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# Road Systems and Betweenness

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# ROAD SYSTEMS AND BETWEENNESS

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Abstract. A *road system* is a collection of subsets of a set—the *roads*—such that every singleton subset is a road in the system and every doubleton subset is contained in a road. The induced ternary (betweenness) relation is defined by saying that a point $c$ *lies between* points $a$ and $b$ if $c$ is an element of every road that contains both $a$ and $b$.

Traditionally, betweenness relations have arisen from a plethora of other structures on a given set, reflecting intuitions that range from the order-theoretic to the geometric and topological. In this paper we initiate a study of road systems as a simple mechanism by means of which a large majority of the classical interpretations of betweenness are induced in a uniform way.

## 1. Introduction

The ancient notion of a point lying between two given points on a geometric line or a totally ordered set has strong intuitive appeal, and has been generalized in a number of interesting directions. In all of these, betweenness is taken to be a ternary relation that satisfies certain conditions, usually expressible via sentences of the first-order language $\mathsf{L}_t$, built up using a single ternary predicate symbol, plus equality (see, e.g., [7] for background). The earliest example of this known to us occurs in the monograph [18] of M. Pasch in 1882, and there is an extensive body of work from various authors since then (see, e.g., [9] for an excellent bibliography of betweenness-related work before 1970), but no definitive axiom system for betweenness has emerged. The reason for this is not particularly surprising; there are many well-studied—often higher-order—structures that naturally give rise to notions of betweenness, but which also behave in their own indiosyncratic ways. These other structures reflect intuitions that range from the order-theoretic to the geometric and topological, and are what we informally refer to here as *pre-betweenness structures.*

In this article we take the intuitive view of a point $c$ lying between points $a$ and $b$ if "$c$ blocks all roads connecting $a$ and $b$". This naïve approach suggests that what we are looking for in a natural pre-betweenness structure is a formalized system of "roads" that "connect" (or, "allow travel from") one point to another.

The essential idea is that a road connecting points $a$ and $b$ of a set $X$ is a special subset of $X$ which contains both points. Many roads may connect $a$ and $b$; if it so happens that a point $c$ is contained in each of them, then we say that $c$ *lies between* $a$ and $b$. The notion of *road system*—i.e., that of a set of points together with a collection of roads connecting them—provides a way of viewing most, if not all,

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of the pre-betweennes structures encountered in the literature as instances of the same general phenomenon.

## 2. Betweenness from Road Systems

More formally, a *road system* is a pair $\langle X, \mathcal{R}\rangle$, where $X$ is a nonempty set and $\mathcal{R}$ is a collection of nonempty subsets of $X$—the *roads*—such that: (1) for each $a \in X$, the singleton set $\{a\}$ is a road (the most efficient way to get from here to here is to stay put); and (2) each two points $a, b \in X$ belong to at least one road (you can always get there from here). We denote by $\mathcal{R}(a, b)$ the set of roads that contain both $a$ and $b$; colloquially, roads in $\mathcal{R}(a, b)$ *connect* the two points.

Each road system $\langle X, \mathcal{R}\rangle$ naturally induces a ternary (betweenness) relation $[\ ,\ ,\ ]_{\mathcal{R}} \subseteq X^3$ as follows: $[a, c, b]_{\mathcal{R}}$ holds precisely when every road containing $a$ and $b$ also contains $c$. Another way of saying this is to define the $\mathcal{R}$-*interval* $[a, b]_{\mathcal{R}}$ to be $\bigcap \mathcal{R}(a, b)$, and to take $[a, c, b]_{\mathcal{R}}$ to mean that $c \in [a, b]_{\mathcal{R}}$. We call the elements $a, b$ *bracket points* for the interval $[a, b]_{\mathcal{R}}$, noting that it may well be the case for an interval to have more than one set of bracket points (see Subsection 3.2 below).

If $\langle X, [\ ,\ ,\ ]\rangle$ is any ternary relational structure, we use interval notation just as above: for fixed $a, b \in X$, $[a, b] := \{x \in X : [a, x, b] \text{ holds in } X\}$. A ternary relation on $X$ is defined to be an *R-relation* if it is induced by a road system on $X$.

Some obvious $\mathsf{L}_t$-formulable facts about R-relations are (the universal closures of) the following, where $\to$ and $\wedge$ denote implication and logical conjunction, respectively:

- (R1) Symmetry: $[a, c, b] \to [b, c, a]$ (so $[a, b] = [b, a]$);
- (R2) Reflexivity: $[a, b, b]$ (so—with R1—$[a, b] \supseteq \{a, b\}$);
- (R3) Minimality: $[a, b, a] \to b = a$ (so—with R2—$[a, a] = \{a\}$); and
- (R4) Transitivity: $([a, c, b] \wedge [a, x, c]) \to [a, x, b]$ (so if $c \in [a, b]$, then $[a, c] \subseteq [a, b]$).

By R1 and R2, intervals are guaranteed to contain their bracket points. An interval that comprises at most two points is called *trivial*; trivial intervals containing exactly two points are called *gaps*.

Note that, in the presence of R1, R4 implies that $[a, c] \cup [c, b] \subseteq [a, b]$ whenever $c \in [a, b]$. We will see later that the reverse inclusion does not generally hold for road systems, but that it does hold for a significant class of them. A ternary relation on a set is called *basic* if it satisfies R1, R2, and R3, and is $\tau$-*basic* if it satisfies R4 as well.

We remark that there are many statements which could justifiably be called "transitivity"; see the paper [19] of Pitcher and Smiley for a list that is *exhaustive* in a precise sense. (In particular, our R4 is their axiom $t_2$.)

If we fix $a \in X$ and define the binary relation $\leq_a$ by saying that $x \leq_a y$ just in case $[a, x, y]$ holds, then R2 and R4 say, respectively, that the relation $\leq_a$ is reflexive and transitive in the usual sense, and is hence a pre-order. Also the conjunction of R1, R2, and R3 says that $a$ is the unique $\leq_a$-minimal element of $X$. We will return later to consider when these pre-orders arising from $\tau$-basic relations are also antisymmetric, and hence partial orders.

Another obvious first-order fact about R-relations is the following strengthened transitivity axiom:

- (R5) Strong Transitivity: $([a, c, b] \wedge [a, d, b] \wedge [c, x, d]) \to [a, x, b]$ (so if $c, d \in [a, b]$, then $[c, d] \subseteq [a, b]$).

Strong transitivity appears in [19] as axiom $T_5$. It clearly implies transitivity in the presence if R1 and R2: just replace each occurrence of $d$ with $a$.

We now show that the class of road systems is $\mathsf{L}_t$-*axiomatizable*, meaning that there is a set of $\mathsf{L}_t$-sentences characterizing when a ternary relation on a set is an R-relation.

**Theorem 2.0.1.** *A ternary relation on a set is an R-relation if and only if it satisfies conditions R1—R5; i.e., it is basic and strongly transitive.*

*Proof.* The less trivial direction is to assume the conditions R1—R5 hold for $\langle X, [\ ,\ ,\ ]\rangle$; and it turns out that the naïve choice of taking $\mathcal{R}$ to be $\{[a,b] : a,b \in X\}$ is a good one. From R1 and R2 we know each $[a,b] \in \mathcal{R}$ contains both $a$ and $b$, so every doubleton set from $X$ is contained in a member of $\mathcal{R}$. If $a \in X$, then R3 gives us $\{a\} = [a,a] \in \mathcal{R}$. Thus $\mathcal{R}$ is a road system, and it remains to show that $[a,b]$ always equals the intersection of $\mathcal{R}(a,b)$. Clearly $[a,b] \supseteq \bigcap \mathcal{R}(a,b)$ since $[a,b] \in \mathcal{R}(a,b)$. For the reverse inclusion, suppose $R \in \mathcal{R}(a,b)$ is chosen. Then, by definition, there are $c,d \in X$ with $R = [c,d]$. Since $a,b \in [c,d]$, we infer by R5 that $[a,b] \subseteq [c,d]$. Hence $[a,b] \subseteq \bigcap \mathcal{R}(a,b)$, and the desired equality is established. $\square$

**Remarks 2.0.2.**
(i) In [19] there is a simple example of a $\tau$-basic relation on a five-element set that is not strongly transitive. (Just declare $[a,c,b]$, $[a,d,b]$, and $[c,e,d]$ as the only generating triples in the relation.) Hence being transitive is not enough to ensure that a basic ternary relation is an R-relation (see also Subsections 3.2 and 3.6 below).
(ii) Bearing in mind that R1—R4 involve four variables and that R5 involves five, it would be interesting to see whether R-relations may be $\mathsf{L}_t$-axiomatized using fewer than five variables. This question is related to the topic of [19], but is not answered there. (See also Remark 4.0.6 (ii) below.)
(iii) The road system offered in the proof of Theorem 2.0.1 is merely the most convenient to make the proof work; generally there are many road systems giving rise to the same R-relation. It is clearly always the case that if $\mathcal{R} \subseteq \mathcal{S}$ then $[\ ,\ ,\ ]_{\mathcal{R}} \supseteq [\ ,\ ,\ ]_{\mathcal{S}}$, but it is possible for there to be road systems $\mathcal{R}$ and $\mathcal{S}$ on $X$ such that $[\ ,\ ,\ ]_{\mathcal{R}} = [\ ,\ ,\ ]_{\mathcal{S}}$, with $\mathcal{R} \cap \mathcal{S}$ comprising the singletons of $X$. (See Theorem 4.0.7 and Example 5.0.9 below.)

