Maximum Queue Length of a Fluid Model with a Gaussian Input

By

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## Abstract

A fractional Brownian queueing model, that is, a fluid model with an input of a fractional Brownian motion, was proposed in the 1990s to capture the self-similarity and long-range dependence observed in Internet traffic. Since then, a Gaussian queueing model, which is a queueing model with an input of a continuous Gaussian process, has received much attention.

In this dissertation, a Gaussian queueing model is discussed and the maximum queue length over a time interval $[0, t]$ is analyzed. Under some mild assumptions, it is shown that a limit of the maximum queue length suitably normalized is determined by a suitable function of the asymptotic variance of the Gaussian input. Some Gaussian queueing models, such as a queue with an input of several independent fractional Brownian motions and a queue with an input of an integrated Ornstein-Uhlenbeck process, are discussed as examples. For a fractional Brownian queueing model, the main results extend some related known results in the literature.

The results on the maximum queue length provide insights for the occurrence of large excursions, which are also called congestion events, in a queueing process. In the context of a fractional Brownian queueing model the temporal properties of congestion events, such as the duration and the inter-congestion event time, are analyzed. A new method based on a Poisson clumping approximation is proposed to evaluate these properties. By comparing with simulation results, it is illustrated that the proposed methodology produces satisfying results for estimating the temporal properties of congestion events in a fractional Brownian queueing model.

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# Contents

# Chapter 1

# Introduction

Traditionally Poisson processes are used to model network traffic. A Poisson queueing model, which has an input of a Poisson process, has been extensively analyzed in queueing theory and network modeling. Various queueing performance measures have been studied. However, in the early 1990s researchers with Bellcore observed the phenomena of self-similarity and long-range dependence in LAN traffic [44], which roughly means that the traffic "looks" similar under different time scales and the correlation between packets decays very slowly. The observation is inconsistent with the assumption of independent increments in Poisson processes. Subsequent studies [10], [20], [35], [57] showed that the traditional models seem inadequate for some data networks. The complex features of network traffic, such as self-similarity and long-range dependence, present challenges for performance analysis and network modeling.

In this dissertation a first-in-first-out(FIFO) fluid queueing model with an input of a continuous Gaussian process is analyzed. The main focus is on Gaussian processes which have the properties of self-similarity and long-range dependence. The objective is to gain a better understanding of the impacts of self-similarity and long-range dependence on the queueing performance. The structure of this dissertation is as follows: In Chapter 1, some preliminaries on stochastic process limit, Gaussian queueing model and fractional Brownian motions are briefly discussed. In Chapter 2 and Chapter 3, a fractional Brownian queueing model and a general

Gaussian queueing model are studied, respectively. The maximum queue length of a Gaussian queueing model is analyzed and some related results in the literature are extended. The main results of these two chapters will appear in [18]. In Chapter 4, some temporal properties of congestion events, such as the duration and the inter-congestion event time, in a fractional Brownian queueing model are analyzed. Some results of this chapter has been published in [34].

Throughout the disseration the following notation is used. Let $\Phi(x)$ denote a standard normal distribution, $\bar{\Phi}(x) = 1 - \Phi(x)$ the complement of a standard normal distribution and $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ the density of a standard normal distribution. For $x \geq 0$ the following inequalities hold, see [8, page 242],

$$\frac{1}{\sqrt{2\pi}} \frac{x}{1+x^2} e^{-\frac{x^2}{2}} \leq \bar{\Phi}(x) \leq \frac{1}{\sqrt{2\pi}} x^{-1} e^{-\frac{x^2}{2}}. \tag{1.1}$$

Let "$\stackrel{d}{=}$" denote equality in distribution. For $a \in \mathbb{R}$, let $\lfloor a \rfloor$ denote the integer part of $a$. Suppose $f(x)$ and $g(x)$ are two functions, if $\lim_{x\to\infty} f(x)/g(x) = 1$, then write $f(x) \sim g(x)$.

## 1.1 Stochastic Process Limit

Applying the terminology in [5], let $(\Omega, \mathscr{F}, P)$ be a probability space and $(D, \mathscr{D})$ be a measurable space, where $D$ is a space of functions $x : [0, \infty) \to R$, which are right-continuous and have left limits, $\mathscr{D}$ equipped with Skorokhod topology is the $\sigma$ algebra generated by open sets. If $X$ is a measurable mapping from $(\Omega, \mathscr{F})$ to $(D, \mathscr{D})$, then $X$ is called a $D$ valued random variable. For a real measurable function $f$ on $D$, let

$$E[f(X)] = \int_\Omega f(X(\omega)) P(d\omega).$$

Suppose that $\{X_n\}$ is a sequence of random elements of $D$, then $X_n$ weakly converges to a random variable $X$ of $D$, denoted by $X_n \Rightarrow X$, if and only if $E[f(X_n)] \rightarrow E[f(X)]$ for all bounded, continuous $f$ on $D$.

In network modeling a stochastic process limit, to which a sequence of stochastic processes weakly converge, makes a good approximation for the complex network traffic and provides insight into system performance [7], [27], [63]. For example, a Poisson process, after properly normalized, weakly converges to a Brownian motion. Let $\{N(t), t \geq 0\}$ be a Poisson process with parameter $\lambda$. For $t \geq 0$, define $X_n(t)$ as

$$X_n(t) = \frac{N(\lfloor nt \rfloor) - \lambda \lfloor nt \rfloor}{\sqrt{n\lambda}},$$

where $\lfloor nt \rfloor$ denotes the integer part of $nt$. It can be verified [5, p 146] that

$$X_n \Rightarrow B,$$

where $\{B(t), t \geq 0\}$ is a standard Brownian motion. Intuitively, $N(t)$ can be approximated by a diffusion process. This diffusion approximation has been used as a heavy traffic model, e.g., see [7], [49], [27]. Note that in this example the Poisson process, which has discontinuous sample path, weakly converges to a Brownian motion with continuous sample path.

## 1.2 Fluid Queue with a Gaussian Input

Different from traditional queueing models, in a fluid queueing model, the network traffic is viewed as a stream of fluid; in other words, the input process has continuous sample path. The continuity assumption is a convenient and reasonable idealization for a queueing system where the input and output are of high volumes in an interval of moderate length, that is, a fluid model is appropriate when the individual items are numerous in a network traffic flow for a chosen time scale.

Consider network modeling, a Gaussian process can be viewed as a stochastic process limit of aggregated network traffic. Based on the central limit theorem, it is natural to assume that at each time $t$ the random variable of the input process has a Gaussian distribution. The Gaussian assumption is reasonable when the backbone Internet traffic with thousands of simultaneous traffic flows is considered. A diffusion process, which is driven by a Brownian motion, has been used to model the inputs to fluid queues for a long time. Many results, such as the overflow probability and optimal control of resources, have been derived for diffusion models, see [27] and the references therein.

In [49] Norros proposed a fractional Brownian queueing model, that is, a queueing model with an input of a fractional Brownian motion, which is a generalization of a Brownian motion. A fractional Brownian queueing model is a useful model for analyzing the impact of self-similarity and long-range dependence on the queueing performance. However there are some generic shortcomings in this model. Firstly, since the input process is Gaussian, negative increments, which are not meaningful for a queueing model, can be observed at small time scales. Secondly, the actual Internet traffic is regulated by TCP/IP protocol, which is a closed-loop congestion control mechanism. A fractional Brownian queueing model, which is open-loop as are many queueing models, cannot capture the dynamics of Internet traffic over small time scales, i.e., less than the typical round trip packet time. Although the model has some shortcomings, it can be used to approximate other aspects of Internet traffic under certain circumstances. It has been empirically demonstrated in [20] that a fractional Brownian queueing model is appropriate for the backbone traffic, in which millions of independent flows are highly aggregated, traffic control on a single flow would not dominate the whole traffic and the time scale is larger than the typical round trip time. In recent network measurements [32], it was observed that for small time scales, less than a millisecond, the traffic in the Internet backbone is memoryless or of short memory; while for larger time scales, in milliseconds, the long-range dependence characterizes the backbone traffic. From a practical point of view, see [52], [58], a fractional Brownian queueing model is an approximation of Internet traffic and can produce meaningful results for queueing performance, such as inter-congestion event times and congestion durations, which are in a time scale larger than the typical round trip time.

In practice it has been observed that the Hurst parameter estimated in network does not remain constant. For this reason, besides a fractional Brownian motion, other Gaussian processes have been proposed to model network traffic, such as an aggregation of independent fractional Brownian motions, [15], [46] [55], [59, p 335] and an integrated Ornstein-Uhlenbeck process [12], [13], [14], [15].

### 1.3 Fractional Brownian Motion

Some preliminaries on a fractional Brownian motion are discussed in this subsection.

**Definition 1.3.1** (Self-similarity). *A continuous stochastic process* $\{X(t), t \geq 0\}$ *is called self-similar with self-similarity parameter* $H \in (0, 1)$*, if the process* $\{X(\alpha t), t \geq 0\}$ *and* $\{\alpha^H X(t), t \geq 0\}$ *have the same probability law for each* $\alpha > 0$*.*

**Definition 1.3.2** (Wide sense stationarity). *A process* $\{X(t), t \geq 0\}$ *is called wide sense stationary, if it has a constant mean and the autocorrelation function* $R(s, t) = E[X(s)X(t)]$*, for* $0 \leq s \leq t$*, depends only on the difference* $t - s$*.*

**Definition 1.3.3** (Long-range dependence). *Let* $\{X(k), k = 1, 2, ...\}$ *be a wide sense stationary process with mean* $m$ *and autocorrelation function* $r$*. The process is long-range dependent if as* $k \to \infty$*,*

$$r(k) \sim k^{2H-2} L(k)$$

*where* $H \in (1/2, 1)$ *and* $L$ *is a slowly varying function, that is, for* $x > 0$

$$\lim_{t \to \infty} \frac{L(tx)}{L(t)} = 1.$$

The definition of a standard fractional Brownian motion is cited from [17].

**Definition 1.3.4** (Fractional Brownian motion)**.** *A standard fractional Brownian motion* $\{B^H(t), t \geq 0\}$ *with a Hurst parameter* $H \in (0, 1)$ *is a real-valued Gaussian process such that for* $s, t \geq 0$,

$$\begin{aligned} E[B^H(t)] &= 0, \\ E\left[B^H(s)B^H(t)\right] &= \frac{1}{2}\left(s^{2H} + t^{2H} - |s - t|^{2H}\right). \end{aligned} \tag{1.2}$$

A fractional Brownian motion is a generalization of Brownian motions, since for $H = 1/2$, $\{B^H(t), t \geq 0\}$ reduces to a standard Brownian motion.

**Definition 1.3.5** (Fractional Gaussian noise)**.** *A discrete time fractional Gaussian noise,* $\{Y(k), k = 1, 2, ...\}$, *is the increment process of a fractional Brownian motion, that is,*

$$Y(k) = B^H(k) - B^H(k - 1), \tag{1.3}$$

*where* $\{B^H(t), t \geq 0\}$ *is a standard fractional Brownian motion with* $H \in (0, 1)$.

A fractional Brownian motion is a self-similar Gaussian process with stationary increments. For $H \in (1/2, 1)$, the process has long-range dependence. Some properties of a fractional Brownian motion and a fractional Gaussian noise are given as follows:

**Proposition 1.3.1.** *Let* $\{B^H(t), t \in \mathbb{R}\}$ *be a standard fractional Brownian motion, then*

1. *For each* $t \in \mathbb{R}$, $B^H(t)$ *is a standard normal random variable with mean* 0 *and variance* $|t|^{2H}$.

2. $\{B^H(t), t \in \mathbb{R}\}$ *has stationary increments, i.e., for* $\forall s, t > 0$, $B^H(t+s) - B^H(t) \overset{d}{=} B^H(s)$.

3. $\{B^H(t), t \in \mathbb{R}\}$ *has continuous sample path, but nowhere differentiable. In fact,*

$$\lim_{s \to t} \frac{\left|B^H(t) - B^H(s)\right|}{t-s} = \infty \quad a.s.$$

4. *The sample paths of* $\{B^H(t), t \in \mathbb{R}\}$ *are of unbounded variation, that is,*

$$\lim_{n \to \infty} \sum_{i=1}^{n-1} \left|B^H(t_{i+1}^n) - B^H(t_i^n)\right| = \infty \quad a.s.$$

   *where* $0 \leq t_1^n \leq t_2^n \leq ... \leq t_n^n \leq 1$. *If* $H \in (1/2, 1)$, *the quadratic variation is 0, that is,*

$$\lim_{n \to \infty} \sum_{i=1}^{n-1} \left|B^H(t_{i+1}^n) - B^H(t_i^n)\right|^2 = 0 \quad a.s.$$

   *For* $H \in (0, 1/2)$, *the quadratic variation is infinite.*

5. $\{B^H(t), t \in \mathbb{R}\}$ *is a self-similar process with parameter* $H$, *that is, the process* $\{B^H(\alpha t), t \geq 0\}$ *and* $\{\alpha^H B^H(t), t \geq 0\}$ *have the same probability law for all* $\alpha > 0$.

6. *The autocorrelation function* $r(k) = E[Y(1)Y(k)]$ *of a fractional Gaussian noise behaves as a power function for large* $k$, *that is,* $r(k) \sim H(2H-1)k^{2H-2}$ *as* $k \to \infty$.

For $H \in (1/2, 1)$, $\{B^H(t), t \geq 0\}$ has the property of long-range dependence in the sense that $\sum_{k=1}^{\infty} r(k) = \infty$ where $r(k)$ is the autocorrelation function of the corresponding fractional Gaussian noise $\{Y(k), k = 1, 2, ...\}$. Figure 1.1 shows sample paths of fractional Brownian motions and the corresponding fractional Gaussian noises for different Hurst parameters. It can be observed that the sample paths of a fractional Brownian motion with a large Hurst parameter, e.g, $H = 0.8$, are smoothier than those with a small Hurst parameter $H = 0.2$. The sample pahts are generated with the method proposed in [11], also see [4]. Many other methods for simulating fractional Brownian motions can be found in the literature, for example, a method based on fast Fourier transform [56], an approximation method based on aggregation of independent ON/OFF sources with heavy tailed ON and OFF periods [64], and a method based on random midpoint displacement method [42], [53], etc.

The distribution of the maximum of a fractional Brownian motion over an interval $[0, t]$, i.e., $\sup_{0\le s\le t} B^H(s)$, is unknown in general, except for the case $H = 1/2$. An upper and lower bounds of $E[\sup_{0\le s\le t} B^H(s)]$ for $H \in [1/2, 1)$ are derived in [54], which is cited as Theorem 1.3.1. Let $B(\mu, \eta)$ denote a beta function, that is,

$$B(\mu, \eta) = \int_0^1 x^{\mu-1}(1-x)^{\eta-1}dx = \frac{\Gamma(\mu)\Gamma(\eta)}{\Gamma(\mu+\eta)}, \tag{1.4}$$

where $\Gamma(\cdot)$ is a gamma function. Let $c_1$ be a constant such that

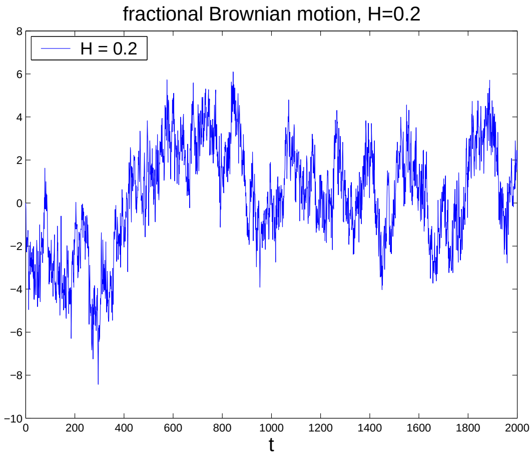
$$c_1 = \left[2H(2H-1)(2-2H)B\left(H - \frac{1}{2}, 2-2H\right)\right]^{-\frac{1}{2}}. \tag{1.5}$$

**Theorem 1.3.1.** *For any $p > 0$ and $H \in (1/2, 1)$,*

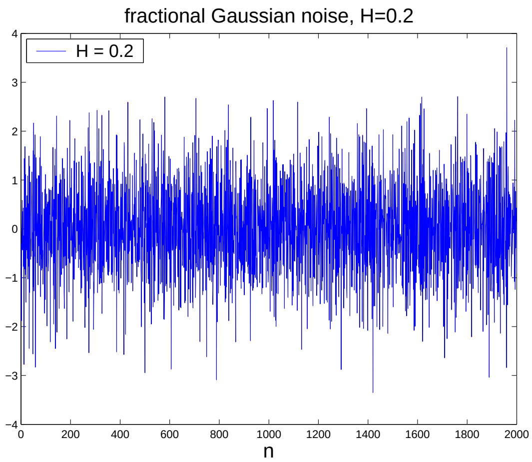
$$\gamma_{p,H} t^{pH} \le E\left[\left(\sup_{0\le s\le t} B^H(s)\right)^p\right] \le (8H)^p c_1^p C_p t^{pH}, \tag{1.6}$$

*where $\gamma_{p,H}$ is a constant depending on $p$ and $H$, $c_1$ is defined in (1.5) and $C_p$ is the constant in the Burkholder-Davis-Gundy inequality.*

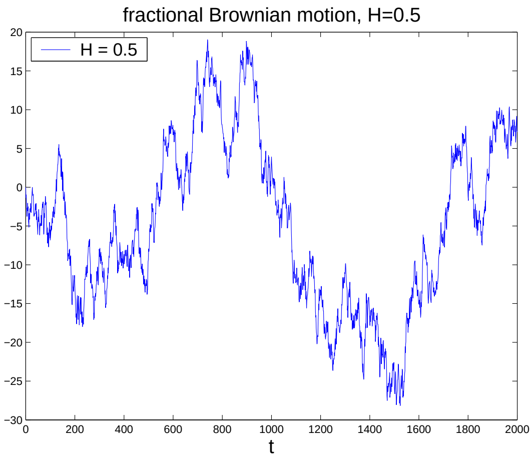
*Proof.* The proof can be found in [54]. □



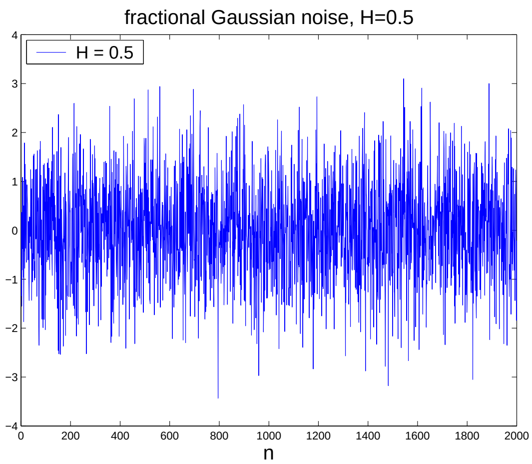

(a) fractional Brownian motion, $H = 0.2$



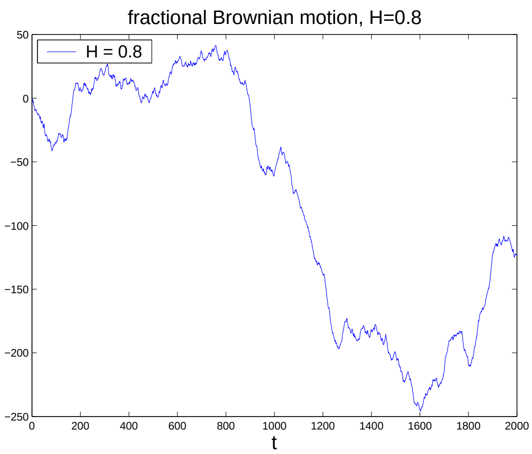

(b) fractional Gaussian noise, $H = 0.2$



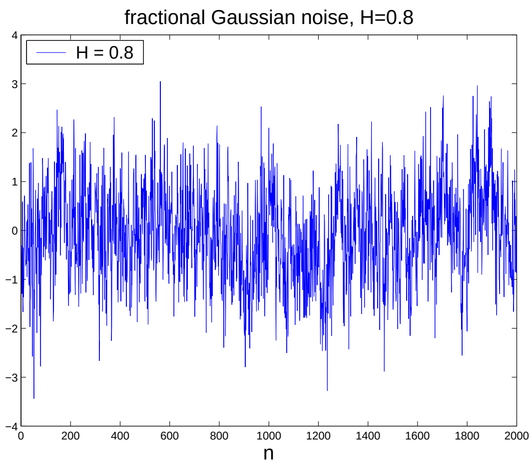

(c) fractional Brownian motion, $H = 0.5$




(d) fractional Gaussian noise, $H = 0.5$




(e) fractional Brownian motion, $H = 0.8$




(f) fractional Gaussian noise, $H = 0.8$

Figure 1.1: Sample paths of fractional Brownian motions and fractional Gaussian noises with different Hurst parameters

This section is concluded with a Girsanov formula for a fractional Brownian motion. The Girsanov formula for a Brownian motion is widely used in mathematical finance, control theory and telecommunications. A similar Girsanov formula is also obtained for a fractional Brownian motion. The following results are cited from [51]. Define a process $\{M_t, t \geq 0\}$ as

$$M_t = c_M \int_0^t s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H} dB^H(s),$$

where $c_M = \left[H(2H-1)B\left(\frac{3}{2}-H, H-\frac{1}{2}\right)\right]^{-1}$, and $B(\cdot,\cdot)$ is a beta function defined in (1.4). It can be verified that the process $\{M_t, t \geq 0\}$ is a centered Gaussian process, a martingale, and has independent increments. Its variance function is given by

$$E[M_t^2] = 2c_1^2 t^{2-2H},$$

where $c_1$ is defined in (1.5). Let

$$G_t(M) = \exp\left(M_t - c_1^2 t^{2-2H}\right),$$

then $G_t(M)$ is a martingale with expectation 1. For $a \in \mathbb{R}$, let $P_a$ be a probability measure on $(\Omega, \mathscr{F}_t, P)$ defined as

$$\frac{dP_a}{dP} = G_t\left(aM\right).$$

**Theorem 1.3.2.** *With respect to the measure $P_a$, the process $B^H$ is a fractional Brownian motion with drift $a$, i.e., the distribution of $B^H$ with respect to $P_a$ is the same as the distribution of $B^H(t) + at$ with respect to $P = P_0$.*

*Proof.* The proof can be found in [51]. □

# Chapter 2

# Fractional Brownian Queueing Model

In this chapter a fractional Brownian queueing model is introduced, some results on this model in the literature are reviewed, and some new results on the maximum queue length are presented. The structure of this chapter is as follows: In Section 2.1 some preliminaries on a fractional Brownian queueing model are given. In Section 2.2 the maximum queue length over a time interval $[0, t]$ is discussed, some results in the literature on this maximum random variable are reviewed, and the main results of this chapter are given. Section 2.3 is devoted to the proof of the main result, Theorem 2.2.4.

## 2.1 Preliminary

A fluid queueing model with a fractional Brownian motion as input was proposed by Norros [49], Figure 4.2. A fractional Brownian motion $\{B^H(t), t \geq 0\}$, which is a self-similar Gaussian process with stationary increments, is used to capture the self-similarity and the long-range dependence in the input traffic. The queueing model is a FIFO queue with a fixed service rate. Let $A(t) = mt + \sigma B^H(t)$ be the cumulated arrivals up to time $t$, where $m$ is the mean input rate, $\sigma$ is a real coefficient, and $B^H(t)$ is a standard fractional Brownian motion with a Hurst parameter $H \in (0, 1)$. Let $\mu$ denote the service rate and $c = \mu - m$ be the surplus rate. For the stability of the queue, it is assumed that $c > 0$.


Figure 2.1: A queue with a fractional Brownian input, $A(t) = mt + \sigma B^H(t)$

### 2.1.1 Queue Length Process

Let $Q = \{Q(t), t \geq 0\}$ denote the queue length process. In the literature, the process $Q$ is also called a workload process, a storage process or a buffer content process. The next proposition gives an expression for $Q(t)$.

**Proposition 2.1.1.** *Let $Q(0) \geq 0$ denote the initial queue length. Then for $t \geq 0$, the queue length $Q(t)$ can be expressed as*

$$Q(t) = \sigma B^H(t) - ct + \max\left\{ \sup_{0 \leq s \leq t} \big(-\sigma B^H(s) + cs\big), Q(0) \right\}. \tag{2.1}$$

*In general, $Q(t)$ can be written in term of $Q(s)$, $0 \leq s \leq t$, as*

$$Q(t) = \sigma B^H(t) - ct + \max\left\{ \sup_{s \leq r \leq t} \big(-\sigma B^H(r) + cr\big), Q(s) - \big(\sigma B^H(s) - cs\big) \right\}.$$

*Proof.* The first part can be verified by the Skorokhod representation, which can be found in [36, page 210], see also [27, page 18-20], [38]. For the second part, since

$$\sup_{0 \leq r \leq t} \big(-\sigma B^H(r) + cr\big) = \max\left\{ \sup_{0 \leq r \leq s} \big(-\sigma B^H(r) + cr\big), \sup_{s \leq r \leq t} \big(-\sigma B^H(r) + cr\big) \right\},$$

it is obtained from (2.1) that

$$\begin{aligned} Q(t) = \sigma B^H(t) - ct - \big(\sigma B^H(s) - cs\big) + \big(\sigma B^H(s) - cs\big) + \\ \max\left\{ \max\left( \sup_{0 \leq r \leq s} \big(-\sigma B^H(r) + cr\big), Q(0) \right), \sup_{s \leq r \leq t} \big(-\sigma B^H(r) + cr\big) \right\}. \end{aligned}$$

Following (2.2), $Q(s) = \sigma B^H(s) - cs + \max\left\{\sup_{0\le r\le s}\left(-\sigma B^H(r) + cr\right), Q(0)\right\}$, so

$$\begin{aligned} Q(t) &= \sigma B^H(t) - ct - (\sigma B^H(s) - cs) \\ &\qquad + \max\left\{Q(s), \sigma B^H(s) - cs + \sup_{s\le r\le t}\left(-\sigma B^H(r) + cr\right)\right\} \\ &= \sigma B^H(t) - ct + \max\left\{Q(s) - (\sigma B^H(s) - cs), \sup_{s\le r\le t}\left(-\sigma B^H(r) + cr\right)\right\}. \end{aligned}$$

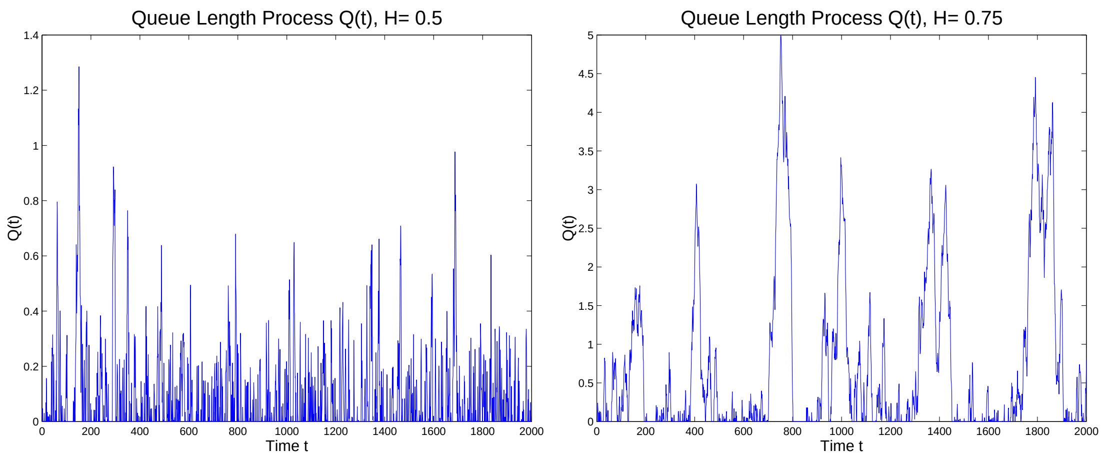
□

**Remark 2.1.1.** *Proposition 2.1.1 illustrates that for $s \le t$, the value of $Q(t)$ is determined by $Q(s)$ and $\{B^H(r), r \in [s,t]\}$. From (2.1), it can be verified that if $Q(0) = 0$, then the queue length $Q(t)$ for $t \ge 0$ can be expressed as,*

$$Q(t) = \sigma B^H(t) - ct + \sup_{0\le s\le t}\left(-\sigma B^H(s) + cs\right). \qquad (2.2)$$

*Throughout this chapter, $\{Q(t), t \ge 0\}$ denotes a queue length process which is initially empty, i.e., $Q(0) = 0$.*

Figure 2.2 shows sample paths of $\{Q(t), t \ge 0\}$ for $\sigma = 1$, $c = 1$ and different Hurst parameters.




