UNIQUENESS OF RESPONSIVE VOTING EQUILIBRIUM

by

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Abstract

I consider a voting model in which voters receive private signals about a state variable that affects the utility of voters. There is a continuum of signals, normally distributed conditional on the state variable. I characterize a sufficient condition under which there does not exist any asymmetric equilibria. Therefore, for any plurality rule, the unique responsive equilibrium is symmetric.

Keywords: Responsive Bayesian-Nash equilibrium, stability, global attractiveness, asymmetric information, strategic voting, pivotal, information aggregation, collective decision making.

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Contents

1 Introduction

There is a large and growing literature on strategic voting by asymmetrically informed voters. This literature revisits the information aggregation problem by committee/jury voting and elections, first studied by Condorcet (1785). One typical example is the jury problem. In that setting, there is a jury to decide whether to convict or acquit a defendant. The two states of the world are "defendant is guilty" and "defendant is innocent". The prior probability of one of these states to occur is common knowledge to all jurors. Before voting, each juror receives a private noisy signal that is correlated with the state of the world. In traditional models, it is assumed that each juror votes "naively," i.e., vote according to his signal.[1] However, Austen-Smith and Banks (1996) argue that it is not rational for a voter to vote naively. In particular, they show that each voter must condition his decision not only on his private information but also on what must be true about others' private information when his vote affects the outcome. This optimal updating and voting by voters is called "strategic voting." Strategic voting by asymmetically informed decision makers led to a large literature with many applications in economics, political science and financial economics. Feddersen and Pesendorfer (1996, 1997 and 1998) study large elections and jury voting. Persico (2004) extends this literature by allowing endogenous information production in committees. Yılmaz (2000) analyzes role of strategic voting in corporate control contests. Similary, Maug and Yılmaz (2002) study multi-class voting in Chapter 11 bankruptcy proceedings.

The voting games in general suffer from multiplicity of equilibria due to the

[1] See, e.g., Young (1988).

standard problem of collective decision making, i.e., everyone votes for the same alternative and thus no single voter can affect the outcome, making it a weak best response to vote for the same alternative. However, the multiplicity problem persists even if we restrict attention to responsive equilibria in which there is strictly positive probability of being pivotal.[2] In particular, standard models of strategic voting produce multiple equilibria, usually one symmetric and many asymmetric equilibria. Overwhelming majority of papers focus on the symmetric equilibrium whereas a smaller number of papers consider only asymmetric equilibria.[3]

The aim of this thesis is to establish a set of sufficient conditions for the existence of a unique responsive equilibrium. We find a set of sufficient conditions over the information structure so that the standard model produces a unique responsive equilibrium. The primary example of the setting that produces this result is quite natural: each voter's private signal is drawn from a normal distribution and the mean of the normal distribution depends on the fundamental variable "state" of the economy. The unique equilibrium is symmetric. Therefore, our results may be viewed as providing a set of sufficient conditions for ruling out the existence of asymmetric equilibria.

In terms of contribution, this thesis is most related to Duggan and Martinelli (2001) who finds sufficient conditions that lead to a unique responsive equilibrium under unanimity rule. Our results complement theirs by extending their result to any plurality rule in a setting where signals are normally distributed conditional on

[2]The term "responsive" is defined earlier by Feddersen and Pesendorfer (1998).

[3]For example, Feddersen and Pesendorfer (1996, 1997 and 1998), Yılmaz (2000) and Maug and Yılmaz (2002) restrict attention to symmetric equilibrium. On the other hand, Persico (2004) study asymmetric equilibria.

the state. In terms of modeling choices, both papers use an information structure that involves continuum of signals.[4] This is in contrast to the standard information structure used in earlier models that only allowed discrete signals. The leading examples of these models are Austen-Smith and Banks (1996), Feddersen and Pesendorfer (1996, 1997 and 1998), Maug and Yılmaz (2002), and Persico (2004).

Consider the following decision problem of 7 voters who have to choose between status quo, Q, and the alternative, A, under simple majority rule. There are two states of the world, $\omega \in \{0,1\}$, and state $\omega = 1$ occurs with probability $\frac{1}{5}$. Voters prefer A if the state is 1 and Q otherwise. In particular, the voters' preference is given by $u(A, \omega = 1) = u(Q, \omega = 0) > u(A, \omega = 0) = u(Q, \omega = 1)$. Each voter receives a signal s given by a random variable. The possible values of s are h and l. Each signal is independently drawn with conditional probabilities :
$P(s = h|\omega = 1) = P(s = l|\omega = 0) = \frac{2}{3}$ and $P(s = h|\omega = 0) = P(s = l|\omega = 1) = \frac{1}{3}$. Each voter updates his posterior belief: $P(\omega = 1|s = h) = \frac{1}{3}$ and $P(\omega = 1|s = l) = \frac{1}{9}$ which can be calculated as follows by Bayes' formula.

$$\begin{aligned} P(\omega = 1|s = h) &= \frac{P(s = h|\omega = 1)P(\omega = 1)}{P(s = h|\omega = 1)P(\omega = 1) + P(s = h|\omega = 0)P(\omega = 0)} \\ &= \frac{\frac{2}{3}\frac{1}{5}}{\frac{2}{3}\frac{1}{5} + \frac{1}{3}\frac{4}{5}} = \frac{1}{3}. \end{aligned}$$

[4] In this sense, our model is also closely related to Yılmaz (2000)

Likewise,

$$\begin{aligned} P(\omega = 1|s = l) &= \frac{P(s = l|\omega = 1)P(\omega = 1)}{P(s = l|\omega = 1)P(\omega = 1) + P(s = l|\omega = 0)P(\omega = 0)} \\ &= \frac{\frac{1}{3}\frac{1}{5}}{\frac{1}{3}\frac{1}{5} + \frac{2}{3}\frac{4}{5}} = \frac{1}{9}. \end{aligned}$$

All voters vote simultaneously. Let $P(\omega = 1|ah, bl)$ be the probability that state of the world being 1 conditional on the total number of signal h received is a, and the total number of signal l received is b.

Each voter knows only his signal but can calculate all the probabilities, e.g.,

$$\begin{aligned} P(\omega = 1|3h, 4l) &= \frac{P(3h, 4l|\omega = 1)P(\omega = 1)}{P(3h, 4l|\omega = 1)P(\omega = 1) + P(3h, 4l|\omega = 0)P(\omega = 0)} \\ &= \frac{\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^4 C(7,3)\frac{1}{5}}{\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^4 C(7,3)\frac{1}{5} + \left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^4 C(7,3)\frac{4}{5}} = \frac{1}{9}, \\ P(\omega = 1|4h, 3l) &= \frac{P(4h, 3l|\omega = 1)P(\omega = 1)}{P(4h, 3l|\omega = 1)P(\omega = 1) + P(4h, 3l|\omega = 0)P(\omega = 0)} \\ &= \frac{\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^3 C(7,4)\frac{1}{5}}{\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^3 C(7,4)\frac{1}{5} + \left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^3 C(7,4)\frac{4}{5}} = \frac{1}{3}, \\ P(\omega = 1|5h, 2l) &= \frac{P(5h, 2l|\omega = 1)P(\omega = 1)}{P(5h, 2l|\omega = 1)P(\omega = 1) + P(5h, 2l|\omega = 0)P(\omega = 0)} \\ &= \frac{\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^2 C(7,5)\frac{1}{5}}{\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^2 C(7,5)\frac{1}{5} + \left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^2 C(7,5)\frac{4}{5}} = \frac{2}{3} \end{aligned}$$

