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THE TRANSMUTED WEIBULL-PARETO DISTRIBUTION

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ABSTRACT

A new generalization of the Weibull-Pareto distribution called the transmuted Weibull-Pareto distribution is proposed and studied. Various mathematical properties of this distribution including ordinary and incomplete moments, quantile and generating functions, Bonferroni and Lorenz curves and order statistics are derived. The method of maximum likelihood is used for estimating the model parameters. The flexibility of the new lifetime model is illustrated by means of an application to a real data set.

KEY WORDS

Transmuted family, Weibull-Pareto Distribution, Bonferroni and Lorenz curves, Order Statistics, Likelihood Estimation.

1. INTRODUCTION

Recently, several families of distributions have been proposed via extending common families, by adding one or more parameters to the baseline model, of continuous distributions. These new families provide more flexibility in modeling and analyzing real life data in many applied areas. For that reason the statistical literature contains a good number of new families. For example, the generalized transmuted-G family proposed by Nofal et al. (2015), the transmuted exponentiated generalized-G class defined by Yousof et al. (2015), the transmuted geometric-G family introduced by Afify et al. (2016a) and the Kumaraswamy transmuted-G family introduced by Afify et al. (2016b). Another example is the Weibull-G (W-G for short) class defined by Bourguignon et al. (2014). Using the W-G family, Tahir et al. (2015) defined and studied the Weibull-Pareto (WPa) distribution extending the Pareto (Pa) distribution.

The cumulative distribution function (cdf) of the WPa distribution is given (for $x > \theta > 0$) by

$$G(x) = 1 - \exp\left\{-\left[\left(\frac{x}{\theta}\right)^{\alpha} - 1\right]^{b}\right\}, \tag{1.1}$$

where θ is a positive scale parameter and α and b are positive shape parameters.

The corresponding probability density function (pdf) is given by

$$g(x) = \frac{b\alpha}{\theta^{\alpha}} x^{\alpha-1}\left[\left(\frac{x}{\theta}\right)^{\alpha} - 1\right]^{b-1} \exp\left\{-\left[\left(\frac{x}{\theta}\right)^{\alpha} - 1\right]^{b}\right\}. \tag{1.2}$$

In this paper, we define and study a new distribution by adding one extra shape parameter in equation (1.2) to provide more flexibility to the generated model. In fact, based on the *transmuted-G* (TG) family of distributions proposed by Shaw and Buckley (2007), we construct a new model called the *transmuted Weibull-Pareto* (TWPa) distribution and give a comprehensive descriptions of some of its mathematical properties. We hope that the new model will attract wider applications in reliability, engineering and other areas of applications.

Recently, many authors used the TG family to construct new distributions. For example, Afify et al. (2015a) introduced the transmuted Marshall-Olkin Fréchet, Afify et al. (2015b) proposed the transmuted Weibull Lomax, Afify et al. (2014) defined the transmuted complementary Weibull geometric, Khan and King (2013) introduced the transmuted modified Weibull and Aryal and Tsokos (2011) proposed the transmuted Weibull distributions.

For an arbitrary baseline cdf $G(x)$, Shaw and Buckley (2007) defined the TG family with cdf and pdf given by

$$F(x) = G(x)\left[1 + \lambda - \lambda G(x)\right] \tag{1.3}$$

and

$$f(x) = g(x)\left[1 + \lambda - 2\lambda G(x)\right], \tag{1.4}$$

respectively, where $|\lambda| \le 1$. The TG density is a mixture of the baseline density and the exponentiated-G (Exp-G) density with power parameter two. For $\lambda = 0$, (1.3) gives the baseline distribution.

The rest of the paper is outlined as follows. In Section 2, we define the TWPa distribution and provide the graphical presentation of its pdf and hazard rate function (hrf). A useful mixture representation for its pdf and cdf is provided in Section 3. Section 4 provides the mathematical properties including ordinary and incomplete moments, quantile and generating functions, Bonferroni, Lorenz and Zenga curves, moments of residual life and moments of the reversed residual life are derived. In Section 5, the order statistics and their moments are discussed. The probability weighted moments (PWMs) are discussed in Section 6. Certain characterizations are presented in Section 7. The maximum likelihood estimates (MLEs) for the model parameters are demonstrated in Section 8. In Section 9, simulation results to assess the performance of the proposed maximum likelihood estimation method are discussed. The TWPa distribution is applied to a real data set to illustrate its usefulness in Section 10. Finally, some concluding remarks are given in Section 11.

2. THE TWPa DISTRIBUTION

By inserting Equation (1.1) into Equation (1.3), the cdf of the TWPa model is given (for $x > \theta$) by

$$F(x) = \left(1 - \exp\left\{-\left[\left(\frac{x}{\theta}\right)^{\alpha} - 1\right]^{b}\right\}\right)\left(1 + \lambda \exp\left\{-\left[\left(\frac{x}{\theta}\right)^{\alpha} - 1\right]^{b}\right\}\right). \qquad (2.1)$$

The corresponding pdf is obtained by

$$f(x) = \frac{b\alpha}{\theta^{\alpha}} x^{\alpha-1}\left[\left(\frac{x}{\theta}\right)^{\alpha} - 1\right]^{b-1} \exp\left\{-\left[\left(\frac{x}{\theta}\right)^{\alpha} - 1\right]^{b}\right\}\left(1 - \lambda + 2\lambda \exp\left\{-\left[\left(\frac{x}{\theta}\right)^{\alpha} - 1\right]^{b}\right\}\right), \qquad (2.2)$$

where θ is a scale parameter and α, b and λ are positive shape parameters. The random variable X is said to have a TWPa distribution, denoted by $X \sim$ TWPa $(\alpha, \theta, b, \lambda)$, if its cdf is given by Equation (2.1). It is clear that Equation (2.2) reduced to the WPa model for $\lambda = 0$. The TWPa distribution due to its flexibility in accommodating all forms of the hrf as shown from Figure 2 seems to be an important model that can be used.

A physical interpretation of the cdf of TWPa model is possible if we take a system consisting of two independent components functioning independently at a given time. So, if the two components are connected in parallel, the overall system will have the TWPa cdf with $\lambda = -1$.

Another motivation for the TWPa distribution follows by taking two iid random variables, say Z_1 and Z_2, with cdf

$$G(x) = 1 - \exp\left\{-\left[\left(\frac{x}{\theta}\right)^{\alpha} - 1\right]^{b}\right\}.$$

Let $Z_{1:2} = \min(Z_1, Z_2)$ and $Z_{2:2} = \max(Z_1, Z_2)$. Now, consider the random variable X defined by

$$X = \begin{cases} Z_{1:2}, \text{ with probability } \dfrac{1+\lambda}{2}; \\ Z_{2:2}, \text{ with probability } \dfrac{1-\lambda}{2}. \end{cases}$$

Then, the cdf of X is given by (2.1).

Figure 1 provide some plots of the TWPa density curves for different values of the parameters α, θ, λ and b. Plots of the hrf of TWPa for selected parameter values are given in Figure 2. Some possible shapes for the TWPa cdf are displayed in Figure 3.