(a) $H = 0.5$, $\sigma = 1$, $c = 1$ (b) $H = 0.75$, $\sigma = 1$, $c = 1$

Figure 2.2: Sample paths of queue length processes

The following lemma shows that $Q(t)$ equals $\sup_{0\le s\le t}\left(\sigma B^H(s) - cs\right)$ in distribution.

**Lemma 2.1.1.** *Let $Q(t)$ be given as in (2.2), then* $Q(t) \overset{d}{=} \sup_{0\le s\le t}\big(\sigma B^H(s) - cs\big)$.

*Proof.* From (2.2) and by substitution $r = t - s$, it follows that

$$\begin{aligned} Q(t) &= \sigma B^H(t) - ct + \sup_{0\le s\le t}\big(-\sigma B^H(s) + cs\big) \\ &= \sup_{0\le s\le t}\big(\sigma B^H(t) - \sigma B^H(s) - c(t-s)\big) \\ &= \sup_{0\le r\le t}\big(\sigma B^H(t) - \sigma B^H(t-r) - cr\big)\,. \end{aligned}$$

Since $B^H$ has stationary increments, $Q(t) \overset{d}{=} \sup_{0\le s\le t}\big(\sigma B^H(s) - cs\big)$. □

**Remark 2.1.2.** *For each $t$, $Q(t)$ and $\sup_{0\le s\le t}(\sigma B^H(s)-cs)$ are called the transient state queue length. Let $Q(\infty) \overset{d}{=} \lim_{t\to\infty}\sup_{0\le s\le t}\big(\sigma B^H(s) - cs\big)$. Since $\lim_{t\to\infty}\frac{B^H(t)}{t} = 0$ a.s.,*

$$Q(\infty) \overset{d}{=} \sup_{s\ge 0}\big(\sigma B^H(s) - cs\big) \tag{2.3}$$

*is a well-defined random variable and is called the steady state queue length.*

### 2.1.2 Transient State Queue Length

Since the fractional Brownian model was proposed, most research has focused on the properties of steady state queue length, i.e., $Q(\infty) \overset{d}{=} \sup_{s\ge 0}\big(\sigma B^H(s) - cs\big)$. It has been noted that the transient state queue length, $Q(t) \overset{d}{=} \sup_{0\le s\le t}(\sigma B^H(s)-cs)$ can also be applied to model some situations in network systems [19], [67]. The exact asymptotic distribution of the transient state queue length, i.e., $\lim_{b\to\infty} P(Q(t) > b)$, is discussed in [19]. Here an upper bound is derived for $E[Q(t)]$ based on Theorem 1.3.1 and the Girsanov formula, i.e. Theorem 1.3.2.

**Theorem 2.1.1.** *Let $Q(t) \overset{d}{=} \sup_{0\le s\le t}\big(\sigma B^H(s) - cs\big)$ be the transient state queue length. Let $E$ be the expectation w.r.t. the measure $P$ and $c_1$, $C_2$ be defined in*

*Theorem 1.3.1, then*

$$E\left[Q(t)\right] \le 8Hc_1\sqrt{C_2}\sigma t^H \exp\left(\frac{c_1^2 c^2 t^{2-2H}}{2\sigma^2}\right).$$

*Proof.* Since $Q(t) \stackrel{d}{=} \sup_{0\le s\le t}(\sigma B^H(s) - cs)$, it follows that $E[Q(t)] = \sigma E\left[\sup_{0\le s\le t}\left(B^H(s) - \frac{c}{\sigma}s\right)\right]$. Then from Theorem 1.3.2 and Theorem 1.3.1, it follows that

$$\begin{aligned}
E\left[\sup_{0\le s\le t}\left(B(s) - \frac{c}{\sigma}s\right)\right] &= E_{P_{\frac{c}{\sigma}}}\left[\sup_{0\le s\le t} B^H(s)\right] \\
&= E\left[G_t\left(\frac{c}{\sigma}M\right)\sup_{0\le s\le t} B^H(s)\right] \\
&\le \sqrt{E\left[G_t^2\left(\frac{c}{\sigma}M\right)\right]}\sqrt{E\left[\left(\sup_{0\le s\le t} B_s^H\right)^2\right]} \\
&\le \sqrt{\exp\left(\frac{c_1^2 c^2 t^{2-2H}}{\sigma^2}\right)}\sqrt{(8H)^2 c_1^2 C_2 t^{2H}} \\
&= 8Hc_1\sqrt{C_2}t^H \exp\left(\frac{c_1^2 c^2 t^{2-2H}}{2\sigma^2}\right).
\end{aligned}$$

□

## 2.2 Maximum Queue Length

Let $\{Q(t), t \ge 0\}$ denote the queue length process where the queue is fed with a fractional Brownian motion and let $M(t)$ be the maximum of the queue length in $[0, t]$, that is,

$$M(t) = \max_{0\le s\le t} Q(s). \tag{2.4}$$

The maximum random variable $M(t)$ has been analyzed for different queueing models [2], [9], [31], and is applied to estimate the overflow probability [25], [68]. In the context of renewal processes, e.g., a Brownian queueing model, some asymptotic properties on the maximum queue length are analyzed in [25]. In [68] the authors discussed the maximum queue length of a fractional Brownian model using the self-similarity of a fractional Brownian motion and the properties of Gaussian processes. In this chapter, this maximum random variable is revisited. Some results in [25] and [68] are extended.

To study the maximum of the queue length process, it is convenient to introduce a stationary version of the process $\{Q(t), t \geq 0\}$. Let $\{W^H(t), t \in \mathbb{R}\}$ be a fractional Brownian motion defined on the real line, that is, for $s, t \in \mathbb{R}$,

$$\begin{aligned} E[W^H(t)] &= 0, \\ E[W^H(s)W^H(t)] &= \frac{1}{2}\left[|s|^{2H} + |t|^{2H} - |s-t|^{2H}\right]. \end{aligned}$$

For a queue with an input $mt + \sigma W^H(t)$ and a service rate $\mu$, the queue length process $\{\tilde{Q}(t), t \in \mathbb{R}\}$ can be expressed as, see [37], [38],

$$\tilde{Q}(t) = \sigma W^H(t) - ct + \sup_{u \leq t}\left(-\sigma W^H(u) + cu\right),$$

where $c = \mu - m$. It can be verified that for all $t \in \mathbb{R}$, $\tilde{Q}(t) \stackrel{d}{=} \sup_{r \geq 0}\left(\sigma W^H(r) - cr\right)$. Given the value of $\tilde{Q}(0)$, $\tilde{Q}(t)$, $t \geq 0$, can be written as

$$\tilde{Q}(t) = \sigma W^H(t) - ct + \max\left\{\sup_{0 \leq u \leq t}\left(-\sigma W^H(u) + cu\right), \tilde{Q}(0)\right\}.$$

From the process $\{\tilde{Q}(t), t \in \mathbb{R}\}$, a stationary version of the queue length process $Q(t)$ can be obtained, cf. [37] [68], which is $Q^* = \{Q^*(t), t \geq 0\}$ such that

**(i)** $Q^*(t) \stackrel{d}{=} Q(\infty)$ for $t \geq 0$,

**(ii)** For $t \geq 0$,

$$Q^*(t) = \sigma B^H(t) - ct + \max\left\{Q^*(0), \sup_{0 \leq s \leq t}\left(-\sigma B^H(s) + cs\right)\right\}. \tag{2.5}$$

**Remark 2.2.1.** *Recall that* $Q(t) = \sigma B^H(t) - ct + \sup_{0\le s\le t}\big(-\sigma B^H(s) + cs\big)$*, it follows from (2.5) that for all* $t \ge 0$*,* $Q^*(t) \ge Q(t)$*.*

Let $M^*(t)$ be the maximum of the queue length process $Q^*$ over an interval $[0, t]$, that is,

$$M^*(t) = \max_{0\le s\le t} Q^*(s), \tag{2.6}$$

The main result on the two random variables, $M(t)$ and $M^*(t)$, is given in Theorem 2.2.4. Before the main result is presented, some properties of the fractional Brownian model and some results in the literature on these two maximum random variables are reviewed.

### 2.2.1 Results in the Literature

Since the fractional Brownian model was proposed by Norros [49], this model have been studied extensively. Many results can be found in the literature, such as, methods on parameter estimations [4], [41], asymptotic properties of the overflow probability [16], [29], [48], and limit results on the maximum queue length [30], [68]. Here some of these results, which will be used in the proofs, are reviewed.

For a fractional Brownian model, the overflow probability $P(Q(\infty) > b)$ is unknown in general, except for the case $H = 1/2$. In this case, $\{B^H(t), t \ge 0\}$ reduces to a standard Brownian motion $\{B(t), t \ge 0\}$ and the overflow probability is given by $P(Q(\infty) > b) = P(\sup_{t\ge 0} \sigma B(t) - ct > b) = e^{-2bc/\sigma^2}$. For $H \ne 1/2$, some asymptotic results are available. The following logarithmic asymptotic result is cited from [16].

**Proposition 2.2.1.** *For* $H \in (0, 1)$*, let* $Q(\infty)$ *be given in (2.3), then*

$$\lim_{b\to\infty} \frac{\log P\left(Q(\infty) > b\right)}{b^{2-2H}} = -\theta, \tag{2.7}$$

*where*

$$\theta = \frac{c^{2H}}{2\sigma^2 H^{2H}(1-H)^{2-2H}}. \tag{2.8}$$

**Remark 2.2.2.** *The above result is derived based on a large deviations approach. Similar results can also be found in [50], [51]. The exact asymptotic overflow probability, i.e.,* $\lim_{b\to\infty} P(Q(\infty) > b)$*, has been obtained with different methods in [29], [48].*

The maximum queue length of a Brownian model, i.e., $H = 1/2$, is discussed in [25]. Since a Brownian motion has independent increments, there exists a renewal structure in the queue length process. Based on renewal theory, the following result is obtained in [25].

**Theorem 2.2.1.** *For* $H = 1/2$*, let* $M^*(t)$ *and* $M(t)$ *be defined in (2.6) and (2.4), respectively. Then*

$$\lim_{t\to\infty} \frac{M^*(t)}{\log t} = \frac{1}{\theta} \quad a.s. \tag{2.9}$$

$$\lim_{t\to\infty} \frac{M(t)}{\log t} = \frac{1}{\theta} \quad a.s., \tag{2.10}$$

*and in* $L^p$ *for* $p \in [1,\infty)$ *where* $\theta$ *is given in (2.8).*

It is shown in [25] that the limit result (2.9) and (2.10) not only holds for a Brownian queueing model, but also holds for a general queueing model which has a renewal structure in the queue length process. However for a queue with a long-range dependent input, such as a fractional Brownian motion with $H \in (1/2, 1)$, it is unclear that there is any renewal structure in the queue length process. So the results in [25] cannot be applied. The maximum queue length with a fractional Brownian input is discussed in [68] and the following result is derived.

**Theorem 2.2.2.** *For* $H \in (1/2, 1)$*, let* $M^*(t)$ *and* $M(t)$ *be defined in (2.6) and*

*(2.4), respectively. Then*

$$\lim_{t\to\infty} \frac{M^*(t)}{(\log t)^\beta} = \left(\frac{1}{\theta}\right)^\beta, \tag{2.11}$$

$$\lim_{t\to\infty} \frac{M(t)}{(\log t)^\beta} = \left(\frac{1}{\theta}\right)^\beta \tag{2.12}$$

*in $L^p$ for $p \in [1, \infty)$ where $\theta$ is given in (2.8) and*

$$\beta = \frac{1}{2-2H}. \tag{2.13}$$

**Remark 2.2.3.** *The result implies that for an input $\{B^H(t), t \geq 0\}$ with $H \in (1/2, 1)$, the maximum queue length over $[0, t]$ grows as $(\log t)^\beta$. With a proper normalization, the maximum queue length $M(t)$ converges to a constant in $L^p$ for $p \in [1, \infty)$.*

**Remark 2.2.4.** *Note that for $H \in (0, 1)$, $\beta > 1/2$.*

In this chapter, it is shown that the above limits (2.11) and (2.12) also hold almost surely. To prove this main result, the following limit theorem from [30] is needed.

**Theorem 2.2.3.** *Let $M^*(t)$ be defined in (2.6). For $H \in (0, 1)$,*

$$\lim_{t\to\infty} P\left(M^*(t) \leq b(t) + xa(t)\right) = \exp\left(-e^{-x}\right), \tag{2.14}$$

*where*

$$b(t) = \frac{1}{\theta^\beta}(\log t)^\beta + \left[\frac{\left(\frac{1}{2H} - \beta^2\right)\log\left(\frac{\log t}{\theta}\right)}{\theta^\beta} + \frac{\beta \log c_2}{\theta^\beta}\right](\log t)^{-\beta(1-2H)}, \tag{2.15}$$

$$a(t) = \frac{\beta}{\theta^\beta}(\log t)^{-\beta(1-2H)}, \tag{2.16}$$

*and $c_2$ is a positive constant in terms of $c, H$, cf. [30].*

**Remark 2.2.5.** *Theorem 2.2.3 shows that as $t \to \infty$, $\frac{M^*(t)-b(t)}{a(t)} \Rightarrow X$ in which $X$ is a random variable with a Gumbel distribution, that is, $P(X \leq x) = \exp(-e^{-x})$.*

*Note that this result holds for $\{B^H(t), t \geq 0\}$ with $H \in (0,1)$. In particular, for $H = 1/2$, $b(t) = \frac{1}{\theta}\log t + \frac{\log c_2}{\theta}$ and $a(t) = \frac{1}{\theta}$.*

### 2.2.2 Main Results

The main result of this chapter is presented in the following theorem.

**Theorem 2.2.4.** *For $H \in (0,1)$, let $M^*(t)$ and $M(t)$ be defined in (2.6) and (2.4), respectively. Then*

$$\lim_{t\to\infty} \frac{M^*(t)}{(\log t)^\beta} = \left(\frac{1}{\theta}\right)^\beta \quad a.s., \tag{2.17}$$

$$\lim_{t\to\infty} \frac{M(t)}{(\log t)^\beta} = \left(\frac{1}{\theta}\right)^\beta \quad a.s \tag{2.18}$$

*and in $L^p$ for $p \in [1,\infty)$ where $\theta$ and $\beta$ are given in (2.8) and (2.13), respectively.*

*Proof.* The proof is given in Section 2.3. □

**Remark 2.2.6.** *Theorem 2.2.4 extends the results in Theorem 2.2.1 and 2.2.2 in two directions: (i) the Hurst parameter $H$ can take any value in $(0,1)$; (ii) the limit results of (2.11) and (2.12) hold almost surely.*

Let $\tau_b$ be the first time that the queue length process $Q$ reaches a level $b$, that is,

$$\tau_b = \inf\{t : Q(t) \geq b\}. \tag{2.19}$$

It follows from this definition that $\{\tau_b \leq t\} = \{M(t) \geq b\}$.

**Corollary 2.2.1.** *Let $\tau_b$ be defined in (2.19), then*

$$\lim_{b\to\infty} \frac{\log \tau_b}{b^{2-2H}} = \theta \quad a.s. \tag{2.20}$$

*where $\theta$ is given in (2.8).*

*Proof.* From Theorem 2.2.4, it follows that for $\varepsilon > 0$, there exists $t_1 = t_1(\varepsilon)$ such that for $t \geq t_1$,

$$M(t) < \left(\frac{1+\varepsilon}{\theta}\log t\right)^{\beta} \quad a.s.$$

Since as $b \to \infty$, $\tau_b \to \infty$ $a.s.$, for $\tau_b > t_1$, $b \leq M(\tau_b) < \left(\frac{1+\varepsilon}{\theta}\log \tau_b\right)^{\beta}$. So for sufficiently large $b$,

$$\frac{\log \tau_b}{b^{2-2H}} > \frac{\theta}{1+\varepsilon} \quad a.s.$$

Similarly there exists $t_2 = t_2(\varepsilon)$ such that for $t \geq t_2$, $M(t) > \left(\frac{1-\varepsilon}{\theta}\log t\right)^{\beta}$ $a.s.$ Then $\frac{\log \tau_b}{b^{2-2H}} < \frac{\theta}{1-\varepsilon}$ $a.s.$ for sufficiently large $b$. Since $\varepsilon > 0$ is arbitrary, the corollary follows. □

**Theorem 2.2.5.** *Let $M^*(t)$ be defined in (2.6). Then for $H \neq 1/2$, as $t \to \infty$,*

$$\frac{M^*(t)}{(\log t)^{\beta}} = \left(\frac{1}{\theta}\right)^{\beta} + O\left(\frac{\log(\log t)}{\log t}\right) \tag{2.21}$$

*where $\theta$ is given in (2.8) and $\beta$ is given in (2.13). For $H = 1/2$, as $t \to \infty$,*

$$\frac{M^*(t)}{\log t} = \frac{1}{\theta} + O\left(\frac{1}{\log t}\right). \tag{2.22}$$

*Proof.* From Theorem 2.2.3, it follows that as $t \to \infty$,

$$\frac{M^*(t) - b(t)}{a(t)} \Rightarrow X,$$

where $X$ is a random variable with a Gumbel distribution, i.e. $P(X \leq x) = \exp\left(-e^{-x}\right)$ for $x \in \mathbb{R}$. Substituting $a(t)$ and $b(t)$, after simplification, it is obtained that as $t \to \infty$,

$$\frac{\theta^\beta}{\beta}\log t\left(\frac{M^*(t)}{(\log t)^\beta}-\frac{1}{\theta^\beta}-\frac{\left(\frac{1}{2H}-\beta^2\right)\log(\log t)}{\theta^\beta\log t}-\frac{\xi}{\log t}\right)\Rightarrow X,$$

where $\xi$ is a constant in terms of $c$, $\sigma$ and $H$. The result (2.21) follows from Theorem 2.2.4. Similarly, (2.22) can be derived. □

**Remark 2.2.7.** *Theorem 2.2.5 gives the convergence rate at which $M^*(t)/(\log t)^\beta$ converges to $1/\theta^\beta$. For $H = 1/2$, the result coincides with [25, Theorem 3.1].*

As discussed in [25], [68], [69], the maximum queue length $M(t)$ can be applied to estimate the overflow probability $P(Q(\infty) > b)$, which is important for the admission control in network systems. Recall that by Proposition 2.2.1, the logarithmic asymptotic overflow probability is essentially determined by $H$ and $\theta$. Assume that the value of $H$ is known (there are many methods for estimating this parameter), following Theorem 2.2.4, $\theta$ can be consistently estimated by using the maximum queue length $M(t)$, that is, $\lim_{t\to\infty}\frac{M(t)^{2-2H}}{\log t}=\frac{1}{\theta}$ *a.s.*

## 2.3 Proof of the Main Results

In this section, the almost sure convergence of Theorem 2.2.4 is proved. The $L^p$ convergence will be shown in the next chapter. It will be demonstrated that besides a fractional Brownian input, the $L^p$ convergence holds for a queue with a general Gaussian input, see Theorem 3.2.1 and Section 3.3.

An upper bound and a lower bound are proved in Proposition 2.3.1 and 2.3.2, respectively, from which the almost sure convergence is concluded. Similar to the arguments in [68], the limit result is first shown for $M^*(t)$, then the proof is extended to $M(t)$.

**Proposition 2.3.1.** *Let $M^*(t)$ be defined in (2.6). Let $\theta$ and $\beta$ be given in (2.8)* *and (2.13), respectively. Then*

$$\limsup_{t\to\infty} \frac{M^*(t)}{(\log t)^\beta} \leq \left(\frac{1}{\theta}\right)^\beta \quad a.s. \tag{2.23}$$

*Proof.* Recall that $M^*(t) = \sup_{0\leq s\leq t} Q^*(s)$, then

$$M^*(t) \leq \max_{0\leq n\leq \lfloor t\rfloor} \left( \sup_{n\leq s\leq n+1} Q^*(s), \right). \tag{2.24}$$

For $s \in [n, n+1]$,

$$\begin{aligned} Q^*(s) =& \max(\sigma B^H(s) - cs + \sup_{n\leq r\leq s} \left(-\sigma B^H(r) + cr\right), \\ & Q^*(n) + \sigma B^H(s) - cs - \left(\sigma B^H(n) - cn\right)), \end{aligned}$$

it is obtained that

$$\begin{aligned} &\sup_{n\leq s\leq n+1} Q^*(s) \\ &\leq Q^*(n) + \sup_{n\leq s\leq n+1} (\sigma B^H(s) - cs) + \sup_{n\leq s\leq n+1} (-\sigma B^H(s) + cs). \end{aligned} \tag{2.25}$$

Combining (2.24) and (2.25),

$$\begin{aligned} & M^*(t) \\ \leq & \max_{0\leq n\leq \lfloor t\rfloor} \left( Q^*(n) + \sup_{n\leq s\leq n+1} \left(\sigma B^H(s) - cs\right) + \sup_{n\leq s\leq n+1} \left(-\sigma B^H(s) + cs\right) \right) \\ \leq & \max_{0\leq n\leq \lfloor t\rfloor} Q^*(n) + \max_{0\leq n\leq \lfloor t\rfloor} \left( \sup_{n\leq s\leq n+1} \sigma B^H(s) - cn + \sup_{n\leq s\leq n+1} \left(-\sigma B^H(s)\right) + c(n+1) \right) \\ \leq & \max_{0\leq n\leq \lfloor t\rfloor} Q^*(n) + \max_{0\leq n\leq \lfloor t\rfloor} \left( \sup_{n\leq s\leq n+1} \sigma B^H(s) - \sigma B^H(n) \right) \\ & \quad + \max_{0\leq n\leq \lfloor t\rfloor} \left( \sup_{n\leq s\leq n+1} \left(-\sigma B^H(s) + \sigma B^H(n)\right) \right) + c. \end{aligned} \tag{2.26}$$

From (2.26), it follows that

$$\frac{M^*(t)}{(\log t)^\beta} \le \frac{\max_{0\le n\le \lfloor t\rfloor} Q^*(n)}{(\log\lfloor t\rfloor)^\beta} + \frac{\max_{0\le n\le \lfloor t\rfloor}\left(\sup_{n\le s\le n+1}\sigma B^H(s) - \sigma B^H(n)\right)}{(\log\lfloor t\rfloor)^\beta}$$
$$+ \frac{\max_{0\le n\le \lfloor t\rfloor}\left(\sup_{n\le s\le n+1}\left(-\sigma B^H(s) + \sigma B^H(n)\right)\right)}{(\log\lfloor t\rfloor)^\beta} + \frac{c}{(\log\lfloor t\rfloor)^\beta}. \quad (2.27)$$

It is claimed that

$$\limsup_{t\to\infty} \frac{\max_{0\le n\le \lfloor t\rfloor} Q^*(n)}{(\log\lfloor t\rfloor)^\beta} \le \left(\frac{1}{\theta}\right)^\beta \quad a.s., \quad (2.28)$$

and as $t \to \infty$,

$$\frac{\max_{0\le n\le \lfloor t\rfloor}\left(\sup_{n\le s\le n+1}\left(\sigma B^H(s) - \sigma B^H(n)\right)\right)}{(\log\lfloor t\rfloor)^\beta} \xrightarrow{a.s.} 0. \quad (2.29)$$

From (2.27) and the two claims (2.28), (2.29), the proposition (2.23) follows. □

To verify the claims (2.28) and (2.29), two technical lemmas are needed.

**Lemma 2.3.1.** *Let $\theta$ and $\beta$ be given in (2.8) and (2.13), respectively. Let $\delta \in (0,1)$ be fixed, then for sufficiently large $n$,*

$$\frac{Q^*(n)}{(\log n)^\beta} \le \left(\frac{1+\delta}{\theta}\right)^\beta \quad a.s. \quad (2.30)$$

*Proof.* By the Borel-Cantelli lemma, it suffices to prove that $\sum_{n=1}^\infty P\left(Q^*(n) \ge \left(\frac{1+\delta}{\theta}\log n\right)^\beta\right) < \infty$.

$$\sum_{n=1}^\infty P\left(Q^*(n) \ge \left(\frac{1+\delta}{\theta}\log n\right)^\beta\right) = \sum_{n=1}^\infty e^{\log P\left(Q^*(n)\ge\left(\frac{1+\delta}{\theta}\log n\right)^\beta\right)}$$
$$= \sum_{n=1}^\infty e^{\frac{1+\delta}{\theta}\log n \frac{\log P\left(Q^*(n)\ge\left(\frac{1+\delta}{\theta}\log n\right)^\beta\right)}{\frac{1+\delta}{\theta}\log n}}. \quad (2.31)$$

Recall that $Q^*(n) \stackrel{d}{=} Q(\infty)$ for all $n$. By Proposition 2.2.1, $\frac{\log P(Q^*(n)\ge\left(\frac{1+\delta}{\theta}\log n\right)^\beta)}{\frac{1+\delta}{\theta}\log n} \to$

$-\theta$ as $n \to \infty$. Choose $\varepsilon \in \left(0, \frac{\delta}{2(1+\delta)}\theta\right)$, there exists $N$ such that for $n \geq N$,

$$\frac{\log P\left(Q^*(n) \geq \left(\frac{1+\delta}{\theta}\log n\right)^\beta\right)}{\frac{1+\delta}{\theta}\log n} < -\theta + \varepsilon.$$

So from (2.31),

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left(Q^*(n) \geq \left(\frac{1+\delta}{\theta}\log n\right)^\beta\right) \\
&\leq \sum_{n=1}^{N} e^{\left(\frac{1+\delta}{\theta}\log n\right)\frac{\log P\left(Q^*(n)\geq\left(\frac{1+\delta}{\theta}\log n\right)^\beta\right)}{\frac{1+\delta}{\theta}\log n}} + \sum_{N+1}^{\infty} e^{\left(\frac{1+\delta}{\theta}\log n\right)(-\theta+\varepsilon)} \\
&\leq N + \sum_{N+1}^{\infty} e^{\left(\frac{1+\delta}{\theta}\log n\right)\left(-\theta+\frac{\delta}{2(1+\delta)}\theta\right)} \\
&= N + \sum_{N+1}^{\infty} e^{-\left(1+\frac{\delta}{2}\right)\log n} \quad < \infty.
\end{aligned}$$

□

Fix an $\omega \in \Omega$ for which (2.30) holds, then there exists $K(\omega)$ such that

$$\begin{aligned}
\frac{\max_{0\leq n\leq\lfloor t\rfloor} Q^*(n,\omega)}{(\log\lfloor t\rfloor)^\beta} &\leq \frac{\max_{0\leq n\leq K(\omega)} Q^*(n,\omega)}{(\log\lfloor t\rfloor)^\beta} + \max_{K(\omega)\leq n\leq\lfloor t\rfloor} \frac{Q^*(n,\omega)}{(\log n)^\beta} \\
&\leq \frac{\max_{0\leq n\leq K(\omega)} Q^*(n,\omega)}{(\log\lfloor t\rfloor)^\beta} + \left(\frac{1+\delta}{\theta}\right)^\beta.
\end{aligned}$$

Let $t \to \infty$ and $\delta$ be arbitrarily small, the claim (2.28) is proved. The claim (2.29) is shown in the following lemma.

