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Exploring the Relationship between K-8 Prospective Teachers' Algebraic Thinking Proficiency and the Questions They Pose during Diagnostic Algebraic Thinking Interviews

Leigh A. van den Kieboom
Marquette University, leigh.vandenkieboom@marquette.edu

Marta T. Magiera
Marquette University, marta.magiera@marquette.edu

John C. Moyer
Marquette University, john.moyer@marquette.edu

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Exploring the relationship between K-8 prospective teachers' algebraic thinking proficiency and the questions they pose during diagnostic algebraic thinking interviews

Leigh A. van den Kieboom
College of Education, Marquette University, Milwaukee, USA
Marta T. Magiera
Department Mathematics, Statistics, and Computer Science, Marquette University, Milwaukee, USA
John C. Moyer
Department Mathematics, Statistics, and Computer Science, Marquette University, Milwaukee, USA

Abstract

In this study, we explored the relationship between prospective teachers' algebraic thinking and the questions they posed during one-on-one diagnostic interviews that focused on investigating the algebraic thinking of middle school students. To do so, we evaluated prospective teachers' algebraic thinking proficiency across 125 algebra-based tasks and we analyzed the characteristics of questions they posed during the interviews. We found that prospective teachers with lower algebraic thinking proficiency did not ask any probing questions. Instead, they either posed questions that simply accepted and affirmed student responses or posed questions that guided the students toward an answer without probing student thinking. In contrast, prospective teachers with higher algebraic thinking proficiency were able to pose probing questions to investigate student thinking or help students clarify their thinking. However, less than half of their questions were of this probing type. These results suggest that prospective teachers' algebraic thinking proficiency is related to the types of questions they ask to explore the algebraic thinking of students. Implications for mathematics teacher education are discussed.

Introduction

Over the past several decades, mathematics education reform initiatives have placed a great deal of emphasis on understanding students' mathematical thinking (Common Core State Standards Initiative [CCSSI] 2010; Grouws 1992; Kilpatrick et al. 2001; Lester 2007; National Council of Teachers of Mathematics [NCTM] 2000). Research has shown that teachers who focus on students' mathematical thinking improve the teaching–learning process in multiple ways. Teachers who routinely elicit and interpret their students' mathematical thinking, for example, are better positioned to support student learning and increase student achievement in mathematics (Clarke 2008; Doerr 2006; Jacobs et al. 2007). Teachers who value and encourage students to explain their thinking create a more productive classroom learning environment (Sowder 2007). Finally, teachers who attend to and use students' mathematical thinking to guide their instruction increase their own knowledge of mathematical content (Fennema et al. 1996).

One way to stimulate student thinking and thus support student learning is to pose well-formulated questions. Good questions engage students in determining evidence, predicting outcomes, drawing conclusions, justifying solutions, or making decisions, and thus create opportunities for deeper learning (Herbal-Eisenmann and Breyfogle 2005; National Council of Teachers of Mathematics [NCTM] 1991; Vacc 1993). Carefully formulated questions encourage students to become independent thinkers, develop the ability to reason logically, and communicate their ideas to others. Questions also provide a window through which teachers can gain access to and assess their students' understanding and learning of mathematical ideas (Carpenter et al. 2000). While the importance of posing well-formulated questions to uncover students' mathematical thinking and support student learning is widely acknowledged (National Council of Teachers of Mathematics [NCTM] 1991, 2000; Carpenter et al. 2000), a substantial portion of the existing body of research simply describes the types and purposes of teacher questioning (Franke et al. 2009; Kazemi and Stipek 2001; Moyer and Milewicz 2002; Sahin and Kulm 2008). Little attention has been paid to how teacher knowledge of mathematics content shapes the kinds of questions teachers pose. Thus, the purpose of this study was to explore the relationship between teachers' mathematics content knowledge and the types of questions they pose to investigate students' mathematical thinking. More specifically, our goal was to examine how prospective teachers' own proficiency with the aspect of algebraic thinking known as Building Rules to Represent Functions relates to the types of questions they pose to investigate student thinking about algebra-based concepts.

Teacher knowledge

Teacher knowledge, also referred to as professional knowledge, has been identified as an important factor that influences the outcomes of teacher practice (Borko and Putnam 1996; Hill et al. 2005; National Research Council [NCR] 2001; Sowder and Schappelle 1995). Building on the work of Shulman and colleagues who conceptualized the core dimensions of teacher knowledge as comprising subject matter content knowledge (CK), pedagogical content knowledge (PCK), and general pedagogical knowledge (PK) (e.g., Shulman 1986; Grossman and Richert 1988), researchers in mathematics education (e.g., Ball et al. 2008; Kunter et al. 2013; Tatto et al. 2008) have created different models to delineate the domain-specific professional knowledge involved in teaching mathematics. While these models stem from a common theoretical perspective, researchers are not unanimous in their descriptions and subdivisions of the domains of teacher knowledge. For instance, Ball et al. (2008) refer to mathematics content knowledge (CK) as "mathematical knowledge for teaching," i.e., the mathematics knowledge that mathematics teachers need to be effective in their work. Depending on the level of teaching, this may be limited to, for example, the knowledge of the mathematics directly related to the elementary curriculum. Baumert and Kunter (2013), on the other hand, argue that the content knowledge needed for teaching school mathematics does not depend on the level taught. They describe mathematics content knowledge as profound understanding of the mathematics that is taught to K-12 students. Tatto et al. (2008) interpret mathematics content knowledge as factual knowledge of the mathematics directly related to the level being taught and conceptual knowledge of the organization and structure of the mathematics 3–4 years beyond the grade level taught. These researchers argue that mathematics content knowledge (CK) is not sufficient for effective teaching. Furthermore, all these researchers have adopted Shulman's (1986) construct of pedagogical content knowledge (PCK), a type of knowledge that is distinct from mathematics content knowledge (CK) but dependent on it. However, they each characterize PCK somewhat differently. Ball et al. (2008) describe PCK as including the (1) knowledge of content and students, (2) knowledge of content and teaching, and (3) knowledge of content and curriculum. Baumert and Kunter (2013) delineate PCK into three somewhat different dimensions: (1) knowledge of mathematical tasks and their potential, (2) knowledge of student mathematical thinking (e.g., conceptions, misconceptions, errors, and strategies), and (3) knowledge of mathematical explanations and representations. Tatto et al. (2008) view PCK as (1) knowledge of curriculum, (2) knowledge of planning for mathematics teaching and learning, and (3) knowledge of enacting teaching and learning. Although these interpretations of CK and PCK appear different, they share a common underlying notion that the knowledge that supports the work of teaching is complex.

Subject-specific CK and PCK are supported by general pedagogical knowledge (PK), which Baumert and Kunter (2013) describe as general knowledge about theories and methods of instruction, student learning, classroom management, and assessment. Each of these researchers accepts that PK is a type of knowledge unto itself and one of the core components of professional knowledge. Only Baumert and Kunter (2013) who broaden PK to include the psychological as well as the general pedagogical aspects of knowledge (PPK) sub-divide PPK into (1) classroom processes (i.e., management, teaching methods, and assessment) and (2) student heterogeneity (i.e., student learning processes and individual student characteristics). All three researchers accept that mathematics teachers draw upon all these forms of knowledge to guide and inform their practice. However, specific to teacher practice, Baumert and Kunter (2013) expressly differentiate two aspects of the core dimensions of professional knowledge (CK, PCK, and PPK), namely a formal aspect (theoretical knowledge) and a practical aspect (knowledge in action). Thus, as consistent with these researchers, but more explicitly with Baumert and Kunter, we hypothesize that the *practical* pedagogical knowledge of assessment (PK/PPK) that teachers use to pose questions that uncover their students' mathematical thinking is mediated by both *practical* CK (knowledge in action of mathematics content to be taught to students) and *practical* PCK (knowledge in action of pedagogical content related to student mathematical thinking and strategies). Next, we discuss the ways in which researchers have examined teacher questioning.

Teacher questioning

Research has shown that teachers have difficulty posing questions through which they can gain access to students' mathematical thinking (Buschman 2001; Mewborn and Huberty 1999). However, to date, the field lacks comprehensive explanations about the possible relationship between questioning and the different forms of teacher knowledge. The desire to understand why effective questioning is so difficult has largely focused on the *general pedagogical knowledge* (PK) aspects of questioning such as the types and purposes of teacher questioning (Sahin and Kulm 2008; Franke et al. 2009; Kazemi and Stipek 2001), or the impact teacher questioning has on student thinking, learning, and achievement (Harrop and Swinson 2003; Kazemi and Stipek 2001; Winne 1979). For instance, Sahin and Kulm (2008), Franke et al. (2009), and Kazemi and Stipek (2001) each broadly categorized the types and purposes of questions teachers pose. Although they use different terminology, they all describe how teachers pose questions to (a) prompt students for a predetermined answer about a specific mathematical fact or definition, (b) guide students toward a solution by providing clues that help students take the next steps in finding a solution to a problem, and (c) probe students to investigate their thinking or help them clarify it.

Findings from this body of research include the following: (1) teachers predominantly formulated and posed questions that called for predetermined factual answers and rarely asked the more desirable probing questions through which they could investigate student thinking (Sahin and Kulm 2008); (2) probing questions assisted students even more than guiding questions in formulating complete and coherent explanations (Franke et al. 2009); and (3) teachers who pose questions that press students to explain their mathematical thinking enhance their students' ability to communicate, reason, and justify (Kazemi and Stipek 2001). From the extant research, it is reasonable to hypothesize that teachers need specialized knowledge of mathematics content to engage students in meaningful thinking about mathematical concepts through questioning. However, none of these studies investigated the role that teachers' knowledge of mathematics content (CK) plays in enabling good questioning ability (PCK).

