Continuous selections from the Pareto correspondence and non-manipulability in exchange economies

Biung-Ghi Ju*

University of Kansas, *Department of Economics*

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Abstract

In n-agent exchange economies, we show that all *efficient* and *continuous* rules are "diagonally dictatorial" over the restricted domain of linear preferences and, in the 2-good case, over the domain of homothetic preferences. The diagonal dictator receives the entire endowment whenever all agents have an identical preference. We show that (fully) dictatorial rules are the only rules satisfying, in addition, *veto-proofness*, the requirement that if truth-telling ever leads to the worst outcome for an agent, he shouldn't be able to escape it, by misrepresenting his preference. The same conclusion holds replacing *veto-proofness* with stronger notions of non-manipulability, *veto-proofness** (no one can escape from the worst outcome or switch to the best outcome), *weak strategy-proofness* (no one can increase his bundle), and *strategy-proofness*. We extend these results to any larger domain imposing *non-bossiness* (no one can affect others' bundles without affecting his own).

Keywords: Efficiency, continuity, veto-proofness, strategy-proofness, exchange economy.

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*Department of Economics, 1300 Sunnyside Avenue, Summerfield Hall, University of Kansas, KS 66045, USA. E-mail: `bgju@ku.edu`. Tel.: (785) 864-2860. Fax.:(785) 864-5270. I am grateful to William Thomson for helpful comments and suggestions. I also thank John Duggan, Lars Ehlers, Sidartha Gordon, James Schummer, and an anonymous referee for comments. All remaining errors are mine.

1 Introduction

In exchange economies, an allocation rule, or simply a *rule*, associates with each profile of preferences a *single* feasible allocation. An allocation is (Pareto) *efficient* if no one can be made better off without anyone else being made worse off. The *Pareto correspondence* maps each preference profile into the set of all *efficient* allocations. An *efficient* rule is a selection from the Pareto correspondence. We study continuous selections from the Pareto correspondence, or *efficient* and *continuous* rules, and, in particular, these rules satisfying the following notions of non-manipulability.

Strategy-proofness is the requirement that no one can benefit by misrepresenting his preference, independently of others' representations (Gibbard, 1973, Satterthwaite, 1975). We introduce three weaker notions of non-manipulability, which pertain to three cases of *partially* strategic agents. The first case is that each agent behaves cooperatively representing his preference truthfully, unless he is treated worst. *Veto-proofness* is the requirement that if truth-telling ever leads to the worst outcome for an agent, he shouldn't be able to escape it, by misrepresenting his preference. A stronger notion, *veto-proofness**, pertains to the case that each agent represents his preference truthfully, unless he can escape from the worst outcome or can switch to the best outcome. The last notion, *weak strategy-proofness*, pertains to the case that each agent represents his preference truthfully, unless he can increase his bundle in terms of vector dominance relation.

We show that over the restricted domain of "linear preferences" (preferences with linear utility function), every *efficient* and *continuous* rule is *diagonally dictatorial*, that is, there exists an agent who receives the social endowment, whenever all agents have the same preference. Therefore, for any domain including the linear domain, there exists no *efficient* and *continuous* rule satisfying any one of the following standard equity criteria, *equal treatment of equals* (any two agents with the same preference should be treated the same), *no-envy* (Foley, 1967; no one should prefer any of others' bundles to his own), etc.[1] We next show that over the linear domain, a rule is *efficient, continuous,* and *veto-proof* if and only if it is *dictatorial*, that is, there exists an agent who always receives the social endowment, leaving nothing for anyone else. The same result holds replacing *veto-proofness* with any one of *veto-proofness**, *weak strategy-proofness*, and *strategy-proofness.* We extend this result over the linear domain to any larger domain with the additional requirement of *non-bossiness* (no one can affect others'

[1]See Thomson (1995) for an extensive treatment of fairness in economic environments.

bundle without affecting his own bundle) introduced by Satterthwaite and Sonnenschein (1981). Given any domain including the linear domain, we show that a rule is *efficient, continuous, veto-proof** (or *weakly strategy-proof*), and *non-bossy* if and only if it is dictatorial. Since all linear preferences are "homothetic", this result applies to the domain of *homothetic* preferences, or the homothetic domain, which is widely considered in various applications (Chipman, 1974, and Chipman and Moore, 1973). In the *2-good case*, we establish the following two stronger results over the homothetic domain: we show that every *efficient* and *continuous* rule is *diagonally dictatorial* and that a rule is *efficient, continuous,* and *veto-proof* if and only if it is *dictatorial.* The second result holds replacing *veto-proofness* with *veto-proofness** or *weak strategy-proofness* or *strategy-proofness.*

For the unrestricted domain of abstract social choice, the Gibbard-Satterthwaite Theorem (Gibbard, 1973, Satterthwaite, 1975) states that if a rule is *strategy-proof* and "onto", then it is dictatorial.[2] Since in economic applications, preferences are subject to a variety of restrictions, the theorem does not apply.[3] However a number of studies have brought out similar difficulties in satisfying *efficiency* and *strategy-proofness* in standard exchange economies.[4] In particular, for the 2-agent case, under "classical" assumptions on preferences, Zhou (1991) shows that a rule is *efficient* and *strategy-proof* if and only if it is dictatorial. Schummer (1997) strengthens this result by establishing it over the two restricted domains of homothetic preferences and linear preferences, respectively. Ju (2002) identifies general domain conditions leading to this impossibility result.

There are well-known difficulties in extending this negative result in the 2-agent case to the n-agent case: see Zhou (1991) and Kato and Ohseto (2001) for *conjectures* in the n-agent case. We contribute to this line of research by showing that over each of the two restricted domains, the linear domain and the 2-good homothetic domain, when *continuity* is required additionally, the negative result extends to the n-agent case.

In the n-agent case, important contributions have been made in the three recent works by Serizawa (2000a), Serizawa (2000b), and Serizawa and Wey-

[2]A rule is *onto* if its range is equal to the set of all social alternatives. In the abstract social choice model, if a rule is *Pareto efficient,* then it is onto. Moreover, as is well-known (see Mas-Colell, Whinston, and Green, 1995), if a rule is *onto* and *strategy-proof*, then it is *Pareto efficient.*

[3]In exchange economies, preferences are assumed to satisfy "no-consumption-externality" (individuals are not affected by others' consumption) and "monotonicity".

[4]References are Hurwicz (1972), Dasgupta, Hammond, and Maskin (1979), Hurwicz and Walker (1990), Zhou (1991a), Schummer (1997), and Serizawa (1998). See also the extensive surveys by Sprumont (1995a), Barberà (2001), and Thomson (2001)

mark (2002). Serizawa (2000a) shows that there is no rule satisfying *efficiency, individual rationality,* and *strategy-proofness.* Replacing *individual rationality* with a much weaker axiom, called "minimum consumption guarantee", Serizawa and Weymark (2002) establish an even stronger impossibility. Serizawa (2000b) does not impose *continuity* but considers a stronger non-manipulability notion, "pairwise strategy-proofness" associated with preference misrepresentation by groups consisting of at most two agents.

Continuity is desirable for the following practical reason. In order to make a choice, we need information about preferences. However, even if people are willing to reveal their own true preferences, for a number of reasons, they may not be known accurately. If a rule is not *continuous*, the choice it makes may be vulnerable to such inaccuracy.[5]

In the 2-agent and 2-good case, Sprumont (1995b) characterizes *continuous* and *strategy-proof* rules over the same homothetic domain as ours. He shows that any *continuous* and *strategy-proof* rule has the following dictatorial feature: there exists an agent and an exogenously determined strictly convex subset of consumption space such that the rule picks the best point for the agent in the set. Our results show that *in the n-agent case*, even if *strategy-proofness* is replaced with *efficiency*, we still have "diagonal dictatorship", which implies strong violation of most standard equity criteria.

Continuity and *strategy-proofness* are studied also by Satterthwaite and Sonnenschein (1981), yet on the domain consisting of only *strictly* convex preferences. They impose in addition, "continuous differentiability" and *non-bossiness.* The domain of production economies with convex technologies is one of the domains they consider, and this domain includes exchange economies. Their result for this domain implies that there exists no rule that is *efficient*, *continuously differentiable*, *strategy-proof*, and *non-bossy.* However they do not study consequences of dropping *non-bossiness* or weakening continuous differentiability to *continuity* or weakening *strategy-proofness.*[6]

[5] In the implementation literature, one of the desirable properties of a game form is *continuity* of its outcome function (See Postlewaite and Wettstein, 1989). *Continuity* guarantees the robustness of equilibrium to small misspecifications of strategies. Outcome functions are often closely related with rules they implement. In particular, in the direct revelation mechanism, the rule coincides with the outcome function.

[6] In various models, central *strategy-proof* rules are not differentiable. For example, in public good economies with single-peaked preference considered by Moulin (1980), the "generalized Condorcet-winner rules", which are the only rules satisfying *Pareto efficiency, anonymity,* and *strategy-proofness,* are not differentiable. In private good economies with single-peaked preferences considered by Sprumont (1991), the "uniform rule", which is the only rule satisfying

Chichilnisky (1979, 1980, 1982), Chichilnisky and Heal (1983), and Zhou (1997) study *continuous* "preference aggregation rule", which is a function mapping each preference profile into a social "preference". These studies exhibit difficulties of satisfying *continuity* together with other equity criteria such as "unanimity" (similar to our *efficiency*) and "anonymity" (similar to *equal treatment of equals*). Although their results are established for "preference aggregation rules" and in the topological social choice model, our result on diagonal dictatorship has a similar flavor to theirs.

This paper is composed of five sections. In Section 2, we define our model and basic concepts. In Section 3, we establish several useful lemmas. Our main results are in Section 4. We conclude with a few remarks in Section 5.

2 The model and basic concepts

Let l be the number of goods, $l \geq 2$, and $\Omega \in \mathbb{R}^l_+$ be the social endowment. Let $N \equiv \{1, \cdots, n\}$ be the set of agents. Let $Z \equiv \{z \in \mathbb{R}^{l \cdot n}_+ : \sum_N z_i = \Omega\}$ be the set of feasible allocations and $Z_0 \equiv \{z_i \in \mathbb{R}^l_+ : 0 \leqq z_i \leqq \Omega\}$ the set of possible consumption bundles for each agent.

Each agent has a ***preference***, a complete and transitive binary relation over $\mathbb{R}^l_+$. Preferences are continuous*, strictly monotonic,*[7], and convex. For each preference R_0, we use P_0 and I_0 to denote its strict relation and indifference relation, respectively. Let $\mathcal{R}$ be a family of admissible preferences. Since we keep the social endowment fixed, an ***economy*** can be identified by a profile of preferences in $\mathcal{R}$. An *allocation rule,* or simply a ***rule****,* is a function $\varphi \colon \mathcal{R}^N \to Z$ associating with each economy a *single* feasible allocation.