and so on. Each voter votes without knowing others' signals or choices. But each voter knows that his vote will not affect his expected utility if the outcome is already certain given the other 6 votes, i.e. if he is not pivotal. Observe that informative voting, i.e., voters with signals h voting for A and others voting for Q, is not an Bayesian-Nash equilibrium. The voters' set of strategies is called Bayesian-Nash equilibrium if, holding all the other voters' strategies constant, no voter can obtain higher expected utility by choosing a different strategy. In other words, given others' strategies, no voter should have a reason to regret his strategy. For contradiction, let us assume that informative voting is an equilibrium. Consider a voter with signal h. His vote matters only when he is pivotal. Given the equilibrium behavior he knows that there must be $3h$ and $3l$ signals received by others in the case when he is pivotal. In that case, there are $4h$ and $3l$ signals including his signal. But given $P(\omega = 1|4h, 3l) = \frac{1}{3}$ as calculated above, he is better off voting for Q not for A, since his expected utility would be higher with Q. [5] However, it turns out that the problem in this situation is that there are too many equilibria. We must note here that the equilibrium strategy profile at which the voters who receive exactly the same signal would vote for the same alternative is called symmetric equilibrium. It is called asymmetric equilibrium otherwise. There is one symmetric equilibrium in which voters with signal l vote for Q and the others randomize between Q and A. More importantly, there are 21 asymmetric equilibria with 5 informative votes: 2 voters vote for Q independent of their signal. The remaining 5 voters vote informatively. Each voter knows that $P(\omega = 1|3h, 2l) = \frac{1}{3}$, $P(\omega = 1|4h, l) = \frac{2}{3}$. Each one of the 5 knows that there must be 3 h and 1 l signal received by other 4 voters when he is pivotal. Therefore, it is optimal (strictly better) for him to vote for A if and only if his signal is h. Each one of the 2 voters who always vote for Q knows that there

[5] This point is first made by Austen-Smith and Banks (1996).

must be 3 h and 2 l signals received by other 5 voters when he is pivotal. If his signal is l, then $P(\omega = 1|3h, 3l) = \frac{1}{5}$ so it is optimal to vote for Q. If his signal is h, then $P(\omega = 1|4h, 2l) = \frac{1}{2}$ so it is weakly optimal (indifferent between two alternatives, i.e. expected utilities are equal) to vote for Q. Each of these asymmetric equilibria aggregates the information of 5 voters but wastes 2 signals.

The standard information structure used in earlier models has been discrete signals similar to the example above. The leading examples of these models are Austen-Smith and Banks (1996), Feddersen and Pesendorfer (1996, 1997 and 1998), Maug and Yılmaz (2002), and Persico (2004). Feddersen and Pesendorfer (1998) characterize the unique symmetric equilibrium. The significant difference between our model and that of Feddersen and Pesendorfer (1998) is that we use a continuum of signals as opposed to a binary signal structure. In this sense, our model is more closely related to Duggan and Martinelli (2001) and Yılmaz (2000). Duggan and Martinelli (2001) show that continuum of signals leads to a unique responsive equilibrium under unanimity rule. Our results complement theirs by extending the result to any plurality rule in a setting where signals are normally distributed conditional on the state.

2 Model

Let $\omega \in \Omega = \{0, 1\}$ be the true state of the world. Let I denote the set of n voters. We assume $n > 2$. $\omega \in \Omega$ is unknown to the voters, but there is a common prior belief. The prior probability of state $\omega = 1$ is denoted by $\lambda \in (0, 1)$. There are two candidates, Q and A.

Each voter is privately informed. In particular, conditional on the true state of the world each voter, i, receives a signal $s_i \in S_i = \mathbb{R}$ independently drawn from an identical distribution, with probability density functions $f(s)$ and $g(s)$ for $\omega = 1$ and $\omega = 0$, respectively. $F(s)$ and $G(s)$ stand for the cumulative distributions. For most of our results, we assume f and g are normal.

The voters simultaneously vote after they each observe a private signal. Given that Q and A are two candidates, set of actions for voter i, $\mathcal{A}_i$, is denoted by {vote for A, vote for Q}, which we simplify as $\{A, Q\}$. A (pure) strategy of a player i, $\sigma_i : S_i \rightarrow \mathcal{A}_i$ is a measurable function. In particular, $\sigma_i : \mathbb{R} \rightarrow \{A, Q\}$. The set of strategies for player i is denoted by Σ_i. Let strategy profile of all players is $\sigma = (\sigma_1, \ldots, \sigma_n)$.

Let n_A stand for the number of votes candidate A receives. The candidate A needs at least m votes, i.e., $n_A \geq m$ in order to be elected. Otherwise Q is elected.

Given the true state of the world, voters have identical preferences . In particular, the voters prefer Q if state is $\omega = 0$ and A otherwise. Now that we have defined the preferences and the voting rule, we can define the payoff function for each player i as u_i where $u_i : \Omega \times \{A, Q\}^n \rightarrow \mathbb{R}$.

Next, we define our equilibrium concept:

Definition 1 *A profile of strategies $(\sigma_1^\star, \ldots, \sigma_n^\star)$ is called a pure strategy (Bayesian) Nash equilibrium if and only if for each player i and each $s_i \in S_i$, $\sigma_i^\star(s_i)$ is a solution to the maximization problem*

$$\max_{\sigma_i \in \Sigma_i} E[u_i(\omega, \sigma_1^\star(s_1), ..., \sigma_{i-1}^\star(s_{i-1}), \sigma_i(s_i), \sigma_{i+1}^\star(s_{i+1}), ..., \sigma_n^\star(s_n))|s_i] \ .$$

2.1 Responsive Equilibrium

Let $p(\omega, \sigma_{-i})$ stand for the probability of i being pivotal given a strategy profile and a true state, i.e., the outcome would be different if one voter from the winning alternative is to vote for the losing alternative. We say strategy profile $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n)$ is responsive if $p(\omega, \sigma_{-i}) > 0$, for all $i \in I$. We will analyze the responsive Nash equilibrium of this game.

3 Analysis

Some notational conventions necessary for following the rest of the thesis are:

1. In general, all vectors have dimension n, the number of voters; and will be denoted with bold face small case letters.
2. Vectors with dimension $n-1$ will be denoted by notation s_{-i}, meaning the entry $i \in I$ not included.
3. Same notational conventions will be used for vector valued functions.
4. Vector functions with arguments of dimension $n-1$ will sometimes be denoted as $\mathbf{v}(\mathbf{s})$ for notational convenience.
5. Scalar variables will be denoted with regular small case letters.
6. "$\triangleq$" should read "is defined as".

Lemma 1 *In equilibrium every voter will have a cut-off strategy* $\sigma_i^\star(s_i)$ *such that a voter with a signal below his cut-off votes for* Q*, otherwise he votes for* A*, that is,* $\sigma_i^\star(s_i) = \begin{cases} Q & if \quad s_i < s_i^\star \\ A & if \quad s_i > s_i^\star \end{cases}$.