**Lemma 2.3.2.** *Suppose that $\sigma > 0$ and $\{B^H(t), t \geq 0\}$ is a standard fractional Brownian motion with $H \in (0,1)$. Let $\beta$ be defined in (2.13). Then*

$$\lim_{t\to\infty} \frac{\max_{0\leq n\leq\lfloor t\rfloor}\left(\sup_{n\leq s\leq n+1}\sigma B^H(s) - \sigma B^H(n)\right)}{(\log\lfloor t\rfloor)^\beta} = 0 \quad a.s. \tag{2.32}$$

*Proof.* By [26, Theorem 3.1], it is sufficient to show that for any $\varepsilon > 0$,

$$\sum_{\lfloor t \rfloor = 0}^{\infty} P\left( \frac{\max_{0 \le n \le \lfloor t \rfloor} \left( \sup_{n \le s \le n+1} \sigma B^H(s) - \sigma B^H(n) \right)}{(\log \lfloor t \rfloor)^{\beta}} > \varepsilon \right) < \infty.$$

Since $\sup_{n \le s \le n+1} \sigma B^H(s) - \sigma B^H(n) \overset{d}{=} \sup_{0 \le s \le 1} \sigma B^H(s)$, it follows that

$$\begin{aligned}
&\sum_{\lfloor t \rfloor = 0}^{\infty} P\left( \frac{\max_{0 \le n \le \lfloor t \rfloor} \left( \sup_{n \le s \le n+1} \sigma B^H(s) - \sigma B^H(n) \right)}{(\log \lfloor t \rfloor)^{\beta}} > \varepsilon \right) \\
&\le \sum_{\lfloor t \rfloor = 0}^{\infty} (\lfloor t \rfloor + 1) P\left( \frac{\sup_{0 \le s \le 1} \sigma B^H(s)}{(\log \lfloor t \rfloor)^{\beta}} > \varepsilon \right) \\
&= \sum_{\lfloor t \rfloor = 0}^{\infty} (\lfloor t \rfloor + 1) P\left( \sup_{0 \le s \le 1} \sigma B^H(s) > \varepsilon (\log \lfloor t \rfloor)^{\beta} \right). \qquad (2.33)
\end{aligned}$$

From (3.5) and (2.33), it can be derived that

$$\begin{aligned}
&\sum_{\lfloor t \rfloor = 0}^{\infty} P\left( \frac{\max_{0 \le n \le \lfloor t \rfloor} \left( \sup_{n \le s \le n+1} \sigma B^H(s) - \sigma B^H(n) \right)}{(\log \lfloor t \rfloor)^{\beta}} > \varepsilon \right) \\
&\le \sum_{\lfloor t \rfloor = 0}^{\infty} (\lfloor t \rfloor + 1)\, 4 \exp\left( -\frac{C_{G,\gamma} \varepsilon^2}{\sigma^2} (\log \lfloor t \rfloor)^{2\beta} \right) \\
&= \sum_{\lfloor t \rfloor = 0}^{\infty} 4 \exp\left( -\frac{C_{G,\gamma} \varepsilon^2}{\sigma^2} (\log \lfloor t \rfloor)^{2\beta} + \log\left( \lfloor t \rfloor + 1 \right) \right). \qquad (2.34)
\end{aligned}$$

Since $\beta > 1/2$ by definition (2.13), there exists $M$ such that for all $\lfloor t \rfloor > M$,

$$-\frac{C_{G,\gamma} \varepsilon^2}{\sigma^2} (\log \lfloor t \rfloor)^{2\beta} + \log\left( \lfloor t \rfloor + 1 \right) \le -2\beta \log \lfloor t \rfloor.$$

From (2.34), it follows that

$$
\begin{aligned}
&\sum_{\lfloor t\rfloor=0}^{\infty} P\left(\frac{\max_{0\le n\le\lfloor t\rfloor}\left(\sup_{n\le s\le n+1}\sigma B^H(s)-\sigma B^H(n)\right)}{(\log\lfloor t\rfloor)^{\beta}}>\varepsilon\right)\\
&\le \sum_{\lfloor t\rfloor=0}^{M} 4\exp\left(-\frac{C_{G,\gamma}\varepsilon^2}{\sigma^2}(\log\lfloor t\rfloor)^{2\beta}+\log\left(\lfloor t\rfloor+1\right)\right)+\sum_{\lfloor t\rfloor=M}^{\infty} 4\exp\left(-2\beta\log\lfloor t\rfloor\right)\\
&\le \sum_{\lfloor t\rfloor=0}^{M} 4\exp\left(-\frac{C_{G,\gamma}\varepsilon^2}{\sigma^2}(\log\lfloor t\rfloor)^{2\beta}+\log\left(\lfloor t\rfloor+1\right)\right)+\sum_{\lfloor t\rfloor=M}^{\infty} 4\lfloor t\rfloor^{-2\beta}\\
&<\infty.
\end{aligned}
$$

□

**Proposition 2.3.2.** *Let $M^*(t)$ be defined in (2.6) and $\theta$, $\beta$ be given in (2.8) and (2.13), respectively. Then*

$$\liminf_{t\to\infty}\frac{M^*(t)}{(\log t)^{\beta}}\ge\left(\frac{1}{\theta}\right)^{\beta}\quad a.s. \tag{2.35}$$

*Proof.* From the definition of $M^*(t)$, i.e. (2.6), it can be observed that

$$\frac{M^*(t)}{(\log t)^{\beta}}\ge\frac{M^*(\lfloor t\rfloor)}{(\log(\lfloor t\rfloor+1))^{\beta}}=\frac{M^*(\lfloor t\rfloor)}{(\log\lfloor t\rfloor)^{\beta}}\frac{(\log\lfloor t\rfloor)^{\beta}}{(\log(\lfloor t\rfloor+1))^{\beta}}. \tag{2.36}$$

It is claimed that for almost all $\omega\in\Omega$, there exists $t_0(\omega)$ such that for $t\ge\lfloor t_0(\omega)\rfloor$,

$$\frac{M^*(\lfloor t\rfloor)}{(\log\lfloor t\rfloor)^{\beta}}\ge\left(\frac{1-\delta}{\theta}\right)^{\beta}. \tag{2.37}$$

Let $t\to\infty$ and $\delta$ be arbitrarily small, from (2.36) and (2.37), the proposition follows. □

The claim (2.37) is shown in the following lemma.

**Lemma 2.3.3.** *Let the conditions of Proposition 2.3.2 be satisfied. Let $\delta\in(0,1)$ be* *fixed. Then for almost every $\omega \in \Omega$, there exists $n_0(\omega)$ such that for $n \geq n_0(\omega)$,*

$$\frac{M^*(n,\omega)}{(\log n)^\beta} \geq \left(\frac{1-\delta}{\theta}\right)^\beta .$$

*Proof.* It is sufficient to check that $\sum_{n=1}^{\infty} P\left(M^*(n) \leq \left(\frac{1-\delta}{\theta}\log n\right)^\beta\right) < \infty$. For a fractional Brownian queueing model, it is known from [30, Equation (23)]that there exists $t_0 < \infty$ such that for $t \geq t_0$,

$$P\left(M^*(t) \leq u(t)\right) \leq \exp\left(-\frac{c_2 t\,(u(t))^h}{2} e^{-\theta(u(t))^{2-2H}}\right), \tag{2.38}$$

where $u(t)$ is a function in terms of $t$, $h = \frac{2(1-H)^2}{H} - 1$ and $c_2$ is a positive constant in terms of $c, H$. Then from (2.38), for the fixed $\delta$ and a sufficiently large $n$, that is, for all $n \geq \lfloor t_0 \rfloor + 1$,

$$P\left(M^*(n) \leq \left(\frac{1-\delta}{\theta}\log n\right)^\beta\right) \leq \exp\left(-\frac{c_2}{2}\left(\frac{1-\delta}{\theta}\right)^{\beta h} n^\delta (\log n)^{\beta h}\right). \tag{2.39}$$

Thus it can be obtained that

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left(M^*(n) \leq \left(\frac{1-\delta}{\theta}\log n\right)^\beta\right)\\
&\leq (\lfloor t_0 \rfloor + 1) + \sum_{n=\lfloor t_0 \rfloor+1}^{\infty} P\left(M^*(n) \leq \left(\frac{1-\delta}{\theta}\log n\right)^\beta\right)\\
&\leq (\lfloor t_0 \rfloor + 1) + \sum_{n=\lfloor t_0 \rfloor+1}^{\infty} \exp\left(-\frac{c_2}{2}\left(\frac{1-\delta}{\theta}\right)^{\beta h} n^\delta (\log n)^{\beta h}\right) < \infty.
\end{aligned}$$

□

*Proof of Theorem 2.2.4.* The result (2.17) follows from Proposition 2.3.1 and 2.3.2. In the following, the proof is extended to $M(t)$, i.e. (2.18). For the upper bound, recall that for all $t \geq 0$, $Q(t) \leq Q^*(t)$, which implies that $M(t) \leq M^*(t)$ for $t \geq 0$, then

$$\limsup_{t\to\infty} \frac{M(t)}{(\log t)^\beta} \le \left(\frac{1}{\theta}\right)^\beta \quad a.s. \tag{2.40}$$

For the lower bound, rewrite $M^*(t)$ as

$$\begin{aligned} M^*(t) &= \max_{0\le s\le t} Q^*(s) \\ &= \max_{0\le s\le t} \left\{ \max\left(Q^*(0) + \sigma B^H(s) - cs, Q(s)\right)\right\} \\ &= \max\left( \max_{0\le s\le t} \left(Q^*(0) + \sigma B^H(s) - cs\right), M(t)\right). \end{aligned}$$

It follows that

$$\begin{aligned} M(t) &\ge M^*(t) - Q^*(0) - \max_{0\le s\le t} \left(\sigma B^H(s) - cs\right) \\ &\ge M^*(t) - Q^*(0) - \max_{s\ge 0} \left(\sigma B^H(s) - cs\right). \end{aligned}$$

Since $Q^*(0) < \infty$ a.s and $\max_{s\ge 0}\left(\sigma B^H(s) - cs\right) < \infty$ a.s, it follows from (2.35) that

$$\liminf_{t\to\infty} \frac{M(t)}{(\log t)^\beta} \ge \left(\frac{1}{\theta}\right)^\beta \quad a.s. \tag{2.41}$$

Therefore from (2.40) and (2.41), (2.18) follows. □

# Chapter 3

## Maximum Queue Length for a Queue with a Gaussian Input

In this chapter, the properties of the maximum queue length for a general Gaussian queueing model are discussed. The input of a Gaussian queueing model can be a general Gaussian process, such as a heterogeneous fractional Brownian motion (an aggregation of independent fractional Brownian motions), or a Gaussian integrated process, which are popular Gaussian models for network traffic. A fractional Brownian queueing model discussed in the previous chapter is a special case of a general Gaussian queue. The assumptions for the input process are given in Section 3.1. The main results of this chapter are as follows:

1. For a Gaussian process, which satisfies the assumptions **A1-A4** given in Section 3.1, the asymptotic property of the maximum queue length is determined by the asymptotic variance of the Gaussian process, see Theorem 3.2.1;

2. The first hitting time that the queue length process reaches a high level $b$ is also determined by the asymptotic variance of the Gaussian input, see Corollary 3.2.2;

3. For a queue with a heterogeneous fractional Brownian input, the maximum queue length is dominated asymptotically by the fractional Brownian motions with the largest Hurst parameter.

The main results extend the findings in [68] in the following directions: (i) the results cover a queue with an input of a heterogeneous fractional Brownian motion, of which a fractional Brownian motion is a special case; (ii) for a fractional Brownian queueing model, the Hurst parameter of the input process can take any value in $(0, 1)$; (iii) the results can be applied to a queue with a Gaussian integrated process, cf. [12], [15].

## 3.1 Preliminary

In this section, some preliminaries of a general Gaussian queueing model and the assumptions on the input process are introduced.

### 3.1.1 Fluid Queueing Model

A general Gaussian queueing model is a FIFO queue which has a fixed service rate and infinite buffer size. Let

$$A(t) = mt + Y(t) \tag{3.1}$$

denote the accumulated input to the queueing model up to time $t$, where $m > 0$ is the mean input rate, $Y = \{Y(t), t \geq 0\}$ is a continuous stochastic process. The service rate is denoted by $\mu$, which is a constant. Let $c = \mu - m$ be the surplus rate. For the stability of the queue, it is assumed that $c > 0$. Let $Q = \{Q(t), t \geq 0\}$ denote the queue length process.

**Proposition 3.1.1.** *Let $Q(0) \geq 0$ denote the initial queue length. Then for $t \geq 0$ the queue length $Q(t)$ can be expressed as*

$$Q(t) = Y(t) - ct + \max\left\{\sup_{0 \leq s \leq t} \left(-Y(s) + cs\right), Q(0)\right\}. \tag{3.2}$$

*In general, $Q(t)$ can be written in terms of $Q(s)$, $0 \leq s \leq t$, as*

$$Q(t) = Y(t) - ct + \max\left\{\sup_{s \leq r \leq t} \left(-Y(r) + cr\right), Q(s) - (Y(s) - cs)\right\}.$$

The proof is similar to Proposition 2.1.1 and is omitted. Proposition 3.1.1 illustrates that for $s \le t$, $Q(t)$ is determined by $Q(s)$ and $\{Y(r), r \in [s,t]\}$. From 3.2, it can be verified that if the queue is empty at time 0, i.e., $Q(0) = 0$, then for $t \ge 0$, $Q(t)$ can be expressed as,

$$Q(t) = Y(t) - ct + \sup_{0 \le s \le t} (-Y(s) + cs). \tag{3.3}$$

Throughout this chapter, let $\{Q(t), t \ge 0\}$ denote the queue length process that the queue is initially empty.

### 3.1.2 Queueing Model with a Gaussian Input

Consider a queue with an input $A$, given in (3.1), where the process $Y$ is Gaussian. Let $v^2(t) = Var(Y(t))$ denote the variance of $Y(t)$. Suppose that $Y$ satisfies the following assumptions:

**A1** The process $\{Y(t), t \ge 0\}$ is a centered Gaussian process with stationary increments, that is, for $t \ge 0$ and $r \ge 0$, $E[Y(t)] = 0$ and $Y(t+r) - Y(t) \stackrel{d}{=} Y(r) - Y(0)$.

**A2** The variance function $v^2(t)$ satisfies that $v^2(0) = 0$ and there exist constants $\sigma > 0$, $H \in (0,1)$ such that as $t \to \infty$,

$$v^2(t) \sim \sigma^2 t^{2H}, \tag{3.4}$$

that is, $v^2(t)$ is regularly varying at infinity with index $2H$. In other words, $\sigma^2 t^{2H}$ is the asymptotic variance of $Y(t)$. Note that since the Gaussian process $Y$ has stationary increments, its covariance is determined by its variance function $v^2(\cdot)$, that is, for $s, t \ge 0$,

$$E[Y(s)Y(t)] = \frac{1}{2}\left[v^2(s) + v^2(t) - v^2(|t-s|)\right].$$

**A3** For $t \in [0,1]$, there exist constants $G > 0$ and $\gamma \in (0,2]$ such that

$$v^2(t) \leq G t^{\gamma}.$$

Based on [43, Lemma12.2.1], there exists a constant $C_{G,\gamma} > 0$, which is only dependent on $G$ and $\gamma$, such that for all $x$,

$$P\left(\sup_{0\leq s\leq 1} Y(s) > x\right) \leq 4\exp\left(-C_{G,\gamma}x^2\right). \tag{3.5}$$

**A4** For $\Delta > 0$ and $k = 0, 1, ...$, let $\rho_{\Delta}(k)$ be defined as

$$\rho_{\Delta}(k) = \frac{E\left[Y(\Delta)\left(Y(k\Delta+\Delta) - Y(k\Delta)\right)\right]}{v^2(\Delta)}. \tag{3.6}$$

There exists a positive definite function $f(k)$ such that

(i) $f(0) = 1$

(ii) $f(k)\log(k) \to 0$ as $k \to \infty$

(iii) for sufficiently large $\Delta$, $\rho_{\Delta}(k) \leq f(k)$, that is, there exists $\Delta_0$ such that for all $\Delta \geq \Delta_0$, $\rho_{\Delta}(k) \leq f(k)$.

**Remark 3.1.1.** *In Section 3.3, it is shown that the assumptions **A1-A4** are general to cover most Gaussian processes applied to model network traffic, such as, a heterogeneous fractional Brownian motion and an integrated Ornstein-Uhlenbeck processes.*

The following lemma shows that under the assumption **A1**, $Q(t)$ has the same distribution as $\sup_{0\leq s\leq t}(Y(s) - cs)$.

**Lemma 3.1.1.** *Let $Q(t)$ be given as in (3.3). If the process $Y$ satisfies **A1**, then*

$$Q(t) \stackrel{d}{=} \sup_{0\leq s\leq t}\left(Y(s) - cs\right).$$

*Proof.* Let $r = t - s$. From (3.3) it follows that

$$\begin{aligned} Q(t) &= Y(t) - ct + \sup_{0 \le s \le t} (-Y(s) + cs) \\ &= \sup_{0 \le s \le t} (Y(t) - Y(s) - c(t-s)) \\ &= \sup_{0 \le r \le t} (Y(t) - Y(t-r) - cr) . \end{aligned}$$

Since $Y$ has stationary increments, $Q(t) \stackrel{d}{=} \sup_{0 \le s \le t} (Y(s) - cs)$. □

For each $t$, $Q(t)$ and $\sup_{0 \le s \le t}(Y(s) - cs)$ are called the transient state queue length. Let $Q(\infty) \stackrel{d}{=} \lim_{t\to\infty} \sup_{0 \le s \le t}(Y(s) - cs)$. It is verified in Lemma 3.5.1 that for $Y$ satisfying **A1-A3**, $\lim_{t\to\infty} \frac{Y(t)}{t} = 0$ *a.s.* So $Q(\infty) \stackrel{d}{=} \sup_{s \ge 0}(Y(s) - cs)$ is a well-defined random variable and is called the steady state queue length.

### 3.1.3 Stationary Version of $\{Q(t), t \ge 0\}$

To study the maximum of the queue length process, it is convenient to introduce a stationary version of the process $\{Q(t), t \ge 0\}$. Let $\{\tilde{Y}(t), t \in \mathbb{R}\}$ be a Gaussian process with stationary increments, for $s, t \in \mathbb{R}$,

$$\begin{aligned} E[\tilde{Y}(t)] &= 0, \\ E[\tilde{Y}(s)\tilde{Y}(t)] &= \frac{1}{2}\left[v^2(|s|) + v^2(|t|) - v^2(|s-t|)\right] , \end{aligned}$$

where $v^2(t)$ is the variance function of $Y(t)$ in **A2**. For a queue with an input $mt + \tilde{Y}(t)$ and a service rate $\mu$, the queue length process $\{\tilde{Q}(t), t \in \mathbb{R}\}$ can be expressed as, see [37], [38],

$$\tilde{Q}(t) = \tilde{Y}(t) - ct + \sup_{u \le t} \left(-\tilde{Y}(u) + cu\right) ,$$

where $c = \mu - m$. It can be verified that for $\forall t \in \mathbb{R}$, $\tilde{Q}(t) \stackrel{d}{=} \sup_{r \geq 0} \left( \tilde{Y}(r) - cr \right)$. Given the value of $\tilde{Q}(0)$, $\tilde{Q}(t)$, $t \geq 0$, can be written as

$$\tilde{Q}(t) = \tilde{Y}(t) - ct + \max \left\{ \sup_{0 \leq u \leq t} \left( -\tilde{Y}(u) + cu \right), \tilde{Q}(0) \right\}.$$

From the process $\{\tilde{Q}(t), t \in \mathbb{R}\}$, a stationary version of the queue length process $Q(t)$, denoted by $Q^* = \{Q^*(t), t \geq 0\}$, can be obtained, see [37] [68],

**(i)** $Q^*(t) \stackrel{d}{=} Q(\infty)$ for $t \geq 0$,

**(ii)** For $t \geq 0$,

$$Q^*(t) = Y(t) - ct + \max \left\{ \sup_{0 \leq s \leq t} \left( -Y(s) + cs \right), Q^*(0) \right\}. \tag{3.7}$$

## 3.2 Main Results

Let $M^*(t)$ be the maximum of the queue length process $Q^*$ over the interval $[0, t]$ and $M(t)$ the maximum of $Q$ over $[0, t]$, i.e.,

$$M^*(t) = \max_{0 \leq s \leq t} Q^*(s), \tag{3.8}$$

$$M(t) = \max_{0 \leq s \leq t} Q(s). \tag{3.9}$$

Let $\tau_b$ be the first passage time that the queue length process $Q$ reaches a level $b$, that is,

$$\tau_b = \inf\{t \geq 0 : Q(t) \geq b\}. \tag{3.10}$$

Note that $\{\tau_b \leq t\} = \{M(t) \geq b\}$. The main results are given in Theorem 3.2.1 and Corollary 3.2.2.

It is known that the properties of the maximum queue length are closely related to $Q(\infty)$, the asymptotic distribution of the steady state queue length, see [1], [25],

[43], [68]. The following proposition gives a property of the asymptotic distribution of $Q(\infty)$.

**Proposition 3.2.1.** *Let* $Q(\infty) \stackrel{d}{=} \sup_{s\geq 0}(Y(s)-cs)$ *be the steady state queue length. Suppose that the process* $Y$ *satisfies the assumptions* ***A1-A3****, then*

$$\lim_{b\to\infty} \frac{\log P(Q(\infty) > b)}{b^{2-2H}} = -\theta,$$

*where*

$$\theta = \frac{c^{2H}}{2\sigma^2 H^{2H}(1-H)^{2-2H}}, \tag{3.11}$$

$\sigma$ *and* $H$ *are the constants in* ***A2****.*

This proposition generalizes the result in Proposition 2.2.1 and shows that for sufficiently large $b$, the logarithm of the overflow probability, $\log P(Q(\infty) > b)$, is essentially determined by $\theta$ and $H$.

**Theorem 3.2.1.** *Let* $M^*(t)$, $M(t)$ *be defined in (3.8) and (3.9), respectively. Let* $Q^*(t)$ *and* $Q(t)$ *be defined in (3.7) and (3.3), respectively. Assume that* $\{Y(t), t \geq 0\}$ *satisfies the assumptions* ***A1-A4****. Then*

$$\lim_{t\to\infty} \frac{M^*(t)}{(\log t)^{\beta}} = \left(\frac{1}{\theta}\right)^{\beta},$$
$$\lim_{t\to\infty} \frac{M(t)}{(\log t)^{\beta}} = \left(\frac{1}{\theta}\right)^{\beta},$$

*in* $L^p$ *for* $p \in [1,\infty)$ *where* $\theta$ *is given in (3.11) and*

$$\beta = \frac{1}{2-2H}, \tag{3.12}$$

$H$ *is the constant in* ***A2****.*

It can be verified that a fractional Brownian motion satisfies the assumption **A1-**

**A4**, see Section 3.3. So Theorem 3.2.1 generalizes the similar result in [68], i.e., Theorem 2.2.2.

**Corollary 3.2.1.** *Assume that the constant $H$ in* ***A2*** *is known, then*

$$\lim_{t\to\infty} \frac{M(t)^{2-2H}}{\log t} = \frac{1}{\theta}$$

*in probability.*

This corollary verifies that the parameter $\theta$ can be consistently estimated with the maximum random variable $M(t)$, provided the value of $H$ is given.

**Corollary 3.2.2.** *Assume that the process $\{Y(t), t \geq 0\}$ satisfies the assumptions* ***A1-A4****. Let $\tau_b$ be the first passage time that given in (3.10). Then*

$$\lim_{b\to\infty} \frac{\log \tau_b}{b^{2-2H}} = \theta,$$

*in probability where $H$ is the constant in* ***A2*** *and $\theta$ is given in (3.11).*

The main result, Theorem 3.2.1, shows that for a queue with a Gaussian input, the asymptotic variance of the Gaussian process determines the asymptotic properties of the maximum random variables, $M^*(t)$ and $M(t)$. As an application, the result can be used to estimate the steady state overflow probability, i.e., $P(Q(\infty) > b)$, which is useful in admission control for high-speed network. From Proposition 3.2.1, the overflow probability is essentially determined by the constant $\theta$ and $H$. Suppose that the process $Y$ is observable and the value of $H$ can be estimated with methods in [4] and [41], then Corollary 3.2.1 asserts that $\theta$ can be consistently estimated. Therefore the overflow probability can be roughly estimated in logarithmic sense.

Comparing to the result in Chapter 2, it can be observed that for a fractional Brownian model, the normalized maximum queue length $M(t)/(\log t)^\beta$ converges to a constant both almost surely and in $L^p$. For a queue with a general Gaussian input, only $L^p$ convergence is proved. It is reasonable to expect that for a general

Gaussian queue the convergence should also hold almost surely. In fact, it can be shown that for a queue with a general Gaussian input $Y$ which satisfies **A1-A4**,

$$\limsup_{t\to\infty} \frac{M^*(t)}{(\log t)^\beta} \leq \left(\frac{1}{\theta}\right)^\beta \quad a.s.$$

This is a generalization of Proposition 2.3.1. To verify this result, simply change $\sigma B^H$ in the proof of Proposition 2.3.1 to $Y$, the other parts remain unchanged. In order to show almost sure convergence, future research is needed to obtain a generalization of Proposition 2.3.2.

## 3.3 Examples

### 3.3.1 A Queue with a Heterogeneous Fractional Brownian Input

Let $\{Y(t), t \geq 0\}$ be a heterogeneous fractional Brownian motion, that is,

$$Y(t) = \sum_{i=1}^{N} \sigma_i B^{H_i}(t) \tag{3.13}$$

where $\sigma_i$, $1 \leq i \leq N$, are variance coefficients and $\{B^{H_i}(t), t \geq 0\}$, $1 \leq i \leq N$, are independent standard fractional Brownian motions with Hurst parameters $H_i \in (0,1)$, respectively. Let $\mathcal{J} \subset \{1, ..., N\}$ be the set of all indices $j$ such that $H_j = \max_{1\leq i\leq N} \{H_i\}$ and

$$\sigma = \sqrt{\sum_{i\in\mathcal{J}} \sigma_i^2}. \tag{3.14}$$

Consider a queue with an input $mt + Y(t)$, where $m$ is the mean input rate, and a constant service rate $\mu$. From (3.3), the queue length process can be written as $Q(t) = Y(t) - cs + \sup_{0\leq s\leq t}(-Y(s) + cs)$, where $c = \mu - m$ is the surplus rate. Applying Theorem 3.2.1, the following result is obtained.

**Theorem 3.3.1.** *Let $Y$ be a heterogeneous fractional Brownian motion given as* *in (3.13). Let $Q(t)$ and $M(t)$ be defined as in (3.3) and (3.9), respectively. Let $H = \max_{1 \le i \le N} H_i$ and $\sigma$ be as in (3.14). Then*

$$\lim_{t \to \infty} \frac{M(t)}{(\log t)^{\beta}} = \left( \frac{1}{\theta} \right)^{\beta},$$

*in $L^p$ for $p \in [1, \infty)$ where $\beta$ and $\theta$ are given by (3.12) and (3.11), respectively.*

According to Theorem 3.3.1, for a queue with an aggregated fractional Brownian input, the asymptotic behavior of $M(t)$ only depends on the largest Hurst parameter. For example, suppose that $Y(t) = 0.99B^{H_1}(t) + 0.01B^{H_2}(t)$ where $\{B^{H_1}(t), t \ge 0\}$ and $\{B^{H_2}(t), t \ge 0\}$ are independent fractional Brownian motions with $H_1 = 0.55$ and $H_2 = 0.95$, respectively. Since the coefficient of $B^{H_1}$ is relatively large, when the transient behavior is considered, the component of $B^{H_1}$ dominates the queueing performance, that is, the component of $B^{H_2}$ can be ignored. However, when the asymptotic behavior is discussed, by Theorem 3.3.1, the maximum queue length will be dominated by the component of $B^{H_2}$. Therefore, even though the coefficient of $B^{H_2}$ is relatively small, when large time periods are considered, the component of $B^{H_2}$ is not negligible. In this example, since the coefficient of $B^{H_2}$ is small, the convergence of the maximum queue length is slow and may be difficult to observe from simulations.

It can be observed that Theorem 2.2.2 is a special case of Theorem 3.3.1 with $N = 1$. Thus the $L^p$ convergence of Theorem 2.2.4 is proved. To prove Theorem 3.3.1, it is sufficient to verify that the process $Y(t)$ satisfies the assumptions **A1-A4**. Since fractional Brownian motions are Gaussian processes with stationary increments, **A1** is satisfied. For **A2**, the variance of $Y(t)$ is $v^2(t) = \sum_{i=1}^{N} \sigma_i^2 t^{2H_i}$. Notice that $v^2(t) \sim \sigma^2 t^{2H}$ where $H = \max_{1 \le i \le N} H_i$ and $\sigma$ is given in (3.14), so **A2** is satisfied. Since for $t \in [0, 1]$,

$$v^2(t) = \sum_{i=1}^{N} \sigma_i^2 t^{2H_i} \le t^{2H_{min}} \sum_{i=1}^{N} \sigma_i^2,$$

where $H_{min} = \min_{1\le i\le N} H_i$, **A3** is satisfied for $G = \sum_{i=1}^N \sigma_i^2$ and $\gamma = 2H_{min}$. For **A4**, if $H \ge 1/2$, then for any $\Delta > 0$ and $k = 1, 2, ...$,

$$\begin{aligned}
\rho_\Delta(k) &= \frac{\sum_{i=1}^N \sigma_i^2 \Delta^{2H_i}\left[(k+1)^{2H_i} - 2k^{2H_i} + (k-1)^{2H_i}\right]}{2\sum_{i=1}^N \sigma_i^2 \Delta^{2H_i}} \\
&\le \frac{\sum_{i=1}^N \sigma_i^2 \Delta^{2H_i}\left[(k+1)^{2H} - 2k^{2H} + (k-1)^{2H}\right]}{2\sum_{i=1}^N \sigma_i^2 \Delta^{2H_i}} \\
&= \frac{1}{2}\left[(k+1)^{2H} - 2k^{2H} + (k-1)^{2H}\right] \\
&:= f(k).
\end{aligned}$$

Let $f(0) = 1$. Notice that for $H \ge 1/2$, the covariance of the stationary standard normal sequence $\{\tilde{Z}_n = B^H(n+1) - B^H(n), k = 0, 1, ...\}$ is $f(k)$.

