_{\kappa+1}[f^{\leftarrow}[cl_{zf}[X \times \{h_\xi^{\alpha, \eta} : \xi \in \gamma\}]] \setminus \{\kappa\}$.

At a limit stage $\gamma \in \eta + 1$ define $h_\gamma^{\alpha, \eta} = \sup\{h_\xi^{\alpha, \eta} : \xi \in \gamma\}$.

Now let $H_\alpha^\eta = \{h_\gamma^{\alpha, \eta} : \gamma \in \eta \text{ and } \gamma \text{ is a successor ordinal}\}$. Then $cl_{\kappa+1}H_\alpha^\eta = \{h_\xi^{\alpha, \eta} : \xi \in \eta + 1\}$. □

Note 82. $cl_{\kappa+1}H_\alpha^\eta$ is homeomorphic to $\eta + 1$.

Proof. Define $g_\alpha : \eta + 1 \rightarrow cl_{\kappa+1}H_\alpha^\eta$ to be the function $g_\alpha(\xi) = h_\xi^{\alpha,\eta}$. The function g_α is obviously one-to-one and onto.

The function g_α is open because: $g_\alpha[(\xi_1, \xi_2)] = (h_{\xi_1}^{\alpha,\eta}, h_{\xi_2}^{\alpha,\eta}) \cap cl_{\kappa+1}H_\alpha^\eta$.

The function g_α is continuous because: $g_\alpha^{-1}[(h_{\xi_1}^{\alpha,\eta}, h_{\xi_2}^{\alpha,\eta}) \cap cl_{\kappa+1}H_\alpha^\eta] = (\xi_1, \xi_2)$. \square

Note 83. $X \times cl_{\kappa+1}H_\alpha^\eta \cong X \times cl_{\kappa+1}H_\beta^\eta$, for α and $\beta \in \kappa$.

Proof. Define $g_{\alpha,\beta} : cl_{\kappa+1}H_\alpha^\eta \rightarrow cl_{\kappa+1}H_\beta^\eta$ to be the function $g_{\alpha,\beta}(x, \xi) = (x, g_\beta(g_\alpha^{-1}(\xi)))$. $g_{\alpha,\beta}$ is a homeomorphism. \square

Note 84. Using Fact 68, for a free z -ultrafilter \mathcal{F} on $X \times (\eta + 1)$, and $\{\mathcal{F}_{(\alpha)} : \alpha \in \kappa\}$ defined on collection $\{X \times H_\alpha^\eta : \alpha \in \kappa\}$ as in Fact 68, $\bar{f}(e(y_{\mathcal{F}}, \kappa))$ is the complete accumulation point of the set $\{\bar{f}(\mathcal{F}_{(\alpha)}) : \alpha \in \kappa\}$.

Fact 85. There exists a $\gamma_\eta^* \in \kappa$ such that for every collection of free z -ultrafilters $\{\mathcal{F}_{(\alpha)} : \alpha \in \kappa\}$ on $\{X \times cl_{\kappa+1}H_\alpha^\eta : \alpha \in \kappa\}$, where each $\mathcal{F}_{(\alpha)}$ is a free z -ultrafilter on $X \times cl_{\kappa+1}H_\alpha^\eta$, such that for $\beta > \gamma_\eta^*$, if $\bar{f}(\cap_{F \in \mathcal{F}_{(\beta)}} cl_{\beta Y} F \cap cl_{\beta Y}(X \times H_\beta^\eta)) =: \bar{f}(\mathcal{F}_{(\beta)}) \in Z$ then $\bar{f}(\mathcal{F}_{(\beta)}) = f(x_y, \alpha)$ for some $y \in C_1$ or $\bar{f}(\mathcal{F}_{(\beta)}) = f(x_y, \gamma_y)$ for some $y \in C_2$.