Proof. Let $\sigma^\star$ stand for an equilibrium strategy profile. Recall that in a responsive equilibrium, $p(\omega, \sigma_{-i}^\star)$, is always strictly positive for all i. Let $W_i(s_i, \sigma_{-i}^\star)$ be the expected utility difference for a voter with signal s_i between voting for A and voting for Q.

$$\begin{aligned} W_i(s_i, \sigma_{-i}^\star) &= E[u_i(\omega, A, \sigma_{-i}^\star)|S = s_i] - E[u_i(\omega, Q, \sigma_{-i}^\star)|S = s_i] \\ &= \frac{\sum_{\omega\in\Omega}[u_i(\omega, A, \sigma_{-i}^\star) - u_i(\omega, Q, \sigma_{-i}^\star)]\beta(\omega|s)p(\omega, \sigma_{-i}^\star)}{\sum_{\omega\in\Omega}\beta(\omega|s)p(\omega, \sigma_{-i}^\star)}. \end{aligned} \tag{1}$$

Note that there exists an $\underline{s}$ such that for $s \in (-\infty, \underline{s})$, $W_i(s, \sigma_{-i}^\star) < 0$. Similarly, for sufficiently large s, we have $W_i(s, \sigma_{-i}^\star) > 0$. Furthermore, $W_i(s, \sigma_{-i}^\star)$ is strictly increasing in s for all $s \in \mathbb{R}$ given that $\beta(1|s)$ is strictly increasing and $\beta(0|s)$ is strictly decreasing in the same interval. By continuity, there exists a $s^\star$ such that $W_i(s, \sigma_{-i}^\star) = 0$. For $W_i(\cdot) > 0$ the voter is better off by voting for A. Similarly, for $W_i(\cdot) < 0$ the voter prefers Q. $\square$

Definition 2

$$v(s_{-i}) = \frac{\sum_{\substack{M \in 2^{I-i} \\ |M| = m-1}} \prod_{k\notin M} G(s_k) \prod_{k\in M}[1 - G(s_k)]}{\sum_{\substack{M \in 2^{I-i} \\ |M| = m-1}} \prod_{k\notin M} F(s_k) \prod_{k\in M}[1 - F(s_k)]}, \tag{2}$$

where s_{-i} is the vector $(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)'$ with dimension $n-1$. $s_j \in \mathbb{R}\ \forall j \in n$, and $M = \{i_1, i_2, \ldots, i_{m-1}\}$ where $i_1, i_2, \ldots, i_{m-1} \in \{1, 2, \ldots, i-1, i+1, \ldots, n\}$. Transpose of a vector $\mathbf{s}$ is denoted by $\mathbf{s}'$.

Proposition 1 *The cut-off strategies of n voters are identified by the solutions to the system of equations at the equilibrium*

$$\frac{f(s_i^\star)}{g(s_i^\star)} = v(s_{-i}^\star) \tag{3}$$

for $i = 1, \ldots, n$.

Proof.

At the equilibrium, there exists a cut-off strategy by Lemma 1. Suppose, $s_i^\star$ is the cut-off point for voter i. Hence, expected utility of voting A with signal $s_i^\star$ should be equal to the expected utility of voting Q with signal $s_i^\star$.

$$\begin{aligned} & u_i(1,A)p(1,\sigma_{-i}^\star)\beta_i(1|s_i^\star) + u_i(0,A)p(0,\sigma_{-i}^\star)\beta_i(0|s_i^\star) \\ = \; & u_i(0,Q)p(0,\sigma_{-i}^\star)\beta_i(0|s_i^\star) + u_i(1,Q)p(1,\sigma_{-i}^\star)\beta_i(1|s_i^\star) \end{aligned}$$

Then,

$$\begin{aligned} p(1,\sigma_{-i}^\star)\beta(1|s_i^\star)[u_i(1,A) \;&-\; u_i(1,Q)] \\ &= \; p(0,\sigma_{-i}^\star)\beta(0|s_i^\star)[u_i(0,Q) - u_i(0,A)]. \end{aligned}$$

It is assumed that $u_i(1,A) > u_i(1,Q)$ and $u_i(0,Q) > u_i(0,A)$.

Hence

$$\frac{u_i(1,A)-u_i(1,Q)}{u_i(0,Q)-u_i(0,A)}\frac{\beta(1|s_i^\star)}{\beta(0|s_i^\star)}=\frac{p(0,\sigma_{-i}^\star)}{p(1,\sigma_{-i}^\star)}. \tag{4}$$

By definition,

$$\begin{aligned}\beta(1|s_i^\star) &\equiv \frac{\lambda f(s_i^\star)}{\lambda f(s_i^\star)+(1-\lambda)g(s_i^\star)},\\ \beta(0|s_i^\star) &\equiv \frac{(1-\lambda)g(s_i^\star)}{\lambda f(s_i^\star)+(1-\lambda)g(s_i^\star)},\end{aligned}$$

and $\lambda \equiv P(\omega = 1)$ is the unconditional probability that $\omega = 1$ will prevail, which is known to all voters. We now have

$$\delta\frac{f(s_i^\star)}{g(s_i^\star)} = \frac{p(0,\sigma_{-i}^\star)}{p(1,\sigma_{-i}^\star)} \triangleq v(s_{-i}^\star)$$

where

$$\delta \triangleq \frac{\lambda}{1-\lambda}\frac{u_i(1,A)-u_i(1,Q)}{u_i(0,Q)-u_i(0,A)}, \tag{5}$$

$\forall i \in I$.

$\square$

Next the function v will be factorized by first multiplying it by one,

$$v(s_{-i}) = \frac{\sum_{\substack{M \in 2^{I-i} \\ |M| = m-1}} \prod_{k \notin M} G(s_k) \prod_{k \in M} [1 - G(s_k)]}{\sum_{\substack{M \in 2^{I-i} \\ |M| = m-1}} \prod_{k \notin M} F(s_k) \prod_{k \in M} [1 - F(s_k)]} \times \frac{\prod_{j \in I-i} \frac{1}{1-G(s_j)} [1 - G(s_j)]}{\prod_{j \in I-i} \frac{1}{F(s_j)} F(s_j)}, \tag{6}$$

and then rearranging terms:

$$v(s_{-i}) = \frac{\sum_{\substack{M \in 2^{I-i} \\ |M| = m-1}} \prod_{k \in M} \frac{G(s_k)}{1-G(s_k)}}{\sum_{\substack{M \in 2^{I-i} \\ |M| = m-1}} \prod_{k \notin M} \frac{1-F(s_k)}{F(s_k)}} \times \prod_{j \in I-i} \frac{1 - G(s_j)}{F(s_j)}. \tag{7}$$

The first factor will be renamed as $w(s_{-i})$ and the second as $k(s_{-i})$. Since $\frac{G(s)}{1-G(s)}$ is increasing and $\frac{1-F(s)}{F(s)}$ is decreasing, w is strictly increasing in each of its $n-1$ arguments. Also, $\frac{1-G(s)}{F(s)}$ is strictly decreasing so k is strictly decreasing in each of its $n-1$ arguments.