The following two restricted families of preferences are important. A preference R_0 is ***homothetic*** if for all $x, y \in \mathbb{R}^l_+$ and all $\alpha \in \mathbb{R}_+$, $x \; I_0 \; y$ implies $\alpha x \; I_0 \; \alpha y$. Let $\mathcal{R}_H$ be the class of homothetic preferences. A preference R_0 is ***linear*** if it is represented by a vector $p_0 \in \mathbb{R}^l_{++}$ as follows: for all $x, y \in \mathbb{R}_+$, $x \; R_0 \; y$ if and only if $p_0 \cdot x \geq p_0 \cdot y$. Let $\mathcal{R}_L$ be the class of linear preferences. Clearly, $\mathcal{R}_L \subset \mathcal{R}_H$. We call $\mathcal{R}^N_H$ and $\mathcal{R}^N_L$ the ***homothetic domain*** and the ***linear domain****,* respectively. Note that since preferences are strictly monotonic, for all homothetic preference, every indifference curve intersects with each axis. Thus,

Pareto efficiency, *anonymity*, and *strategy-proofness*, is not differentiable.

[7]A preference relation R_i is *strictly monotonic* if $z_i \geq z'_i$ implies $z_i \; P_i \; z'_i$, where the vector inequality $z_i \geq z'_i$ means that each component of z_i is weakly larger than each component of z'_i and $z_i \neq z'_i$.

for example, the "Cobb-Douglas" preferences are not members of $\mathcal{R}_H$. However, this feature is not crucial for our result as explained in Section 5.

For each economy $R \in \mathcal{R}^N$, an allocation $z \in Z$ is (Pareto) ***efficient*** if there is no feasible allocation that makes at least one agent better off without making anyone else worse off. Let $P(R)$ be the set of all *efficient* allocations for R. We call $P(\cdot)$, the ***Pareto correspondence***. A rule $\varphi\colon \mathcal{R}^N \to Z$ is ***efficient*** if it is a selection from the Pareto correspondence, that is, for all $R \in \mathcal{R}^N$, $\varphi(R) \in P(R)$.

Domain $\mathcal{R}_H$ is endowed with the metric $\rho\colon \mathcal{R}_H \times \mathcal{R}_H \to \mathbb{R}_+$ defined as follows. For all $x, y \in \mathbb{R}^l$, let $\overrightarrow{x,y} \equiv \{x + \alpha(y-x) : \alpha \in \mathbb{R}_+\}$ and $\overline{x,y} \equiv \{x+\alpha(y-x) : \alpha \in [0,1]\}$. For all $R_0 \in \mathcal{R}_H$ and all $q \in \Delta^{l-1}$, let $r(q, R_0) \in \mathbb{R}^l_+$ be the point of intersection of the ray, $\overrightarrow{0,q}$, and the indifference curve of R_0 through $(1, \cdots, 1)$ (note that $r(q, R_0)$ is well-defined because preferences are strictly monotonic). For all $R_0, R_0' \in \mathcal{R}_H$, let $\rho(R_0, R_0') \equiv \sup_{q \in \Delta^{l-1}} ||r(q, R_0) - r(q, R_0')||$. For convenience, over the subdomain $\mathcal{R}_L$, we use the following equivalent metric $d\colon \mathcal{R}_L \times \mathcal{R}_L \to \mathbb{R}_+$: for all $R_i, R_i' \in \mathcal{R}_L$, let $p_i \in \Delta^{l-1}$ represent R_i and $p_i' \in \Delta^{l-1}$ represent R_i'. Let $d(R_i, R_i') \equiv \|p_i - p_i'\|$. The topologies of $\mathcal{R}_L^N$ and $\mathcal{R}_H^N$ are the product topologies corresponding to the metric topologies of $\mathcal{R}_L$ and $\mathcal{R}_H$.

In what follows, for any family of preferences, denoted by $\mathcal{R}$, when $\mathcal{R}$ contains $\mathcal{R}_H$ (or $\mathcal{R}_L$), we assume that $\mathcal{R}$ is endowed with a topology inducing the above metric topology in the subspace $\mathcal{R}_H$ (or $\mathcal{R}_L$, respectively) and that $\mathcal{R}^N$ is endowed with the product topology associated with the topology on $\mathcal{R}$. We refer readers to Kannai (1970) for a construction of topology on spaces of preferences.

Given a topology of $\mathcal{R}^N$, we define ***continuity*** of an allocation rule over $\mathcal{R}^N$ in the standard way. We study *efficient* and *continuous* rules, or *continuous selections from the Pareto correspondence*, which are immune to the following kinds of manipulative behavior.

Most well-known is manipulation via preference misrepresentation. A rule $\varphi\colon \mathcal{R}^N \to Z$ is ***strategy-proof*** if no one can ever benefit by misrepresenting his preference, independently of others' representations; that is, for all $i \in N$ and all $R \in \mathcal{R}^N$, there exists no $R_i' \in \mathcal{R}$ such that $\varphi_i(R_i', R_{-i})\ P_i\ \varphi_i(R_i, R_{-i})$.

We next define three weaker notions of non-manipulability, which pertain to three types of *partially* strategic behavior of agents. First is the type in which each agent behaves cooperatively, truthfully representing his preference, *except when he is treated worst*. Each agent misrepresents his preference only to "veto" the worst outcome, namely, the zero bundle. In this case, the following weakening of *strategy-proofness* is important. A rule $\varphi\colon \mathcal{R}^N \to Z$ is ***veto-proof*** if for all

$i \in N$ and all $R \in \mathcal{R}^N$ with $\varphi_i(R) = 0$, there exists no $R_i' \in \mathcal{R}$ such that $\varphi_i(R_i', R_{-i})\ P_i\ \varphi_i(R_i, R_{-i})$.

Second is the type in which each agent truthfully represents his preference, unless he can switch to the best outcome or escape the worst outcome, by misrepresenting his preference. Since preferences are strictly monotonic, Ω is the best outcome and 0 is the worst outcome. In this case, the following weakening of *strategy-proofness* is important. A rule $\varphi\colon \mathcal{R}^N \to Z$ is ***veto-proof***** if for all $i \in N$ and all $R \in \mathcal{R}^N$, there exists no $R_i' \in \mathcal{R}$ such that (i) $\varphi_i(R_i', R_{-i}) = \Omega\ P_i\ \varphi_i(R)$ or (ii) $\varphi_i(R_i', R_{-i})\ P_i\ 0 = \varphi_i(R_i, R_{-i})$.

Third is the type in which each agent truthfully represents his preference, unless he can increase his bundle in terms of vector dominance relation, by misrepresenting his preference. In this case, the following weakening of *strategy-proofness* is important. A rule $\varphi\colon \mathcal{R}^N \to Z$ is ***weakly strategy-proof*** if for all $i \in N$ and all $R \in \mathcal{R}^N$, there exists no $R_i' \in \mathcal{R}$ such that $\varphi_i(R_i', R_{-i}) \geq \varphi_i(R)$.

Clearly, every rule satisfies *veto-proofness* if no allocation in its range has any zero component, that is, no one ever receives the zero bundle. Similarly, every rule satisfies *veto-proofness** if no one ever receives either the zero bundle or Ω. Since preferences are strictly monotonic, *strategy-proofness* implies *weak strategy-proofness*, which implies *veto-proofness**, which implies *veto-proofness.*

Our main results show that all rules satisfying the above requirements have the following "dictatorial" feature. A rule is ***dictatorial*** if there exists an agent who always gets his most preferred bundle in Z_0. Since preferences are strictly monotonic, a rule $\varphi\colon \mathcal{R}^N \to Z$ is dictatorial if and only if there exists $i \in N$ such that for all $R \in \mathcal{R}^N$, $\varphi_i(R) = \Omega$. We call such an agent the ***dictator****.* A preference profile $R \in \mathcal{R}^N$ is ***diagonal*** if there exists $R_0 \in \mathcal{R}$ such that $R = (R_0, \cdots, R_0)$. A rule $\varphi\colon \mathcal{R}^N \to Z$ is ***diagonally dictatorial*** if there exists an agent who receives his most preferred bundle in Z_0 at every diagonal preference profile. We call such an agent the ***diagonal dictator****.*

Every diagonally dictatorial rule violates the following standard equity criteria. A rule $\varphi\colon \mathcal{R}^N \to Z$ satisfies ***equal treatment of equals*** if for all $i, j \in N$ and all $R \in \mathcal{R}^N$ with $R_i = R_j = R_0$, $\varphi_i(R)\ I_0\ \varphi_j(R)$. It satisfies ***no-envy****,* (Foley, 1967) if for all $i, j \in N$ and all $R \in \mathcal{R}^N$, $\varphi_i(R)\ R_i\ \varphi_j(R)$. It meets the ***equal division lower bound property*** if for all $i \in N$ and all $R \in \mathcal{R}^N$, $\varphi_i(R)\ R_i\ \Omega/n$.[8]

[8] We refer readers to Thomson (1995) for an extensive treatment of these equity criterion in various economic environments including exchange economies.

3 Useful lemmas

We will show later that every *continuous* and *efficient* rule selects allocations on the boundary of a predetermined "truncated (polyhedral) cone", whenever there exist at least two identical agents. In this section, we study implications of this property in conjunction with *continuity* and *veto-proofness.*

We use the following notation. For all l linearly independent vectors $a_1, \cdots, a_l \in \mathbb{R}^l$ and all $k \in \{1, \cdots, l\}$, let $\langle a_1, \cdots, a_k \rangle \equiv \{x \in \mathbb{R}^l : x = \lambda_1 a_1 + \cdots + \lambda_k a_k$, for some $\lambda_1, \cdots, \lambda_k \in \mathbb{R}\}$ be the space generated by the k vectors, $a_1, \cdots, a_k$. Let $\langle a_1, \cdots, a_k \rangle_+ \equiv \{x \in \mathbb{R}^l : x = \lambda_1 a_1 + \cdots + \lambda_k a_k$, for some $\lambda_1, \cdots, \lambda_k \in \mathbb{R}_+\}$ be the set of positive linear combinations of the k vectors, a_1, $\cdots$, a_k.[9] Since a_1, $\cdots$, a_l are linearly independent, then for all $x \in \mathbb{R}^l$, there exists a unique list of numbers, $\lambda_1, \cdots, \lambda_l$, such that $x = \lambda_1 a_1 + \cdots + \lambda_l a_l$. For each $X \subseteq \mathbb{R}^l$, let $intX$ be the interior of X and $int_{\langle a_1, \cdots, a_k \rangle} X$ be the relative interior of X in the subspace $\langle a_1, \cdots, a_k \rangle$. Note that $int_{\langle a_1, \cdots, a_k \rangle} \langle a_1, \cdots, a_k \rangle_+ = \{x \in \mathbb{R}^l :$ for some strictly positive $\lambda_1, \cdots, \lambda_k \in \mathbb{R}_{++}$, $x = \lambda_1 a_1 + \cdots + \lambda_k a_k\}$.