Proof. There are three cases:

Case 1) If $|\{\bar{f}(\mathcal{F}_{(\alpha)}) \in Z : \alpha \in \kappa\}| = \kappa$, then $y_{\mathcal{F}} \in C_1$:

To prove that $y_{\mathcal{F}} \in C_2$, we need to show $|\{\bar{f}(e(y_{\mathcal{F}}, \xi)) \in Z : \xi \in \kappa\}| = \kappa$. Now again since $cof(\kappa) > |X|$, there exists $x_{\mathcal{F}}$ such that $|\{\xi : \bar{f}(\mathcal{F}_{(\xi)}) = f(x_{\mathcal{F}}, \alpha_\xi) \text{ for some } \xi \in \kappa\}| = \kappa$. For any $\mu \in \kappa \setminus \sup(\{\beta^*\} \cup \pi_{(\kappa+1)}[\bar{f}^{-1}[\bar{f}[e[C_2 \times B]]]])$, let

$$E_\mu := \{\xi \in (\mu, \kappa) : \bar{f}(\mathcal{F}_{(\xi)}) = f(x_{\mathcal{F}}, \alpha_\xi)\}.$$

Pick $|X|^+$ elements of E_μ and denote them as K , in such a way that at every successor ordinal $\varsigma = \rho + 1$, $\xi_\varsigma \in (\max\{\xi_\rho, \alpha_{\xi_\rho}\}, \kappa) \cap E$ where $\bar{f}(\mathcal{F}_{(\xi_\rho)}) = f(x_{\mathcal{F}}, \alpha_{\xi_\rho})$

and $\alpha_{\xi_\zeta} \in (\max\{\xi_\rho, \alpha_{\xi_\rho}\}, \kappa)$, where $\bar{f}(\mathcal{F}_{(\xi_\zeta)}) = f(x_{\mathcal{F}}, \alpha_{\xi_\zeta})$. At limit stage ζ , let $\xi_\zeta \in E \cap (\sup\{\xi_\rho : \rho < \zeta\}, \kappa)$. By the same technique as above, $e(y, \sup K)$ has large neighbourhoods in $\{\mathcal{F}_{(\xi)} : \xi \in K\}$. Therefore, $\{f(x_{\mathcal{F}}, \alpha_\rho) : \rho \in K\}$ converges to $\bar{f}(e(y_{\mathcal{F}}, \sup K))$. Thus $\bar{f}(e(y_{\mathcal{F}}, \sup K)) = f(x_{\mathcal{F}}, \sup K)$ and this proves that $y \in C_1$ and $x_{\mathcal{F}} = x_{y_{\mathcal{F}}}$. (Because for all $\xi > \beta^*$ and the fact that $\xi > \beta_{(\sup \pi_X \bar{f}^{-1}[\bar{f}[e[C_2 \times B]])]}$ and $y \in \beta X \setminus X$, $\bar{f}(e(y, \xi)) \in Z \setminus \bar{f}[e[C_2 \times B]]$ implies $y \in C_1$ and $\bar{f}(e(y, \xi)) = f(x_y, \xi')$.)

Now since $e(y_{\mathcal{F}}, \kappa)$ has large neighbourhoods in the set $\{\mathcal{F}_{(\alpha)} : \alpha \in \kappa\}$, for every $x \neq x_{y_{\mathcal{F}}}$ in X , there exists $\phi_{(\mathcal{F}, x)}$ such that for all $\alpha > \phi_{(\mathcal{F}, x)}$ $\bar{f}(\mathcal{F}_{(\alpha)}) \neq f(x, \xi)$ for any $\xi \in \kappa$.

Using the fact that $e(y_{\mathcal{F}}, \lambda)$ has large neighbourhoods in the set $\{\mathcal{F}_{(\alpha)} : \alpha \in \lambda\}$ for $\lambda > |X|$, and by the way of contradiction, if $|\{\alpha : \bar{f}(\mathcal{F}_{(\alpha)}) = f(x_{y_{\mathcal{F}}}, \gamma)\}| = \kappa$, then for κ many ξ 's, $e(y_{\mathcal{F}}, \xi)$ has a large neighbourhoods in a subset of $\{\mathcal{F}_{(\alpha)} : \bar{f}(\mathcal{F}_{(\alpha)}) = f(x_{y_{\mathcal{F}}}, \gamma)\}$ of size $|X|^+$ which means for κ many ξ 's $\bar{f}(e(y_{\mathcal{F}}, \xi)) = f(x_{y_{\mathcal{F}}}, \gamma)$, a contradiction to Fact 73(a)(iv).

So $|\{\alpha \in \kappa : \bar{f}(\mathcal{F}_{(\alpha)}) = f(x_{y_{\mathcal{F}}}, \gamma)\}| < \kappa$ for all $\gamma \in \kappa + 1$.