## 3. Some Classical Pre-Betweenness Structures

In this section we survey some classical interpretations of betweenness arising from order-theoretic and geometric structures. We show that all of these ternary relations are basic, and that the vast majority of them are R-relations.

### 3.1. General Partial Orders.

If $\langle X, \leq\rangle$ is a partially ordered set, then one may define, as does Birkhoff [5], the betweenness interpretation $[\ ,\ ,\ ]_P$ on $X$ by the condition that $[a,c,b]_P$ holds for $\langle a,b,c\rangle \in X^3$ just in case either $a \leq c \leq b$ or $b \leq c \leq a$ is true. (Note: In the sequel, when we define betweenness relations in a manner other than that given by road systems, we use subscripts as a mnemonic. Thus the subscripted letter "P" just means "partially ordered set in the Birkhoff interpretation".)

As it stands, this interpretation agrees with that given in the Huntington-Kline paper [12] when the partial ordering is total; but it does not define a basic ternary relation on $X$ otherwise, as R2 clearly fails when there are incomparable elements.

However, if we modify the Birkhoff interpretation by defining $[a, c, b]_O$ to hold just in case $([a, c, b]_P \vee (c = a) \vee (c = b))$ does (where $\vee$ is logical disjunction), then we clearly have an R-relation whose collection of order intervals of the form $[a, b]_O := [a, b]_P \cup \{a, b\}$ provides an inducing road system. So under the modified interpretation, what used to be empty intervals now become gaps.

One of the main results of [12] is that one may characterize—using a finite list of $\mathsf{L}_t$-axioms that each involve no more than four variables—when a ternary relation is the betweenness relation naturally induced by a total ordering.

Birkhoff [5] gives as exercises several $\mathsf{L}_t$-sentences that follow from properties of general partial orders, and Altwegg [2] later provides an infinite $\mathsf{L}_t$-axiomatization of the class of partially ordered sets under the Birkhoff interpretation.

Düntsch and Urquhart [8] do likewise for betweenness interpretations induced in the same way by reflexive antisymmetric binary relations, no transitivity implied. They show the Altwegg axiomatization result to be a corollary of theirs; they also show that in any such axiomatization, it is necessary that infinitely many variables be used.

While it is true that $[\ ,\ ,\ ]_O$ is first-order definable from $[\ ,\ ,\ ]_P$, the opposite is not the case: both a total ordering and an antichain ordering—i.e., no two elements are comparable—on a two-element set give rise to the same interpretation for $[\ ,\ ,\ ]_O$, but not for $[\ ,\ ,\ ]_P$. In spite of this, it seems quite likely that the axiomatizability conclusions in [8] may be extended to hold for the modified interpretation.

3.2. **Doubly Directed Partial Orders.** If $\langle X, \leq \rangle$ is a partially ordered set and $d \leq e$, then the order interval $[d, e]_O$ defined above is just $\{x \in X : d \leq x \leq e\}$. We call the ordering *doubly directed* if, for each $a, b \in X$, there are $d, e \in X$ with $d \leq e$ and $a, b \in [d, e]_O$; i.e., the ordering is directed both downwardly and upwardly. So, in doubly directed partial orders, each two-element set is contained in at least one order interval, and we define $[a, c, b]_D$ to mean that $c$ is contained within each order interval also containing $a$ and $b$. Thus, by taking $\mathcal{R}$ to be the class of order intervals $[c, d]_O$ with $c \leq d$, we obtain a road system inducing $[\ ,\ ,\ ]_D$.

Note that this interpretation of betweenness *enhances* the one given by Birkhoff [5], in the sense that there is agreement when the bracket points are $\leq$-comparable. As far as we know, there is no $\mathsf{L}_t$-axiomatization of doubly directed partial orderings under this interpretation.

When the doubly directed partial ordering is a lattice, the interval $[a, b]_D$ is just $[a \sqcap b, a \sqcup b]_O$ (where $\sqcap$ and $\sqcup$ are the lattice operations of meet and join, respectively). Here is an illustration of how intervals may not determine their sets of bracket points, as $[a, b]_D$ always equals $[a \sqcap b, a \sqcup b]_D$.

However, there is another notion of betweenness for lattices, itself well known [19], and the two notions agree exactly when the lattice is distributive. Note that, in general lattices, the inequalities $(a \sqcap b) \sqcup (c \sqcap b) \leq (a \sqcup c) \sqcap b$ and $(a \sqcap c) \sqcup b \leq (a \sqcup b) \sqcap (c \sqcup b)$ hold. Hence, if we define $[a, c, b]_G$ to mean that $(a \sqcap c) \sqcup (b \sqcap c) = c = (a \sqcup c) \sqcap (b \sqcup c)$, we may easily see that $[a, c, b]_G$ implies $[a, c, b]_D$. The definition of $[\ ,\ ,\ ]_G$ is due to Glivenko [10, 11], who was studying metric lattices. He showed that this purely lattice-theoretic condition coincides with the metric definition of betweenness (see Subsection 3.6 below). It is easy to show that $[\ ,\ ,\ ]_G$ is a basic ternary relation, but—from Theorem 9.3 in [19]—strong transitivity holds just in case the lattice is distributive (i.e., when $[\ ,\ ,\ ]_G = [\ ,\ ,\ ]_D$). Even R4 is not guaranteed: By Theorem 9.1 in [19], transitivity holds for $[\ ,\ ,\ ]_G$ if and only if the lattice is modular (i.e., satisfies the condition $c \leq b \rightarrow (c \sqcup (a \sqcap b) = (c \sqcup a) \sqcap b))$. In view of Theorem 2.0.1, this immediately gives us:

**Corollary 3.2.1.** *The basic ternary relation on a lattice arising from the Glivenko interpretation of betweenness is an R-relation precisely when the lattice is distributive.*

3.3. **Trees.** Another betweenness interpretation enhancing the order-theoretic one used in [12] and [5] arises when the partial ordering $\langle X, \leq \rangle$ is a *tree*; i.e., the ordering is downwardly directed, but no two order-incomparable elements have a common upper bound. In trees the predecessors of any element form a totally ordered subset, or *chain*. If $a, b, c \in X$, define $[a, c, b]_T$ to hold just in case $c \in [d, a]_O \cup [d, b]_O$ for every common lower bound $d$ of $a, b$. By taking $\mathcal{R}(a, b)$ to consist of sets $V(a, b, d) := [d, a]_O \cup [d, b]_O$ for $d \leq a, b$, we clearly have a road system inducing $[\ ,\ ,\ ]_T$. Betweenness relations arising from trees are extensively studied in the Adeleke-Neumann paper [1].

3.4. **Branch Sets of Trees.** This topic is also explored in [1]. Here we define our betweenness relation not on the tree ordering $\langle Y, \leq \rangle$ itself, but on the set $X = \mathfrak{B}(Y)$ of its *branches*; i.e., maximal chains. If $\alpha, \beta, \gamma \in X$, define $[\alpha, \gamma, \beta]_{\mathfrak{B}}$ to hold just in case $\alpha \cap \gamma \supseteq \alpha \cap \beta$ (equivalently, just in case $\gamma \cap \beta \supseteq \alpha \cap \beta$). In this way the interval $[\alpha, \beta]_{\mathfrak{B}}$ consists of all branches that "coalesce" with $\alpha$ and $\beta$ above where $\alpha$ and $\beta$ "coalesce" with each other. If we view branches also as leaf nodes of $Y$ (so $X \subseteq Y$), then $[\alpha, \gamma, \beta]_{\mathfrak{B}}$ holds exactly when $[\alpha, c, \beta]_T$ holds for some common lower bound $c$ of $\alpha, \gamma$ (i.e., some $c \in \alpha \cap \gamma$). By taking our roads to consist of sets $V(a) := \{\alpha \in X : a \in \alpha\}$, $a \in Y$, we obtain a road system that induces $[\ ,\ ,\ ]_{\mathfrak{B}}$.

3.5. **Real Vector Spaces.** If $X$ is a vector space over the real field $\mathbb{R}$ and $a, b \in X$ are two vectors, the *linear betweenness relation* is given by $[a, c, b]_L$ just in case $c = ta + (1 - t)b$ for some $0 \leq t \leq 1$; i.e., $c$ is a convex linear combination of $a$ and $b$.

**Proposition 3.5.1.** *A linear betweenness relation is an R-relation.*

*Proof.* A linear betweenness relation is clearly basic; all we need to show is that R5 holds, and then use Theorem 2.0.1. (This will show that letting $\mathcal{R}$ consist of the intervals $[a, b]_L$ will yield an inducing road system.) Indeed, suppose $a$ and $b$ are given and $c, d \in [a, b]_L$. Then we have $c = sa + (1 - s)b$ and $d = ta + (1 - t)b$ for some $0 \leq s, t \leq 1$. And if $x \in [c, d]_L$, say $x = uc + (1 - u)d$, $0 \leq u \leq 1$, we then have $x = va + (1 - v)b$, where $v = us + (1 - u)t$. Since $0 \leq s, t, u \leq 1$, so is $v$, and hence $x \in [a, b]_L$. □

3.6. **Metric Spaces.** If $\langle X, \rho \rangle$ is a metric space—so $\rho : X^2 \rightarrow [0, \infty)$; and $\rho(x, y) = \rho(y, x)$, $\rho(x, y) = 0$ iff $x = y$, and $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ universally hold (see, e.g., [15])—then the *strict metric betweenness relation* is defined by saying $[a, c, b]_{SM}$ holds just in case $a \neq c \neq b$ and $\rho(a, b) = \rho(a, c) + \rho(c, b)$. This notion of betweenness was first studied by Menger [14]; we modify the definition—as is done in [19]—by including R2 instead of its negation. That is, we define the *metric betweenness relation* by saying $[a, c, b]_M$ holds just in case $\rho(a, b) = \rho(a, c) + \rho(c, b)$. Hence we have the straightforward interdefinability

$$[a, c, b]_M \Leftrightarrow ([a, c, b]_{SM} \vee (c = a) \vee (c = b))$$

$[a,c,b]_{SM} \Leftrightarrow ([a,c,b]_M \wedge (c \neq a) \wedge (c \neq b))$.