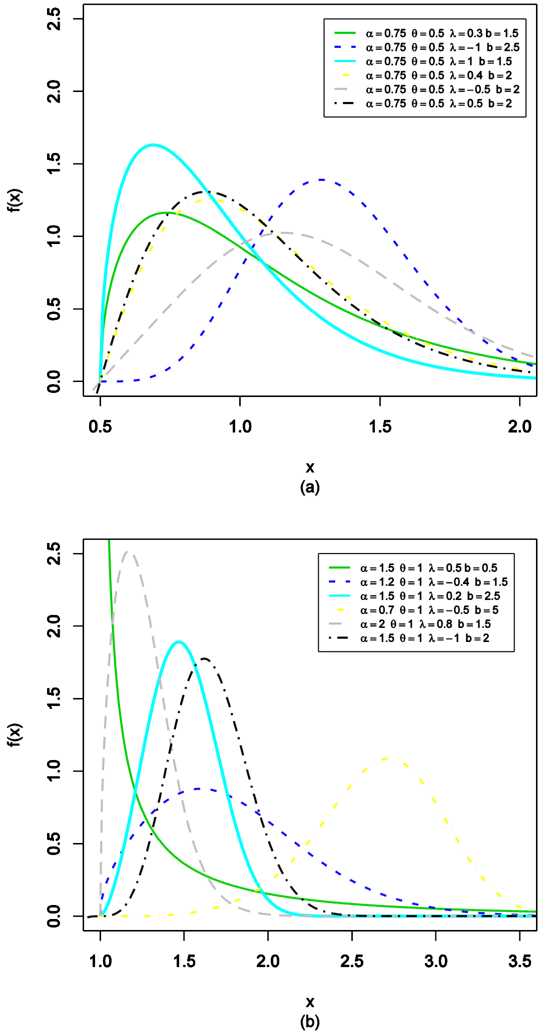

Figure 1: Plots of the TWPa Density Function for some Parameter Values

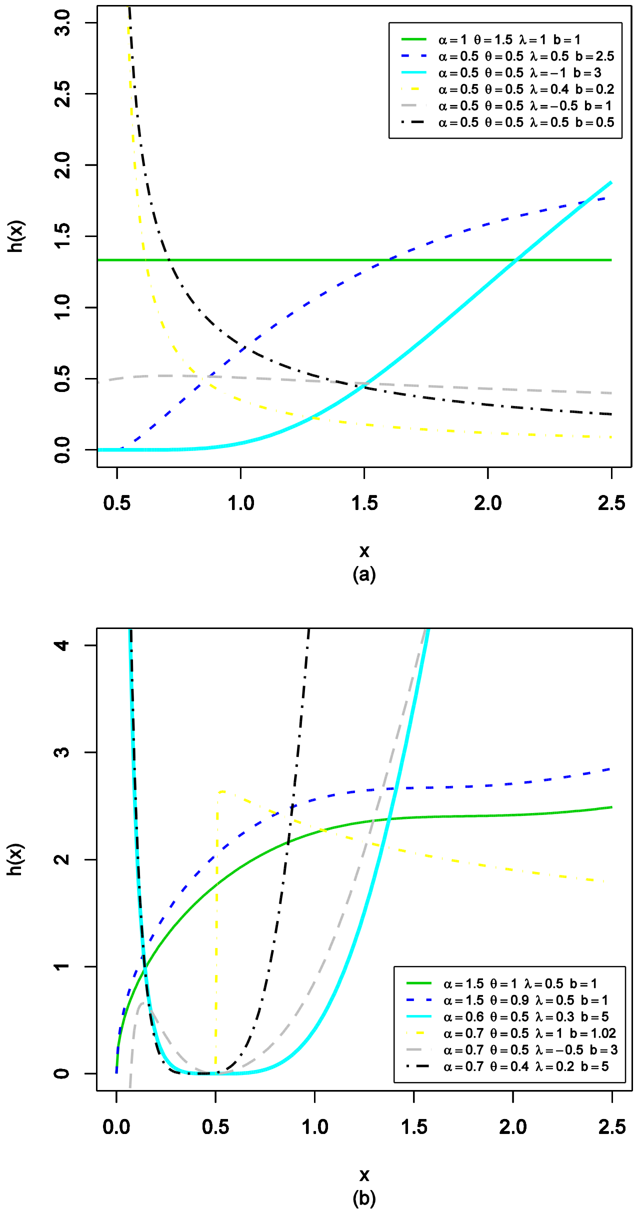

Figure 2: Plots of the hrf of the TWPa for some Parameter Values

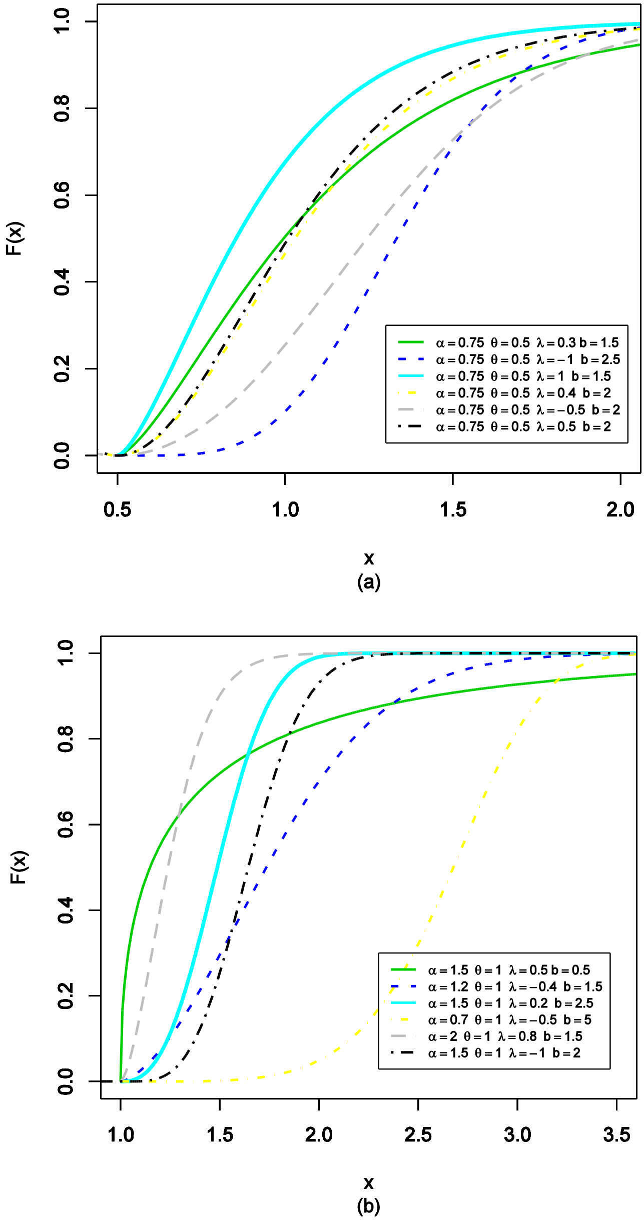

Figure 3: Plots of the cdf of the TWPa for some Parameter Values

3. MIXTURE REPRESENTATION

The TWPa density function given in Equation (2.2) can be expressed as

$$f(x)=(1+\lambda)bg(x)\frac{G(x)^{b-1}}{\overline{G}(x)^{b+1}}\exp\left\{-\left(\frac{G(x)}{\overline{G}(x)}\right)^{b}\right\}$$
$$-2\lambda bg(x)\frac{G(x)^{b-1}}{\overline{G}(x)^{b+1}}\exp\left\{-\left(\frac{G(x)}{\overline{G}(x)}\right)^{b}\right\}+2\lambda bg(x)\frac{G(x)^{b-1}}{\overline{G}(x)^{b+1}}\exp\left\{-2\left(\frac{G(x)}{\overline{G}(x)}\right)^{b}\right\}.$$