Therefore, there exists a $\phi_{(\mathcal{F}, \kappa)}$ such that for $\alpha \in (\phi_{(\mathcal{F}, \kappa)}, \kappa)$ $\bar{f}(\mathcal{F}_{(\alpha)}) \neq f(x_{y_{\mathcal{F}}}, \kappa)$ and for $f(x_\xi, \alpha_\xi) \in \bar{f}[e[C_2 \times B]]$ there exists $\phi_{(\mathcal{F}, f(x_\xi, \alpha_\xi))}$ such that for $\alpha \in (\phi_{(\mathcal{F}, f(x_\xi, \alpha_\xi))}, \kappa)$, $\bar{f}(\mathcal{F}_{(\alpha)}) \neq f(x_\xi, \alpha_\xi)$.

Let

$$\Phi_{\mathcal{F}} := \sup(\{\phi_{(\mathcal{F}, x)} : x \in X \setminus \{x_{y_{\mathcal{F}}}\}\} \cup \{\phi_{(\mathcal{F}, f(x_\xi, \alpha_\xi))} : f(x_\xi, \alpha_\xi) \in \bar{f}[e[C_2 \times B]]\} \cup \{\phi_{(\mathcal{F}, \kappa)}\}).$$

Now for $\xi > \Phi_{\mathcal{F}}$ if $\bar{f}(\mathcal{F}_{(\xi)}) \in Z$, then $\bar{f}(\mathcal{F}_{(\xi)}) = f(x_{y_{\mathcal{F}}}, \gamma)$ for some $\gamma \in \kappa$.

Case 2) $|\{\bar{f}(\mathcal{F}_{(\alpha)}) \in Z : \alpha \in \kappa\}| < \kappa$ but $|\{\xi : \bar{f}(\mathcal{F}_{(\xi)}) \in Z\}| = \kappa$ then we claim $y_{\mathcal{F}} \in C_2$:

First we need to show $|\{\xi : \bar{f}(e(y_{\mathcal{F}}, \xi)) \in Z\}| = \kappa$ and $|\{\bar{f}(e(y_{\mathcal{F}}, \xi)) : \bar{f}(e(y_{\mathcal{F}}, \xi)) \in Z\}| < \kappa$.

Since $|\{\bar{f}(\mathcal{F}_{(\xi)}) : \bar{f}(\mathcal{F}_{(\xi)}) \in Z\}| < \kappa$ but $|\{\xi : \bar{f}(\mathcal{F}_{(\xi)}) \in Z\}| = \kappa$, there exists $f(x_{\mathcal{F}}, \alpha_{\mathcal{F}})$ such that $|\{\xi : \bar{f}(\mathcal{F}_{(\xi)}) = f(x_{\mathcal{F}}, \alpha_{\mathcal{F}})\}| = \kappa$. For any $\mu \in \kappa \setminus \sup(\{\beta^*\} \cup \pi_X \bar{f}^+[\bar{f}[e[C_2 \times B]]])$, let $E_\mu := \{\xi \in (\mu, \kappa) : \bar{f}(\mathcal{F}_{(\xi)}) = f(x_{\mathcal{F}}, \alpha_{\mathcal{F}})\}$.

Denote $|X|^+$ elements of E_μ as subset K in this manner: At every successor stage $\zeta = \rho + 1$, let $\xi_\zeta = \min((\xi_\rho, \kappa) \cap E_\mu)$. At limit stage ζ , let $\xi_\zeta = \min(E_\mu \cap (\sup\{\xi_\rho : \rho \in \zeta\}, \kappa))$. Now $e(y_{\mathcal{F}}, \sup K)$ has large neighbourhoods in $\{\xi : \bar{f}(\mathcal{F}_{(\xi)}) : \xi \in K\}$. Therefore, $\bar{f}(e(y_{\mathcal{F}}, \sup K)) = f(x_{\mathcal{F}}, \alpha_{\mathcal{F}})$ and this shows $y \in C_2$ and $(x_{y_{\mathcal{F}}}, \alpha_{y_{\mathcal{F}}}) = (x_{\mathcal{F}}, \alpha_{\mathcal{F}})$. (Because for all $\xi > \max(\beta^*, \beta_{(\sup \pi_X \bar{f}^+[\bar{f}[e[C_2 \times B]]])})$ and for all $y \in \beta X \setminus X$, $\bar{f}(e(y, \xi)) \in \bar{f}[e[C_2 \times B]]$ implies $y \in C_2$ and $\bar{f}(e(y, \xi)) = f(x_y, \alpha_y)$.)