If $H < 1/2$, then by Lemma 3.5.3,

$$\rho_\Delta(k) = \frac{\sum_{i=1}^N \sigma_i^2 \Delta^{2H_i}\left[(k+1)^{2H_i} - 2k^{2H_i} + (k-1)^{2H_i}\right]}{2\sum_{i=1}^N \sigma_i^2 \Delta^{2H_i}} < 0 := f(k).$$

Let $\tilde{Z}_n = B(n+1) - B(n)$ where $\{B(t), t \ge 0\}$ is a standard Brownian motion. It can be observed that the covariance of $\{\tilde{Z}_n, n = 0, 1, ...\}$ is given by $f(k)$. Thus for $H \in (0, 1)$, $f(k)\log k \to 0$ as $k \to \infty$. Therefore **A1-A4** are satisfied, following Theorem 3.2.1, Theorem 3.3.1 is proved.

### 3.3.2 A Queue with a Gaussian Integrated Input

In this subsection, a queue with a Gaussian integrated input is discussed. Let $\{Y(t), t \ge 0\}$ be a Gaussian integrated process, such that,

$$Y(t) = \int_0^t Z(s)ds \tag{3.15}$$

where $\{Z(t), t \geq 0\}$ is an Ornstein-Uhlenbeck process, that is, a centered stationary Gaussian process with covariance function

$$R(t) = r^2 \exp(-\alpha |t|) \tag{3.16}$$

for $r$, $\alpha > 0$. The process $Y$, defined in (3.15), has been used to model network traffic in [13], [15], [40].

**Theorem 3.3.2.** *Let $Y$ be a Gaussian integrated process given in (3.15). Let $Q$ be the queue length process and $M(t)$ be the maximum queue length in $[0, t]$, as defined in (3.2) and (3.9), respectively. Then*

$$\lim_{t\to\infty} \frac{M(t)}{\log t} = \frac{r^2}{c\alpha}$$

*in $L^p$ for $p \in [1, \infty)$.*

To show this theorem, it suffices to check that the process $Y$, defined in (3.15), satisfies the assumptions **A1-A4**. For **A1**, it is known that a Gaussian integrated process $Y$ is a centered Gaussian process with stationary increments [13], [15]. The variance of $Y(t)$ is

$$\begin{aligned} v^2(t) &= \int_0^t \int_0^t E\left[Z_s Z_u\right] dsdu = 2r^2 \int_0^t \int_u^t e^{-\alpha(s-u)} dsdu \\ &= \frac{2r^2}{\alpha}\left(t + \frac{1}{\alpha}e^{-\alpha t} - \frac{1}{\alpha}\right). \end{aligned} \tag{3.17}$$

From (3.17), as $t \to \infty$, it is obtained that $v^2(t) \sim \frac{2r^2}{\alpha}t$. So **A2** is satisfied for $\sigma^2 = \frac{2r^2}{\alpha}$ and $H = 1/2$. For $t \in [0, 1]$, it can be verified that

$$t + \frac{1}{\alpha}e^{-\alpha t} - \frac{1}{\alpha} \leq \frac{\alpha}{2}t^2.$$

So $v^2(t) \leq r^2t^2$ and **A3** holds. For **A4**, it is claimed that for sufficiently large $\Delta$,

$$\rho_\Delta(k) = \frac{E\left[Y(\Delta)\left(Y(k\Delta+\Delta) - Y(k\Delta)\right)\right]}{v^2(\Delta)} \leq e^{-k}. \tag{3.18}$$

Since $e^{-k}\log k \to 0$ as $k \to \infty$, **A4** holds. Theorem 3.3.2 follows from Theorem 3.2.1.

In the following, the claim (3.18) is verified. From definition,

$$\rho_\Delta(k) = \frac{v^2(k\Delta+\Delta) - 2v^2(k\Delta) + v^2(k\Delta-\Delta)}{v^2(\Delta)}. \tag{3.19}$$

Combining (3.17) and (3.19), after simplification, it is obtained that

$$\rho_\Delta(k) = \frac{2}{\alpha}\frac{e^{-\alpha k\Delta}\left(e^{\alpha\Delta} - 2 + e^{-\alpha\Delta}\right)}{\Delta + \frac{1}{\alpha}e^{-\alpha\Delta} - \frac{1}{\alpha}}. \tag{3.20}$$

Since for large $\Delta$, $-2 + e^{-\alpha\Delta} \leq 0$. From (3.20), it is obtained that

$$\rho_\Delta(k) \leq \frac{2}{\alpha}\frac{e^{-\alpha\Delta(k-1)}}{\Delta + \frac{1}{\alpha}e^{-\alpha\Delta} - \frac{1}{\alpha}} \leq \frac{2}{\alpha}\frac{e^{-\alpha\Delta(k-1)}}{\Delta - \frac{1}{\alpha}} = \frac{2e}{\alpha}\frac{e^{-\alpha(k-1)\left(\Delta - \frac{1}{\alpha}\right)}}{\Delta - \frac{1}{\alpha}}e^{-k}.$$

So for $k = 1, 2, ...$ and large $\Delta$, $\rho_\Delta(k) \leq e^{-k}$.

## 3.4 Proofs

Proposition 3.2.1 is proved by applying the result in [16], where the asymptotic probability, i.e., $\lim_{b\to\infty} b^{-\alpha}\log P(Q(\infty) > b)$ for some $\alpha > 0$, is derived based on a large deviations method. The proposition follows [16, Corollary 2.3] if the following hypotheses are verified.

**B1** There exist functions $a, w : [0, \infty) \to [0, \infty)$ which are increasing and have limits of infinity, such that for each $\xi \in R$, the function defined as a limit by

$$\lambda(\xi) = \lim_{t\to\infty} \frac{\log E\left[\exp\left(\xi \frac{w(t)}{a(t)}(Y(t) - ct)\right)\right]}{w(t)}$$

exists in $[-\infty, \infty]$;

**B2** There exists a $\xi > 0$ for which $\lambda(\xi) < 0$;

**B3** There exists an increasing function $h : [0, \infty) \to [0, \infty)$ such that the limit

$$g(\xi) = \lim_{t\to\infty} \frac{w\left(a^{-1}(t/\xi)\right)}{h(t)}$$

exists for each $\xi > 0$, where $a^{-1}(t) = \sup\{s \geq 0 : a(s) \leq t\}$;

**B4** Let $X_t = Y(t) - ct$. For $n \in \mathbb{Z}$, let

$$X_n^* = \sup_{0\leq r<1} X_{n+r}. \tag{3.21}$$

Then

$$\limsup_{n\to\infty} \frac{\log E\left[e^{\xi w(n)(X_n^* - X_n)/a(n)}\right]}{w(n)} = 0,$$

for all $\xi > 0$.

Define a function $\lambda^*(x)$ by

$$\lambda^*(x) = \sup_{\xi\in R} \left(\xi x - \lambda(\xi)\right), \tag{3.22}$$

which is called Fenchel-Legendre or Cramer transform of $\lambda$. It is known from [16, Corollary 2.3] that if $\lambda^*(x)$ is continuous, **B1-4** are satisfied, and in particular, **B1** is satisfied for $a(t) = t^\alpha$, $w(t) = t^\gamma$ where $\alpha, \gamma > 0$, then

$$\lim_{b\to\infty} b^{-\gamma/\alpha} \log P(Q(\infty) > b) = -\inf_{\xi>0} \xi^{-\gamma/\alpha}\lambda^*(\xi). \tag{3.23}$$

*Proof of Proposition 3.2.1.* In the following, it is verified that for properly chosen functions $a(t)$ and $w(t)$, the hypotheses **B1-4** are satisfied. Consequently the result (3.23) for a queue with a general Gaussian input can be derived. For **B1, B2**, let $w(t) = t^{2-2H}$ and $a(t) = t$, then

$$\begin{aligned}
\lambda(\xi) &= \lim_{t\to\infty} \frac{\log E\left[\exp\left(\xi \frac{w(t)(Y(t)-ct)}{a(t)}\right)\right]}{w(t)} \\
&= \lim_{t\to\infty} \frac{\log E\left[\exp\left(\xi t^{1-2H}(Y(t)-ct)\right)\right]}{t^{2-2H}} \\
&= \lim_{t\to\infty} \frac{\log\left(\exp\left(-\xi c t^{2-2H} + \frac{1}{2}\xi^2 t^{2-4H} v^2(t)\right)\right)}{t^{2-2H}} \\
&= \lim_{t\to\infty} \frac{-\xi c t^{2-2H} + \frac{1}{2}\xi^2 t^{2-4H} v^2(t)}{t^{2-2H}} \\
&= \frac{1}{2}\xi^2\sigma^2 - c\xi.
\end{aligned}$$

The last equality is obtained since $v(t) \sim \sigma t^H$ by assumption **A2**. Thus **B1** and **B2** are satisfied. To verify **B3**, since $a^{-1}(t) = \sup\{s \in [0,\infty); a(s) \le t\} = t$, let $h(t) = t^{2-2H}$, then

$$g(\xi) = \lim_{t\to\infty} \frac{w(t/\xi)}{h(t)} = \lim_{t\to\infty} \frac{(t/\xi)^{2-2H}}{t^{2-2H}} = \xi^{2H-2}.$$

For **B4**, from the definition of $X_n^*$, i.e. (3.21), and the stationarity of the increments of $Y$, it follows that

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{\log E\left[\exp\left(\xi n^{1-2H}\left(X_n^* - X_n\right)\right)\right]}{n^{2-2H}} \\
&= \limsup_{n\to\infty} \frac{\log E\left[\exp\left(\xi n^{1-2H} \sup_{0\le r\le 1}(Y(n+r) - c(n+r) - Y(n) + cn)\right)\right]}{n^{2-2H}} \\
&= \limsup_{n\to\infty} \frac{\log E\left[\exp\left(\xi n^{1-2H} \sup_{0\le r\le 1}(Y(n+r) - Y(n) - cr)\right)\right]}{n^{2-2H}} \\
&\le \limsup_{n\to\infty} \frac{\log E\left[\exp\left(\xi n^{1-2H} \sup_{0\le r\le 1}(Y(n+r) - Y(n))\right)\right]}{n^{2-2H}} \\
&= \limsup_{n\to\infty} \frac{\log E\left[\exp\left(\xi n^{1-2H} \sup_{0\le r\le 1} Y(r)\right)\right]}{n^{2-2H}}
\end{aligned}$$

Based on Lemma 3.5.2, the hypothesis **B4** is satisfied. Since

$$\lambda^*(x) = \sup_{\xi\in R}(\xi x - \lambda(\xi)) = \sup_{\xi\in R}\left(\xi x - \frac{1}{2}\xi^2\sigma^2 + \xi c\right) = \frac{(x+c)^2}{2\sigma^2},$$

the proposition follows from [16, Corollary 2.3]. □

*Proof of Theorem 3.2.1.* Following the arguments in [68], the result is first proved for $M^*(t)$, then it is extended to $M(t)$ naturally. The proof consists of three steps. The following results, (3.24) and (3.25), are proved in **Step I** and **II**, respectively. For a fixed $\delta \in (0,1)$,

$$\lim_{t\to\infty} P\left(M^*(t) \ge \left(\frac{1-\delta}{\theta}\log t\right)^{\beta}\right) = 1, \tag{3.24}$$

$$\lim_{t\to\infty} P\left(M^*(t) \ge \left(\frac{1+\delta}{\theta}\log t\right)^{\beta}\right) = 0. \tag{3.25}$$

Following (3.24) and (3.25), It can be concluded that $\lim_{t\to\infty} M^*(t)/(\log t)^\beta = \left(\frac{1}{\theta}\right)^\beta$ in probability. In **Step III**, the uniform integrability of $(M^*(t)/(\log t)^\beta)^p$ is proved for $p \in [1,\infty)$, which completes the proof of the theorem.

**Step I** In this step, (3.24) is verified. Let $\delta \in (0,1)$ be fixed. For brevity, let

$$\alpha(t) = \left(\frac{1-\delta}{\theta}\log t\right)^{\beta}. \tag{3.26}$$

Fix $\Delta \in (0,t)$, from the definition of $Q^*$, it follows that

$$\begin{aligned} Q^*(t) &\geq Y(t) - ct - \inf_{0\leq s\leq t}(Y(s) - cs) \\ &\geq Y(t) - ct - Y(t-\Delta) + c(t-\Delta) \\ &= Y(t) - Y(t-\Delta) - c\Delta. \end{aligned} \tag{3.27}$$

Consequently,

$$\begin{aligned} &P\left(M^*(t) \geq \alpha(t)\right) \\ &= P\left(\sup_{0\leq s\leq t} Q^*(s) \geq \alpha(t)\right) \\ &\geq P\left(\sup_{1\leq k\leq \lfloor t/\Delta\rfloor} Q^*(k\Delta) \geq \alpha(t)\right) \\ &\geq P\left(\sup_{1\leq k\leq \lfloor t/\Delta\rfloor} Y(k\Delta) - Y\left(k\Delta - \Delta\right) - c\Delta \geq \alpha(t)\right) \\ &= P\left(\sup_{1\leq k\leq \lfloor t/\Delta\rfloor} Y(k\Delta) - Y\left(k\Delta - \Delta\right) \geq \alpha(t) + c\Delta\right). \end{aligned} \tag{3.28}$$

For $k = 1, 2, ...,$ let

$$Z_k^{\Delta} = \frac{Y\left(k\Delta\right) - Y\left(k\Delta - \Delta\right)}{v(\Delta)}. \tag{3.29}$$

From (3.28), it can be obtained that

$$P\left(M^*(t) \geq \alpha(t)\right) \geq P\left(\sup_{1\leq k\leq \lfloor t/\Delta\rfloor} Z_k^{\Delta} \geq \frac{\alpha(t) + c\Delta}{v(\Delta)}\right). \tag{3.30}$$

Choose $\varepsilon \in (0, \delta]$ and let $\Delta$ be dependent on $t$ such that

$$\Delta_t = \left( \frac{2\sigma^2(1-\varepsilon)}{c^2} H^2 \log t \right)^{\beta}. \tag{3.31}$$

Then (3.30) can be written as

$$P\left(M^*(t) \geq \alpha(t)\right) \geq P\left( \sup_{1 \leq k \leq \lfloor t/\Delta_t \rfloor} Z_k^{\Delta_t} \geq \frac{\alpha(t) + c\Delta_t}{v(\Delta_t)} \right). \tag{3.32}$$

By **A4** there exists a function $f(k), k = 0, 1, ...$ such that for large $\Delta_t$ (large $t$), and $j = 1, 2, ...$

$$cov(Z_j^{\Delta_t}, Z_{j+k}^{\Delta_t}) = \rho_{\Delta_t}(k) \leq f(k).$$

Let $\{\tilde{Z}_k, k = 1, 2, ...\}$ be a stationary standard normal sequence such that the covariance of $\tilde{Z}_k$ is determined by $f(k)$. By the Slepian inequality, for $t$ sufficiently large, it follows from (3.32) that

$$P\left( \sup_{1 \leq k \leq \lfloor t/\Delta_t \rfloor} Z_k^{\Delta_t} \geq \frac{\alpha(t) + c\Delta_t}{v(\Delta_t)} \right) \geq P\left( \sup_{1 \leq k \leq \lfloor t/\Delta_t \rfloor} \tilde{Z}_k \geq \frac{\alpha(t) + c\Delta_t}{v(\Delta_t)} \right). \tag{3.33}$$

Note that (3.33) holds for all sufficiently large $\Delta_t$ (sufficiently large $t$), that is, there exists a $t_0$, (3.33) holds for all $t \geq t_0$.

Next it is claimed that for sufficiently large $t$,

$$\frac{\alpha(t) + c\Delta_t}{v(\Delta_t)} \leq \sqrt{2\left(1 - \frac{\varepsilon}{2}\right) \log t}. \tag{3.34}$$

Since $v(\Delta_t) \sim \sigma \Delta_t^H$ by **A2**, there exists $0 \leq \gamma < H$ such that for sufficiently large $t$,

$$v(\Delta_t) \geq \sigma \Delta_t^H - \Delta_t^\gamma.$$

So for all sufficiently large $t$, it can verified that

$$\frac{\alpha(t) + c\Delta_t}{v(\Delta_t)} \leq \frac{\alpha(t) + c\Delta_t}{\sigma\Delta_t^H - \Delta_t^\gamma} \leq \frac{\alpha(t) + c\Delta_t}{\sigma\Delta_t^H} + \frac{\frac{2}{\sigma}\left(\alpha(t) + c\Delta_t\right)\Delta_t^{\gamma - H}}{\sigma\Delta_t^H}.$$

From Lemma 3.5.5, it can be obtained that $\frac{\alpha(t)+c\Delta_t}{\sigma\Delta_t^H} \leq \sqrt{2(1-\varepsilon)\log t}$. Similarly the following can be verified. $\frac{\frac{2}{\sigma}(\alpha(t)+c\Delta_t)\Delta_t^{\gamma-H}}{\sigma\Delta_t^H} \leq A(\log t)^{(1/2+\beta(\gamma-H))^+}$ for some constant $A$ which is dependent on $c, \varepsilon, \sigma$. Since $\gamma < H$, for sufficiently large $t$,

$$\begin{aligned} &\frac{\alpha(t) + c\Delta_t}{\sigma\Delta_t^H} + \frac{2\frac{\xi}{\sigma}\left(\alpha(t) + c\Delta_t\right)\Delta_t^{\gamma-H}}{\sigma\Delta_t^H} \\ &\leq \sqrt{2(1-\varepsilon)\log t} + A(\log t)^{(1/2+\beta(\gamma-H))^+} \\ &\leq \sqrt{2\left(1 - \frac{\varepsilon}{2}\right)\log t}. \end{aligned}$$

So the claim (3.34) is proved.

Let $n = \lfloor \frac{t}{\Delta_t} \rfloor$ and $t_n = \left\{ t : \frac{t}{\Delta_t} = n \right\}$. Note that from the definition of $t_n$, it can verified that $\lfloor \frac{t}{\Delta_t} \rfloor = n$ if and only if $t_n \leq t < t_{n+1}$. Then for sufficiently large $t \in [t_n, t_{n+1})$, the following inequalities are obtained

$$\frac{\alpha(t) + c\Delta_t}{v(\Delta_t)} \leq \sqrt{2\left(1 - \frac{\varepsilon}{2}\right)\log t} \leq \sqrt{2\left(1 - \frac{\varepsilon}{2}\right)\log t_{n+1}}.$$

So from (3.33),

$$P\left(\sup_{1 \leq k \leq \lfloor t/\Delta_t \rfloor} \tilde{Z}_k \geq \frac{\alpha(t) + c\Delta_t}{v(\Delta_t)}\right) \geq P\left(\sup_{1 \leq k \leq n} \tilde{Z}_k \geq \sqrt{2\left(1 - \frac{\varepsilon}{2}\right)\log t_{n+1}}\right). \tag{3.35}$$

Theorem 3.5.1, i.e. [43, Theorem 4.3.3], is applied to show that the right hand side of (3.35) approaches to 1. Let $u_n$ be defined as

$$u_n = \sqrt{2\left(1 - \frac{\varepsilon}{2}\right) \log t_{n+1}}. \tag{3.36}$$

Following Theorem 3.5.1, to show (3.35) approaches 1, it is sufficient to show that $n(1-\Phi(u_n)) \to \infty$ as $n \to \infty$. Recall that for $x \geq 0$, $\bar{\Phi}(x) \geq \frac{x}{\sqrt{2\pi}(1+x^2)} e^{-x^2/2}$, then

$$n(1 - \Phi(u_n)) \geq n \frac{u_n}{\sqrt{2\pi}(1+u_n^2)} \exp\left(-\frac{u_n^2}{2}\right).$$

Since $u_n \to \infty$ as $n \to \infty$, there exists $n_0$ such that for all $n > n_0$, $u_n > 1$. For $n > n_0$,

$$n(1 - \Phi(u_n)) \geq n \frac{1}{2\sqrt{2\pi} u_n} \exp\left(-\frac{u_n^2}{2}\right). \tag{3.37}$$

From (3.36), it follows that $e^{-u_n^2/2} \geq t_{n+1}^{-1+\varepsilon/2}$. Thus, from (3.37),

$$\begin{aligned} n(1 - \Phi(u_n)) &\geq n \frac{1}{2\sqrt{2\pi} u_n} t_{n+1}^{-1+\varepsilon/2} \\ &= n \frac{1}{2\sqrt{2\pi} u_n} \left(\frac{t_{n+1}}{\Delta_{t_{n+1}}}\right)^{-1} \Delta_{t_{n+1}}^{-1} t_{n+1}^{\varepsilon/2} \\ &= \frac{n}{n+1} \frac{t_{n+1}^{\varepsilon/2}}{2\sqrt{2\pi} \Delta_{t_{n+1}} u_n}. \end{aligned}$$

From (3.31) and (3.36), it can be observed that $\Delta_{t_{n+1}} = C_1(\log t_{n+1})^\beta$ and $u_n = C_2(\log t_{n+1})^{1/2}$ for some positive constants $C_1$ and $C_2$, respectively. Then as $n \to \infty$,

$$\frac{n}{n+1} \frac{t_{n+1}^{\varepsilon/2}}{2\sqrt{2\pi} \Delta_{t_{n+1}} u_n} \sim \frac{t_{n+1}^{\varepsilon/2}}{2\sqrt{2\pi} C_1 (\log t_{n+1})^\beta C_2 (\log t_{n+1})^{1/2}} \to \infty.$$

Thus the expression (3.24) is verified.

**Step II** The expression (3.25) is verified in this step. Let $V_i = \sup_{i-1\le s<i} Q^*(s)$, then $M^*(t) \le \max_{1\le i\le t} V_i$, since $M^*(t) = \sup_{0\le s\le t} Q^*(s)$. By the stationarity of $Q^*$, it follows that

$$P\left(M^*(t) \ge \left(\frac{1+\delta}{\theta}\log t\right)^{\beta}\right) \le tP\left(V_1 \ge \left(\frac{1+\delta}{\theta}\log t\right)^{\beta}\right).$$

To verify (3.25), it is necessary to show that the right hand side of the above inequality approaches to 0, that is, $\lim_{t\to\infty} tP\left(V_1 \ge \left(\frac{1+\delta}{\theta}\log t\right)^{\beta}\right) = 0$. Since

$$\begin{aligned} V_1 &\le Q^*(0) + \sup_{0\le s\le 1}\left((Y(s)-cs) - \inf_{0\le r\le s}(Y(r)-cr)\right) \\ &\le Q^*(0) + \sup_{0\le s\le 1}(Y(s)-cs) - \inf_{0\le s\le 1}(Y(s)-cs), \end{aligned}$$

then

$$\begin{aligned} &P\left(V_1 \ge \left(\frac{1+\delta}{\theta}\log t\right)^{\beta}\right) \\ \le &P\left(Q^*(0) + \sup_{0\le s\le 1}(Y(s)-cs) - \inf_{0\le s\le 1}(Y(s)-cs) \ge \left(\frac{1+\delta}{\theta}\log t\right)^{\beta}\right). \end{aligned} \tag{3.38}$$

Since $(1+\delta)^{\beta} \ge (1+\delta/2)^{\beta} + \delta/10$, Lemma 3.5.4, for $\beta > 1/2$ and $0<\delta<1$, it is obtained from (3.38) that

$$\begin{aligned} &P\left(V_1 \ge \left(\frac{1+\delta}{\theta}\log t\right)^{\beta}\right) \\ \le\ &P\left(Q^*(0) \ge \left(\frac{1+\delta/2}{\theta}\log t\right)^{\beta}\right) + P\left(\sup_{0\le s\le 1}(Y(s)-cs) \ge \frac{\delta}{20}\left(\frac{\log t}{\theta}\right)^{\beta}\right) \\ &+ P\left(-\inf_{0\le s\le 1}(Y(s)-cs) \ge \frac{\delta}{20}\left(\frac{\log t}{\theta}\right)^{\beta}\right) \\ \le\ &P\left(Q^*(0) \ge \left(\frac{1+\delta/2}{\theta}\log t\right)^{\beta}\right) + 2P\left(\sup_{0\le s\le 1} Y(s) \ge \frac{\delta}{20}\left(\frac{\log t}{\theta}\right)^{\beta} - c\right). \end{aligned}$$

Let $L_1$ and $L_2$ represent the above two terms respectively, that is, $L_1 = P\left(Q^*(0) \geq \left(\frac{1+\delta/2}{\theta}\log t\right)^\beta\right)$ and $L_2 = P\left(\sup_{0\leq s\leq 1} Y(s) \geq \frac{\delta}{20}\left(\frac{\log t}{\theta}\right)^\beta - c\right)$. So it is necessary to show that $tL_i \to 0$ as $t \to \infty$ for $i = 1, 2$. From Lemma 3.5.6, it follows that $tL_2 \to 0$ as $t \to \infty$. To show $tL_1 \to 0$, it is equivalent to show that $\log t + \log P\left(Q^*(0) \geq \left(\frac{1+\delta/2}{\theta}\log t\right)^\beta\right) \to -\infty$. Following Proposition 3.2.1, it is obtained as $t \to \infty$,

$$\begin{aligned}
&\log t + \log P\left(Q^*(0) \geq \left(\frac{1+\delta/2}{\theta}\log t\right)^\beta\right)\\
&= \log t\left[1 + \frac{\log P\left(Q^*(0) \geq \left(\frac{1+\delta/2}{\theta}\log t\right)^\beta\right)}{\log t}\right]\\
&\sim \log t\left[1 + (-1 - \frac{\delta}{2})\right]\\
&\to -\infty.
\end{aligned}$$

**Step III** In this step, the uniform integrability of $\left(\frac{M^*(t)}{(\log t)^\beta}\right)^p$ is proved. It is sufficient to show that for $p \in (1, \infty)$

$$\sup_{t\geq e} E\left[\frac{M^*(t)}{(\log t)^\beta}\right]^p < \infty.$$

Let

$$K_0 = \inf\left\{x : \frac{\log P\left(Q^*(0) > x\right)}{x^{2-2H}} \leq -\frac{\theta}{2}\right\}. \tag{3.39}$$

Following Proposition 3.2.1, the constant $K_0$ is finite, that is, $K_0 < \infty$. Let

$y = x^{1/p}$, for $t \geq e$,

$$\begin{aligned}
E\left[\frac{M^*(t)}{(\log t)^\beta}\right]^p &= \int_0^\infty P\left(\left(\frac{M^*(t)}{(\log t)^\beta}\right)^p > x\right) dx \\
&= \int_0^\infty P\left(\frac{M^*(t)}{(\log t)^\beta} > y\right) py^{p-1} dy \\
&= \int_0^\infty P\left(M^*(t) > y(\log t)^\beta\right) py^{p-1} dy.
\end{aligned}$$

Let $K = \max\{K_0, (4/\theta)^\beta), 4\sigma_Y, 2(c + a_Y)\}$, then $K < \infty$ and

$$\begin{aligned}
E\left[\frac{M^*(t)}{(\log t)^\beta}\right]^p &= \int_0^{3K} P\left(M^*(t) > y(\log t)^\beta\right) py^{p-1} dy \\
&\quad + \int_{3K}^\infty P\left(M^*(t) > y(\log t)^\beta\right) py^{p-1} dy \\
&\leq (3K)^p + \int_{3K}^\infty P\left(M^*(t) > y(\log t)^\beta\right) py^{p-1} dy.
\end{aligned}$$

Let $L_3 = \int_{3K}^\infty P\left(M^*(t) > y(\log t)^\beta\right) py^{p-1} dy$. Recall that in **Step II**, for all $x > 0$, $P\left(M^*(t) > x\right) \leq tP\left(Q^*(0) + \max_{0\leq s\leq 1}(Y(s) - cs) - \min_{0\leq s\leq 1}(Y(s) - cs) > x\right)$. Then

$$\begin{aligned}
L_3 &\leq \int_{3K}^\infty tP\left(Q^*(0) + \max_{0\leq s\leq 1}(Y(s) - cs) - \min_{0\leq s\leq 1}(Y(s) - cs) > y(\log t)^\beta\right) py^{p-1} dy \\
&\leq \int_{3K}^\infty tP\left(Q^*(0) > \frac{y(\log t)^\beta}{3}\right) py^{p-1} dy \\
&\quad + \int_{3K}^\infty tP\left(\max_{0\leq s\leq 1}(Y(s) - cs) > \frac{y(\log t)^\beta}{3}\right) py^{p-1} dy \\
&\quad + \int_{3K}^\infty tP\left(-\min_{0\leq s\leq 1}(Y(s) - cs) > \frac{y(\log t)^\beta}{3}\right) py^{p-1} dy \\
&\leq \underbrace{\int_{3K}^\infty tP\left(Q^*(0) > \frac{y(\log t)^\beta}{3}\right) py^{p-1} dy}_{L_{3,1}} \\
&\quad + 2\underbrace{\int_{3K}^\infty tP\left(\max_{0\leq s\leq 1} Y(s) > \frac{y(\log t)^\beta}{3} - c\right) py^{p-1} dy}_{L_{3,2}}.
\end{aligned}$$

It is shown in Lemma 3.5.7 and 3.5.8 that $L_{3,1} < \infty$ and $L_{3,2} < \infty$ with the choice of $K$, respectively. Therefore it is obtained that $\sup_{t \geq e} E\left[\frac{M^*(t)}{(\log t)^\beta}\right]^p < \infty$.