Given a subset $\mathcal{R}_0 \subseteq \mathcal{R}$, a rule φ: $\mathcal{R}^N \to Z$ satisfies ***Property B over*** $\boldsymbol{\mathcal{R}_0^N}$, if there exist l linearly independent vectors, $a_1, \cdots, a_l \in \mathbb{R}^l_+$, such that
(i) $\Omega \in int \langle a_1, \cdots, a_l \rangle_+$;
and for all $R \in \mathcal{R}_0^N$, if for all $i \in N$, $\varphi_i(R) = \lambda_{i1} a_1 + \cdots + \lambda_{il} a_l$ for some $\lambda_{i1}, \cdots, \lambda_{il} \in \mathbb{R}$,
(ii) for all $i, j \in N$ with $i \neq j$, all $k, m \in \{1, \cdots, l\}$ with $k \neq m$, and all $R \in \mathcal{R}_0^N$ with $R_i = R_j$,

$$\lambda_{ik} = \lambda_{im} = 0, \text{ or } \lambda_{jk} = \lambda_{jm} = 0, \text{ or } (\lambda_{ik} + \lambda_{jk}, \lambda_{im} + \lambda_{jm}) \notin \mathbb{R}^2_{++};^{10}$$

(iii) for all $I \subseteq N$, if for all $i, j \in I$, $R_i = R_j$ and $\sum_I \varphi_i(R) = \Omega$, then for all $i \in I$, $(\lambda_{i1}, \cdots, \lambda_{il}) \geqq 0$ (that is, $\varphi_i(R) \in \langle a_1, \cdots, a_l \rangle_+$).

When a rule satisfies Property B over the entire domain, we say that the rule satisfies ***Property B***. For all $k \in \{1, \cdots, l\}$, let e_k be the *unit vector* that has 1 in its k^{th} component and 0 in every other component. Note that when $a_1, \cdots, a_l$ are unit vectors, parts (i) and (iii) hold trivially. In this case, part (ii) says that whenever two agents have the same preferences, at least one of them receives 0 or both of them receive some positive amounts of only one and the same good.

Property B imposes a severe restriction on the choice, when there is a group I of agents who have identical preferences and who consume the entire social

[9] Such a set is called "polyhedral cone".

[10] The last part $(\lambda_{ik} + \lambda_{jk}, \lambda_{im} + \lambda_{jm}) \notin \mathbb{R}^2_{++}$ is equivalent to $(\lambda_{ik} + \lambda_{jk}) a_k + (\lambda_{im} + \lambda_{jm}) a_m \notin int_{\langle a_k, a_m \rangle} \langle a_k, a_m \rangle_+$.

endowment in the aggregate. In this case, by part (ii), no two agents in I can consume, in the aggregate, a bundle in the interior of $\langle a_1, \cdots, a_l \rangle_+$. Moreover, part (ii) imposes similar restrictions for all two dimensional subspaces. On the other hand, by part (i), the aggregate bundle Ω (consumed by agents in I) lies in the interior of $\langle a_1, \cdots, a_l \rangle_+$. Since by part (iii), individual bundles should lie in $\langle a_1, \cdots, a_l \rangle_+$, the only way to satisfy all three parts is to give a single agent in I the entire endowment Ω and all others the zero bundle, as shown by the next lemma. An allocation $z \in Z$ is ***extreme*** if it is one of the following l allocations, $(\Omega, 0, \cdots, 0), (0, \Omega, 0, \cdots, 0), \cdots, (0, \cdots, 0, \Omega)$.

Lemma 1. *Let $\mathcal{R}_0 \subseteq \mathcal{R}$. Assume that $\varphi \colon \mathcal{R}^N \to Z$ satisfies Property B over $\mathcal{R}_0^N$. If $I \subseteq N$ and $R \in \mathcal{R}_0^N$ are such that for all $i, j \in I$, $R_i = R_j$ and $\sum_{i \in I} \varphi_i(R) = \Omega$, then $\varphi(R)$ is extreme.*

Proof. Suppose that φ satisfies Property B with respect to l linearly independent vectors $a_1, \cdots, a_l \in \mathbb{R}_+^l$. Without loss of generality, we may assume $\Omega = a_1 + \cdots + a_l$. Let $I \subseteq N$ and $R \in \mathcal{R}_0^N$ be given as above. Let $z \equiv \varphi(R)$. By part (iii) of Property B, for all $i \in I$, $z_i \in \langle a_1, \cdots, a_l \rangle_+$. Then for each $i \in I$, there exist $\lambda_{i1}, \cdots, \lambda_{il} \geq 0$ such that $z_i = \lambda_{i1} a_1 + \cdots + \lambda_{il} a_l$. If $\lambda_{ik} > 1$ for some $i \in I$ and $k \in \{1, \cdots, l\}$, then $\sum_{j \in I \backslash i} z_j = \sum_{j \in N \backslash i} \left(\lambda_{jk} a_k + \sum_{m \neq k} \lambda_{jm} a_m \right) = \Omega - z_i = (1 - \lambda_{ik}) a_k + \sum_{m \neq k} (1 - \lambda_{im}) a_m$. Since $a_1, \cdots, a_l$ are linearly independent, $\sum_{j \in N \backslash i} \lambda_{jk} = 1 - \lambda_{ik} < 0$, contradicting the fact that all λ_{jk}'s are non-negative. Therefore, for all $j \in I$ and all $k \in \{1, \cdots, l\}$, $\lambda_{jk} \in [0, 1]$.

Since $\sum_{i \in I} z_i = \Omega \in int \langle a_1, \cdots, a_l \rangle_+$, there exist $i \in I$ and $k \in \{1, \cdots, l\}$ such that $\lambda_{ik} > 0$. We first show $\lambda_{ik} = 1$. Suppose, by contradiction, $\lambda_{ik} < 1$. Then since $\sum_{j \in I} \lambda_{jk} = 1$, there exists $j \in I \backslash i$ such that $\lambda_{jk} > 0$. Take any $m \neq k$. By part (ii) of Property B, $\lambda_{im} = 0$. Then since $\sum_{h \in I} \lambda_{hm} = 1$, there exists $h \in I \backslash i$ such that $\lambda_{hm} > 0$. This contradicts part (ii) of Property B. Therefore, $\lambda_{ik} = 1$.

Let $m \in \{1, \cdots, l\} \backslash k$. We only have to show $\lambda_{im} = 1$. If $\lambda_{im} < 1$, then since $\sum_{i \in I} z_i = \Omega$, there is $j \in I \backslash i$ such that $\lambda_{jm} > 0$. Then since $\lambda_{ik} > 0$, we have a contradiction to part (ii) of Property B. Therefore $\lambda_{im} = 1$. ∎

We now show that if a *continuous* rule satisfies Property B over the entire domain $\mathcal{R}^N$ and the domain is *connected*, then the rule is diagonally dictatorial.

Lemma 2. *If a* continuous *rule satisfies Property B and the domain is connected, then it is diagonally dictatorial.*

Proof. Let $\mathcal{R}^N$ be a connected domain. Assume that $\varphi \colon \mathcal{R}^N \to Z$ is *continuous* and satisfies Property B. Then by Lemma 1, for all diagonal profiles R, $\varphi(R)$ is extreme. Let $h\colon \mathcal{R} \to Z$ be defined as follows: for all $R_0 \in \mathcal{R}$, $h(R_0) \equiv \varphi(R_0, \cdots, R_0)$. By *continuity* of φ, h is continuous. Since φ chooses an extreme allocation at all diagonal profiles, h has a finite range. Now we can apply the fact that if a continuous function defined over a connected space has a finite range in a Hausdorff space, then it is constant. Thus, h is constant, that is, there exists an agent $i \in N$ such that for all $R_0 \in \mathcal{R}$, $h_i(R_0) = \Omega$. Therefore agent i is the diagonal dictator and φ is diagonally dictatorial. ∎

Remark 1. Suppose that $\mathcal{R}^N$ is not connected but is a union of connected subdomains. Then every *continuous* rule satisfying Property B over $\mathcal{R}^N$ is diagonally dictatorial over each of the subdomains. However the diagonal dictators may vary from subdomain to subdomain.

Adding *veto-proofness*, we obtain:

Lemma 3. *If a* continuous *and* veto-proof *rule satisfies Property B and the domain is connected, then it is dictatorial.*

Proof. Let $\mathcal{R}^N$ be a connected domain. Assume that $\varphi\colon \mathcal{R}^N \to Z$ satisfies Property B and is *continuous* and *veto-proof.* By Lemma 2, φ is diagonally dictatorial. Let $i^* \in N$ be the diagonal dictator. Without loss of generality, let $i^* = 1$. We prove that for all $R \in \mathcal{R}^N$, $\varphi(R) = (\Omega, 0, \cdots, 0)$.

For all $R \in \mathcal{R}^N$, let $H(R) \equiv \{i \in N : R_i \neq R_1\}$ be the set of agents who have different preferences from agent 1's. Note that $H(R) \subseteq N \backslash \{1\}$ and $0 \leq |H(R)| \leq n-1$. In what follows, we prove the following statement, referred to as $S(k)$, by an induction argument with respect to $k \in \{0, 1, \cdots, n-1\}$: for all $R \in \mathcal{R}^N$, if $|H(R)| \leq k$, $\varphi(R) = (\Omega, 0, \cdots, 0)$.

The first step of the induction argument, $S(0)$, follows directly from Lemma 2. Now suppose $S(m)$ for $m \in \{1, \cdots, n-2\}$. In order to prove $S(m+1)$, we use the following claim.

Claim 1. For all $R \in \mathcal{R}^N$ with $|H(R)| = m+1$, if $i \in H(R)$, then $\varphi_i(R) = 0$.