Now define $J := \{f(x_\xi, \alpha_\xi) : \bar{f}(\mathcal{F}_{(\xi)}) = f(x_\xi, \alpha_\xi)\}; |J| < \kappa$. Since $e(y_{\mathcal{F}}, \kappa)$ has large neighbourhoods in $\{\mathcal{F}_{(\xi)} : \xi \in \kappa\}$, for $f(x_\xi, \alpha_\xi) \in J \setminus \{f(x_{y_{\mathcal{F}}}, \alpha_{y_{\mathcal{F}}})\}$, there exists $\phi_{(\mathcal{F}, (x_\xi, \alpha_\xi))}$ such that for all $\beta > \phi_{(\mathcal{F}, (x_\xi, \alpha_\xi))}$, $\bar{f}(\mathcal{F}_{(\beta)}) \neq f(x_\xi, \alpha_\xi)$. Let $\phi_{\mathcal{F}} = \sup\{\phi_{(\mathcal{F}, (x_\xi, \alpha_\xi))} : f(x_\xi, \alpha_\xi) \in J \setminus \{f(x_{y_{\mathcal{F}}}, \alpha_{y_{\mathcal{F}}})\}\}$.

Now for all $\beta > \phi_{\mathcal{F}}$, if $\bar{f}(\mathcal{F}_{(\beta)}) \in Z$, then $\bar{f}(\mathcal{F}_{(\beta)}) = f(x_{y_{\mathcal{F}}}, \alpha_{y_{\mathcal{F}}})$.

Case 3) $|\{\xi : \bar{f}(\mathcal{F}_{(\xi)}) \in Z\}| < \kappa$, then we claim there exists $\Phi_{\mathcal{F}} \in \kappa$ such that $\bar{f}(\mathcal{F}_{(\alpha)}) \notin Z$, for all $\alpha > \Phi_{\mathcal{F}}$.

Since at every level α , there are at most $2^{2^{|X|}}$ many z -ultrafilters on $X \times cl_{\kappa+1} H_\alpha^\eta$, we can say that there exists $\gamma_\eta^* = \sup\{\Phi_{\mathcal{F}} : \mathcal{F} \text{ } z\text{-ultrafilter on } X \times (\eta + 1)\}$. \square

3.2.5 Using a Cover to Build Two Disjoint Closed Sets

Now let $\{U_\xi : \xi \in \eta\}$ be any cover of $f[X \times \{\delta\}] \cup (cl_{\beta Z} \bar{f}[e[C_2 \times \{\delta\}]] \cap Z)$.

We pick $\min H_\alpha^\eta$ larger than γ_η^* to build a family of decreasing closed sets on $f[X \times H_\alpha^\eta]$ whose closure is disjoint from closure of $f[X \times \{\sup H_\alpha^\eta\}]$.

Claim 86. Let $\{U_\xi : \xi \in \eta\}$ be an open cover of $f[X \times \{\gamma\}] \cup cl_{\beta Z} \bar{f}[e[C_2 \times \{\gamma\}]]$ for $\gamma \in B$. There exists finitely many U_ξ that cover $cl_{\beta Z} \bar{f}[e[C_2 \times \{\gamma\}]]$, denote that finite union by W , and for $\alpha > \gamma_\eta^*$ let

$$D_1 := f[cl_{X \times (\kappa+1)}(\cup_{\xi \in \eta} ((X \setminus (\pi_X[f^{-1}[(\cup_{\delta \in \xi} U_\delta \cup W) \cap f[X \times \{\gamma\}]])]) \times \{h_{\xi+1}^{\alpha, \eta}\})))]$$

and $D_2 := f[X \times \{\sup H_\alpha^\eta\}]$. Then $cl_Z D_1 \cap cl_Z D_2 = \emptyset$.