Combining **Step I, II** and **III**, the proof for $M^*(t)$ is complete.

In the following, the result is extended to $M(t)$. Notice that from (3.7), it follows that for all $t \geq 0$, $Q(t) \leq Q^*(t)$. Consequently, $M(t) \leq M^*(t)$ for all $t \geq 0$. In **Step I**, replacing (3.27) with

$$Q(t) = Y(t) - ct - \inf_{0 \leq s \leq t} (Y(s) - cs),$$

the rest remains unchanged. For **Step II** and **III**, since $M(t) \leq M^*(t)$ for all $t \geq 0$, it is obtained that

$$P\left(M(t) \geq \left(\frac{1+\delta}{\theta} \log t\right)^\beta\right) \to 0,$$
$$\sup_{t \geq e} E\left[\frac{M(t)}{(\log t)^\beta}\right]^p < \infty.$$

Thus the proof is complete. □

*Proof of Corollary 3.2.2.* It is sufficient to show that for $\varepsilon > 0$, as $b \to \infty$,

$$P\left(\frac{\log \tau_b}{\theta b^{2-2H}} > 1 + \varepsilon\right) \to 0, \tag{3.40}$$

$$P\left(\frac{\log \tau_b}{\theta b^{2-2H}} < 1 - \varepsilon\right) \to 0. \tag{3.41}$$

The upper bound (3.41) can be verified as follows: let $\varepsilon < 1/2$ and $\delta = \varepsilon/(1-\varepsilon) < 1$, then

$$P\left(\frac{\log \tau_b}{\theta b^{2-2H}} < 1 - \varepsilon\right) = P\left(\log \tau_b < \frac{\theta b^{2-2H}}{1+\delta}\right) = P\left(\tau_b < e^{\frac{\theta b^{2-2H}}{1+\delta}}\right). \tag{3.42}$$

Let $t = e^{\frac{\theta b^{2-2H}}{1+\delta}}$, then $b = \left(\frac{(1+\delta)\log t}{\theta}\right)^\beta$. Since $\{M(t) \geq b\} = \{\tau_b \leq t\}$, from (3.42) and **Step I** in the proof of Theorem 3.2.1,

$$P\left(\frac{\log \tau_b}{\theta b^{2-2H}} < 1-\varepsilon\right) = P\left(M(t) > \left(\frac{(1+\delta)\log t}{\theta}\right)^{\beta}\right) \to 0.$$

Let $\delta = \frac{\varepsilon}{1+\varepsilon}$, the lower bound (3.40) can be verified similarly. □

### 3.5 Appendix

The following theorem is cited from [43, Theorem 4.3.3].

**Theorem 3.5.1.** *Let $\{Z_n\}$ be a standardized stationary normal sequence with covariance $\{\rho_n\}$ satisfying the condition $\rho_n \log n \to 0$. Let $\{u_n\}$ be a sequence of numbers. Then for $0 \le \tau \le \infty$, $P\left(\sup_{1\le k\le n} Z_k \le u_n\right) \to e^{-\tau}$ if and only if $n\left(1-\Phi(u_n)\right) \to \tau$.*

**Lemma 3.5.1.** *Suppose that a process $\{Y(t), t \ge 0\}$ satisfies the assumptions **A1-3**, then*

$$\lim_{t\to\infty} \frac{Y(t)}{t} = 0 \quad a.s.$$

*Proof.* First it is shown that $\lim_{n\to\infty} \frac{Y(n)}{n} = 0$ *a.s.*. By [26, Theorem 3.1], it is sufficient to show that for any $\varepsilon > 0$, $\sum_{n=1}^{\infty} P\left(\left|\frac{Y(n)}{n}\right| > \varepsilon\right) < \infty$.

$$\sum_{n=1}^{\infty} P\left(\left|\frac{Y(n)}{n}\right| > \varepsilon\right) = \sum_{n=1}^{\infty} P\left(|Y(n)| > n\varepsilon\right) = \sum_{n=1}^{\infty} P\left(\left|\frac{Y(n)}{v(n)}\right| > \frac{n\varepsilon}{v(n)}\right), \quad (3.43)$$

where $v^2(n)$ is the variance of $Y(n)$ by **A2**. Since $v(n) \sim \sigma n^H$, there exists $N < \infty$, such that $n > N$, $v(n) \le 2\sigma n^H$. From (3.43), it follows that

$$\begin{aligned}\sum_{n=1}^{\infty} P\left(\left|\frac{Y(n)}{v(n)}\right| > \frac{n\varepsilon}{v(n)}\right) &\le \sum_{n=1}^{N} P\left(\left|\frac{Y(n)}{v(n)}\right| > \frac{n\varepsilon}{v(n)}\right) + \sum_{n=N+1}^{\infty} P\left(\left|\frac{Y(n)}{v(n)}\right| > \frac{n\varepsilon}{2\sigma n^H}\right) \\ &\le \sum_{n=1}^{N} P\left(\left|\frac{Y(n)}{v(n)}\right| > \frac{n\varepsilon}{v(n)}\right) + \sum_{n=N+1}^{\infty} 2\bar{\Phi}\left(\frac{n^{1-H}\varepsilon}{2\sigma}\right) \\ &< \infty.\end{aligned}$$

Thus $\lim_{n\to\infty} \frac{Y(n)}{n} = 0\ a.s.$ Next step is to show that $\lim_{t\to\infty} \frac{Y(t)}{t} = 0\ a.s.$ For $t \geq 1$, it can be obtained that

$$\begin{aligned}\left|\frac{Y(t)}{t}\right| &\leq \frac{|Y(t) - Y(\lfloor t \rfloor)|}{\lfloor t \rfloor} + \frac{|Y(\lfloor t \rfloor)|}{\lfloor t \rfloor} \\ &\leq \frac{\sup_{0\leq r\leq 1} Y(\lfloor t \rfloor + r) - Y(\lfloor t \rfloor)}{\lfloor t \rfloor} - \frac{\inf_{0\leq r\leq 1} Y(\lfloor t \rfloor + r) - Y(\lfloor t \rfloor)}{\lfloor t \rfloor} + \frac{Y(\lfloor t \rfloor)}{\lfloor t \rfloor}.\end{aligned}$$

Let $n = \lfloor t \rfloor$, then

$$\left|\frac{Y(t)}{t}\right| \leq \frac{\sup_{0\leq r\leq 1} Y(n+r) - Y(n)}{n} - \frac{\inf_{0\leq r\leq 1} Y(n+r) - Y(n)}{n} + \frac{Y(n)}{n}.$$

Since $\lim_{n\to\infty} \frac{Y(n)}{n} = 0\ a.s.$, it suffices to show that

$$\lim_{n\to\infty} \frac{\sup_{0\leq r\leq 1} Y(n+r) - Y(n)}{n} = 0 \quad a.s., \tag{3.44}$$

$$\lim_{n\to\infty} -\frac{\inf_{0\leq r\leq 1} Y(n+r) - Y(n)}{n} = 0 \quad a.s. \tag{3.45}$$

For $\varepsilon > 0$, since $Y$ has stationary increments by **A1** and from (3.5), it follows that

$$\begin{aligned}\sum_{n=1}^{\infty} P\left(\sup_{0\leq r\leq 1} Y(n+r) - Y(n) > n\varepsilon\right) &= \sum_{n=1}^{\infty} P\left(\sup_{0\leq r\leq 1} Y(r) > n\varepsilon\right) \\ &\leq \sum_{n=1}^{\infty} 4\exp\left(-C_{G,\gamma}(n\varepsilon)^2\right) \\ &< \infty.\end{aligned}$$

So (3.44) is obtained. Similarly, (3.45) can be verified. Thus $\lim_{t\to\infty} \frac{Y(t)}{t} = 0$ a.s. □

**Lemma 3.5.2.** *Suppose that a process $\{Y(t), t \geq 0\}$ satisfies the assumptions **A1-3**, then for all $\xi > 0$,*

$$E\left[\exp\left(\xi \sup_{0\leq s\leq 1} Y(s)\right)\right] \leq 1 + \frac{4\sqrt{\pi}\xi}{\sqrt{C_{G,\gamma}}} \exp\left(\frac{\xi^2}{4C_{G,\gamma}}\right).$$

*Proof.* For all $\xi > 0$,

$$E\left[\exp\left(\xi \sup_{0\le s\le 1} Y(s)\right)\right] = \int_0^\infty P\left(\exp\left(\xi \sup_{0\le s\le 1} Y(s)\right) > x\right) dx$$
$$= \int_0^\infty P\left(\sup_{0\le s\le 1} Y(s) > \frac{\log x}{\xi}\right) dx$$
$$\le 1 + \int_1^\infty P\left(\sup_{0\le s\le 1} Y(s) > \frac{\log x}{\xi}\right) dx.$$

From (3.5), it follows that

$$E\left[\exp\left(\xi \sup_{0\le s\le 1} Y(s)\right)\right] \le 1 + \int_1^\infty 4\exp\left(-C_{G,\gamma}\frac{(\log x)^2}{\xi^2}\right) dx.$$

Let $y = \frac{\sqrt{C_{G,\gamma}}}{\xi}\log x$. By substitution,

$$E\left[\exp\left(\xi \sup_{0\le s\le 1} Y(s)\right)\right] \le 1 + \frac{4\xi}{\sqrt{C_{G,\gamma}}}\exp\left(\frac{\xi^2}{4C_{G,\gamma}}\right)\int_0^\infty \exp\left(-\left(y - \frac{\xi}{2\sqrt{C_{G,\gamma}}}\right)^2\right) dy.$$

Since

$$\int_0^\infty \exp\left(-\left(y - \frac{\xi}{2\sqrt{C_{G,\gamma}}}\right)^2\right) dy \le \int_{-\infty}^\infty e^{-y^2} dy = 2\int_0^\infty e^{-y^2} dy = \sqrt{\pi},$$

the proof is complete. □

**Lemma 3.5.3.** **(i)** *Let $H \in (0, 1/2)$, then for $k \ge 1$,*

$$(k+1)^{2H} - 2k^{2H} + (k-1)^{2H} \le 0.$$

**(ii)** *Let $H_1$, $H_2 \in [1/2, 1)$. Suppose that $H_1 \le H_2$, then for $k \ge 1$,*

$$(k+1)^{2H_1} - 2k^{2H_1} + (k-1)^{2H_1} \le (k+1)^{2H_2} - 2k^{2H_2} + (k-1)^{2H_2}.$$

*Proof.* For (i), let $f(k) = k^{2H}$. Observe that $f(k)$ is concave for $H \in (0, 1/2)$. The first part follows from $f(k+1) - f(k) \leq f(k) - f(k-1)$.

For (ii), let $g(H) = (k+1)^{2H} - 2k^{2H} + (k-1)^{2H}$ and $H \in [1/2, 1)$. It is sufficient to show that for $k \geq 1$, $g(H)$ increases with respect to $H$.

$$\begin{aligned} g'(H) &= 2(k+1)^{2H}\log(k+1) - 4k^{2H}\log(k) + 2(k-1)^{2H}\log(k-1) \\ &= 2\left[(k+1)^{2H}\log(k+1) - k^{2H}\log(k)\right] - 2\left[k^{2H}\log(k) - (k-1)^{2H}\log(k-1)\right]. \end{aligned}$$

Let $h(k) = k^{2H}\log(k)$. To show that $g'(H) \geq 0$, i.e. $g(H)$ increases, it is sufficient to show that $h(k)$ is a convex function. It can be obtained from the second derivative of $h(k)$.

$$\begin{aligned} h'(k) &= 2Hk^{2H-1}\log(k) + k^{2H-1}, \\ h''(k) &= 2H(2H-1)k^{2H-2}\log(k) + 2Hk^{2H-2} + (2H-1)k^{2H-2}. \end{aligned}$$

It can be seen that for $H \geq 1/2$, $h''(k) \geq 0$. So the lemma follows. □

**Lemma 3.5.4.** *For $\beta \geq 1/2$ and $\delta \in (0,1)$,*

$$(1+\delta)^\beta \geq \left(1 + \frac{\delta}{2}\right)^\beta + \frac{\delta}{10}.$$

*Proof.* Let $\delta > 0$ be fixed. Let $f_\delta(\beta) = (1+\delta)^\beta - \left(1+\frac{\delta}{2}\right)^\beta - \frac{\delta}{10}$. For $\beta \geq 1/2$, it can be derived that

$$f'_\delta(\beta) = \log(1+\delta)(1+\delta)^\beta - \log\left(1+\frac{\delta}{2}\right)\left(1+\frac{\delta}{2}\right)^\beta > 0.$$

So $f_\delta(\beta)$ increases with respect to $\beta$. It follows that $f_\delta(\beta) \geq (1+\delta)^{1/2} - \left(1+\frac{\delta}{2}\right)^{1/2} - \frac{\delta}{10}$ for $\beta \geq 1/2$. Let $g(\delta) = (1+\delta)^{1/2} - \left(1+\frac{\delta}{2}\right)^{1/2} - \frac{\delta}{10}$. Need to show that $g(\delta)$

increases with respect to $\delta$ for $\delta \in (0,1)$.

$$g'(\delta) = \frac{1}{2\sqrt{1+\delta}} - \frac{1}{4\sqrt{1+\frac{\delta}{2}}} - \frac{1}{10} \geq \frac{1}{2\sqrt{2}} - \frac{1}{4} - \frac{1}{10} > 0.$$

Since $g(0) = 0$, then $g(\delta) \geq 0$ for $\delta \in (0,1)$. □

**Lemma 3.5.5.** *Let $\varepsilon \in (0, \delta]$, then*

$$\frac{\alpha(t) + c\Delta_t}{\sigma \Delta_t^H} \leq \sqrt{2(1-\varepsilon)\log(t)}.$$

*Proof.* Substituting $\theta$ and $\Delta_t$, which are given in (3.11) and (3.31), respectively, from (3.36) and (3.26), it follows that

$$\begin{aligned}\frac{\alpha(t) + c\Delta_t}{\sigma \Delta_t^H} &= \frac{\frac{(1-\delta)^\beta}{\theta^\beta}(\log t)^\beta + c\Delta_t}{\sigma \Delta_t^H} \\ &= \frac{(1-\delta)^\beta 2^\beta H^{2\beta H}(1-H)(\log t)^\beta + 2^\beta (1-\varepsilon)^\beta H^{2\beta}(\log t)^\beta}{2^{\beta H}(1-\varepsilon)^{\beta H} H^{2\beta H}(\log t)^{\beta H}}.\end{aligned}$$

Since $1 - \delta \leq 1 - \varepsilon$ and the definition of $\beta$, (3.12),

$$\begin{aligned}\frac{\alpha(t) + c\Delta_t}{\sigma \Delta_t^H} &\leq \frac{(1-\varepsilon)^\beta 2^\beta H^{2\beta H}(1-H)(\log t)^\beta + 2^\beta (1-\varepsilon)^\beta H^{2\beta}(\log t)^\beta}{2^{\beta H}(1-\varepsilon)^{\beta H} H^{2\beta H}(\log t)^{\beta H}} \\ &= \sqrt{2(1-\varepsilon)\log t}.\end{aligned}$$

□

**Lemma 3.5.6.** *Let $\alpha > 0$, $\eta \geq 0$ be constants. Let $\beta$ be defined in (3.12). Suppose that $Y(t)$ satisfies **A1-3**, then* $\lim_{t\to\infty} tP\left(\sup_{0\leq s\leq 1} Y(s) \geq \alpha (\log t)^\beta - \eta\right) = 0$.

*Proof.* There exists a $t_0$ such that for $t \geq t_0$, $\alpha (\log t)^\beta - \eta \geq \frac{\alpha}{2} (\log t)^\beta$. So for $t \geq t_0$

and from (3.5), it follows that

$$\begin{aligned} tP\left(\sup_{0\le s\le 1} Y(s) \ge \alpha (\log t)^{\beta} - \eta\right) &\le tP\left(\sup_{0\le s\le 1} Y(s) \ge \frac{\alpha}{2} (\log t)^{\beta}\right) \\ &\le 4t \exp\left(-\frac{\alpha^2 C_{G,\gamma}}{4} (\log t)^{2\beta}\right) \\ &= 4 \exp\left(-\frac{\alpha^2 C_{G,\gamma}}{4} (\log t)^{2\beta} + \log t\right) \\ &= 4 \exp\left(-(\log t)\left(\frac{\alpha^2 C_{G,\gamma}}{4} (\log t)^{2\beta-1} - 1\right)\right). \end{aligned}$$

Since $\beta > 1/2$, the lemma follows. □

**Lemma 3.5.7.** *Let $\beta$ be the constant defined in (3.12). For $\forall t \ge e$ and $K = \max\{K_0, (4/\theta)^{\beta}\}$ where $K_0$ is defined in (3.39), then*

$$\int_{3K}^{\infty} ty^{p-1} P\left(Q^*(0) > \frac{y}{3}(\log t)^{\beta}\right) dy < \infty. \tag{3.46}$$

*Proof.* Rewrite $P\left(Q^*(0) > \frac{y}{3}(\log t)^{\beta}\right)$ as

$$P\left(Q^*(0) > \frac{y}{3}(\log t)^{\beta}\right) = \exp\left(\left(\frac{y}{3}\right)^{1/\beta} \log t \; \frac{\log P\left(Q^*(0) > (y/3)(\log t)^{\beta}\right)}{(y/3)^{1/\beta}\log t}\right).$$

Since $y/3 \ge K \ge K_0$ and $\log t \ge 1$, from (3.39), $\frac{\log P\left(Q^*(0) > (y/3)(\log t)^{\beta}\right)}{(y/3)^{1/\beta}\log t} \le -\theta/2$. Then

$$\begin{aligned} \int_{3K}^{\infty} ty^{p-1} P\left(Q^*(0) > \frac{y}{3}(\log t)^{\beta}\right) dy &\le \int_{3K}^{\infty} ty^{p-1} \exp\left(-\frac{\theta}{2}\left(\frac{y}{3}\right)^{1/\beta} \log t\right) dy \\ &= \int_{3K}^{\infty} y^{p-1} \exp\left(-\frac{\theta}{2}\left(\frac{y}{3}\right)^{1/\beta} \log t + \log t\right) dy. \end{aligned}$$

Since $K \ge (4/\theta)^{\beta}$, for $y \ge 3K$ it is derived that $-\frac{\theta}{2}\left(\frac{y}{3}\right)^{1/\beta} + 1 \le -\frac{\theta}{4}\left(\frac{y}{3}\right)^{1/\beta}$. So let

$z = y^{1/\beta}$,

$$\begin{aligned}\int_{3K}^{\infty} ty^{p-1} P\left(Q^*(0) > \frac{y}{3}(\log t)^{\beta}\right) dy &\leq \int_{3K}^{\infty} y^{p-1} \exp\left(-\frac{\theta}{4}\left(\frac{y}{3}\right)^{1/\beta} \log t\right) dy \\ &= \int_{(3K)^{1/\beta}}^{\infty} \beta z^{\beta-1} z^{\beta(p-1)} \exp\left(-\frac{\theta z}{4 \cdot 3^{1/\beta}} \log t\right) dz \\ &\leq \frac{3^p \cdot 4^{\beta p} \beta}{\theta^{\beta p} (\log t)^{\beta p}} \Gamma(\beta p) < \infty.\end{aligned}$$

□

**Lemma 3.5.8.** *Let $\eta \geq 0$ be constant. For $t \geq e$, $K = \max\{K_0, 2\eta, \sqrt{8/C_{G,\gamma}}, 1/3\}$,*

$$\int_{3K}^{\infty} ty^{p-1} P\left(\sup_{0 \leq s \leq 1} Y(s) \geq \frac{y}{3}(\log t)^{\beta} - \eta\right) dy < \infty.$$

*Proof.* Since $K \geq 2\eta$ and $\log t \geq 1$, then for $y \geq 3K$, $\frac{y}{3}(\log t)^{\beta} - \eta \geq \frac{y}{6}(\log t)^{\beta}$. Based on (3.5), it follows that

$$\begin{aligned}P\left(\sup_{0 \leq s \leq 1} Y(s) \geq \frac{y}{3}(\log t)^{\beta} - \eta\right) &\leq P\left(\sup_{0 \leq s \leq 1} Y(s) \geq \frac{y}{6}(\log t)^{\beta}\right) \\ &\leq 4 \exp\left(-C_{G,\gamma} \frac{y^2}{36} (\log t)^{2\beta}\right). \qquad (3.47)\end{aligned}$$

So from (3.47),

$$\begin{aligned}&\int_{3K}^{\infty} ty^{p-1} P\left(\sup_{0 \leq s \leq 1} Y(s) \geq \frac{y}{3}(\log t)^{\beta} - \eta\right) dy \\ \leq &\int_{3K}^{\infty} 4y^{p-1} t \exp\left(-C_{G,\gamma} \frac{y^2}{36} (\log t)^{2\beta}\right) dy \\ = &\int_{3K}^{\infty} 4y^{p-1} \exp\left(-C_{G,\gamma} \frac{y^2}{36} (\log t)^{2\beta} + \log t\right) dy.\end{aligned}$$

Since $\beta > 1/2$ and $t \geq e$, it follows that

$$\begin{aligned}
&\int_{3K}^{\infty} ty^{p-1} P\left(\sup_{0\leq s\leq 1} Y(s) \geq \frac{y}{3}(\log t)^{\beta} - \eta\right) dy \\
&\leq \int_{3K}^{\infty} 4y^{p-1} \exp\left(-C_{G,\gamma}\frac{y^2}{36}(\log t) + \log t\right) dy \\
&= \int_{3K}^{\infty} 4y^{p-1} \exp\left(-(\log t)\left(C_{G,\gamma}\frac{y^2}{36} - 1\right)\right) dy.
\end{aligned}$$

Since $y \geq 3K \geq 3\sqrt{8/C_{G,\gamma}}$, then $C_{G,\gamma}\frac{y^2}{36} - 1 \geq C_{G,\gamma}\frac{y^2}{72}$. So from the above expression,

$$\int_{3K}^{\infty} ty^{p-1} P\left(\sup_{0\leq s\leq 1} Y(s) \geq \frac{y}{3}(\log t)^{\beta} - \eta\right) dy \leq \int_{3K}^{\infty} 4y^{p-1} \exp\left(-C_{G,\gamma}\frac{y^2}{72}(\log t)\right) dy.$$

Let $z = y^2$, by substitution,

$$\begin{aligned}
&\int_{3K}^{\infty} ty^{p-1} P\left(\sup_{0\leq s\leq 1} Y(s) \geq \alpha y(\log t)^{\beta} - \eta\right) dy \\
&\leq 2\int_{(3K)^2}^{\infty} z^{\frac{p-2}{2}} \exp\left(-\frac{C_{G,\gamma}(\log t)}{72} z\right) dz \\
&\leq 2\left(\frac{C_{G,\gamma}(\log t)}{72}\right)^{-\frac{p}{2}} \Gamma\left(\frac{p}{2}\right) < \infty.
\end{aligned}$$

□

# Chapter 4

# Congestion Events in a Fractional Brownian Model

## 4.1 Introduction

Congestion events in communication networks cause packet losses, and it is well known that these losses occur in bursts [33] [70]. Furthermore the frequency and the duration of these congestion events significantly influence the perceived network performance [6] [65]. The Internet Engineering Task Force has defined measurement-based QoS metrics [39] aimed at characterizing packet loss patterns. Measured packet traces [33] [66] have been used to create models for the temporal dependence of packet loss. These models assume a specific packet loss process, e.g., one that transits between different states, such as a no-loss state and a loss state. However, transforming network traffic parameters directly into predictions of the properties of congestion events will aid network design and provide a useful indication of QoS. The properties to be considered here include the rate, the duration, and the magnitude of the delay induced by congestion events. An approach for determining the rate of congestion events for some standard traffic models, such as M/M/1, M/D/1 is presented in [24]. In this chapter the approach is extended in two directions: (1) to a fluid queueing model with a self-similar input and (2) to include additional properties of congestion events.

In the early 1990s, researchers with Bellcore observed the phenomena of self-similarity and long-range dependence in LAN traffic [44], which roughly means that the traffic "looks" similar under different time scales and the correlation between packets decays very slowly. This observation is inconsistent with the short range dependence assumption in traditional traffic models, such as the Poisson process and other Markov models. Subsequent studies [10], [57] showed that the traditional models seem inadequate for data networks. Since then, many other traffic models have been proposed, such as fractal point processes [60] and multifractal models [21]. In 1994, Norros [49] proposed a fluid queueing model with a fractional Brownian motion as input, that is, a fractional Brownian queueing model. A fluid model whose input is not packetized is suitable for modeling high speed networks. For example, a fluid model is used to analyze high-precision router measurement in [28]. A fractional Brownian motion for suitable values of the Hurst parameter process has the properties of self-similarity and long-range dependence. By analyzing the origin of self-similarity and long-range dependence in network traffic, it was shown in [61] that the superposition of a family of homogeneous ON/OFF traffic sources with heavy tailed ON and OFF periods, with proper scaling, weakly converges to a fractional Brownian motion plus a linear component. The superposition of traffic sources is well-suited to the network core, which has thousands of simultaneous traffic flows. It has been observed that long-range dependence is a property of the backbone traffic [35]. Recent network measurements [62] also justify the applicability of a fractional Brownian motion, which is a Gaussian process, as a traffic model for aggregated network traffic. Thus the main focus is on the characteristics of congestion events in a fractional Brownian model.

The primary contribution of this chapter is the development of methodologies for evaluating the expectations of the properties of congestion events in a fractional Brownian model. The structure of this chapter is as follows: In Section 4.2, a congestion event is defined, some preliminaries on the fractional Brownian model and the Poisson clumping approximation are given. In Section 4.3, the definition of a conditioned fBm is introduced and some properties of the process are discussed.

An approximation for a congestion event based on a conditioned fBm is proposed in Section 4.4. The temporal properties of congestion events and approximation methods are discussed in Section 4.5. Comparisons between the evaluations made by the proposed methodologies and simulations are presented in Section 4.6. Finally, some conclusions are drawn in Section 4.7.

## 4.2 Preliminaries

In this section, a congestion event is defined and some preliminaries on a fractional Brownian model and the Poisson clumping approximation are given for future analysis.

### 4.2.1 Congestion Events

Let $\{Q(t), t \in \mathbb{R}\}$ be a queue length process. A busy period from $t_1$ to $t_2$ is a period such that $Q(t_1) = Q(t_2) = 0$ but $Q(t) > 0$ for all $t \in (t_1, t_2)$. In a busy period from $t_1$ to $t_2$, a congestion event with a level $b$ is defined to occur at time $t_b$ if $t_b$ is the first time that the process $Q(t)$ reaches a fixed level $b$. The congestion event ends at time $t_2$, i.e. the first time the queue becomes empty after $t_b$. Two congestion events are shown in Figure 4.1. Given this definition of a congestion event, the process $Q(t)$ can reenter the level $b$ multiple times during one congestion event. The premise of this work is that, for a large $b$, a congestion event as defined here results in a burst of packet losses.