As with linear betweenness, one might reasonably expect that an appropriate road system for metric betweenness comes about by letting $\mathcal{R}$ consist of the metric intervals $[a,b]_M$. Surprisingly, this approach does not work in general: in [14] it is shown that while strict metric betweenness is transitive—and hence metric betweenness is $\tau$-basic—strong transitivity does not necessarily hold. (See the "railroad space", [14], p. 80). This means that $[\ ,\ ,\ ]_M$ is not always an R-relation. And while $\mathcal{R}$ is a perfectly good road system, it is possible for $[a,b]_M$ to properly contain $\bigcap \mathcal{R}(a,b)$.

Smiley [20] considers betweenness relations that arise from multiple structures on the same underlying set. For example, if a real vector space has a norm $\|\ \|$, then it has the metric $\rho(a,b) := \|a-b\|$, and hence the two betweenness notions $[\ ,\ ,\ ]_L$ and $[\ ,\ ,\ ]_M$. Betweenness in the linear sense generally entails betweenness in the metric, but the converse is true exactly when the normed space is *rotund*; i.e., when $\|a\|+\|b\| = \|a+b\|$ for nonzero $a, b$ happens exactly when one of the two is a non-negative scalar multiple of the other. Hence, using Proposition 3.5.1, we obtain a weak analogue of Corollary 3.2.1.

**Corollary 3.6.1.** *In a rotund normed vector space, the ternary relation arising from the Menger interpretation of betweenness is an R-relation.*

## 4. Betweenness from Additive Road Systems

A road system is *additive* if the union of two overlapping roads is a road. Additive road systems may be regarded as arising from topological intuitions rather than geometric; indeed any connected topological space may be seen as a road system, with its connected subsets for roads. We will have more to say about topological interpretations of betweenness in Section 6 below.

A ternary relation on $X$ is defined to be an *AR-relation* if it is induced by an additive road system on $X$. An important first-order betweenness feature of AR-relations is the following:

(R6) Disjunctivity: $[a,x,b] \rightarrow ([a,x,c] \vee [c,x,b])$ (so for any $a,b,c \in X$, $[a,b] \subseteq [a,c] \cup [c,b]$).

To see this, suppose $\mathcal{R}$ is an additive road system and that $x \notin [a,c]_{\mathcal{R}} \cup [c,b]_{\mathcal{R}}$. Then there are $R \in \mathcal{R}(a,c)$ and $S \in \mathcal{R}(c,b)$ such that $x$ is in neither $R$ nor $S$. Since $c \in R \cap S$, we have $R \cup S \in \mathcal{R}(a,b)$. Since $x \notin R \cup S$, we know $x \notin [a,b]_{\mathcal{R}}$.

The ternary relation $\langle X, [\ ,\ ,\ ]\rangle$ is *weakly disjunctive* if $[a,b] \subseteq [a,c] \cup [c,b]$ for any $c \in [a,b]$. Since the reverse inclusion is automatic for all $\tau$-basic relations, we have the equality $[a,b] = [a,c] \cup [c,b]$, for $c \in [a,b]$, under the added assumption of weak disjunctivity.

**Proposition 4.0.1.** *Every weakly disjunctive $\tau$-basic relation is an R-relation.*

*Proof.* In view of Theorem 2.0.1, what we need to show is that weakly disjunctive $\tau$-relations $\langle X, [\ ,\ ,\ ]\rangle$ are strongly transitive. So suppose $c, d \in [a,b]$, with $x \in [c,d]$. By weak disjunctivity, either $c \in [a,d]$ or $c \in [d,b]$. If $c \in [a,d]$, then $x \in [a,d]$, by R4 (modulo R1). A second application of R4 gives $[a,d] \subseteq [a,b]$, and so $x \in [a,b]$. If $c \in [d,b]$, we make the same argument to infer $x \in [a,b]$. $\square$

Here are some examples of R-relations which are not weakly disjunctive.

**Examples 4.0.2.** (i) *Lattice Road Systems:* For lattice $\langle X, \sqcup, \sqcap \rangle$, we take the road system $\mathcal{R} = \{[a \sqcap b, a \sqcup b]_O : a, b \in X\}$, as in Subsection 3.2. The induced R-relation $[\ ,\ ,\ ]_D$ is obviously weakly disjunctive if the lattice is a chain, but is not otherwise: Indeed, if $a, b \in X$ are order-incomparable, then we have $a \sqcup b \in [a, b]_D$, but $a \sqcap b \in [a, b]_D \setminus ([a, a \sqcup b]_D \cup [a \sqcup b, b]_D)$.

(ii) *Metric Road Systems:* If $\langle X, \rho \rangle$ is a metric space, we saw in Subsection 3.6 that $\mathcal{R} = \{[a, b]_M : a, b \in X\}$ is not necessarily a road system because R5 may fail. However, even if it succeeds, weak disjunctivity may still fail: Consider the unit circle $X = \{a \in \mathbb{R}^2 : \|a\| = 1\}$. For $a, b \in X$, define $\rho(a, b) := 2 \arcsin(\frac{1}{2}\|a - b\|)$. This gives the "shortest arc" distance from $a$ to $b$, and it is easy to see that R5 holds for this metric space. However, if $a$ and $b$ are antipodal points, then the $\rho$-induced interval $[a, b]_M$ is all of $X$. So if $c$ is any third point, then $[a, c]_M \cup [c, b]_M$ is a proper subset.

The R-relations arising from trees are disjunctive.

**Examples 4.0.3.** (i) *Tree Road Systems:* If $\langle X, \leq \rangle$ is a tree ordering, we saw in Subsection 3.3 that the associated road system $\mathcal{R}$ consists of the sets $V(a, b, d) := [d, a]_O \cup [d, b]_O$, $d \leq a, b$, and the intervals are of the form $[a, b]_T = \bigcap\{V(a, b, d) : d \leq a, b\}$. For any $a \in X$, let $\overline{a}$ denote the order ideal $\{x \in X : x \leq a\}$. Given $a, b, c \in X$, the three order ideals $\overline{a} \cap \overline{b}$, $\overline{a} \cap \overline{c}$, and $\overline{b} \cap \overline{c}$ form a chain of sets under inclusion, so there are six possible ways to arrange them in order. However, if $\overline{a} \cap \overline{c} \subseteq \overline{b} \cap \overline{c} \subseteq \overline{a} \cap \overline{b}$, say, and $d$ is a common lower bound of $b$ and $c$, then $d$ is also a lower bound of $a$. Hence we have $\overline{a} \cap \overline{c} = \overline{b} \cap \overline{c}$, and the six original arrangements collapse to three.

So if $a, b, c$ are given and we wish to show $[a, b]_T \subseteq [a, c]_T \cup [c, b]_T$, then there are three cases to consider, dictated by which of the ideals $\overline{a} \cap \overline{b}$, $\overline{a} \cap \overline{c}$, $\overline{b} \cap \overline{c}$ includes the others. Now suppose $x \notin [a, c]_T \cup [c, b]_T$. Then there are $d \in \overline{a} \cap \overline{c}$ and $e \in \overline{b} \cap \overline{c}$ such that $x$ is in neither $[d, a]_O \cup [d, c]_O$ nor $[e, b]_O \cup [e, c]_O$. Since $d$ and $e$ are in the chain $\overline{c}$, they are comparable. If $\overline{a} \cap \overline{b}$ contains the other two (equal) ideals, then both $d$ and $e$ are common lower bounds of $a$ and $b$. If $d \leq e$ then we have $x \notin [e, a]_O \cup [e, b]_O = V(a, b, e)$; if $e \leq d$, then $x \notin V(a, b, d)$. In either instance, we have $x \notin [a, b]_T$.

Each of the remaining two cases is handled the same as the other, so suppose $\overline{a} \cap \overline{c} \supseteq \overline{b} \cap \overline{c} = \overline{a} \cap \overline{b}$. We know $e$ is a common lower bound of $a$ and $b$, so if $d \leq e$ we may argue as above to infer $x \notin [a, b]_T$. If $e \leq d$, then we have $[e, a]_O = [e, d]_O \cup [d, a]_O$ and $[e, c]_O = [e, d]_O \cup [d, c]_O$. From these two equalities and what we assume above about $x$, we conclude $x \notin [e, a]_O \cup [e, b]_O = V(a, b, e)$; so $x \notin [a, b]_T$.

