7-17-2017

Merging Peg Solitaire in Graphs

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INVOLVE 11:1 (2018)
dx.doi.org/10.2140/involve.2018.11.53

Merging peg solitaire on graphs

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(Communicated by Anant Godbole)

Peg solitaire has recently been generalized to graphs. Here, pegs start on all but one of the vertices in a graph. A move takes pegs on adjacent vertices x and y, with y also adjacent to a hole on vertex z, and jumps the peg on x over the peg on y to z, removing the peg on y. The goal of the game is to reduce the number of pegs to one.

We introduce the game *merging peg solitaire on graphs*, where a move takes pegs on vertices x and z (with a hole on y) and merges them to a single peg on y. When can a configuration on a graph, consisting of pegs on all vertices but one, be reduced to a configuration with only a single peg? We give results for a number of graph classes, including stars, paths, cycles, complete bipartite graphs, and some caterpillars.

1. Introduction

Peg solitaire on graphs has recently been introduced as a generalization of peg solitaire on geometric boards [Avis and Deza 2001; Beeler and Hoilman 2011]. Peg solitaire on graphs is played on a simple connected graph G and begins with a starting configuration consisting of pegs in all vertices but one; the remaining vertex is said to have a *hole*. A move involves finding vertices x, y, and z with x and y adjacent and y and z adjacent with pegs on x and y only, and jumping the peg from x over y and into z (while removing the peg at y); see Figure 1.

If there is some starting configuration of pegs and some combination of moves that reduces the number of pegs to one, we say the graph is *solvable*; if the graph is solvable for every starting configuration then we say the graph is *freely solvable*.

Recently, several variations on peg solitaire were introduced. One variant, called fool's solitaire [Beeler and Rodriguez 2012] tries to maximize the number of pegs left in the game when no more moves can be made. A second variant, called reversible peg solitaire [Engbers and Stocker 2015], asks which graphs are solvable if both moves and reverse moves are allowed.

MSC2010: 05C57.
Keywords: peg solitaire, games on graphs, graph theory.

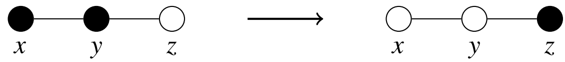

Figure 1. A move in peg solitaire on graphs.

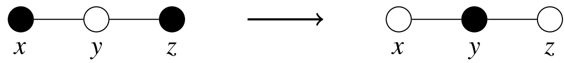

Figure 2. A move in merging peg solitaire on graphs.

In this paper, we introduce a new variation on peg solitaire, called *merging peg solitaire on graphs*, by using a different move. We again consider vertices x, y, and z with x and y adjacent and y and z adjacent. However, now we start with pegs on vertices x and z only, and the new move merges those two pegs to a single peg on y; see Figure 2.

For a fixed simple connected graph G and some initial configuration of pegs — occupying all but a single vertex — the goal of the game is to use this move to reduce the number of pegs to one. If this is possible for some initial configuration, we again say that the graph is *solvable*, and if it is possible for any initial configuration we say that the graph is *freely solvable*. The main question that we ask is the following. Given a fixed simple connected graph G, is G solvable, and if so, is G freely solvable?

Notice that the merging move is the only other symmetric way of reducing exactly two pegs in a path on 3 vertices, P_3, to exactly one peg where each vertex must change from peg to hole or vice versa. In this way, this new game may be viewed as a restricted version of Lights Out on graphs, a game where the *entire* closed neighborhood of a vertex flips all states (here pegs/holes). In this formulation, we are allowed to flip the states of all vertices in a P_3 subgraph if the endpoints of the P_3 have pegs and the center has a hole. For a survey of Lights Out, see, e.g., [Fleischer and Yu 2013].

The game is also similar to graph rubbling (see, e.g., [Belford and Sieben 2009] for an introduction to graph rubbling) in that the moves allowed are nearly identical, but the end goal of the game is quite different. Indeed, in graph rubbling, a number of pebbles (pegs) are placed on some vertices, and the allowable move removes two pebbles at vertices v and w adjacent to a vertex u while an extra pebble is added at u. The goal of graph rubbling is to use the least number of pebbles m so that any vertex is reachable from any pebble distribution of the m pebbles. In addition to the goal of merging peg solitaire on graphs being different, our game also does not allow for multiple pebbles on the vertices (and so, in particular, forces $v \neq w$).