Few studies exist describing prospective teachers' questioning abilities or the strategies teacher educators could use to help prospective teachers learn to question students effectively (Moyer and Milewicz 2002; Nicol 1999). McDonough et al. (2002) hypothesized that prospective teachers may benefit from conducting one-on-one diagnostic interviews of students, provided the prospective teachers subsequently use them to analyze and reflect on their questioning abilities. Furthermore, McDonnough and colleagues proposed using these types of activities to study prospective teachers' questioning abilities. Moyer and Milewicz (2002) situated their study in the context of those types of interviews, revealing that prospective teachers rarely asked more effective probing questions to investigate student thinking and strategies. Instead, their prospective teachers either posed check listing questions, asking one factual question after another without regard for student thinking, or instructing questions, giving clues or hints to guide students toward an answer. These findings appear to support Nicol's (1999) observations that prospective teachers have difficulty understanding what questions to ask students as well as a limited understanding of the significance and purpose of questioning. Similar to studies with practicing teachers, the research on prospective teachers' questioning is limited to investigating the relationship between questioning and the more *general aspects of pedagogical knowledge* (PK). Limited attention has been placed so far on the domain-specific mathematics content knowledge (CK) needed for effective questioning.

Algebraic thinking

A growing number of mathematics educators, teachers, and policymakers argue that the elementary and middle school mathematics curriculum should emphasize algebraic thinking (e.g., CCSSI 2010; Cuoco et al. 1996; National Mathematics Advisory Panel 2008). Although the term algebraic thinking is conceptualized in multiple ways in the mathematics education literature, all existing perspectives share the common viewpoint that algebra-related concepts and skills should be taught with a focus on reasoning and sense making (i.e., Cuoco et al. 1996; Driscoll 1999, 2001 Kaput and Blanton 2005; Kieran 1996, 2004; Kieran and Chalouh 1993; Swafford and Langrall 2000). To help focus our work with prospective teachers, we selected and used Driscoll's (1999, 2001) framework and descriptions of algebraic thinking. Driscoll interpreted algebraic thinking as thinking about quantitative situations in ways that make the relationship between variables obvious. He identified algebraic thinking as habits of mind: Building Rules to Represent Functions and Abstracting from Computation, both linked through the overarching habit of Doing and Undoing Procedures and Operations. Because a major goal of middle school algebra is to enable students to "…express a relationship [rule] between the two quantities in question and to interpret components of the relationship" (CCSSI 2010, p. 52), and because K-8 students should learn algebraic ideas as a "set of concepts and competencies tied to the representation of quantitative relationships and as a style of thinking for formalizing patterns, functions, and generalizations," (NCTM 2000, p. 223), we intentionally narrowed the scope of our work with prospective teachers to Driscoll's (2001) habit of mind Building Rules to Represent Functions.

Driscoll operationalized the thinking processes that underlie Building Rules to Represent Functions in terms of seven features: Organizing Information, Predicting Patterns, Chunking Information, Describing a Rule, Different Representations, Describing Change, and Justifying a Rule (Table 1). We used his operational description of these features to support our study of the relationship between prospective teachers' own proficiency with functional relationships and the types of questions they pose to investigate middle school students' thinking about functional relationships. In this article, we narrow the meaning of the phrase "algebraic thinking" to connote these seven features, which characterize Building Rules to Represent Functions.

Table 1 Features of the algebraic habit of mind Building Rules to Represent Functions (adapted from Driscoll 2001)

Organizing Information	Thinking characterized by proficiency at Organizing Information in ways useful for uncovering patterns, relationships, and rules that define them
Predicting a Pattern	Thinking characterized by proficiency at discovering and making sense of regularities in a given situation
Chunking the information	Thinking characterized by proficiency at looking for repeating chunks of information that reveal how a pattern works
Describing a Rule	Thinking characterized by proficiency at describing the steps of a procedure or rule explicitly or recursively without specific inputs
Different Representations	Thinking characterized by proficiency at thinking about and trying Different Representations of the problem to uncover different information about the problem
Describing Change	Thinking characterized by proficiency at Describing Change in a process or a relationship explicitly as a functional relationship between variables
Justifying a Rule	Thinking characterized by proficiency at justifying why a rule works for *any* number

Carraher and Schliemann (2007) emphasized that a research agenda related to algebraic thinking should include studies that provide a sound understanding of how to prepare teachers to introduce algebraic thinking in K-8 mathematics classrooms. Research shows that teacher knowledge of mathematics, including algebra, is often dominated by an unwarranted preoccupation with the importance of symbol manipulation. This preoccupation interferes with an appreciation of the trend toward fostering students' algebraic thinking (e.g., Ball 1990; Ma 1999; Mewborn 2003; Van Dooren et al. 2002). Teachers' perspectives on algebraic thinking are often strongly influenced by their own experiences with traditional, symbol-oriented school algebra. To teach algebra-based concepts in ways that foster algebraic thinking in K-8 students, teachers need to know the symbolic procedures of algebra, but more importantly, they must understand the ideas of algebraic thinking that underlie algebraic symbolism and procedures (e.g., analyzing and formalizing patterns, describing change in processes and relationships, generalizing from repeated calculations). Without this understanding, they cannot take advantage of opportunities to foster algebraic thinking in their students.

Research questions

As we have pointed out, mathematics education researchers (e.g., Ball et al. 2008; Baumert and Kunter 2013; Tatto et al. 2008) argue that teachers need both mathematics content knowledge (CK) and pedagogical content knowledge (PCK) to be effective in their work. Baumert et al. (2010) further recommend that studies of teacher knowledge should not only distinguish between CK and PCK but also identify the relationship between them. The research reported in this article responds to this recommendation by examining the relationship between prospective teachers' own facility with algebraic thinking (one aspect of CK) and the types of questions they pose (one aspect of PCK) when investigating middle school students' mathematical thinking that supports Building Rules to Represent Functions. Toward this end, our research questions were as follows:

1. How proficiently can prospective teachers use the seven features of Building Rules to Represent Functions to solve algebra-based tasks?
2. What types of questions do prospective teachers pose when conducting one-on-one diagnostic algebraic thinking interviews to investigate middle school students' algebraic thinking?
3. What is the relationship between prospective teachers' facility with the seven features of Building Rules to Represent Functions and the types of questions they pose to uncover middle school students' mathematical thinking?

Method

Context of the study

Our semester-long study was conducted at a large private university situated in the Midwestern United States in a mathematics content course and in a pedagogy course whose curricula were coordinated. The participants included 18 prospective teachers and 18 middle school students from a nearby school selected as the site for the field experience. The prospective teachers were in their third or fourth year of their 4-year teacher education program, all candidates for a grades 1–8 teaching license. The prospective teachers had all previously completed two mathematics courses for elementary teachers, one of which was also coordinated with a pedagogy course. Drawing on recommendations from the existing research (e.g., Baumert et al. 2010; Philipp et al. 2007), we designed an instructional intervention to support prospective teachers in simultaneously strengthening their mathematics content knowledge (algebraic thinking) and their questioning skills.

Mathematics content course

The mathematics content course was taught in the mathematics department and emphasized the mathematical knowledge needed for teaching grades 6–8 algebra. The instructional focus of the first 6 weeks was on algebraic thinking. The course was designed with two goals in mind: (1) to strengthen the prospective teachers' proficiency in algebraic thinking by explicitly focusing on the features of the habit of mind Building Rules to Represent Functions, and (2) to introduce prospective teachers to the instructional strategies used to engage middle school students in algebraic thinking. In the context of a variety of algebra-based problems, prospective teachers examined multiple solutions and representations of mathematical ideas, and explored connections among various algebraic concepts. They also engaged in analyzing the different features of Building Rules to Represent Functions evident in middle school students' written work and reflected on mathematical situations that have the potential to foster algebraic thinking in students. These activities were designed to prepare the prospective teachers for their work with students during follow-up field experiences.

Pedagogy course

The concurrent pedagogy course, which consisted of 2 weeks of university instruction followed by a 14-week field experience, was taught in the College of Education. The university instruction was designed to prepare the prospective teachers to conduct one-on-one diagnostic interviews with middle school students in the field experience. Course activities assisted the prospective teachers in understanding that the purpose of a one-on-one diagnostic interview is to investigate student thinking. Accordingly, the prospective teachers analyzed an illustrative video-recording and transcript of an algebraic thinking interview to examine the potential of different types of questions for learning about and investigating students' mathematical thinking. They also practiced formulating, recognizing, and classifying various types of questions teachers might ask to gain access to students' mathematical thinking.

Once in the field experience, each prospective teacher prepared, conducted, and transcribed two audio-recorded, problem-based, one-on-one diagnostic algebraic thinking interviews with one selected middle school student. Each interview lasted approximately 45 min. The goal of the interviews was for the prospective teachers to use probing questioning to investigate their student's algebraic thinking while the student solved two algebra-based tasks. In preparation for the interviews, the prospective teachers were asked to select two of the seven algebra-based tasks that the content instructor had selected from among the tasks previously solved by the prospective teachers. Each of the tasks had the potential to elicit all seven features of the habit of mind Building Rules to Represent Functions. To maximize the prospective teachers' chances to conduct interviews that would uncover their students' algebraic thinking, the cooperating teachers selected middle school students who performed at grade level, demonstrated the ability to solve an algebra-based task, and communicated their mathematical thinking and reasoning. In addition, each prospective teacher participated in a 30-min debriefing interview with a trained researcher after completing their first algebraic thinking interview (see "Appendix 1" for Debriefing Interview Questions). Prior to the debriefing interview, each prospective teacher was asked to transcribe their first algebraic thinking interview and find and highlight the questions they posed during the interview. The purpose of the debriefing interview was twofold: (1) to help the prospective teachers reflect on their questioning patterns and the effectiveness of their questions in uncovering student's mathematical thinking, and (2) to help the prospective teachers recognize specific aspects of their questioning practice (types of questions) they need to improve to make their second interview more effective.