For the rest of the thesis, I will concentrate on a specific distribution, namely the normal distribution: $s|(\omega = 0) \sim N(\mu_o, \tau^2)$, $s|(\omega = 1) \sim N(\mu_1, \tau^2)$ with the corresponding densities $f(s) = (\sqrt{2\pi}\tau)^{-1} \exp\left(\frac{-(s-\mu_1)^2}{2\tau^2}\right)$ and $g(s) = (\sqrt{2\pi}\tau)^{-1} \exp\left(\frac{-(s-\mu_o)^2}{2\tau^2}\right)$ where $\mu_o < \mu_1$. The proof may also be generalized to distributions from the exponential family with density

$$f(s;\theta) = B(s)C(\theta) \exp\left(\sum_{j=1}^{k} \omega(\theta) q_i(s)\right), \tag{8}$$

and with infinite support, $s \in \mathbb{R}$, so that the summand on the right hand side of the equation below is linear:

$$\ln \frac{f(s;\theta_1)}{g(s;\theta_o)} = \sum_{j=1}^{k} [\omega(\theta_1) - \omega(\theta_o)] q_j(s). \tag{9}$$

Lemma 2 *Let f and g be normal density functions. Then f/g is strictly increasing whenever $\mu_o < \mu_1$ where μ_o and μ_1 are the means of f and g respectively.*

Proof.

$$\begin{aligned} \frac{f(s)}{g(s)} &= \exp\left(\frac{-(s-\mu_1)^2 + (s-\mu_o)^2}{2\tau^2}\right) \\ &= \exp\left(\frac{\mu_1 - \mu_o}{\tau^2} s - \frac{\mu_1^2 - \mu_o^2}{2\tau^2}\right). \end{aligned} \tag{10}$$

$\frac{\mu_1 - \mu_o}{\tau^2}$ in the above exponential is positive by our hypothesis, so $\frac{f(s)}{g(s)}$ is increasing $\forall s \in \mathbb{R}$. Furthermore, $\lim_{s \to -\infty} \frac{f(s)}{g(s)} = 0$ and $\lim_{s \to +\infty} \frac{f(s)}{g(s)} = +\infty$.

Without loss of generality, for the rest of the thesis, I will assume a strictly increasing likelihood ratio: $\mu_o < \mu_1$.

Theorem 1 *Brouwer (1912). Let $K \subset \mathbb{R}^n$ be nonempty, compact and convex. Then each continuous map $\psi : K \to K$ has at least one fixed point.*

Theorem 2 *Luce (1991).*

If g is a differentiable density function and G is the corresponding cumulative distribution function, and $-\frac{d}{dt} \ln g(t)$ is strictly increasing, then the hazard function of g, $h_g^-(t) \triangleq \frac{g(t)}{1-G(t)}$ is also strictly increasing.

Proof. Define $\psi^-(t) \triangleq -\frac{d}{dt}\ln g(t) = -\frac{g'(t)}{g(t)}$ which is strictly increasing. Observe that

$$\begin{aligned} \frac{\int_t^\infty \psi^-(x)g(x)dx}{\int_t^\infty g(x)dx} &= \frac{-\int_t^\infty \frac{g'(x)}{g(x)}g(x)dx}{\int_t^\infty g(x)dx} \\ &= \frac{g(t)}{1-G(t)} \\ &= h_g^-(t). \end{aligned} \tag{11}$$

Taking the derivative of the left hand side of (11) yields

$$g(t)\int_t^\infty [\psi^-(x) - \psi^-(t)]g(x)dx \Bigg/ \left[\int_t^\infty g(x)dx\right]^2, \tag{12}$$

which is strictly positive since ψ^- is strictly increasing and so for $t < x$, $\psi^-(x) - \psi^-(t) > 0$. Thus the hazard function h_g^- is strictly increasing.

□

Theorem 3 *If f is a differentiable density function and F is the corresponding cumulative distribution function, such that $f(t) \to 0$ as $F(t) \to 0$ and $\frac{d}{dt}\ln f(t)$ is strictly decreasing, then the function, $h_f^+(t) \triangleq \frac{f(t)}{F(t)}$ is also strictly decreasing.*

Proof. Define $\psi^+(t) \triangleq \frac{d}{dt}\ln f(t) = \frac{f'(t)}{f(t)}$ which is strictly decreasing. Observe that

$$\begin{aligned} \frac{\int_{-\infty}^t \psi^+(x)f(x)dx}{\int_{-\infty}^t f(x)dx} &= \frac{\int_{-\infty}^t \frac{f'(x)}{f(x)}f(x)dx}{\int_{-\infty}^t f(x)dx} \\ &= \frac{f(t)}{F(t)} \\ &= h_f^+(t). \end{aligned} \tag{13}$$

Taking the derivative of the left hand side of (13) yields

$$f(t)\int_{-\infty}^{t}[\psi^{+}(t)-\psi^{+}(x)]f(x)dx \Bigg/ \left[\int_{-\infty}^{t}f(x)dx\right]^{2}, \tag{14}$$

which is strictly negative since ψ^{+} is strictly decreasing and so for $x < t$, $\psi^{+}(t)-\psi^{+}(x) < 0$. Thus the function h_f^{+} is strictly decreasing.

□

For the normal densities assumed, h_g^{-} is the hazard function, and it is strictly increasing by Theorem 2 since $-\frac{d}{dt}\ln g(t) = (t-\mu_o)/\tau^2$ is strictly increasing. By Theorem 3, $h_f^{+}(x) \triangleq \frac{f(x}{F(x)}$ is strictly decreasing for the assumed normal density since $\frac{d}{dt}\ln f(t) = -(t-\mu_1)/\tau^2$ is strictly decreasing.

□

Going back to the system (3) we can write it again using our specified normal densities f and g, and their corresponding cumulative distribution functions F and G.

$$\begin{aligned} & w_i(s_{-i})k_i(s_{-i}) = \delta\exp\left(\frac{\mu_1-\mu_o}{\tau^2}\left[s_i-\frac{\mu_1+\mu_o}{2}\right]\right) \\ \Rightarrow\ & \frac{\mu_1+\mu_o}{2}-\frac{\tau^2}{\mu_1-\mu_o}\ln\delta+\frac{\tau^2}{\mu_1-\mu_o}\ln w_i(s_{-i})+\frac{\tau^2}{\mu_1-\mu_o}\ln k_i(s_{-i}) = s_i \end{aligned} \tag{15}$$

I will need new definitions here for simplification purposes:

Definition 3 *1.* $\bar{\mathbf{k}}(\mathbf{s}) \triangleq \frac{\mu_1+\mu_o}{2}-\frac{\tau^2}{\mu_1-\mu_o}\ln\delta+\frac{\tau^2}{\mu_1-\mu_o}\ln\mathbf{k}(\mathbf{s})$

2. $\bar{\mathbf{w}}(\mathbf{s}) \triangleq \frac{\tau^2}{\mu_1-\mu_o}\ln\mathbf{w}(\mathbf{s})$.

3. $\hat{k}(s) \triangleq \bar{k}(s, \ldots, s)$.

For notational convenience let $\chi(s) \triangleq 1/h_g^-(s) = [1-G(s)]/g(s)$, $\kappa(s) \triangleq 1/h_f^+(s) = F(s)/f(s)$, and $\eta_\xi(s) \triangleq -\frac{d}{ds}\ln\xi(s) = -\frac{\xi'(s)}{\xi(s)}$ for any density ξ.