2. Preliminary results

In this section we describe some preliminary results for various classes of graphs. As usual, we let P_n and C_n denote the path and cycle on n vertices, respectively. The complete bipartite graph with $V = X \cup Y$, where $|X| = m$ and $|Y| = n$, is denoted $K_{m,n}$; when $m = 1$ we refer to the complete bipartite graph as a *star*. A vertex of degree one is a *pendant* vertex. We begin with several useful lemmas.

Lemma 2.1. *Let G be a graph and suppose that the only holes on the vertices of G are on pendant vertices. Then there are no available moves.*

Proof. Any move requires two pegs on distinct vertices, both adjacent to the vertex with a hole. □

The next results follow from Lemma 2.1.

Lemma 2.2. *Let G be a graph. If G has any pendant vertices, then G is not freely solvable.*

Corollary 2.3. *Let T be a tree. Then T is not freely solvable.*

Next, we show that a star on at least 4 vertices is not solvable.

Theorem 2.4. *Fix $n > 2$. The star $K_{1,n}$ is not solvable.*

Proof. Let $G = K_{1,n}$. If the hole starts on a pendant vertex, then there are no available moves by Lemma 2.1. If the hole starts on the center, then a single move will leave exactly two holes on two pendant vertices. Again, by Lemma 2.1, there are no more available moves. Since $n > 2$, there are at least two pegs remaining. □

We already know that trees are not freely solvable. For the games of peg solitaire on graphs and reversible peg solitaire on graphs, not all paths are solvable [Beeler and Hoilman 2011; Engbers and Stocker 2015]; in particular, P_5 is not solvable in either of those two games. In contrast, for merging peg solitaire on graphs all paths are solvable.

Theorem 2.5. *If $n \geq 2$, the path P_n is solvable, and furthermore if an initial configuration can be reduced to a single vertex, then the initial hole must start on a vertex adjacent to a pendant vertex.*

Proof. We induct on n, with the base case $n = 2$ clear. Let the vertices of the path be labeled $1, \ldots, n$. By Lemma 2.1, the hole cannot start on vertex 1 or on vertex n. If the hole starts on vertex 2, then one move creates holes on vertices 1 and 3 only. By considering the vertices $2, \ldots, n$ we have a path on $n - 1$ vertices with a hole second from one end. Therefore we are done by induction.

Suppose the hole is on vertex i with $2 < i < n - 1$. After the first move, there are holes on vertices $i - 1$ and $i + 1$. Suppose next that the pegs on vertices i and $i - 2$ merge to a peg on $i - 1$, leaving a configuration with holes on vertices $i - 2$, i, and

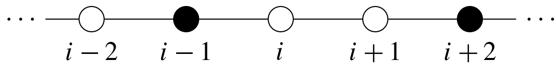

Figure 3. The configuration after the first two moves.

$i+1$; see Figure 3. The only move available is to merge pegs into $i-2$ and iterate this process, producing a graph with pegs on vertex 2 and on vertices $i+1, \dots, n$. By the assumption on i, at least two pegs remain.

The other possible second move produces a similar result, and so no set of moves can reduce the path to a configuration with a single peg unless the hole starts on a vertex adjacent to a pendant vertex. $\square$

Part of the proof of Theorem 2.5 will be useful later, but in the following (slightly) generalized form. We start with a definition.

Definition 2.6. In a configuration of pegs on a graph G, an *empty bridge* is a pair of adjacent degree 2 vertices, joined by a cut-edge, both of which have holes.

Lemma 2.7. *Suppose that G is a graph and some configuration of pegs and holes on the graph has an empty bridge and a nonzero number of pegs on either side of the empty bridge. Then G is not solvable from that configuration.*

Proof. Suppose that the empty bridge consists of vertices u and v. To solve the graph from this configuration, a peg must be moved to either vertex u or vertex v, since there are a nonzero number of pegs on either side of the empty bridge. But any move that puts a peg on u requires a prior peg on v, and any move that puts a peg on v requires a prior peg on u. $\square$

Since any graph containing a spanning solvable subgraph must also be solvable, we have the following result.

Theorem 2.8. *Let $n > 2$. The n-cycle C_n is freely solvable.*

The cycle is *freely* solvable since given a hole on any vertex of the cycle we can choose a spanning path so that the hole is adjacent to a pendant vertex of the path.

Corollary 2.9. *If G is Hamiltonian, then G is freely solvable.*

Let us consider other graph classes. By Corollary 2.9, complete graphs are freely solvable. The behavior of nonstar complete bipartite graphs is more interesting.