Data sources and data collection

Data for this study were collected in the mathematics content course and in the field experience of the pedagogy course. Data collected in the content course consisted of the prospective teachers' written solutions to 125 algebra-based tasks completed for homework, during class, or on performance assessments. Data collected in the field experience of the pedagogy course consisted of transcripts (n = 36) of two one-on-one diagnostic algebraic thinking interviews prospective teachers conducted during week 8 (interview 1) and week 12 (interview 2) of the 16-week semester course.

Data analysis

To establish the reliability and validity of our data analysis processes, the three authors jointly developed scoring rubrics and then independently applied them to code the data. The sets of independent results were then compared, and specific examples were cited to clarify the rubrics and negotiate coding agreement to 100 %.

Rating scale for prospective teachers' task solutions

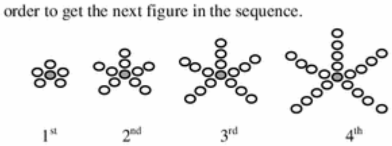
To analyze the prospective teachers' written solutions for each of the 125 tasks, we first identified the specific features of Building Rules to Represent Functions (Table 1) encouraged by each task. Then, we rated each prospective teacher's written task solutions in terms of how well they demonstrated the use of each feature. While each of the 125 tasks had the *potential* to elicit all seven features of Building Rule to Represent Functions, the task statements were not always explicit about all seven features. Accordingly, we only rated the prospective teachers' solutions on the features that were explicitly elicited by the statement of the task. To assess the prospective teachers' algebraic thinking proficiency, we rated how well the prospective teachers used a given feature of Building Rules to Represent Functions as (3) proficient, (2) emerging, or (1) not evident (see "Appendix 2" for rubric).

On a given task, we rated the prospective teachers' use of a specific feature as (3) *proficient* if the written solution revealed thinking *characteristic* of that feature, if the feature was carried out correctly, and if it directly linked to the context of the problem. On a given task, we rated a prospective teacher's use of a specific feature as (2) *emerging* if the written solution articulated thinking *characteristic* of that feature, but the thinking was incorrect or was not directly linked the context of the problem. Finally, on a given task, we rated a prospective teacher's use of a specific feature as (1) *not evident* if the problem explicitly encouraged using that feature, but none of the feature's characteristics appeared in the written solution.

Example analysis of a participant's task solution

Included in Fig. 1 is an example of a task that could support thinking related to all seven features of Building Rules to Represent Functions (although not all seven features are explicitly elicited). We selected Mary's solution (Fig. 2) to illustrate our task analysis process because her solution shows some evidence related to each of the seven features of interest.

Assume that a sequence of circles in the figure below continues by adding one circle to each of the 5 "arms" of a figure in order to get the next figure in the sequence.

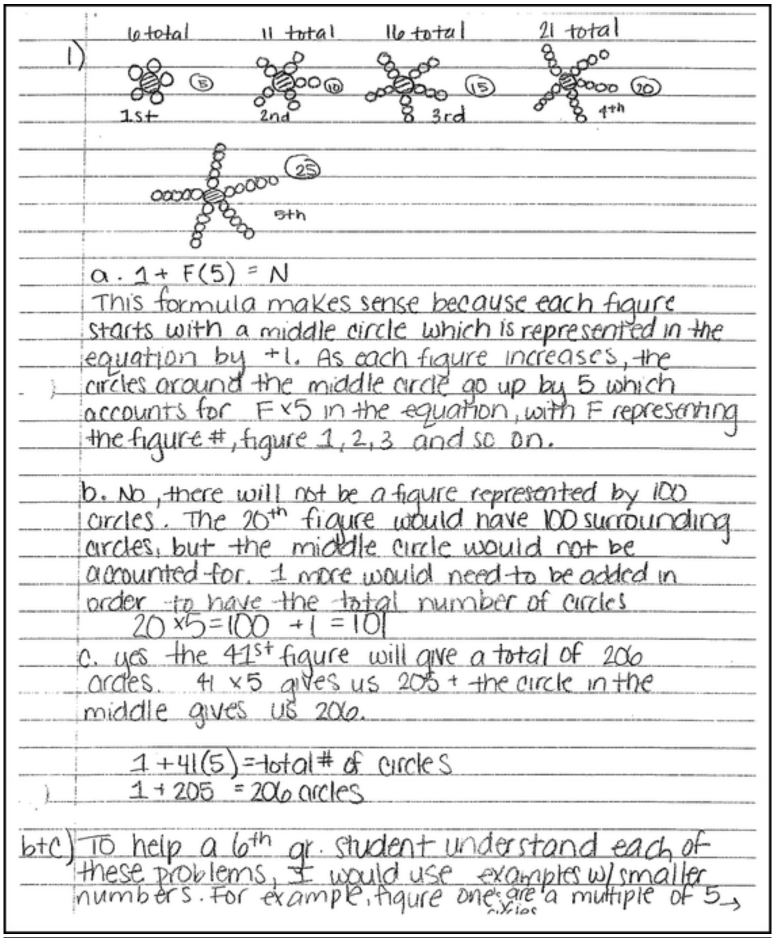


(a) Find a formula for the number of circles making up the Nth figure. Explain why your formula makes sense by relating it to the structure of the figures.

(b) Will there be a figure in the sequence that is made of 100 circles? If yes, which one? If no, why not? Determine the answer to these questions algebraically and in a way that a student in elementary school might be able to do.

(c) Will there be a figure in the sequence that is made of 206 circles? If yes, which one? If no, why not?

Fig. 1 Sequence of circles task (from Beckmann 2007, p. 739)



1) 6 total 11 total 16 total 21 total

1st 2nd 3rd 4th 5th

a. 1 + F(5) = N
This formula makes sense because each figure starts with a middle circle which is represented in the equation by +1. As each figure increases, the circles around the middle circle go up by 5 which accounts for F×5 in the equation, with F representing the figure #, figure 1, 2, 3 and so on.

b. No, there will not be a figure represented by 100 circles. The 20th figure would have 100 surrounding circles, but the middle circle would not be accounted for. 1 more would need to be added in order to have the total number of circles
20 ×5 = 100 + 1 = 101

c. yes the 41st figure will give a total of 206 circles. 41 × 5 gives us 205 + the circle in the middle gives us 206.

1 + 41(5) = total # of circles
1 + 205 = 206 circles

b+c) To help a 6th gr. student understand each of these problems, I would use examples w/ smaller numbers. For example, figure one's circles are a multiple of 5

Fig. 2 Mary’s solution to the sequence of circles task

Organizing Information. We assessed Mary’s use of the Organizing Information feature in the Sequence of Circles task as (3) proficient. Part (a) of Mary’s solution documents that she used a sequence of figures to organize the information found in the problem. She labeled her drawing of the pattern with the information needed to derive a formula for the number of circles in any figure: the figure number, the number of circles that surround each center, and the total number of circles. Mary directly linked the circle in the center of the figure, as well as the number of circles surrounding it, to the rule she derived. Her way of organizing the information supported her discovery of the problem’s regularity as evident in her rule.

Predicting a Pattern. We assessed Mary’s use of the Predicting a Pattern feature in this problem as (3) proficient. She demonstrated proficiency in Predicting a Pattern in part (a), and she showed her knowledge of how the pattern works by determining whether the pattern would generate figures made of 100 circles in part (b) or 206 circles in part (c). Despite a concern about her use of the equal sign, in part (b) (20 × 5 = 100 + 1 = 101), her accompanying written explanation supplied evidence of her proficiency in predicting how the observed regularity works:

> The 20th figure would have 100 surrounding circles, but the middle circle would not be accounted for. [One] 1 more would need to be added in order to have the total number of circles

Chunking Information. We assessed Mary’s use of the Chunking Information feature in this problem as (3) proficient. Her written explanation in part (a) provides evidence that she searched for and examined repeating chunks of information that revealed how the pattern works. She clearly identified that each successive figure needs a chunk of five additional circles as evidenced by her labeling of the diagram.

Describing a Rule. In her description of the rule for finding the number of circles in the Nth figure, $1 + F(5) = N$, Mary explained that 1 represents the middle circle in each figure, while the number of circles needed to form the five arms are represented by $F \times 5$. Given that Mary contextualized each part of her equation without using specific inputs, we scored her use of the Describing a Rule feature in this problem as (3) proficient.

Different Representations. Mary's written solution shows that she considered four Different Representations: figures, words, formulas, and numbers. In her labeling of each figure, she linked the figure number to its corresponding total number of circles and to the number of circles needed to create the arms. In part (a), she described the pattern with words and with an algebraic formula. Given Mary's flexibility in representing the problem situation, we assessed her use of Different Representations as (3) proficient.

Describing Change. Since the statement of the task does not explicitly elicit a description of the change in the number of circles that resulted from changing one figure to the next, we did not expect Mary to explicitly describe this change. However, had the task explicitly asked for a description of this change, we would have assessed Mary's description of change as (2) emerging, because she did not explicitly describe that the change of +1 in the figure number is associated with the change of +5 in the total number of circles in the figure. Although the labels of her diagram suggest that she was thinking about the functional relationship between the figure number and the total number of circles included in each diagram, her written description does not explicitly identify the change in the *total* number of circles that accompanies a change from one figure to the next:

> This formula makes sense because each figure starts with a middle circle which is represented in the equation by +1. As each figure increases, the circles around the middle of the circle go up by 5 which accounts for $F \times 5$ in the equation.