Formally as in [3], let $(\Omega, \mathscr{F}, P)$ be a probability space and $\theta_t$ be a measurable flow on $(\Omega, \mathscr{F})$ which is invariant under $P$. Let $t_b^{(i)}$ denote the beginning of the $i$-th congestion event, such that, $-\infty \leq ... < t_b^{(-1)} < t_b^{(0)} \leq 0 < t_b^{(1)} < t_b^{(2)} < ... \leq \infty$. Let $t_1^{(i)}$ and $t_2^{(i)}$ be the corresponding beginning and end of the busy period in which the $i$-th congestion event occurs. Let $N = \{t_b^{(i)}, i \in \mathbb{Z}\}$ denote the set of the beginning times of congestion events, then $\{N, \theta_t, P\}$ forms a stationary marked point process, where the paths of congestion events are viewed as marks. Let $P_N^0$ be the associated

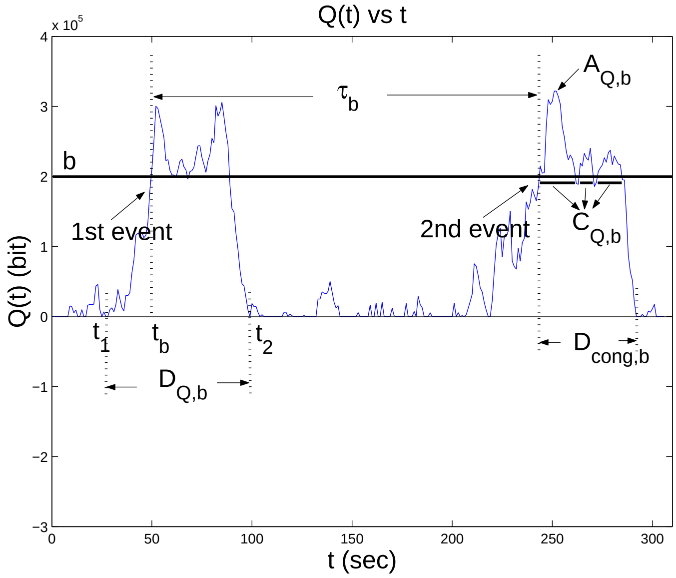



Figure 4.1: Example of workload process and definitions of random variables of interest

Palm probability defined as

$$P_N^0(A) = \frac{1}{E\left[N(C)\right]} E\left[\int_C (1_A \circ \theta_s) N(ds)\right],$$

where $A \in \mathscr{F}$, $N(C)$ denotes the number of points in a Borel set $C$ and $1_A$ is an indicator function. Let $E$ represent the expectation with respect to $P$, and let $E^0$ represent the expectation with respect to $P_N^0$.

The inter-congestion event time between the $i$-th and the $(i+1)$-th congestion events is denoted by $\tau_b^{(i+1)} = t_b^{(i+1)} - t_b^{(i)}$. For studying the properties of an arbitrary congestion event, the superscripts are omitted to simplify the notation. Then the mean inter-congestion event time is $E^0\left[\tau_b\right]$. As shown in [24], $E^0\left[\tau_b\right]$ (or the rate $1/E^0\left[\tau_b\right]$) is a useful QoS metric. The other metrics of an arbitrary congestion event are $E^0[C_{Q,b}]$, the mean sojourn time that $Q(t)$ spends above threshold $b$ in a congestion event; $E^0[D_{cong,b}]$, the mean duration of a congestion event, i.e., the time from $t_b$ to $t_2$; $E^0[D_{Q,b}]$, the mean duration of a busy period containing a congestion event, i.e., the time from $t_1$ to $t_2$; and $E^0[A_{Q,b}]$ which is the mean peak queue length of a congestion event. In a study of high precision router measurements [28], it is demonstrated that the $(D_{Q,b}, A_{Q,b})$ pairs can be used to describe a busy period in which the queue length exceeds a congestion threshold $b$. The set of metrics, $E^0[\tau_b]$, $E^0[C_{Q,b}]$, $E^0[D_{cong,b}]$, $E^0[D_{Q,b}]$, $E^0[A_{Q,b}]$ can be used to characterize the nature of congestion events.

### 4.2.2 Fractional Brownian Queueing Model

As in [49], a fractional Brownian motion, which is a Gaussian process with stationary increments, is used to model network traffic to capture the self-similarity and the long-range dependence, Figure 4.2.

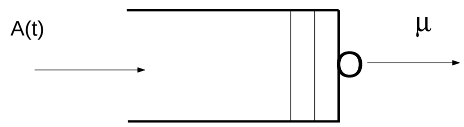



Figure 4.2: A queueing model with Fractional Brownian input, $A(t) = mt + \sigma B^H(t)$

Let $A(t) = mt + \sigma B^H(t)$ be the cumulated arrivals up to time $t$, where $m$ is the mean input rate ($bps$), $\sigma$ stands for the variance coefficient ($bit$), and $\{B^H(t), t \in \mathbb{R}\}$ is a standard fractional Brownian motion with Hurst parameter $H \in [1/2, 1)$. An input traffic, modeled by $A(t)$, is determined by the parameters ($m$, $\sigma$, $H$). At time $t$, the queue length $Q_o(t)$ can be expressed as, see Section 2.1 of Chapter 2 and the references therein, $Q_o(t) = A(t) - \mu t - \inf_{s \le t}(A(s) - \mu s)$, where $\mu$ is a fixed service rate in ($bps$). Then $Q_o(t)$ can be written as

$$Q_o(t) = \sigma B^H(t) - (\mu - m)t - \inf_{s \le t} \left( \sigma B^H(s) - (\mu - m)s \right), \tag{4.1}$$

where $\mu - m$ is the surplus rate. For the stability of the queue, it is assumed that $\mu - m > 0$.

Consider a scaled $Q_o(t)$, which is defined as $Q(t) = Q_o(t)/\sigma$. It can be observed that the temporal properties of the congestion events of $Q_o(t)$ with a level $b_o$ are the same as those of the congestion events of $Q(t)$ with a level $b = b_o/\sigma$. Therefore to study the properties of congestion events of a queue with an input $(m, \sigma, H)$ and a service rate $\mu$, it is equivalent to study the corresponding scaled queue length process $Q(t)$,

$$Q(t) = B^H(t) - ct - \inf_{s \leq t}(B^H(s) - cs), \tag{4.2}$$

where $c = (\mu - m)/\sigma$ stands for the scaled surplus rate.

### 4.2.3 Poisson Clumping Approximation

Following [24], the Poisson clumping approximation [1] is used to find the inter-congestion event time. For a threshold $b$, the overflow probability, $P\left(Q(0) \geq b\right)$, and the mean sojourn time of $Q(t)$ above the threshold in a congestion event, $E^0[C_{Q,b}]$, are applied to evaluate the mean inter-congestion event time as

$$E^0[\tau_b] \approx \frac{E^0[C_{Q,b}]}{P\left(Q(0) \geq b\right)}. \tag{4.3}$$

Note that for a fractional Brownian traffic, the probability $P\left(Q(0) \geq b\right)$ can be approximated using the result in [29]. Thus the problem reduces to finding $E^0[C_{Q,b}]$. By applying the Poisson clumping approximation, it is assumed that the congestion events are rare and the dependence among the events are small. These assumptions are reasonable for the case studied here. When $b$ is large, the congestion events are rare and far apart. Although $B^H(t)$ has long range dependence, the dependence among congestion events are small. The Poisson clumping approximation is validated with simulations, some of which are shown in Figure 4.8$a$, 4.8$b$, 4.8$c$. These results indicate that the Poisson clumping approximation can be used to evaluate the mean inter-congestion event time.

### 4.3 Conditioned Fractional Brownian Motion

To simplify the analysis of congestion events in a fractional Brownian model, a concept of a conditioned fractional Brownian motion is introduced.

**Proposition 4.3.1.** *For a fixed constant $r$ and $t, s \geq 0$, let*

$$\sigma_r(s,t) = \frac{1}{2}[s^{2H} + t^{2H} - |t-s|^{2H}] - \frac{M_r(t)M_r(s)}{4r^{2H}}, \tag{4.4}$$

*where*

$$M_r(t) = (t+r)^{2H} - t^{2H} - r^{2H}, \tag{4.5}$$

*then $\sigma_r(s,t)$ is positive definite.*

*Proof.* Since $\sigma_r(s,t) = \sigma_r(t,s)$ is a symmetric function, it is sufficient to show that the for any $t_1, t_2, ..., t_n \geq 0$, the $n \times n$ symmetric matrix $M$, where

$$M = \begin{bmatrix} \sigma_r(t_1,t_1) & \sigma_r(t_1,t_2) & \cdots & \sigma_r(t_1,t_n) \\ \sigma_r(t_2,t_1) & \sigma_r(t_2,t_2) & \cdots & \sigma_r(t_2,t_n) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_r(t_n,t_1) & \sigma_r(t_n,t_2) & \cdots & \sigma_r(t_n,t_n) \end{bmatrix},$$

is positive definite. Let $\Sigma_{11}$ be an $n \times n$ matrix such that

$$\Sigma_{11} = \begin{bmatrix} t_1^{2H} & \cdots & \cdots & \frac{1}{2}[t_1^{2H} + t_n^{2H} - |t_1 - t_n|^{2H}] \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{1}{2}[t_1^{2H} + t_n^{2H} - |t_1 - t_n|^{2H}] & \cdots & \cdots & t_n^{2H} \end{bmatrix},$$

$\Sigma_{21} = \left[\frac{1}{2}M_r(t_1) \quad \frac{1}{2}M_r(t_2) \quad \cdots \quad \frac{1}{2}M_r(t_n)\right]$, $\Sigma_{12}$ be the transpose of $\Sigma_{21}$ and $\Sigma_{22} = r^{2H}$. Let $\{B^H(t), t \geq 0\}$ be a fractional Brownian motion with Hurst parameter $H$.

It can be observed that

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

is the covariance matrix of $\{B^H(t_1 + r) - B^H(r), \cdots, B^H(t_n + r) - B^H(r), B^H(r)\}$, and $M = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$ is the Schur complement of $\Sigma_{22}$ in $\Sigma$. Since $\Sigma$ is positive definite, then $M$ is also positive definite. □

According to this proposition, there exists a Gaussian process such that the covariance is given by $\sigma_r(s,t)$. This Gaussian process with certain mean function is defined as a conditioned fractional Brownian motion.

**Definition 4.3.1** (Conditioned fractional Brownian motion)**.** *A conditioned fractional Brownian motion, denoted by $\{\tilde{B}^H(t; r, d), t \geq 0\}$, is a Gaussian process with mean $\mu_{r,d}(t)$*

$$\mu_{r,d}(t) = \frac{M_r(t)}{2r^{2H}} d. \tag{4.6}$$

*and covariance $\sigma_r(s,t)$ given in (4.4), where $r$ and $d$ are two constants.*

The process $\{\tilde{B}^H(t; r, d), t \geq 0\}$ is called a conditioned fractional Brownian motion, since $\mu_{r,d}(t) = E[B^H(t + r) - B^H(r)|B^H(r) = d]$ and $\sigma_r(s,t) = Cov(B^H(s + r) - B^H(r), B^H(t + r) - B^H(r)|B^H(r) = d)$. For $H = 1/2$, a conditioned fractional Brownian motion $\{\tilde{B}^H(t; r, d), t \geq 0\}$ reduces to a standard Brownian motion $\{B(t), t \geq 0\}$. The following two lemmas give some properties of the mean $\mu_{r,d}(\cdot)$ and the covariance function $\sigma_r(\cdot, \cdot)$ of a conditioned fractional Brownian motion. For brevity, let

$$m_r(t) = (t + r)^{2H-1} - t^{2H-1}. \tag{4.7}$$

**Lemma 4.3.1.** *For $H \in [1/2, 1)$, $r > 0$ and $t \geq 0$, the following results hold.*

*1.* $m_r(t) \leq r^{2H-1}$,

*2.* $M_r(t) \leq 2Htr^{2H-1}$, $M_r(t) \leq 2Hrt^{2H-1}$,

*3.* $M_r(t) \geq 2Htm_r(t)$,

*4.* $ct - \mu_{r,d}(t)$ *increases with respect to* $t$.

*Proof.*

1. It can be shown that $m_r(t)$ is a decreasing function for $H \in (\frac{1}{2}, 1)$. The lemma follows the observation that $m_r(0) = r^{2H-1}$.

2. Since $M_r(t) = 2H(2H-1)\int_0^t \int_0^r (s+u)^{2H-2} ds du$, then

$$M_r(t) = 2H \int_0^t [(r+u)^{2H-1} - u^{2H-1}]du \leq 2H \int_0^t r^{2H-1} du = 2Htr^{2H-1}$$

   Similarly, it can be shown that $M_r(t) \leq 2Hrt^{2H-1}$.

3. Notice that $M_r'(t) = 2Hm_r(t)$, with Taylor's expansion,

$$M_r(t) = M_r(0) + 2Hm_r(\xi)t = 2Hm_r(\xi)t$$

   where $\xi \in [0, t]$. Since $m_r(t)$ is decreasing with respect to $t$, the result follows.

4. With the above results, it can be verified that the first derivative of $ct - \mu_{r,d}(t)$ is positive.

□

**Lemma 4.3.2.** *1. For a fixed* $t$, $\sigma_r(s,t)$ *increases with respect to* $s$,

*2.* $\sigma_r^2(t)$ *increases with respect to* $t$.

*Proof.* For the first part, it is sufficient to show that $\frac{d\sigma_r(s,t)}{ds} \geq 0$ for fixed $t$.

$$\frac{d\sigma_r(s,t)}{ds} = H\left[s^{2H-1} - (s-t)^{2H-1}\right] - H\frac{M_r(t)m_r(s)}{2r^{2H}}$$

Since $M_r(t) \leq 2Htr^{2H-1}$ by Lemma 4.3.1, it follows that

$$
\begin{aligned}
\frac{d\sigma_r(s,t)}{ds} &\geq H\left\{s^{2H-1} - (s-t)^{2H-1} - \frac{2Htr^{2H-1}}{2r^{2H}} m_r(s)\right\} \\
&= H\left\{s^{2H-1} - (s-t)^{2H-1} - \frac{Ht}{r}\left[(s+r)^{2H-1} - s^{2H-1}\right]\right\} \\
&= Hs^{2H-1}\left\{1 - \left(1 - \frac{t}{s}\right)^{2H-1} - \frac{Ht}{r}\left[\left(1+\frac{r}{s}\right)^{2H-1} - 1\right]\right\} \\
&= Hs^{2H-1}\left\{\left(1 - \frac{t}{s} + \frac{t}{s}\right)^{2H-1} - \left(1 - \frac{t}{s}\right)^{2H-1} - \frac{Ht}{r}\left[\left(1+\frac{r}{s}\right)^{2H-1} - 1\right]\right\}.
\end{aligned}
$$

Expand $1 = (1-t/s+t/s)^{2H-1}$ at $1-t/s$, then $1 = (1-t/s)^{2H-1} + (2H-1)\xi_1^{2H-2}t/s$ where $\xi_1 \in (1-t/s, 1)$, and expand $(1+r/s)^{2H-1}$ at 1, then $(1+r/s)^{2H-1} = 1 + (2H-1)\xi_2^{2H-2}r/s$, where $\xi_2 \in (1, 1+r/s)$, thus,

$$
\begin{aligned}
\frac{d\sigma_r(s,t)}{ds} &= Hs^{2H-1}\left\{(2H-1)\xi_1^{2H-2}\frac{t}{s} - \frac{Ht}{r}(2H-1)\xi_2^{2H-2}\frac{r}{s}\right\} \\
&= H(2H-1)s^{2H-2}t\left(\xi_1^{2H-2} - H\xi_2^{2H-2}\right) \\
&\geq H(2H-1)s^{2H-2}t(\xi_1^{2H-2} - H), \quad \textit{since} \quad \xi_2^{2H-2} \leq 1 \\
&> 0, \quad \textit{since} \quad \xi_1^{2H-2} \geq 1.
\end{aligned}
$$

Thus $\sigma_r(s,t)$ is increasing with respect to $s$. Next it is shown that $\frac{d\sigma_r^2(t)}{dt} \geq 0$, which implies that $\sigma_r^2(t)$ increases with respect to $t$. Since $M_r(t) \leq 2Hrt^{2H-1}$ and $m_r(t) \leq r^{2H-1}$,

$$
\begin{aligned}
\frac{d\sigma_r^2(t)}{dt} &= 2Ht^{2H-1} - H\frac{M_r(t)m_r(t)}{r^{2H}} \\
&\geq 2Ht^{2H-1} - H\frac{2Hrt^{2H-1}r^{2H-1}}{r^{2H}} \\
&= 2H(1-H)t^{2H-1} > 0.
\end{aligned}
$$

□

Define a process $\{X(t), t \geq 0\}$, such that,

$$X(t) = \tilde{B}^H(t; r_b, d_b) - ct + b, \tag{4.8}$$

which is a conditioned fractional Brownian motion with a negative drift, Figure 4.3, where $c > 0$, $b > 0$ are constants and

$$r_b = \frac{bH}{c(1-H)}, \tag{4.9}$$

$$d_b = b + cr_b. \tag{4.10}$$

Note that when $t = 0$, $X(0) = b$. The process $\{X(t), t \geq 0\}$ will be used to

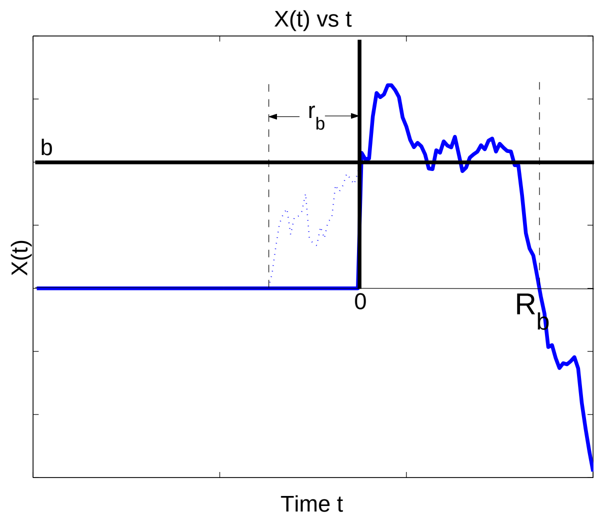



Figure 4.3: Process $X(t)$

approximate a congestion event with a level $b$ in Section 4.4.

Although the parameters $r$ and $d$ of a conditioned fractional Brownian motion can take any values, here only a special case is considered, that is, $r = r_b$ and $d = d_b$ given in (4.9) and (4.10), respectively. The justification for choosing the values will be given in Section 4.4. Note that $r_b$ and $d_b$ are uniquely determined by $b$, $c$ and $H$. Rewriting $\mu_{r_b,d_b}(t)$ in terms of $r_b$ with (4.9), (4.10) and denoting $\mu_{r_b,d_b}(t)$ with

$\mu_{r_b}(t)$, it is obtained that

$$\mu_{r_b}(t) = \frac{M_{r_b}(t)}{2r_b^{2H-1}} \frac{c}{H}. \tag{4.11}$$

To simplify notation, write $\tilde{B}^H(t; r_b, d_b)$ as $\tilde{B}^H(t; r_b)$.

Let $R_b$ be the first time that $X(t)$ reaches 0, that is,

$$R_b = \inf_{t \geq 0} \{t : X(t) \leq 0\}. \tag{4.12}$$

Notice that the two events $\{R_b \leq u\}$ and $\{\inf_{0 \leq s \leq u} X(s) \leq 0\}$ are equivalent. Let $C_{X,b}$ be the time that $X(t)$ spends above $b$ in $[0, R_b]$, see Figure 4.3. The properties of $C_{X,b}$ and $R_b$ are discussed in the following subsections.

### 4.3.1 An Upper Bound of $E[C_{X,b}]$

The sojourn time $C_{X,b}$, that is, the time $X(t)$ spends above $b$ in $[0, R_b]$, can be written as

$$C_{X,b} = \int_0^{R_b} 1_{[b,\infty)} \left(X(t; r_b, d_b)\right) dt.$$

Let $\bar{\Phi}(\cdot) = 1 - \Phi(\cdot) = \frac{1}{\sqrt{2\pi}} \int_\cdot^\infty e^{-\xi^2/2} d\xi$ and

$$U_{C_b} = \int_0^\infty \bar{\Phi} \left( \frac{ct - \mu_{r_b,d_b}(t)}{\sqrt{\sigma_{r_b}^2(t)}} \right) dt. \tag{4.13}$$

Then $U_{C_b}$ is an upper bound of $E[C_{X,b}]$, since

$$\begin{aligned} E[C_{X,b}] &\leq E \int_0^\infty 1_{[b,\infty)} \left( b + \tilde{B}^H(t; r_b, d_b) - ct \right) dt \\ &= U_{C_b}. \end{aligned}$$

For $H = 1/2$, the process $X(t)$ reduces to a standard Brownian motion with a negative drift. By the dominated convergence theorem, it can be verified that $\lim_{b\to\infty} E[C_{X,b}] = U_{C_b}$ (for $H = 1/2$, it can be verified from (4.13) that $U_{C_b}$ is a constant which is independent of $b$). Then for the Brownian case, as $b$ is large, $E[C_{X,b}] \approx U_{C_b}$. Motivated by the case of $H = 1/2$, an ad-hoc approximation is proposed for $H \in (1/2, 1)$, that is,

$$E[C_{X,b}] \approx U_{C_b}. \tag{4.14}$$

Unfortunately, this approximation cannot be justified with limit argument. But as it will be shown in Section 4.6, this approximation produces useful results on evaluating properties of congestion events in a fractional Brownian model.

### 4.3.2 Properties of $E[R_b]$

In this subsection, the expectation of $R_b$, i.e., $E[R_b]$, is discussed. The exact $E[R_b]$ is unknown except for the case $H = 1/2$, where $E[R_b] = b/c$. For $H \in (1/2, 1)$, both upper and lower bounds of $E[R_b]$ are derived in Theorem 4.3.1 and 4.3.2, respectively.

**Theorem 4.3.1** (An upper bound of $E[R_b]$)**.** *Let $R_b$ be defined in (4.12). For $c > 0$, $b > 0$,*

$$E[R_b] \leq \int_0^\infty \bar{\Phi}\left(\frac{ct - b - \mu_{r_b}(t)}{\sqrt{\sigma^2_{r_b}(t)}}\right) dt, \tag{4.15}$$

*where $\bar{\Phi}(x) = 1 - \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{\frac{-\xi^2}{2}} d\xi$ and $\mu_{r_b}(t)$, $\sigma^2_{r_b}(t)$ are as given in (4.11) and (4.4), respectively.*

*Proof.* By the definition of $R_b$, the two events $\{R_b \geq t\} = \{\inf_{0\leq s\leq t} X(s) \geq 0\}$ are equivalent. Then

$$
\begin{aligned}
P(R_b \geq t) &= P\left(\inf_{0\leq s\leq t} X(s) \geq 0\right) \\
&= P\left(\inf_{0\leq s\leq t} \left(b + \tilde{B}^H(s; r_b) - cs\right) \geq 0\right) \\
&\leq P\left(b + \tilde{B}^H(t; r_b) - ct \geq 0\right) \\
&= P\left(\tilde{B}^H(t; r_b) \geq ct - b\right) \\
&= \bar{\Phi}\left(\frac{ct - b - \mu_{r_b}(t)}{\sqrt{\sigma^2_{r_b}(t)}}\right),
\end{aligned}
$$

Since $E[R_b] = \int_0^\infty P(R_b \geq t)dt$, following the above inequality, the upper bound is obtained. □

Let $U_{R_b}$ denote the upper bound of $E[R_b]$, that is,

$$
U_{R_b} = \int_0^\infty \bar{\Phi}\left(\frac{ct - b - \mu_{r_b}(t)}{\sqrt{\sigma^2_{r_b}(t)}}\right) dt. \tag{4.16}
$$

**Remark 4.3.1.** *For $H = 1/2$, it is known [27, page 14] that*

$$
P\left(R_b \geq t\right) = \Phi\left(\frac{b - ct}{\sqrt{t}}\right) - e^{2cb}\Phi\left(\frac{-b - ct}{\sqrt{t}}\right), \tag{4.17}
$$

*and $E[R_b] = b/c$. For a large $b$, it is claimed that $U_{R_b} \approx E[R_b]$ in the sense that the relative error defined as $err = \frac{U_{R_b} - E[R_b]}{E[R_b]} \to 0$ as $b \to \infty$. From (4.16), it is obtained that for $H = 1/2$, $U_{R_b} = \int_0^\infty \bar{\Phi}\left(\frac{ct-b}{\sqrt{t}}\right) dt = \int_0^\infty \Phi\left(\frac{b-ct}{\sqrt{t}}\right) dt$. So from (4.17), it* *follows that*

$$
\begin{aligned}
U_{R_b} - E[R_b] &= \int_0^\infty e^{2cb}\bar{\Phi}\left(\frac{b+ct}{\sqrt{t}}\right)dt \\
&\leq \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{2cb}\frac{\sqrt{t}}{b+ct}e^{-\frac{(b+ct)^2}{2t}}dt \\
&= \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{\sqrt{t}}{b+ct}e^{-\frac{(b-ct)^2}{2t}}dt.
\end{aligned}
$$

*Since* $\frac{\sqrt{t}}{b+ct} \leq \frac{1}{2\sqrt{bc}}$ *and from Lemma 4.8.1, as* $b \to \infty$*,*

$$
U_{R_b} - E[R_b] \leq \frac{1}{\sqrt{2\pi}}\frac{1}{2\sqrt{bc}}\int_0^\infty e^{-\frac{(b-ct)^2}{2t}}dt \sim \frac{1}{2c^2}.
$$

*Therefore* $err = \frac{U_{R_b}-E[R_b]}{E[R_b]} \to 0$ *as* $b \to \infty$*, since* $E[R_b] = b/c$*.*

**Remark 4.3.2.** *For* $H > 1/2$*, the exact* $E[R_b]$ *is unknown. But it will be illustrated by deriving a lower bound of* $E[R_b]$ *that for a large* $b$*,* $U_{R_b} \approx E[R_b]$ *in the sense that* $err = \frac{U_{R_b}-E[R_b]}{E[R_b]} \to 0$ *as* $b \to \infty$*.*

Let $\alpha_0$ be a constant such that

$$
\alpha_0 = 1 - \max_{s\geq 0}\frac{\left((s+1)^{2H} - s^{2H} - 1\right)\left((s+1)^{2H-1} - s^{2H-1}\right)}{2s^{2H-1}}. \tag{4.18}
$$

**Theorem 4.3.2** (A lower bound of $E[R_b]$)**.**

$$
E\left[R_b\right] \geq \int_0^\infty w^{-1}(v)p(v)dv, \tag{4.19}
$$

*where* $w(v) = \sigma_{r_b}^2(v)$ *and* $w^{-1}(v)$ *is the inverse function of* $w(v)$*, that is,* $w^{-1}(w(v)) = v$*. The function* $p(v)$ *is defined as*

$$
p(v) = \frac{b + \mu_{r_b}(g(v)) - cg(v) - v\left[\mu'_{r_b}(g(v))g'(v) - cg'(v)\right]}{v^{3/2}}\phi\left(\frac{b+\mu_{r_b}(g(v)) - cg(v)}{\sqrt{v}}\right), \tag{4.20}
$$

*where* $\phi(x)$ *is the standard normal density, that is,* $\phi(x) = (2\pi)^{-1/2}\exp(-x^2/2)$,

$$g(v) = \alpha_0^{-\frac{1}{2H}} v^{\frac{1}{2H}}, \tag{4.21}$$

*and* $\alpha_0$ *is given in (4.18).*

*Proof.* Since $R_b = \inf_{t\geq 0}\{t : X(t) \leq 0\}$, it follows that

$$\begin{aligned} P\left(R_b \geq u\right) &= P\left(\inf_{t\in(0,u)} X(t) > 0\right) \\ &= P\left(-\sup_{t\in(0,u)} -X(t) > 0\right) \\ &= P\left(\sup_{t\in(0,u)} -\tilde{B}^H(t;r_b) + ct - b < 0\right) \\ &= P\left(\sup_{t\in(0,u)} -\tilde{B}^H(t;r_b) + ct < b\right). \end{aligned} \tag{4.22}$$

Recall that for $s, t \geq 0$, the mean of $\tilde{B}^H(t;r_b)$ is $\mu_{r_b}(t) = \frac{M_{r_b}(t)}{2r_b^{2H-1}}\frac{c}{H}$, and the covariance is $\sigma_{r_b}(s,t) = Cov(\tilde{B}^H(t;r_b), \tilde{B}^H(s;r_b)) = \frac{1}{2}[s^{2H} + t^{2H} - |t-s|^{2H}] - \frac{M_{r_b}(t)M_{r_b}(s)}{4r_b^{2H}}$. Let $\{B(t), t \geq 0\}$ be a standard Brownian motion and let

$$w(t) = \sigma_{r_b}^2(t) = t^{2H} - \frac{[M_{r_b}(t)]^2}{4r_b^{2H}} \tag{4.23}$$

denote the variance of $\tilde{B}^H(t;r_b)$. Define a random process $\{Y(t), t \geq 0\}$ which is a scaled Brownian motion with a drift, that is, $\{Y(t) = B(w(t)) - \mu_{r_b}(t), t \geq 0\}$. So $\{Y(t), t \geq 0\}$ is a Gaussian process, for $\forall s, t \geq 0$, it has mean $E[Y(t)] = -\mu_{r_b}(t)$ and covariance $Cov(Y(s), Y(t)) = \min(w(s), w(t))$. Based on Lemma 4.3.2, for

$s \geq t \geq 0$, it is obtained that

$$E[-\tilde{B}^H(t;r_b) + ct] = E[Y(t) + ct],$$
$$Var(-\tilde{B}^H(t;r_b) + ct) = Var(Y(t) + ct),$$
$$Cov(-\tilde{B}^H(t;r_b) + ct, -\tilde{B}^H(s;r_b) + cs) \geq Cov(Y(t) + ct, Y(s) + cs).$$

By the Slepian inequality, it is obtained that

$$P\left(\sup_{t\in(0,u)} -\tilde{B}^H(t;r_b) + ct < b\right) \geq P\left(\sup_{t\in(0,u)} Y(t) + ct < b\right). \tag{4.24}$$

Let $w^{-1}(t)$ be the inverse function of $w(t)$, that is, $w^{-1}(w(t)) = t$. Combining (4.22) and (4.24), it follows that

$$\begin{aligned} P(R_b \geq u) &\geq P\left(\sup_{t\in(0,u)} B(w(t)) - \mu_{r_b}(t) + ct < b\right), \quad let\ v = w(t) \\ &= P\left(\sup_{v\in(0,w(u))} B(v) - \mu_{r_b}(w^{-1}(v)) + cw^{-1}(v) < b\right). \end{aligned} \tag{4.25}$$

In Lemma 4.8.2, it is verified that $w^{-1}(v) \leq g(v)$ for $\forall v \geq 0$, and since $-\mu_{r_b}(t) + ct$ is increasing with respect to $t$, it is obtained that $-\mu_{r_b}(w^{-1}(v)) + cw^{-1}(v) \leq -\mu_{r_b}(g(v)) + cg(v)$ for $\forall v \geq 0$. From (4.25),

$$P(R_b \geq u) \geq P\left(\sup_{v\in(0,w(u))} B(v) - \mu_{r_b}(g(v)) + cg(v) < b\right). \tag{4.26}$$

Let $T_b = \inf\{v \geq 0 : B(v) - \mu_{r_b}(g(v)) + cg(v) \geq b\}$ be the first time that the Brownian motion $\{B(v), v \geq 0\}$ reaches the boundary $b + \mu_{r_b}(g(v)) - cg(v)$, then from (4.26),

$$P(R_b \geq u) \geq P(T_b \geq w(u)) = P\left(w^{-1}(T_b) \geq u\right).$$

Thus

$$E\left[R_b\right] \geq E\left[w^{-1}(T_b)\right]. \tag{4.27}$$

Based on the integral equation of the density of $T_b$, cf. [22], if the boundary $b + \mu_{r_b}(g(v)) - cg(v)$ is convex, then the function $p(v)$ given in (4.20) is a lower bound of the density of $T_b$.