Lemma 3 $\hat{k}'(s) < -(n-1)$, $\forall s \in \mathbb{R}$.

Proof. Claim: $\hat{k}'(s) = -(n-1)\frac{\tau^2}{\mu_1-\mu_o}[h_f^+(s) + h_g^-(s)] < -(n-1)$. $\eta_\xi(s) = (s-\mu)/\tau^2$ for the normal distribution where μ and τ are the mean and standard deviation of the distribution, respectively. By Glaser (1980),

$$\frac{\chi'(s)}{\chi(s)} = -\frac{g(s)}{G(s)} - \frac{g'(s)}{g(s)} \Rightarrow \chi'(s) = \left[-\frac{1}{\chi(s)} + \eta_g(s)\right]\chi(s) = -1 + \chi(s)\eta_g(s) < 0, \quad (16)$$

$$\frac{\kappa'(s)}{\kappa(s)} = \frac{f(s)}{F(s)} - \frac{f'(s)}{f(s)} \Rightarrow \kappa'(s) = \left[\frac{1}{\kappa(s)} + \eta_f(s)\right]\kappa(s) = 1 + \kappa(s)\eta_f(s) > 0. \quad (17)$$

Inequalities (16) and (17) above follow from Theorem 2 that h_f^+, and from Theorem 3 that h_g^- are strictly monotone, and so are their reciprocals. The inequalities then take the form

$$\chi(s)(s-\mu_o)/\tau^2 < 1 \quad \Rightarrow \quad (s-\mu_o)/\tau^2 < 1/\chi(s) \equiv h_g^-(s), \qquad (18)$$

$$\kappa(s)(s-\mu_1)/\tau^2 > -1 \quad \Rightarrow \quad (\mu_1 - s)/\tau^2 < 1/\kappa(s) \equiv h_f^+(s). \qquad (19)$$

Adding the two inequalities side by side we get,

$$\frac{\mu_1 - \mu_o}{\tau^2} < h_f^+(s) + h_g^-(s). \qquad (20)$$

Rearranging and then multiplying both sides by $-(n-1)$ we get

$$-(n-1) > -(n-1)\frac{\tau^2}{\mu_1 - \mu_o}[h_f^+(s) + h_g^-(s)] \equiv \hat{k}'(s). \tag{21}$$

$\square$

Definition 4 *Let (X, d) be a metric space. A mapping $H : X \to X$ is expanding if there is $\gamma > 1$ such that $d(H(s), H(t)) \geq \gamma d(s, t)$, for all $s, t \in X$, $s \neq t$.*

Lemma 4 *$\iota\hat{k}$ is expanding for some finite $\iota > \frac{1}{n-1}$.*

Proof. By Lemma 3, $\iota\hat{k}'(s) < -(n-1)\iota$. By mean value theorem, for any $s, t \in \mathbb{R}$ there is $\varsigma \in \mathbb{R}$ such that $\iota\hat{k}'(\varsigma) = \frac{\iota\hat{k}(s) - \iota\hat{k}(t)}{s-t} < -(n-1)\iota$. Without loss of generality, take $s > t$. The inequality above implies $\iota\hat{k}(t) - \iota\hat{k}(s) > \iota(n-1)(s-t)$. Defining $d(a, b) = |a - b|$ as the Euclidean distance, we have $d(\iota\hat{k}(s), \iota\hat{k}(t)) > \iota(n-1)d(s, t)$. Since $(n-1)\iota > 1$ by definition of ι, $\iota\hat{k}$ is expanding. $\square$

Definition 5 *Let (X, d) be a metric space. A mapping $H : X \to X$ is a contraction if there is $\alpha < 1$ such that $d(H(s), H(t)) \leq \alpha d(s, t)$, for all $s, t \in X$, $s \neq t$.*

Theorem 4 *(Banach contraction theorem) Let (X, d) be a complete metric space and $\Phi : X \to X$ is a contraction on X. Then Φ admits one and only one fixed point $x^\star$ in X and $\Phi^N(x_o) \to x^\star$ as $N \to \infty$ for any initial $x_o \in X$. Particularly, $d(\Phi^N(x), x^\star) \leq \frac{\alpha^N}{1-\alpha} d(x, \Phi(x))$ for all $x \in X$ and $N \in \mathbb{Z}^+$.*

Theorem 5 *Let (X, d) be a complete metric space and $H : X \to X$ continuous and onto. If H is expanding, then*

1. *H is $1-1$ and onto,*

2. *H^{-1} exists and is a contraction,*

3. *H has a unique fixed point.*

Proof.

1. Suppose H is not $1-1$. Then there should be $s, t \in X$, $s \neq t$ such that $H(s) = H(t) \Rightarrow d(H(s), H(t)) = 0$. This contradicts expanding condition. So H is $1-1$ and onto.

2. H is $1-1$ onto and is continuous, so it has an inverse $H^{-1} : X \to X$. Put $s = H^{-1}(z)$, $t = H^{-1}(r)$, $s, t, r, z \in X$. The condition for expansion becomes

$$d(z, r) \geq \gamma d(H^{-1}(z), H^{-1}(r)) \tag{22}$$

$$\Rightarrow \quad \frac{1}{\gamma} d(z, r) \geq d(H^{-1}(z), H^{-1}(r)). \tag{23}$$

   Since $1/\gamma < 1$, H^{-1} is a contraction.

3. By Banach's contraction theorem, H^{-1} has a unique fixed point. Now suppose $s^\star$ is the unique fixed point of H^{-1}: $s^\star = H^{-1}(s^\star) \Rightarrow H(s^\star) = s^\star$. Let $s^{\star\star}$ be another fixed point of H: $s^{\star\star} = H(s^{\star\star}) \Rightarrow s^{\star\star} = H^{-1}(s^{\star\star})$, in turn, implies $s^\star = s^{\star\star}$ by the uniqueness of the fixed point for H^{-1}. So H has a unique fixed point.

□

Definition 6 $\bar{\mathbf{k}}(\mathbf{s}) \triangleq \begin{pmatrix} \frac{\mu_1+\mu_o}{2} + \frac{\tau^2}{\mu_1-\mu_o}\sum_{j\neq 1} \ln\frac{1-G(s_j)}{F(s_j)} \\ \frac{\mu_1+\mu_o}{2} + \frac{\tau^2}{\mu_1-\mu_o}\sum_{j\neq 2} \ln\frac{1-G(s_j)}{F(s_j)} \\ \vdots \\ \frac{\mu_1+\mu_o}{2} + \frac{\tau^2}{\mu_1-\mu_o}\sum_{j\neq n} \ln\frac{1-G(s_j)}{F(s_j)} \end{pmatrix}.$

Lemma 5 $\bar{\mathbf{k}}^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ *exists.*

Proof [6]. Claim 1: $\bar{\mathbf{k}}$ is onto. Take any $(y_1, \ldots, y_n) \in \mathbb{R}^n$. Then we can find $(z_1, \ldots, z_n) \in \mathbb{R}^n$ such that

$$z_i = \frac{y_i - \frac{\mu_1+\mu_o}{2}}{\frac{\tau^2}{\mu_1-\mu_2}}, \tag{24}$$

for all $i = 1, \ldots, n$.