Proof. Let $R \in \mathcal{R}^N$ be given as above. Let $z \equiv \varphi(R)$. For each $i \in H(R)$, let $R'_i = R_1$. Then $|H(R'_i, R_{-i})| = m$ and so by the induction hypothesis, $\varphi_i(R'_i, R_{-i}) = 0$. Hence for all $i \in H(R)$, if $z_i \neq 0$, then agent i with preference R'_i can escape the worst outcome 0 by reporting R_i, contradicting *veto-proofness.* □

To complete the final step, let $\bar{R} \in \mathcal{R}^N$ be such that $|H(\bar{R})| = m+1$. Let $j \in H(\bar{R})$ (since $m \geq 1$, there exists such j). For all $R'_j \in \mathcal{R}$, let $g(R'_j) \equiv$

$\varphi(R'_j, \bar{R}_{-j})$. If $R'_j = \bar{R}_1$, then $|H(R'_j, \bar{R}_{-j})| = m$. Hence by the induction hypothesis, $g(R'_j) = (\Omega, 0, \cdots, 0)$. If $R'_j \neq \bar{R}_1$, then $|H(R'_j, \bar{R}_{-j})| = m + 1$. Hence by Claim 1, $\sum_{i \in N \setminus H(R'_j, \bar{R}_{-j})} \varphi_i(R'_j, \bar{R}_{-j}) = \Omega$. Since all agents in $N \setminus H(R'_j, \bar{R}_{-j})$ have the same preferences, then, by Lemma 1, $g(R'_j)$ is an extreme allocation. Therefore g has a finite range containing $(\Omega, 0, \cdots, 0)$. Since g is continuous and $\mathcal{R}$ is connected, the range of g is a singleton set $\{(\Omega, 0, \cdots, 0)\}$. Therefore $\varphi(\bar{R}) = g(\bar{R}_j) = (\Omega, 0, \cdots, 0)$. ∎

Remark 2. When the domain $\mathcal{R}^N$ is not connected, Lemma 3 does not hold. Let $\mathcal{R}^*_a, \mathcal{R}^*_b \subseteq \mathcal{R}$ constitute a *disconnection* of $\mathcal{R} \equiv \mathcal{R}^*_a \cup \mathcal{R}^*_b$. Let $N \equiv \{1, 2, 3\}$. Define a rule φ as follows: for all $R \in \mathcal{D}$, (i) if $R_3 \in \mathcal{R}^*_a$, $\varphi(R) \equiv (\Omega, 0, 0)$ and (ii) if $R_3 \in \mathcal{R}^*_b$, $\varphi(R) \equiv (0, \Omega, 0)$. Then φ satisfies Property B over $\mathcal{R}^N$ and is *continuous*, and *strategy-proof*. However, φ is not dictatorial.

Let A be an arbitrary index set. A family of subsets of $\mathcal{R}$, $\{\mathcal{R}_\alpha \subseteq \mathcal{R} : \alpha \in A\}$, is ***union-dense*** if $\cup_{\alpha \in A} \mathcal{R}_\alpha$ is dense in $\mathcal{R}$. We next show that if $\mathcal{R}$ has a union-dense family of connected subsets, $\{\mathcal{R}_\alpha \subseteq \mathcal{R} : \alpha \in A\}$, then every *continuous* rule that satisfies Property B over $\mathcal{R}^N_\alpha$ for all $\alpha \in A$ is diagonally dictatorial.

Lemma 4. *Assume that $\mathcal{R}$ is connected and has a union-dense family of connected subsets, $\{\mathcal{R}_\alpha \subseteq \mathcal{R} : \alpha \in A\}$. Then every* continuous *rule satisfying Property B over $\mathcal{R}^N_\alpha$ for all $\alpha \in A$, is diagonally dictatorial.*

Proof. Let $\varphi \colon \mathcal{R}^N \to Z$ be *continuous* and satisfy Property B over $\mathcal{R}^N_\alpha$ for all $\alpha \in A$. Then by Lemma 2, for all $\alpha \in A$, φ is diagonally dictatorial over $\mathcal{R}^N_\alpha$. Since $\{\mathcal{R}_\alpha : \alpha \in A\}$ is union-dense, then for each $R_0 \in \mathcal{R}$, there exists a sequence of preferences $(R^n_0 : n \in \mathbb{N})$ in $\cup_{\alpha \in A} \mathcal{R}_\alpha$, converging to R_0. Since for all $n \in \mathbb{N}$, $\varphi(R^n_0, \cdots, R^n_0)$ is extreme and φ is *continuous*, then $\varphi(R_0, \cdots, R_0)$ is also extreme. Therefore φ chooses an extreme allocation at all diagonal profiles in $\mathcal{R}^N$. For all $R_0 \in \mathcal{R}$, let $h(R_0) \equiv \varphi(R_0, \cdots, R_0)$. Then the range of h is a finite set of extreme allocations. Since φ is *continuous*, h is also continuous. Therefore since $\mathcal{R}$ is connected, the range of h is a singleton. That is, there exists $i \in N$ such that for all $R_0 \in \mathcal{R}$, $\varphi_i(R_0, \cdots, R_0) = \Omega$. ∎

Adding *veto-proofness*, we obtain:

Lemma 5. *Assume that $\mathcal{R}$ is connected and has a union-dense family of connected subsets, $\{\mathcal{R}_\alpha \subseteq \mathcal{R} : \alpha \in A\}$. Then every* continuous *and* veto-proof *rule that satisfies Property B over $\mathcal{R}^N_\alpha$ for all $\alpha \in A$ is dictatorial.*

Proof. Let $\varphi\colon \mathcal{D} \to Z$ satisfy Property B over $\mathcal{R}_\alpha^N$ for each $\alpha \in A$ and be *continuous* and *veto-proof.* By Lemma 3, for all $\alpha \in A$, φ is dictatorial over $\mathcal{R}_\alpha^N$. Since $\cup_{\alpha \in A}\mathcal{R}_\alpha^N$ is dense in $\mathcal{R}$ and φ is *continuous*, then the range of φ is composed of finite extreme allocations. By *continuity* of φ and connectedness of $\mathcal{R}$, we conclude that the range is a singleton: that is, φ is dictatorial. ∎

4 The main results

We show that every *efficient* and *continuous* rule over the linear domain satisfies Property B. Therefore, by Lemmas 2 and 3, we obtain the following results. Over the *linear domain*, (i) every *efficient* and *continuous* rule is diagonally dictatorial (Theorem 1) and (ii) a rule is *efficient*, *continuous*, and *veto-proof* if and only if it is dictatorial (Theorem 2). It follows from the first result that for any domain including the linear domain, there is no *efficient* and *continuous* rule satisfying any of the following equity criteria, *equal treatment of equals*, *no-envy*, and the *equal division lower bound* property (Corollary 1). Also it follows from the second result that a rule over the linear domain is *efficient*, *continuous*, and (*weakly*) *strategy-proof* if and only if it is dictatorial (Corollary 2). Adding "non-bossiness" (to be defined later) introduced by Satterthwaite and Sonnenschein (1981), we show that for any domain including the linear domain, a rule is *efficient, continuous, veto-proof** (or *weakly strategy-proof*), and *non-bossy* if and only if it is dictatorial (Theorem 3). The homothetic domain contains the linear domain and so Corollary 1 and Theorem 3 apply. In the 2-good case, we show that *non-bossiness* in Theorem 3 is redundant and moreover, *veto-proofness** (or *weak strategy-proofness*) can be weakened to *veto-proofness.* More precisely, we show that (i) every *efficient* and *continuous* rule over the homothetic domain is diagonally dictatorial (Theorem 4) and (ii) a rule is *efficient*, *continuous*, and *veto-proof* if and only if it is dictatorial (Theorem 5). *Veto-proofness* in the second result can be replaced with any one of the three requirements, *veto-proofness**, *weak strategy-proofness*, and *strategy-proofness.*

We now show that every *efficient* and *continuous* rule over the linear domain is diagonally dictatorial.

Theorem 1. *Every* efficient *and* continuous *rule over the linear domain is diagonally dictatorial.*

Proof. Since $\mathcal{R}_L$ connected, then by Lemma 2, we only have to show that every *efficient* and *continuous* rule over $\mathcal{R}_L^N$ satisfies Property B.

Let $\varphi\colon \mathcal{R}_L^N \to Z$ be *efficient* and *continuous*. We show that φ satisfies parts (i)-(iii) in the definition of Property B, with respect to $\{e_1, \cdots, e_l\}$. Parts (i) and (iii) are trivially satisfied. To show part (ii), let $R \in \mathcal{R}_L^N$ be such that for two agents i and j with $i \neq j$, $R_i = R_j$. Let $z \equiv \varphi(R)$. We only have to show that for all $k, m \in \{1, \cdots, l\}$ with $k \neq m$, either $z_{ik} = 0$ or $z_{jm} = 0$. Suppose by contradiction that there exist $k, m \in \{1, \cdots, l\}$ such that $k \neq m$, $z_{ik} > 0$, and $z_{jm} > 0$. Without loss of generality, we set $k = 1$ and $m = 2$.

Let $\hat{\Omega} \equiv z_i + z_j$. Then $\hat{\Omega}_1 > 0$ and $\hat{\Omega}_2 > 0$. Let $R_0 \equiv R_i = R_j$. Let $p_0 \in \mathbb{R}^l_{++}$ represent R_0. Let d be a real number satisfying $0 < d < \frac{1}{2}\hat{\Omega}_1$. Let $B(\hat{\Omega}, d) \equiv \{z'_0 \in \mathbb{R}^l_+ : ||z'_0 - \hat{\Omega}|| \leq d\}$ be the closed ball with center $\hat{\Omega}$ and radius d. Let $A \equiv \{z'_i \in Z_0 : z'_{i1} \in [\hat{\Omega}_1 - d, \hat{\Omega}_1 + d]$ or $z'_{i2} = 0\}$ and $B \equiv \{z'_i \in Z_0 : z'_{i1} = 0$ or $z'_{i2} \in [\hat{\Omega}_2 - d, \hat{\Omega}_2 + d]\}$. We show that both z_i and z_j are in $A \cap B$.

For all $\delta > 0$, let $p_0^\delta \equiv (p_{01} + \delta, p_{02}, \cdots, p_{0l})$ and R_0^δ be the linear preference represented by p_0^δ. Let R^δ be a profile such that $R_i^\delta \equiv R_0^\delta$ and for all $h \neq i$, $R_h^\delta \equiv R_h$. Let $z^\delta \equiv \varphi(R^\delta)$. Note that $\lim_{\delta \to 0} R^\delta = R$. Hence by *continuity* of φ, there exists $\bar{\delta} > 0$ such that for all $\delta \in (0, \bar{\delta})$, $z_i^\delta + z_j^\delta \in B(\hat{\Omega}, d)$.

Claim 1. For all $\delta \in (0, \bar{\delta})$, $z_i^\delta \in A$.