Theorem 2.10. *Let $m, n \geq 2$ be integers. If $m - n$ is divisible by 3, then $K_{m,n}$ is freely solvable. If $m - n$ is not divisible by 3, then $K_{m,n}$ is solvable but not freely solvable.*

Proof. Notice that any move results in two pegs becoming holes on one partition class of the graph and a single hole becomes a peg on the other partition class. Therefore if there are p pegs in the partition class of size m and q pegs in the partition class of size n, then the quantity $f(p, q) := (p-q) \mod 3$ is preserved by a move. Notice that a configuration with only a single peg has $f(p, q) = 1$ or $f(p, q) = 2$.

This immediately implies several facts. If $f(m, n) = 1$, then a configuration with the hole on a vertex in the partition class of size m cannot be reduced to a configuration with a single peg, and if $f(m, n) = 2$ then a configuration with the hole starting on a vertex in the partition class of size n cannot be reduced to a configuration with a single peg.

Next, notice that given any $m, n \geq 2$ either $f(m-1, n)$ or $f(m, n-1)$ is nonzero. So suppose that $m, n \geq 2$ and either $f(m, n) = 0$, $f(m, n) = 1$ and the hole starts on a vertex in the partition class of size n, or $f(m, n) = 2$ and the hole starts on a vertex in the partition class of size m. We describe a collection of moves that, when iterated, produces a configuration with a single peg. A *partition move* is a sequence of moves that merges pegs from one partition class into the opposite partition class until either all of the holes on the latter partition class have been filled with pegs or the vertices on the former partition class are all holes (with possibly a single peg left, depending on parity). Each partition move decreases the total number of pegs on the vertices. Note that the iteration requires $m, n \geq 2$ so that partition moves can be made back and forth. This process will terminate when there is a single peg remaining (the terminating state can't have a single peg in each partition class by the assumptions on m and n). If the initial configuration of pegs satisfies $f(p, q) = 1$ ($f(p, q) = 2$, resp.), then the final peg will be in the partition class of size m (n, resp.). □

We also investigate what happens when an edge is added to a star and, more generally, when a matching is added to a star. These graphs were analyzed for peg solitaire on graphs in [Beeler and Hoilman 2012].

Definition 2.11. Given fixed nonnegative integers B and P, the *windmill variant graph*, denoted $W(P, B)$, is the graph on $P + 2B + 1$ vertices obtained by taking a star $K_{1,P+2B}$ and adding a matching of size B on the pendant vertices of the star. We will label the pendant vertices of $W(P, B)$ by $p_1, \ldots, p_P$ and the pendant vertices of $K_{1,P+2B}$ involved in the matching by $b_1, b_2, \ldots, b_{2B}$ so that $b_{2i-1}b_{2i}$ is an edge of $W(P, B)$ for $i = 1, \ldots, B$.

See Figure 4 for an example of a windmill variant graph. Note that if $B = 0$, then $W(P, 0) = K_{1,P}$ and if $P = 0$ then $W(0, B)$ is the *windmill graph*. The vertex corresponding to the center of $K_{1,P+2B}$ is called the *universal vertex* u which is adjacent to B *blades* consisting of two vertices each. We now show that $W(P, B)$ is solvable unless $B = 0$, and $W(0, B)$ is freely solvable. We note that this differs from the results for peg solitaire, where $W(P, B)$ is solvable if and only if $P \leq 2B$

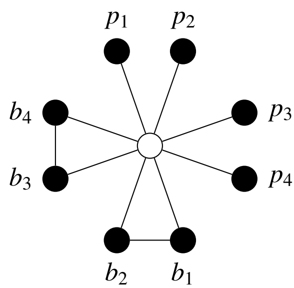

Figure 4. The windmill variant $W(4, 2)$.

and freely solvable if and only if $P \leq 2B - 1$ and $(P, B) \neq (0, 2)$ [Beeler and Hoilman 2012, Theorem 2.2].

Theorem 2.12. *Let P and B be nonnegative integers and let $W(P, B)$ be a windmill variant graph on at least 2 vertices. If $P = 0$, then $W(0, B)$ is freely solvable. If $P \neq 0$ and $B \geq 1$, then $W(P, B)$ is solvable but not freely solvable.*

Proof. Suppose first that $P = 0$ and the hole starts on the center u. If $B = 1$, then the result follows. For $B > 1$, we iteratively eliminate the pegs on distinct blades. We first merge the pegs on b_{2B} and b_1 to a peg on u, and then merge the pegs on u and b_{2B-1} to a peg on b_{2B}. If $B = 2$, we merge the pegs on b_{2B} and b_2 to u and we're finished. If $B > 2$, we have $B - 2$ full blades and pegs on b_2 and b_{2B}. We merge b_2 and b_4 into u, and then u and b_3 to b_4. Doing this last step $B - 2$ times leaves two pegs on distinct blades; we then merge them to u.

