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On Congruence Lattices of Nilsemigroups

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Abstract

We prove that the congruence lattice of a nilsemigroup is modular if and only if the width of the semigroup, as a poset, is at most two, and distributive if and only if its width is one. In the latter case, such semigroups therefore coincide with the nil Δ-semigroups. It is further shown that if a finitely generated nilsemigroup has modular congruence lattice, then the semigroup is finite.

Studying congruence lattices is a well-established direction in semigroup theory. Surveys of this area have been given in [13] and [14]. One of the natural questions here is to characterize semigroups with distributive or modular congruence lattices. This problem has been solved for several classes of semigroups.

Of course the congruence lattice of every group is modular. The characterization of abelian groups whose congruence lattices are distributive is a corollary of the famous result of Ore [16]. Non-abelian groups with distributive congruence lattices had been studied by Pazderski [17] and Maj [12]. The case of semilattices had been considered by Dean and Oehmke [4] and Hamilton [8], who showed that the congruence lattice of a semilattice is modular if and only if it is distributive, and if and only if the semilattice itself is a tree. Fountain and Lockley [5] classified the Clifford semigroups with either modular or distributive congruence lattice and the same authors [6] determined the bands with distributive congruence lattice. Auinger [1] studied strict inverse semigroups with distributive or modular congruence lattices. Bonzini and Cherubini [2] studied the case of inverse ω-semigroups. Regular semigroups with the minimal condition for idempotents and having distributive or modular congruence lattices were characterized by Jones [10]. Hamilton [9] studied the case of commutative cancellative semigroups. The Δ-semigroups – semigroups whose congruence lattices form a chain – have been particularly well studied (see, for instance, [15]).

In the present paper we are interested in the class of nilsemigroups. Recall that a semigroup with zero is called a *nilsemigroup* if, for each of its elements x, there exists a positive integer n such that $x^n = 0$. A little is already known about congruence lattices of nilsemigroups. It follows from [11] that the congruence lattice L of a nilsemigroup is strictly semimodular, i.e. it

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satisfies the following property: for every $a, b, c \in L$, $a \succ b$ implies $a \vee c \succ b \vee c$ or $a \vee c = b \vee c$, where $\succ$ is the covering relation (see [7]). As is noted below, it is easy to see that the class of congruence lattices of nilsemigroups satisfies no nontrivial lattice identity.

Other than in the study of Δ-semigroups, until now nothing has been known in general about nilsemigroups with distributive or modular congruence lattices. We reduce the distributive case to that of Δ-semigroups and then characterize the modular case in terms of the width of the underlying poset of the semigroup.

Let S be a nilsemigroup. It is a well-known fact that S is $\mathscr{J}$-trivial. Thus the natural partial order on the $\mathscr{J}$-classes reduces to a partial order on S itself, given by

$$y \leqslant x \text{ iff there exist } s, t \in S^1 \text{ such that } y = sxt.$$

Let $(X, \leqslant)$ be any poset. The notation $a \parallel b$ indicates that a and b are incomparable under $\leqslant$. A subset $A \subseteq X$ is called an *antichain* [*a chain*] if elements of A are pairwise incomparable [pairwise comparable] under $\leqslant$. The *width* of $(X, \leqslant)$ is the greatest cardinality, if it exists, of any antichain in X.

For the elementary background on semigroups and lattices needed here we refer the reader to [3] and [7] respectively. A lattice $(L, \wedge, \vee)$ is *distributive* if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a, b, c \in L$. Among several characterizations of *modularity* of L is the following: if $a \leqslant c$, then $(a \vee b) \wedge c = a \vee (b \wedge c)$, for all $a, b, c \in L$.

Denote the congruence lattice of a semigroup S by $(\operatorname{Con} S, \cap, \vee)$, or just $\operatorname{Con} S$.

The main results of the paper are the following:

Theorem 1. *Let S be a nilsemigroup. Then the following are equivalent:*
1) $\operatorname{Con} S$ *is distributive;*
2) The poset $(S, \leqslant)$ *is a chain;*
3) $\operatorname{Con} S$ *is a chain.*

Theorem 2. *Let S be a nilsemigroup. Then the following are equivalent:*
1) $\operatorname{Con} S$ *is modular but not distributive;*
2) The poset $(S, \leqslant)$ *has width 2.*

According to Theorem 1, a nilsemigroup has distributive congruence lattice if and only if it is a Δ-semigroup. In the finite case, it follows from earlier work on such semigroups that the semigroup must be cyclic. Following Proposition 2, we include the proof of a slightly more general statement, for completeness.