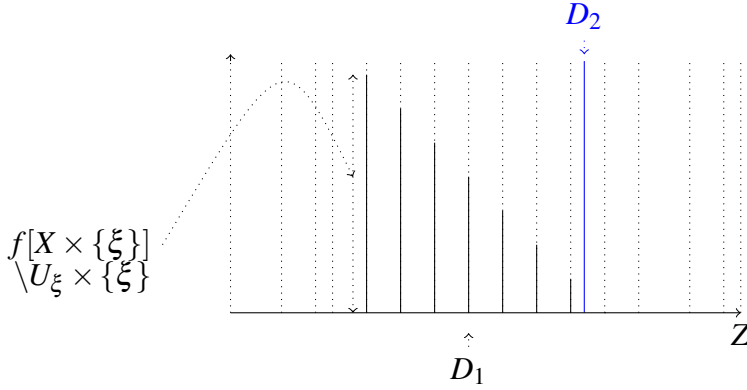


Figure 3.11: An illustration of D_1 and D_2 in the space.

Proof. Since we used $\beta_{(\gamma)}$ in the construction of H_ξ^η 's, $cl_Z D_2 \cap f[X \times [h_0^{\alpha, \eta}, \sup H_\alpha^\eta]] = \emptyset$ and if we show $cl_Z D_1 \subseteq f[X \times [h_0^{\alpha, \eta}, \sup H_\alpha^\eta]]$ we'll be done:

By the Lemma 69 if $f(x, \beta) \in cl_Z D_1 \setminus D_1$ then $f(x, \beta) = \bar{f}(\mathcal{F}_{(\alpha)})$ for some free z -ultrafilter $\mathcal{F}_{(\alpha)}$ on $X \times cl_{\kappa+1} H_\alpha^\eta$.

Now since $\alpha > \gamma_\eta^*$, for $y = y[F]$, where $\{y_{\mathcal{F}}\} = \cap_{F \in \mathcal{F}_{(\alpha)}} cl_{\beta X} \pi_X(F)$ we have two cases:

Case 1) $y \in C_2$: $y \in \pi_{(X \cup C_2)}[\bar{f}^{-1}[W]]$ so there exists V open in βX such that $y \in V$ and $V \cap X = \pi_X[f^{-1}[W]]$ then $V \cap cl_{\beta X} \pi_X[f^{-1}[D_1 \cap (X \times \{h_\xi^{\alpha, \eta}\})]] = \emptyset$ for all $\xi \in \eta$; therefore, $y \notin cl_{\beta X} \pi_X[f^{-1}[D_1]]$ but $f^{-1}[D_1] \in \mathcal{F}_{(\alpha)}$, a contradiction.

Case 2) $y \in C_1$: Now if $\mathcal{F}_{(\alpha)}$ is in the closure of less than η many $X \times \{h_\xi^{\alpha, \eta}\}$'s then by the construction of H_α^η and the restriction by γ_η^* on elements of C_1 it stays within the sets above. If $\mathcal{F}_{(\alpha)}$ is not in the closure of less than η many of $X \times \{h_\xi^{\alpha, \eta}\}$'s then notice that there exists ξ such that $y \in \pi_{(X \cup C_1)}[\bar{f}^{-1}[U_\xi]]$ which implies there exists an open set $V \ni y$ in βX such that $V \cap X = \pi_X[f^{-1}[U_\xi]]$. Therefore, $V \cap cl_{\beta X} \pi_X[f^{-1}[D_1 \cap f[X \times \{h_\zeta^{\alpha, \eta}\}]] = \emptyset$, for $\zeta \in (\xi, \eta)$. Which means $y \notin cl_{\beta X} \pi_X[f^{-1}[D_1] \setminus \cup\{X \times \{h_\zeta^{\alpha, \eta}\} : \zeta \in [0, \xi + 1)\}]$ But $f^{-1}[D_1] \setminus \cup\{X \times \{h_\zeta^{\alpha, \eta}\} : \zeta \in [0, \xi + 1)\} \in \mathcal{F}_{(\alpha)}$, a contradiction. \square

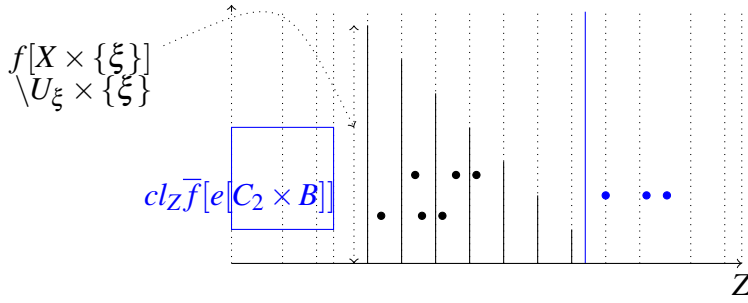


Figure 3.12: Closure of D_1 and D_2 in Z

Now, we apply the normality of Z .