Justifying a Rule. Mary's written description in part (a) acceptably justifies why her rule, $1 + F(5) = N$, predicts the total number of circles in any figure. She linked each term of the rule to the sequence of figures: the +1 to the middle circle and the $F \times 5$ to the 5 arms in each figure. Thus, with respect to substantiating the rule in the context of the problem, we assessed Mary's justification as (3) proficient.

To provide a measure of the prospective teachers' algebraic thinking proficiency with respect to each feature, we computed the mean scores on each feature across the collection of analyzed tasks. This yielded seven feature scores (AT-feature scores) for each prospective teacher. We then computed the average of the seven AT-feature scores to provide a measure of each prospective teacher's algebraic thinking proficiency (AT-composite score).

Analysis of algebraic thinking interview transcripts

The qualitative analysis of the interview transcripts was conducted in two steps. First, each transcript was divided into questioning segments. Consistent with Franke et al. (2009), we defined a questioning segment as a portion of the transcript that begins with the interviewer asking a question and ends when the interviewer moves on to explore a different mathematical issue related to the original question, or poses a new question that addresses a different mathematical task. Second, drawing on Moyer and Milewicz' (2002) framework for analyzing prospective teachers' questioning, we analyzed questions identified within each segment to categorize the overall intent of the segment as check listing, instructing, or probing. We illustrate our analysis with excerpts from Beth, Eric, and Leslie's interviews, conducted using the letter V task (Fig. 3).

1. If the pattern continues, how many yellow blocks will be contained in the next letter V?
2. How many blocks would be in the 15th figure in the sequence? How did you figure out your answer?
3. How could you figure out the number of blocks in any letter V in this pattern?
4. Can you a build a letter V that follows the pattern and uses 36 blocks?
5. Would any of the letter V's in this pattern have an even number of blocks? Why or why not?

Fig. 3 Letter V task (Educational development Center 2008)

Check listing. Consistent with Moyer and Milewicz (2002), a questioning segment was categorized as Check listing if, within that segment, the prospective teacher asked one question after another without any attempt to instruct a student or probe the thinking behind a given response. The following transcript from Beth's (B) interview of a middle school student (S) illustrates a check listing segment:

1. B: Read the next problem
2. S: Can you build a letter V that uses 36 blocks and follows the pattern? [Task prompt]
3. B: Do you see a pattern?
4. S: They're odd
5. B: What does that tell you?
6. S: That you can't
7. B: Okay, write that down. You're right about that. Let's do the last one

Beth did not ask questions to investigate the student's thinking. For instance, after asking, "Do you see a pattern?" and being told, "They're odd," she did not probe the student about the thinking related to the pattern, but instead posed a third question: "What does that tell you?" After affirming the student's response, Beth moved on to the next problem without gaining access to the thinking that led to the student's conclusion that the letter V cannot be made with 36 blocks.

Instructing. Using Moyer and Milewicz' (2002) framework, we categorized a questioning segment as Instructing, if within that segment, the prospective teacher predominantly asked questions with a goal of guiding the student toward the answer. Eric's excerpt below illustrates this type of questioning:

1. E: Can you build a letter V that follows the same pattern and uses 36 blocks? [Task prompt]. It means that you use 36 blocks. So the answer would be 36. Would that work for this pattern?
2. S: I don't get it
3. E: Can you build a figure that would have 36 of them in the V?
4. S: It's going by two, 36
5. E: So could there be 36?
6. S: 19, I mean 18
7. E: We're trying to say yes or no if there can be a figure made of 36. Look at what we have. We have one, three, five, seven. So do you think the answer can be 36?
8. S: No because you take off one

Eric posed the initial question three different times (lines 1, 3, 5) to bring the student's attention to the fact that the V pattern generates an odd number sequence and then suggested the answer: "We're just trying to say yes or no if there can be a figure made of 36. Look at what we have here. We have one, three, five, seven, so do you think the answer can be 36?" (line 7).

Continuing with the interview Eric asked:

9. E: Good. What about 36?
10. S: You start with one
11. E: You take off one because you start off with one. But then, is there something about the number 36, that you know it won't work? What?
12. S: It's going by two
13. E: It's going by two. So the two's are what? Are they odd or even?
14. S: Odd. Two is even and then one, three, five, are odd
15. E: And look at how this is going
16. S: Yeah, odd
17. E: So you know from 36 since it's even it won't fit, right?
18. S: Yeah, I was trying to say that
19. E: Alright, so you got that. So do you understand that?
20. S: Yeah because 36 is going by the pattern
21. E: By the figure times two and then what do you do?
22. S: Subtract one
23. E: So 36 won't work because?
24. S: It's even
25. E: It's even and you'd have to subtract one. Okay

In this segment, the majority of Eric's questions (line 11, 13, 17, 21, and 23) provided hints for the student. The intent of Eric's questions was to help the student solve the task rather than to investigate the student's thinking. The question in line 17 even provides the expected answer for the student to accept.

Probing. Following Moyer and Milewicz's (2002) framework, we categorized questioning segments as probing if the questions within the segment investigated the thinking that led to the student's response or encouraged the student to further clarify their response. By posing probing questions, the prospective teachers demonstrated that they had listened to the student's responses to determine whether they were correct, incorrect, complete, or incomplete, and that they wanted to further explore the student's thinking or encourage the student to clarify their response. Leslie's (L) transcript segment below illustrates how she used probing questions to investigate her student's (S) thinking:

1. L: Can you find the letter V that follows the pattern and uses 36 blocks? [Task prompt]
2. S: No
3. L: Why not?
4. S: Because all of these numbers are odd numbers and 36 is an even number
5. L: Why are they all odd numbers?
6. S: Because odd numbers are not divisible by 2
7. L: But looking at the pattern, why can't it ever be an even number?
8. S: Because if you start out with an odd number, or if you just keep adding, or an odd plus an even equals an odd and you're adding an even number to the odd number
9. L: Good job. Read the next question

Leslie followed up on her student's answers (lines 2, 6), formulating questions (lines 3, 5, and 7) that would help her to understand how the student arrived at a conclusion. Not satisfied with the response, "Because odd numbers are not divisible by 2" (line 6), Leslie pressed the student further for an explanation that would allow her to understand the links her student made between the answer and the context of the problem (line 7).

Analysis of questioning patterns

We examined the patterns of questioning segments across all transcripts by analyzing the proportions of check listing, instructing, and probing segments. To examine the relationship between prospective teachers' algebraic thinking proficiency and the types of questions they posed to investigate students' algebraic thinking, we clustered the prospective teachers into two groups: those with the highest overall AT-composite scores (top 27 %) and those with the lowest overall AT-composite scores (bottom 33 %) and examined questioning patterns of the prospective teachers in each group. We used a Mann–Whitney *U* test to determine whether a significant difference existed between the means of the AT-composite scores for the two groups. We then further examined the transcripts of prospective teachers in the high and low group by analyzing the differences in the proportion of check listing, instructing, and probing segments across their first and second interview. In the section that follows, we present the results of our analysis by research question.

Results

Research question #1: algebraic thinking proficiency

Table 2 presents the means and standard deviations of the AT-composite and AT-feature scores for all prospective teachers and for the high and low groups. The overall AT-composite score across all prospective teachers was M¯M¯ = 2.46 (max. 3), SD = 0.34. The means of the AT-composite scores in the high (n = 5) and low (n = 6) groups were M¯M¯ = 2.74 (max 3) with SD = 0.11 and M¯M¯ = 2.23 (max 3) with SD = 0.16, respectively. A Mann–Whitney *U* test, conducted by rank-ordering the AT-composite scores and comparing the ranks within both groups, confirmed a significant difference between the means of the AT-composite scores for the two groups, U = 0.000, $p < 0.01$, with the sum of the ranks equal to 45 for the high group and 21 for the low group.

Table 2 Prospective teachers' AT-feature and AT-composite mean scores

	All (n = 18)	High (n = 5)	Low (n = 6)
AT-composite score	M¯M¯ = 2.46	M¯M¯ = 2.74	M¯M¯ = 2.23
	SD = 0.34	SD = 0.11	SD = 0.16
Organizing Information	M¯M¯ = 2.56	M¯M¯ = 2.92	M¯M¯ = 2.33
	SD = 0.30	SD = 0.11	SD = 0.22
Predicting Patterns	M¯M¯ = 2.54	M¯M¯ = 2.92	M¯M¯ = 2.27
	SD = 0.29	SD = 0.07	SD = 0.16
Chunking Information	M¯M¯ = 2.39	M¯M¯ = 2.77	M¯M¯ = 2.00
	SD = 0.39	SD = 0.14	SD = 0.28
Different Representations	M¯M¯ = 2.50	M¯M¯ = 2.81	M¯M¯ = 2.08
	SD = 0.42	SD = 0.13	SD = 0.35
Describing a Rule	M¯M¯ = 2.58	M¯M¯ = 2.79	M¯M¯ = 2.42
	SD = 0.22	SD = 0.15	SD = 0.15
Describing Change	M¯M¯ = 2.46	M¯M¯ = 2.50	M¯M¯ = 2.24
	SD = 0.31	SD = 0.23	SD = 0.43
Justifying a Rule	M¯M¯ = 2.20	M¯M¯ = 2.39	M¯M¯ = 1.93
	SD = 0.41	SD = 0.48	SD = 0.43

As illustrated in Table 2, the highest AT-feature mean score for all prospective teachers was for Describing a Rule (M¯M¯ = 2.58) and the lowest was for Justifying a Rule (M¯M¯ = 2.20). When comparing prospective teachers in the high and low groups, the prospective teachers in the high group demonstrated higher algebraic thinking proficiency on *all* of the seven features. The low group's greatest mean was for Describing a Rule (M¯M¯ = 2.42), while their smallest mean was for Justifying a Rule (M¯M¯ = 1.93). The high group's smallest mean was likewise for Justifying a Rule (M¯M¯ = 2.39), but their greatest means were for Organizing Information (M¯M¯ = 2.92) and Predicting a Pattern (M¯M¯ = 2.92). The mean of the low group for Describing a Rule (M¯M¯ = 2.42) was higher than the mean of the high group for Justifying a Rule (M¯M¯ = 2.39).