In the infinite case, the commutative Δ-semigroups were completely described by Tamura [21] and Schein [19, 20]. Specializing to the case of nilsemigroups, the following complete description is obtained. Here the semigroups Q and R are the Rees quotients of the semigroup $\mathbb{R}^+$ of positive real numbers, under addition, modulo the ideals $[1, \infty)$ and $(1, \infty)$, respectively.

Corollary 1. *An infinite, commutative nilsemigroup has distributive congruence lattice if and only if it is isomorphic to a subsemigroup G of Q or of R with the property that whenever $x \in G$ and $x + y \in G \backslash 0$, then $y \in G$.*

Corollary 2. *Let S be a nilsemigroup such that* $\operatorname{Con} S$ *is modular. If S is finitely generated, then it is finite. If S is not cyclic, then it is generated by two elements a and b, say, and the poset $\{a^2, ab, ba, b^2\}$ has width at most two.*

Examples of infinite nilsemigroups with congruence lattices that are modular but not distributive are easily constructed. For instance, the 0-direct union of any two infinite, totally ordered, nilsemigroups is a nilsemigroup of width two. Corollary 1 provides a wealth of candidates from which to build.

The following elementary fact is a part of semigroup folklore.

Lemma 1. *Let S be a nilsemigroup, $a, b \in S$ and $a \neq 0$. Then $a > ab$ and $a > ba$.*

Lemma 2. *Let S be a nilsemigroup, $\theta \in \operatorname{Con} S$, $(a, b) \in \theta$ and $a > b$. Then $(a, 0) \in \theta$.*

Proof. Since $a > b$, then $b = sat$ for some $s, t \in S^1$ and either $s \neq 1$ or $t \neq 1$. By assumption, there exists n such that $s^n = 0$ or $t^n = 0$, respectively. Then

$$(b, sbt) = (sat, sbt) \in \theta, (sbt, s^2bt^2) \in \theta, \ldots, (s^{n-1}bt^{n-1}, 0) \in \theta.$$

By transitivity, $(a, 0) \in \theta$. □

Denote by Part_n the full partition lattice of a set with n elements. It is well known that Part_n is non-distributive for $n \geqslant 3$ and non-modular for $n \geqslant 4$. Moreover, the class of all such finite partition lattices satisfies no proper lattice identity [18]. Observe that the congruence lattice of any n-element null (or 'zero') semigroup is Part_n itself. Since such semigroups are nilsemigroups, it follows that the class of congruence lattices of finite nilsemigroups satisfies no nontrivial lattice identity.

Proposition 1. *Let S be a nilsemigroup for which $(S, \leqslant)$ contains an antichain of size n. Then* $\operatorname{Con} S$ *has a sublattice isomorphic to* Part_{n+1}*.*

Proof. Let A be an n-element antichain in $(S, \leqslant)$ and

$$K = \{x \in S \mid x < a \text{ for some } a \in A\}.$$

It follows from Lemma 1 that K is an ideal of S. Consider the Rees quotient $T = S/K$. The set $I = A \cup \{0\}$ forms an $(n+1)$-element ideal in T. Let π be a partition of I. If $(x, y) \in \pi$, then for every $t \in T$, $tx = xt = yt = ty = 0$. Therefore the congruence generated by π has the form $\pi \cup \Delta_T$, where Δ_T denotes the equality relation on T. Thus we have the mapping $f\colon \mathrm{Part}_{n+1} \to \operatorname{Con} T$ defined by $\pi \mapsto \pi \cup \Delta_T$. It is easy to verify that f is a lattice embedding. Therefore Part_{n+1} is isomorphic to a sublattice of $\operatorname{Con} T$, which in turn is isomorphic to a filter of $\operatorname{Con} S$. So Part_{n+1} is isomorphic to a sublattice of $\operatorname{Con} S$. □

In view of the properties of partition lattices cited earlier, that (1) implies (2) in Theorems 1 and 2 is immediate. We now turn to the converses. In the case of distributivity, this is already included in the proof of [15, Theorem 1.56]. Since it is easy, we include a proof for completeness.