Claim 87. Let $\kappa > 2^{2^{|\kappa|^+}}$ and $f : X \times (\kappa + 1) \rightarrow Z$ be a condensation onto a normal space Z . Then $f[X \times \{\gamma\}] \cup cl_{\beta Z} \bar{f}[e[C_2 \times \{\gamma\}]]$ is countably paracompact for $\gamma \in B$.

Proof. Let $\{U_i : i \in \omega\}$ be an open cover of the space $f[X \times \{\gamma\}] \cup cl_{\beta Z} \bar{f}[e[C_2 \times \{\gamma\}]]$, we enumerate U_i such that the first n elements cover $cl_{\beta Z} \bar{f}[e[C_2 \times \{\gamma\}]]$ and denote $\cup_{i \in n} U_i$ by W . Define D_1 and D_2 as in Claim 86 and $A_i := f[X \times \{h_i^{\alpha, \omega}\}] \setminus f[\pi_X[f^{-1}[\cup_{\zeta \in i} U_\zeta \cup W]] \times \{h_i^{\alpha, \omega}\}]$.

Since $cl_Z D_1 \cap cl_Z D_2 = \emptyset$, there exists a function $g : Z \rightarrow [0, 1]$ such that $g[D_2] = \{0\}$ and $g[D_1] = \{1\}$. Then let

$$B_{ij} := f[\pi_X[f^{\leftarrow}[g^{\leftarrow}[0, 1 - 1/j] \cap f[X \times \{h_i^{\alpha, \omega}\}]] \times \{\gamma\}]] \cup cl_{\beta Z}(\bar{f}[e[C_2 \times \{\gamma\}]] \cap \bar{g}^{\leftarrow}[0, 1 - 1/j])$$

and define $V_i = U_i$ for $0 \leq i < n$ and $V_{i+n} := U_{i+n} \setminus \cup_{j < i} B_{ji}$ for $i \geq 0$. Then $\{V_i : i \in \omega\}$ is a locally finite refinement of the cover.

Here is how $cl_Z D_1$ and $cl_Z D_2$ look like for countable cover. □

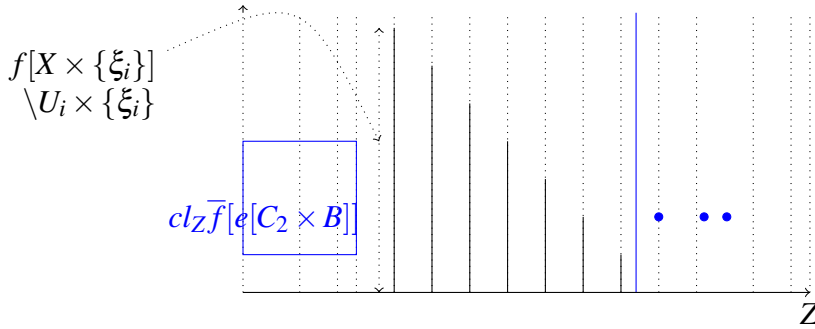


Figure 3.13: An illustration of $cl_Z D_1$ and $cl_Z D_2$ for a countable cover

By utilizing the next claim we get the final result:

Claim 88. Let $Y \cup C$ be a normal space where C is compact and $Y \cap C = \emptyset$. If $Y \cup C$ is countably paracompact, then there exists a coarser topology on Y that is countably paracompact.

Proof. Define a topology on Y as following: Pick $y \in Y$ and let $\tau' := \{U \cap Y : y \in U \in \tau(Y \cup C) \text{ implies } U \supseteq C\}$.

τ' is obviously a coarser topology on Y . We need to show it's countably paracompact and normal.