(ii) *Branch Set Road Systems:* Let $\langle Y, \leq \rangle$ be a tree ordering, with $X = \mathfrak{B}(Y)$. Then the associated road system $\mathcal{R}$ consists of the sets $V(a) := \{\alpha \in X : a \in \alpha\}$, $a \in Y$. If $a, b \in Y$ and $\alpha \in V(a) \cap V(b)$, then both $a$ and $b$ belong to the chain $\alpha$, and hence either $a \leq b$ or $b \leq a$. This tells us that $V(a) \cup V(b)$ is either $V(a)$ or $V(b)$, and therefore that $\mathcal{R}$ is additive.

In light of Theorem 2.0.1, it is natural to ask whether every disjunctive (or maybe weakly disjunctive) R-relation is an AR-relation. The following example shows that weak disjunctivity is not enough.

**Example 4.0.4.** Suppose $X$ is a real vector space, with associated road system consisting of intervals $[a, b]_L$. We first show weak disjunctivity holds: indeed fix

$c \in [a,b]_L$, say $c = sa+(1-s)b$, for $0 \leq s \leq 1$; and let $x \in [a,b]$, say $x = ta+(1-t)b$, for $0 \leq t \leq 1$. If either $s$ or $t$ is 0 or 1, or if $s = t$, there is nothing to prove; so we first assume $0 < t < s < 1$. Then $0 < \frac{t}{s} < 1$ and $x = \frac{t}{s}c + (1 - \frac{t}{s})b$, so $x \in [c,b]_L$. In the event $s < t$, we have $x \in [a,c]_L$; hence $[a,b]_L \subseteq [a,c]_L \cup [c,b]_L$ whenever $c \in [a,b]_L$.

Now assume the vector space dimension of $X$ is $\geq 2$. Then there exist three points $a,b,c \in X$ such that $[a,b]_L \cap ([a,c]_L \cup [b,c]_L) = \{a,b\}$. Since $[a,b]_L$ contains points other than $a$ and $b$, this shows the failure of disjunctivity. So no additive road system can induce $\langle X, [\ ,\ ,\ ]_L\rangle$.

But full disjunctivity does turn out to be enough.

**Theorem 4.0.5.** *An R-relation on a set is disjunctive if and only if each of its inducing road systems is contained in an inducing road system that is additive. In particular, a ternary relation on a set is an AR-relation if and only if it satisfies axioms R1—R6; i.e., it is a disjunctive R-relation.*

*Proof.* We have already seen that additive road systems induce AR-relations, so we prove the converse.

Given an AR-relation $\langle X, [\ ,\ ,\ ]\rangle$, let $\mathcal{R}$ be a road system inducing $[\ ,\ ,\ ]$. $\mathcal{R}$ may easily fail to be additive (see Remark 4.0.6 (i) below). Recognizing this, let $\mathcal{R}^*$ consist of all $R \subseteq X$ such that there is a nonempty finite family $\{R_i : i \in I\}$ of roads from $\mathcal{R}$ such that $R = \bigcup_{i\in I} R_i$, and whenever $I = J \cup K$ with $J \neq \emptyset \neq K$, we have $(\bigcup_{i\in J} R_i) \cap (\bigcup_{i\in K} R_i) \neq \emptyset$. The family $\{R_i : i \in I\}$ representing $R \in \mathcal{R}^*$ is said to satisfy the *intersection property.*

Clearly $\mathcal{R}^* \supseteq \mathcal{R}$; and since every superfamily of a road system is also a road system, it remains to show: (1) $\mathcal{R}^*$ is additive; and (2) $\mathcal{R}^*$ induces $[\ ,\ ,\ ] = [\ ,\ ,\ ]_{\mathcal{R}}$.

Ad (1): Suppose $R, S \in \mathcal{R}^*$ have nonempty intersection; we need to show $R \cup S \in \mathcal{R}^*$. By how $\mathcal{R}^*$ is defined, we may write $R = \bigcup_{1\leq i\leq m} R_i$, $S = \bigcup_{m+1\leq i\leq n} R_i$, where $1 \leq m < n$, each $R_i$ is a road in $\mathcal{R}$, $1 \leq i \leq n$, and both families $\{R_1, \ldots, R_m\}$ and $\{R_{m+1}, \ldots, R_n\}$ satisfy the intersection property. Fix $p, q$ such that $1 \leq p \leq m < q \leq n$ and $R_p \cap R_q \neq \emptyset$, and suppose $\{1, \ldots, n\} = J \cup K$, with $J \neq \emptyset \neq K$. Our goal is to show $(\bigcup_{i\in I} R_i) \cap (\bigcup_{i\in J} R_i) \neq \emptyset$. If $r \in J \cap K$, we clearly have $(\bigcup_{i\in J} R_i) \cap (\bigcup_{i\in K} R_i) \supseteq R_r \neq \emptyset$, so we may assume $J$ and $K$ form a partition of $\{1, \ldots, n\}$. If $p$ and $q$ lie in different blocks of the partition $\{J, K\}$, then $(\bigcup_{i\in J} R_i) \cap (\bigcup_{i\in K} R_i) \supseteq R_p \cap R_q \neq \emptyset$. If $p$ and $q$ lie in the same block, say, $p, q \in J$, let $J_m = J \cap \{1, \ldots, m\}$ and $K_m = K \cap \{1, \ldots, m\}$. Then $J_m \neq \emptyset$ because $p \in J_m$. If $K_m \neq \emptyset$ also, then $\{J_m, K_m\}$ forms a partition of $\{1, \ldots, m\}$ and $(\bigcup_{i\in J} R_i) \cap (\bigcup_{i\in K} R_i) \supseteq (\bigcup_{i\in J_m} R_i) \cap (\bigcup_{i\in K_m} R_i) \neq \emptyset$ because $\{R_1, \ldots, R_m\}$ satisfies the intersection property. If $K_m$ is empty, then—because $q \in J$—we know $\{J \cap \{m+1, \ldots, n\}\}, K\}$ forms a partition of $\{m+1, \ldots, n\}$. Now use the fact that $\{R_{m+1}, \ldots, R_n\}$ satisfies the intersection property to infer $(\bigcup_{i\in J} R_i) \cap (\bigcup_{i\in K} R_i) \neq \emptyset$. Thus $R \cup S \in \mathcal{R}^*$.

Ad (2): Let $a, b \in X$. Since $[a,b] = [a,b]_{\mathcal{R}} = \bigcap \mathcal{R}(a,b)$ and $\mathcal{R} \subseteq \mathcal{R}^*$, we immediately have $[a,b] \supseteq [a,b]_{\mathcal{R}^*} = \bigcap \mathcal{R}^*(a,b)$. So it remains to show that $[a,b] \subseteq [a,b]_{\mathcal{R}^*}$; and for this it suffices to prove the

Claim. If $R \in \mathcal{R}^*(a,b)$, then $[a,b] \subseteq R$.

For each $R \in \mathcal{R}^*$, let $\|R\|$ be the least number $n \geq 1$ such that $R$ may be written as the union $R = \bigcup_{i\in I} R_i$ of roads from $\mathcal{R}$, where $\{R_i : i \in I\}$ satisfies the intersection property.

In the interests of obtaining a contradiction, suppose $R \in \mathcal{R}^*$ is a "least counterexample" to our Claim; namely that there exist $a, b \in R$ with $[a, b] \nsubseteq R$, but whenever $R' \in \mathcal{R}^*$ is such that $\|R'\| < \|R\|$, it follows that $[x, y] \subseteq R'$ for all $x, y \in R'$.

Since $[a, b] \nsubseteq R$, we know $R \notin \mathcal{R}$; so we may write $R$ as a union $\bigcup_{i \in I} R_i$ of members of $\mathcal{R}$, where $|I| = \|R\| \geq 2$ and $\{R_i : i \in I\}$ satisfies the intersection property. Pick $j, k \in I$, with $a \in R_j$ and $b \in R_k$. If there is some $x \in R_j \cap R_k$, then disjunctivity ensures that $[a, b] \subseteq [a, x] \cup [x, b] \subseteq R_j \cup R_k \subseteq R$, and we have a contradiction right away.

Thus it must be the case that $R_j$ and $R_k$ are disjoint, and therefore that $\{R_j, R_k\}$ does not satisfy the intersection property. This means that $I$ properly contains $\{j, k\}$.

Let $S = \bigcup_{i \in I \setminus \{j,k\}} R_i$. It is quite possible for $\{R_i : i \in I \setminus \{j, k\}\}$ to fail to satisfy the intersection property. In that case there is a partition $\{J', K'\}$ of $I \setminus \{j, k\}$ such that $(\bigcup_{i \in J'} R_i) \cap (\bigcup_{i \in K'} R_i) = \emptyset$. Continuing inductively in this way—since our index sets are finite—we obtain a partition $\{I_1, \ldots, I_m\}$ if $I \setminus \{j, k\}$ such that $\{R_i : i \in I_l\}$ has the intersection property, $1 \leq l \leq m$, and the unions $T_l = \bigcup_{i \in I_l} R_i$ are pairwise disjoint.

If some $T_l$ misses both $R_j$ and $R_k$, then $J = I_l$ and $K = I \setminus I_l$ are nonempty, and give a partition of $I$ such that $(\bigcup_{i \in J} R_i) \cap (\bigcup_{i \in K} R_i) = \emptyset$. This contradicts the assumption that $\{R_i : i \in I\}$ satisfies the intersection property.