Proof. Since $z_i^\delta, z_j^\delta \in \mathbb{R}^l_+$ and $z_i^\delta + z_j^\delta \in B(\hat{\Omega}, d)$, $z_{i1}^\delta \leq \hat{\Omega}_1 + d$. Hence if we assume $z_i^\delta \notin A$, then $z_{i1}^\delta < \hat{\Omega}_1 - d$ and $z_{i2}^\delta > 0$. Since $z_i^\delta + z_j^\delta \in B(\hat{\Omega}, d)$, $z_{j1}^\delta > 0$. Since $z_{i2}^\delta > 0$, $z_{j2}^\delta < \Omega_2$.

In the following, we define a feasible allocation that Pareto dominates z^δ. Let $z'_i \equiv (z_{i1}^\delta + \rho, z_{i2}^\delta - \rho \cdot p_{01}/p_{02}, z_{i3}^\delta, \cdots, z_{il}^\delta)$ and $z'_j \equiv (z_{j1}^\delta - \rho, z_{j2}^\delta + \rho \cdot p_{01}/p_{02}, z_{j3}^\delta, \cdots, z_{jl}^\delta)$. For all $h \in N \backslash \{i, j\}$, let $z'_h = z_h^\delta$. Note that $\sum_N z'_i = \sum_N z_i^\delta = \Omega$. Since $z_{i1}^\delta \leq \hat{\Omega}_1 - d < \Omega_1, z_{i2}^\delta > 0, z_{j1}^\delta > 0$, and $z_{j2}^\delta < \Omega_2$, there exists $\rho > 0$ such that z' is feasible. By definition of R^δ, it is easy to show that $z'_i \; P_i^\delta \; z_i^\delta$ and for all $h \neq i$, $z'_h \; I_h^\delta \; z_h^\delta$. Hence z' Pareto dominates z^δ. This contradicts *efficiency* of φ. $\square$

By *continuity* of φ, $\lim_{\delta \to 0} z_i^\delta = z_i$. Since for all $\delta \in (0, \bar{\delta})$, $z_i^\delta \in A$ and A is closed, $z_i \in A$.

For all $\delta > 0$, let $\hat{p}_0^\delta \equiv (p_{01} - \delta, p_{02}, \cdots, p_{0l})$ and let $\hat{R}_0^\delta$ be the linear preference represented by $\hat{p}_0^\delta$. Let $\hat{R}^\delta$ be the profile defined by $\hat{R}_i^\delta \equiv \hat{R}_0^\delta$ and for all $h \neq i$, $\hat{R}_h^\delta \equiv \hat{R}_h$. Let $\hat{z}^\delta \equiv \varphi(\hat{R}^\delta)$. Note that $\lim_{\delta \to 0} \hat{R}^\delta = R$. Hence by *continuity* of φ, there exists $\hat{\delta} > 0$ such that for all $\delta \in (0, \hat{\delta})$, $\hat{z}_i^\delta + \hat{z}_j^\delta \in B(\hat{\Omega}, d)$. Now using the same argument as in Claim 1, we show that for all $\delta \in (0, \hat{\delta})$, $\hat{z}_i^\delta \in B$. Therefore by *continuity* of φ and closedness of B, we have $z_i \in B$.

Therefore $z_i \in A \cap B$. In order to prove $z_j \in A \cap B$, we apply the same argument as above.

Since $z_i \in A \cap B$, either (i) $z_{i1} = 0$ and $z_{i2} = 0$, or (ii) $z_{i1} \in [\hat{\Omega}_1 - d, \hat{\Omega}_1 + d]$

and $z_{i2} \in [\hat{\Omega}_2 - d, \hat{\Omega}_2 + d]$. Since $z_{i1} > 0$ by the initial assumption, (i) does not hold. Therefore $z_{i1} \in [\hat{\Omega}_1 - d, \hat{\Omega}_1 + d]$ and $z_{i2} \in [\hat{\Omega}_2 - d, \hat{\Omega}_2 + d]$.

Similarly, we show that $z_{j1} \in [\hat{\Omega}_1 - d, \hat{\Omega}_1 + d]$ and $z_{j2} \in [\hat{\Omega}_2 - d, \hat{\Omega}_2 + d]$. Hence $z_{i1} + z_{j1} \geq 2\hat{\Omega}_1 - 2d$. Therefore since $d < \frac{1}{2}\hat{\Omega}_1$, then $z_{i1} + z_{j1} \geq 2\hat{\Omega}_1 - 2d > \hat{\Omega}_1 = z_{i1} + z_{j1}$. This is a contradiction. ∎

If an allocation rule is diagonally dictatorial, it violates standard equity properties such as *equal treatment of equals*, *no-envy*, the *equal division lower bound property*, etc.

Corollary 1. *Given any domain containing the linear domain, there exists no* efficient *and* continuous *rule satisfying any one of the three equity criteria,* equal treatment of equals*,* no-envy*, and the* equal division lower bound property*.*

The impossibility is because of extreme choices for diagonal preference profiles. And the set of diagonal preference profiles is a "measure zero" subset of the linear domain. Thus, despite the impossibility, there may exist *efficient* and *continuous* rules that behave "nicely" over the set of non-diagonal profiles.[11] Our next result is that if we require *veto-proofness* in addition, we cannot escape dictatorship. The proof is immediate from Lemma 3 and Proof of Theorem 1.

[11]For example, in the two agents and two goods economy, let $\varepsilon > 0$ be an arbitrary real number and $\mathcal{D}_\varepsilon \equiv \{(R_1, R_2) \in \mathcal{R}_L^2 : ||p_1 - p_2|| > \varepsilon$, where $p_1 \in \Delta^1$ and $p_2 \in \Delta^1$ represent R_1 and R_2 respectively$\}$. For each $p_1, p_2 \in \Delta^1$ with $p_1 \neq p_2$, let $z(p_1, p_2)$ be the *efficient* allocation that is indifferent for agent 1 to the equal division (thus when $p_{11} > p_{21}$, $z_1(p_1, p_2)$ is the intersection of the 1's indifference curve through $\Omega/2$ and the "lower right corner" of the Edgeworth box; when $p_{11} < p_{21}$, $z_1(p_1, p_2)$ is on the "upper left corner" of the Edgeworth box). For each $\delta \in [0, \varepsilon]$, let $z_1^{lr}(p_1, \delta)$ be a bundle on the lower right corner of the Edgeworth box such that $z_1^{lr}(p_1, \delta)$ moves continuously from $z_1(p_1, p_2)$ to Ω as δ changes from ε to 0, for some p_2 with $p_{21} < p_{11}$. Thus, $z_1^{lr}(p_1, 0) = \Omega$ and $z_1^{lr}(p_1, \varepsilon) = z_1(p_1, p_2)$. Similarly, for each $\delta \in [0, \varepsilon]$, let $z_1^{ul}(p_1, \delta)$ be a bundle on the upper left corner such that $z_1^{ul}(p_1, \delta)$ moves continuously from $z_1(p_1, p_2)$ to Ω as δ changes from ε to 0, for some p_2 with $p_{21} > p_{11}$.

Now define φ as follows. For all $(R_1, R_2) \in \mathcal{D}_\varepsilon$, $\varphi_1(R_1, R_2) \equiv z_1(p_1, p_2)$, where $p_1, p_2 \in \Delta^1$ represent R_1 and R_2 respectively. When $||p_1 - p_2|| \leq \varepsilon$,

$$\varphi_1(R_1, R_2) \equiv \begin{cases} z_1^{lr}(p_1, ||p_1 - p_2||), \text{ if } p_{11} \geq p_{21}; \\ z_1^{ul}(p_1, ||p_1 - p_2||), \text{ if } p_{11} < p_{21}. \end{cases}$$

For each (R_1, R_2), let $\varphi_2(R_1, R_2) \equiv \Omega - \varphi_1(R_1, R_2)$. By definition, φ satisfies *efficiency* and *continuity*. Clearly, over $\mathcal{D}_\varepsilon$, φ satisfies the *equal division lower bound property*. However, φ is diagonally dictatorial, since it gives Ω to agent 1 at each diagonal profile. Thus, although φ is diagonally dictatorial, φ violates the *equal division lower bound property* only over $\mathcal{R}_L^2 \backslash \mathcal{D}_\varepsilon$, which can be made arbitrarily small by choosing sufficiently small $\varepsilon > 0$.

Theorem 2. *A rule over the linear domain is* efficient*,* continuous*, and* veto-proof *if and only if it is dictatorial.*

Since *veto-proofness** [or (*weak*) *strategy-proofness*] implies *veto-proofness*, we have the following corollary:

Corollary 2. (i) *A rule over the linear domain is* efficient*,* continuous*, and* veto-proof* *if and only if it is dictatorial.* (ii) *A rule over the linear domain is* efficient*,* continuous*, and* (weakly) strategy-proof *if and only if it is dictatorial.*

Examples 1-3 below establish independence of the three requirements in each of Theorem 2 and Corollary 2. For simplicity, we assume $N = \{1, 2, 3\}$ and $l = 2$. Then we may represent each linear preference R_i by a number $p_i \in \mathbb{R}_{++}$, where $(p_i, 1)$ is the normal vector of indifference curves of R_i. In the following examples, we use p_i instead of R_i.

Example 1. Define φ as follows: (i) if $p_3 \geq 1$, $\varphi(p_1, p_2, p_3) \equiv (\Omega, 0, 0)$, (ii) if $p_3 < 1$, $\varphi(p_1, p_2, p_3) \equiv (0, \Omega, 0)$. Then φ satisfies *efficiency* and *strategy-proofness.*

Example 2. Define φ as follows: (i) if $p_1 > \max\{p_2, p_3\}$, $\varphi_1(p_1, p_2, p_3) \equiv (\Omega_1, 0) + \frac{\max\{p_2,p_3\}}{p_1} \cdot (0, \Omega_2)$ and for all $i = 2, 3$, $\varphi_i(p_1, p_2, p_3) \equiv \frac{1}{2}\left(1 - \frac{\max\{p_2,p_3\}}{p_1}\right) \cdot (0, \Omega_2)$, (ii) if $p_1 < \min\{p_2, p_3\}$, $\varphi_1(p_1, p_2, p_3) \equiv \frac{p_1}{\min\{p_2,p_3\}} \cdot (\Omega_1, 0) + (0, \Omega_2)$ and for all $i = 2, 3$, $\varphi_i(p_1, p_2, p_3) \equiv \frac{1}{2}\left(1 - \frac{p_1}{\min\{p_2,p_3\}}\right) \cdot (\Omega_1, 0)$, and (iii) in all other cases, $\varphi(p_1, p_2, p_3) \equiv (\Omega, 0, 0)$. It is easy to show that φ satisfies *efficiency* and *continuity.*

Example 3. Constant allocation rules satisfy *continuity* and *strategy-proofness* trivially.