Define $h : Y \cup C \rightarrow (Y, \tau')$ by $h(x) = x$ if $x \in Y$ and $h(c) = y$ if $c \in C$. Note that h is continuous. Now let $\{U_i : i \in \omega\}$ be an τ' -open cover of Y , such that $y \in U_0$. $\{h^{\leftarrow}[U_i] : i \in \omega\}$ is an open cover of $Y \cup C$ and has a locally finite refinement $\{V_i : i \in \omega\}$. Let $W_0 = V_0 \cup \cup \mathcal{V}$ where \mathcal{V} is a finite collection of V_i 's which covers the compact set

$\{y\} \cup C$. Now $\{h[W_0 \cap Y]\} \cup \{h[V_i \cap Y \setminus \{y\}] : i \in \omega \setminus \{0\}\}$ is the desired locally finite refinement of the original cover. Note that singletons are obviously closed in τ' so to show τ' is normal, it suffices to show that disjoint closed sets can be separated by disjoint open sets. Let A and B be closed sets in τ' . We have two cases:

Case 1) $y \notin A$ and $y \notin B$. Then A and B are closed in τ . There exists U and V , disjoint open sets in τ , such that $A \subseteq U$ and $B \subseteq V$. Then $A \subseteq U \setminus (\{y\} \cup C) \in \tau'$ and $B \subseteq V \setminus (\{y\} \cup C) \in \tau'$.

Case 2) $y \in A$. Then $A \cup C$ and B are closed in τ . Thus there exists U and V , disjoint open sets in τ , such that $A \cup C \subseteq U$ and $B \subseteq V$. Then $A \subseteq U \cap Y \in \tau'$ and $B \subseteq V \in \tau'$.

□

It is necessary to prove Claim 88 as Weiss [Wei] has constructed a locally compact Dowker space X , whose one-point compactification is a normal, countably paracompact space but X is not countably paracompact.

Finally, we note that Pavlov [Pav99] proved that for every compact space K , there exists a space X such that $X \times K$ can not be condensed onto a normal space. In our case, there exists a space X such that $X \times \mathbb{I}$ does not have a coarser normal topology. Therefore, there exists a space X such that X does not have coarser countably paracompact topology. That is, our conclusion is not vacuous.

Chapter 4

Questions and Research Possibilities

In this final chapter, we present a few questions the author will continue investigating. First, in the second chapter, we showed that realcompact right-separated spaces have realcompact subspaces of size κ , where κ is a cardinal less than the size of the space and the continuum and we showed that for a realcompact space X with pseudocharacter less than or equal to ω_1 , X has a realcompact subspace of size ω_1 .

So what is remaining to investigate is:

Question 89. Is it true that any uncountable realcompact space has a realcompact subspace of size ω_1 ?

If, as we suspect, the answer to Question 89 is negative, we ask again with a stronger hypothesis

Question 90. Is it true that any uncountable Lindelöf space has a realcompact subspace of size ω_1 ?

Since in Chapter 2, we also showed that Continuum Hypothesis implies that every realcompact space has a realcompact subspace of size ω_1 , a counterexample to either of the Questions 89 and 90 can be only constructed in a model in which $2^\omega > \omega_1$ is true. That is, a negative answer to either of the questions will result in independence of the

question from *ZFC*. Also this space, if it existed, will have the following properties: X is hereditary separable. It is not hereditary Lindelöf. Its pseudocharacter exceeds ω_1 . It has CRV property. It contains no uncountable realcompact right-separated subspace.

In chapter 2, we also showed that a compact space of size larger than continuum has a realcompact subset of size continuum. The question remains:

Question 91. Is it true that any realcompact space of size larger than continuum has a realcompact subspace of size continuum?

A counterexample to Question 91 should have the following properties. Every subset of it has a dense subset of size $< \mathfrak{c}$. It is $< \mathfrak{c}$ real-valued. It is not hereditary Lindelöf. It contains no realcompact right-separated subspace of size \mathfrak{c} .

In Chapter 3, we showed for different ordinal less than the size of the space X , there exists a good subset of $\kappa + 1$ which has all the properties needed for Claim 86. We need to investigate whether there exists a closed such subset or at least such subset with some convergent property. This investigation can lead us to answering the following question.

Question 92. Is it true that for $\kappa = (2^{2^{|X|}})^+$, if $X \times (\kappa + 1)$ condenses onto a normal space, then X can be condensed onto η -paracompact space, where $\eta > \omega$?

Of course, if Question 92 was true for all $\eta \leq |X|$ and the topology that witnesses the truth of the statement was the same for all $\eta \leq |X|$, then X can be condensed onto a paracompact space and the following question will be answered.

Question 93. Is it true that for $\kappa = (2^{2^{|X|}})^+$, $X \times (\kappa + 1)$ condenses onto a normal space iff X condenses onto a paracompact space?