In Lemma 4.8.3, it is shown that $b + \mu_{r_b}(g(v)) - cg(v)$ is convex. Thus following (4.27), the theorem is proved. □

Let $L_{R_b}$ denote the lower bound of $E[R_b]$, that is,

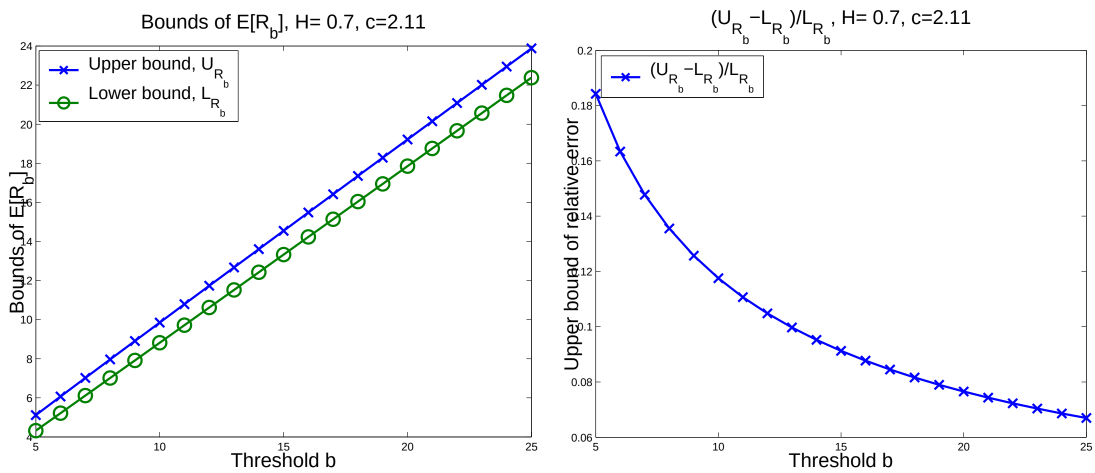
$$L_{R_b} = \int_0^{\infty} w^{-1}(v)p(v)dv,$$

where $w(v)$ and $p(v)$ are defined in Theorem (4.3.2). Approximating $E[R_b]$ with $U_{R_b}$, let the relative error be defined as

$$err = \frac{U_{R_b} - E[R_b]}{E[R_b]}.$$

It is illustrated in Figure 4.4 that for $H = 0.7$ and $c = 2.11$, an upper bound of the relative error, $(U_{R_b} - L_{R_b})/L_{R_b}$, decays to 0 as $b$ becomes large. Similar phenomena are observed for different values of $H$ and $c$. Thus for $H \in (1/2, 1)$, the upper bound, $U_{R_b}$, can be used as an approximation of $E[R_b]$ for large $b$, i.e.,

$$E[R_b] \approx U_{R_b}. \tag{4.28}$$



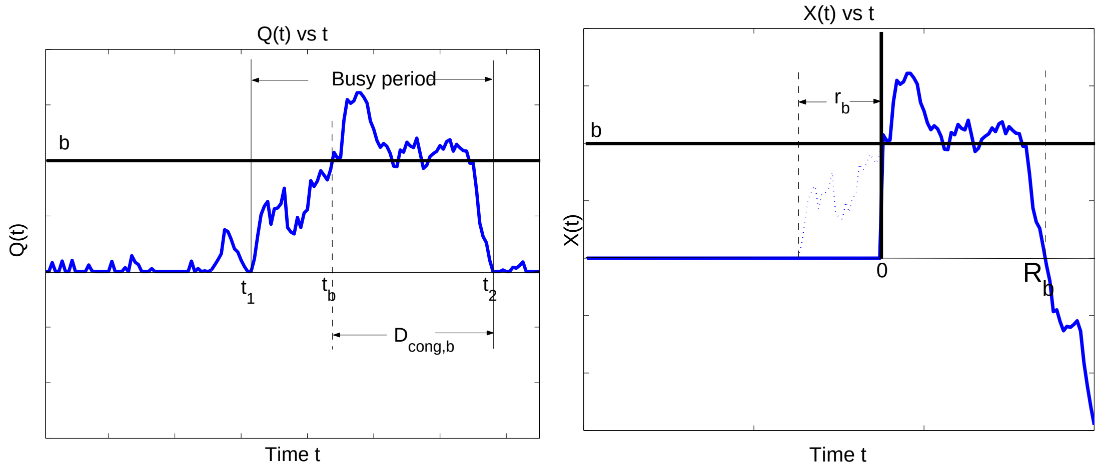

(a) Bounds of $E[R_b]$ (b) Error bound

Figure 4.4: Bounds of $E[R_b]$ and the bound of the relative error

## 4.4 Busy Periods Containing Congestion Events

The busy periods of a fractional Brownian model have been discussed in [51], and recently in [45], where the busy periods are defined as the periods that the queue is not empty. It is different from the busy periods discussed here. Note that a busy period hereafter always means a busy period containing a congestion event defined in Section 4.2. A busy period from $t_1$ to $t_2$ is shown in Figure 4.5$a$, where $t_b$ is the first time that the queue reaches a level $b$ in the busy period, and $t_2$ is the first time that the queue returns to 0 after $t_b$. The time $t_b$ separates one busy period into two parts, $[t_1, t_b]$ and $[t_b, t_2]$. The objective is to evaluate the mean inter-congestion event time $E[\tau_b]$ with the Poisson clumping approximation. So from (4.3), it is necessary to find $E^0[C_{Q,b}]$, which is the mean time that the queue spends above the level $b$ in a congestion event, Figure 4.5$a$. It will be demonstrated that the problem can be simplified by approximating $Q(t)$ in $[t_b, t_2]$ with a process $X(t)$, which is a conditioned fBm with a negative drift, Figure 4.5$b$.

**Proposition 4.4.1.** *Let $t_1$ and $t_2$ be the end points of a busy period. Let $t_b \in [t_1, t_2]$ be the first time that $Q(t)$ reaches a level $b$. Then for $t \in [t_1, t_2]$, $Q(t)$ can be rewritten*




(a) A busy period of $Q(t)$

(b) Approximation process $X(t)$

Figure 4.5: A busy period of $Q(t)$ from $t_1$ to $t_2$, and the approximation process $X(t)$

*as*

$$Q(t) = B^H(t) - B^H(t_1) - c(t - t_1), t \in [t_1, t_b] \tag{4.29}$$

$$Q(t) = b + B^H(t) - B^H(t_b) - c(t - t_b), t \in [t_b, t_2]. \tag{4.30}$$

*Proof.* From the given condition, $Q(t_1) = 0$, $Q(t_b) = b$ and $Q(s) > 0$ for $s \in (t_1, t_2)$. From (4.2), it can be verified that for $\forall t \in (t_1, t_2)$,

$$B^H(t_1) - ct_1 = \inf_{s \le t} \left( B^H(s) - cs \right).$$

Then based on (4.2), for $t \in [t_1, t_b]$,

$$\begin{aligned} Q(t) &= B^H(t) - ct - \inf_{s \le t} \left( B^H(s) - cs \right) \\ &= B^H(t) - B^H(t_1) - c(t - t_1). \end{aligned}$$

Similarly, $Q(t) = b + [B^H(t) - B^H(t_b)] - c(t - t_b)$, for $t \in [t_b, t_2]$. □

**Remark 4.4.1.** *In $[t_1, t_b]$, $Q(t)$ increases from 0 to the level $b$. Since $Q(t_b) = b$, from (4.29), the increment of the fractional Brownian motion in $[t_1, t_b]$ is $B^H(t_b) -$*

$B^H(t_1) = b + c(t_b - t_1)$. *For the period of* $[t_b, t_2]$, *recall that if* $t_b$ *is a constant,* $\{B^H(t) - B^H(t_b), t \in [t_b, \infty)\}$ *is equivalent to* $\{B^H(t), t \in [0, \infty)\}$ *in distribution. This is the motivation for approximating the period* $[t_b, t_2]$ *of* $Q(t)$ *with a conditioned fractional Brownian motion with a negative drift, that is, the process* $\{X(t), t \geq 0\}$ *defined in (4.8).*

For a large $b$, the congestion events are rare. Since "rare events occur in the most likely way" and the most probable sample path of $Q(t)$ found in [51] spends time $bH/c(1-H)$ increasing from 0 to a large fixed level $b$, then the constants $r_b$ defined in (4.9) and $d_b$ defined in (4.10) are used to represent the time $t_b - t_1$ and the increment of the fractional Brownian motion in $[t_1, t_b]$, respectively.

The part $[t_b, t_2]$ of a busy period of $Q(t)$ is approximated by $[0, R_b]$ of $X(t)$, Figure 4.5. Let $C_{X,b}$ denote the sojourn time that $X(t)$ spends above the level $b$ in the period of $[0, R_b]$. The idea is to approximate $E^0[C_{Q,b}]$ with $E[C_{X,b}]$, that is,

$$E^0[C_{Q,b}] \approx E[C_{X,b}]. \tag{4.31}$$

Since the process $X(t)$ is not related to the point process $N$ in Section 4.2.1, the expectation of $C_{X,b}$ is denoted with $E[C_{X,b}]$, which is with respect to $P$.

**Remark 4.4.2.** *The time interval* $[0, R_b]$ *of* $X(t)$ *is used to approximate the part* $[t_b, t_2]$ *of a busy period. This approximation has some inherent shortcomings. The parameters* $r_b$ *and* $d_b$ *of* $X(t)$ *are used to represent* $t_b - t_1$ *and the corresponding increment of the fractional Brownian motion, respectively. However, they cannot capture the property that* $Q(t)$ *is less than* $b$ *and strictly positive in* $(t_1, t_b)$, *i.e.,* $0 < Q(t) < b$, *for all* $t \in (t_1, t_b)$. *And for a fixed* $b$, $r_b$ *is a constant, but* $t_b - t_1$ *is obviously a random variable. Thus* $\{Q(t), t \in (t_b, t_2)\}$ *is not equivalent to a conditioned fractional Brownian motion. As an approximation,* $X(t)$ *cannot exactly capture all the characteristics of a congestion event. However its use simplifies the analysis and produces useful results.*

## 4.5 Approximations for Temporal Properties of Congestion Events

### 4.5.1 Mean Sojourn Time and Inter-congestion Event Time

By (4.28), the mean sojourn time $E^0[C_{Q,b}]$ is approximated with $E[C_{X,b}]$. From Section 4.3, $U_{C_b}$ defined in (4.13) is used to approximate $E[C_{X,b}]$. Thus

$$E^0[C_{Q,b}] \approx E[C_{X,b}] \approx U_{C_b}. \tag{4.32}$$

Combining (4.3) and (4.32), the mean inter-congestion event time $E^0[\tau_b]$ can be expressed as

$$E^0[\tau_b] \approx \frac{U_{C_b}}{P(Q(0) \geq b)}. \tag{4.33}$$

Even though several approximations were applied to obtain (4.33), the above analysis successfully predicts trends observed from simulations. The method provides better predictions for the inter-congestion event time than directly using the reciprocal of the tail probability, $1/P(Q(0) \geq b)$, as will be discussed in Section 4.6.

### 4.5.2 Mean Duration of Congestion Events

As shown in Figure 4.1, a congestion event starts at time $t_b$ and ends at $t_2$. Let $D_{cong,b} = t_2 - t_b$ denote the duration time of a congestion event. Since the period $[t_b, t_2]$ of $Q(t)$ is approximated by $[0, R_b]$ of $X(t)$ and from (4.28), $E^0[D_{cong,b}]$ can be expressed as

$$E^0[D_{cong,b}] = E^0[t_2 - t_b] \approx E[R_b] \approx U_{R_b}. \tag{4.34}$$

### 4.5.3 Mean Duration of Busy Periods

Let $D_{Q,b}$ denote the duration of a busy period in which a congestion event occurs. The mean duration is $E^0[D_{Q,b}] = E^0[t_2 - t_1]$. From Figure 4.1, $E^0[D_{Q,b}]$ can be written as $E^0[D_{Q,b}] = E^0[D_{cong,b}] + E^0[t_b - t_1]$. Recall that $t_b - t_1$ is approximated with a constant $r_b$, which can be evaluated with (4.9). Thus, combining (4.34),

$$E^0[D_{Q,b}] \approx E^0\left[D_{cong,b}\right] + r_b \approx U_{R_b} + r_b. \tag{4.35}$$

### 4.5.4 Mean Amplitude

The busy periods in a network router have been previously modeled by triangles in [28], so a triangle is used to approximate a busy period in Figure 4.6. The triangle

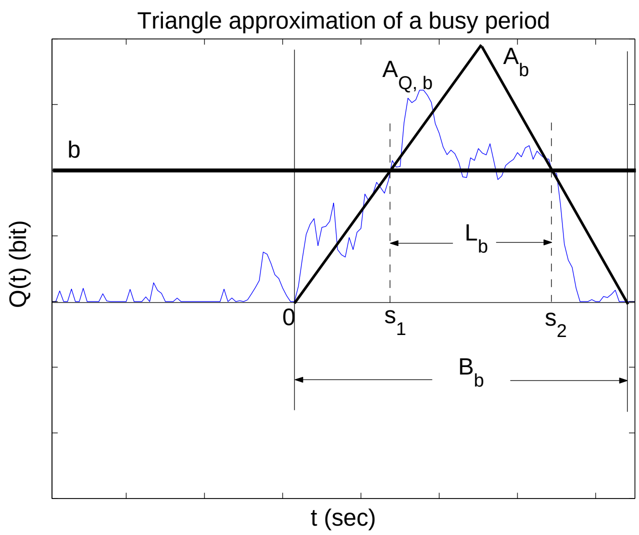



Figure 4.6: Triangle approximation of a busy period

has a base of $B_b$, crosses the level $b$ at $s_1$ and $s_2$. The mean amplitude of a busy period, $E^0[A_{Q,b}]$, can be approximated with the height of the triangle $A_b$. Let $L_b$ denote $s_2 - s_1$. Note that $L_b$ is the length that the triangle stays above the level $b$, $E^0[C_{Q,b}]$, the mean sojourn time of a congestion event, is applied to approximate $L_b$, that is, $L_b \approx E^0[C_{Q,b}]$. The base $B_b$ is approximated with the mean duration time of a busy period $E^0[D_{Q,b}]$. With simple geometry, it can be derived that $A_b = b\frac{B_b}{B_b - L_b} \approx b\frac{E^0[D_{Q,b}]}{E^0[D_{Q,b}] - E^0[C_{Q,b}]}$. Combining (4.32) and (4.35),

$$E^0[A_{Q,b}] \approx A_b \approx b\frac{U_{R_b} + r_b}{U_{R_b} + r_b - U_{C_b}}. \tag{4.36}$$

### 4.6 Numerical Comparison

So far the properties of congestion events of a scaled queue length process $Q = \{Q(t), t \in \mathbb{R}\}$ has been discussed.

Here the properties of congestion events of the original queue length process, given in (4.1), is briefly discussed. Recall that $Q_o = \{Q_o(t), t \in \mathbb{R}\}$ is the queue length process which has an input ($m$, $\sigma$, $H$) and a service rate $\mu$. To study the congestion events in $Q_o$, first transform the process $Q_o$ to the corresponding scaled queue length process $Q$, as given in (4.2). Then the temporal properties of congestion events with a level $b_o$ in $Q_o$ is equal to the corresponding temporal properties of congestion events with a level $b = b_o/\sigma$ in the scaled process $Q$. For example, the mean sojourn time of congestion events with a level $b_o$ in $Q_o$ is equal to $E^0[C_{Q,b}]$ where $b = b_o/\sigma$. Similar results hold for the mean inter-congestion event time, the mean duration of a congestion event and the mean duration of a busy period. But note that in the process $Q_o$, the mean amplitude of congestion events with a level $b_o$ is equal to $\sigma E^0[A_{Q,b}]$.

In the following, the properties of congestion events and the corresponding busy periods are evaluated. Evaluations based on the above analysis are compared with simulation results. Fractional Brownian motions are generated with the algorithm proposed in [11]. For $H \in [0.5, 0.79]$, 20 traces of fractional Brownian motion are generated, each trace has $2^{24}$ samples; for $H = 0.85$, 80 traces are generated, each has $2^{22}$ samples [1]. The parameters $H$ and $c$ are varied to modify the long-range intensity and the scaled surplus rate. The relative error of the approximations is reported, which is defined as $\frac{||x-\hat{x}||}{||x||}$, where $x$ is the simulation result, $\hat{x}$ is the corresponding approximation and $||\cdot||$ is the Euclidean norm.

Different values for $\Delta t$, the time between consecutive samples, are used in simulations. On one hand, the $\Delta t$ is necessary to be small so that the sojourn and duration

[1] The simulations were performed on a computer with two Intel Xeon Processors running at 2.8 GHz with 2GB RAM. The memory capacity combined with the numerical limitation of the algorithm in [11] limited the sample size to $2^{24}$ for $H \in [0.5, 0.79]$, and $2^{22}$ for $H = 0.85$.

times can be measured accurately; on the other hand, to collect enough congestion events, the whole trace ($2^{24}$ or $2^{22}$ samples) needs to represent a time series which is in the order of hours.

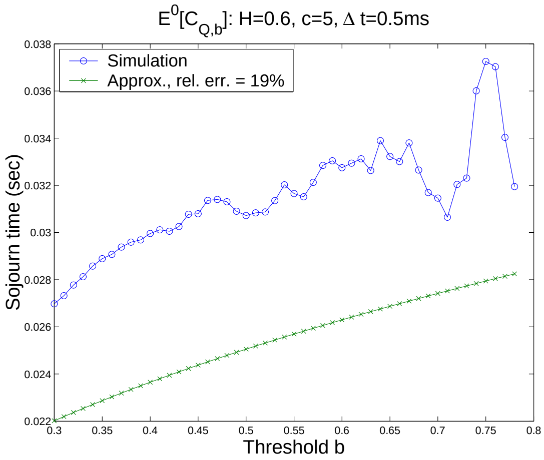
For a fixed simulation length when the threshold $b$ increases, fewer and fewer congestion events occur (the events become rare). For example, under the conditions $H = 0.85$, $c = 3.5$, for $b = 0.05$, there are over 50000 congestion events, but for $b = 0.25$, only about 600 events over 80 traces can be collected. Consequently, fluctuations can be noticed for large $b$ in the simulation results, see Figure 4.7$a$, 4.7$b$, 4.7$c$, 4.9$a$.

### 4.6.1 Mean Sojourn Time $E^0[C_{Q,b}]$

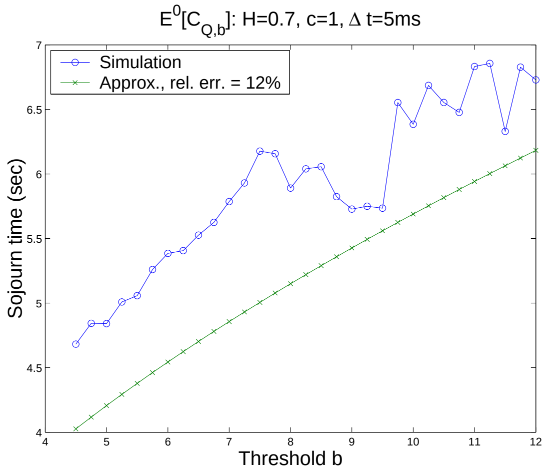
The comparisons between the predicted and simulated $E^0[C_{Q,b}]$ are shown in Figure 4.7. The approximation results follow the trends as a function of the surplus rate $c$ and the Hurst parameter $H$, the relative errors range from 10% to 20%. The errors are partly caused by $r_b$. It is observed that $r_b$ overestimates $E^0[t_b - t_1]$, i.e., the time that the queue builds up from 0 to $b$ in a busy period. Fluctuations, which are caused by small sample sizes, can be observed for large $b$, Figure 4.7$a$, 4.7$b$, 4.7$c$. Similar phenomena have been observed in network router performance measurements, e.g., Figure 13 in [28].

### 4.6.2 Mean Inter-Congestion Event Time $E^0[\tau_b]$

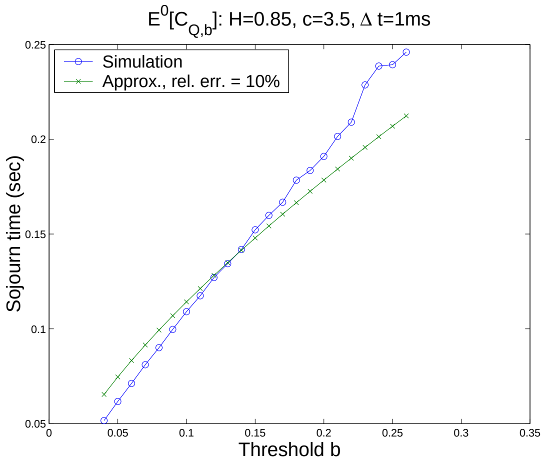
The approximation given in (4.33) is compared with the simulation results and another approximation method, $1/P(Q(0) \geq b)$, the reciprocal of the overflow probability. As shown in Figure 4.8, the approximation (4.33) outperforms $1/P(Q(0) \geq b)$ in most cases. Notice that for different parameter sets, $E^0[\tau_b]$ may increase or decrease with respect to $H$. For example, when $b = 2.9$, $c = 1.5$, $E^0[\tau_b]$ decreases versus $H$ as shown in Figure 4.8$e$; but for $b = 0.95$, $c = 3$, $E^0[\tau_b]$ increases in Figure 4.8$f$. In both cases, the approximation results can follow the observed trends. In Figure



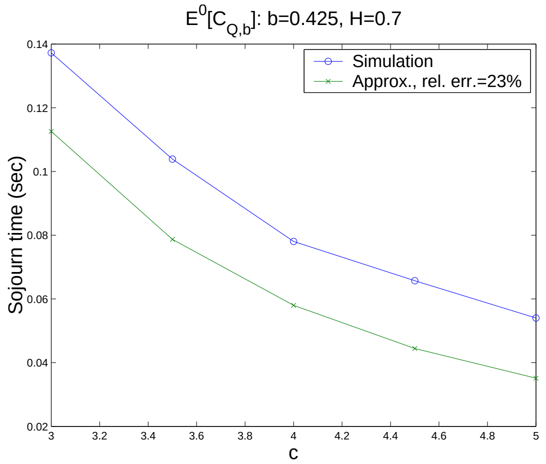

(a) Sojourn time vs $b$



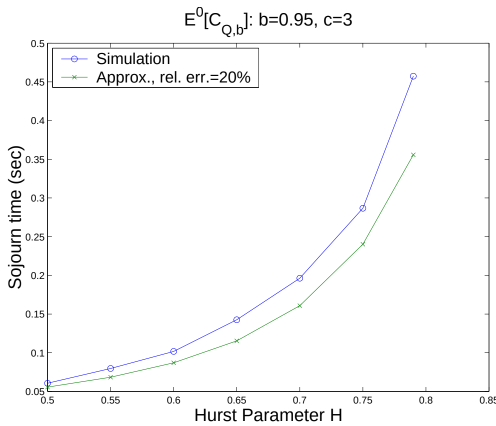

(b) Sojourn time vs $b$




(c) Sojourn time vs $b$

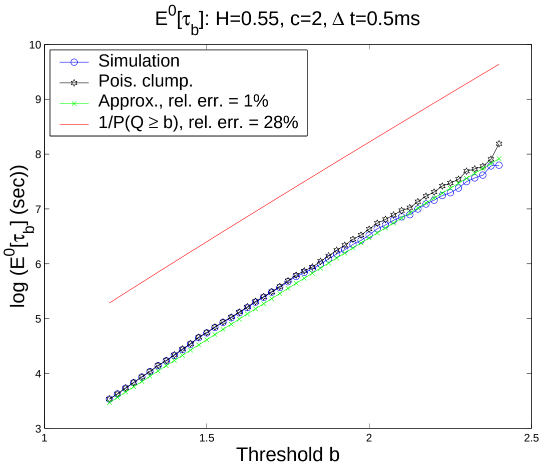



(d) Sojourn time vs $c$

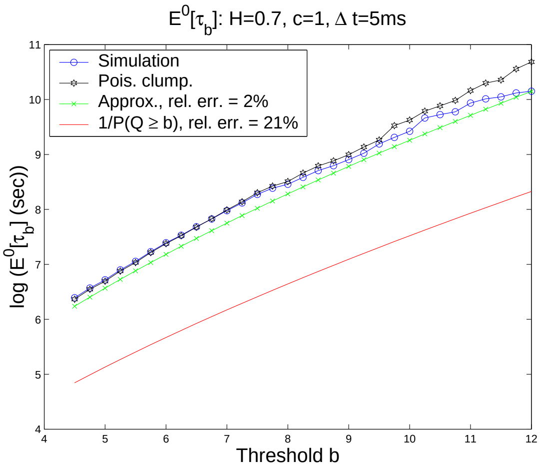



(e) Sojourn time vs $H$

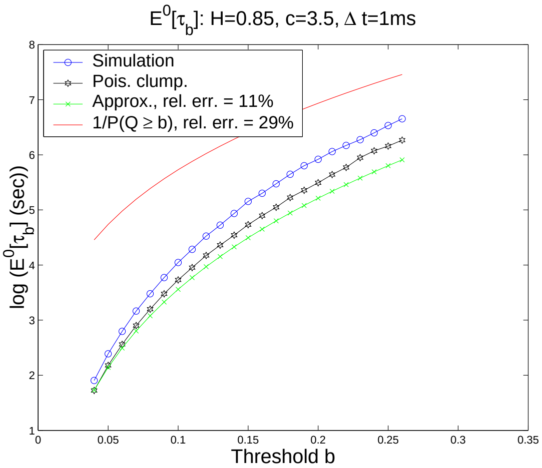
Figure 4.7: Comparison of mean sojourn time



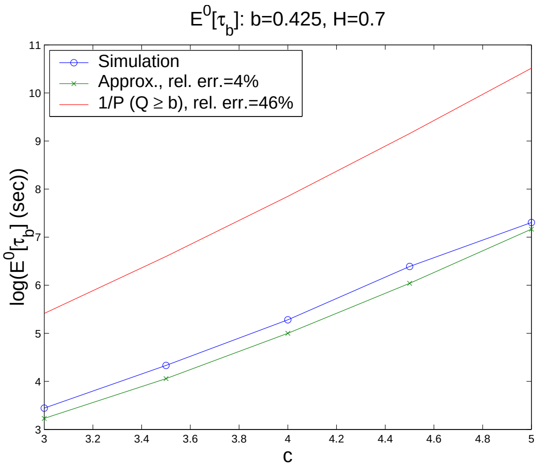

(a) Inter-congestion time vs $b$



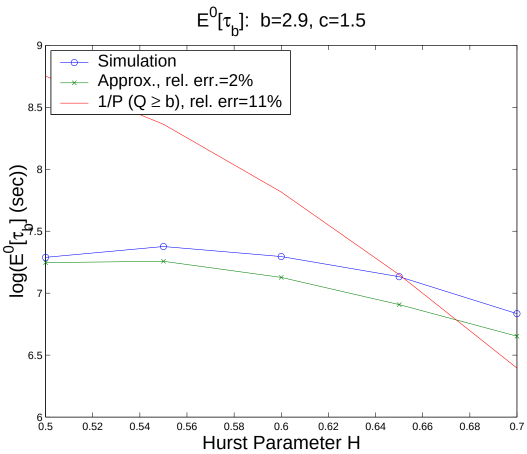

(b) Inter-congestion time vs $b$



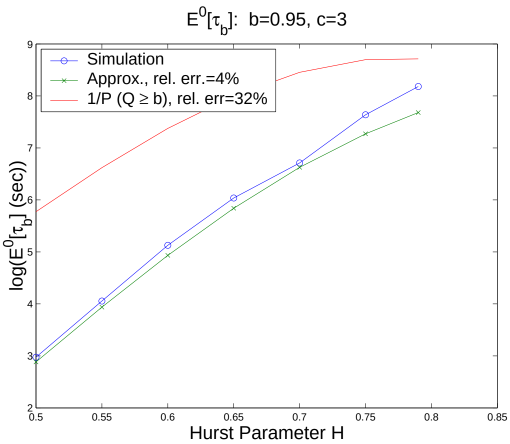

(c) Inter-congestion time vs $b$




(d) Inter-congestion time vs $c$




(e) Inter-congestion time vs $H$




(f) Inter-congestion time vs $H$

Figure 4.8: Comparison of mean inter-congestion time

4.8$a$, 4.8$b$ and 4.8$c$, the Poisson clumping approximation, given in (4.3), is validated; both $E^0[C_{Q,b}]$ and $P(Q(0) \geq b)$ in (4.3) are measured from the simulations.