So each $T_l$ must intersect at least one of $R_j, R_k$. Let $J' = \bigcup\{I_l : T_l \cap R_j \neq \emptyset\}$, $K' = \bigcup\{I_l : T_l \cap R_k \neq \emptyset\}$. Then clearly $J' \cup K' = I \setminus \{j, k\}$. Hence, if no $T_l$ intersects both $R_j$ and $R_k$, we have $J' \cap K' = \emptyset$; and $J = J' \cup \{j\}$ and $K = K' \cup \{k\}$ give a partition of $I$ such that $(\bigcup_{i \in J} R_i) \cap (\bigcup_{i \in K} R_i) = \emptyset$. Again a contradiction.

We have now shown that there is some $T_l$ intersecting both $R_j$ and $R_k$; say $x \in T_l \cap R_j$ and $y \in T_l \cap R_k$. But $T_l \in \mathcal{R}^*$ and $\|T_l\| < \|R\|$, and our induction hypothesis tells us that $[x, y] \subseteq T_l$. So using disjunctivity twice, we have $[a, b] \subseteq [a, x] \cup [x, b] \subseteq [a, x] \cup [x, y] \cup [y, b] \subseteq R_j \cup T_l \cup R_k \subseteq R$. This last contradiction completes the proof.

The second sentence of the theorem follows immediately from the argument above, together with Theorem 2.0.1. □

**Remarks 4.0.6.** (i) In Example 4.0.3(i), the road system $\mathcal{R}$ consisting of sets of the form $V(a, b, d)$, $d \leq a, b$, induces an AR-relation while not in general being additive. In this case we may describe the sets in $\mathcal{R}^*$ as being of the form $V(a_1, \ldots, a_n, d)$, where $d$ is a common lower bound of $\{a_1, \ldots, a_n\}$ and
$V(a_1, \ldots, a_n, d) := \bigcup_{1 \leq i \leq n} [d, a_i]_O$.

As we saw in Example 4.0.3 (ii), the roads $V(a)$, $a \in Y$, do provide an additive road system for $\mathfrak{B}(Y)$.

(ii) In Remark 2.0.2 (ii) we asked whether being an R-relation may be $\mathsf{L}_t$-axiomatized using at most four variables. In view of Proposition 4.0.1 and Theorems 2.0.1, 4.0.5, being an AR-relation is so axiomatizable.

If $\mathcal{R}$ and $\mathcal{S}$ are road systems on a set, write $\mathcal{R} \leq_B \mathcal{S}$ to mean that $[\ ,\ ,\ ]_{\mathcal{R}} \supseteq [\ ,\ ,\ ]_{\mathcal{S}}$. $\mathcal{R}$ and $\mathcal{S}$ are *B-equivalent* if $[\ ,\ ,\ ]_{\mathcal{R}} = [\ ,\ ,\ ]_{\mathcal{S}}$.

The construction $\mathcal{R} \mapsto \mathcal{R}^*$ described in the proof of Theorem 4.0.5 works for any road system $\mathcal{R}$. $\mathcal{R}^*$ is always an additive road system; it contains $\mathcal{R}$, and hence we always have $\mathcal{R} \leq_B \mathcal{R}^*$. Theorem 4.0.5 then tells us that $\mathcal{R}$ and $\mathcal{R}^*$ are B-equivalent precisely when $[\ ,\ ,\ ]_{\mathcal{R}}$ is disjunctive.

The following result justifies the designation of $\mathcal{R}^*$ as the *additive closure* of $\mathcal{R}$.

**Theorem 4.0.7.** *For any road system $\mathcal{R}$, $\mathcal{R}^*$ is an additive road system that contains $\mathcal{R}$. If $\mathcal{S}$ is an additive road system and $\mathcal{R} \subseteq \mathcal{S}$ (resp., $\mathcal{R} \leq_B \mathcal{S}$), then $\mathcal{R}^* \subseteq \mathcal{S}$ (resp., $\mathcal{R}^* \leq_B \mathcal{S}$). Moreover, if $\mathcal{R}$ and $\mathcal{S}$ are road systems with $\mathcal{R} \leq_B \mathcal{S}$, then $\mathcal{R}^* \leq_B \mathcal{S}^*$.*

*Proof.* Suppose road system $\mathcal{R}$ is contained in road system $\mathcal{S}$, where $\mathcal{S}$ is additive. To show $\mathcal{R}^* \subseteq \mathcal{S}$, we use induction on the number $\|R\|$ for $R \in \mathcal{R}^*$ (as defined in the proof of Theorem 4.0.5). If $R \in \mathcal{R}^*$ and $\|R\| = 1$, then $R \in \mathcal{R}$. Hence $R \in \mathcal{S}$ by assumption. Now suppose $\|R\| = n > 1$ and that $R' \in \mathcal{S}$ for every $R' \in \mathcal{R}^*$ with $\|R'\| < n$. We write $R = R_1 \cup \cdots \cup R_n$, where each $R_i$ is in $\mathcal{R}$, $n = \|R\|$, and $\{R_1, \ldots, R_n\}$ satisfies the intersection property. Set $S = R_2 \cup \cdots \cup R_n$. Then, as in the proof of Theorem 4.0.5, $\{2, \ldots, n\}$ has a partition $\{I_1, \ldots, I_m\}$, where $\{R_i : i \in I_l\}$ has the intersection property for each $1 \leq l \leq m$, and the unions $T_l = \bigcup_{i \in I_l} R_i$ are pairwise disjoint.

Now for each $1 \leq l \leq m$, we have $T_l \in \mathcal{R}^*$, $\|T_l\| < n$, and hence $T_l \in \mathcal{S}$, by our induction hypothesis.

Also $R_1$ must intersect each $T_l$. Indeed, if it happens that $R_1 \cap T_l = \emptyset$ for some $1 \leq l \leq m$, then $J = I_l$ and $K = I \setminus I_l$ are both nonempty (note that $1 \in K$). Thus $\{J, K\}$ provides a partition of $I$ such that $(\bigcup_{i \in J} R_i) \cap (\bigcup_{i \in K} R_i) = \emptyset$, contradicting the assumption that $\{R_i : i \in I\}$ satisfies the intersection property.

Since $R_1$ and $T_1, \ldots, T_m$ all belong to the additive road system $\mathcal{S}$ and $R_1$ intersects each $T_l$, $1 \leq l \leq m$, it follows that $R = R_1 \cup T_1 \cup \cdots \cup T_m \in \mathcal{S}$.

Next suppose $\mathcal{R} \leq_B \mathcal{S}$, where $\mathcal{S}$ is additive. We know that for any $a, b \in X$ and any $R \in \mathcal{R}(a, b)$, we have $[a, b]_{\mathcal{S}} \subseteq R$, and we wish to show that this is also true for any $R \in \mathcal{R}^*(a, b)$. To do this, we mimic the induction part of the proof of Theorem 4.0.5, using the fact that $[\ ,\ ,\ ]_{\mathcal{S}}$ is disjunctive. The exact same argument—with only trivial changes—works.

Finally suppose $\mathcal{R}$ and $\mathcal{S}$ are road systems, such that $\mathcal{R} \leq_B \mathcal{S}$. Then $\mathcal{R} \leq_B \mathcal{S}^*$ because $\mathcal{S} \subseteq \mathcal{S}^*$. Since $\mathcal{S}^*$ is additive, we infer $\mathcal{R}^* \leq_B \mathcal{S}^*$ by the preceding argument. $\square$

**Example 4.0.8.** Suppose $\langle X, \mathcal{R} \rangle$ is a road system, with $a, b \in X$. If disjunctivity fails to the extent that for all $x \in [a, b]_{\mathcal{R}} \setminus \{a, b\}$ there is some $c \in X$ with $x \notin [a, c]_{\mathcal{R}} \cup [c, b]_{\mathcal{R}}$, then $[a, b]_{\mathcal{R}^*} = \{a, b\}$; i.e., it is trivial.

Harking back to Example 4.0.4: If $X$ is a real vector space of dimension $\geq 2$ and $\mathcal{R}$ is the road system of intervals $[a, b]_L$, then disjunctivity fails in this sense for all pairs; so $\mathcal{R}^*$-intervals are all trivial.

Harking back to Example 4.0.2(i): If $X$ is a lattice and $\mathcal{R}$ is the road system of order intervals (so $[a, b]_D = [a \sqcap b, a \sqcup b]_O$), then disjunctivity may easily fail in this sense: suppose, for example, that $a \leq b$ and $[a, b]_D = [a, b]_O$ is such that for each $x \in [a, b]_D \setminus \{a, b\}$ there is some $c \in [a, b]_D$ order-incomparable with $x$. Then $[a, b]_{\mathcal{R}^*}$ is trivial.

## 5. Betweenness from Separative Road Systems

A road system $\langle X, \mathcal{R} \rangle$ is *separative* if for any $a, b, c \in X$ with $b \neq c$, there is some $R \in \mathcal{R}$ such that either $a, b \in R$ but $c \notin R$ or $a, c \in R$ but $b \notin R$. A ternary relation on $X$ is an *SR-relation* (resp., *SAR-relation*) if it is induced by a separative (resp., separative additive) road system on $X$.

Recall from above the pre-orders $\leq_a$ on $\tau$-basic relations; reflexive because of R2 and transitive because of R4. They become antisymmetric—and hence partial orders—in the presence of the following condition:

(R7) Antisymmetry: $([a, b, c] \wedge [a, c, b]) \to b = c$ (so for any $a, b, c \in X$, if $c \in [a, b]$ and $b \in [a, c]$, then b=c).