Using Equations (1.1) and (1.2) and after a power series, the we have

$$f(x)=(1+\lambda)b\frac{\alpha}{x}\left(\frac{x}{\theta}\right)^{-\alpha}\sum_{k,j=0}^{\infty}\frac{(-1)^{k}\,\Gamma\left((k+1)b+j+1\right)}{k!j!\Gamma\left((k+1)b+1\right)}\left[1-\left(\frac{x}{\theta}\right)^{-\alpha}\right]^{bk+b+j-1}$$
$$-2\lambda b\frac{\alpha}{x}\left(\frac{x}{\theta}\right)^{-\alpha}\sum_{k,j=0}^{\infty}\frac{(-1)^{k}\,\Gamma\left((k+1)b+j+1\right)}{k!j!\Gamma\left((k+1)b+1\right)}\left[1-\left(\frac{x}{\theta}\right)^{-\alpha}\right]^{bk+b+j-1}$$
$$+2\lambda b\frac{\alpha}{x}\left(\frac{x}{\theta}\right)^{-\alpha}\sum_{k,j=0}^{\infty}\frac{(-2)^{k}\,\Gamma\left((k+1)b+j+1\right)}{k!j!\Gamma\left((k+1)b+1\right)}\left[1-\left(\frac{x}{\theta}\right)^{-\alpha}\right]^{bk+b+j-1}. \qquad (3.1)$$

Using the generalized binomial series expansion and after some algebra, the TWPa density can be expressed as

$$f(x)=\sum_{k,j=0}^{\infty}\frac{(-1)^{k}\,b\Gamma\left((k+1)b+j+1\right)\left[1+\lambda-2\lambda+2^{k+1}\lambda\right]}{k!j!\Gamma\left((k+1)b+1\right)\left[(k+1)b+j\right]}$$
$$\times\left[(k+1)b+j\right]\frac{\alpha}{x}\left(\frac{x}{\theta}\right)^{-\alpha}\left[1-\left(\frac{x}{\theta}\right)^{-\alpha}\right]^{(k+1)b+j-1},$$

or equivalently

$$f(x)=\sum_{k,j=0}^{\infty}\upsilon_{k,j}\;h_{\alpha,\theta,(k+1)b+j}(x), \qquad (3.2)$$

where

$$\upsilon_{k,j}=\frac{(-1)^{k}\,b\Gamma\left((k+1)b+j+1\right)\left[1+\lambda\left(2^{k+1}-1\right)\right]}{k!j!\left[(k+1)b+j\right]\Gamma\left((k+1)b+1\right)}.$$

and

$$h_{\alpha,\theta,((k+1)b+j)}(x)=\left[(k+1)b+j\right]\frac{\alpha}{x}\left(\frac{x}{\theta}\right)^{-\alpha}\left[1-\left(\frac{x}{\theta}\right)^{-\alpha}\right]^{(k+1)b+j-1}$$

is the exponentiated Pareto (EPa) density with parameters α,θ and $\left(k+1\right)b+j$. This means that the TWPa density can be expressed as a mixture of EPa densities. So, several of its properties can be derived form those of the EPa model.

Integrating (3.2), the cdf of TWPa can be expressed as

$$F(x) = \sum_{k,j=0}^{\infty} \upsilon_{k,j}\, H_{\alpha,\theta,(k+1)b+j}(x),$$

where $H_{\alpha,\theta,(k+1)b+j}(x)$ is the cdf of EPa with parameters α, θ and $\left(k+1\right)b+j$.

4. PROPERTIES

Established algebraic expansions to determine some structural properties of the TWPa distribution can be more efficient than computing those directly by numerical integration of its density function. The mathematical properties of the TWPa distribution including ordinary and incomplete moments, factorial moments, quantile and generating functions, Bonferroni, Lorenz and Zenga curves, residual life function and reversed residual life function are provided in this section.

4.1 Moments

Using (2.2), the r th moment of X, denoted by μ_r', is given by

$$\mu_r' = \sum_{k,j=0}^{\infty} \upsilon_{k,j} E\left[Y_{\alpha,\theta,(k+1)b+j}^{r}\right],$$

where $E\left[Y_{\alpha,\theta,(k+1)b+j}^{r}\right] = \int_0^{\infty} x^r h_{\alpha,\theta,(k+1)b+j}(x)dx$. Then, we obtain (for $r \le \alpha$)

$$\mu_r' = \sum_{k,j=0}^{\infty} \left[\left(k+1\right)b+j\right]\theta^r \upsilon_{k,j} B\left(1-\frac{r}{\alpha},\left(k+1\right)b+j\right), \tag{4.1}$$

where $B\left(m,n\right) = \int_0^1 t^{m-1}\left(1-t\right)^{n-1} dt$ is the beta function.

The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships.

Corollary 1

Using the relation between the central moments and non-centeral moments, we can obtain the n th *central moment of a TWPa random variable, denoted by* M_n*, as follows*

$$M_n = E\left(X-\mu\right)^n = \sum_{r=0}^{n} \binom{n}{r} \left(-\mu_1'\right)^{n-r} E\left(X^r\right),$$

then,

$$M_n = \sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} \left(\mu_1'\right)^{n-r} \mu_r'$$

and cumulants (κ_n) of X are obtained from (4.1) as

$$\kappa_n = \mu_n' - \sum_{r=0}^{n-1} \binom{n-1}{r-1} (\kappa_r)(\mu_{n-r}'),$$

where $\kappa_1 = \mu_1'$ hence $\kappa_2 = \mu_2' - \mu_1'^2, \kappa_3 = \mu_3' - 3\mu_2'\mu_1' + \mu_1'^3$ etc. The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships.

4.2 Quantile and Generating Functions

The quantile function (qf) of X, where $X \sim$ TWPa $(\alpha, \theta, \lambda, b)$, is obtained by inverting (2.1) to obtain $x_u = F^{-1}(u)$ as

$$x_u = \theta \left\{ 1 + \left[-\log \left(\frac{\lambda - 1 + \sqrt{(1+\lambda)^2 - 4\lambda u}}{2\lambda} \right) \right]^{1/b} \right\}^{1/\alpha}.$$

Simulating the TWPa random variable is straightforward. If U is a uniform variate on the unit interval $(0,1)$, then the random variable $X = x_U$ follows (2.2).