Research question #2: questioning

Table 3 summarizes the questioning segments identified in each interview for our 18 prospective teachers as a whole, and for the high and low algebraic thinking groups. Across all transcripts, we identified and analyzed 236 questioning segments. Disaggregating all questioning segments by type revealed drastic differences among the proportions: 49 % check listing, 33 % instructing, and 18 % probing. A *z* test for the difference in proportions revealed that as a whole, the transcripts of the prospective teachers included significantly fewer probing questioning segments than the less desirable check listing ($z = 7.53$; $p < 0.05$) and instructing segments ($z = 3.78$; $p < 0.05$).

Table 3 Prospective teachers' questioning segments interviews #1 and #2

AT groups	Check listing segments interview #, *n* (%)			Instructing segments interview #, *n* (%)			Probing segments interview #, *n (%)*			Total segments interview #, *n* (%)
	#1	#2	Total	#1	#2	Total	#1	#2	Total	Total
All PSTs (*n* = 18)	57 (49 %)	59 (51 %)	116 (49 %)	30 (38 %)	48 (62 %)	78 (33 %)	21 (50 %)	21 (50 %)	42 (18 %)	236 (100 %)
High (*n* = 5)	10 (43 %)	13 (57 %)	23 (33 %)	7 (35 %)	13 (65 %)	20 (29 %)	11 (42 %)	15 (58 %)	26 (38 %)	69 (100 %)
Low (*n* = 6)	25 (60 %)	17 (40 %)	42 (62 %)	10 (38 %)	16 (62 %)	26 (38 %)	0 (0 %)	0 (0 %)	0 (0 %)	68 (100 %)

Comparing questioning segments from interview 1 to segments from interview 2 for *all* prospective teachers revealed a lack of statistically significant differences between the proportions of check listing and probing questioning segments while the proportion of instructing questioning segments in the transcripts of interview 2 increased ($z = 3.78$; $p < 0.05$). There were no statistically significant differences in the proportions of probing questions asked by the prospective teachers in their two interviews. This result is surprising because each prospective teacher was debriefed for 30 min by a trained researcher after their first student interview. The purpose of the debriefing was to assist the prospective teachers to analyze and reflect on their questioning during their first interviews with an eye toward improving it during their second interviews.

Our prospective teachers' limited use of probing questions reinforces the finding of other researchers (e.g., Moyer and Milewicz 2002; Nicol 1999), namely that prospective teachers generally have an overall difficulty in asking probing questions, even after being specifically coached on how to do so. Our goal was to examine the possible explanations for this difficulty by exploring the relationship between prospective teachers' algebraic thinking proficiency (CK) and the types of questions they pose to investigate student's algebraic thinking (PCK).

Research question #3: The relationship between algebraic thinking and questioning

Comparing the transcripts of prospective teachers in the high and low algebraic thinking groups revealed approximately the same number of questioning segments (69 high; 68 low) in both groups (Table 3). In the transcripts of prospective teachers in the high group, we found no statistically significant differences between the proportions of check listing, instructing, or probing segments. In contrast, the transcripts of prospective teachers in the low group included a significantly greater number of check listing than instructing ($z = 2.86$; $p < 0.05$), instructing than probing ($z = 6.40$; $p < 0.05$), and check listing than probing segments

($z = 10.45$; $p < 0.05$). This pattern alone serves to document the limited ability of prospective teachers in the low group to pose questions to uncover student thinking (i.e., probing questions).

There was no statistically significant difference between the proportion of instructing segments in the transcripts of the two groups. By way of contrast, however, the transcripts of the prospective teachers in the high group included significantly fewer check listing segments than those of the prospective teachers in the low group: 33 versus 62 % ($z = 3.53$; $p < 0.05$). Moreover, the transcripts of prospective teachers in the low group did not contain any probing segments, but 38 % of the high group's questioning segments were probing, and this difference was statistically significant ($z = 6.46$; $p < 0.05$). This result further indicates that prospective teachers' algebraic thinking proficiency is related to the questions they pose while conducting one-on-one diagnostic interviews to elicit students' algebraic thinking.

Zooming in

We wanted to provide further insights into the relationship between prospective teachers' algebraic thinking proficiency and the questions they pose during one-on-one diagnostic interviews. To do so, we probed more deeply into the questioning differences between the high and low groups on *specific* features of algebraic thinking exhibited in their interviews of students working on a single common task.

Zooming in on one task

As explained in the methods section, the prospective teachers were asked to select two of seven suggested tasks for their algebraic thinking interviews. Because almost all of the prospective teachers in the high and low groups used the letter V task (Fig. 3) for their first algebraic thinking interview (10 out of 11), we use the transcripts from this task to point out the interesting differences we found in the questioning segments of the two groups.

Table 4 shows that the interview transcripts of the prospective teachers in the high group who used the letter V task included significantly fewer check listing segments compared to the transcripts of the prospective teachers in the low group: 35 % compared with 68 % ($z = 2.50$, $p < 0.05$). While the transcripts of the low group did not include any probing segments, 39 % of the high group did ($z = 4.15$, $p < 0.05$). There was no statistically significant difference in the proportion of instructing segments identified in the transcripts of the two groups.

Table 4 Characteristics of high and low AT groups' questioning segments

Prospective teachers (PSTs)	Check listing segments *n* (%)	Instructing segments *n* (%)	Probing segments *n* (%)	Total segments *n* (%)
High AT group				
PST #H1	2 (33 %)	1 (17 %)	3 (50 %)	6 (100 %)
PST #H2	1 (17 %)	1 (17 %)	4 (66 %)	6 (100 %)
PST #H3	3 (60 %)	1 (20 %)	1 (20 %)	5 (100 %)
PST #H4	1 (20 %)	3 (60 %)	1 (20 %)	5 (100 %)
PST #H5	3 (50 %)	1 (17 %)	2 (33 %)	6 (100 %)
Aggregated high	10 (35 %)	7 (26 %)	11 (39 %)	28 (100 %)
Low AT group				
PST #L1	3 (75 %)	1 (25 %)	0 (0 %)	4 (100 %)
PST #L3[a]	3 (60 %)	2 (40 %)	0 (0 %)	5 (100 %)
PST #L4	3 (60 %)	2 (40 %)	0 (0 %)	5 (100 %)
PST #L5	5 (83 %)	1 (17 %)	0 (0 %)	6 (100 %)
PST #L6	3 (60 %)	2 (40 %)	0 (0 %)	5 (100 %)
Aggregated low	17 (68 %)	8 (32 %)	0 (0 %)	25 (100 %)

[a]Only prospective teachers who selected the letter V task for their first interview are included; PST #L2 is missing

Zooming in on specific features

We further examined the high and low groups' questioning on the letter V task, focusing on Predicting a Pattern and Justifying a Rule. We decided to highlight the prospective teachers' questioning patterns on these two features because the prospective teachers' stronger ability to predict a pattern differed sharply from their weaker ability to justify a rule (Table 2). We use transcript segments of PSTs #H1, #H4, #H5, and #L3 to illustrate our observations.

Predicting Patterns. Consider the following transcript segment from PST #H4's interview, which illustrates how PST #H4's own ability to predict a pattern ($\bar{M}\bar{M}$ = 2.80) relates to her questioning ability. Despite her student's (S) relatively quick indication that she knew how to predict the pattern, PST #H4 used probing questions to further investigate her student's thinking.

1. S: How many blocks would be in the fifteenth figure in the sequence? How did you figure out your answer? [Task prompt]
2. PST #H4: Can you tell me what you think the question is asking you?
3. S: To figure out how many blocks will be in the 15th V
4. PST #H4: What are you thinking about this one?
5. S: How many blocks?
6. PST #H4: How many blocks, right. So how might you go about answering that problem?
7. S: I would add two more fifteen times until I finished the pattern
8. PST #H4: Why do you think that would be helpful to figure out the 15th?
9. S: Basically because it continues the pattern
10. PST #H4: Why don't you go ahead and show that for me. (Student begins a chart on her article with each figure number and the total number of blocks in each figure.)
11. PST #H4: How many are in your 5th figure?
12. S: Nine
13. PST #H4: What about your 6th figure?
14. S: Eleven
15. PST #H4: What do you notice about the numbers you're saying?
16. S: You're adding two each time
17. PST #H4: And what is the 15th figure? How many will that one have?
18. S: Twenty-nine
19. PST #H4: Now was there any trick you used to get up to the 15th figure?
20. S: Uh-huh
21. PST #H4: What was it?
22. S: Adding two each time
23. PST #H4: Adding two, absolutely. Do you think you can use this pattern to find the number of boxes in any figure number? Go ahead and look down at number 3 for me. Can you read that out loud?
24. S: How could you figure out the number of blocks in any letter V in this pattern? [Task prompt]

When PST #H4's student explained that she added two more fifteen times until she finished the pattern (line 7), PST #H4 did not accept this response as an indication that her student could predict the pattern. Instead, PST #H4 probed the thinking behind her student's response, first asking her student why adding two would be helpful (line 8) and then asking her student to show how the pattern continued (line 10). As the student created a chart to record her work, PST #H4, asked about the number of yellow blocks in the 5th and 6th figures (lines 11 and 13), and then further probed her student's thinking about the pattern, asking the student what she noticed about the numbers on the chart (line 15). PST #H4 followed up on her student's successful response that the 15th figure would need 29 blocks, requested further explanation about the thinking behind this result (lines 19 and 21), and proceeded to ask her student to use the pattern to find the number of blocks in any figure (line 23).