Theorem 2 and Corollary 2 pertain to the linear domain. However, for any larger domain, these two results are still applicable at least locally over the linear domain. Considering the following additional requirement introduced by Satterthwaite and Sonnenschein (1981), we extend the two results to any domain including the linear domain. A rule $\varphi \colon \mathcal{R}^N \to Z$ is ***non-bossy*** if no one can affect others' bundles without affecting his own bundle, that is, for all $R \in \mathcal{R}^N$ and all $i \in N$, there is no $R'_i \in \mathcal{R}$ such that $\varphi_i(R'_i, R_{-i}) = \varphi_i(R)$ and $\varphi_{-i}(R'_i, R_{-i}) \neq \varphi_{-i}(R)$.

Theorem 3. *Given any domain including the linear domain,* (i) *a rule is* efficient*,* continuous*,* veto-proof**, and* non-bossy *if and only if it is dictatorial*; (ii) *a rule is* efficient*,* continuous*,* weakly strategy-proof*, and* non-bossy *if and only if it is dictatorial.*

Proof. We only have to show (i). Let $\mathcal{R} \supseteq \mathcal{R}_L^N$ be a family of preferences. Let $\varphi \colon \mathcal{R}^N \to Z$ be *efficient, continuous*, *veto-proof**, and *non-bossy.* Then by Theorem 2, there exists $i \in N$ such that for all $R \in \mathcal{R}_L^N$, $\varphi_i(R) = \Omega$. Let $R \in \mathcal{R}^N$. We only have to show $\varphi_i(R) = \Omega$.

Let $\bar{R} \in \mathcal{R}_L^N$. Then $\varphi_i(\bar{R}) = \Omega$. Let $j \in N \backslash i$. Since $\varphi_j(\bar{R}) = 0$, then by *veto-proofness**, $\varphi_j(R_j, \bar{R}_{-j}) = 0$. By *non-bossiness*, $\varphi(R_j, \bar{R}_{-j}) = \varphi(\bar{R})$. Applying the same argument successively to each agent in $N \backslash i$, we show $\varphi(\bar{R}_i, R_{-i}) = \varphi(\bar{R})$. Then $\varphi_i(\bar{R}_i, R_{-i}) = \Omega$. Applying *veto-proofness** to agent i, $\varphi_i(R_i, R_{-i}) = \Omega$. ∎

Remark 3. (i) In the 2-agent case, *non-bossiness* is redundant. Given any domain including the linear domain, a rule is *efficient,* continuous*,* and *veto-proof** [or, *weakly strategy-proof*] if and only if it is dictatorial.
(ii) When *veto-proofness** or *weak strategy-proofness* is replaced with *strategy-proofness*, similar result holds without *continuity.* Serizawa (2000b) shows that a rule is *efficient, strategy-proof ,* and *non-bossy* if and only if it is dictatorial.

The homothetic domain contains the linear domain and so Corollary 1 and Theorem 3 apply. Moreover, in the 2-good case, we establish stronger results.

We consider the following union-dense family of connected subsets of the homothetic domain. We will show later that every *efficient* and *continuous* rule satisfies Property B over each of these subsets.

We first introduce useful notation. Let $\varepsilon > 0$. If $\Omega - (\varepsilon, 0) \in \mathbb{R}_+^2$, then let $a_1^\varepsilon \equiv \Omega - (\varepsilon, 0)$. Otherwise, let $a_1^\varepsilon \equiv (0, \Omega_2)$. If $\Omega - (0, \varepsilon) \in \mathbb{R}_+^2$, then let $a_2^\varepsilon \equiv \Omega - (0, \varepsilon)$. Otherwise, let $a_2^\varepsilon \equiv (\Omega_1, 0)$.

The two vectors, a_1^ε and a_2^ε, generate a positive cone $C^\varepsilon \equiv \{\alpha a_1^\varepsilon + \beta a_2^\varepsilon : \alpha, \beta \in \mathbb{R}_+\}$.

Let $X \subseteq \mathbb{R}_+^2$ and $R_0, R_0' \in \mathcal{R}_H$. We say that two preferences, R_0 *and* R_0', *coincide on* X if the two preferences order every two bundles in X in the same way. Let $x \in X$. A vector $p \in \mathbb{R}_+^2$ is a *supporting normal vector at* x *for* R_0 if p is normal to the hyperplane through x, which supports the upper contour set (at x) of R_0, formally, for all $y \in \mathbb{R}_+^2$ with $y \; R_0 \; x$, $p \cdot y \geq p \cdot x$. We say that two preferences R_0 *and* R_0' *have identical supporting normal vectors on* X if for all $x \in X$, the set of supporting normal vectors at x for R_0 is equal to the set of supporting normal vectors at x for R_0'. We say that R_0 is *strictly convex over* X if for all x, $x' \in X$ and all $\lambda \in (0,1)$, if $\lambda x + (1-\lambda)x' \in X$ and $x' \; R_0 \; x$, then $(\lambda x + (1-\lambda)x') \; P_0 \; x$.

Next we define a class of homothetic preferences that are locally linear over the cone and strictly convex outside the cone. Formally, let $\mathcal{R}_{H,\varepsilon} \subseteq \mathcal{R}_H$ be the class of preferences defined as follows: for all $R_0 \in \mathcal{R}_H$, $R_0 \in \mathcal{R}_{H,\varepsilon}$ if and only if (i) R_0 is strictly convex on $\mathbb{R}^2_+ \backslash C^\varepsilon$, (ii) there exists a linear preference $R'_0 \in \mathcal{R}_L$ such that R_0 coincide with R'_0 on C^ε, and (iii) the indifference curves of R_0 have kinks along both of the two rays, $\overrightarrow{0, a^\varepsilon_1}$ and $\overrightarrow{0, a^\varepsilon_2}$.

Let $\mathcal{R}_{H,sc}$ be the class of all strictly convex preferences in $\mathcal{R}_H$. Note that for all $R_0 \in \mathcal{R}_{H,sc}$, there exists a unique preference in $\mathcal{R}_{H,\varepsilon}$ that coincides with R_0 on $\mathbb{R}^2_+ \backslash C^\varepsilon$. However for each $R_0 \in \mathcal{R}_{H,\varepsilon}$, there are many preferences in $\mathcal{R}_{H,sc}$ that coincide with R_0 on $\mathbb{R}^2_+ \backslash C^\varepsilon$. Hence we may say that $\mathcal{R}_{H,\varepsilon}$ is smaller than $\mathcal{R}_{H,sc}$.

We first show that if an *efficient* allocation has i-th component that lies outside C^ε and two preferences in $\mathcal{R}_{H,\varepsilon}$ of agents i and j have identical supporting normal vectors outside C^ε, then their bundles are along the same ray through the origin.[12]

Lemma 6. *Let z be* efficient *for $R \in \mathcal{R}^N_{H,\varepsilon}$. For all $i, j \in N$, if R_i and R_j have identical supporting normal vectors over $\mathbb{R}^2_+ \backslash C^\varepsilon$ and $z_i \notin C^\varepsilon$, then there exists $\alpha \in \mathbb{R}_+$ such that $z_j = \alpha z_i$.*

Proof. Let $z \in Z$ be *efficient* for $R \in \mathcal{R}^N_{H,\varepsilon}$. Let $i, j \in N$. Assume that R_i and R_j have identical supporting normal vectors over $\mathbb{R}^2_+ \backslash C^\varepsilon$ and $z_i \notin C^\varepsilon$.

Claim 1. For all $x \in \mathbb{R}^l_+$ such that $x \neq z_i$ and $x\ I_i\ z_i$, there exists no vector that is a supporting normal vector for R_i both at x and at z_i.

Proof. Let $x\ I_i\ z_i$. Suppose to the contrary that there exists $p \in \mathbb{R}^l_+$ such that (i) for all $y \in \mathbb{R}^l_+$ with $y\ R_i\ z_i$, $p \cdot y \geq p \cdot z_i$ and (ii) for all $y \in \mathbb{R}^l_+$ with $y\ R_i\ x$, $p \cdot y \geq p \cdot x$. Then since $z_i\ I_i\ x$, $p \cdot z_i = p \cdot x$. Since R_i is convex, for all $\delta \in (0, 1)$, $[\delta x + (1 - \delta) z_i]\ I_i\ z_i$. Since C^ε is closed, there exists $\delta^* \in (0, 1)$ such that for all $\delta \in [0, \delta^*]$, $\delta x + (1-\delta) z_i \notin C^\varepsilon$. Therefore for all $\lambda \in (0, 1)$, $\lambda(\delta^* x + (1-\delta^*) z_i) + (1-\lambda) z_i \notin C^\varepsilon$, $\delta^* x + (1-\delta^*) z_i \notin C^\varepsilon$, and $[\lambda(\delta^* x + (1 - \delta^*) z_i) + (1 - \lambda) z_i]\ I_i\ [\delta^* x + (1 - \delta^*) z_i]\ I_i\ z_i$. This contradicts that $R_i \in \mathcal{R}_{H,\varepsilon}$ is strictly convex over $\mathbb{R}^2_+ \backslash C^\varepsilon$. □

Next we show that $z_j \in \overrightarrow{0, z_i}$. Suppose to the contrary that z_j is not along the ray $\overrightarrow{0, z_i}$. Then, there exist $\alpha, \beta \in \mathbb{R}_{++}$ such that $z_i\ I_i\ \alpha z_j$ and $z_j\ I_j\ \beta z_i$. Since $z_j \notin \overrightarrow{0, z_i}$, $z_i \neq \alpha z_j$ and $z_j \neq \beta z_i$. Since z is *efficient* for R, there is $p \in \mathbb{R}^l_+$ that supports R_i at z_i and R_j at z_j. Then by homotheticity, p supports R_i both at z_i and at αz_j, contradicting Claim 1. ∎

In the 2-good case, the following three simple facts hold: (i) $\mathcal{R}_H$ is path-connected, (ii) for all $\varepsilon > 0$, $\mathcal{R}_{H,\varepsilon}$ is path-connected, and (iii) $\cup_{\varepsilon>0} \mathcal{R}^\varepsilon_H$ is dense

[12]Schummer (1997) uses a similar fact for strictly convex and homothetic preferences.

in $\mathcal{R}_H$. We sketch the proofs in Appendix. Using the three facts and applying Lemma 4, we show that every *efficient* and *continuous* rule over $\mathcal{R}_H^N$ is diagonally dictatorial.