Here, we will provide two formulas for the moment generating function (mgf) of the TWPa distribution. The mgf is defined by $M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E\left(X^r\right)$. Then

$$M_X(t) = \sum_{k,j,r=0}^{\infty} \left[(k+1)b + j \right] \frac{(t\theta)^r}{r!} \upsilon_{k,j} B\left(1 - \frac{r}{\alpha}, (k+1)b + j \right).$$

The second can be computed using Maple. For $t \le 0$, $p > 0$ and $q > 0$ Let $J(q,p,t) = \int_q^{\infty} x^{-p} e^{tx} dx$. Using this software, we can obtain

$$J(q,p,t) = (-1)^p q \left[-\frac{\pi \csc(p\pi)}{tq\Gamma(p)} - \frac{p\Gamma(-p)}{qt} + \frac{e^{tq}}{(-tq)^{p+1}} + \frac{p\gamma(p,-tq)}{tq} \right]$$

and

$$M_X(t) = \sum_{m=0}^{\infty} d_{k,j} J\left[\theta, (m+1)\alpha + 1, t \right],$$

where $\gamma(z,s) = \int_s^{\infty} y^{z-1} e^{-y} dy$ is the complementary incomplete gamma function and

$$d_{k,j} = \alpha \theta^{(m+1)\alpha} (-1)^m \sum_{k,j=0}^{\infty} \upsilon_{k,j} \left[(k+1)b + j \right] \binom{(k+1)b + j - 1}{m}.$$

4.3 Incomplete Moments

The s th incomplete moments, say $\varphi_s(t)$, is given by

$$\varphi_s(t) = \int_0^t x^s f(x)\,dx.$$

Using equation (3.2) and the lower incomplete beta function, we obtain (for $s \le \alpha$)

$$\varphi_s(t) = \theta^s \sum_{k,j=0}^{\infty} \upsilon_{k,j}\left[(k+1)b+j\right] B_t\left(\frac{\alpha-s}{\alpha},(k+1)b+j\right), \tag{4.2}$$

where $B_z(a,b) = \int_0^z w^{a-1}(1-w)^{b-1}\,dw$ is the incomplete beta function. The first incomplete moment of the TWPa distribution can be obtained by setting $s=1$ in (4.2).

The main application of the first incomplete moment refers to Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance and medicine. The answers to many important questions in economics require more than just knowing the mean of the distribution, but its shape as well. This is obvious not only in the study of econometrics but in other areas as well.

The important application of the first incomplete moment is related to the Lorenz and Bonferroni curves. These curves are very useful in economics, reliability, demography, insurance and medicine. The Lorenz curve, say $L_F(x)$, and Bonferroni curve, say $B\left[F(x)\right]$, are defined, respectively, by

$$L_F(x) = \frac{1}{E(X)}\int_0^x tf(t)\,dt \quad \text{and} \quad B\left[F(x)\right] = \frac{1}{E(X)F(x)}\int_0^x tf(t)\,dt = \frac{L_F(x)}{F(x)}.$$

Then,

$$L_F(x) = \frac{\sum_{k,j=0}^{\infty} \upsilon_{k,j} B_t\left(1-\frac{1}{\alpha},(k+1)b+j\right)}{\sum_{k,j=0}^{\infty} \upsilon_{k,j} B\left(1-\frac{r}{\alpha},(k+1)b+j\right)}$$

and

$$B\left[F(x)\right] = \frac{\sum_{k,j=0}^{\infty} \upsilon_{k,j} B_t\left(1-\frac{1}{\alpha},(k+1)b+j\right)}{F(x)\sum_{k,j=0}^{\infty} \upsilon_{k,j} B\left(1-\frac{r}{\alpha},(k+1)b+j\right)}.$$

Another application of the first incomplete moment is related to mean residual life and mean waiting time given by $m_1(t) = \left[1-\varphi_1(t)\right]/R(t)-t$ and $M_1(t) = t-\left[\varphi_1(t)/F(t)\right]$, respectively.

4.4 Residual Life Function

Several functions are defined related to the residual life. The failure rate function, mean residual life function and the left censored mean function, also called vitality function. It is well known that these three functions uniquely determine $F(x)$.

Moreover, the moments of the residual life, $m_n(t) = E[(X-t)^n \mid X > t]$, $n = 1, 2, ...$, uniquely determine $F(x)$. The nth moment of the residual life of X is given by

$$m_n(t) = \frac{1}{1-F(t)} \int_t^{\infty} (x-t)^n \, dF(x).$$

Then, we can write (for $r < \alpha$)

$$m_n(t) = \frac{1}{1-F(t)} \sum_{r=0}^{n} \binom{n}{r} (-t)^{n-r} \int_t^{\infty} x^r \, dF(x),$$

$$m_n(t) = \frac{1}{R(t)} \sum_{r=0}^{n} \sum_{k,j=0}^{\infty} \left[(k+1)b + j \right] (-t)^{r-n} \, \theta^r \upsilon_{k,j} \binom{n}{r}$$
$$\times \left\{ B\left(\frac{\alpha - r}{\alpha}, (k+1)b + j \right) - B_t\left(\frac{\alpha - r}{\alpha}, (k+1)b + j \right) \right\}.$$

Another interesting function is the mean residual life (MRL) function or the life expectation at age x defined by $m_1(x) = E\left[(X-x) \mid X > x\right]$, which represents the expected additional life length for a unit which is alive at age x. The MRL of the TWPa distribution can be obtained by setting $n = 1$ in the last equation.

4.5 Reversed Residual Life Function

The moments of the reversed residual life, $M_n(t) = E\left[(t-X)^n \mid X \le t\right]$ for $t > 0$, $n = 1, 2, ...$ uniquely determine $F(x)$. We obtain

$$M_n(t) = \int_0^t (t-x)^n \, dF(x).$$

Therefore, the nth moment of the reversed residual life of X given that $r < \alpha$ becomes

$$M_n(t) = \frac{1}{F(t)} \sum_{r=0}^{n} \binom{n}{r} (-1)^r \, t^{n-r} \int_0^t x^r \, dF(x),$$

$$M_n(t) = \frac{1}{F(t)} \sum_{r=0}^{n} \sum_{k,j=0}^{\infty} \binom{n}{r} \frac{(-\theta)^r}{t^{r-n}} \left[(k+1)b + j \right] \upsilon_{k,j} B_t\left(1 - \frac{r}{\alpha}, (k+1)b + j \right).$$

The mean inactivity time (MIT) or mean waiting time (MWT) also called the mean reversed residual life function is defined by $M_1(t) = E[(t-X) \mid X \le t]$, and it represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0, x)$. The MIT of X can be obtained by setting $n = 1$ in the last equation.

5.ORDER STATISTICS

If $X_1, X_2, ..., X_n$ is a random sample of size n from the TWPa distribution and $X_{(1)}, X_{(2)}, ..., X_{(n)}$ are the corresponding order statistics, then the pdf of ith order statistic denoted by $f_{i:n}(x)$ is given by

$$f_{i:n}(x) = \frac{f(x)}{\mathrm{B}(i, n-i+1)} \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} F^{i+j-1}(x). \tag{5.1}$$

Using Equations (2.1) (2.2) and (5.1), we can write

$$f(x) F(x)^{i+j-1} = \sum_{k,l=0}^{\infty} t_{k,l} h_{\alpha,\theta,(k+1)b+l}(x).$$

where

$$t_{k,l} = \sum_{m,w=0}^{\infty} \frac{(-1)^{k+m+w} b\lambda^m (1+\lambda)^{i+j-m-1} \Gamma((k+1)b+l+1)}{k!l![(k+1)b+l](w+1)^{-k} \Gamma((k+1)b+1)} \times \binom{i+j-1}{m} \binom{i+j+m-1}{w} \left[(1-\lambda)(w+1)^k + 2\lambda(w+2)^k\right].$$