The next transcript segment from PST #H5's interview also illustrates how PST #H5's own high ability to Predict a Pattern (M¯M¯ = 2.96) relates to her questioning. Unlike PST #H4's student, PST #H5's student was unsure about how to find the number of blocks in the 15th figure in the sequence.

1. PST #H5: How many blocks would be in the fifteenth figure in the sequence? How did you figure out your answer? [Task prompt]
2. S: Nine times five?
3. PST #H5: Nine times five. And why do you think that?
4. S: Wait, you just add two? I don't know
5. PST #H5: Okay, let's go back to what you just did and see if that helps. What did you do here?
6. S: I drew the fifth one. So the sixth one would be (adds two blocks) nine and so the fifteenth one would be… (adds blocks to the figure and then counts the number of blocks on each side of the figure) twenty-nine
7. PST #H5: Is this what you're saying the fifteenth figure would look like?
8. S: Yeah

The segment documents that PST #H5 consistently attended to her student's thinking. For instance, when the student stated that she multiplied nine times five to find the number of blocks in the 15th figure in the sequence (line 2), PST #H5 asked her student to explain why. When the student responded that she did not know (line 4), PST #H5 posed another question (line 5), prompting her student to describe the strategy she had previously used. PST #H5 went beyond simply accepting the correct answer of 29 blocks and continued to probe the student by asking her to discuss how the regularity in the 5th, 6th, and 15th figures could help her to predict the pattern:

9. PST #H5: How many blocks on this side [of the letter V]?
10. S: Ten plus the nine before
11. PST #H5: Why do you think that you would add ten blocks, why did you do that?
12. S: Because there were nine blocks and then I added an extra ten because it asks how many would be in the fifteenth
13. PST #H5: Could you explain to me why you added ten?
14. S: Because there was only nine here and then there was five on one side, so I added ten more, because it's the fifteenth figure. If you keep going, ten more it's the fifteenth figure and so then I added ten more
15. PST #H5: So how do you find the number of blocks in the fifteenth [letter V] figure?
16. S: So there is fourteen on this side [of the letter V] not counting this one, and Then there is fifteen [on this side of the letter V], and that equals twenty-nine
17. PST #H5: Okay, so are you looking at one side [of the letter V] and there's fourteen on this side and fifteen on the other side and then it equals 29?
18. S: Yeah. Fifteen on one side plus fourteen on the other equals twenty-nine
19. PST #H5: Okay, I see what you did. Can you read number three for me?

PST #H5's persistent questioning prompted the student to explain how she arrived at the correct answer (lines 9, 11, 13, 15). The student's explanation showed that she thought about the solution by looking at the sides separately, but also in tandem. The student explained in line 14 that converting the 5th figure to the 15th figure required adding 10 blocks to the side of the 5th figure that already had 5 blocks (resulting in 15 blocks on one side). PST #H5 paraphrased her student's response, noting that her student considered 14 blocks on one side and 15 on the other side to produce an answer of 29 (line 17).

A sharply different pattern of questioning was evident in the following transcript excerpt from a prospective teacher in the low group (PST #L3), who like PST #H5 encountered incorrect thinking from her student (S) about the pattern. Unlike the prospective teachers in the high group, PST #L3's ability to Predict a Pattern was weak

(M¯M¯ = 2.22). In contrast to PST #H5 and PST #H4 who followed up on their students' responses with probing questions, PST #L3 simply asked one question after another without fully attending to her student's thinking:

1. PST #L3: How could you figure out the number of blocks in the fifteenth figure? How did you figure out your answer? [Task prompt]
2. S: 30
3. PST #L3: How did you get 30?
4. S: I multiplied 15 by 2 to get 30
5. PST #L3: You're really close. You just have to think about it. Remember that it's going up by two
6. S: 29
7. PST #L3: Okay, how did you get that without writing that all out? Because first you said 15 times 2 is 30 but then you said the answer is 29
8. S: I subtracted 30 minus 1 is 29
9. PST #L3: Good

PST #L3 first read the task prompt (line 1) and then asked her student how she produced her answer (line 3). As consistent with a check listing pattern of questioning, each time the student responded to a question, PST #L3 simply accepted her student's response and moved on without further probing the student's thinking.

Justify a rule. Consider the following excerpts to compare the questioning of three prospective teachers from the high group (PST #H1, PST #H4, PST #H5). While each of these prospective teachers' demonstrated overall high algebraic thinking proficiency, PST #H1's mean on the Justify a Rule feature was higher (M¯M¯ = 3.00) than PST #H4's (M¯M¯ = 2.18) and PST #H5's

(M¯M¯ = 1.72). The first segment shows how PST #H1 (highest justification score) posed several questions to ask the student to expound on her correct response to the task, think about how to generate a recursive and explicit rule for the pattern, and justify why her rule works. Each time her student provided a justification (lines 4, 6, and 8), PST #H1 followed up with a question to further investigate what her student understood about the pattern.

1. PST #H1: Would any of the letter V's in the pattern have an even number of blocks? Why or why not? [Task prompt]
2. S: No.
3. PST #H1: Why wouldn't they?
4. S: Because an odd plus an even equals an odd.
5. PST #H1: But looking at the pattern, why can't the figure ever have an even number?
6. S: Because, all of these numbers [the number of blocks in each letter V figure] are odd numbers. If you start out with an odd number and you just keep adding two more each time, it will always equal an odd number.
7. PST #H1: Are you always adding an even number to the odd number?
8. S: Yes, so that's why there will never be one with an even number of blocks. If you start out with an odd number, and odd number plus an even number equals an odd number.
9. PST #H1: Okay and do you always start with an odd number?
10. S: Yes.

PST #H1 did not finish the interview by accepting her student's last answer (line 10), but further probed her student's ability to justify the rule. In the earlier part of the interview, PST #H1's student found the correct number of blocks in the 15th figure by using a recursive rule to add two more blocks to each successive figure. In the excerpt that follows, we show how PST #H1 investigated whether her student was able to "see if there's another way to get the number of blocks in any figure" (line 11), asking her student to determine the number of blocks in the 500th figure by formulating an explicit rule in closed form:

11. PST #H1: Okay, let's see if there's another way to get the number of blocks in any figure. What if you had to find the 500th V? Then what would you do?
12. S: No I don't understand. I don't know
13. PST #H1: You can think about it for a few minutes. What would help?
14. S: I guess I could draw a chart to keep track
15. PST #H1: Okay, do what works. Whatever you want
16. S: I'll label it like this…(writes figure number on one column, number of blocks on the other column)
17. PST #H1: Okay, that works
18. S: I don't know if this has anything to do with it but it keep separating. The range in the two numbers keeps changing like the range between the two numbers is 1, 2, 3, and if it's the 500th figure it would be…
19. PST #H1: How does that help you figure out the pattern?
20. S: It will help that you would know that by 500 it would be 499

After her student constructed a chart to organize the information, PST #H1 further prompted her student to think about how her approach might help find the number of blocks in the 500th figure.

21. PST #H1: So how would you get the 500th figure? How many blocks?
22. S: You would add 500 and 499
23. PST #H1: Why would you add 500 and 499?
24. S: You subtract one every time
25. PST #H1: Minus one from what?
26. S: You would minus 1 from the figure number to get the range. And that would equal adding them together and that would be the number of blocks in the next figure.
27. PST #H1: Okay. You can find the range of the figure and the number of blocks then add them together to get the number of blocks. What is the range between the figures?
28. S: You minus one from the figure number to find out. You subtract 1 from the figure number to get the range and then add the range and the figure number together to get the number of blocks
29. PST #H1: Do you think that will work for every figure in the pattern?
30. S: Yes
31. PST #H1: Why do you think so?
32. S: Because you can always subtract one from the something
33. PST #H1: Do you want to try it for the 15th figure?
34. S: Okay. You minus 1 from the figure to get the range which would be 14 then you add 15 and 14 together so you add the figure number and the range and that equals 29 which is the number of blocks
35. PST #H1: Good job. I didn't even think about that way to do it

PST #H1 asked her student to justify why the addition worked (line 23). In an effort to help her student to further clarify her thinking, PST #H1 questioned how thinking about the "range" (i.e., "difference") between the figure number and the number of blocks (lines 27, 29, 31) helped her student derive a rule. After listening to her student's explanation about why her rule worked, PST #H1 was satisfied enough with her student's generalization and justification to move on and ask her to use her generalization to find the 15th figure (line 33).

The above questioning segment highlights PST #H1's ability to pose questions that not only investigated her student's ability to justify a rule but also to clarify her response. PST #H1 attended to and probed how her student discovered the regularities in the pattern, ultimately using these regularities to describe a recursive and explicit rule. We contrast PST #H1's questioning with PST #H4's, who was also in the high group, but with a lower justification score (M¯M¯ = 2.18) than PST #H1 (M¯M¯ = 3.00). PST #H4's mean for the Justify a Rule feature was lower than any of her means for the other features of Building Rules to Represent Functions. Unlike PST #H1 whose questioning segments are illustrated above, PST #H4 did not ask probing questions to investigate the thinking behind her student's response. Instead, each time her student responded, PST #H4 affirmed her student's answer.