And there also exists $(x_1, \ldots, x_n) \in \mathbb{R}^n$ such that

$$x_j = \frac{1}{n-1}\sum_{i=1}^{n} z_i - z_j, \tag{25}$$

$j = 1, \ldots, n$.

Let $x_j = \ln\frac{1-G(s_j)}{F(s_j)}$. Since $\ln\frac{1-G(s_j)}{F(s_j)}$ is onto on $\mathbb{R}$, $\exists s_j \in \mathbb{R}$ such that $x_j = \ln\frac{1-G(s_j)}{F(s_j)}$. Hence $\bar{\mathbf{k}}$ is onto.

Claim 2: $\bar{\mathbf{k}}$ is 1 -1.

Take any $\mathbf{s}$, $\mathbf{s}' \in \mathbb{R}$ such that

$$\begin{aligned} & \bar{\mathbf{k}}(\mathbf{s}) = \bar{\mathbf{k}}(\mathbf{s}'), \qquad (26) \\ \Rightarrow \quad & \sum_{j\neq i} \ln\frac{1-G(s_j)}{F(s_j)} = \sum_{j\neq i} \ln\frac{1-G(s_j')}{F(s_j')} \end{aligned}$$

[6]This proof has been significantly improved by Yaozhong Hu.

$\forall i = 1, \ldots, n.$

Hence

$$\sum_{j=1}^{n} \ln \frac{1-G(s_j)}{F(s_j)} - \ln \frac{1-G(s_i)}{F(s_i)} = \sum_{j=1}^{n} \ln \frac{1-G(s'_j)}{F(s'_j)} - \ln \frac{1-G(s'_i)}{F(s'_i)}, \tag{27}$$

$\forall i = 1, \ldots, n.$

Subclaim: $\sum_{j=1}^{n} \ln \frac{1-G(s_j)}{F(s_j)} = \sum_{j=1}^{n} \ln \frac{1-G(s'_j)}{F(s'_j)}$

$$\begin{aligned}
& \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \ln \frac{1-G(s_j)}{F(s_j)} - \ln \frac{1-G(s_i)}{F(s_i)} \right) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \ln \frac{1-G(s'_j)}{F(s'_j)} - \ln \frac{1-G(s'_i)}{F(s'_i)} \right) \\
\Leftrightarrow \quad & (n-1) \sum_{j=1}^{n} \ln \frac{1-G(s_j)}{F(s_j)} = (n-1) \sum_{j=1}^{n} \ln \frac{1-G(s'_j)}{F(s'_j)}, \\
\Leftrightarrow \quad & \sum_{j=1}^{n} \ln \frac{1-G(s_j)}{F(s_j)} = \sum_{j=1}^{n} \ln \frac{1-G(s'_j)}{F(s'_j)}.
\end{aligned} \tag{28}$$

Thus by (27) and (28), $\ln \frac{1-G(s_i)}{F(s_i)} = \ln \frac{1-G(s'_i)}{F(s'_i)}$, $\forall i = 1, \ldots, n$. Then $s_i = s'_i$ $\forall i = 1, \ldots, n$ since $\ln \frac{1-G(s_i)}{F(s_i)}$ is $1-1$. Therefore $\bar{\mathbf{k}}$ is $1-1$.

By Claim 1 and Claim 2, $\bar{\mathbf{k}}$ has an inverse. $\square$

Theorem 6 *Meyers (1967). Let X be a complete metric space and $\phi : X \to X$ be a map satisfying*

1. *(Existence) There exists a $\xi \in X$ such that $\phi(\xi) = \xi$,*

2. *(Global attraction) $\phi^r(x) \to \xi$ as $r \to \infty$ for all $x \in X$, and*

3. *(Stability) There exists an open neighborhood U of ξ such that to each open neighborhood V of ξ, there corresponds a natural number r' such that for all $r \geq r'$, $\phi^r(U) \subset V$,*

then for each $\alpha \in (0,1)$ there is an equivalent complete metric ρ_α on X such that $\rho_\alpha(\phi(x), \phi(y)) \leq \alpha \rho_\alpha(x,y)$ for all $x, y \in X$.

Lemma 6 *For each $\alpha \in (0,1)$ there exists an equivalent metric ρ_α on $\mathbb{R}^n$ such that*

$$\rho_\alpha(\mathbf{\bar{k}^{-1}}(\mathbf{s_1}/\iota),\ \mathbf{\bar{k}^{-1}}(\mathbf{s_2}/\iota)) \leq \alpha \rho_\alpha(\mathbf{s_1}, \mathbf{s_2}).$$

Proof.

1. By Theorem 1 there exists $s^\star$ which is a fixed point of $\iota\hat{k} : \mathbb{R} \to \mathbb{R}$ which is continuous, and of $\iota\bar{\mathbf{k}} : \mathbb{R}^n \to \mathbb{R}^n$ which is also continuous.

2. By Lemma 5 $\bar{\mathbf{k}}^{-1}(\mathbf{s}/\iota)$ exists, and by Theorem 5 $\hat{k}^{-1}(s/\iota)$ is a contraction since $\iota\hat{k}$ is expanding. For any vector $\mathbf{s^o} \in \mathbb{R}^n$, take $s^o_{\min} = argmin\{s^o_1 - s^\star, \ldots, s^o_n - s^\star\}$, and $s^o_{\max} = argmax\{s^o_1 - s^\star, \ldots, s^o_n - s^\star\}$. Also, $\mathbf{s^o} = (s^o_1, \ldots, s^o_n)$. Since $\hat{k}^{-1}(s/\iota)$ is decreasing in every argument, $\hat{k}^{-1}(s_{\max}/\iota) \leq \bar{k}^{-1}_i(\mathbf{s^o}/\iota) \leq \hat{k}^{-1}(s_{\min}/\iota)$ for all $i \in I$. When we recursively generate $\mathbf{s}^t = \bar{\mathbf{k}}^{-1}(\mathbf{s}^{t-1}/\iota)$ starting from $\mathbf{s^o}$, we will get

$$\hat{k}^{-1}(s^{t-1}_{max}/\iota) \leq \hat{k}^{-1}(s^t_{max}/\iota) \leq \bar{k}^{-1}_i(\mathbf{s^t}/\iota) \leq \hat{k}^{-1}(s^t_{min}/\iota) \leq \hat{k}^{-1}(s^{t-1}_{min}/\iota), \quad (29)$$

for all $i \in I$. $\hat{k}^{-1}(s/\iota)$ is a contraction. By Banach's contraction theorem, $s^t_{\min}, s^t_{\max} \to s^\star$ as $t \to \infty$ which implies, $\mathbf{s^t} \to \mathbf{s}^\star$ as $t \to \infty$. $\bar{\mathbf{k}}^{-1}(\mathbf{s}/\iota)$ is globally attracting.