The q th moment of $X_{i:n}$ is

$$E\left(X_{i:n}^q\right) = \sum_{k,l=0}^{\infty} \sum_{j=0}^{n-1} \frac{(-1)^j}{\mathrm{B}(i, n-i+1)} \binom{n-1}{j} t_{k,l} E\left[Y_{\alpha,\theta,(k+1)b+l}^q\right]. \tag{5.2}$$

Using (4.1), we can write

$$E\left(X_{i:n}^q\right) = \sum_{k,l=0}^{\infty} \sum_{j=0}^{n-1} \frac{(-1)^j \binom{n-1}{j}}{\mathrm{B}(i, n-i+1)} [(k+1)b+l] \theta^q B\left(1-\frac{q}{\alpha}, (k+1)b+l\right).$$

Based upon the moments in Equation (5.2), we can derive explicit expressions for the L-moments of X as infinite weighted linear combinations of the means of suitable TWPa distribution. They are linear functions of expected order statistics defined by

$$\lambda_r = \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} E(X_{r-k:r}), r \geq 1.$$

The first four L-moments are given by: $\lambda_1 = E(X_{1:1})$, $\lambda_2 = \frac{1}{2} E(X_{2:2} - X_{1:2})$, $\lambda_3 = \frac{1}{3} E(X_{3:3} - 2X_{2:3} + X_{1:3})$ and $\lambda_4 = \frac{1}{4} E(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4})$. One simply can obtain the λ's for X from equation (5.2) with $q = 1$.

6. PROBABILITY WEIGHTED MOMENTS

The PWMs are expectations of certain functions of a random variable. They can be derived for any random variable whose ordinary moments exist. The PWM approach can be used for estimating parameters of any distribution whose inverse form cannot be expressed explicitly.

The (s,r) th PWM of X, say $\rho_{s,r}$, is defined by

$$\rho_{s,r} = E\left\{X^s F(X)^r\right\} = \int_{-\infty}^{\infty} x^s F(x)^r f(x)dx.$$

Using Equations (2.1) and (2.2), we can write

$$f(x)F(x)^r = \sum_{m,w,k,j=0}^{\infty} \frac{b(-1)^{k+m+w}(w+1)^k \Gamma\left((k+1)b+j+1\right)}{k!j!\Gamma\left((k+1)b+1\right)\left[(k+1)b+j\right]} \lambda^m (1+\lambda)^{r-m} \binom{r}{m}\binom{r+m}{w}$$

$$\times\left[(1-\lambda)(w+1)^k + 2\lambda(w+2)^k\right]\left[(k+1)b+j\right]\frac{\alpha}{x}\left(\frac{x}{\theta}\right)^{-\alpha}\left[1-\left(\frac{x}{\theta}\right)^{-\alpha}\right]^{(k+1)b+j-1}.$$

Then, we have

$$f(x)F(x)^r = \sum_{k,j=0}^{\infty} d_{k,j} h_{\alpha,\theta,(k+1)b+j}(x),$$

where

$$d_{k,j} = \sum_{m,w=0}^{\infty} \frac{(-1)^{k+m+w} b\lambda^m (1+\lambda)^{r-m} \Gamma\left((k+1)b+j+1\right)}{k!j!\left[(k+1)b+j\right](w+1)^{-k} \Gamma\left((k+1)b+1\right)}$$

$$\times\binom{r}{m}\binom{r+m}{w}\left[(1-\lambda)(w+1)^k + 2\lambda(w+2)^k\right].$$

Then, the (s,r) th PWM of X can be expressed as

$$\rho_{s,r} = \sum_{k,j=0}^{\infty} d_{k,j} E\left[Y^s_{\alpha,\theta,(k+1)b+j}\right].$$

Using (4.1), we can write

$$\rho_{s,r} = \sum_{k,j=0}^{\infty} d_{k,j}\left[(k+1)b+j\right]\theta^s B\left(1-\frac{s}{\alpha},(k+1)b+j\right).$$

7. CHARACTERIZATIONS

The problem of characterizing a distribution is an important problem in various fields which has recently attracted the attention of many researchers. These characterizations have been established in many different directions. This section deals with two characterizations of TWPa distribution. These characterizations are based on (i) a simple relationship between two truncated moments and (ii) on conditional expectation of a function of the random variable. It should be mentioned that for our characterization (i), the cdf need not have a closed form. We believe, due to the nature of the cdf of TWPa, there may not be other possible characterizations of this distribution than the ones presented here.

7.1 Characterizations Based on Two Truncated Moments

In this subsection we present characterizations of TWPa distribution in terms of a simple relationship between two truncated moments. Our first characterization result borrows from a theorem due to Glänzel (1987), see Theorem A below. We refer the interested reader to Glänzel (1990) for a proof of Theorem A. Note that the result holds also when the interval H is not closed. Moreover, as mentioned above, it could be also applied when the cdf F does not have a closed form. As shown in Glänzel (1990), this characterization is stable in the sense of weak convergence.

Theorem 7.1.

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a given probability space and let $H = [d, e]$ be an interval for some $d < e$ ($d = -\infty$, $e = \infty$ might as well be allowed). Let $X : \Omega \to H$ be a continuous random variable with the distribution function F and let g and h be two real functions defined on H such that

$$\mathrm{E}\left[g(X) \mid X \geq x\right] = \mathrm{E}\left[h(X) \mid X \geq x\right]\eta(x), \quad x \in H,$$

is defined with some real function η. Assume that g, $h \in C^1(H)$, $\eta \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $h\eta = g$ has no real solution in the interior of H. Then F is uniquely determined by the functions g, h and η, particularly

$$F(x) = \int_a^x C \left| \frac{\eta'(u)}{\eta(u)h(u) - g(u)} \right| \exp(-s(u))\, du,$$

where the function s is a solution of the differential equation $s' = \eta' h / (\eta h - g)$ and C is the normalization constant, such that $\int_H dF = 1$.

Proposition 7.1.

Let $X : \Omega \to (\theta, \infty)$ be a continuous random variable and let

$$h(x) = \left[1 - \lambda + 2\lambda \exp\left\{-\left[\left(\frac{x}{\theta}\right)^{\alpha} - 1\right]^{b}\right\}\right]^{-1}$$

and

$$g(x) = h(x)\exp\left\{-\left[\left(\frac{x}{\theta}\right)^{\alpha} - 1\right]^{b}\right\} \text{for } x > \theta.$$

The random variable X belongs to TWPa family (2.2) if and only if the function η defined in Theorem 7.1 has the form

$$\eta(x) = \frac{1}{2}\exp\left\{-\left[\left(\frac{x}{\theta}\right)^{\alpha} - 1\right]^{b}\right\}, \quad x > \theta. \tag{7.1}$$

Proof.

Let X be a random variable with density (2.2), then

$$(1 - F(x))E[h(x) \mid X \geq x] = \exp\left\{-\left[\left(\frac{x}{\theta}\right)^{\alpha} - 1\right]^{b}\right\}, x > \theta,$$

and

$$(1 - F(x))E[g(x) \mid X \geq x] = \frac{1}{2}\exp\left\{-2\left[\left(\frac{x}{\theta}\right)^{\alpha} - 1\right]^{b}\right\}, \quad x > \theta,$$

and finally

$$\eta(x)h(x) - g(x) = -\frac{1}{2}h(x)\exp\left\{-\left[\left(\frac{x}{\theta}\right)^{\alpha} - 1\right]^{b}\right\} < 0, \text{ for } x > \theta.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)h(x)}{\eta(x)h(x) - g(x)} = b\alpha\theta^{-\alpha}x^{\alpha-1}\left[\left(\frac{x}{\theta}\right)^{\alpha} - 1\right]^{b-1}, x > \theta,$$

and hence

$$s(x) = \left[\left(\frac{x}{\theta}\right)^{\alpha} - 1\right]^{b}, \quad x > \theta.$$

Now, in view of Theorem 7.1, X has density (2.2).