If $P = 0$ and the hole starts on a blade, say b_2, then we merge the pegs on u and b_1 to a peg on b_2. If $B = 1$ we're done, so suppose $B > 1$. Now ignoring the blade b_1b_2, we have a graph with $B - 1$ blades with the hole on u, which we can solve by the previous paragraph and end with the peg on u. We then merge the pegs on u and b_2 to a peg on b_1.

Now suppose that $P \geq 1$. By Lemma 2.1, in this case $W(P, B)$ is not freely solvable. We show that if $B = 1$ and $P \geq 1$, then $W(P, 1)$ is solvable. Since for $B \geq 1$ and $P' = P + 2(B - 1)$, $W(P', 1)$ is a spanning subgraph of $W(P, B)$, this proves the result.

Start with the hole on b_2, and merge the pegs on u and b_2 to a peg on b_1. Then merge the pegs on two pendant vertices to a peg on u, and subsequently merge the pegs on u and b_1 to a peg on b_2. Iteratively merge the pegs on two pendant vertices to a peg on u then merge the peg on u with the peg on the blade to the hole on the blade. This process stops when there are 0 pegs or 1 peg remaining on the pendant vertices. If there are 0 pegs remaining, then we are done. If there is 1 peg remaining, then merge with the peg on the blade to a peg on u. □

3. Double stars and caterpillars

Knowing whether or not a given tree is solvable would be extremely helpful in determining whether or not a connected graph is solvable or not; in particular, any connected graph with a solvable spanning tree would necessarily be solvable. Since stars are not solvable but paths are solvable, a natural first step in classifying the solvable trees is to describe when a caterpillar is solvable.

Definition 3.1. Let $n \geq 1$ be given, and let $p_1, \ldots, p_n$ be nonnegative integers. A *caterpillar* on $n + p_1 + \cdots + p_n$ vertices consists of a path on n vertices so that the i-th vertex on the path has p_i pendant vertices attached to it. We will denote this caterpillar by $P_n(p_1, \ldots, p_n)$.

See Figure 5 for an example of a caterpillar. Note also that $P_1(n)$ is isomorphic to the star $K_{1,n}$ and $P_n(0, \ldots, 0)$ is isomorphic to the path P_n.

We will prove that a large family of caterpillars are solvable and also fully classify the solvability of some special types of caterpillars. To do so, we start with a special type of caterpillar. A *double star* is a caterpillar of the form $P_2(m, n)$ — see Figure 6 — and the two vertices from the path are its *centers*.

Theorem 3.2. *Let $m, n \geq 1$. If $|m - n| \leq 1$, then the double star $P_2(m, n)$ is solvable. If $|m - n| > 1$ then the double star $P_2(m, n)$ is not solvable.*

Also, if $m = n$ and the hole starts on center vertex u, then the final peg is on v.

We note that in peg solitaire $P_2(m, n)$ (with $m \geq n$) is solvable if and only if $m \leq n + 1$ and $n \neq 1$ and freely solvable if and only if $m = n$ and $n \neq 1$ [Beeler and Hoilman 2012, Theorem 3.1].

Proof. We must start with the hole on one of the two center vertices u or v; without loss of generality assume the hole starts on u, where u has m pendant vertices. If

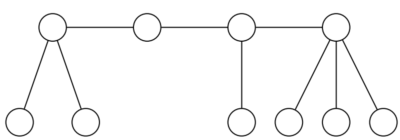

Figure 5. The caterpillar $P_4(2, 0, 1, 3)$.

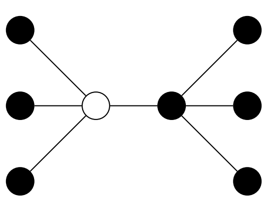

Figure 6. The graph $P_2(3, 3)$.

the pegs on two pendant vertices are merged to a peg on u, then by Lemma 2.1 no more moves are possible, and since there is a peg on v this move will never produce a graph with a single peg remaining. Therefore the only move that allows for future moves is to merge the peg on v and the peg on a pendant vertex of u to a peg on u. We then repeat by merging the peg on u and the peg on a pendant vertex of v to a peg on v. Continuing in this way, we remove the same number of pegs from the pendant vertices of u and v, so if $m = n$ this process terminates with a peg only on v.