The following is now obvious.

**Proposition 5.0.1.** *A road system is separative if and only if its induced R-relation is antisymmetric.*

And, as an immediate consequence of Proposition 5.0.1 and Theorem 4.0.5, we obtain

**Corollary 5.0.2.** *An AR-relation on a set is antisymmetric if and only if each of its inducing road systems is contained in an inducing road system that is additive and separative.*

**Remarks 5.0.3.** (i) Antisymmetry—our terminology—appears in many betweenness studies as a fundamental axiom. (See, e.g., [1, 5, 8, 12, 19]. In [19] it is condition $\beta$ (closure).) It is an easy exercise to verify that antisymmetry holds for the betweenness interpretations arising from linear and metric structures; as for the order-theoretic ones, the verdict is mixed: $[\ ,\ ,\ ]_G$ is antisymmetric, by Lemma 8.2 in [19]; consequently, $[\ ,\ ,\ ]_D$ is antisymmetric for distributive lattices because it coincides with $[\ ,\ ,\ ]_G$ then. As for trees, it is easy to show $[\ ,\ ,\ ]_T$ is antisymmetric, but that $[\ ,\ ,\ ]_{\mathfrak{B}}$ is not. (If $\alpha, \beta$, and $\gamma$ all have the same pairwise intersection, then each is between the other two.) In [3, 4] we investigate betweenness in the context of continuum theory, where antisymmetry often fails as well. So from our point of view this condition is not axiomatic; rather it is as honored in the breach as in the observance.

(ii) From Remark 4.0.6 (i), there is a nonadditive road systems $\langle X, \mathcal{R}\rangle$ which induces an AR-relation. Theorem 4.0.5 then tells us that $\mathcal{R}^*$, an additive road system on $X$, also induces $[\ ,\ ,\ ]_{\mathcal{R}}$. In particular, we may infer that additivity in a road system—in contrast with separativity (Proposition 5.0.1)—cannot be captured just using information—first-order or otherwise—about its induced AR-relation.

**Proposition 5.0.4.** *If $\langle X, [\ ,\ ,\ ]\rangle$ is a $\tau$-basic relation that is antisymmetruc and weakly disjunctive, then for each $a \in X$ the ordering $\leq_a$ is a tree ordering with least element $a$ (the* root *of the tree).*

*Proof.* Fix $a \in X$. By antisymmetry, plus conditions R1—R4, $\leq_a$ is a partial ordering with least element $a$. It remains to show it is a tree ordering. Downward directedness is immediate, so suppose $c \leq_a b$ and $d \leq_a b$ both hold. We need to show that either $c \leq_a d$ or $d \leq_a c$. But we have $c, d \in [a, b]$. Thus, by weak disjunctivity, either $d \in [a, c]$, in which case $d \leq_a c$, or $d \in [c, b]$. In the first case, we are done; in the second, we wish to show that $c \in [a, d]$. But if $c \notin [a, d]$, then—by weak disjunctivity again—$c \in [d, b]$. But then, by antisymmetry, we have $c = d$, a contradiction. $\square$

For any ternary relation $\langle X, [\ ,\ ,\ ]\rangle$ and $a, b \in X$, define the binary relation $\leq_{ab}$ to be the restriction of $\leq_a$ to the interval $[a, b]$.

**Proposition 5.0.5.** *In $\tau$-basic relations that are antisymmetric and weakly disjunctive, the orderings $\leq_{ab}$ are total orderings. Moreover, $\leq_{ba}$ is the relation-inverse of $\leq_{ab}$.*

*Proof.* That each $\leq_{ab}$ is a total ordering is obvious from the proof of Proposition 5.0.4. For the rest, suppose $c, d \in [a, b]$. We wish to show that $c \leq_{ab} d$ if and only if $d \leq_{ba} c$. But the first condition is that $c \in [a, d]$ and the second is that $d \in [b, c] = [c, b]$. If the second condition fails, then weak disjunctivity forces $d \in [a, c]$, and we infer again—by antisymmetry—that $c = d$. □

Antisymmetry has several interesting variants.

**Theorem 5.0.6.** *For a $\tau$-basic relation that is weakly disjunctive, the following conditions are equivalent:*

(i) (R7) *Antisymmetry.*
(ii) (R7.1) *Slenderness: the property that if $c \in [a, b]$, then $[a, c] \cap [c, b] = \{c\}$.*
(iii) (R7.2) *Reciprocity: the property that if $c, d \in [a, b]$ and $c \in [a, d]$, then $d \in [c, b]$.*
(iv) (R7.3) *Uniqueness of Centroids: the property that the set $[abc] := [a, b] \cap [b, c] \cap [a, c]$—the* centroids *of $\{a, b, c\}$—has at most one element.*
(v) (R7.4) *Uniqueness of Bracket Point Sets: the property that if $[a, b] = [c, d]$, then $\{a, b\} = \{c, d\}$.*

*Proof.* Let $\langle X, [\ ,\ ,\ ]\rangle$ be a weakly disjunctive ternary relation satisfying R1—R4, and assume (i) holds. Reciprocity is merely a restatement that the total ordering $\leq_{ba}$ is the relation-inverse of $\leq_{ab}$, so (iii) holds by virtue of the proof of Proposition 5.0.5.

Still assuming antisymmetry, suppose $c \in [a, b]$ and $d \in [a, c] \cap [c, b]$. Then $d \in [a, b]$ as well; hence, by reciprocity, we have $c \in [d, b]$. But then $d = c$, by antisymmetry. Hence (ii) holds.

Suppose (ii) holds, and assume $c \in [a, b]$ and $b \in [a, c]$. Since $[a, c] = [a, b]$, we have—by slenderness—that $\{b\} = [a, b] \cap [b, c] = [a, c] \cap [c, b] = \{c\}$. Thus (i) holds.

Next, suppose (iii) holds for $\langle X, [\ ,\ ,\ ]\rangle$, and that $b \in [a, c]$ and $c \in [a, b]$ are both true. Pick $d = b$. Then $c, d \in [a, b]$ and $d \in [a, c]$; so—by reciprocity—we have $c \in [d, b] = [b, b] = \{b\}$. Hence (i) holds.

So we now have the equivalence of (i), (ii), and (iii); next we show the equivalence of (iv) with any one of the first three.

Start by showing (iv) implies slenderness. Indeed, suppose $c \in [a, b]$, with $d \in [a, c] \cap [c, b]$. Since both $[a, c]$ and $[c, b]$ are contained in $[a, b]$, we have that $c, d \in [abc]$. by (iv) we infer that $b = c$, so slenderness follows from uniqueness of centroids.

Assuming (i), (ii), and (iii), suppose $d_1$ and $d_2$ are centroids of $\{a, b, c\}$. Since $d_1, d_2 \in [a, b]$, Proposition 5.0.5 tells us that either $d_1 \in [a, d_2]$ or $d_2 \in [d_1, b]$. Without loss of generality, let us suppose $d_1 \in [a, d_2]$. By reciprocity, we now have $d_2 \in [d_1, b]$. We also know $d_1 \in [b, c]$. Since $d_2 \in [b, d_1]$, we infer $d_1 \in [d_2, c]$, again by reciprocity. Lastly, we know that $d_2 \in [a, c]$, so—reciprocity again—we have $d_2 \in [a, d_1]$. But now $d_1 = d_2$ by direct application of antisymmetry.

Finally we show the equivalence of (i),...,(iii) and (v). Clearly (v) implies antisymmetry; just take $d = a$. Then, given $[a, c] = [a, b]$, $c = b$ immediately follows.

Now suppose $[a,b]=[c,d]$ and (i),...,(iii) hold. Then we have $c,d\in[a,b]$; so, by weak disjunctivity, either $d\in[a,c]$ or $d\in[c,b]$. In the first case, $[c,d]\subseteq[a,c]\subseteq[a,b]$, and so $[a,c]=[a,b]$. Thus $b=c$ by (i), and so $[d,b]=[a,b]$. A second appeal to (i) gives $a=d$; so $\{a,b\}=\{c,d\}$ in this case.

If it happens that $d\in[c,b]$, then reciprocity implies that $c\in[a,d]$. Argue as above to obtain $b=d$ and $a=c$. Hence $\{a,b\}=\{c,d\}$ in this case too. □

**Remark 5.0.7.** R7.4 is not an automatic consequence of R7, as witnessed by the lattice interpretation $[\ ,\ ,\ ]_D$ of betweenness (see Subsection 3.2 and Example 4.0.2(i)). This shows the necessity of weak disjunctivity in the base assumption of Theorem 5.0.6.

As we saw in Proposition 5.0.4, if $\langle X,[\ ,\ ,\ ]\rangle$ is a $\tau$-basic relation that is antisymmetric and weakly disjunctive, and $a\in X$ is a fixed root point, then the binary relation $\le_a$ defines a tree ordering on $X$. Using the notation of Subsection 3.3, we then have an SAR-relation $[\ ,\ ,\ ]_T$ induced by this tree structure; and a natural question to ask is how the tree notion of betweenness relates to the original one. First we note that, by Example 4.0.3(i), the original $\tau$-basic relation—automatically an R-relation, by Proposition 4.0.1—must be disjunctive (not just weakly so) if there is to be any hope that the two notions coincide.