Corollary 7.1.

Let $X : \Omega \to (\theta, \infty)$ be a continuous random variable and let $h(x)$ be as in Proposition 7.1. The pdf of X is (6) if and only if there exist functions g and η defined in Theorem 7.1 satisfying the differential equation

$$\frac{\eta'(x)h(x)}{\eta(x)h(x)-g(x)} = b\alpha\theta^{-\alpha}x^{\alpha-1}\left[\left(\frac{x}{\theta}\right)^{\alpha}-1\right]^{b-1}, \; x>\theta. \tag{7.2}$$

The general solution of the differential equation in Corollary 7.1 is

$$\eta(x) = \exp\left\{\left[\left(\frac{x}{\theta}\right)^{\alpha}-1\right]^{b}\right\}\left\{\begin{array}{l} -\int b\alpha\theta^{-\alpha}x^{\alpha-1}\left[\left(\frac{x}{\theta}\right)^{\alpha}-1\right]^{b-1} \\ \qquad \times\exp\left\{-\left[\left(\frac{x}{\theta}\right)^{\alpha}-1\right]^{b}\right\}\left(h(x)\right)^{-1}g(x)\,dx+D \end{array}\right\},$$

where D is a constant. Note that a set of functions satisfying the differential equation (7.2) is given in Proposition 7.1 with $D=0.$ However, it should be also noted that there are other triplets (h,g,η) satisfying the conditions of Theorem 7.1.

7.2 Characterizations Based on Conditional Expectation of a Function of the Random Variable

In this subsection we present a characterization result in terms of a function of the random variable X. The following Proposition has appeared in our previous work which will be used to characterize TWPa distribution

Proposition 7.2.

Let $X:\Omega\to(a,b)$ be a continuous random variable with cdf F and corresponding pdf f. Let $\psi(x)$ be a differentiable function greater than 1 on (a,b) such that $\lim_{x\to a^{+}}\psi(x)=1$ and $\lim_{x\to b^{-}}\psi(x)=1+c$. Then, for $0<c<1$,

$$E\left[\psi(X)\mid X\le x\right]=c+(1-c)\psi(x), \tag{7.3}$$

if and only if

$$F(x)=\left(\frac{\psi(x)-1}{c}\right)^{\frac{1-c}{c}}. \tag{7.4}$$

Remark 7.1.

Taking, e.g., $(a,b)=(\theta,\infty)$ and

$$\psi(x)=1+c\left\{\left(1-\exp\left\{-\left[\left(\frac{x}{\theta}\right)^{\alpha}-1\right]^{b}\right\}\right)\left(1+\lambda\exp\left\{-\left[\left(\frac{x}{\theta}\right)^{\alpha}-1\right]^{b}\right\}\right)\right\}^{\frac{c}{1-c}}.$$

Proposition 7.3 gives a characterization of TWPa distribution.

8. ESTIMATION

The maximum likelihood estimators (MLEs) for the parameters of the TWPa distribution are discussed in this section. Let $x_1,...,x_n$ be a random sample of of size n from the TWPa($x;\upsilon$) distribution, where υ is the unknown parameter vector $\upsilon=\left(\alpha,\theta,b,\lambda\right)^T$.

Then, the log-likelihood function for the parameters vector υ, say ℓ, can be expressed as

$$\ell = n\ln b+n\ln\alpha-n\alpha\ln\theta+\left(\alpha-1\right)\sum_{i=1}^{n}\ln x_i-\sum_{i=1}^{n}s_i^b$$
$$+\left(b-1\right)\sum_{i=1}^{n}\ln s_i+\sum_{i=1}^{n}\ln\left(1-\lambda+2\lambda p_i\right),$$

where $s_i=\left(\frac{x_i}{\theta}\right)^{\alpha}-1$ and $p_i=e^{-s_i^b}$.

Assuming θ known, therefore the score vector components, $\mathrm{U}\left(\upsilon\right)=\frac{\partial\ell}{\partial\upsilon}=\left(\frac{\partial\ell}{\partial b},\frac{\partial\ell}{\partial\alpha},\frac{\partial\ell}{\partial\lambda}\right)^T$ are given by

$$\frac{\partial\ell}{\partial\alpha}=\frac{n}{\alpha}-n\ln\theta+\sum_{i=1}^{n}\ln x_i+\left(b-1\right)\sum_{i=1}^{n}\frac{z_i}{s_i}-b\sum_{i=1}^{n}z_i s_i^{b-1}-2\lambda b\sum_{i=1}^{n}\frac{z_i\, p_i\, s_i^{b-1}}{1-\lambda+2\lambda p_i},$$

$$\frac{\partial\ell}{\partial b}=\frac{n}{b}+\sum_{i=1}^{n}\ln s_i-\sum_{i=1}^{n}s_i^b\ln s_i-\sum_{i=1}^{n}\frac{p_i s_i^b\ln s_i}{1-\lambda+2\lambda p_i}$$

and

$$\frac{\partial\ell}{\partial\lambda}=\sum_{i=1}^{n}\frac{2p_i-1}{1-\lambda+2\lambda p_i},$$

where $z_i=\left(\frac{x_i}{\theta}\right)^{\alpha}\ln\left(\frac{x_i}{\theta}\right)$.

We can find the estimates of the unknown parameters by setting the score vector to zero, $\mathrm{U}\left(\hat{\upsilon}\right)=0$, and solving them simultaneously to obtain the ML estimators $\hat{\alpha},\hat{b}$ and $\hat{\lambda}$. These equations cannot be solved analytically and statistical software can be used to solve them numerically by means of iterative techniques such as the Newton-Raphson algorithm. For the four-parameter TWPa distribution all the second order derivatives exist.

For interval estimation of the model parameters, we require the 3×3 observed information matrix, $J\left(\upsilon\right)=\left\{J_{rs}\right\}\left(\text{for } r,s=\alpha,b,\lambda\right)$, whose elements are given by

$$J_{\alpha\alpha} = \frac{-n}{\alpha^2} + (b-1)\sum_{i=1}^{n}\frac{z_i}{s_i}\left[\ln\left(\frac{x_i}{\theta}\right)-1\right] - b\sum_{i=1}^{n}\left[(b-1)z_i s_i^{b-1} + s_i^b \ln\left(\frac{x_i}{\theta}\right)\right]$$

$$-2\lambda b\sum_{i=1}^{n}\frac{p_i\left[(b-1)z_i s_i^{b-1} + s_i^b \ln\left(\frac{x_i}{\theta}\right)\right] - bz_i p_i s_i^{2b-1}}{(1-\lambda+2\lambda p_i)^2},$$