If $m + 1 = n$, then after the first move we have the same number of pegs on the pendant vertices of u and v with the hole on v, and so the double star is solvable by the previous argument. If $m = n + 1$, then we start with the hole on v and the previous argument shows that the graph is solvable.

Now suppose that $|m - n| \geq 2$. Notice that each move that allows for future move alternates reducing the number of pegs on pendant vertices of u by 1 and the number of pegs on pendant vertices of v by 1; without loss of generality assume the hole starts on u. If $m < n$, then removing the last peg on a pendant vertex of u leaves pegs on u and $n - m + 1$ pendant vertices of v. Then the final remaining move merges two of these pegs to a peg on v, and no further moves are possible. If $m > n$ and the hole starts on u, then removing the last peg on a pendant vertex of v leaves pegs on v and $n - m$ pendant vertices of u. Merging two of these pegs leaves $n - m$ pegs remaining with no further moves available. □

Next, we see what happens to solvability when we subdivide the edge between the center vertices of a double star.

Definition 3.3. Fix an integer $k \geq 3$ and positive integers m and n. A *path-k double star* is the graph $P_k(m, 0, \ldots, 0, n)$.

See Figure 7 for an example of a path-3 double star. Recall that by Corollary 2.3 no tree is freely solvable. In what follows, we fully classify the solvability of path-k double stars. We are unaware of any results in peg solitaire for path-k double stars when $k > 2$.

Theorem 3.4. *Fix nonnegative integers m and n and let $P_3(m, 0, n)$ be a path-3 double star. Then $P_3(m, 0, n)$ is solvable if $\lfloor \frac{1}{2}(m - 1) \rfloor \leq n \leq 2m + 2$ and is not solvable otherwise.*

Proof. As before we cannot start with a hole on a pendant vertex; assume that the graph has nonpendant vertices u, w, and v with u having m pendants attached to it and v having n pendants attached to it.

Suppose first that the hole starts on u. Merging two pendant pegs results in no further moves, so the only move is to merge pegs on a pendant vertex and w to a peg on u, leaving one fewer peg on the pendants of u and a hole on w. The initial

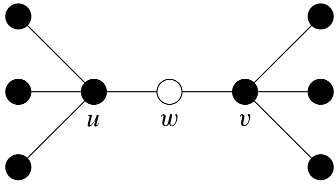

Figure 7. The graph $P_3(3, 0, 3)$.

move when the hole starts on v is similar. It remains to analyze the situation where a hole starts on w only, and we note that if $P_3(a, 0, b)$ is solvable with initial hole on w, then $P_3(a+1, 0, b)$ is solvable with initial hole on v and $P_3(a, 0, b+1)$ is solvable with initial hole on w.

Now, with a hole on w, the only available move is to merge the pegs on u and v to a peg on w, creating holes on u and v. Focusing on the hole at u, either two pegs on pendant vertices of u can merge to a peg on u or a peg on a pendant vertex of u and the peg on w can merge to a peg on u. Two similar moves are possible at v, but these moves cannot be made independently. If two pendant pegs merge to u and two pendant pegs merge to v, then no further moves can be made. So suppose that w and a pendant peg merge to u. Then the only available move merges two pegs on pendants of v to a peg on v. A similar result follows from merging w and a pendant peg to v.

This shows that if the holes are on w and pendant vertices only, then the only sets of moves that allow for future moves result in the removal of two pegs on pendant vertices from u (v, resp.), the removal of one peg on a pendant vertex from v (u, resp.), and a configuration where the only holes are on w and pendant vertices again.

Next, it is useful to see which graphs $P_3(m, 0, n)$ are solvable with initial hole on w for small values of m and n. Since we can effectively reduce one of m and n by 2 and the other by 1 (by viewing holes on pendant vertices as deleted vertices), we only need to check the solvability of $P_3(m, 0, 0)$, $P_3(0, 0, n)$, and $P_3(1, 0, 1)$ with initial hole on w. The only graphs that are solvable are $P_3(0, 0, 0)$, $P_3(1, 0, 0)$, and $P_3(0, 0, 1)$, as $P_3(1, 0, 1)$ is not solvable by Theorem 2.5 and $P_3(m, 0, 0)$ for $m > 1$ is, after one move, essentially a star with $m+1$ pendants and so is not solvable by Theorem 2.4.

Suppose that we:

(1) complete x sets of moves that remove 2 pegs on pendant vertices of u and 1 peg on a pendant vertex of v;

(2) complete y sets of moves that remove 1 peg on a pendant vertex of u and 2 pegs on pendant vertices of v; and

(3) end with $P_3(0, 0, 0)$, $P_3(1, 0, 0)$, or $P_3(0, 0, 1)$ and a hole on w.