### 4.6.3 Mean Duration of Congestion Events $E^0[D_{cong,b}]$

It is shown in Figure 4.9 that the approximation, given in (4.34), is close to the simulation results of $E^0[D_{cong,b}]$, the relative errors are around 10%. In all situations, as shown in Figure 4.9$d$, 4.9$e$, the approximations follow the trends of the simulation results.

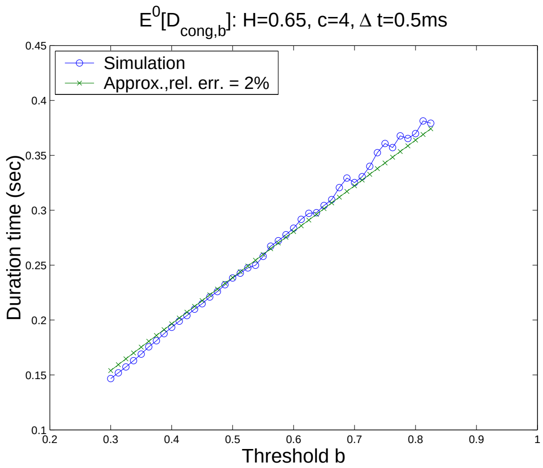
### 4.6.4 Mean Duration of Busy Periods $E^0[D_{Q,b}]$

In Figure 4.10, the mean durations of busy periods observed from simulations are compared with the approximation (4.35). Noticed that $r_b$, given in (4.9), overestimates the mean time that the queue increases from 0 to $b$. Thus $E^0[D_{Q,b}]$ is overestimated by the approximation. However, the approximation results follow the observed trends, the relative errors are from 10% to 30%.

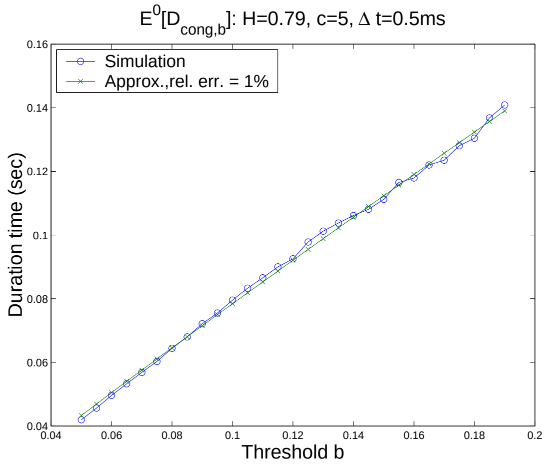
### 4.6.5 Mean Amplitude $E^0[A_{Q,b}]$

From the simulation results, it is observed that the mean amplitude follows a linear trend as a function of the threshold $b$. As shown in Figure 4.11, the approximations underestimate $E^0[A_{Q,b}]$. Based on (4.36), the underestimation is caused by the overestimation of $E^0[D_{Q,b}]$. But again the approximations follow the simulation trends, the relative errors are around 10%. The errors in the approximations are partly from $r_b$, which overestimates the time that $Q(t)$ increases from 0 to $b$. If better knowledge of $r_b$ is known, the approximation results can be improved.

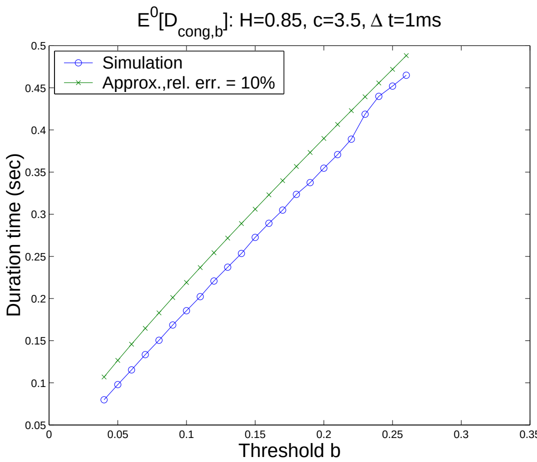
To illustrate an application of the proposed methodology, suppose that a link capacity is to be chosen for a conferencing teleservice. The requirement of an error free interval for audio and video multimedia conferencing teleservices is given as 30 minutes [23], i.e., the average inter-congestion event time $E^0[\tau_b]$ is 1800 seconds,



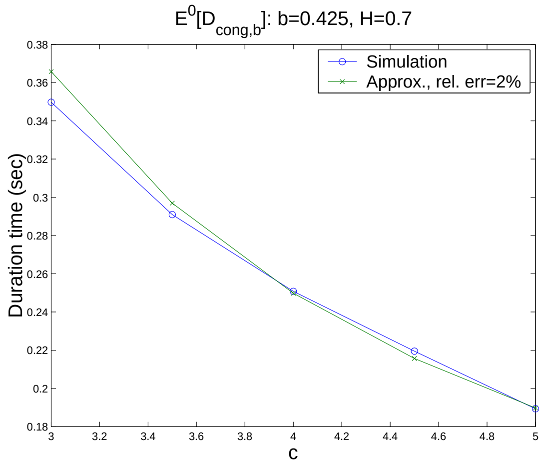

(a) Duration of congestions vs $b$



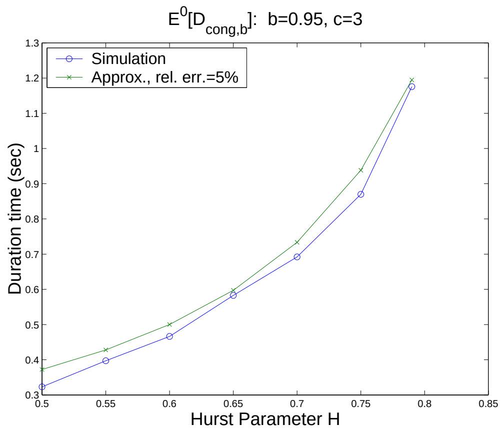

(b) Duration of congestions vs $b$




(c) Duration of congestions vs $b$

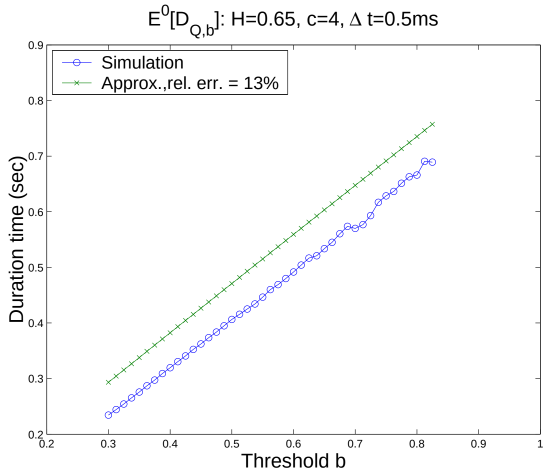



(d) Duration of congestions vs $c$

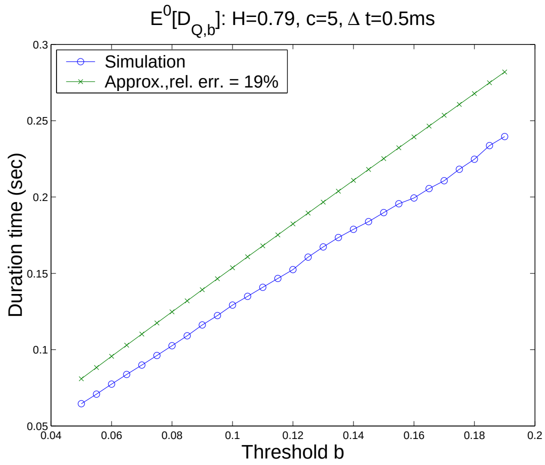



(e) Duration of congestions vs $H$

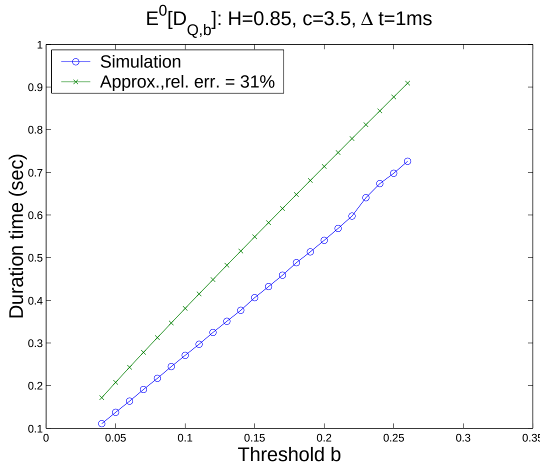
Figure 4.9: Comparison of mean duration of congestion events



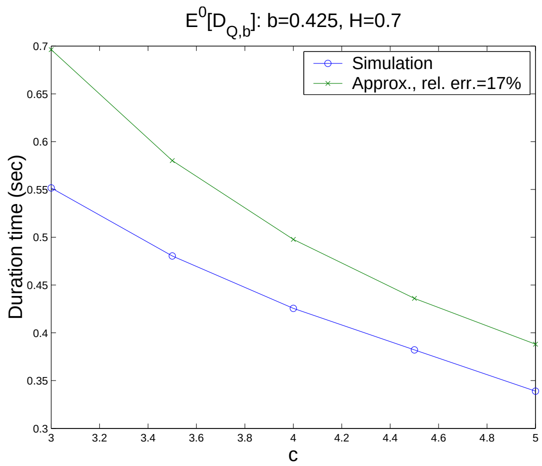

(a) Duration of busy periods vs $b$



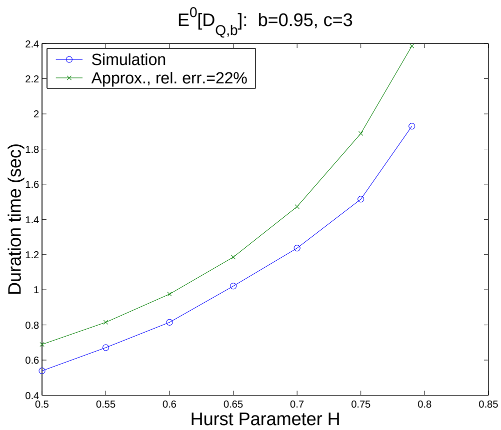

(b) Duration of busy periods vs $b$




(c) Duration of busy periods vs $b$

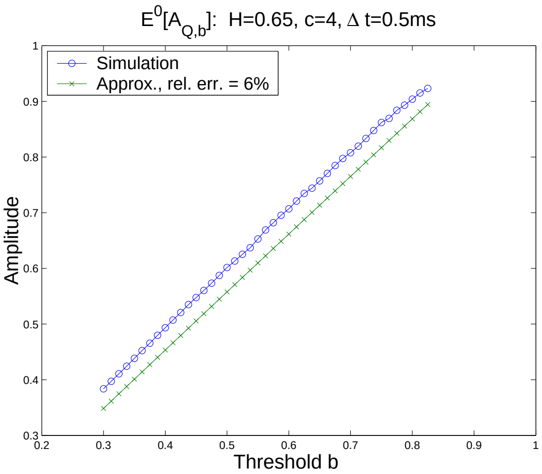



(d) Duration of busy periods vs $c$

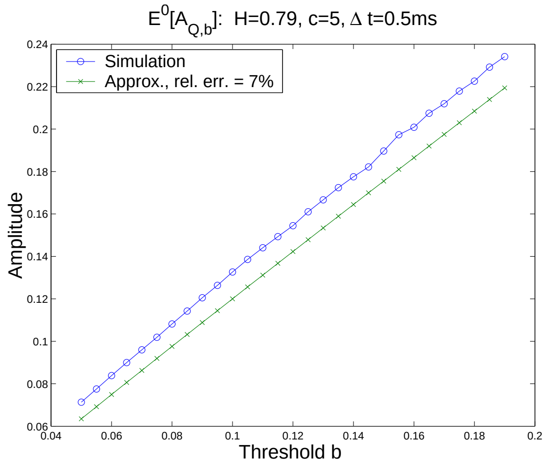



(e) Duration of busy periods vs $H$

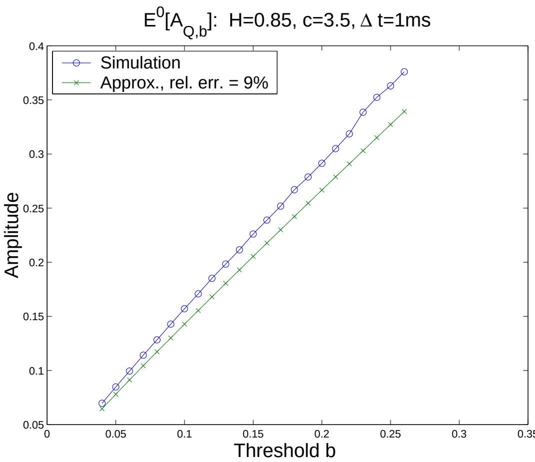
Figure 4.10: Comparison of mean duration of busy periods



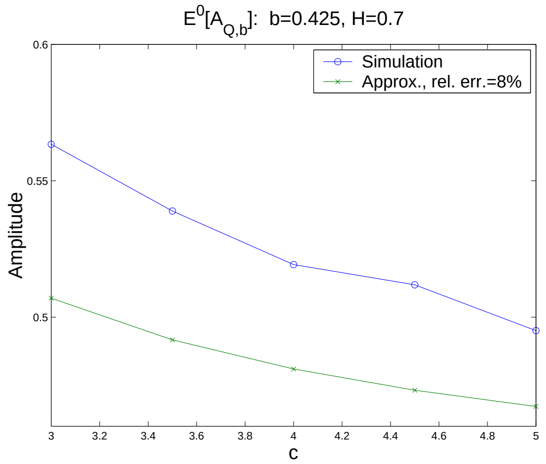

(a) Amplitude vs $b$



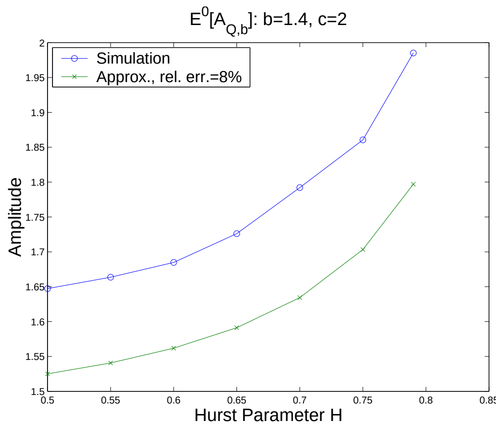

(b) Amplitude vs $b$




(c) Amplitude vs $b$




(d) Amplitude vs $c$




(e) Amplitude vs $H$

Figure 4.11: Comparison of mean amplitude of congestion events

$\log(E^0[\tau_b]) \approx 7.5$. Then for a fractional Brownian motion traffic characterized by $m = 100Mbps$, $\sigma = 10^7 bit$, $H = 0.75$, the proposed method indicates that for a congestion level of $b = 5.5Mb$, a link capacity $\mu = 140Mbps$ (traffic load $\rho = m/\mu \approx 0.7$) would be required to ensure the average congestion free interval of 30 minutes, and in this case, $E^0[C_{Q,b}] \approx 90ms$, $E^0[D_{cong,b}] \approx 395ms$, $E^0[D_{Q,b}] \approx 800ms$, $E^0[A_{Q,b}] \approx 6.2Mb$.

## 4.7 Conclusion

It has been recognized that the frequency and the duration of congestion events significantly impact user-perceived performance. Previous efforts have focused on measurement-based approaches to determine the frequency and duration of these events. However, for network design, techniques are needed to predict the congestion events given the nature of traffic. This chapter provides new techniques to approximate several properties of congestion events, the rate, duration, and amplitude given a fractional Brownian motion traffic. The technique to approximate the rate outperforms the reciprocal of the overflow probability, i.e., $1/P(Q(0) \geq b)$, and follows the trends observed from simulations. As in [24], the approach for predicting the rate of congestion events can be directly extended to determine the expected rate of congestion events for an end-to-end flow that passes through several queues. Congestion events at each queue along a path can be assumed to be independent and rare, so an end-to-end flow will experience the sum of the congestion events along the path. The inter-congestion event time $E^0[\tau_b]$ (or its rate $1/E^0[\tau_b]$), which can be easily understood by network users, is a useful QoS metric for network design. The other metrics of congestion events, such as the sojourn time above a threshold, the duration, and the amplitude, give additional insights into the nature of congestion events.

These results can be extended in several areas. The accuracy of the techniques developed here can be improved. The properties of busy periods whose durations are larger than a fixed $T$, discussed in [45], [52], are interesting problems for further study. Other self-similar traffic models need to be considered, such as the Levy processes. To understand fully the impacts of self-similar traffic on networks, these processes need to be analyzed and additional methodologies developed.

## 4.8 Appendix

Some results used in the chapter are proved in this section as lemmas.

For particular forms of $S(x, u)$, the asymptotic of the integral as $u \to \infty$,

$$\int_0^\infty \exp\left(S(x,u)\right) dx$$

can be discussed through the saddle point method, cf. [12], [47] and the references therein. Let $x_0(u)$ denotes the point at which the function $S(x, u)$ of $x$ achieves its maximum over $[0, \infty)$. Denote for some suitable chosen function $q(u)$

$$U(x_0(u)) = \left\{ x : |x - x_0(u)| \le q(u) \left| S''_{xx}(x_0(u), u) \right|^{-1/2} \right\}.$$

The following result is cited from [12].

**Theorem 4.8.1.** *Suppose that*

*(a) there exists a function $q(u) \to \infty$ as $u \to \infty$ such that*

$$S''_{xx}(x,u) = S''_{xx}(x_0(u), u)[1 + o(1)]$$

*as $u \to \infty$ uniformly for $x \in U(x_0(u))$,*

*(b) $S''_{xx}(x, u) < 0$ for all $x, u$,*

*(c) $\lim_{u\to\infty} x_0(u)\sqrt{|S''_{xx}(x_0(u), u)|} = \infty$.*

*Then as $u \to \infty$,*

$$\int \exp\left(S(x,u)\right) dx \sim \sqrt{-\frac{2\pi}{S''_{xx}(x_0(u),u)}} \exp\left(S\left(x_0(u),u\right)\right).$$

**Lemma 4.8.1.** *For $c > 0$, as $u \to \infty$,*

$$\int_0^\infty \exp\left(-\frac{(u-ct)^2}{2t}\right) dt \sim \sqrt{\frac{2\pi u}{c^3}}.$$

*Proof.* Let $S(x,u) = -\frac{(u-cx)^2}{2x}$, then $x_0(u) = \frac{u}{c}$ and

$$\begin{aligned} S'_x(x,u) &= \frac{u^2}{2x^2} - \frac{c^2}{2}, \\ S''_{xx}(x,u) &= -\frac{u^2}{x^3}, \\ S''_{xx}\left(x_0(u),u\right) &= -\frac{c^3}{u}. \end{aligned}$$

Let $q(u) = u^{\frac{1}{3}}$, then

$$\begin{aligned} U\left(x_0(u)\right) &= \{x : |x - x_0(u)| \le q(u)c^{-\frac{3}{2}}\} \\ &= \left\{x : \frac{u}{c} - u^{\frac{5}{6}}c^{-\frac{3}{2}} \le x \le \frac{u}{c} + u^{\frac{5}{6}}c^{-\frac{3}{2}}\right\}. \end{aligned}$$

It can be verified that

- $S''_{xx}(x,u) = S''_{xx}(x_0(u),u)[1 + o(1)]$ as $u \to \infty$ uniformly for $x \in U(x_0(u))$;
- $S''_{xx}(x,u) < 0$ for all $x, u$;
- $\lim_{u\to\infty} x_0(u)\sqrt{|S''_{xx}(x_0(u),u)|} = \infty$.

Since $S(x_0(u),u) = 0$, by Theorem 4.8.1, as $u \to \infty$,

$$\int_0^\infty \exp\left(S(x,u)\right) dx \sim \sqrt{-\frac{2\pi}{S''_{xx}(x_0(u),u)}} \exp\left(S(x_0(u),u)\right) = \sqrt{\frac{2\pi u}{c^3}}.$$

□

**Lemma 4.8.2.** *Let $w(t)$ and $g(t)$ be defined in Theorem 4.3.2. Then for all $t \geq 0$,*

$$w^{-1}(t) \leq g(t). \tag{4.37}$$

*Proof.* To show $w^{-1}(t) \leq g(t)$, it is sufficient to show that $w(t) \geq g^{-1}(t)$, where $g^{-1}(t)$ is the inverse function of $g(t)$, that is, $g^{-1}(g(t)) = t$. From the definition of $g(t)$, it is obtained that $g^{-1}(t) = \alpha_0 t^{2H}$, where $\alpha_0$ is defined in (4.18).

Since $w(t) - \alpha_0 t^{2H} = 0$ for $t = 0$, it suffices to show that $w(t) - \alpha_0 t^{2H}$ increases with respect to $t$, i.e., $w'(t) - 2H\alpha_0 t^{2H-1} \geq 0$.

$$\begin{aligned}
& w'(t) - 2H\alpha_0 t^{2H-1} \\
=& 2H(1-\alpha_0)t^{2H-1} - \frac{H\left[(t+r)^{2H} - t^{2H} - r^{2H}\right]\left[(t+r)^{2H-1} - t^{2H-1}\right]}{r^{2H}} \\
=& 2H(1-\alpha_0)r^{2H-1}\left(\frac{t}{r}\right)^{2H-1} \\
& - Hr^{2H-1}\left[\left(\frac{t}{r}+1\right)^{2H} - \left(\frac{t}{r}\right)^{2H} - 1\right]\left[\left(\frac{t}{r}+1\right)^{2H-1} - \left(\frac{t}{r}\right)^{2H-1}\right]
\end{aligned}$$

Let $s = t/r$, then $w'(t) - 2H\alpha_0 t^{2H-1}$ can be written as

$$Hr^{2H-1}\left\{2(1-\alpha_0)s^{2H-1} - \left[(s+1)^{2H} - s^{2H} - 1\right]\left[(s+1)^{2H-1} - s^{2H-1}\right]\right\},$$

which implies the inequality (4.37) holds. □

**Lemma 4.8.3.** *Let $\mu_{r_b}(\cdot)$ and $g(v)$ be defined as in (4.11) and (4.21), respectively. Suppose then $b > 0$ and $c > 0$, then $b + \mu_{r_b}(g(v)) - cg(v)$ is a convex function.*

*Proof.* Let $h(v) = b + \mu_{r_b}(g(v)) - cg(v)$ and $\beta = \alpha_0^{-\frac{1}{2H}}$, where $\alpha_0$ is given in (4.18), then

$$h(v) = b + \frac{c}{2Hr_b^{2H-1}}\left[\left(\beta t^{\frac{1}{2H}} + r_b\right)^{2H} - r_b^{2H} - \left(\beta t^{\frac{1}{2H}}\right)^{2H}\right] - c\beta t^{\frac{1}{2H}}.$$

The first derivative of the function is

$$h'(v) = \frac{c}{2Hr_b^{2H-1}}\left[\left(\beta t^{\frac{1}{2H}} + r_b\right)^{2H-1} \beta t^{\frac{1}{2H}-1} - \beta^{2H}\right] - \frac{c\beta}{2H} t^{\frac{1}{2H}-1}.$$

The second derivative is given by

$$\begin{aligned}
h''(v) = &= \frac{c\beta}{2Hr_b^{2H-1}}(\frac{1}{2H}-1)t^{\frac{1}{2H}-2}(\beta t^{\frac{1}{2H}} + r_b)^{2H-2}\left[(\beta t^{\frac{1}{2H}} + r_b) - \beta t^{\frac{1}{2H}}\right] \\
&- \frac{c\beta}{2H}(\frac{1}{2H}-1)t^{\frac{1}{2H}-2} \\
&= \frac{c\beta}{2Hr_b^{2H-1}}(\frac{1}{2H}-1)t^{\frac{1}{2H}-2}(\beta t^{\frac{1}{2H}} + r_b)^{2H-2} r_b - \frac{c\beta}{2H}(\frac{1}{2H}-1)t^{\frac{1}{2H}-2} \\
&= \frac{c\beta}{2H}(\frac{1}{2H}-1)t^{\frac{1}{2H}-2}\left(\frac{(\beta t^{\frac{1}{2H}} + r_b)^{2H-2}}{r_b^{2H-2}} - 1\right) \geq 0.
\end{aligned}$$

□

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