1. PST #H4: Why don't you go ahead and read number 5 [fifth part of the problem] for me?
2. S: Would any of the letter V's in this pattern have an even number of blocks? Why or why not?
3. PST #H4: What do you think?
4. S: No, because if it did it would be an uneven V
5. PST #H4: Okay, if it did it would be an uneven V. I like that a lot. Go ahead and write that down

In the context of the letter V task, a complete justification of whether any of the letter V's in the pattern would have an even number of blocks would include and explanation about the relationship between the number of blocks and the figure number. Despite that her student's justification was not completely clear (line 4), PST #H4 accepted her student's response and moved on.

In a similar fashion to PST #H4, PST #H5 (with the lowest justification score in the high group) also accepted the response her student provided and finished the interview, despite that the response did not provide a complete justification of the rule. Like PST #H4, PST #H5's mean for the Justify a Rule feature (M¯M¯ = 1.72) was lower than any of her means for the other features of Building Rules to Represent Functions.

1. PST #H5: So let's move onto number 5 [fifth part of the problem]. Would any of the letter V's in this pattern have an even number of blocks, why or why not? [Task prompt]
2. S: No it wouldn't it would always have to be odd because if it was even there would be one less on one of the sides
3. PST #H5: Perfect! Okay, so we are done with the worksheet

PST #H5 simply accepted her student's answer and then concluded the interview without posing any follow-up questions to probe the thinking behind the student's response (check listing). Like PST #H4, PST #H5 failed to ask probing questions to further investigate whether her student understood the relationship between the number of blocks and the figure number. Moreover, earlier in the interview (see lines 9–19 of PST #H5's excerpt in the *Predicting Patterns* section), when asked to predict a pattern to find the number of blocks in the 15th figure, PST #H5's student explained that she thought about the V pattern as 14 blocks on one side and 15 on the other. Despite that earlier in the interview, PST #H5's student offered a different way of thinking about the V pattern, PST #H5 praised her student's answer (line 3) and concluded the interview without any further investigation. The student's justification, however, revealed two possible interpretations that warranted clarification. It could be that the student thought that the number of blocks in the V pattern could never be even because she viewed the pattern as having an even number of blocks on both sides with one additional block on the bottom (the vertex of the V). On the other hand, stating that "there would be one less on one of the sides" (line 2) could also mean that she thought the V pattern could never be even because removing one block from the top of either side would result in two sides (one of which included the vertex of the V) with an even number of blocks. Given the ambiguity in the student's response, the goal of PST #H5's diagnostic interview would have been better accomplished by further investigating her thinking.

Discussion and implications

To date, the research related to teacher questioning focuses largely on the general pedagogical knowledge aspects of questioning, documenting the types and purposes of questioning (Sahin and Kulm 2008; Franke et al. 2009), the impact teacher questioning has on student thinking, learning, and achievement (Harrop and Swinson 2003; Kazemi and Stipek 2001; Winne 1979), and the difficulties prospective teachers have posing questions to explore student thinking about mathematics (Moyer and Milewicz 2002; Nicol 1999). While these studies support our understanding of teacher questioning, they fall short in exploring the nuances of teacher questioning ability such as the links between teacher knowledge of mathematics content and the types of questions posed to uncover students' mathematical thinking. Thus, the goal of this study was to help provide an understanding of the relationship between teacher knowledge of mathematics content and questioning. Toward this end, this study explored (1) one aspect of prospective teachers' mathematics content knowledge, namely their algebraic thinking proficiency using the habit of mind Building Rules to Represent Functions; (2) the types of questions prospective teachers pose to investigate students' algebraic thinking during one-on-one diagnostic interviews; and (3) the relationship between the two.

Prospective teachers' mathematics content knowledge

Teachers need a profound understanding of the mathematics they teach (Ball et al. 2008; Kunter et al. 2013; Tatto et al. 2008). To teach algebra-based concepts to K-8 students, they need a profound understanding of the ideas of algebraic thinking that underlie the symbols and processes of algebra (Cuoco et al. 1996; Driscoll 1999, 2001; Kaput and Blanton 2005; Kieran 1996, 2004; Kieran and Chalouh 1993; Swafford and Langrall 2000). The results of our study provide insights about prospective teachers' algebraic thinking ability and their readiness for fostering algebraic thinking in the K-8 students.

While our prospective teachers were overall relatively proficient in using the features of the habit of mind Building Rules to Represent Functions to solve a variety of algebra-based tasks, their proficiency varied widely across the features. Across the seven features of Building Rules to Represent Functions, the prospective teachers' ability to Describe a Rule was the highest while their ability to Justify a Rule lagged behind their ability to use the other features. The prospective teachers' weaker ability to use the Justify a Rule feature is consistent with the findings of Castro (2004) and Morris (2010) who also reported prospective teachers' low ability to justify a rule. Our results, therefore, provide mathematics teacher educators with additional evidence regarding the need to strengthen prospective teachers' ability to justify algebraic rules.

Before prospective teachers can foster algebraic thinking in their future students, they need a strong understanding of each of the characteristics (i.e., features) of effective algebraic thinking. Strengthening prospective teachers' ability to justify a rule, then, is an obvious goal in positioning prospective teachers to understand this feature well enough to support their students' proficiency with this feature. In a larger context, prospective teachers' ability to justify a rule must be strengthened to prepare them to assume the role of the teacher who is able to provide mathematical explanations for their students (Ball et al. 2008; Baumert and Kunter 2013; Tatto et al. 2008). Informally explaining and generalizing why a rule or procedure works in the context of a specific algebra-based task comprises the ability to proficiently use the Justify a Rule feature with less mature students. More generally, however, teachers must also be able to more formally explain (justify) to more mature students why the symbols, rules, and procedures of algebra and algebra-based concepts work for all algebra-based tasks. Thus, strengthening prospective teachers' ability to justify (at varying levels of formalism depending on the grade level) may need to be an explicit focus of the entire teacher education curriculum and not limited to a single course on algebraic thinking. This approach aligns quite well with the Common Core State Standards Document (CCSSI 2010) and the National Council of Teachers of Mathematics (NCTM 2000), which recommend integrating algebra and algebraic thinking throughout the K-8 curriculum. A more comprehensive approach that consistently engages prospective teachers in informally and formally Justifying a Rule would help prospective teachers develop this useful habit of mind as well as their overall ability to explain and justify why and how the rules and procedures of algebra work.

Our examination of prospective teachers' algebraic thinking, one aspect of their mathematics content knowledge (CK), reinforces other research findings about prospective teachers' algebraic thinking proficiency.

Comparing the algebraic thinking proficiency of prospective teachers in the high and low groups revealed that across the seven features of Building Rules to Represent Functions, the high group's greatest means were for Organizing Information and Predicting a Pattern. These greater means indicate that the prospective teachers in the high group are most proficient at identifying, organizing, uncovering, and *making sense of* the regularities found in a problem, an important characteristic of algebraic thinking. In contrast, the low group's greatest mean was for Describing a Rule, a feature that relates to the procedural aspects of symbolic algebra rather than the sense making aspects of algebraic thinking. This result reinforces what others (e.g., Ball 1990; Ma 1999; Mewborn 2003; Van Dooren et al. 2002) reported about prospective teachers' understanding of algebraic topics; specifically, prospective teachers' limited knowledge of algebra is often restricted to facility with algebraic symbols and procedures. We hypothesize, then, that prospective teachers in the high group are more prepared to use algebraic thinking to support student learning of algebra-related concepts. This hypothesis raises concern about the preparedness of prospective teachers in the low group to use algebraic thinking to teach algebra-based concepts to their students. In order to be able to teach algebra-related concepts to K-8 students, teachers need to know the thinking that underlies algebraic rules and procedures. Mathematics teacher educators may want to use algebra-based tasks that explicitly focus on patterns, relationships, and the rules that define them to help prospective teachers develop their capacity for understanding and making sense of the rules and procedures that underlie algebra-based concepts.

Prospective teachers' questioning ability

McDonough et al. (2002) recommended using one-on-one diagnostic interviews of students to help prospective teachers' improve their questioning ability. During such one-on-one diagnostic algebraic thinking interviews with middle school students, the prospective teachers in our study rarely formulated probing questions (18 %) to investigate their students' algebraic thinking. Instead, the prospective teachers predominantly asked check listing questions (49 % of the time), prompting students for an answer, and moving from one question to the next with little attention to their students' algebraic thinking. About 33 % of all questioning segments were instructing, meaning that the prospective teachers formulated questions to guide or teach their students despite the clearly stated goals for the diagnostic interviews. Neither check listing nor instructing questions helped the prospective teachers gain a deep insight into their students' algebraic thinking. In addition, despite debriefing the prospective teachers between their first and second interviews in an attempt to help them improve the effectiveness of their questioning, the proportion of check listing and probing questioning segments remained the same while the proportion of instructing questioning segments increased.

While the overall pattern of prospective teachers' questioning revealed by our analysis was consistent with Moyer and Milewicz (2002) and Nicol (1999), our work provides several new directions to consider while using one-on-one diagnostic interviews as a mean of supporting prospective teachers' in learning how to pose effective questions.