Proposition 2. *Let S be a nilsemigroup such that $(S, \leqslant)$ is a chain. Then the ideals of S are totally ordered and every congruence is a Rees ideal congruence. Hence* $\operatorname{Con} S$ *is a chain.*

Proof. That the ideals are totally ordered is clear. Now let $\theta \in \operatorname{Con} S$ and denote by I the θ-class of 0. Clearly I is an ideal of S. If $a \in S$ and the θ-class of a is not a singleton, then by Lemma 2, $a \in I$. That is, θ is the Rees ideal congruence modulo I. □

In the introduction, it was noted that every finite nilsemigroup, whose congruence lattice is a chain, is cyclic. Again, we include the proof for completeness and for comparison with the modular case. In fact we show that any *finitely generated* nilsemigroup S such that $(S, \leqslant)$ is a chain must be finite cyclic. For suppose that $x_1 > x_2 > \cdots > x_k$ is an irredundant generating set for S, with $k > 1$. Since $x_2 < x_1$, $x_2 = sx_1t$ for some $s, t \in S^1$, not both 1. But then s and t, if not 1, must be power of x_1 and so the same is true of x_2 itself, contradicting the assumption. Thus S is cyclic. But if S is infinite, it has the finite cyclic groups as quotients and therefore their congruence lattices as filters in $\operatorname{Con} S$. Hence S is finite cyclic. (Finiteness is also a consequence of Corollary 2.)

The converse argument in the case of modularity is somewhat more complex.

Proposition 3. *The congruence lattice of any nilsemigroup S such that $(S, \leqslant)$ has width two is modular.*

Proof. Let ρ, θ, τ be congruences on S such that $\rho \subseteq \tau$. Let $(a, b) \in (\rho \vee \theta) \cap \tau$. It must be shown that $(a, b) \in \rho \vee (\theta \cap \tau)$.

There is a sequence $a = x_0, x_1, \ldots, x_k, x_{k+1} = b$ such that $(x_i, x_{i+1}) \in \rho \cup \theta$ for each i. We may proceed by induction on k, the case $k = 1$ being obvious. In fact, since $\rho \subseteq \tau$, it may be assumed that $a \; \theta \; x_1$ and $x_k \; \theta \; b$, so that $k \geq 2$. It may, further, be assumed that each pair (x_i, x_{i+1}) belongs to exactly one of ρ and θ.

First suppose one of a and b is zero, say $b = 0$. It follows from Lemma 2 that the θ-class of x_k is an ideal of S. Observe next that if $x_i < a$ for any i, $1 \leqslant i \leqslant k$, then (by the same lemma) $a \; \tau \; x_i \; \tau \; 0$ and the induction hypothesis applies. Now consider $x_{k-1} \; \rho \; x_k$. If these elements are comparable, then $x_{k-1} \; \rho \; 0$ and the induction hypothesis applies. Otherwise, by the width hypothesis, a is comparable with, and thus necessarily less than, one of the two. If $a < x_k$, then $a \; \theta \; 0$ (and so $(a, 0) \in \theta \cap \tau$). So $a < x_{k-1}$. Write $a = sx_{k-1}t$ for some $s, t \in S^1$, not both 1. (We shall omit mention of this qualification in similar situations below and in the next proof.) Then $a \; \rho \; sx_kt \; \theta \; 0$ provides a shorter sequence and the induction hypothesis again applies.

In the general case, suppose a and b are comparable. By Lemma 2, $(a, 0), (b, 0) \in (\rho \vee \theta) \cap \tau$ and so the previous case completes the argument.

Otherwise $a \parallel b$. Suppose a and x_1 are comparable and, also, that x_k and b are comparable. Then $a \; \theta \; 0 \; \theta \; b$ and so $(a, b) \in \theta \cap \tau$. Without loss of generality, it may therefore be assumed that $a \parallel x_1$ and, by the width hypothesis, x_1 and b are therefore comparable.

Suppose that $x_1 > b$. Note that since $(x_1, b) \in \rho \vee \theta$, then by Lemma 2, $(b, 0) \in \rho \vee \theta$. Thus if it should happen that $a \; \tau \; 0$ or $b \; \tau \; 0$, then the proof of that special case applies. If x_1 and x_2

are comparable, then $x_1 \; \rho \; 0$ and so $x_1 \; \rho \; b$, resulting in a shorter sequence. So $x_1 \parallel x_2$ and, by the width hypothesis, x_2 and a are comparable.