3. Let $\epsilon_o > 0$ and $U = \{\mathbf{s} | \sup_{\mathbf{s}\in U} \rho_\alpha(\mathbf{s}, \bar{\mathbf{k}}^{-1}(\mathbf{s}/\iota) < \epsilon_o) \subset \mathbb{R}^n$. Let $V = \{\mathbf{s} | \rho_\alpha(\mathbf{s}^\star, \mathbf{s}) \leq \epsilon) \subset \mathbb{R}^n$ be a ball with center $\mathbf{s}^\star$ and radius ϵ. By Banach's contraction theorem, and by (29) there will be an $N_V \in \mathbb{Z}^+$ such that for all $\mathbf{s} \in U$, and $N > N_V$, $\rho_\alpha((\bar{\mathbf{k}}^{-1})^{\mathbf{N}}(\mathbf{s}/\iota), \mathbf{s}^\star) \leq \rho_\alpha((\hat{k}^{-1})^N(s_m/\iota), s^\star) \leq \frac{\alpha^N}{1-\alpha}\rho_\alpha(s_m, \hat{k}^{-1}(s_m/\iota)) < \epsilon$ so that $(\bar{\mathbf{k}}^{-1})^{\mathbf{N}}(\mathbf{s}/\iota)$ is in V. By s_m we mean $argmax_{i\in I}\{|s_i - s^\star|\}$. Therefore $\bar{\mathbf{k}}^{-1}(\mathbf{s}/\iota)$ is stable.

$\bar{\mathbf{k}}^{-1}(\mathbf{s}/\iota)$ is a continuous self mapping of $\mathbb{R}^n$, a metrizable topological space, and satisfies 1. of Meyers' converse Banach theorem since it has a fixed point $\mathbf{s}^\star$. $\bar{\mathbf{k}}^{-1}(\mathbf{s}/\iota)$ satisfies 2. of the same theorem since the symmetric fixed point $\mathbf{s}^\star$ is globally attracting and 3. since the symmetric fixed point $\mathbf{s}^\star$ is stable. Therefore lemma follows by Meyers' converse Banach theorem. $\square$

Definition 7 *R is the set $\{\rho_\alpha \in \mathcal{M} | \rho_\alpha(\bar{\mathbf{k}}^{-1}(\mathbf{s_1}/\iota),\ \bar{\mathbf{k}}^{-1}(\mathbf{s_2}/\iota)) \leq \alpha\rho_\alpha(\mathbf{s_1}, \mathbf{s_2}), \alpha \in (0, 1)\}$ where $\mathcal{M}$ is the set of all metrics.*

Lemma 7 *$\exists\gamma > 1$ such that $\rho_\alpha(\iota\bar{\mathbf{k}}(\mathbf{s_1}),\ \iota\bar{\mathbf{k}}(\mathbf{s_2})) \geq \gamma\rho_\alpha(\mathbf{s_1}, \mathbf{s_2})$, $\forall \mathbf{s_1}, \mathbf{s_2} \in \mathbb{R}^n$.*

Proof. By Lemma 6 we know that $\rho_\alpha(\bar{\mathbf{k}}^{-1}(\mathbf{s_1}/\iota),\ \bar{\mathbf{k}}^{-1}(\mathbf{s_2}/\iota)) \leq \alpha\rho_\alpha(\mathbf{s_1}, \mathbf{s_2})$. Let $\gamma = 1/\alpha$ and $\mathbf{t_1} = \bar{\mathbf{k}}^{-1}(\mathbf{s_1}/\iota)$ and $\mathbf{t_2} = \bar{\mathbf{k}}^{-1}(\mathbf{s_2}/\iota)$. The inequality above becomes $\gamma\rho_\alpha(\mathbf{t_1},\ \mathbf{t_2}) \leq \rho_\alpha(\iota\bar{\mathbf{k}}(\mathbf{t_1}),\ \iota\bar{\mathbf{k}}(\mathbf{t_2}))$, $\forall \mathbf{t_1},\ \mathbf{t_2} \in \mathbb{R}$. Therefore $\iota\bar{\mathbf{k}}$ is expanding since $\bar{\mathbf{k}}^{-1}(\mathbf{s}/\iota)$ is onto. $\square$

Definition 8 *Two metrics $|\cdot|$ and $||\cdot||$ on the same space X are called equivalent if there are constants $a, b \in \mathbb{R}^+$ such that $a|x| \leq ||x|| \leq b|x|$, $\forall x \in X$.*

Lemma 8 $\bar{\mathbf{v}}(\mathbf{s}) = \bar{\mathbf{k}}(\mathbf{s}) + \bar{\mathbf{w}}(\mathbf{s}) \equiv \frac{\mu_1+\mu_o}{2} + \frac{\tau^2}{\mu_1-\mu_o} \ln \mathbf{v}(\mathbf{s})$ *is also expanding with respect to* sup *norm on* $\mathbb{R}^n$*, where* $\bar{\mathbf{w}}(\mathbf{s}) \triangleq \frac{\tau^2}{\mu_1-\mu_o} \ln \mathbf{w}(\mathbf{s})$ *provided that* $n > \min_{\rho_\alpha \in R} 1 + \frac{b}{a}\alpha$*.* R *is the set defined in Definition* 7*.*

Proof. By Lemma 7 there exists a metric ρ_α where $\iota\bar{\mathbf{k}}$ is expanding. Since Euclidean metric and the sup norm on $\mathbb{R}^n$ are also equivalent, the sup norm and ρ_α are equivalent. By definition of equivalence of metrics, there are $a, b \in \mathbb{R}^+$ such that

$$a\rho_\alpha(\mathbf{s}, \mathbf{t}) \leq \max_{1 \leq i \leq n} |s_i - t_i| \leq b\rho_\alpha(\mathbf{s}, \mathbf{t}). \tag{30}$$

I need to show that

$$c \max_{1 \leq i \leq n} |\ln v(s_{-i}) - \ln v(t_{-i})| \quad \geq \quad \gamma' \max_{1 \leq j \leq n} |s_j - t_j|. \tag{31}$$

where $c \triangleq \frac{\tau^2}{\mu_1-\mu_o}$ and for some $\gamma' > 1$.

Assume without loss of generality that $w_i(s_{-i}) \geq w_i(t_{-i})$ for the particular i that maximizes the left hand side of the inequality below. For $\bar{\mathbf{k}}(\mathbf{s}) + \bar{\mathbf{w}}(\mathbf{s})$ we have

$$\iota c \max_{1 \leq i \leq n} |\ln k(s_{-i}) + \ln w(s_{-i}) - \ln k(t_{-i}) - \ln w(t_{-i})| = \iota c \max_{1 \leq i \leq n} \left| \ln \frac{k(s_{-i})w(s_{-i})}{k(t_{-i})w(t_{-i})} \right| \tag{32}$$

But

$$\begin{aligned}
\iota c \max_{1\le i\le n}\left|\ln\frac{k(s_{-i})w(s_{-i})}{k(t_{-i})w(t_{-i})}\right| &\ge \iota c \max_{1\le i\le n}\left|\ln\frac{k(s_{-i})w(s_{-i})}{k(t_{-i})w(s_{-i})}\right| \qquad (33)\\
&\ge \iota c \max_{1\le i\le n}\left|\ln\frac{k(s_{-i})}{k(t_{-i})}\right| = \iota c \max_{1\le i\le n}|\ln k(s_{-i}) - \ln k(t_{-i})|\\
&\ge a\rho_\alpha(\iota\bar{\mathbf{k}}(\mathbf{s}), \iota\bar{\mathbf{k}}(\mathbf{t})) \ge a\gamma\rho_\alpha(\mathbf{s},\mathbf{t})\\
&\ge \frac{a}{b}\gamma \max_{1\le j\le n}|s_j - t_j|.
\end{aligned}$$

which implies

$$c \max_{1\le i\le n}|\ln k(s_{-i}) + \ln w(s_{-i}) - \ln k(t_{-i}) - \ln w(t_{-i})| \ge \frac{a}{b\iota}\gamma \max_{1\le j\le n}|s_j - t_j|. \qquad (34)$$

$\bar{\mathbf{v}}$ is expanding if $\frac{a\gamma}{b\iota} > 1$ that is $\exists\iota > 0$ such that $\frac{b}{a\gamma} < \frac{1}{\iota} < n-1$.