Note that the next example shows we can not achieve much stronger conclusions than the one proposed in Question 93.

Example 94. $\mathbb{Q} \times K$ is a normal space for any compact space K , because \mathbb{Q} is paracompact. But \mathbb{Q} does not condense onto any compact, Hausdorff space.

$\mathbb{Q} \subset \mathbb{I}$ and the unit interval \mathbb{I} is hereditary Lindelöf, hence hereditary paracompact. Thus by Tamano's theorem $\mathbb{Q} \times K$ is normal for any compact set K . It suffices to show that \mathbb{Q} does not condense onto a compact space. By the way of contradiction, let $f: \mathbb{Q} \rightarrow K$ be a condensation, where K is a compact space. Enumerate $K = \{q_i : i \in \omega\}$. Let U_0 be an open set in K such that $cl_K U_0 \cap \{q_0\} = \emptyset$. At stage $i + 1$, let $U_{i+1} \subseteq U_i$ be an open set in K such that $cl_K U_{i+1} \cap \{q_{i+1}\} = \emptyset$. Note that an open set U_{i+1} exists by the fact that K is Hausdorff and since K has coarser topology than \mathbb{Q} , every open set is infinite. Now $\bigcap_{i \in \omega} cl_K U_i = \emptyset$, a contradiction to the compactness of K .

On the other hand, Question 92 and/or 93 might be answered in negative by finding the following counterexamples.

Question 95. Is there a space X , such that $X \times (\kappa + 1)$ condenses onto a normal space but X does not condense onto a η -paracompact space, where $\kappa = (2^{2^{|X|}})^+$ and $\eta \leq |X|$?

Question 96. Is there a space X such that $X \times (\kappa + 1)$ condenses onto a normal space but X does not condense onto a paracompact space, where $\kappa = (2^{2^{|X|}})^+$?

Note that if a space fulfills the requirements of Question 92, then the same space satisfies the requirements of Question 93 but the other way around is not true.

Finally in the same chapter, one might ask whether it is possible to lower the size of κ . That is:

Question 97. Let X be a Tychonoff space. Is there an ordinal $\kappa < (2^{2^{|X|}})^+$ such that $X \times (\kappa + 1)$ has a coarser normal topology implies X has a coarser countably paracompact topology?

Among all the questions in Chapter 3 Question 93 or its counterexample, Question 96, interests the author the most. For they can settle whether or not the condensation variation of Tamano's theorem, without any extra assumption, exists.

Bibliography

- [AO] Ostaszewski, A.J., *On countably compact, perfectly normal spaces*, J. London Math. Soc., (2)14, 505-516, 1975. Cited on 20
- [Buz] Buzjakova, Rushan Z., *A criterion that a pseudocompact space condenses onto a compact space*, Questions and Answers in General topology, 2(15), 167-172, 1997. Cited on 3, 7, 28
- [Dow] Dowker, C. H., *On countably paracompact spaces*, Canadian J. Math., (3)219-224, 1951. Cited on 6, 27
- [GJ] Gilman and Jerison, *Rings of Continuous Function*. Cited on 9, 12, 18, 23
- [Kun] Przymusiński, Teodor C., *Product of normal spaces*, Hand Book of set-theoretic Topology, 1984. Cited on 28, 29
- [Pav99] Pavlov, Oleg, *Condensation of Cartesian Products*, *Commentationes Mathematicae Universitatis Carolinae*, Vol. 40(1999), No. 2, 355-365. Cited on 7, 28, 59
- [PW] Porter, J. R. and Woods, R. G., *Extensions and Absolutes of Hausdorff Spaces*, Springer-Verlag, 1988. Cited on 9, 12
- [Rud] Rudin, Mary Ellen, *A normal space X for which $X \times I$ is not normal*, Fund. Math. 73(1971), 179-186, MR 45:2660. Cited on 6, 27
- [Swa00] Swardson, Mary Anne, *Realcompact subspaces of size ω_1* , *Topology Appl.*, 100(1):85–90, 2000. Special issue in honor of Howard H. Wicke. Cited on 3, 17
- [Tam] Tamano, Hisahiro, *On paracompactness*, Pacific J. Math., (10)1043-1047, 1960. Cited on 27
- [Wei] Weiss, W., *Small Dowker spaces*, Pacific J. Math. 94 (2)485–492, 1981. Cited on 59