If $x_2 < a$, then, since we may write $b = sx_1t$, we have $b \; \rho \; sx_2t < x_2 < a$. From $a \; \tau \; b$ it then follows that $a \; \tau \; 0$.

Alternatively, $x_2 > a$ and we may write $a = qx_2r \; \rho \; qx_1r < x_1$. Since $a \parallel x_1$, $a \neq qx_1r$. Thus if a and qx_1r are comparable, then by Lemma 2, $a \; \rho \; 0$ and so $a \; \tau \; 0$. Otherwise, $a \parallel qx_1r$ and, by the width hypothesis, qx_1r and b are comparable. If they are distinct, then $b \; \tau \; a \; \tau \; qx_1r$ implies $b \; \tau \; 0$. If $b = qx_1r$, then $a \; \rho \; b$. This concludes the analysis in the case that $x_1 > b$.

Now suppose $x_1 < b$, $x_1 = sbt$, say, and consider x_k. If x_k and b are comparable, then $b \; \theta \; 0$ and so $b \; \theta \; x_1 \; \theta \; a$. So we may assume $x_k \parallel b$. Then after applying the analysis for the case $x_1 > b$ to the case $x_k > a$, it remains to consider $x_k < a$. Now $a \; \theta \; x_1 = sbt \; \theta \; sx_kt < x_k < a$. Thus $a \; \theta \; 0$ and $a \; \theta \; x_k \; \theta \; b$. This completes the proof. □

Now we show how Theorems 1 and 2 imply Corollary 2.

Proof. Since S is finitely generated, then it has an irredundant generating set $B = \{a_1, \ldots, a_n\}$. Suppose $a_i < a_j$ for some i, j. Then there exist s, t such that $a_i = sa_jt$. By irredundancy, any expression for sa_jt as a product from B must involve a_i, which by Lemma 1 implies $a_i = 0$, contradicting irredundancy. Thus B is an antichain under $\leqslant$. By Theorems 1 and 2, $|B| = 1$ or $|B| = 2$. If $|B| = 1$, then S is a cyclic nilsemigroup, which is finite. Let $|B| = 2$ and $B = \{a, b\}$, for convenience of notation. Then every element of S can be written as $a^{k_0}b^{l_1}a^{k_1}b^{l_2} \ldots a^{k_n}b^{l_{n+1}}$ for some $k_i, l_i > 0$ for $1 \leqslant i \leqslant n$, and $k_0, l_{n+1} \geqslant 0$.

Suppose that $ab = ba$. Then every element of S can be represented as a^kb^l for suitable k, l. Since $a^n = b^m = 0$ for some n, m, then S can contain only finitely many elements.

Otherwise, without loss of generality we assume $ab \nleqslant ba$. Thus $ab \neq 0$. Let $a^2 \geqslant ab$. Then there exist s, t such that $ab = sa^2t$. By Lemma 1, $ab = a^p$ for some p. Then every element of S can be written as $b^\ell a^k$ for suitable k, ℓ, which, as before, means that S is finite. The case $b^2 \geqslant ab$ is similar.

Now let $a^2 \ngeqslant ab$, $b^2 \ngeqslant ab$. Suppose that $ba < ab$. Then $ba = sabt$. If s and t (if not 1) are not respectively powers of a and of b, then by Lemma 1, $ba = 0$. In either event, we can write every element of S in the form a^kb^ℓ, so S is finite.

Finally, let $ab \parallel ba$. We can assume that $a^2 \ngeqslant ba$ and $b^2 \ngeqslant ba$ (if not, then we have the same arguments for ba as we had before for ab). Since S has width two, then $a^2 \leqslant c$ and $b^2 \leqslant d$ for $c, d \in \{ab, ba\}$. Without loss of generality we may suppose that $a^2 < ab$ and $a^2 \nless b^2$. Now either $a^2 = 0$ or from $a^2 = sabt$ it then follows from Lemma 1 that $a^2 = a^ibaba \ldots ba^j$ for some $i, j \in \{0, 1\}$.

Similarly, $b^2 = uabv$ or $b^2 = ubav$ and either $b^2 = 0$ or, using the alternatives for a^2 just stated, $b^2 = a^fbaba \ldots ba^g$ for some $f, g \in \{0, 1\}$. Therefore, in all cases every element of S can be written in the latter form and, since $(ab)^n = 0$ for some n, S is finite. □

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