$$J_{\alpha b} = \sum_{i=1}^{n}\frac{z_i}{s_i} - \sum_{i=1}^{n} z_i s_i^{b-1}(1+b\ln s_i) - (2\lambda)^2\sum_{i=1}^{n}\frac{bz_i p_i^2 s_i^{2b-1}\ln s_i}{(1-\lambda+2\lambda p_i)^2}$$

$$-2\lambda\sum_{i=1}^{n}\frac{z_i p_i s_i^{b-1}\left(1+b\ln s_i - bs_i^b \ln s_i\right)}{1-\lambda+2\lambda p_i},$$

$$J_{\alpha\lambda} = 2b\sum_{i=1}^{n}\frac{2\lambda z_i p_i^2 s_i^{b-1} - (1-\lambda+2\lambda p_i) z_i p_i s_i^{b-1}}{(1-\lambda+2\lambda p_i)^2},$$

$$J_{bb} = \frac{-n}{b^2} - \sum_{i=1}^{n} s_i^b (\ln s_i)^2 - \sum_{i=1}^{n}\frac{p_i s_i^b (\ln s_i)^2 (1-s_i^b)}{1-\lambda+2\lambda p_i} - 2\lambda\sum_{i=1}^{n}\frac{p_i^2 s_i^{2b}(\ln s_i)^2(1-s_i^b)}{(1-\lambda+2\lambda p_i)^2},$$

$$J_{b\lambda} = \sum_{i=1}^{n}\frac{-(2p_i-1)^2}{(1-\lambda+2\lambda p_i)^2} \quad \text{and} \quad J_{\lambda\lambda} = \sum_{i=1}^{n}\frac{-(2p_i-1)^2}{(1-\lambda+2\lambda p_i)^2}.$$

Under standard regularity conditions, the multivariate normal $N_3\left(0, J(\hat{\upsilon})^{-1}\right)$ distribution can be used to construct approximate confidence intervals for the model parameters. Here, $J(\hat{\upsilon})$ is the total observed information matrix evaluated at $\hat{\upsilon}$. Therefore, approximate $100(1-\phi)\%$ confidence intervals for α, b and λ can be determined as:

$\hat{\alpha} \pm Z_{\frac{\phi}{2}}\sqrt{\hat{J}_{\alpha\alpha}}$, $\hat{\lambda} \pm Z_{\frac{\phi}{2}}\sqrt{\hat{J}_{\lambda\lambda}}$, and $\hat{b} \pm Z_{\frac{\phi}{2}}\sqrt{\hat{J}_{bb}}$, where $Z_{\frac{\phi}{2}}$ is the upper ϕ th percentile of the standard normal distribution.

9. SIMULATION STUDY

Here, we assess the performance of the maximum likelihood estimation procedure for estimating the TWPa parameters using Monte Carlo simulation. An ideal technique for simulating from (5) is the inversion method. For different combinations of α, θ, b and λ samples of sizes $n =$ 100, 200, 500 and 1000 are generated from the TWPa model. We repeated the simulation $k = 100$ times and calculated the MLEs and the standard deviations of the parameter estimates. We use three combinations for the parameter values (I: $\alpha = \theta = b = 1.5$ and $\lambda = -1$, II: $\alpha = 1.5$, $\theta = b = 2$ and $\lambda = 0$ and III: $\alpha = \lambda = 1$, $\theta = 2$ and $b = 3$). The empirical results are given in Table 1. It is evident that the estimates are quite stable and are close to the true value of the parameters for these sample sizes. Additionally, as the sample size increases, the biases and the standard deviations of the MLEs decrease as expected.

Table 1
MLEs and Standard Deviations for various Parameter Values

Sample size (n)	**Estimated Values (Standard Deviations)**			
	$\hat{\alpha}$	$\hat{\theta}$	$\hat{b}$	$\hat{\lambda}$
I	1.498091143	1.502948857	1.501953638	–0.997782415
	(0.11582947)	(0.172767349)	(0.243121613)	(0.144500353)
100	1.491903012	1.48957008	1.527788397	–0.999536076
	(0.199914811)	(0.301097609)	(0.381782447)	(0.199173712)
200	1.491958427	1.501891389	1.509888485	–0.992963765
	(0.149228829)	(0.198243384)	(0.309084222)	(0.20302651)
500	1.502175325	1.497094523	1.499390957	–0.994431539
	(0.119792874)	(0.179465332)	(0.278695995)	(0.159880471)
1000	1.497894409	1.507425395	1.499064534	–1.000246218
	(0.098759361)	(0.151490124)	(0.198275817)	(0.119637727)
II	1.500885689	2.00032535	1.997983274	–0.003727487
	(0.121687918)	(0.228695465)	(0.139224288)	(0.145980986)
100	1.50426267	1.974097335	1.97696601	–0.014608287
	(0.288560488)	(0.519928598)	(0.413326812)	(0.521138173)
200	1.495429289	2.00483308	1.983424014	–0.017654112
	(–0.017654112)	(0.290681421)	(0.305144854)	(0.300017145)
500	1.502350282	1.98916705	2.00784619	–0.004494032
	(0.10061869)	(0.204113398)	(0.097860177)	(0.1008336)
1000	1.500906974	2.007625756	1.998065395	0.000529191
	(0.100826821)	(0.199465995)	(0.099311977)	(0.100231728)
III	1.002653476	2.004604317	2.997469654	1.009583677
	(0.270238689)	(0.175851467)	(0.300490017)	(0.182163635)
100	0.972887849	1.990449446	2.98928733	0.967372304
	(0.490687527)	(0.300195793)	(0.960050826)	(0.487313015)
200	0.998436465	2.058020528	2.977709622	1.112873677
	(0.421358094)	(0.187659576)	(0.499413199)	(0.403941241)
500	1.021321137	1.999576707	3.011564071	0.990865319
	(0.299676141)	(0.100350456)	(0.301259786)	(0.199220749)
1000	0.997139611	1.997850366	2.995192685	1.002505993
	(0.203251199)	(0.198805918)	(0.194364415)	(0.09876462)

The bold values are combined results of (100 + 200 + 500 + 1000 = 1800).

10. APPLICATION

In this section, we provide an application of the TWPa distribution to show its flexibility and importance. We shall compare the TWPa model with other related models, namely Weibull Pareto (WPa), McDonald Lomax (McL) (Lemonte and Cordeiro 2013), transmuted Weibull Lomax (TWL) (Afify et al. 2015) and transmuted complementary Weibull geometric (Afify et al. 2014) distributions. The density functions (for $x > 0$) associated to these models are given by

• The McL pdf given by

$$f(x) = \frac{\alpha\gamma}{\theta B\left(a\gamma^{-1}, b\right)}\left(1+\frac{x}{\theta}\right)^{-(\alpha+1)}\left[1-\left(1+\frac{x}{\theta}\right)^{-\alpha}\right]^{a-1}\left\{1-\left[1-\left(1+\frac{x}{\theta}\right)^{-\alpha}\right]^{\gamma}\right\}^{b-1}.$$

• The TWL pdf given by

$$f(x) = \frac{ab\alpha}{\theta}\left(1+\frac{x}{\theta}\right)^{b\alpha-1}\exp\left\{-a\left[\left(1+\frac{x}{\theta}\right)^{\alpha}-1\right]^{b}\right\}$$

$$\times\left[1-\left(1+\frac{x}{\theta}\right)^{-\alpha}\right]^{b-1}\left(1-\lambda+2\lambda\exp\left\{-a\left[\left(1+\frac{x}{\theta}\right)^{\alpha}-1\right]^{b}\right\}\right).$$

• The TCWG pdf given by

$$f(x) = \alpha\theta\beta(\beta x)^{\theta-1}e^{-(\beta x)^{\theta}}\left[\alpha+(1-\alpha)e^{-(\beta x)^{\theta}}\right]^{-3}\left[\alpha(1-b)-(\alpha-\alpha\lambda-\lambda-1)e^{-(\beta x)^{\theta}}\right].$$

The parameters of the above densities are all positive real numbers except for the TWL and TCWG distributions for which $|\lambda| \le 1$.