Theorem 4. *In the 2-good case, every* efficient *and* continuous *rule over the homothetic domain is diagonally dictatorial.*

Proof. By Lemma 4, we only have to show that every *efficient* and *continuous* rule over the homothetic domain satisfies Property B over $\mathcal{R}_{H,\varepsilon}^N$, for all $\varepsilon > 0$.

Let $\varphi\colon \mathcal{R}_H^N \to Z$ be *efficient* and *continuous*. We show that φ satisfies conditions (i)-(iii) of Property B over $\mathcal{R}_{H,\varepsilon}^N$, associated with the two linearly independent vectors, a_1^ε and a_2^ε. Clearly, $\Omega \in int\, \langle a_1^\varepsilon, a_2^\varepsilon \rangle_+$. Let $R \in \mathcal{R}_{H,\varepsilon}^N$ and $z \equiv \varphi(R)$. Assume that there exist $i, j \in N$ such that $i \neq j$ and $R_i = R_j$. Let $R_0 \equiv R_i = R_j$. Let $p_0 \in \mathbb{R}_{++}$ and R_0 coincide with the linear preference represented by $(p_0, 1) \in \mathbb{R}_{++}^2$ on C^ε. We only have to show that $z_i = 0$ or $z_j = 0$ or $z_i + z_j \notin int\, \langle a_1^\varepsilon, a_2^\varepsilon \rangle_+$.

The proof is by contradiction. Suppose to the contrary that $z_i \neq 0$, $z_j \neq 0$, and $z_i + z_j \in int\, \langle a_1^\varepsilon, a_2^\varepsilon \rangle_+ (= int(C^\varepsilon))$. Let $z_i + z_j \equiv \hat{\Omega}$. Since $\hat{\Omega} \in int(C^\varepsilon)$, there exists $\bar{\rho}$ such that for all $\rho \in (0, \bar{\rho})$, the closed ball $B(\hat{\Omega}, \rho)$ is contained in C^ε.

Our proof makes use of the following geometric objects. For all $B \subseteq \mathbb{R}^2$, let $comp\,(B) \equiv \{x \in \mathbb{R}^l : \text{for some } y \in B,\ x \leqq y\}$. Let $\rho \in (0, \bar{\rho})$ and $Z_0^\rho \equiv Z_0 \cap comp\left(B(\hat{\Omega}, \rho)\right)$. Let H_1 be the ray $\overrightarrow{0, a_2^\varepsilon}$ and $d_1 \in \mathbb{R}^2$ be a vector such that $d_1 \cdot a_2^\varepsilon = 0$ and for all $z_0 \in C^\varepsilon$, $d_1 \cdot z_0 \geq 0$. Let H_2 be the ray $\overrightarrow{0, a_1^\varepsilon}$ and $d_2 \in \mathbb{R}^2$ be a vector such that $d_2 \cdot a_1^\varepsilon = 0$ and for all $z_0 \in C^\varepsilon$, $d_2 \cdot z_0 \geq 0$. Note that since a_1^ε and a_2^ε are linearly independent, d_1 and d_2 are linearly independent also and that for all $z_0 \in \mathbb{R}_+^2$, there exist $\alpha, \beta \in \mathbb{R}_+$ such that $z_0 = \alpha d_1 + \beta d_2$.

Clearly, there exists a unique bundle $b_1^\rho \in B(\hat{\Omega}, \rho)$ such that for all $z_0 \in B(\hat{\Omega}, \rho)$, $d_1 \cdot z_0 \geq d_1 \cdot b_1^\rho$. Also, there exists a unique bundle $b_2^\rho \in B(\hat{\Omega}, \rho)$ such that for all $z_0 \in B(\hat{\Omega}, \rho)$, $d_2 \cdot z_0 \geq d_2 \cdot b_2^\rho$.[13] Let $H_1^\rho \equiv \{z_0 \in Z_0^\rho : d_1 \cdot z_0 \geq d_1 \cdot b_1^\rho\}$ be the intersection of Z_0^ρ and the half space above the line through b_1^ρ, which is normal to d_1. Let $H_2^\rho \equiv \{z_0 \in Z_0^\rho : d_2 \cdot z_0 \geq d_2 \cdot b_2^\rho\}$ be the intersection of Z_0^ρ and the half space below the line through b_2^ρ, which is normal to d_2.

[13]Since $B(\hat{\Omega}, \rho)$ is compact and strictly convex, the following (a) and (b) have unique solutions.

$$\text{(a)} \quad \max_{z_0 \in B(\hat{\Omega}, \rho)} d_1 \cdot z_0,$$
$$\text{(b)} \quad \min_{z_0 \in B(\hat{\Omega}, \rho)} d_2 \cdot z_0.$$

The solution for (a) is b_1^ρ and the solution for (b) is b_2^ρ.

In the following argument, we first prove that for sufficiently small $\rho > 0$, we have $z_i \in \left(\overrightarrow{0, a_1^\varepsilon} \cup H_1^\rho\right) \cap \left(\overrightarrow{0, a_2^\varepsilon} \cup H_2^\rho\right)$. We next derive a contradiction based on this fact.

Since R_0 has kinks along both two rays, $\overrightarrow{0, a_1^\varepsilon}$ and $\overrightarrow{0, a_2^\varepsilon}$, then for some $\delta^o > 0$, if $\delta \in (0, \delta^o)$, there exists $R_0^\delta \in \mathcal{R}_{H,\varepsilon}$ such that R_0^δ and R_0 have identical supporting normal vectors over $\mathbb{R}_+^2 \backslash C^\varepsilon$,[14] and R_0^δ coincides with the linear preference represented by $(p_0 - \delta, 1)$ over C^ε. Let $\delta \in (0, \delta^o)$. Let $R_i^\delta \equiv R_0^\delta$ and for all $h \neq i$, let $R_h^\delta \equiv R_h$. Let $z^\delta \equiv \varphi(R^\delta)$.

By *continuity* of φ, there exists $\bar{\delta} \in (0, \delta^o)$ such that for all $\delta \in (0, \bar{\delta})$, $z_i^\delta + z_j^\delta \in B(\hat{\Omega}, \rho)$.

Claim 1. For all $\delta \in (0, \bar{\delta})$, *we have* $z_i^\delta, z_j^\delta \in C^\varepsilon \cap Z_0^\rho$.

Proof. Since $z_i^\delta + z_j^\delta \in B(\hat{\Omega}, \rho)$, clearly we have $z_i^\delta, z_j^\delta \in Z_0^\rho$. We only have to show that $z_i^\delta, z_j^\delta \in C^\varepsilon$. Since $B(\hat{\Omega}, \rho) \subseteq C^\varepsilon$, we have $z_i^\delta + z_j^\delta \in C^\varepsilon$. By definition, R_i^δ and $R_j^\delta \equiv R_0$ have identical supporting normal vectors on $\mathbb{R}_+^2 \backslash C^\varepsilon$. Since φ is *efficient,* z^δ is an *efficient* allocation. Therefore if $z_i^\delta \notin C^\varepsilon$, then by Lemma 6, there exists $\alpha \geq 0$, $z_j^\delta = \alpha z_i^\delta$. Then $z_i^\delta + z_j^\delta \notin C^\varepsilon$, contradicting $z_i^\delta + z_j^\delta \in B(\hat{\Omega}, \rho) \subseteq C^\varepsilon$. Therefore $z_i^\delta \in C^\varepsilon$.

Using the same argument, we show that $z_j^\delta \in C^\varepsilon$. □

Claim 2. For all $\delta \in (0, \bar{\delta})$, *we have* $z_i^\delta \in \overrightarrow{0, a_1^\varepsilon} \cup H_1^\rho$.

Proof. Assume $z_j^\delta \neq 0$. If $z_j^\delta \in \overrightarrow{0, a_1^\varepsilon}$, then by *efficiency,* $z_i^\delta \in \overrightarrow{0, a_1^\varepsilon}$. So $z_i^\delta + z_j^\delta \in \overrightarrow{0, a_1^\varepsilon}$, contradicting $z_i^\delta + z_j^\delta \in B(\hat{\Omega}, \rho)$. Hence if $z_j^\delta \neq 0$, then $z_j^\delta \notin \overrightarrow{0, a_1^\varepsilon}$. Therefore we distinguish three cases below.

Case 1. $z_j^\delta = 0$. In this case, $z_i^\delta \in B(\hat{\Omega}, \rho)$. Hence by definition of b_1^ρ, $d_1 \cdot z_i^\delta \geq d_1 \cdot b_1^\rho$. Therefore, $z_i^\rho \in H_1^\rho$.

Case 2. $z_j^\delta \neq 0$ *and* $z_j^\delta \in int(C^\varepsilon) \cap Z_0^\rho$. In this case, by *efficiency,* $z_i^\delta \in \overrightarrow{0, a_1^\varepsilon}$.

Case 3. $z_j^\delta \neq 0$ *and* $z_j^\delta \in \overrightarrow{0, a_2^\varepsilon} \cap Z_0^\rho$. In this case, there exists $\alpha > 0$ such that $z_j^\delta = \alpha a_2^\varepsilon$. Let $\Omega^\delta \equiv z_i^\delta + z_j^\delta$. Then $\Omega^\delta \in B(\hat{\Omega}, \rho)$ and $z_i^\delta = \Omega^\delta - \alpha a_2^\varepsilon$. Hence by definition of b_1^ρ, $d_1 \cdot \Omega^\delta \geq d_1 \cdot b_1^\rho$. Therefore $d_1 \cdot z_i^\delta = d_1 \cdot \Omega^\delta - \alpha(d_1 \cdot a_2^\varepsilon) = d_1 \cdot \Omega^\delta \geq d_1 \cdot b_1^\rho$. Hence $z_i^\delta \in H_1^\rho$.

Therefore in all three cases, $z_i^\delta \in \overrightarrow{0, a_1^\varepsilon} \cup H_1^\rho$. □

By *continuity* of φ, $\lim_{\delta \to 0} z_i^\delta = z_i$. Therefore since $\overrightarrow{0, a_1^\varepsilon} \cup H_1^\rho$ is closed, $z_i \in \overrightarrow{0, a_1^\varepsilon} \cup H_1^\rho$.

[14] Even if R_0^δ and R_0 have identical supporting normal vectors over $\mathbb{R}_+^2 \backslash C^\varepsilon$, R_0^δ is not necessarily equal to R_0 over $\mathbb{R}_+^2 \backslash C^\varepsilon$.