First, we think our debriefing interviews were less effective than they could have been because we asked the prospective teachers to simply find and highlight the questions they posed during their first interview. We did not provide the prospective teachers with specific guidance about how to analyze and reflect on their own questioning patterns. We think that this omission may have been the reason that the debriefing interviews failed to assist the prospective teachers in improving their questioning ability from their first interview to the next. We now ask prospective teachers to transcribe their one-on-one diagnostic interviews and code each of the questions found on their transcripts as check listing, instructing, and probing. Coding questions assists the prospective teachers to take ownership of their own questioning, think about the patterns of questioning found in their transcripts, and evaluate the kinds of questions that are most productive in uncovering student thinking.

Second, we provide more guidance for prospective teachers about how to use their one-on-one diagnostic interviews to develop their knowledge of mathematics content and student thinking and strategies. To do this, we ask prospective teachers to use their transcript to identify a missed opportunity, a time in the interview in which they did not understand the mathematics involved in their students' thinking or strategy but moved on to the next part of the task anyway. We also ask prospective teachers to explain what they do not understand about the mathematics involved in their students' thinking or strategy and direct the prospective teachers to develop a series of questions that might help the student to be more explicit about their thinking or strategy. This kind of guidance assists prospective teachers in concurrently attending to mathematics content (CK) and questioning *about* student mathematical thinking and strategies (PCK) when analyzing and reflecting on a one-on-one diagnostic interview.

The relationship between mathematics content knowledge and questioning

Mathematics content knowledge (CK) is an important prerequisite in establishing pedagogical content knowledge (PCK) (e.g., Ball et al. 2008; Baumert and Kunter 2013; Tatto et al. 2008). Teacher preparation programs should concurrently emphasize both CK and PCK to help prospective teachers develop professional knowledge in support of their practice. Sahin and Kulm (2008) hypothesized that teachers need a deep knowledge of mathematics content to pose questions that engage students in meaningful thinking about mathematical concepts. In our study, we uncovered that the types of questions our prospective teachers posed to examine their students' algebraic thinking (one aspect of PCK) might closely relate to their own algebraic thinking proficiency (one aspect of CK). The prospective teachers in the high algebraic thinking group formulated some probing questions during their interviews (38 %), but the prospective teachers in the low group never asked probing questions in their interviews (0 %). Simply put, it may be that the low group's insufficient proficiency with the various features of Building Rules to Represent Functions embedded in their interview tasks may have discouraged them from posing questions to uncover their students' thinking about those same features. We conjecture that the low group of prospective teachers did not ask probing questions to further investigate their students' thinking because doing so would run the risk that their student might ask a question or suggest a strategy the prospective teacher did not understand. Instead, the prospective teachers in the low group either affirmed their students' responses and moved on or used instructing questions to guide their student to solve the problem using a strategy they understood. Prospective teachers in the high group, on the other hand, were more often able (and willing) to pose probing questions to investigate their students' understanding of the different features of Building Rules to Represent Functions.

A fine-grained analysis of questioning segments from the letter V task provided additional insights about the relationship between prospective teachers' use of probing questions and their algebraic thinking proficiency. We found that prospective teachers in the high group with weaker proficiency in using a given feature of Building Rules to Represent Functions had difficulty posing probing questions about that same feature. For example, the prospective teachers in the high group whose own ability to predict a pattern was high had a good ability to use probing questions to uncover their students' ability to predict a pattern. In contrast, prospective teachers in the high group whose proficiency with Justifying a Rule was weak had a weak ability to use probing questions to investigate their students' understanding of Justifying a Rule. This result provides further support for Baumert et al.'s (2010) findings that content knowledge (CK) is necessary for developing pedagogical content knowledge (PCK), and therefore, insufficient content knowledge (CK) is related to deficits in pedagogical content knowledge (PCK). In our study, we found that without a profound understanding of the Justify a Rule feature (CK), even the high ability prospective teachers were unable to pose the types of questions (PCK) that would assist them in investigating their students' ability to justify a rule. This result can help mathematics teacher educators modify the activities they use to help their prospective teachers become effective questioners. For example, in our integrated mathematics content and pedagogy course, when we discuss the one-on-one diagnostic algebraic thinking interview assignment, we are now more explicit about the connection between the prospective teachers' knowledge of the different features of Building Rules to Represent Functions and their ability to pose probing questions to investigate students' algebraic thinking.

Final remarks

We acknowledge that our study has many limitations. A relatively small number of participants reliance on only written solutions as the measure of prospective teachers' algebraic thinking, the limited type of problems, and the parameters we used to choose the students that were interviewed are some of the reasons to interpret our results with caution. However, we believe that our study provides an important detail about the relationship between one aspect of CK (algebraic thinking proficiency) and one aspect of PCK (questioning about mathematical thinking), which can inform the work of mathematics educators in preparing prospective teachers to effectively teach mathematics. Knowing how to investigate students' mathematical thinking through questioning is an important ability that prospective teachers need to develop. Supporting this ability by focusing on strengthening prospective teachers' own mathematics content knowledge (CK) with their pedagogical content knowledge (PCK) should be one of the goals of teacher education programs. Our study draws added attention to the importance of helping prospective teachers develop both mathematics content and pedagogical content knowledge in a symbiotic way that informs, builds upon, and strengthens one another.

The design of our study does not allow us to do more than hypothesize about the potential directions for teacher education in the area of teacher knowledge and questioning. Our work suggests that a relationship exists between algebraic thinking proficiency and questioning. Additional research is needed to determine the nature and extent of the limitations imposed on prospective teachers' use of probing questions due to a lack of mathematics content knowledge (CK) or pedagogical content knowledge (PCK).

Appendix 1: Debriefing interview questions

1. What questions did you pose for your student during your first interview?
2. Why did you ask these questions?
3. How did the student react to the questions you posed?
4. Thinking about your interview experience, what questions do you wish you had asked the student during the first interview? Why?
5. Would you change the way you questioned your student? Why or why not?
6. Are there any questions you would like to include in your second interview? Why?

Appendix 2: Rubric for assessing prospective teachers' use of Building Rules to Represent Functions

	Not Evident (1)	Emerging (2)	Proficient (3)
Organizing Information	The solution does not indicate that the prospective teacher organized the information in the problem in a way that is useful for discovering the underlying patterns and relationships	The solution indicates that the prospective teacher organized the information in the problem in a way that is useful for discovering the underlying patterns and relationships; BUT, the organizational scheme used is not explicitly connected to the context of the problem (e.g., problem information is organized in a table but table entries are not contextualized, i.e., their meaning	The solution indicates that the prospective teacher organized the information in the problem in a way that is useful for discovering underlying patterns and relationships; AND, the organizational scheme used is explicitly connected to the context of the problem (e.g., uses a table to organize information in the problem and clearly relates

		explained with links to the context of the problem)	table entries to the context of the problem)
Predicting Patterns	The solution does not indicate the prospective teacher's understanding of how the pattern works; OR, the pattern is identified incorrectly	The solution indicates the prospective teacher's understanding of how the pattern works (e.g., terms beyond the perceptual field are identified correctly, explicit or recursive rule that describes the pattern is correct); BUT, the pattern or discovered regularities are not explicitly connected to the context of the problem	The solution indicates the prospective teacher's understanding of how the pattern works (e.g., terms beyond the perceptual field are identified correctly, explicit or recursive rule that describes the pattern is correct); AND, the pattern or discovered regularities are explicitly connected to the context of the problem
Chunking Information	The solution does not indicate that the prospective teacher identified repeated chunks of information that explain how the pattern works, OR repeated chunks of information in the pattern are identified incorrectly	The solution indicates that the prospective teacher identified repeated chunks of information to explain how the pattern works; BUT, the identified repeated chunks are not explicitly connected to the context of the problem	The solution indicates that the prospective teacher identified repeated chunks of information that explain how the pattern works; AND, the identified repeated chunks of information are explicitly connected to the context of the problem
Describe a rule	The solution does not indicate that the prospective teacher identified and described the steps of a rule through which the relationship embedded in the problem can be represented	The solution indicates that the prospective teacher described the rule (verbal or symbolic) to represent the uncovered relationship; BUT the rule is not explicitly connected to the context of the problem	The solution indicates that the prospective teacher described the rule (verbal or symbolic) to represent the uncovered relationship; AND, the rule is explicitly connected to the context of the problem
Different Representations	The solution does not indicate that the prospective teachers used different verbal, numerical, graphical, or algebraic representations to uncover different information about the problem	The solution indicates that the prospective teacher used Different Representations (e.g., verbal, numerical graphical, or algebraic) to uncover and explore information embedded in the problem; BUT, the representations used are not explicitly connected to the context of the problem (e.g., uses a list of numbers without contextualizing their meaning)	The solution indicates that the prospective teacher used Different Representations (e.g., verbal, numerical graphical, or algebraic) to uncover and explore information embedded in the problem; AND, the representations used are explicitly connected to the context of the problem
Describing Change	The solution does not indicate that the prospective teacher considered change in a process or relationship as a function of the relationship between variables in the problem, i.e., change in the input variable with respect to the change in the output variable	The solution indicates that the prospective teacher described the change in a process or relationship as a function of the relationship between variables in the problem, (i.e., change in the input variable with respect to the change in the output variable); BUT the described change is not explicitly connected to the context of the problem	The solution indicates that the prospective teacher described the change in a process or relationship as a function of the relationship between variables in the problem (i.e., change in the input variable with respect to the change in the output variable); AND, the described change is explicitly connected to the context of the problem
Justifying a Rule	The solution does not indicate that the	The solution indicates that the prospective teacher explained	The solution indicates that the prospective teacher explained

	prospective teacher explained why the rule found in the problem works for any number	why the rule found in the problem works for any number. The justification is not explicitly connected to the context of the problem	why the rule found in the problem works for any number. The justification is explicitly connected to the context of the problem

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