We make use of the data set of gauge lengths of 10 mm from Kundu and Raqab (2009). This data set consists of 63 observations: 1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020. These data have been used by Afify et al. (2015) to fit the exponentiated transmuted generalized Rayleigh distribution.

We consider some criteria like $-2\hat{\ell}$ (where $\hat{\ell}$ is the maximized log-likelihood), *AIC* (Akaike information criterion) and *CAIC* (the consistent Akaike information criterion), *HQIC* (Hannan-Quinn information criterion) and *BIC* (Bayesian information criterion) in order to compare the distributions. In general, the smaller the values of these statistics, the better the fit to the data.

Table 2 provides the MLEs and their corresponding standard errors (SEs) of the model parameters and the numerical values of the $-2\hat{\ell}$, AIC, $CAIC$, $HQIC$ and BIC. These numerical results are obtained using the Mathcad program.

It is shown from Table 2 that the TWPa model has the lowest values for the $-2\hat{\ell}$, AIC, $CAIC$, $HQIC$ and BIC statistics among all fitted models (except for $HQIC$ and BIC of the WPa distribution). So, the TWPa model could be chosen as the best model. Figure 4 displays the fitted pdf and cdf for the TWPa model and Figure 5 displays the QQ-plot and estimated survival function of the TWPa distribution. It is clear from these plots that the TWPa provides good fit to this data set.

Table 2
MLEs, their Standard Errors (SEs) and Goodness-of-Fit Statistics

Model	**Estimates**	SEs	$-2\hat{\ell}$	AIC	$CAIC$	$HQIC$	BIC
TWPa	$\hat{\alpha} =$ 0.1885	0.065	119.282	127.282	127.972	130.654	135.855
	$\hat{\theta} =$ 0.0909	0.115					
	$\hat{b} =$ 14.4535	5.252					
	$\hat{\lambda} =$ 0.7288	0.284					
WPa	$\hat{\alpha} =$ 0.1834	0.102	121.790	127.790	128.197	130.319	134.219
	$\hat{\theta} =$ 0.0755	0.159					
	$\hat{b} =$ 13.9522	7.705					
TWL	$\hat{\alpha} =$ 0.3922	0.339	119.688	129.688	130.741	133.903	140.404
	$\hat{\theta} =$ 0.6603	1.174					
	$\hat{\alpha} =$ 0.5287	3.32					
	$\hat{b} =$ 8.4451	4.397					
	$\hat{\lambda} =$ 0.7364	0.286					
McL	$\hat{\alpha} =$ 45.9249	59.312	130.597	140.597	141.65	144.812	151.313
	$\hat{\theta} =$ 48.3024	63.047					
	$\hat{\alpha} =$ 18.1192	8.855					
	$\hat{b} =$ 195.4633	123.217					
	$\hat{\gamma} =$ 353.1435	375.678					
TCWG	$\hat{\alpha} =$ 0.2022	0.217	126.895	134.895	135.585	138.267	143.468
	$\hat{\theta} =$ 3.3482	0.783					
	$\hat{\beta} =$ 0.3876	0.069					
	$\hat{\lambda} =$ -0.0001	0.496					

11. CONCLUSIONS

In this paper, We propose a new four-parameter model, called the *transmuted Weibull-Pareto* (TWPa) distribution, which extends the Weibull-Pareto (WPa) distribution introduced by Tahir et al. (2015). We provide some of its mathematical properties. The TWPa density function can be expressed as a mixture of exponentiated Pareto densities. We derive explicit expressions for the ordinary and incomplete moments, quantile and generating functions, Lorenz, Bonferroni and Zenga curves, moments of residual life and moments of the reversed residual life. We also obtain the density function of order statistics and their moments. Further, the Probability weighted moments are investigated and certain characterization results are provided. We discuss the maximum likelihood estimation of the model parameters. The proposed distribution is applied to a real data set and provides a better fit than several nested and non-nested models.

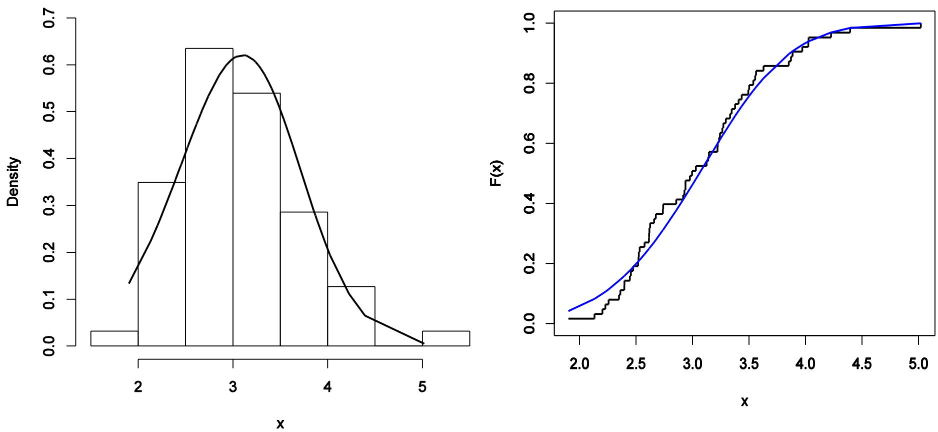

Figure 4: Estimated pdf and cdf of the TWPa Distribution

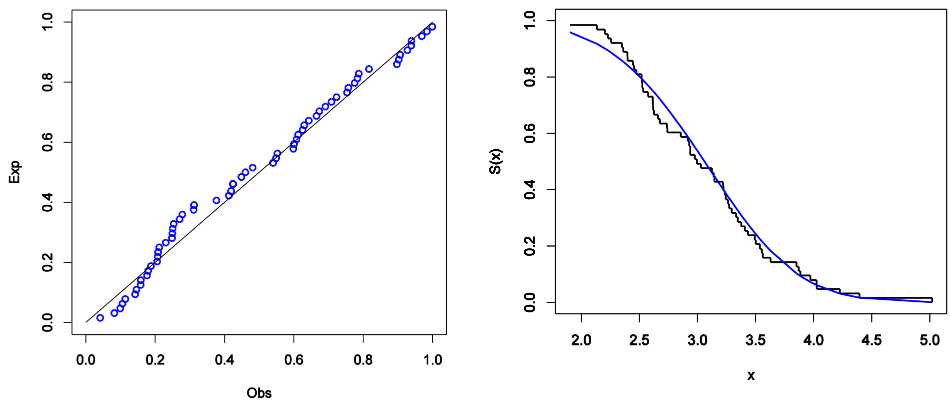

Figure 5: QQ-Plot and Estimated Survival Function of the TWPa Distribution

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