If the initial hole started on w, then $2x + y$ pegs were removed from the pendant vertices of u and $x + 2y$ pegs were removed from the pendant vertices of v. If the initial hole started on u (v, resp.) then $2x+y+1$ ($2x+y$, resp.) pegs were removed from the pendant vertices of u and $x+2y$ ($x+2y+1$, resp.) pegs were removed from the pendant vertices of v. We analyze the possible values of m and n that are solvable by considering both where the hole starts and also which of $P_3(0, 0, 0)$, $P_3(1, 0, 0)$, or $P_3(0, 0, 1)$ remains.

For these fixed values of x and y, if $P_3(0, 0, 0)$ remains, then we have $(m, n) = (2x+y, x+2y)$, $(2x+y+1, x+2y)$, or $(2x+y, x+2y+1)$. If $P_3(0, 0, 1)$ remains, then $(m, n) = (2x+y, x+2y+1)$, $(2x+y+1, x+2y+1)$, or $(2x+y, x+2y+2)$. If $P_3(1, 0, 0)$ remains, then $(m, n) = (2x + y + 1, x + 2y)$, $(2x + y + 2, x + 2y)$, or $(2x + y + 1, x + 2y + 1)$. By the above arguments, these are the only solvable values for m and n.

Now, suppose $m > 0$ is fixed. What values of n (as a function of m) are solvable? For n to be maximized, we take $m = 2x + y$ and $n = x + 2y + 2$ where $x = 0$ and $y = m$. Then we have $n = 2m + 2$; therefore $n \le 2m + 2$. Symmetrically we have $m \le 2n + 2$, so $\lfloor \frac{1}{2}(m - 1)\rfloor \le n$. To show that all values of n in that range are possible, note that for a given m there are values of x and y with $2x + y = m$. But for each x and y pair, we have, as possible values for n, $x + 2y$, $x + 2y + 1$, and $x + 2y + 2$. This shows that all values $\lfloor \frac{1}{2}(m + 1)\rfloor \le n \le 2m + 2$ are possible. But we can have $n = \lfloor \frac{1}{2}(m - 1)\rfloor$ by taking $m = 2x + y + 2$ or $m = 2x + y + 1$ (depending on parity) and $n = x + 2y$ where $x = \lfloor \frac{1}{2}(m - 1)\rfloor$ and $y = 0$. □

Theorem 3.5. *Fix nonnegative integers m and n. Then the graph $P_4(m, 0, 0, n)$ is solvable if*:

(1) $m = n$, *or*

(2) *m is even and $n = m + 1, m + 2, m + 3,$ or $m + 4$, or*

(3) *n is even and $m = n + 1, n + 2, n + 3,$ or $n + 4$,*

and is not solvable otherwise.

Proof. As before we cannot start with a hole on a pendant vertex; assume that the graph has nonpendant vertices u, w_1, w_2 and v with m pendant vertices on u, n pendant vertices on v, and u adjacent to w_1; see Figure 8.

Suppose first that the hole starts on u. Merging two pegs on pendant vertices results in no further possible moves, so the only move is to merge a peg on a pendant vertex and the peg on w_1 to a peg on u, leaving one fewer peg on the pendants of u and a hole on w_1. When the hole starts on v the analysis is similar. So we again consider the initial hole starting on w_1 (with a similar analysis of the hole at w_2 following immediately), and have that if $P_4(a, 0, 0, b)$ is solvable with initial hole

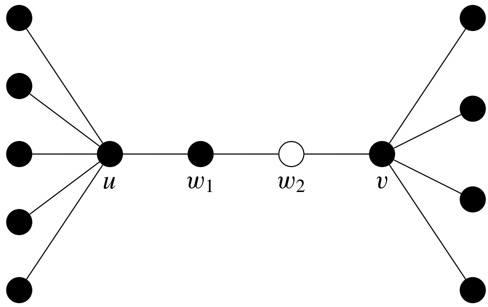

Figure 8. The graph $P_4(5, 0, 0, 4)$.

on w_1 (w_2, resp.), then $P_4(a+1, 0, 0, b)$ ($P_4(a, 0, 0, b+1)$, resp.) is solvable with initial hole on u (v, resp.).

Suppose then that the only hole on a nonpendant vertex is on w_1. The available move is to merge pegs on w_2 and u to a peg on w_1. If we then merge pegs on a pendant of u and w_1 to a peg on u, we create an empty bridge which is not solvable by Lemma 2.7.