Since R_0 has kinks along both two rays, $\overrightarrow{0, a_1^\varepsilon}$ and $\overrightarrow{0, a_2^\varepsilon}$, then for some $\delta^* > 0$, if $\delta \in (0, \delta^*)$, there exists $\hat{R}_0^\delta \in \mathcal{R}_{H,\varepsilon}$ such that $\hat{R}_0^\delta$ and R_0 have identical supporting normal vectors over $\mathbb{R}_+^2 \backslash C^\varepsilon$ and $\hat{R}_0^\delta$ coincides with the linear preference represented by $(p_0 + \delta, 1)$ on C^ε. Let $\delta \in (0, \delta^*)$. Let $\hat{R}_i^\delta = \hat{R}_0^\delta$ and for all $h \neq i$, let $\hat{R}_h^\delta = R_h$. Let $\hat{z}^\delta = \varphi(\hat{R}^\delta)$. Now using $\hat{R}^\delta$ and $\hat{z}^\delta$ and following the same arguments as above, we prove that $z_i \in \overrightarrow{0, a_2^\varepsilon} \cup H_2^\rho$.

Therefore $z_i \in \left(\overrightarrow{0, a_1^\varepsilon} \cup H_1^\rho\right) \cap \left(\overrightarrow{0, a_2^\varepsilon} \cup H_2^\rho\right)$. Since for all $k, k' \in \{1, 2\}$ with $k \neq k'$, $\overrightarrow{0, a_k^\varepsilon} \cap H_{k'}^\rho = \varnothing$ and $\overrightarrow{0, a_k^\varepsilon} \cap \overrightarrow{0, a_{k'}^\varepsilon} = \{0\}$, then $z_i \in \{0\} \cup [H_1^\rho \cap H_2^\rho]$. By assumption, $z_i \neq 0$. Therefore $z_i \in H_1^\rho \cap H_2^\rho$, that is, $d_1 \cdot z_i \geq d_1 \cdot b_1^\rho$ and $d_2 \cdot z_i \geq d_2 \cdot b_2^\rho$. Note that $\lim_{\rho \to 0} b_1^\rho = \lim_{\rho \to 0} b_2^\rho = \hat{\Omega}$. Taking limits in both sides of the two inequalities, $d_1 \cdot z_i \geq d_1 \cdot \hat{\Omega}$ and $d_2 \cdot z_i \geq d_2 \cdot \hat{\Omega}$. Since $\hat{\Omega} = z_i + z_j$, $d_1 \cdot z_j \leq 0$ and $d_2 \cdot z_j \leq 0$. Let $\alpha, \beta \in \mathbb{R}_+$ be such that $z_j = \alpha d_1 + \beta d_2$. Then from the two inequalities, we obtain $z_j \cdot z_j \leq 0$. This implies $z_j = 0$, contradicting our assumption. ∎

Using Lemma 5 and the same proof as in Theorem 4, we show that if we require *veto-proofness* in addition, then we cannot escape dictatorship.

Theorem 5. *In the 2-good case, a rule over the homothetic domain is* efficient*,* continuous*, and* veto-proof *if and only if it is dictatorial.*

Since each of *veto-proofness** and (*weak*) *strategy-proofness* implies *veto-proofness*, we have:

Corollary 3. *In the 2-good case,* (i) *a rule over the homothetic domain is* efficient*,* continuous*, and* veto-proof* *if and only if it is dictatorial*; (ii) *a rule over the homothetic domain is* efficient*,* continuous*, and* (weakly) strategy-proof *if and only if it is dictatorial.*

The independence of the requirements in each of Theorem 5 and Corollary 3 can be established easily.

5 Discussion

1. Our diagonal dictatorship results (Theorems 1 and 4) crucially rely on the admissibility of "linear" or "locally linear" preferences. For such preferences, one may easily imagine how difficult it is to select continuously from the Pareto set. However, no earlier study has shown formally what exactly the cost of *continuity* is. This paper offers an answer. To attain *continuity*, we have to pay the cost of *diagonal dictatorship*, which seems to be too high.

The diagonal dictatorship result does not hold on domains consisting of only strictly convex preferences. For example, on the domain of Cobb-Douglas preferences, the "Walrasian rule", which is the rule selecting always the unique Walrasian equilibrium allocation, is *efficient* and *continuous*. Then, one may well wonder whether *efficiency*, *continuity*, and "fairness" are compatible on the full domain of continuous, strictly monotonic, and strictly convex preferences. The answer is left for future study.

2. In the n-agent exchange economies, we showed that only dictatorial rules are *efficient*, *continuous*, and *strategy-proof* over the linear domain and also over the 2-good homothetic domain. In proving these results, we use the diagonal dictatorship feature of *efficient* and *continuous* rules, which is a consequence of allowing linear or "locally linear" preferences. In this sense, our proof is not robust. However, we think that even if only strictly convex preferences are admissible, the same impossibility will apply. This is a natural conjecture following from the well-known *conjecture* by Zhou (1991) and Kato and Ohseto (2001) that the range of every *efficient* and *strategy-proof* rule contains only extreme allocations. Let us call this property the "extreme range property".

3. Recently, an important contribution is made by Serizawa and Weymark (2002). They showed that any *efficient* and *strategy-proof* rule "cannot guarantee everyone a consumption bundle bounded away from the origin", violating the condition of "minimum consumption guarantee". This means that at least one agent receives bundles sufficiently close to the zero bundle, which is an obnoxious feature. However, it is less obnoxious than (diagonal) dictatorship or the extreme range property. This is because violation of *minimum consumption guarantee* does not exclude the possibility of choosing only non-extreme bundles. Thus, their result still leaves a substantial gap between "no minimum consumption guarantee" and the extreme range property, conjectured by Zhou (1991) and Kato and Ohseto (2001). Adding *continuity*, their result implies that at least one agent should receive the zero bundle at some economies, which is still far from (diagonal) dictatorship or the extreme range property.

It is remarkable to note that the proof in Serizawa and Weymark (2002) is quite robust and does not rely on the admissibility of some non-standard or artificial preferences, such as linear preferences or locally linear homothetic preferences that are used in our proofs.

4. The homothetic preference domain $\mathcal{R}_H^N$ does not include preferences whose indifference curves do not intersect with all the axes, such as Cobb-Douglas preferences. However, these preferences are limit points of $\mathcal{R}_H$ (that is, there are

sequences of preferences in $\mathcal{R}_H$, which converge to such preferences). Hence our results can be extended over the larger domain containing $\mathcal{R}_H$ and including such preferences.

A Topological properties of the homothetic domain in the 2-good case

In this section, we sketch the proofs of the following facts on the homothetic domain with two goods.

Fact 1. *$\mathcal{R}_H$ is path-connected.*

Proof. Let $R_0, R_0' \in \mathcal{R}_H$. For all $x \in [0,1]$, let $f(x) \in \mathbb{R}_+$ satisfy $(x, f(x))\ I_0\ (1,0)$. Then the function $f\colon [0,1] \to \mathbb{R}_+$ is well-defined and its graph is the indifference curve of R_0 through $(1,0)$. Since R_0 is strictly monotonic and convex, f is monotone decreasing and convex. Since R_0 is homothetic, the function f completely determines R_0. Hence we may say that f represents R_0. Similarly there exists a monotone decreasing and convex function $f'\colon [0,1] \to \mathbb{R}_+$ that represents R_0'. For all $\lambda \in [0,1]$, let f^λ be defined by $f^\lambda \equiv \lambda f + (1-\lambda) f'$. Then clearly, f^λ is also monotone decreasing and convex and so there exists a preference $R_0^\lambda \in \mathcal{R}_H$ represented by f^λ. Define $\pi\colon [0,1] \to \mathcal{R}_H$ as follows: for all $\lambda \in [0,1]$, $\pi(\lambda) \equiv R_0^\lambda$. It is easy to show that π is a continuous path from R_0 to R_0'. ∎

Fact 2. *For all $\varepsilon > 0$, $\mathcal{R}_{H,\varepsilon}$ is path-connected.*

Proof. For all $R_0 \in \mathcal{R}_{H,\varepsilon}$, there exists a strictly convex preference $\bar{R}_0 \in \mathcal{R}_H$ that coincides with R_0 on $\mathbb{R}^2_{++} \backslash C^\varepsilon$. Let $R_0, R_0' \in \mathcal{R}_{H,\varepsilon}$. Let $\bar{R}_0, \bar{R}_0' \in \mathcal{R}_H$ be the strictly convex preferences that coincide with R_0, R_0' on $\mathbb{R}^2_{++} \backslash C^\varepsilon$, respectively. Then we can define a continuous path $\pi : [0,1] \to \mathcal{R}_H$ from $\bar{R}_0$ to $\bar{R}_0'$ as in the proof of Fact 1. For all $\lambda \in [0,1]$, there exists $R_0^\lambda \in \mathcal{R}_{H,\varepsilon}$ such that R_0^λ coincides with $\pi(\lambda)$ on $\mathbb{R}^2_{++} \backslash C^\varepsilon$. For all $\lambda \in [0,1]$, let $\pi^\varepsilon(\lambda) \equiv R_0^\lambda$. Then it is easy to show that $\pi^\varepsilon : [0,1] \to \mathcal{R}_{H,\varepsilon}$ is a continuous path from R_0 to R_0'. ∎

Fact 3. *$\cup_{\varepsilon>0} \mathcal{R}^\varepsilon_H$ is dense in $\mathcal{R}_H$.*

Proof. Let $R_0 \in \mathcal{R}_H$ be strictly convex. For all $n \in \mathbb{N}$, let $R_0^n \in \mathcal{R}_{H,1/n}$ be such that R_0^n coincides with R_0 over $\mathbb{R}^l_+ \backslash C^{1/n}$. It is clear that R_0^n converges to R_0 as n goes to infinity. Therefore for each strictly convex preference R_0 in $\mathcal{R}_H$, there exists a sequence of preferences in $\cup_{\varepsilon>0} \mathcal{R}_{H,\varepsilon}$ that converges to R_0.

On the other hand, as we show below, the class of strictly convex preferences is dense in $\mathcal{R}_H$.

Let $R_0 \in \mathcal{R}_H$ be convex. Pick a strictly convex preference relation $R_0' \in \mathcal{R}_H$. Let $\pi : [0,1] \to \mathcal{R}_H$ be the continuous path from $R_0^{'}$ to R_0 defined in the proof of Fact 1. It is clear by definition that for all $\lambda \in (0,1)$, $\pi(\lambda)$ is a strictly convex preference relation in $\mathcal{R}_H$. Let $R_0^n \equiv \pi(1/n)$. Then $(R_0^n)_{n\in\mathbb{N}}$ is a sequence of strictly convex preference relations that converges to R_0.

Therefore $\cup_{\varepsilon>0}\mathcal{R}_{H,\varepsilon}$ is dense in $\mathcal{R}_H$. ∎

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