**Corollary 5.0.8.** *Let $\langle X,[\ ,\ ,\ ]\rangle$ be a $\tau$-basic relation that is antisymmetric and disjunctive, with root point $a\in X$. Then each interval $[b,c]$ is contained in the corresponding tree interval $[b,c]_T$ induced by $\le_a$. Furthermore, if the centroid set $[abc]$ is nonempty, then $[b,c]=[b,c]_T$.*

*Proof.* First we note that if $d\le_a b$, then $[d,b]$ equals the order interval $[d,b]_O=\{x\in X: d\le_a x\le_a b\}$. Indeed, suppose $[a,d,b]$ and that $x\in[d,b]$; i.e., $[d,x,b]$ holds. Then we get $[a,x,b]$ by R4 (modulo R1), so $x\le_a b$. Since $d,c\in[a,b]$ and $[d,x,b]$, we have $[a,d,x]$, by antisymmetry (in the form of reciprocity R7.2, see Theorem 5.0.6). Hence $d\le_a x$, and thus $x\in[d,b]_O$. Conversely, if $x\in[d,b]_O$, we know that $[a,d,x]$ and $[a,x,b]$. Since $[a,d,b]$ also, we have $[d,x,b]$, again by reciprocity. So $x\in[d,b]$.

Now suppose $b,c\in X$ and $d\in X$ is arbitrary. Then disjunctivity tells us $[b,c]\subseteq[d,b]\cup[d,c]$. And if also $d\le_a b,c$, then the right-hand side equals $[d,b]_O\cup[d,c]_O=V(b,c,d)$. This shows $[b,c]\subseteq[b,c]_T$.

Finally suppose there is a centroid $d\in[abc]$. Since $d\in[b,c]$, weak disjunctivity tells us $[b,c]=[d,b]\cup[d,c]$. And because $d\in[a,b]\cap[a,c]$, the right-hand side becomes $V(b,c,d)$. This shows that $[b,c]\supseteq[b,c]_T$. □

We do not know whether the centroid existence hypothesis is necessary for equality to hold in Corollary 5.0.8, but the following example shows it cannot be dropped.

**Example 5.0.9.** Let $X$ be a *simple closed curve*; i.e., a homeomorphic copy of the unit circle from Example 4.0.2(ii) (topologized as a subspace of the euclidean plane), and let $\mathcal{C}$ be the road system of connected subsets of $X$. Then (see Theorem 6.1.2 below) $\mathcal{C}$ is separative and additive; hence $[\ ,\ ,\ ]_{\mathcal{C}}$ satisfies the hypothesis of Corollary 5.0.8. But all the intervals in this case are clearly trivial; so if $a\in X$ is any root point, the tree order $\le_a$ is described by saying $x\le_a y$ precisely when either $x=y$ or $x=a$. If $b,c\in X\setminus\{a\}$, then $[b,c]_{\mathcal{C}}=\{b,c\}$, while $[b,c]_T=\{a,b,c\}$

Also worthy of note is that the road system you get from the trivial betweenness relation according to the proof of Theorem 4.0.4 is the collection $\mathcal{R}$ of nonempty subsets of $X$ with at most two points. $\mathcal{R}^*$, then, is the collection of all finite nonempty subsets of $X$. $\mathcal{R}^*$ and $\mathcal{C}$ are as disjoint as possible, though, since thay have only the singleton subsets of $X$ in common.

**Remarks 5.0.10.**
(i) All first-order properties so far considered in this paper are universal; i.e., of the form $\forall\overline{x}\varphi$, where $\varphi$ is quantifier-free. Centroid existence, on the other hand, is our first example of a more complex statement, as its format is universal-existential. Higher complexity $\mathsf{L}_t$-formulas arise in considerations of topological betweenness notions in Hausdorff continua—this being taken up in [3, 4].
(ii) When centroids exist in antisymmetric, weakly disjuctive $\tau$-basic relations, they are unique, by Theorem 5.0.6. This brings up the notion of the centroid as a ternary operation, akin to the median operations studied in [13] and later papers, in connection with graphs and other structures.

In light of Theorem 4.0.7, it is tempting to ask whether every road system has a "separative closure" in a sense analogous to how it has an additive closure. We are grateful to Aisling McCluskey for bringing up this issue and to Jorge Bruno [6] for providing a negative answer.

**Example 5.0.11.** Let $X$ be a set with at least three elements, and fix distinct $b, c \in X$. Let $\mathcal{R}$ consist of all singletons of $X$, along with all nonempty subsets of $X \setminus \{b, c\}$, along with all sets of the form $A \cup \{b, c\}$ for $A \subseteq X \setminus \{b, c\}$. Clearly $\mathcal{R}$ is a road system. And if we take $a \in X \setminus \{b, c\}$, we see that any road containing $a$ must contain both $b$ and $c$ if it contains either. Hence $\mathcal{R}$ is not separative. However, both $\mathcal{R} \cup \{\{a, b\} : a \in X \setminus \{b, c\}\}$ and $\mathcal{R} \cup \{\{a, c\} : a \in X \setminus \{b, c\}\}$ are separative road systems, neither is contained in the other, and neither properly contains a separative road system extension of $\mathcal{R}$.

## 6. Three Topological Pre-Betweenness Structures

In this section we consider three different interpretations of betweenness in a connected topological space. (See, e.g., [15] for topological background.)

If $Y$ is any topological space, recall that a subset of $Y$ that is both open and closed is called *clopen*. $Y$ is *connected* just in case the only nonempty clopen subset of $Y$ is $Y$ itself. If $Y$ is not connected, a *disconnection* of $Y$ is a pair $\langle A, B\rangle$ of disjoint nonempty clopen subsets of $Y$, with $Y = A \cup B$. A *component* of $Y$ is a connected subset of $Y$ that is not properly contained in any other connected subset of $Y$. Each point of $Y$ is contained in a unique component, namely the union of all connected subsets containing the point. The union of two overlapping connected sets is connected; hence the components of a space form a partition of the space. Since the closure of any connected subset is also connected, each component of $Y$ is a closed subset. A point $c$ of a connected space $X$ is a *cut point* if $X \setminus \{c\}$ is disconnected; i.e., has more than one component.

### 6.1. **The C-Interpretation.**

Let $X$ be a connected topological space, and define the ternary relation $[a, c, b]_C$ to hold for $a, b, c \in X$ just in case either $c \in \{a, b\}$ or $a$ and $b$ lie in different components of $X \setminus \{c\}$. This is the *C-interpretation* of betweenness in a connected topological space; a point of $X$ is a cut point if and only if it lies properly between two other points in the C-interpretation.

For the next result, we state an old chestnut concerning connected spaces. It occurs as Theorems 3.3 and 3.4 in [17]; a weak version of it can also be found in [16].

**Lemma 6.1.1.** *If $X$ is a connected topological space, $A$ is a connected subset of $X$, and $B$ is either a component of $X \setminus A$ or a clopen set therein, then $B \cup A$ is connected.*

**Theorem 6.1.2.** *For any connected space $X$, the C-interpretation of betweenness is an SAR-relation.*

*Proof.* Let $\mathcal{C}$ consist of the connected subsets of $X$. Since singletons are connected and $X$ itself is in $\mathcal{C}$, we know $\mathcal{C}$ is a road system. As mentioned above, the union of two overlapping connected sets is connected; so $\mathcal{C}$ is additive.

Next we show $[\ ,\ ,\ ]_C = [\ ,\ ,\ ]_{\mathcal{C}}$. Fixing $a, b \in X$, we show $[a,b]_C = \bigcap \mathcal{C}(a,b)$. Indeed, suppose $c \in [a,b]_C$. If $c \in \{a,b\}$, then $c \in \bigcap \mathcal{C}(a,b)$; otherwise $a$ and $b$ lie in different components of $X \setminus \{c\}$. But then no connected subset of $X \setminus \{c\}$ can be a member of $\mathcal{C}(a,b)$; i.e., we have $c \in \bigcap \mathcal{C}(a,b)$ in this case as well. If $c \notin [a,b]_C$, then $a$ and $b$ lie in the same component $A$ of $X \setminus \{c\}$. Thus $A$ witnesses the fact that $c \notin \bigcap \mathcal{C}(a,b)$.

Finally we show $\mathcal{C}$ is separative by showing $[\ ,\ ,\ ]_C$ to be antisymmetric (see Proposition 5.0.1). So assume $c \in [a,b]_C$, with $c \neq b$. It suffices to show $b \notin [a,c]_C$. If $c \in \{a,b\}$ then $c = a$ and $[a,c]_C = \{c\}$. Thus $b \notin [a,c]_C$. If $c \notin \{a,b\}$, then there are distinct components $A$ and $B$ of $X \setminus \{c\}$ with $a \in A$ and $b \in B$. By Lemma 6.1.1, $A \cup \{c\}$ is a connected set containing both $a$ and $c$, but not $b$. Hence here too, $b \notin [a,c]_C$. □

6.2. **The Q-Interpretation.** In a topological space $Y$, the *quasicomponent* of a point $a \in Y$ is the intersection of all clopen subsets containing $a$. Points $a$ and $b$ lie in different quasicomponents precisely when there is a disconnection $\langle A, B \rangle$ of $Y$ with $a \in A$ and $b \in B$. The quasicomponents of a space also form a partition of the space into closed subsets; every quasicomponent is a union of components. If the space is locally connected—i.e., components of open sets are open—the components and the quasicomponents coincide.