If we first merge the pegs on w_1 and v to a peg on w_2, then we have holes on u, w_1, and v. To avoid creating an empty bridge, we must merge two pegs on pendant vertices of u to a peg on u and merge two pegs on pendant vertices of v to a peg on v. This produces a hole on w_1 and removes two pegs on the pendant vertices of u and two pegs from the pendant vertices of v.

Note that if we instead first merge two pegs on pendant vertices of u to a peg on u, a similar analysis produces the same loss of two pegs from both sets of pendant vertices with a hole on w_1.

Again, we now analyze the small cases of m and n; we see that $P_4(0, 0, 0, 0)$ is solvable with the hole on w_1 or w_2; $P_4(1, 0, 0, 0)$, $P_4(0, 0, 0, 2)$, and $P(0, 0, 0, 3)$ are solvable with the hole on w_2, and $P_4(0, 0, 0, 1)$, $P_4(2, 0, 0, 0)$, and $P_4(3, 0, 0, 0)$ are solvable with the hole on w_1, and by inspection no other graph $P_4(m, 0, 0, n)$ is solvable when one of m or n is 0 or 1 and the hole is on w_1 or w_2.

We now put all of this together. The graphs that are solvable with initial hole on w_1 are $P_4(2x, 0, 0, 2x)$, $P_4(2x, 0, 0, 2x+1)$, $P_4(2x+2, 0, 0, 2x)$ and $P_4(2x+3, 0, 0, 2x)$; the solvable graphs with initial hole on w_2 are $P_4(2x, 0, 0, 2x)$, $P_4(2x+1, 0, 0, 2x)$, $P_4(2x, 0, 0, 2x+2)$ and $P_4(2x, 0, 0, 2x+3)$. This then shows that the graphs that are solvable with initial hole on u are $P_4(2x+1, 0, 0, 2x)$, $P_4(2x+1, 0, 0, 2x+1)$, $P_4(2x+3, 0, 0, 2x)$, and $P_4(2x+4, 0, 0, 2x)$; the graphs that are solvable with initial hole on v are $P_4(2x, 0, 0, 2x+1)$, $P_4(2x+1, 0, 0, 2x+1)$, $P_4(2x, 0, 0, 2x+3)$ and $P_4(2x, 0, 0, 2x+4)$. This gives the result. □

We next show that the remaining nontrivial path-k double stars are not solvable for positive integers m and n.

Theorem 3.6. *Fix positive integers m, n (where both m and n are not 1) and fix an integer $k \geq 5$. Then the graph $P_k(m, 0, \ldots, 0, n)$ is not solvable.*

Proof. Label the vertices of the path (in order) $u = w_0, w_1, w_2, \ldots, w_{k-2}$, and $v = w_{k-1}$. Assume that u has m pendants and v has n pendants.

If the hole starts on w_i for some $i \in \{2, \ldots, k-3\}$, then any two possible consecutive moves produces an empty bridge and thus a configuration that is not solvable by Lemma 2.7.

If the hole starts on w_1, then the first move produces holes on u and w_2. If the next move merges two pegs on pendant vertices of u to a peg on u, then we have a configuration with holes only on pendant vertices of u and on w_2, which as above is not solvable. If the next move instead merges a peg on a pendant of u with the peg on w_1, then we have a configuration with an empty bridge on vertices w_1 and w_2 and so the graph again is not solvable.

Lastly, suppose that the hole starts on u. Then the only move that allows for future moves merges the pegs on a pendant vertex of u and w_1 to a peg on u. But the next move must merge the pegs on u and w_2 to a peg on w_1.

From this configuration, if we merge two pegs on pendants of u to u, then we are left with a configuration with holes only on pendant vertices of u and w_2, which as above is not solvable. So we must merge the pegs on w_1 and w_3 to a peg on w_2.

If we then merge two pegs from the pendants of u to u, then any subsequent move produces a configuration with an empty bridge and so the graph is not solvable. So the only other possible move is to merge the pegs on w_2 and w_4 to a peg on w_3. Now, if $m > 1$ we have a configuration with an empty bridge on vertices w_1 and w_2 and so is not solvable. If $m = 1$, then we can iterate this move through the path until finally we merge pegs on w_{k-3} and w_{k-1} to a peg on w_{k-2}, which leaves pegs on w_{k-2} and the n (where $n > 1$ as $m = 1$) pendant vertices of $v = w_{k-1}$. But the only possible move now merges two pegs from the neighbors of v to v; since there are at least $n+2$ pegs on the neighbors of v, this leaves at least two pegs remaining and no further moves. □

We now provide a large class of caterpillars that are solvable by combining the double star and the path. We are unaware of any results in peg solitaire for this class of caterpillars.