I complete the proof by hypothesis that $n > \min_{\rho_\alpha\in R} 1 + \frac{b}{a}\alpha$, which is finite by the definition of a, b and $\alpha = 1/\gamma$. R is the set defined in Definition 7.

□

Lemma 9 *$s^\star$ is the unique solution for the system* $\mathbf{s} = \frac{\mu_1+\mu_o}{2} - \frac{\tau^2}{\mu_1-\mu_o}\ln\delta + \frac{\tau^2}{\mu_1-\mu_o}\ln\mathbf{v}(\mathbf{s})$.

Proof. $\frac{\mu_1+\mu_o}{2} - \frac{\tau^2}{\mu_1-\mu_o}\ln\delta + \frac{\tau^2}{\mu_1-\mu_o}\ln\mathbf{v}(\mathbf{s})$ is expanding on a complete metric space $\mathbb{R}^n$. Hence it has a unique fixed point by theorem 5.

□

Lemma 10 *$s^\star$ is a solution of* $\mathbf{v}(\mathbf{s}) = \mathbf{f}(\mathbf{s})/\mathbf{g}(\mathbf{s})$.

Proof. The result follows from the previous lemma by taking exponential of both sides. $s^\star$ still satisfies the expression mentioned in this lemma.

$\square$

Lemma 11 *$s^\star$ is the unique solution for the system $\delta \exp\left(\frac{\mu_1-\mu_o}{\tau^2}s_i^\star - \frac{\mu_1^2-\mu_o^2}{2\tau^2}\right) = \mathbf{v}(\mathbf{s})$.*

Proof. Suppose $\mathbf{s}^{\star\star} \neq \mathbf{s}^\star$ is another equilibrium. Then $\mathbf{s}^{\star\star}$ satisfies $\delta \exp\left(\frac{\mu_1-\mu_o}{\tau^2}s_i^{\star\star} - \frac{\mu_1^2-\mu_o^2}{2\tau^2}\right) = v(s_{-i}^{\star\star})$, $\forall i \in I$. Therefore $\frac{\mu_1+\mu_o}{2} - \frac{\tau^2}{\mu_1-\mu_o}\ln\delta + \frac{\tau^2}{\mu_1-\mu_o}\ln v(s_{-i}^{\star\star}) = s_i^{\star\star}$, is satisfied $\forall i \in I$. Since $\mathbf{s}^\star$ is the only such solution, $\mathbf{s}^{\star\star} = \mathbf{s}^\star$.

$\square$

Thus, the proof of the main result has been completed by the previous Lemma. Hence, we can conclude the following theorem.

Theorem 7 *The symmetric equilibrium is unique in the class of pure strategies under $m, n \in \mathbb{Z}^+$ satisfying $0 < m < n$ and $\min_{\rho_\alpha \in R} 1 + \frac{b}{a}\alpha < n < \infty$.*

Proof. This is a direct implication of Lemma 11.

$\square$

The main difficulty throughout the proof is the existence of a measure for $\bar{\mathbf{v}}$ to be expanding. Meyer's theorem tells us that for each α (or γ) there exists an equivalent metric where $\bar{\mathbf{k}}$ is expanding, but we still do not know whether any specific metric, like the sup norm used above, is included in that particular set of metrics. Since the sets $(0,1)$ where α belongs to, or the set $(1,\infty)$ where γ belongs to are uncountable, so is the set of metrics corresponding to these parameters. All these uncountable number of metrics come with a parameter $\frac{a}{b\alpha}$ (or $\frac{a}{b}\gamma$) which needs to be greater than one. Even in an uncountable set R of metrics defined in Definition 7, such a metric may not exist. In order to guarantee the existence of a metric that satisfies $\frac{a}{b\alpha} > 1$ (or $\frac{a}{b}\gamma > 1$), an extra parameter ι is introduced, which connects the slope of $\hat{k}$ to the expansion parameter $\frac{a}{b\iota}\gamma$ of $\bar{\mathbf{v}}$ with the sup norm. Hence it is possible to reach the conclusion: when the number of voters n is large enough to make $\hat{k}$ expand fast enough, then $\bar{\mathbf{v}}$ is expanding. A more concrete method to prove that $\bar{\mathbf{v}}$ is expanding is to construct the metric ρ_α particularly. This is the subject of my future research.

4 Conclusion

In this thesis, we present a set of sufficient conditions that lead to a unique responsive equilibrium in strategic voting models. The sufficient conditions are valid for any plurality rule. Our results complements those of Duggan and Martinelli (2001) who find sufficient conditions for a unique equilibrium under unanimity. Furthermore, considering the fact that the unique equilibrium is symmetric, our results may be viewed as justification of the restriction on symmetric equilibria imposed by overwhelming majority of the earlier literature on strategic voting. focusing on symmetric. Although our current framework does not allow conflict of interests among the electorate, we conjecture that our result extends to the case with heterogenous preferences. We leave this for future research.

My further research continues with another collective decision making mechanism which shares the similar information aggregation model used above. In particular, I have been working the optimality of partial (restricted) tender offers in a takeover contest of a widely held firm. It is well established that a value increasing raider suffers from the free rider problem; each shareholder anticipating a successful takeover has an incentive to hold out whenever the offer price is below the post takeover value. Therefore, a raider cannot profitably takeover without a toehold or private benefits of control as he needs to pay the post takeover value in order to buy the majority of shares. More recent studies have shown that this problem is further exacerbated by asymmetrically informed shareholders. In particular, shareholders who are more pessimistic about the post takeover value are more likely to tender. Consequently, a raider in addition to the free rider problem suffers from a winners curse and ends up paying more than the actual post takeover value whenever he ends up taking over. A takeover may be still profitable due to private benefits of control, but the raider may want to protect himself from overpayment as this eats into his profits. One intuitive conclusion may be the optimality of partial offers so that the raider buys only a fraction of shares that is necessary for control. However, restricted offers are very infrequent in practice. I will explain the reason of this well known empirical fact by showing that partial offers are suboptimal. It is correct that partial offers protect the raider in the sense that he overpays only for a prespecified number of shares as opposed to all shares. However, shareholders correctly anticipate that their shares are more likely to be prorated when more shareholders tender. If more shareholders are tendering, then it must be that more shareholders are pessimistic about the post takeover value. Therefore, they will fail to sell a larger fraction of their shares precisely when the raider is overpaying and end up selling when he is underpaying. Consequently, they have less incentive to tender at a price when the raider uses partial offers. Thus, the raider has to offer a higher price to induce shareholder tendering in a partial tender offer. I characterize the tendering strategies of shareholders and the raiders profit function and show that the cost of higher price dominates the benefits restricting number of overpaid shares by using a similar voting model above.

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