Again we start with a connected topological space $X$, but we now define $[a,c,b]_Q$ to hold for $a, b, c \in X$ just in case either $c \in \{a,b\}$ or $a$ and $b$ lie in different quasicomponents of $X \setminus \{c\}$. This is the *Q-interpretation* of betweenness in a connected topological space, and is generally more restrictive than the C-interpretation. The two interpretations do agree in a locally connected $\mathsf{T}_1$ space; regardless of agreement, though, a point $c \in X$ is a cut point of $X$ if and only if $c$ lies properly between two other points in either the C- or the Q-interpretation.

The Q-interpretation—our terminology—was first studied from the point of view of betweenness by L. E. Ward [21], whose focus was to outline an abstract approach to G H. Whyburn's theory of *cyclic elements* in continuum theory (see [22]). (In the parlance of [22], if $c$ lies properly between $a$ and $b$ in the Q-interpretation, $c$ *separates $a$ from $b$*.) The paper contains no proofs, but instead promises a more comprehensive treatment in a later work (which sadly has never appeared).

Ward defined a *cut point structure* to be a ternary relational structure $\langle X, [\ ,\ ,\ ] \rangle$ satisfying conditions R1, R2, R3, R6, and R7. He claimed that the Q-interpretation of betweenness on a connected topological space defines a cut point structure, and went on to assert that a number of other properties—including R4 (transitivity)—hold in any cut point structure.

The goal of this subsection is an analogue of Theorem 6.1.2 for the Q-interpretation. The difficulty is that we know of no obvious choice for a suitable road system; only that such a system, in general, could not be contained within the family $\mathcal{C}$ of connected subsets. This is where Corollary 5.0.2 proves most useful.

**Theorem 6.2.1.** *For any connected space $X$, the Q-interpretation of betweenness is an SAR-relation.*

*Proof.* The existence of an appropriate road system is guaranteed by Corollary 5.0.2, once we establish that the Q-interpretation satisfies conditions R1—R7. R1 (symmetry), R2 (reflexivity), and R3 (minimality) obviously hold for $\langle X, [\ ,\ ,\ ]_Q\rangle$. Since transitivity always follows from strong transitivity, it is R5, R6 and R7 that we need to concentrate on. It will be convenient to do this in reverse order.

Ad R7 (antisymmetry): Assume $c \in [a,b]_Q$, with $c \neq b$. Proceeding as in the proof of Theorem 6.1.2, we may jump to the assumption that $c \notin \{a,b\}$. Then there is a disconnection $\langle A, B\rangle$ of $X \setminus \{c\}$ with $a \in A$ and $b \in B$. Again we may apply Lemma 6.1.1. Since $A$ is clopen in $X \setminus \{c\}$, we infer that $A \cup \{c\}$ is a connected set containing both $a$ and $c$, but not $b$. This immediately shows that $b \notin [a,c]_C$, and hence that $b \notin [a,c]_Q$.

Ad R6 (disjunctivity): Suppose $[a,x,b]_Q$ holds, with $c \in X$ arbitrary. We wish to show that either $[a,x,c]_Q$ or $[c,x,b]_Q$ holds. If $x \in \{a,b\}$, we are done; so assume there is a disconnection $\langle A,B\rangle$ of $X \setminus \{x\}$ with $a \in A$ and $b \in B$. If $[a,x,c]_Q$ fails, then it must be the case that $c \in A$. Hence the disconnection $\langle A,B\rangle$ witnesses that $[c,x,b]_Q$ indeed holds.

Ad R5 (strong transitivity): Suppose $\langle X, [\ ,\ ,\ ]\rangle$ is any ternary relational structure satisfying R1, R6, and R7. We show R5 necessarily follows.

First we show R4 (transitivity). Suppose $[a,c,b]$ and $[a,x,c]$ are both true; we wish to show that $[a,x,b]$ is true also. By R6 and $[a,x,c]$ we have either $[a,x,b]$ or $[b,x,c]$. In the first case we are done; in the second, we use $[a,c,b]$ and R6 to infer that either $[a,c,x]$ or $[b,c,x]$. But $[a,x,c]$ and $[b,x,c]$ are both true. Hence, by antisymmetry, we have $x = c$. Thus $[a,x,b]$ holds in this case too.

Now we may argue as in Proposition 4.0.1, since only R1, R4, and weak disjunctivity are used. This gives us R5. □

As a corollary to Theorems 6.2.1 and 2.0.1, we see that in a connected topological space the collection $\mathcal{Q}$ of Q-intervals gives us a road system inducing $[\ ,\ ,\ ]_Q$. $\mathcal{Q}$ is automatically separative, but is not necessarily additive. Since $[\ ,\ ,\ ]_Q$ is disjunctive, though, the additive closure $\mathcal{Q}^*$ is additive and separative, and is B-equivalent to $\mathcal{Q}$ (see Theorems 4.0.5 and 4.0.7). What may seem a bit odd is that no members of $\mathcal{Q}^*$, other than singletons, are guaranteed to be connected; hence $\mathcal{Q}^*$ and $\mathcal{C}$ may have very little to do with one another. We may get $\mathcal{C}$ back into the game, however, with the following.

**Corollary 6.2.2.** *Let $X$ be a connected topological space. Then $[\ ,\ ,\ ]_Q$ is an SAR-relation on $X$. And if $\mathcal{R}$ is any road system inducing $[\ ,\ ,\ ]_Q$, then $\mathcal{R} \cup \mathcal{C}$ is also a road system inducing $[\ ,\ ,\ ]_Q$. Consequently, $(\mathcal{R} \cup \mathcal{C})^*$ is a separative, additive, $[\ ,\ ,\ ]_Q$-inducing road system, which also contains $\mathcal{C}$.*

*Proof.* In light of Theorems 4.0.5 and 4.0.7, all we need to show is that $\mathcal{R} \cup \mathcal{C} \leq_B \mathcal{R}$. So pick $a, b \in X$, and let $S \in (\mathcal{R} \cup \mathcal{C})(a, b)$. If $S \in \mathcal{R}$, then $[a, b]_Q \subseteq S$ because $\mathcal{R}$ induces $[\ ,\ ,\ ]_Q$. If $S \in \mathcal{C}$, then $[a, b]_Q \subseteq [a, b]_C \subseteq S$ because $\mathcal{C}$ induces $[\ ,\ ,\ ]_C$. □

6.3. **The K-Interpretation.** For the purposes of this paper, a *continuum* is a connected space that is also compact; a *subcontinuum* of a space is a subset which is a continuum in its subspace topology. (Since we do not assume any separation conditions, subcontinua need not be closed subsets.) A space is *continuumwise connected* if each two of its points are contained in a subcontinuum. (Continuumwise connectedness implies connectedness; the converse is true if the space is compact Hausdorff.) The union of all subcontinua containing a given point is the *continuum component* of the point; each continuum component is contained within a unique component. The union of two overlapping subcontinua is a subcontinuum, so no two continuum components can overlap. A space is continuumwise connected just in case it has exactly one continuum component. Continuum components are connected, but are not necessarily open, closed, or compact.

We now start with a continuumwise connected topological space $X$, and define $[a, c, b]_K$ to hold for $a, b, c \in X$ just in case either $c \in \{a, b\}$ or $a$ and $b$ lie in different continuum components of $X \setminus \{c\}$. This is the *K-interpretation* of betweenness in a continuumwise connected topological space, and is generally less restrictive than the C-interpretation. A point whose complement has more than one continuum component is known as a *weak cut point*; so a point is a weak cut point if and only if it lies properly between two other points in the K-interpretation of betweenness.

**Theorem 6.3.1.** *For any continuumwise connected space $X$, the K-interpretation of betweenness is an AR-relation.*

*Proof.* Let $\mathcal{K}$ consist of the subcontinua of $X$. Then it is routine to show that $\mathcal{K}$ is an additive road system. To show $[a, b]_K = \bigcap \mathcal{K}(a, b)$ for $a, b \in X$, mimic the proof of Theorem 6.1.2. □

Significant in its absence for the K-interpretation of betweenness is antisymmetry; the road system $\mathcal{K}$ is not generally separative, even when the space is a metrizable continuum.

**Example 6.3.2.** $X$ is the $\sin(1/x)$*-curve*, defined to be the union $A \cup S \subseteq \mathbb{R}^2$, where $A = \{0\} \times [-1, 1]$ and $S = \{\langle x, \sin(1/x)\rangle : 0 < x \leq 1\}$. $X$ is an example of a metrizable continuum whose K-interpretation of betweenness is not antisymmetric: if $a \in S$ and $b, c \in A$, then $c \in [a, b]_K$ and $b \in [a, c]_K$; hence $\leq_a$ (see Section 5) fails to be a partial ordering.

On the other hand, if $a \in A$ and $a$ is a cut point of the arc $A$, then $\leq_a$ is a partial ordering which is not a tree ordering: the two noncut points of $A$ are $\leq_a$-incomparable, but any element of $S$ is a common $\leq_a$-upper bound for them.

## 7. Concluding Remarks

The notion of *road system* is a new and very simple way to interpret betweenness uniformly, over a wide spectrum of mathematical environments. We have shown that the vast majority of betweenness interpretations in the literature may each be viewed as being naturally induced by a suitable road system; and have begun a serious investigation of road systems themselves, especially in establishing Theorems 4.0.5 and 4.0.7.

In [3, 4] we take up a more extended study of the topological interpretations of betweenness in the context of Hausdorff continua. In [3] the focus is on when there are no gaps, and in [4] we look at the role of antisymmetry.

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