Theorem 3.7. *Let $t_1, t_2, \ldots, t_{n-1}$ be nonnegative integers where $p_1 = t_1$, $p_n = t_{n-1}$, and $p_i = t_i + t_{i-1}$ for $2 \leq i \leq n-1$. Then the caterpillar $P_n(p_1, p_2, \ldots, p_n)$ is solvable.*

We'll first provide the proof, and then give two specific examples of caterpillars that satisfy the conditions for p_i in Theorem 3.7. We note that this theorem can also incorporate solvable path-k double stars, but for reading ease we state this theorem

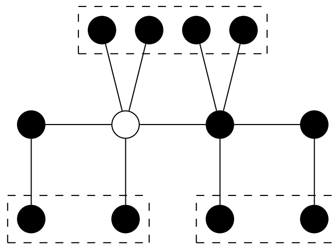

Figure 9. The solvable caterpillar $P_4(1, 3, 3, 1)$. Here $t_1 = 1$, $t_2 = 2$, and $t_3 = 1$.

and proof without adding path-3 double stars or path-4 double stars as intermediate steps. We leave the details of these changes to the reader.

Proof. Let $t_1, \ldots, t_{n-1}$ be any nonnegative integers, $p_1 = t_1$, $p_n = t_{n-1}$, and for $2 \le i \le n-1$ let $p_i = t_i + t_{i-1}$. We need to show that $P_n(p_1, \ldots, p_n)$ is solvable.

Start with the hole on vertex 2 of the path, i.e., the vertex with p_2 pendant vertices. Then focus on the double star that has as its two centers the first two vertices of the path. By Theorem 3.2 we can eliminate pegs on t_1 pendant vertices from vertex 1 and vertex 2 in the path, leaving the hole on vertex 2. Then we merge pegs from vertex 1 and 3 (in the path) to vertex 2. We then focus on the double star that has as its two centers vertex 2 and vertex 3 in the path, noting that vertex 2 has a peg and vertex 3 has a hole. Again, by Theorem 3.2 we eliminate t_2 pendant vertices from each, leaving a peg on vertex 2 and a hole on vertex 3. Then we merge the pegs from vertex 2 and vertex 4 to vertex 3. We iteratively continue until we reach vertex $n-1$ and vertex n; eliminating t_{n-1} vertices from each leaves a peg on vertex $n-1$ and a hole on vertex n. By construction, all pendant vertices have holes, and there is only one peg left on the path. This means that the caterpillar $P_n(p_1, \ldots, p_n)$ is solvable. □

Notice that by solving for each t_i we can find equivalent conditions on the values p_i: for each $i \in [1, n-1]$, $\sum_{j=1}^{i}(-1)^{i-j} p_j$ is nonnegative, and also $p_n = \sum_{j=1}^{n-1}(-1)^{i-j} p_j$.

Several interesting sequences that satisfy this condition include setting $p_i = \binom{n}{i}$ (see, e.g., Figure 9) and, for n even, letting $p_i = c$ for some nonnegative integer c.

4. Related questions and future work

We end our discussion by giving several open problems that can serve as a basis for future investigations. The main open question is to classify all simple connected graphs according to whether they are freely solvable, solvable, or not solvable. A helpful step would be to classify all trees according to whether they are solvable or not. While this might prove difficult, even determining a nice characterization of solvable caterpillars would be interesting. Another possible direction toward the main open question would be to determine which trees of a fixed diameter are solvable (see, e.g., [Walvoort 2013] for results related to peg solitaire on graphs with fixed diameter).

Another interesting question is the following. Let $G_{n,k}$ denote the set of all simple connected graphs on n vertices with k edges. Note that the only graph in $G_{n,n(n-1)/2}$ is solvable, while the star shows that not every graph in $G_{n,n-1}$ is solvable. For fixed n, what is the minimum value of k so that every graph in $G_{n,k}$ is solvable?

Suppose that we wanted to leave the *maximum* number of pegs left so that no further moves can be made; i.e., we wanted to play *merging fool's solitaire on graphs* (for results for fool's solitaire on graphs, see, e.g., [Beeler and Rodriguez 2012; Loeb and Wise 2015]). For a given graph G, determine the maximum number of pegs that can be left when playing merging fool's solitaire on graphs.

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